THE WEAK HARNACK INEQUALITY FOR UNBOUNDED SUPERSOLUTIONS OF EQUATIONS WITH GENERALIZED ORLICZ GROWTH

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ABSTRACT. We study unbounded weak supersolutions of elliptic partial differential equations with generalized Orlicz (Musielak–Orlicz) growth. We show that they satisfy the weak Harnack inequality with optimal exponent provided that they belong to a suitable Lebesgue or Sobolev space. Furthermore, we establish the sharpness of our central assumptions.

1. INTRODUCTION

We prove the weak Harnack inequality for unbounded supersolutions of partial differential equations with generalized Orlicz growth (also known as Musielak–Orlicz growth). A general principle states that only intrinsic Harnack inequalities are possible for PDEs which are not scaling invariant, and we do indeed find that the constant in the weak Harnack inequality depends on the norm of the supersolution. With dependence on the L^s - or $W^{1,s}$ -norms, our result requires that s be sufficiently large, namely, $s \ge \max\{s_1, s_2\}$, where s_1 depends continuity of the generalized Orlicz functional and is shown to be sharp and s_2 depends on the global growth of the functional and does not occur in any previously known special cases. The result is new even for the special case of double phase functionals [6]. Our framework includes also the following special cases in which the weak Harnack inequality was not known before, even for bounded solutions: perturbed variable exponent [20, 36, 42, 43], Orlicz variable exponent [10, 21], degenerate double phase [5, 8], Orlicz double phase [9], variable exponent double phase [40, 46], triple phase [17], and double variable exponent [11, 45, 49].

Let us introduce the context of this paper. Minimizers and (weak) solutions of

$$\inf \int_{\Omega} F(x, \nabla u) \, dx$$
 and $-\operatorname{div}(f(x, \nabla u)) = 0$

have been actively studied during recent years when F or f has generalized Orlicz growth. For instance, solutions with given boudary values exist [14, 22, 25], minimizers or solutions with given boundary values are locally bounded, satisfy Harnack's inequality and belong to $C_{\text{loc}}^{0,\alpha}$ [7, 29, 48] or $C_{\text{loc}}^{1,\alpha}$ [32], quasiminimizers satisfy a reverse Hölder inequality [26], ω -minimizers are locally Hölder continuous [28], minimizers for the obstacle problem are continuous [34] and the boundary Harnack inequality holds for harmonic functions [13]. In most cases the assumptions in these results coincides with optimal assumptions in wellknown special cases. The important special cases are the variable exponent case $F(x,\xi) :=$ $|\xi|^{p(x)}$ [1, 27, 30, 33, 47], the Orlicz case $F(x,\xi) := \varphi(|\xi|)$ [2, 37, 38], and the double phase case $F(x,\xi) := |\xi|^p + a(x)|\xi|^q$ [4, 15, 16]. The surveys [12, 39] include more references of variational problems and partial differential equations of generalized Orlicz growth, while the recent monographs [23, 35] present the theory of the underlying function spaces.

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In [7], the weak Harnack inequality for bounded supersolutions and Harnack's inequality for solutions were proved by Moser's iteration in the generalized Orlicz case. In those results the constants depend on L_{loc}^{∞} -norm of the function. In [4], Harnack's inequality has been proved in the double phase case, and the constant depends on L_{loc}^{∞} -norm of the function.

We want to study unbounded supersolutions, so we need the constants not to depend on the L^{∞} -norm. In the variable exponent case, the constant in the weak Harnack inequality depends on L_{loc}^{t} -norm of the function and t > 0 can be chosen arbitrarily small; furthermore, an example shows that the constant in Harnack's inequality cannot be independent of the function, in contrast to the constant exponent case [30]. We extend these results to the generalized Orlicz case with sharp assumptions on the continuity of φ .

We assume that $f: \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following φ -growth conditions:

(1.1)
$$\nu\varphi(x,|\xi|) \leq f(x,\xi) \cdot \xi \text{ and } |f(x,\xi)| |\xi| \leq \Lambda\varphi(x,|\xi|)$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, and fixed but arbitrary constants $0 < \nu \leq \Lambda$. We are interested in local (weak) supersolutions:

Definition 1.2. A function $u \in W^{1,\varphi}_{\text{loc}}(\Omega)$ is a *supersolution* if

(1.3)
$$\int_{\Omega} f(x, \nabla u) \cdot \nabla h \, dx \ge 0,$$

for all non-negative $h \in W^{1,\varphi}(\Omega)$ with compact support in Ω .

We define the limiting exponent

$$\ell(p) := \frac{p^*}{p'} = \begin{cases} \frac{n}{n-p}(p-1) & \text{if } p < n, \\ \infty & \text{if } p \ge n; \end{cases}$$

the ratio of the Sobolev exponent p^* and the Hölder exponent p'. Since $u_p(x) := |x|^{-\frac{n-p}{p-1}}$ is a supersolution of the *p*-Laplace equation $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ in the unit ball and since $u_p \notin L^{\ell(p)}(B_1)$, we see that $\ell(p)$ is an upper bound on the exponent of integrability of *p*-supersolutions.

The following is a special case of our main result, Theorem 3.9 together with Propositions 4.1 and 4.2. Theorem 3.9 contains also the cases $||u||_{L^{\omega}(B_{2R})} \leq d$ and $||u||_{W^{1,\omega}(B_{2R})} \leq d$ for general $\omega \in \Phi_{w}(\Omega)$. The last statement in the next theorem follows from the example in Section 5. Note that with the choice $s = \infty$ we recover as a special case previously known results for bounded solutions [7] with the correct, (A1-n) assumption.

Theorem 1.4 (The weak Harnack inequality). Suppose φ satisfies (A0), $(aInc)_p$ and $(aDec)_q$, $1 . Let u be a non-negative supersolution to (1.3) in <math>B_{2R}$. Assume that one of the following holds:

- (1) φ satisfies (A1- s_*) and $||u||_{L^s(B_{2R})} \leq d$, where $s_* := \frac{ns}{n+s}$ and $s \in [q-p,\infty]$.
- (2) φ satisfies (A1) and $||u||_{W^{1,\varphi}(B_{2B})} \leq d$.

Then there exist positive constants ℓ_0 and C such that the weak Harnack inequality holds:

$$\left(\int_{B_{2R}} (u+R)^{\ell_0} dx\right)^{\frac{1}{\ell_0}} \leqslant C(\operatorname{essinf}_{B_R} u+R)$$

If (1) holds with $s > \max\{\frac{n}{p}, 1\}(q-p)$ or if (2) holds with $p^* > q$, then the weak Harnack inequality holds for any $\ell_0 < \ell(p)$.

The (A1- s_*) assumption is sharp, since for any $s' < s_*$ if, instead of (1), φ satisfies (A1-s') and $||u||_{L^s(B_{2R})} \leq d$, then the weak Harnack inequality need not hold.

The next result follows by Corollary 5.1 and the counter-example given in Section 5. Corresponding corollaries could be formulated in the other cases of double phase type functionals listed in the beginning of this section.

Corollary 1.5. Let $\varphi(x,t) := t^p + a(x)t^q$ be the double phase functional with $a^{\lambda} \in C^{0,\alpha\lambda}(\Omega)$ for some $\lambda > 0$. Let u be a non-negative supersolution to (1.3) in B_{2R} . If $u \in L^s(\Omega)$ with

 $\alpha \ge (\frac{n}{s}+1)(q-p)$ and $s \in [q-p,\infty],$

then there exist positive constants ℓ_0 and C such that

$$\left(\int_{B_{2R}} (u+R)^{\ell_0} dx\right)^{\frac{1}{\ell_0}} \leqslant C(\operatorname{ess\,inf}_{B_R} u+R).$$

If additionally $s > \max\{\frac{n}{p}, 1\}(q-p)$, then the inequality holds for any $\ell_0 < \ell(p)$.

The bound on α is sharp, since for every $\alpha < (\frac{n}{s}+1)(q-p)$, there exists $a \in C^{0,\alpha}(\Omega)$ for which the weak Harnack inequality does not hold.

As the example in Section 5 shows, the assumption $\alpha \ge (\frac{n}{s} + 1)(q - p)$ is sharp and this restriction has been previously encountered in [44], see also [6]. The assumption $s \ge q - p$ is a consequence of considering supersolutions as it can be omitted if u is a solution. It is especially interesting to note that such a restriction does not occur in the variable exponent case and it is another example that the double phase functional is more subtle than the variable exponent case [15].

Remark 1.6. Compared to the classical Harnack inequality, our estimate contains an extra +R-term. It is not know whether this is necessary, but the same phenomenon occurs when $\varphi(x,t) = t^{p(x)}$ [1, 30, 31], unless the exponent p is assumed to belong to C^1 and slightly different Harnack's inequality is used [33]. In the Orlicz and the double phase cases the extra +R term is not needed [2, 4, 38].

Remark 1.7. In the case of the basic double phase functional, namely, when $\lambda = 1$ in Corollary 1.5, the assumption $\alpha \ge (\frac{n}{s} + 1)(q - p)$ implies $s \ge n(q - p) \ge q - p$ and $s > \frac{n}{p}(q - p)$, since $\alpha \le 1$ and $q \ge 1$. Mizuta, Ohno and Shimomura [41] (see also [24]) have considered the functional $\varphi(x, t) := t^p + (a(x)t)^q$, which corresponds to the case $\lambda = \frac{1}{q}$ above. Also in this case, the first assumption implies the latter two. However, if $\lambda < \frac{1}{q}$, then the first condition can hold while the others do not.

Note that the parameter λ does not impact the restrictions in Corollary 1.5. The reason for this is that only the growth of a at a = 0 is important, see [23, Proposition 7.2.2].

2. PRELIMINARIES

We briefly introduce our assumptions. More information about L^{φ} -spaces can be found in [23]. Almost increasing means that a function satisfies $f(s) \leq Lf(t)$ for all s < t and some constant $L \geq 1$. If there exists a constant C such that $f(x) \leq Cg(x)$ for almost every x, then we write $f \leq g$. If $f \leq g \leq f$, then we write $f \approx g$.

Definition 2.1. We say that $\varphi : \Omega \times [0, \infty) \to [0, \infty]$ is a *weak* Φ -function, and write $\varphi \in \Phi_w(\Omega)$, if the following conditions hold:

- For every measurable function $f : \Omega \to \mathbb{R}$ the function $x \mapsto \varphi(x, f(x))$ is measurable and for every $x \in \Omega$ the function $t \mapsto \varphi(x, t)$ is non-decreasing.
- $\varphi(x,0) = \lim_{t \to 0^+} \varphi(x,t) = 0$ and $\lim_{t \to \infty} \varphi(x,t) = \infty$ for every $x \in \Omega$.
- The function $t \mapsto \frac{\varphi(x,t)}{t}$ is *L*-almost increasing on $(0,\infty)$ with *L* independent of *x*.

If $\varphi \in \Phi_{w}(\Omega)$ is additionally convex and left-continuous, then φ is a *convex* Φ -function, and we write $\varphi \in \Phi_{c}(\Omega)$. If φ does not depend on x, then we omit the set and write $\varphi \in \Phi_{w}$ or $\varphi \in \Phi_{c}$.

We denote $\varphi_A^+(t) = \operatorname{ess\,sup}_{x \in A \cap \Omega} \varphi(x, t)$ and $\varphi_A^-(t) = \operatorname{ess\,inf}_{x \in A \cap \Omega} \varphi(x, t)$. We define several conditions. See Table 1 for an intuition of their meaning in special cases. Let p, q, s > 0 and let $\omega : \Omega \times [0, \infty) \to [0, \infty)$ be almost increasing. We say that $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ satisfies

(A0) if there exists $\beta \in (0, 1]$ such that $\beta \leq \varphi^{-1}(x, 1) \leq \frac{1}{\beta}$ for a.e. $x \in \Omega$, (A1- ω) if there exists $\beta \in (0, 1]$ such that, for every ball B and a.e. $x, y \in B \cap \Omega$,

$$\varphi(x,\beta t) \leqslant \varphi(y,t) \quad \text{when} \quad \omega_B^-(t) \in \left\lfloor 1, \frac{1}{|B|} \right\rfloor;$$

(A1-s) if it satisfies (A1- ω) for $\omega(x, t) := t^s$;

(A1) if it satisfies (A1- φ);

(aInc)_p if $t \mapsto \frac{\varphi(x,t)}{t^p}$ is L_p -almost increasing in $(0,\infty)$ for some $L_p \ge 1$ and a.e. $x \in \Omega$; (aDec)_q if $t \mapsto \frac{\varphi(x,t)}{t^q}$ is L_q -almost decreasing in $(0,\infty)$ for some $L_q \ge 1$ and a.e. $x \in \Omega$. We say that (aInc) holds if (aInc)_p holds for some p > 1, and similarly for (aDec). If in the definition of (aInc)_p we have $L_p = 1$, then we say that φ satisfies (Inc)_p, similarly for (Dec)_q.

$$\begin{array}{c|cccc} \varphi(x,t) := & (\text{A0}) & (\text{A1}) & (\text{A1-}s) & (\text{aInc}) & (\text{aDec}) \\ \hline \varphi(t) & \text{true} & \text{true} & \text{true} & \nabla_2 & \Delta_2 \\ t^{p(x)}a(x) & a \approx 1 & p \in C^{\log} & p \in C^{\log} & p^- > 1 & p^+ < \infty \\ t^{p(x)}\log(e+t) & \text{true} & p \in C^{\log} & p \in C^{\log} & p^- > 1 & p^+ < \infty \\ t^p + a(x)t^q & a \in L^\infty & a \in C^{0,\frac{n}{p}(q-p)} & a \in C^{0,\frac{n}{s}(q-p)} & p > 1 & q < \infty \\ & \text{TABLE 1. Assumptions in special cases} \end{array}$$

We note that $(A1-\omega)$ is a new condition introduced in this article to combine (A1) and (A1-n) as well as other cases. Basically, $(A1-\omega)$ is the appropriate assumption if we have *a priori* information that the solution is in space $W^{1,\omega}$ or the corresponding Lebesgue or Hölder space. The most important cases are $\omega = \varphi$ and $\omega(x,t) = t^s$, but we may, for instance, consider $\omega = \varphi^{1+\epsilon}$ if we have some higher integrability information.

By [23, Section 4.1], (A0) can be stated equivalently as the existence of $\beta \in (0, 1]$ such that $\varphi(x, \beta) \leq 1 \leq \varphi(x, 1/\beta)$ for almost every $x \in \Omega$. If φ satisfies (A0), then (A1) is equivalent to the condition that there exists $\beta \in (0, 1)$ such that

$$\beta \varphi^{-1}(x,t) \leqslant \varphi^{-1}(y,t) \quad \text{when} \quad t \in \left[1, \frac{1}{|B|}\right]$$

for every ball B and a.e. $x, y \in B \cap \Omega$ [23, Section 4.2].

Remark 2.2. Assume that the derivative φ' with respect to the second variable exists. Then

$$\frac{d}{dt}\frac{\varphi(x,t)}{t^p} = \frac{\varphi'(x,t)t^p - pt^{p-1}\varphi(x,t)}{t^{2p}}.$$

If φ satisfies $(Inc)_p$, then the derivative is non-negative and so $\frac{t\varphi'(x,t)}{\varphi(x,t)} \ge p$. Similarly, $\frac{t\varphi'(x,t)}{\varphi(x,t)} \le q$ if φ satisfies $(Dec)_q$. If, on the other hand,

$$p \leqslant \frac{t\varphi'(x,t)}{\varphi(x,t)} \leqslant q$$

then $\frac{d}{dt} \frac{\varphi(x,t)}{t^p}$ is non-negative and $\frac{d}{dt} \frac{\varphi(x,t)}{t^q}$ is non-positive and hence $(\text{Inc})_p$ and $(\text{Dec})_q$ hold. Moreover, the double inequality implies also that φ' satisfies $(a\text{Inc})_{p-1}$ and $(a\text{Dec})_{q-1}$ if φ satisfies $(\text{Inc})_p$ and $(\text{Dec})_q$.

Definition 2.3. Let $\varphi \in \Phi_w(\Omega)$ and define the *modular* ϱ_{φ} for $u \in L^0(\Omega)$, the set of measurable functions in Ω , by

$$\varrho_{\varphi}(u) := \int_{\Omega} \varphi(x, |u(x)|) \, dx.$$

The generalized Orlicz space, also called Musielak-Orlicz space, is defined as the set

$$L^{\varphi}(\Omega) := \left\{ u \in L^{0}(\Omega) \colon \lim_{\lambda \to 0^{+}} \varrho_{\varphi}(\lambda u) = 0 \right\}$$

equipped with the (Luxemburg) norm

$$||u||_{L^{\varphi}(\Omega)} := \inf \left\{ \lambda > 0 \colon \varrho_{\varphi}\left(\frac{u}{\lambda}\right) \leqslant 1 \right\}.$$

If the set is clear from context we abbreviate $||u||_{L^{\varphi}(\Omega)}$ by $||u||_{\varphi}$.

We denote by φ^* the conjugate Φ -function, defined by

$$\varphi^*(x,t) := \sup_{s \ge 0} (st - \varphi(x,s)).$$

By this definition, we have Young's inequality $st \leq \varphi(x, s) + \varphi^*(x, t)$. Hölder's inequality holds in generalized Orlicz spaces for $\varphi \in \Phi_w(\Omega)$ with constant 2 [23, Lemma 3.2.13]:

$$\int_{\Omega} |u| \, |v| \, dx \leq 2 \|u\|_{\varphi} \|v\|_{\varphi^*(\cdot)}.$$

Definition 2.4. A function $u \in L^{\varphi}(\Omega)$ belongs to the *Orlicz–Sobolev space* $W^{1,\varphi}(\Omega)$ if its weak partial derivatives $\partial_1 u, \ldots, \partial_n u$ exist and belong to the space $L^{\varphi}(\Omega)$. For $u \in W^{1,\varphi}(\Omega)$, we define the norm

$$\|u\|_{W^{1,\varphi}(\Omega)} := \|u\|_{L^{\varphi}(\Omega)} + \|\nabla u\|_{L^{\varphi}(\Omega)}$$

Here $\|\nabla u\|_{\varphi}$ is a abbreviation of $\||\nabla u|\|_{\varphi}$. Again, if Ω is clear from context, we abbreviate $\|u\|_{W^{1,\varphi}(\Omega)}$ by $\|u\|_{1,\varphi}$.

We conclude the section by proving an appropriate version of the Caccioppoli inequality. We denote by η a cut-off function in B_R , more precisely, $\eta \in C_0^{\infty}(B_R)$, $\chi_{B_{\sigma R}} \leq \eta \leq \chi_{B_R}$ and $|\nabla \eta| \leq \frac{2}{(1-\sigma)R}$, where $\sigma \in (0, 1)$. Note that the auxiliary function ψ is independent of xin the next lemma. This avoids assumptions regarding the differentiability of ψ in the space variable, but it does mean that the application of the lemma later on is more complicated compared to classical, standard growth cases where one simply choses $\psi = \varphi$.

Lemma 2.5 (Caccioppoli inequality). Suppose $\varphi \in \Phi_w(\Omega)$ satisfies (A0) and $(aDec)_q$, and let $\psi \in \Phi_w$ be differentiable and satisfy (A0), $(Inc)_{p_1}$ and $(Dec)_{q_1}$, $p_1, q_1 \ge 1$. Let u be a non-negative supersolution to equation (1.3) and η be a cut-off function in $B_R \subset \Omega$. For any $\ell > \frac{1}{p_1}$ and $s \ge q$,

$$\int_{B_R} \varphi(x, |\nabla u|) \psi(\frac{u+R}{R})^{-\ell} \eta^s \, dx \leqslant \left(\frac{c(L_q)s\Lambda}{(1-\sigma)(p_1\ell-1)\nu}\right)^q \int_{B_R} \psi(\frac{u+R}{R})^{-\ell} \varphi(x, \frac{u+R}{R}) \eta^{s-q} \, dx.$$

Proof. Let us simplify the notation by denoting $\tilde{u} := u + R$ and $v := \frac{\tilde{u}}{R}$. Since $\nabla u = \nabla \tilde{u}$, we see that \tilde{u} is still a supersolution. Since $v \ge 1$, $\psi(v)^{-\ell} \eta^s \le c_1$ by (A0) and $(\text{Inc})_{p_1}$.

We would like to test equation (1.3) with $h := \psi(v)^{-\ell} \eta^s \tilde{u}$. Let us first check that h is a valid test function, that is $h \in W^{1,\varphi}(B_R)$ and has compact support in Ω . As $\tilde{u} \in L^{\varphi}(B_R)$ and $|h| \leq c_1 \tilde{u}$ it is immediate that $h \in L^{\varphi}(B_R)$. By a direct calculation we have

$$\nabla h = -\ell\psi(v)^{-\ell-1}\eta^s \tilde{u}\psi'(v)\nabla v + s\psi(v)^{-\ell}\eta^{s-1}\tilde{u}\nabla\eta + \psi(v)^{-\ell}\eta^s\nabla\tilde{u}.$$

Note that $\tilde{u}\nabla v = v\nabla \tilde{u}$. We use Remark 2.2 and get

$$\left|\ell\psi(v)^{-\ell-1}\eta^s\psi'(v)v\nabla\tilde{u}\right|\leqslant q_1\ell c_1|\nabla\tilde{u}|\in L^{\varphi}(B_R).$$

For the third term, we obtain $|\psi(v)^{-\ell}\eta^s \nabla \tilde{u}| \leq c_1 |\nabla \tilde{u}| \in L^{\varphi}(B_R)$. The term with $\nabla \eta$ is treated as h itself. Thus $h \in W^{1,\varphi}(B_R)$. Since s > 0 and $\eta \in C_0^{\infty}(B_R)$, h has compact support in Ω and so it is a valid test-function.

We next calculate

$$f(x,\nabla \tilde{u}) \cdot \nabla h = \psi(v)^{-\ell-1} \eta^s [-\ell \psi'(v)v + \psi(v)] f(x,\nabla \tilde{u}) \cdot \nabla \tilde{u} + s \psi(v)^{-\ell} \eta^{s-1} \tilde{u} f(x,\nabla \tilde{u}) \cdot \nabla \eta.$$

Since \tilde{u} is a supersolution, we have $\int_{B_R} f(x, \nabla \tilde{u}) \cdot \nabla h \, dx \ge 0$, which implies with the growth conditions (1.1) that

$$\nu \int_{B_R} \varphi(x, |\nabla \tilde{u}|) \psi(v)^{-\ell-1} \eta^s [\ell \psi'(v)v - \psi(v)] \, dx \leqslant s \Lambda \int_{B_R} \frac{\varphi(x, |\nabla \tilde{u}|)}{|\nabla \tilde{u}|} |\nabla \eta| \, \eta^{s-1} \psi(v)^{-\ell} \tilde{u} \, dx.$$

By Remark 2.2, $p_1\psi(t) \leq \psi'(t)t$ so we conclude that

$$[p_1\ell - 1]\nu \int_{B_R} \varphi(x, |\nabla \tilde{u}|) \psi(v)^{-\ell} \eta^s \, dx \leqslant s\Lambda \int_{B_R} \frac{\varphi(x, |\nabla \tilde{u}|)}{|\nabla \tilde{u}|} |\nabla \eta| \, \eta^{s-1} \psi(v)^{-\ell} \tilde{u} \, dx.$$

Here $p_1\ell - 1$ is positive, since $\ell > \frac{1}{p_1}$. Recalling that $|\nabla n|\tilde{u} \leq \frac{2}{(1-\lambda)^2}\tilde{u} = \frac{2}{1-\lambda^2}v$,

ecalling that
$$|\nabla \eta| \tilde{u} \leq \frac{2}{(1-\sigma)R} \tilde{u} = \frac{2}{1-\sigma} v$$
, we arrive at

$$\int_{B_R} \varphi(x, |\nabla \tilde{u}|) \psi(v)^{-\ell} \eta^s \, dx \leqslant \frac{C}{1 - \sigma} \int_{B_R} \frac{\varphi(x, |\nabla \tilde{u}|)}{|\nabla \tilde{u}|} v \psi(v)^{-\ell} \eta^{s-1} \, dx$$

By Young's inequality [23, (2.4.2)]

$$\frac{\varphi(x,|\nabla \tilde{u}|)}{|\nabla \tilde{u}|} v \leqslant \varphi\left(x,\epsilon^{-\frac{1}{q'}}v\right) + \varphi^*\left(x,\epsilon^{\frac{1}{q'}}\frac{\varphi(x,|\nabla \tilde{u}|)}{|\nabla \tilde{u}|}\right).$$

For the first term on the right hand side we use $(aDec)_q$ of φ . For the second term we first use $(aInc)_{q'}$ of φ^* [23, Proposition 2.4.9] and then $\varphi^*(\frac{\varphi(t)}{t}) \leq \varphi(t)$ (comment after [23, Theorem 2.4.10]), and obtain

$$\frac{\varphi(x,|\nabla \tilde{u}|)}{|\nabla \tilde{u}|}v \lesssim \epsilon^{1-q}\varphi(x,v) + \epsilon\varphi(x,|\nabla u|).$$

Finally, we choose $\epsilon := \frac{1-\sigma}{2C}\eta(x)$ and so

$$\int_{B_R} \varphi(x, |\nabla \tilde{u}|) \psi(v)^{-\ell} \eta^s \, dx$$

$$\leqslant \frac{1}{2} \int_{B_R} \varphi(x, |\nabla \tilde{u}|) \psi(v)^{-\ell} \eta^s \, dx + \frac{C}{(1-\sigma)^q} \int_{B_R} \psi(v)^{-\ell} \varphi(x, v) \eta^{s-q} \, dx.$$

The first term on the right-hand side can be absorbed in the left-hand side. This gives the claim. $\hfill \Box$

3. THE WEAK HARNACK INEQUALITY

The plan of the proof follows the usual scheme of Moser iteration. We first show that the infimum is bounded below by an integral-mean with negative power. We then prove a reverse Hölder-type inequality for positive exponents below a certain threshold. These steps are proved by iteration. Jumping over zero is the final piece and it is accomplished by the John–Nirenberg lemma.

We next define a differentiable approximation of $\varphi_{B_R}^-$ with nice growth properties. We assume that $\varphi \in \Phi_w(\Omega)$ satisfies (aDec) and (aInc)_p, $p \ge 1$.

Definition 3.1. We define an auxiliary weak Φ -function ψ_r by setting $\psi_r(0) := 0$ and

$$\psi_r(t) := \int_0^t \tau^{p-1} \sup_{s \in (0,\tau]} \frac{\varphi_{B_r}^{-}(s)}{s^p} d\tau \quad \text{for } t > 0.$$

Then $\frac{\psi'_r(t)}{t^{p-1}} = \sup_{s \in (0,t]} \frac{\varphi_{B_r}^{-}(s)}{s^p}$ is increasing and positive so that ψ'_r satisfies $(\text{Inc})_{p-1}$ and ψ_r is convex, strictly increasing and satisfies $(\text{Inc})_p$. Since $\varphi_{B_r}^{-}$ satisfies $(\text{aInc})_p$ and (aDec), the integrand is finite in every point, and so ψ_r is continuous. Thus $\psi_r \in \Phi_c$. Further, by (aDec),

$$\psi_r(t) \ge \int_{t/2}^t \tau^{p-1} \sup_{s \in (0,\tau]} \frac{\varphi_{B_r}^-(s)}{s^p} d\tau \gtrsim \frac{\varphi_{B_r}^-(t/2)}{(t/2)^p} \left(\frac{t}{2}\right)^p \approx \varphi_{B_r}^-(t),$$

and since $\varphi_{B_r}^-$ satisfies (aInc)_p we obtain

$$\psi_r(t) \lesssim \int_0^t \tau^{p-1} \frac{\varphi_{B_R}^-(t)}{t^p} d\tau = \frac{1}{p} \varphi_{B_r}^-(t).$$

Thus $\psi_r(t) \approx \varphi_{B_r}^-(t)$. It follows that ψ_r satisfies $(aDec)_q$, and since it is convex, it satisfies (Dec) [23, Lemma 2.2.6]. Therefore, ψ_r satisfies the assumptions of Lemma 2.5 with $p_1 = p$; q_1 is a function of q, but it does not affect the constant. See Proposition 4.1 for a simpler, sufficient condition for (3.3).

Theorem 3.2. Suppose $\varphi \in \Phi_w(\Omega)$ satisfies (A0), $(aInc)_p$ and $(aDec)_q$, $1 \leq p \leq q$. Let u be a nonnegative supersolution to (1.3) in B_R . Let $\omega \in \Phi_w(\Omega)$ satisfy (A0) and (aDec). Assume that φ satisfies (A1- ω), and

(3.3)
$$\omega_{B_R}^{-} \left(\oint_{B_R} \frac{u+R}{R} \, dx \right) \leqslant \frac{d}{|B_R|}$$

for some d > 0. For any $\ell > 0$, there exists a constant $C_{\ell,d} = C(p,q,L_p,L_q,n,\ell,d) > 0$ such that

$$\operatorname{ess\,inf}_{B_{R/2}} u + R \ge C_{\ell,d} \Big(\int_{B_R} (u+R)^{-\ell} \, dx \Big)^{-\frac{1}{\ell}}$$

Proof. Let us assume that $r \in [\frac{R}{2}, R]$ and denote $\tilde{u} := u + R, v := \frac{\tilde{u}}{r}$ and $n' := \frac{n}{n-1}$. Let $\psi_r \in \Phi_c$ be as in Definition 3.1 in the ball B_r , and abbreviate $\psi := \psi_r$. Let s > 0 be a constant that will be fixed later. We use the $W^{1,1}$ -Sobolev–Poincaré inequality for the function $\psi(v)^{-\ell}\eta^s$, where $\eta \in C_0^{\infty}(B_r)$ is a cut-off function as before. We see that $\psi(v)^{-\ell}\eta^s \in W^{1,1}(\Omega)$ and it has a compact support in Ω by the same arguments as in the proof of the Caccioppoli inequality (Lemma 2.5). The Sobolev–Poincaré inequality gives us

$$\left(\int_{B_r} \psi(v)^{-\ell n'} \eta^{sn'} dx\right)^{\frac{1}{n'}} \lesssim \int_{B_r} |\nabla(\psi(v)^{-\ell} \eta^s)| dx$$
$$\leqslant \int_{B_r} \psi(v)^{-\ell-1} \eta^{s-1} [s\psi(v)|\nabla\eta| + \ell\eta |\nabla\psi(v)|] dx.$$

By the definition of ψ and $(aInc)_p$ of φ , $\psi'(t) \approx \frac{1}{t}\varphi_{B_r}^-(t) \leq \frac{1}{t}\varphi(x,t)$ for $x \in B_r$. Thus

$$|\nabla\psi(v)| = |\psi'(v)\frac{1}{r}\nabla\tilde{u}| \lesssim \frac{1}{rv}\varphi(x,v)|\nabla\tilde{u}|$$

almost everywhere in B_r . We use this and the estimate $|\nabla \eta| \leq \frac{2}{(1-\sigma)r}$:

$$(3.4) \ \left(\int_{B_r} \psi(v)^{-\ell n'} \eta^{sn'} \, dx\right)^{\frac{1}{n'}} \leqslant \frac{C(s+\ell)}{r} \int_{B_r} \psi(v)^{-\ell-1} \eta^{s-1} \left[\frac{\psi(v)}{1-\sigma} + \eta \frac{1}{v} \varphi(x,v) |\nabla \tilde{u}|\right] dx.$$

By Young's inequality [23, (2.4.1)] and $\varphi^*(x, \frac{1}{t}\varphi(x, t)) \leq \varphi(x, t)$ [23, p. 35], we have

$$\frac{1}{v}\varphi(x,v)|\nabla \tilde{u}| \leqslant \varphi^*\left(x,\frac{1}{v}\varphi(x,v)\right) + \varphi(x,|\nabla \tilde{u}|) \leqslant \varphi(x,v) + \varphi(x,|\nabla \tilde{u}|).$$

This and the Caccioppoli inequality (Lemma 2.5) for u + R - r in B_r yield

$$\begin{split} \int_{B_r} \psi(v)^{-\ell-1} \eta^s \frac{1}{v} \varphi(x,v) |\nabla \tilde{u}| \, dx &\leq \int_{B_r} \psi(v)^{-\ell-1} \eta^s \big(\varphi(x,v) + \varphi(x,|\nabla \tilde{u}|)\big) \, dx \\ &\leq \frac{C}{(1-\sigma)^q} \int_{B_r} \psi(v)^{-\ell-1} \eta^{s-q} \varphi(x,v) \, dx, \end{split}$$

where we assumed that $(\ell + 1)p > 1$ and $s \ge q$ and used $\eta^s \le \eta^{s-q}$. We next divide (3.4) by r^{n-1} , use this estimate as well as $\eta^{s-1} \le \eta^{s-q}$ and $\psi(t) \le \varphi(x, t)$:

$$\begin{split} \left(\int_{B_r} \psi(v)^{-\ell n'} \eta^{sn'} \, dx \right)^{\frac{1}{n'}} &\leq \frac{C(s+\ell)}{(1-\sigma)^q} \int_{B_r} \psi(v)^{-\ell-1} \eta^{s-1} \psi(v) + \psi(v)^{-\ell-1} \eta^{s-q} \varphi(x,v) \, dx \\ &\lesssim \frac{s+\ell}{(1-\sigma)^q} \int_{B_r} \psi(v)^{-\ell-1} \eta^{s-q} \varphi(x,v) \, dx. \end{split}$$

Let us denote $V_R := (\omega_{B_R}^-)^{-1} (d/|B_R|)$ and $E := \{v(x) < V_R\}$. Since $v \ge 1$, we find by (A0), (aDec) and (A1- ω) that $\varphi(x, v) \approx \psi(v)$ in E. Hence

$$\frac{1}{|B_r|} \int_{B_r \cap E} \psi(v)^{-\ell-1} \eta^{s-q} \varphi(x,v) \, dx \approx \frac{1}{|B_r|} \int_{B_r \cap E} \psi(v)^{-\ell} \eta^{s-q} \, dx.$$

Since φ satisfies $(aDec)_q$ and ψ satisfies $(aInc)_p$, we see that $t \mapsto \psi(t)^{-\ell-1}\varphi(x,t)$ is almost decreasing when $(\ell+1)p \ge q$. Then when $v \ge V_R$ we obtain that

$$\frac{1}{|B_r|} \int_{B_r \setminus E} \psi(v)^{-\ell-1} \eta^{s-q} \varphi(x, v) \, dx \lesssim \frac{1}{|B_r|} \int_{B_r \setminus E} \psi(V_R)^{-\ell-1} \eta^{s-q} \varphi(x, V_R) \, dx$$
$$\lesssim \int_{B_r} \psi(V_R)^{-\ell} \eta^{s-q} \, dx \lesssim \psi(V_R)^{-\ell},$$

again by $\varphi(x, V_R) \approx \psi(V_R)$ (from (A1- ω)). Furthermore, by (3.3) we obtain

$$\int_{B_{r/2}} v \, dx \lesssim \int_{B_R} \frac{u+R}{R} \, dx \lesssim (\omega_R^-)^{-1} \left(\frac{d}{|B_R|}\right) = V_R$$

This and Jensen's inequality for the convex function $t\mapsto \psi(t)^{-\ell}$ yield

$$\psi(V_R)^{-\ell} \lesssim \psi\left(\int_{B_{r/2}} v \, dx\right)^{-\ell} \lesssim \int_{B_{r/2}} \psi(v)^{-\ell} \, dx \lesssim \int_{B_r} \psi(v)^{-\ell} \eta^{s-q} \, dx.$$

Combining the estimates in E and $B_r \setminus E$ and the previous inequality, we have established that

(3.5)
$$\left(\int_{B_r} \psi(v)^{-\ell n'} \eta^{sn'} dx\right)^{\frac{1}{n'}} \leq \frac{C(s+\ell)}{(1-\sigma)^q} \int_{B_r} \psi(v)^{-\ell} \eta^{s-q} dx.$$

Let us choose $s := \ell - (n-1)q$, and suppose that ℓ is so large that $s \ge q$, $(\ell + 1)p \ge q$ and $\ell \ge nq$. Raising both sides of the previous inequality to the power $-\frac{1}{\ell}$ gives

$$\underbrace{\left(\int_{B_r} \psi(v)^{-\ell n'} \eta^{\ell n'-nq} \, dx\right)^{-\frac{1}{n'\ell}}}_{=:\Psi(n'\ell)} \ge \left(\frac{C\ell}{(1-\sigma)^q}\right)^{-\frac{1}{\ell}} \underbrace{\left(\int_{B_r} \psi(v)^{-\ell} \eta^{\ell-nq} \, dx\right)^{-\frac{1}{\ell}}}_{=\Psi(\ell)}$$

Let us then set $\ell = n_k := (n')^k$. For $k \ge k_0$ (so that the required lower bounds on ℓ hold) we use the standard iteration technique. By induction, we obtain that

$$\Psi(n_K) \ge \exp\Big(-\sum_{k=k_0}^{K-1} \frac{\ln n_k}{n_k}\Big) (C(1-\sigma))^{-\sum_{k=k_0}^{K-1} \frac{q}{n_k}} \Psi(n_{k_0})$$

Denote $\gamma := n_{k_0}$. Since $\sum_{k=k_0}^{\infty} \frac{q}{n_k} = \frac{nq}{\gamma} =: \beta < \infty$ as a geometric series and $\sum_{k=k_0}^{\infty} \frac{\ln n_k}{n_k} < \infty$ by comparison with a geometric series, we get

$$\begin{aligned} \underset{B_{\sigma r}}{\operatorname{ess\,inf}} \psi(v) &\geq \operatorname{ess\,inf} \frac{\psi(v)}{\eta} = \lim_{K \to \infty} \Psi(n_K) \\ &\gtrsim (1 - \sigma)^{-\beta} \Psi(\gamma) = (1 - \sigma)^{-\beta} \left(\int_{B_r} \psi(v)^{-\gamma} \eta^{\gamma - nq} \, dx \right)^{-\frac{1}{\gamma}} \\ &\geq (1 - \sigma)^{-\beta} \left(\int_{B_r} \psi(v)^{-\gamma} \, dx \right)^{-\frac{1}{\gamma}} = (1 - \sigma)^{-\beta} \psi \left(\xi^{-1} \left(\int_{B_r} \xi(v) \, dx \right) \right) \\ &\gtrsim \psi \left((1 - \sigma)^{-\frac{\beta}{p}} \xi^{-1} \left(\int_{B_r} \xi(v) \, dx \right) \right), \end{aligned}$$

where $\xi(t) := \psi(t)^{-\gamma}$, and (aInc)_p of ψ was used in the last inequality. As ψ is strictly increasing, this implies

$$\operatorname{essinf}_{B_{\sigma r}} v \gtrsim (1 - \sigma)^{-\frac{\beta}{p}} \xi^{-1} \Big(\int_{B_r} \xi(v) \, dx \Big).$$

Since $t \mapsto \xi(t^{-1/(\gamma q)})$ satisfies (aDec)₁, it is equivalent to a concave function [23, Lemma 2.2.1], and so by Jensen's inequality

$$\operatorname{ess\,inf}_{B_{\sigma r}} v \gtrsim (1-\sigma)^{-\frac{\beta}{p}} \xi^{-1} \Big(\int_{B_r} \xi(v) \, dx \Big) \gtrsim (1-\sigma)^{-\frac{\beta}{p}} \Big(\int_{B_r} v^{-\gamma q} \, dx \Big)^{-\frac{1}{\gamma q}}.$$

Then we recall that $v = \frac{u+R}{r}$ and multiply both sides by r:

(3.6)
$$\operatorname{ess\,inf}_{B_{\sigma r}} u + R \gtrsim (1 - \sigma)^{-\frac{\beta}{p}} \Big(\int_{B_r} (u + R)^{-\gamma q} \, dx \Big)^{-\frac{1}{\gamma q}},$$

where $\frac{R}{2} \leq r \leq R$ and the constant depends only on p, q, L_p , L_q , n, ℓ and d. This is the claim for $\ell = \gamma q$. For exponents larger than γq the claim follows by Hölder's inequality. We have thus established the claim for large exponents.

Finally we show the claim for small exponents. We use the +R to simplify the proof, but this is not essential here. So let $\ell \in (0, \gamma q)$. We observe that

$$\underset{B_{\sigma r}}{\operatorname{ess\,inf}} u + R \ge C(1-\sigma)^{-\frac{\beta}{p}} \Big(\int_{B_{r}} (u+R)^{-\gamma q} \, dx \Big)^{-\frac{1}{\gamma q}} \\ \ge C(1-\sigma)^{-\frac{\beta}{p}} \Big(\int_{B_{r}} (u+R)^{-\ell} \, dx \Big)^{-\frac{1}{\gamma q}} (\operatorname{ess\,inf}_{B_{r}} u + R)^{\frac{\gamma q-\ell}{\gamma q}}.$$

We denote $Q := \frac{\gamma q - \ell}{\gamma q}$, $\Lambda(r) := \operatorname{ess\,inf}_{B_r} u + R$ and $A := (\int_{B_R} (u + R)^{-\ell} dx)^{-1/\ell}$. Thus, for $r \in [\frac{R}{2}, R]$,

$$\Lambda(\sigma r) \ge C(1-\sigma)^{-\frac{\beta}{p}} A^{\frac{\ell}{\gamma q}} \Lambda(r)^Q.$$

Then we set $r := (1 - 2^{-(k+1)})R$ and $\sigma r := (1 - 2^{-k})R$ so that $1 - \sigma \approx 2^{-k}$. With $\Lambda_k := \Lambda((1 - 2^{-k})R)$ and iteration we obtain

$$\Lambda_1 \ge C2^{\frac{\beta}{p}} A^{\frac{\ell}{\gamma_q}} \Lambda_2^Q \ge 2^{\frac{\beta}{p}(1+2Q)} (CA^{\frac{\ell}{\gamma_q}})^{1+Q} \Lambda_3^{Q^2} \ge \dots$$
$$\ge 2^{\frac{\beta}{p} \sum_{k=1}^{\infty} kQ^{k-1}} (CA^{\frac{\ell}{\gamma_q}})^{\sum_{k=0}^{\infty} Q^k} \liminf_{k \to \infty} \Lambda_k^{Q^k}.$$

Since $\Lambda_k \ge R$ and $Q \in (0,1)$, $\liminf_{k\to\infty} \Lambda_k^{Q^k} \ge 1$. Furthermore, $\sum_{k=1}^{\infty} kQ^{k-1} < \infty$ and $\sum_{k=0}^{\infty} Q^k = \frac{1}{1-Q} = \frac{\gamma q}{\ell}$. Hence $\Lambda_1 \gtrsim A$, which is the claim for ℓ .

The previous proof only works for negative ℓ . In fact, the largeness of $-\ell$ and the condition (A1- ω) were only used in the paragraph with (3.5). With some modifications, we can iterate also for some positive exponents. We use the limiting exponent $\ell(p)$ defined in the introduction. See Proposition 4.2 for a simpler, sufficient condition for (3.8).

Proposition 3.7. Suppose $\varphi \in \Phi_w(\Omega)$ satisfies (A0), $(aInc)_p$ and $(aDec)_q$, p, q > 1. Let u be a nonnegative supersolution to (1.3) in B_R . Assume that

(3.8)
$$\int_{B_r} \left(\frac{\varphi(x,v)}{\varphi_{B_r}^-(v)}\right)^\beta dx \leqslant d,$$

for some $\beta > \max\{\frac{n}{p}, 1\}$ and all $B_r \subset B_R$, where $v := \frac{u+r}{r}$. For any $\ell_0 > 0$ and $\ell < \ell(p)$, there exists a constant $C = C(p, q, L_p, L_q, n, \ell_0, \ell, d) > 0$ such that

$$\left(\int_{B_{2R}} (u+R)^{\ell_0} dx\right)^{\frac{1}{\ell_0}} \ge C\psi^{-1}\left(\left(\int_{B_R} \psi(v)^{\frac{\ell}{p}} dx\right)^{\frac{p}{\ell}}\right) \ge C\left(\int_{B_R} (u+R)^{\ell} dx\right)^{\frac{1}{\ell}}.$$

Proof. We proceed as in the previous proof, but use the $W^{1,\gamma}$ -Sobolev inequality instead of the $W^{1,1}$ -version. Here we will eventually take $\gamma \nearrow \min\{p, n\}$. In place of (3.4) we obtain

$$\left(\int_{B_r} \psi(v)^{-\ell\gamma^*} \eta^{s\gamma^*} dx \right)^{\frac{\gamma}{\gamma^*}} \lesssim \int_{B_r} \left(\psi(v)^{-\ell-1} \eta^{s-1} \left[\frac{\psi(v)}{1-\sigma} + \eta \psi'(v) |\nabla \tilde{u}| \right] \right)^{\gamma} dx$$
$$\lesssim \int_{B_r} \psi(v)^{-\ell\gamma} \frac{\eta^{\gamma(s-1)}}{(1-\sigma)^{\gamma}} + \eta^{s\gamma} \psi(v)^{-\ell\gamma-1} \frac{\psi(v)}{v^{\gamma}} |\nabla \tilde{u}|^{\gamma} dx$$

where we already divided by r^{n-1} , and used Remark 2.2 as well as $(a+b)^{\gamma} \approx a^{\gamma} + b^{\gamma}$.

We estimate $\frac{\psi(v)}{v^{\gamma}} |\nabla \tilde{u}|^{\gamma}$ with Young's inequality. Define $\xi(t) := \psi(t^{1/\gamma})$. Then $\xi^{-1}(t) = \psi^{-1}(t)^{\gamma}$ and

$$(\xi^*)^{-1}(t) \approx \frac{t}{\xi^{-1}(t)} = \frac{t}{\psi^{-1}(t)^{\gamma}}$$

since $\xi^{-1}(t)(\xi^*)^{-1}(t) \approx t$ by [23, Theorem 2.4.8]. Hence

$$\xi^*\left(\frac{\psi(t)}{t^{\gamma}}\right) \approx \psi(t)$$

and so by Young's inequality

$$\frac{\psi(v)}{v^{\gamma}} |\nabla \tilde{u}|^{\gamma} \leqslant \xi(|\nabla \tilde{u}|^{\gamma}) + \xi^* \left(\frac{\psi(v)}{v^{\gamma}}\right) \approx \psi(|\nabla \tilde{u}|) + \psi(v) \lesssim \varphi(x, |\nabla \tilde{u}|) + \psi(v).$$

Then we estimate with the Caccioppoli inequality (Lemma 2.5)

$$\int_{B_r} \psi(v)^{-\ell\gamma-1} \eta^{s\gamma} \varphi(x, |\nabla \tilde{u}|) \, dx \lesssim \frac{1}{(1-\sigma)^q} \int_{B_r} \psi(v)^{-\ell\gamma-1} \eta^{s\gamma-q} \varphi(x, v) \, dx$$

provided $\ell \gamma + 1 > \frac{1}{p}$, which means that ℓ can also be negative. Thus we have

$$\left(\int_{B_r} \psi(v)^{-\ell\gamma^*} \eta^{s\gamma^*} \, dx \right)^{\frac{\gamma}{\gamma^*}} \lesssim \int_{B_r} \psi(v)^{-\ell\gamma} \frac{\eta^{\gamma(s-1)}}{(1-\sigma)^{\gamma}} + \psi(v)^{-\ell\gamma-1} \frac{\eta^{s\gamma-q}}{(1-\sigma)^q} \varphi(x,v) + \eta^{s\gamma} \psi(v)^{-\ell\gamma} \, dx \\ \leqslant \frac{1}{(1-\sigma)^q} \int_{B_r} \psi(v)^{-\ell\gamma-1} \eta^{s-q} \varphi(x,v) \, dx,$$

where $\psi(v) \lesssim \varphi(x, v)$, $\eta^{\gamma(s-1)} \leqslant \eta^{s-q}$ and $\eta^{\gamma s} \leqslant \eta^{s-q}$ were used. In contrast to the previous proof, we next use Hölder's inequality

$$\int_{B_r} \psi(v)^{-\ell\gamma-1} \eta^{s-q} \varphi(x,v) \, dx \leqslant \left(\int_{B_r} \psi(v)^{-\ell\gamma\lambda} \eta^{(s-q)\lambda} \, dx \right)^{\frac{1}{\lambda}} \left(\int_{B_r} \left(\frac{\varphi(x,v)}{\psi(v)} \right)^{\lambda'} \, dx \right)^{\frac{1}{\lambda'}}.$$

Since the second factor on the right-hand side is bounded by (3.8) when $\lambda' \in (\frac{n}{\gamma}, \beta]$, we obtain that

$$\left(\int_{B_r} \psi(v)^{-\ell\gamma^*} \eta^{s\gamma^*} \, dx\right)^{\frac{1}{\gamma^*}} \lesssim \left(\int_{B_r} \psi(v)^{-\ell\gamma\lambda} \eta^{(s-q)\lambda} \, dx\right)^{\frac{1}{\gamma^{\lambda}}}$$

Since $\lambda < (\frac{n}{\gamma})' = \frac{n}{n-\gamma}$, we conclude that $\gamma \lambda < \gamma^*$. Therefore, we have obtained a reverse Hölder type inequality, which can be iterated (as in the previous proof) to show that

$$\left(\int_{B_{2r}} v^{\ell_0} dx\right)^{\frac{1}{\ell_0}} \gtrsim \psi^{-1} \left(\left(\int_{B_r} \psi(v)^{\gamma_2} dx\right)^{\frac{1}{\gamma_2}} \right) \geqslant \left(\int_{B_r} v^{p\gamma_2} dx\right)^{\frac{1}{p\gamma_2}}$$

for $\gamma_2 \leq -\ell\gamma^*$ and any $\ell_0 > 0$; the last step is just Jensen's inequality. Since we can choose any value of ℓ with $-\ell < \frac{1}{\gamma p'}$, we have the inequality for any $\gamma_2 < \frac{\gamma^*}{\gamma p'} = \frac{n}{n-\gamma} \frac{p-1}{p}$. Thus letting $\gamma \nearrow \min\{p, n\}$ we can obtain the weak Harnack inequality for exponent up to, but not including, $\ell(p)$.

We are now ready for the proof of the main result, the weak Harnack inequality. Note that here we need to add the (aInc) assumption for φ compared to Theorem 3.2.

Theorem 3.9. Suppose φ satisfies (A0), $(aInc)_p$ and $(aDec)_q$, 1 . Let <math>u be a non-negative solution to (1.3) on B_{2R} . Assume that there exists $\omega \in \Phi_w(\Omega)$ which satisfies (A0) and (aDec) such that one of the following holds:

- (1) φ satisfies (A1- ω) and (3.3) and (3.8), with $\beta = 1$, hold.
- (2) φ satisfies (A1- ω) and $||u||_{W^{1,\omega}(B_R)} \leq d$.

Then there exist $\ell_0 = \ell_0(p, q, L_p, L_q, \beta, d, n) > 0$ *and* $C = C(p, q, L_p, L_q, \beta, d, n) > 0$ *such that*

$$\left(\int_{B_{2R}} (u+R)^{\ell_0} dx\right)^{\frac{1}{\ell_0}} \leqslant C(\operatorname{ess\,inf}_{B_R} u+R).$$

If (3.8) holds with $\beta > \max\{\frac{n}{p}, 1\}$, then we can choose any $\ell_0 < \ell(p)$.

Proof. Let $0 < r \leq \frac{1}{2}R$ and denote $v := \frac{u+r}{r}$ so that $v \approx \frac{u+2r}{2r}$. By the $W^{1,1}$ -Poincaré inequality we get

$$\int_{B_r} \left| \log(u+R) - \int_{B_r} \log(u+R) \, dy \right| \, dx \leqslant r \int_{B_r} \left| \nabla (\log(u+R)) \right| \, dx \approx \int_{B_r} \frac{|\nabla u|}{v} \, dx.$$

Considering the cases $\frac{|\nabla u|}{v} \leq 1$ and $\frac{|\nabla u|}{v} > 1$, we conclude that

$$\frac{|\nabla u|}{v} \lesssim 1 + \frac{\varphi_{B_{2r}}^-(|\nabla u|)}{\varphi_{B_{2r}}^-(v)} \lesssim 1 + \frac{\varphi(x, |\nabla u|)}{\psi_{2r}(v)}$$

where $(aInc)_1$ was used in the case $\frac{|\nabla u|}{v} > 1$, and $\psi_{2r} \approx \varphi_{B_{2r}}^-$ in the last step. Note that ψ_{2r} satisfies $(Inc)_p$. It follows from the Caccioppoli inequality (Lemma 2.5) with $\ell = 1 > \frac{1}{p}$ and s = q that

$$\int_{B_r} \frac{\varphi(x, |\nabla u|)}{\psi_{2r}(v)} dx \lesssim \int_{B_{2r}} \frac{\varphi(x, |\nabla u|)}{\psi_{2r}\left(\frac{u+2r}{2r}\right)} \eta^q dx \lesssim \int_{B_{2r}} \frac{\varphi\left(x, \frac{u+2r}{2r}\right)}{\psi_{2r}\left(\frac{u+2r}{2r}\right)} dx \approx \int_{B_{2r}} \frac{\varphi\left(x, \frac{u+2r}{2r}\right)}{\varphi_{2r}^-\left(\frac{u+2r}{2r}\right)} dx.$$

Then we divide by $|B_r|$, note that $|B_r| \approx |B_{2r}|$, and combine with the previous inequalities to obtain that

$$\int_{B_r} \frac{|\nabla u|}{v} dx \lesssim 1 + \int_{B_r} \frac{\varphi(x, |\nabla u|)}{\psi_{2r}(v)} dx \lesssim 1 + \int_{B_{2r}} \frac{\varphi\left(x, \frac{u+2r}{2r}\right)}{\varphi_{2r}^-\left(\frac{u+2r}{2r}\right)} dx \leqslant 1 + d$$

where (3.8) have been used in the last inequality. Thus we have established $\log(u + R) \in BMO$ under assumption (1).

Next we consider assumption (2) with $\|\nabla u\|_{L^{\omega}(B_{2R})} \leq d$. Define

$$E := \{ x \in B_{2r} : \omega^{-1}(x, \frac{1}{r^n}) < v(x) \}.$$

In $B_r \setminus E$, we have $\varphi_{B_{2r}}^+(v) \approx \varphi_{B_{2r}}^-(v)$ by the (A1- ω) condition of φ and $v \ge 1$. Then we use the Caccioppoli inequality as before, except instead of ψ_r we use a corresponding function with $\psi_{2r} \approx \varphi_{B_{2r}}^+$ and so do not need (3.8).

In E, we use Young's inequality for the Φ -function $\xi := r^n \omega$:

$$\frac{|\nabla u|}{v} \leqslant \xi(x, \frac{1}{\epsilon} |\nabla u|) + \xi^*(x, \frac{\epsilon}{v}) = r^n \omega(x, \frac{1}{\epsilon} |\nabla u|) + r^n \omega^*(x, \frac{\epsilon}{r^n v}),$$

since $\xi^*(t) = r^n \omega^*(t/r^n)$ by [23, Lemma 2.4.3]. For the first term, $\int_{B_r} r^n \omega(x, \frac{1}{\epsilon} |\nabla u|) dx \leq c$ by the assumption $\|\nabla u\|_{L^{\omega}(B_{2R})} \leq d$ and (aDec) of ω . In $E, v > \omega^{-1}(x, \frac{1}{r^n}) \approx \frac{1}{r^n(\omega^*)^{-1}(x, \frac{1}{r^n})}$, since $\omega^{-1}(t)(\omega^*)^{-1}(t) \approx t$ [23, Theorem 2.4.8]. So $\frac{1}{r^n v} \leq (\omega^*)^{-1}(x, \frac{1}{r^n})$, and hence $\omega^*(x, \frac{\epsilon}{r^n v}) \leq \frac{1}{r^n}$ for appropriate $\epsilon > 0$. Therefore

$$\frac{1}{|B_{2r}|} \int_E r^n \omega^*(x, \frac{\epsilon}{r^n v}) \, dx \leqslant \frac{1}{|B_{2r}|} \int_E 1 \, dx \leqslant 1.$$

We have bounded all terms, so $\log(u + R) \in BMO$ also under assumptions (2).

Since $\log(u + R) \in BMO$, the John–Nirenberg lemma says that there exist positive constants ℓ_0 and C such that

$$\left(\int_{B_{2R}} (u+R)^{\ell_0} dx\right) \left(\int_{B_{2R}} (u+R)^{-\ell_0} dx\right) \leqslant C.$$

Let us note that under assumption (2), condition (3.3) holds by the Poincaré inequality and Jensen's inequality, see Proposition 4.1. By the previous inequality and Theorem 3.2 we obtain

$$\left(\int_{B_{2R}} (u+R)^{\ell_0} \, dx\right)^{\frac{1}{\ell_0}} \lesssim \left(\int_{B_{2R}} (u+R)^{-\ell_0} \, dx\right)^{-\frac{1}{\ell_0}} \lesssim \operatorname{ess\,inf}_{B_R} u+R$$

Assume then that (3.8) holds with $\beta > \max\{\frac{n}{p}, 1\}$. Then by Proposition 3.7 we have for any $\ell_0 > 0$ and $\ell < \ell(p)$, that

$$\left(\int_{B_{4R}} (u+2R)^{\ell} \, dx\right)^{\frac{1}{\ell}} \lesssim \left(\int_{B_{2R}} (u+2R)^{\ell_0} \, dx\right)^{\frac{1}{\ell_0}},$$

and hence the claim follows when we combine this with the previous inequality. (With this proof we obtained the ball B_{4R} on the right-hand side. Proving the intermediate results for $B_{\sqrt{2}R}$ give B_{2R} .)

4. Special cases

Let us next investigate conditions (3.3) and (3.8). We define the "Sobolev conjugate" $\omega^{\#} \in \Phi_{w}$ by the condition $(\omega^{\#})^{-1}(t) := t^{-1/n}\omega^{-1}(t)$ for t > 0. Note that for $\omega^{\#}$ to be in Φ_{w} , it must be increasing and so ω must satisfy (Dec)_n, see [23, Lemma 5.2.3]. If $\omega(t) = t^{s}$, then $\omega^{\#}(t) = t^{s^{*}}$, where $s^{*} := \frac{ns}{n-s}$ is the Sobolev exponent. Also s = n is included and in this case $||u||_{\omega^{\#}} = ||u||_{\infty}$. Thus we must not assume that $\omega^{\#}$ satisfies (aDec), which means that extra care must be taken in the next proof regarding constants and inverse functions, but tools for this were established in [23].

Note that (2) implies (1) if we have a Sobolev inequality, but then we have difficulties with the case $s \ge n$ in (2), so we provide separate proofs for the two cases.

Proposition 4.1. Let $B_R \subset \Omega$, $R \leq 1$ and let $\omega \in \Phi_w(\Omega)$ satisfy (A0) and (aDec). Assume that $\varphi \in \Phi_w(\Omega)$ satisfies (A1- ω) and that one of the following hold:

(1)
$$u \in L^{\omega^{\#}}(B_R)$$
 with $||u||_{\omega^{\#}} \leq d$, where $\omega^{\#} \in \Phi_{w}(\Omega)$.
(2) $u \in W^{1,\omega}(B_R)$ with $||u||_{1,\omega} \leq d$.

Then (3.3) holds, for $v := \frac{u+R}{R}$, i.e.

$$\omega_{B_R}^{-} \left(\oint_{B_R} v \, dx \right) \leqslant \frac{d_2}{|B_R|}$$

Proof. By an elementary embedding, $||u||_{W^{1,\omega_B^-}(B)} \leq ||u||_{W^{1,\omega}(\Omega)}$. Therefore it suffices to prove the result for $\omega \in \Phi_w$ (i.e. ω independent of x) and apply this result to $\omega_{B_R}^-$, and similarly in the case of assumption (1).

Let us first use assumption (1). Jensen's inequality [23, Lemma 4.3.1], and (A0) of $\omega^{\#}$ yield that, for sufficiently small d' > 0,

$$\omega^{\#} \left(2d' \oint_{B_R} \frac{u+R}{2} dx \right) \leqslant \oint_{B_R} \omega^{\#} \left(\frac{u+\beta_0}{2} \right) dx \lesssim |B_R|^{-1} + 1 \lesssim |B_R|^{-1}$$

with β_0 from (A0). By [23, Lemma 2.3.9], $(\omega^{\#})^{-1}(\omega^{\#}(t)) \approx t$ or $\omega^{\#}(t) = 0$. In the former case,

$$\int_{B_R} u + R \, dx \lesssim (\omega^{\#})^{-1} (c \, |B_R|^{-1}) = c' R \omega^{-1} (c \, |B_R|^{-1}) \approx R \omega^{-1} (|B_R|^{-1}).$$

Since ω satisfies (aDec), we can move the constants outside. We divide both sides by R and apply ω to obtain $\omega(f_{B_R} v \, dx) \lesssim |B_R|^{-1}$, i.e. (3.3). In the case $\omega^{\#}(t) = 0$ we conclude from (A0) and (Dec)_n of ω that $t \leq d''$ and so

$$\omega\left(\oint_{B_R} v \, dx\right) \leqslant \omega\left(\frac{1}{d'R} 2d' \oint_{B_R} \frac{u+R}{2} \, dx\right) \leqslant \omega\left(\frac{d''}{d'R}\right) \lesssim R^{-n}$$

also by $(Dec)_n$ of ω . Thus we have (3.3) under assumption (1).

For the other case, we first use Hölder's inequality to obtain

$$\int_{B_R} v \, dx \lesssim R^{-n} \Big(\int_{B_R} (u+R)^{n'} \, dx \Big)^{\frac{1}{n'}}.$$

Then we use that B_R is a $W^{1,1}$ -extension domain, extend u + R to \mathbb{R}^n and denote this extension by v. We obtain by the $W^{1,1}$ -Sobolev embedding

$$||u+R||_{L^{n'}(B_R)} \leq ||v||_{L^{n'}(\mathbb{R}^n)} \lesssim ||v||_{W^{1,1}(\mathbb{R}^n)} \lesssim ||u+R||_{W^{1,1}(B_R)}.$$

Combining the above steps, we have

$$\int_{B_R} v \, dx \leqslant \int_{B_R} u + R + |\nabla u| \, dx.$$

Jensen's inequality, (Dec) of ω , (A0) of ω , $R \leq 1$, and $||u||_{1,\omega} \leq d$ yield

$$\begin{split} \omega\Big(\int_{B_R} v\,dx\Big) &\leqslant \omega\Big(\int_{B_R} u + R + |\nabla u|\,dx\Big) \leqslant \int_{B_R} \omega(u + R + |\nabla u|)\,dx\\ &\lesssim \int_{B_R} \omega(u) + \omega(|\nabla u|) + \omega(1)\,dx \lesssim \frac{1}{|B_R|}. \end{split}$$

Note that in the next proposition if we consider case (2) with $\omega = \varphi$, then s = p and $s^* > \max\{\frac{n}{p}, 1\}(q-p)$ is equivalent to $p^* > q$, the condition from Theorem 1.4.

Proposition 4.2. Let $\omega \in \Phi_w(\Omega)$ satisfy (A0), (aDec), and (aInc)_s, $s \ge 1$. Let $\varphi \in \Phi_w(\Omega)$ satisfy (A0), (A1- ω) and (aDec) and assume that one of the following hold:

(1) $u \in L^{\omega^{\#}}(B_R)$ with $||u||_{\omega^{\#}} \leq d$, where $\omega^{\#} \in \Phi_{w}(\Omega)$. (2) $u \in W^{1,\omega}(B_R)$ with $||u||_{1,\omega} \leq d$.

If $s^* \ge \beta(q-p)$, then (3.8) holds, i.e.

$$\int_{B_r} \left(\frac{\varphi(x,v)}{\varphi_{B_r}^-(v)}\right)^\beta dx \leqslant C$$

for and $B_r \subset B_R$ and $v := \frac{u+r}{r}$.

Proof. Fix $B_r \subset B_R$. As in the previous proof, it suffices to consider $\omega \in \Phi_w$ independent of x. Points where $u(x) \leq r$ make only a constant contribution to the integral, so we may assume that u > r and take $v = \frac{u}{r}$ for simplicity. Here we used (A0) and (aDec) of φ .

We denote $V_r := \omega^{-1}(|B_r|^{-1})$ and $E := \{v < V_r\}$. Then $\omega(V_r) \leq \frac{1}{|B_r|}$ and (A1- ω) of φ yields

$$\frac{1}{|B_r|} \int_{B_r \cap E} \left(\frac{\varphi(x,v)}{\varphi_{B_r}^-(v)}\right)^\beta dx \approx \frac{1}{|B_r|} \int_{B_r \cap E} \left(\frac{\varphi(x,v)}{\varphi(x,v)}\right)^\beta dx \leqslant 1;$$

we used v > 1 and (A0) for the lower bound of (A1- ω). This holds in both case (1) and (2), since these assumption were not used yet.

In $B_r \setminus E$ we have

$$\frac{\varphi(x,v)}{\varphi_{B_r}^-(v)} \lesssim \frac{\varphi(x,V_r)}{\varphi_{B_r}^-(V_r)} (\frac{v}{V_r})^{q-p} \approx (\frac{v}{V_r})^{q-p}$$

by $(aInc)_p$, $(aDec)_q$ and $(A1-\omega)$. Consider first the case $||u||_{\omega^{\#}} \leq d$. By the definition of $(w^{\#})^{-1}$, we obtain that $(w^{\#})^{-1}$ satisfies $(aDec)_{\frac{1}{2}-\frac{1}{2}}$ and hence $w^{\#}$ satisfies $(aInc)_{\frac{sn}{n-s}}$ and

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thus also $(aInc)_{\beta(q-p)}$. Therefore $t \mapsto \omega^{\#}(t^{1/(\beta(q-p))})$ satisfies $(aInc)_1$, and hence a Jensen-type inequality [23, Lemma 4.3.1] yields

$$\left(\int_{B_r} (\frac{v}{V_r})^{\beta(q-p)} dx \right)^{\frac{1}{\beta(q-p)}} \leqslant \frac{1}{rV_r} (\omega^{\#})^{-1} \left(\int_{B_r} \omega^{\#}(u) dx \right) \lesssim \frac{(\omega^{\#})^{-1}(|B_r|^{-1})}{rV_r}$$
$$= \frac{|B_r|^{1/n} \omega^{-1}(|B_r|^{-1})}{rV_r} \approx 1$$

where we used $\varrho_{\omega^{\#}}(u) \leq C$ and the definition of V_r . This completes the estimate in case (1).

In the case $\|\tilde{u}\|_{1,\omega} \leq d$, the Hölder inequality (with $\beta(q-p) \leq s^*$), the Sobolev inequality and the Jensen inequality (with $(aInc)_p$) give

$$\left(\int_{B_r} (\frac{v}{V_r})^{\beta(q-p)} dx \right)^{\frac{1}{\beta(q-p)}} \leqslant \frac{1}{V_r} \left(\int_{B_r} (\frac{u}{r})^{s^*} dx \right)^{\frac{1}{s^*}} \lesssim \frac{1}{V_r} \left(\int_{B_r} u^s + |\nabla u|^s dx \right)^{\frac{1}{s}}$$
$$\lesssim \frac{1}{V_r} \omega^{-1} \left(\int_{B_r} \omega(u) + \omega(|\nabla u|) dx \right) \lesssim \frac{\omega^{-1}(|B_r|^{-1})}{V_r} = 1. \square$$

We can now state the weak Harnack inequality for the variable exponent case as a corollary. For notation and terminology we refer to [18]. Note that here we can choose any s > 0.

Corollary 4.3. Let $\varphi(x,t) := t^{p(x)}$ be the variable exponent functional and let u be a nonnegative solution to (1.3) on B_{2R} . We assume that p is log-Hölder continuous with constant c_{\log} , $1 < p^- \leq p^+ < \infty$, and $||u||_s \leq d$ for some $s \in (0,\infty]$. Then for every $0 < \ell_0 < \ell(p_{B_R}^-)$, there exists $C = C(p^-, p^+, \ell_0, c_{\log}, d, n)$ and R_0 such that

$$\left(\int_{B_{2R}} (u+R)^{\ell_0} dx\right)^{\frac{1}{\ell_0}} \leqslant C(\operatorname{ess\,inf} u+R) \quad \text{for all } r \leqslant R_0.$$

Proof. We check that the assumptions of Theorem 3.9 are fulfilled. Define $s_* := \frac{ns}{n+s}$ so that $(s_*)^* = s$ and let $\omega(t) := t^{s_*}$. Now (A1- s_*) reads as $\beta^{p(x)}t^{p(x)-p(y)} \leq 1$ for $t \in [1, |B|^{-\frac{1}{s_*}}]$. Since p is log-Hölder continuous, this holds (see [23, Section 7.1] for details). Thus by Proposition 4.1 (1) condition (3.3) holds.

Let $\beta := n > \frac{n}{p_{B_R}^-}$. We choose R_0 so small that $n(p_{B_R}^+ - p_{B_R}^-) < s$ for $R \leq R_0$. Then $\omega^{\#}(t) = t^s$ satisfies $(aInc)_{\beta(p_{B_R}^+ - p_{B_R}^-)}$, and hence (3.8) holds by Proposition 4.2 (1). Thus the exponent can be chosen up to $\ell(p_{B_r}^-)$.

5. THE DOUBLE PHASE CASE AND COUNTER-EXAMPLES

Let us study the double phase case. Note that when $s = \infty$ in the next result we obtain the special case of bounded supersolutions from [7].

Corollary 5.1. Let $\varphi(x,t) := t^p + a(x)t^q$ be the double phase functional and let u be a nonnegative supersolution to (1.3) on B_{2R} . We assume that $a \in C^{0,\alpha}$ and $||u||_s \leq d$, $s \in (0,\infty]$. If $\alpha \geq (\frac{n}{s} + 1)(q - p)$ and $s \geq q - p$, then there exist positive constants ℓ_0 and C, such that

$$\left(\int_{B_{2R}} (u+R)^{\ell_0} dx\right)^{\frac{1}{\ell_0}} \leqslant C(\operatorname{essinf}_{B_R} u+R)$$

Furthermore, if $s > \max\{\frac{n}{p}, 1\}(q-p)$, then the inequality holds for any $\ell_0 < \ell(p)$.

Proof. Let us show that assumption (1) of Theorem 3.9 is fulfilled. Let $\omega(t) := t^{s_*}$, where $s_* = \frac{ns}{n+s}$. Let $x, y \in B_R$ and $t \in [1, |B_R|^{-1/s_*}]$. Then

$$\varphi(x,t) = t^p + a(x)t^q \leqslant t^p + (a(y) + cR^{\alpha})t^q \leqslant (1 + cR^{\alpha}t^{q-p})\varphi(y,t).$$

Since $t \leq R^{-\frac{n}{s_*}}$, the coefficient is bounded provided $\alpha - \frac{n}{s_*}(q-p) \geq 0$. Since $\frac{n}{s_*} = \frac{n}{s} + 1$, this is $\alpha \geq (\frac{n}{s} + 1)(q-p)$. We have proved that φ satisfies (A1- s_*). Now $(\omega^{\#})^{-1}(t) = t^{-1/n}\omega^{-1}(t) = t^{-1/n+1/s_*} = t^{1/s}$, and thus $\varrho_{\omega^{\#}}(u) = \varrho_{L^s}(u) \leq d^s$. Hence (3.3) holds by Proposition 4.1. Let us then consider (3.8). Since $\omega^{\#}(t) = t^s$, it satisfies (alnc)_{q-p} provided that $s \geq q-p$ and (alnc)_{$\beta(q-p)$} for $\beta > \max\{\frac{n}{p}, 1\}$ provided that $s > \max\{\frac{n}{p}, 1\}(q-p)$. Thus (3.8) follows by Proposition 4.2 and the claims follow by Theorem 3.9.

We will next give an example that the weak Harnack inequality need not hold if $\alpha < (\frac{n}{s} + 1)(q - p)$. We consider the one-dimensional case and focus on the role of s. In the double phase (or (p,q)-growth) case Fonseca, Malý and Mingione [19] have given more sophisticated counter-examples for s = p, which is related to the assumption (A1) in this case. See also [3]. However, to the best of our knowledge, examples for $s \neq p$ have not been considered before.

Let $\varphi \in \Phi_{\mathbf{w}}(\mathbb{R})$ be defined by $\varphi(x,0) := 0$ and

$$\varphi'(x,t) := \max\{t^{p-1}, a(x)t^{q-1}\},\$$

so that $\varphi(x,t) \approx \max\{t^p, a(x)t^q\} \approx t^p + a(x)t^q$, the double phase functional. Let u be a solution of $\left(\varphi'(x, |u'|)\frac{u'}{|u'|}\right)' = 0$ on the interval (a, b). We assume that $\lim_{x\to a^+} u(x) < \lim_{x\to b^-} u(x)$, so u is increasing and $\frac{u'}{|u'|} = 1$. Then the differential equation reduces to $\varphi'(x, u') \equiv c$, i.e.

$$u'(x) = \begin{cases} c^{\frac{1}{p-1}}, & \text{when } c^{-\frac{q-p}{p-1}} \ge a(x), \\ (c/a(x))^{\frac{1}{q-1}}, & \text{otherwise.} \end{cases}$$

We further assume that $a(x) := \max\{-x, 0\}^{\alpha}$. Since a is decreasing, we obtain that

$$u'(x) = \begin{cases} c^{\frac{1}{p-1}}, & \text{when } x \ge -x_0, \\ (c|x|^{-\alpha})^{\frac{1}{q-1}}, & \text{when } x < -x_0, \end{cases}$$

for $x_0 := c^{-\frac{1}{\alpha}\frac{q-p}{p-1}}$. Some solutions are illustrated in Figure 1 for different values of c with zero left boundary values at x = -1.

For c > 0 and $r > x_0$, we next consider a solution with $u(-x_0 - 2r) = 0$. When $\frac{\alpha}{q-1} \neq 1$ we have, for $\varrho \in [0, r]$ and $\alpha_2 := 1 - \frac{\alpha}{1-q}$,

$$u(-x_0-\varrho) = c^{\frac{1}{q-1}} \int_{-x_0-2r}^{-x_0-\varrho} |x|^{-\frac{\alpha}{q-1}} dx = c^{\frac{1}{q-1}} \frac{(x_0+2r)^{\alpha_2} - (x_0+\varrho)^{\alpha_2}}{\alpha_2} \approx c^{\frac{1}{q-1}} r^{\alpha_2}$$

Furthermore, $u(-x_0 + \varrho) = u(-x_0) + \varrho c^{\frac{1}{p-1}}$ since the derivative is constant on $(-x_0, \infty)$. With $c^{\frac{1}{p-1}-\frac{1}{q-1}} = c^{\frac{q-p}{(p-1)(q-1)}} = x_0^{-\frac{\alpha}{q-1}}$, we calculate

$$\frac{u(-x_0+r)}{u(-x_0)} = 1 + \alpha_2 \frac{rc^{\frac{1}{p-1}-\frac{1}{q-1}}x_0^{-\alpha_2}}{(1+2r/x_0)^{\alpha_2}-1} = 1 + \alpha_2 \frac{r/x_0}{(1+2r/x_0)^{\alpha_2}-1}.$$

Since $\alpha_2 < 1$, we see that the constant in the Harnack inequality in $B(-x_0, r)$ blows up if $r/x_0 \to \infty$.

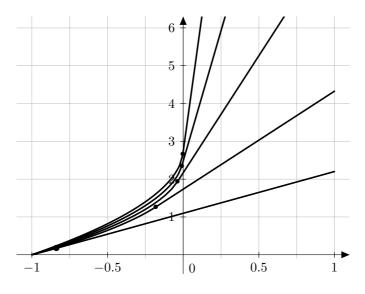


FIGURE 1. Solution for $c \in \{1.01, 1.1, 1.2, 1.3, 1.4\}$ in [-1, 1]. The parameters are p = 1.1, q = 2 and $\alpha = 0.5$. The right boundary values have been partly cut away but they are in the range [2, 32]. The dots indicate x_0 .

We next estimate the L^s -norm of u when s > 0. For $\varrho \in [\frac{r}{2}, 2r]$, $u(-x_0 + \varrho) = u(-x_0) + \varrho c^{\frac{1}{p-1}} \approx r c^{\frac{1}{p-1}}$. Since $u \ge 0$ is increasing and since only large values of the function are important when s > 0, we find that

$$\|u\|_{L^{s}(B(-x_{0},2r))} \approx c^{\frac{1}{p-1}}r^{1+\frac{1}{s}} = x_{0}^{1+\frac{1}{s}-\frac{\alpha}{q-p}}(\frac{r}{x_{0}})^{1+\frac{1}{s}} \quad \text{and} \quad \left(\int_{B(-x_{0},r)} u^{s} dx\right)^{\frac{1}{s}} \approx c^{\frac{1}{p-1}}r.$$

Similarly, if $x \in [-x_0 - r, -x_0]$, then by the earlier formula $u(x) \approx c^{\frac{1}{q-1}}r^{\alpha_2}$ and, since only small values of the function are important when -s < 0, we obtain that

$$\left(\int_{B(-x_0,r)} u^{-s} dx\right)^{-\frac{1}{s}} \approx c^{\frac{1}{q-1}} r^{\alpha_2}.$$

We can therefore say in this example that

$$\operatorname{ess\,sup}_{B(-x_0,r)} u = u(-x_0 + r) \approx \left(\oint_{B(-x_0,r)} u^s \, dx \right)^{\frac{1}{s}}$$

and

$$\operatorname{ess\,inf}_{B(-x_0,r)} u = u(-x_0 - r) \approx \left(\oint_{B(-x_0,r)} u^{-s} \, dx \right)^{-\frac{1}{2}}$$

for every s > 0, with constant depending on s but not on c or r. It follows that a possible failure in the Harnack inequality is due to the passing over zero.

As was mentioned before, the failure of the weak Harnack inequality happens if $\frac{r}{x_0} \to \infty$, for instance if we choose $r := x_0 \log \frac{1}{x_0}$. Furthermore,

$$\|u\|_{L^{s}(B(-x_{0},2r))} \approx x_{0}^{1+\frac{1}{s}-\frac{\alpha}{q-p}} (\frac{r}{x_{0}})^{1+\frac{1}{s}} = x_{0}^{1+\frac{1}{s}-\frac{\alpha}{q-p}} (\log \frac{1}{x_{0}})^{1+\frac{1}{s}}$$

remains bounded as $x_0 \to 0$ if $1 + \frac{1}{s} - \frac{\alpha}{q-p} > 0$. Since n = 1, this is equivalent to $\alpha < (1 + \frac{n}{s})(p - q)$, the complement of the inequality in Corollary 1.5. This shows the sharpness of the (A1-s_{*}) assumption in Theorem 1.4.

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In addition, we have $\varphi(x, u') \approx u' \varphi'(x, u') = cu'$. Thus

$$\int_{B(-x_0,2r)} \varphi(x,u') \, dx = c(u(-x_0+2r)-u(-x_0-2r)) = cu(-x_0+2r) \approx c^{\frac{p}{p-1}}r = x_0^{1-\frac{p\alpha}{q-p}} \frac{r}{x_0}.$$

Here we see that the Harnack inequality does not hold with uniform constant even though the $W^{1,\varphi}$ -norm of u remains bounded provided that $\frac{p\alpha}{q-p} < 1$. On the other hand, for the double phase functional (A1) is equivalent to $\frac{p\alpha}{q-p} \ge 1$, and we showed above that the weak Harnack inequality holds in this case. This proves the sharpness of the (A1) assumption in Theorem 1.4.

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