Degrees of Infinite Words, Polynomials and Atoms*

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Abstract. Our objects of study are finite state transducers and their power for transforming infinite words. Infinite sequences of symbols are of paramount importance in a wide range of fields, from formal languages to pure mathematics and physics. While finite automata for recognising and transforming languages are well-understood, very little is known about the power of automata to transform infinite words.
We use methods from linear algebra and analysis to show that there is an infinite number of atoms in the transducer degrees, that is, minimal non-trivial degrees.

1 Introduction

The transformation realised by finite state transducers induces a partial order of degrees of infinite words: for words $v, w \in \Delta^{\mathbb{N}}$, we write $v \geq w$ if v can be transformed into w by some finite state transducer. If $v \geq w$, then v can be thought of as *at least as complex as* w. This complexity comparison induces equivalence classes of words, called *degrees*, and a partial order on these degrees, that we call *transducer degrees*.

The ensuing hierarchy of degrees is analogous to the recursion theoretic *degrees of unsolvability*, also known as *Turing degrees*, where the transformational devices are Turing machines. The Turing degrees have been widely studied in the 60's and 70's. However, as a complexity measure, Turing machines are too strong: they trivialise the classification problem by identifying all computable infinite words. Finite state transducers give rise to a much more fine-grained hierarchy.

We are interested in the structural properties of the hierarchy of transducer degrees. In this paper, we investigate the existence of atom degrees. An *atom degree* is a minimal non-trivial degree, that is, a degree that is directly above the bottom degree without interpolant degree.

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Our contribution In [6] and [5] it has been proven that the degree of the words $\langle n \rangle$ and $\langle n^2 \rangle$ are atoms. Surprisingly, we find that this does not hold for $\langle n^3 \rangle$. In particular, we show that the degree of $\langle n^k \rangle$ is *never* an atom for $k \geq 3$ (see Theorem 5.4). On the other hand, we prove that for every $k > 0$, there exists a unique atom among the degrees of words $\langle p(n) \rangle$ for polynomials $p(n)$ of order k (see Theorem 6.9). (To avoid confusion between two meanings of *degrees*, namely *degrees of words* and *degrees of polynomials*, we speak of the *order* of a polynomial.) We moreover show that this atom is the infimum of all degrees of polynomials $p(n)$ of order k.

Further related work The paper [9] discusses complexity hierarchies derived from notions of reduction. The paper [7] gives an overview over the subject of transducer degrees and compares them with the well-known Turing degrees [14,10]. Restricting the transducers to output precisely one letter in each step, we arrive at Mealy machines. These gives rise to an analogous hierarchy of Mealy degrees that has been studied in [3,12]. The structural properties of this hierarchy are very different from the transducer degrees, see further [7].

2 Preliminaries

Let Σ be an alphabet. We write ε for the empty word, Σ^* for the set of finite words over Σ, and let $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. The set of infinite words over Σ is $\Sigma^{\mathbb{N}} = \{\sigma \mid \sigma : \mathbb{N} \to \Sigma\}$ and we let $\Sigma^\infty = \Sigma^* \cup \Sigma^{\mathbb{N}}$. Let $u, w \in \Sigma^\infty$. Then u is called a *prefix of* w, denoted $u \sqsubseteq w$, if $u = w$ or there exists $u' \in \Sigma^\infty$ such that $uu' = w$.

A *sequential finite state transducer* (FST) [2,13], a.k.a. *deterministic generalised sequential machine (DGSM)*, is a finite automaton with input letters and finite output words along the edges.

Definition 2.1. A *sequential finite state transducer* $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ consists of a finite *input alphabet* Σ, a finite *output alphabet* Γ, a finite set of *states* Q, an *initial state* $q_0 \in Q$, a *transition function* $\delta : Q \times \Sigma \to Q$, and an *output function* $\lambda : Q \times \Sigma \to \Gamma^*$. Whenever the alphabets Σ and Γ are clear from the context, we write $A = \langle Q, q_0, \delta, \lambda \rangle$.

We only consider sequential transducers and will simply speak of finite state transducers henceforth.

Definition 2.2. Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be a finite state transducer. We homomorphically extend the transition function δ to $Q \times \Sigma^* \to Q$ by: for $q \in Q$, $a \in \Sigma$, $u \in \Sigma^*$ let $\delta(q, \varepsilon) = q$ and $\delta(q, au) = \delta(\delta(q, a), u)$. We extend the output function λ to $Q \times \Sigma^\infty \to \Gamma^\infty$ by: for $q \in Q$, $a \in \Sigma$, $u \in \Sigma^\infty$, let $\lambda(q, \varepsilon) = \varepsilon$ and $\lambda(q, au) = \lambda(q, a) \cdot \lambda(\delta(q, a), u)$.

3 Transducer Degrees

In this section, we explain how finite state transducers give rise to a hierarchy of degrees of infinite words, called transducer degrees. First, we formally introduce the transducibility relation $\geq$ on words as realised by finite state transducers.

Definition 3.1. Let $w \in \Sigma^{\mathbb{N}}$, $u \in \Gamma^{\mathbb{N}}$ for finite alphabets Σ, Γ. Let $A = \langle \Sigma, \Gamma, Q, q_0, \delta, \lambda \rangle$ be a finite state transducer. We write $w \geq_{\mathrm{A}} u$ if $u = \lambda(q_0, w)$. We write $w \geq u$, and say that u is a *transduct* of w, if there exists a finite state transducer A such that $w \geq_{\mathrm{A}} u$.

Note that the transducibility relation $\geq$ is a pre-order. It thus induces a partial order of 'degrees', the equivalence classes with respect to $\geq \cap \leq$. We denote equivalence using $\equiv$. It is not difficult to see that every word over a finite alphabet is equivalent to a word over the alphabet $\mathbf{2} = \{0, 1\}$. For the study of transducer degrees it suffices therefore to consider words over the latter alphabet.

Definition 3.2. Define the equivalence relation $\equiv \; = \; (\geq \cap \leq)$. The *(transducer) degree* $w^{\equiv}$ of an infinite word w is the equivalence class of w with respect to $\equiv$, that is, $w^{\equiv} = \{u \in \mathbf{2}^{\mathbb{N}} \mid w \equiv u\}$. We write $\mathbf{2}^{\mathbb{N}}/_{\equiv}$ to denote the set of degrees $\{w^{\equiv} \mid w \in \mathbf{2}^{\mathbb{N}}\}$.

The *transducer degrees* form the partial order $\langle \mathbf{2}^{\mathbb{N}}/_{\equiv}, \geq \rangle$[4] induced by the pre-order $\geq$ on $\mathbf{2}^{\mathbb{N}}$, that is, for words $w, u \in \mathbf{2}^{\mathbb{N}}$ we have $w^{\equiv} \geq u^{\equiv} \iff w \geq u$.

The *bottom degree* $\mathbf{0}$ of the transducer degrees is the least degree of the hierarchy, that is, the unique degree $\mathfrak{a} \in \mathbf{2}^{\mathbb{N}}/_{\equiv}$ such that $\mathfrak{a} \leq \mathfrak{b}$ for every $\mathfrak{b} \in \mathbf{2}^{\mathbb{N}}/_{\equiv}$. The bottom degree $\mathbf{0}$ consists of the ultimately periodic words, that is, words of the form $uvvv\cdots$ for finite words u, v where $v \neq \varepsilon$.

An atom is a degree that has only $\mathbf{0}$ below itself.

Definition 3.3. An *atom* is a minimal non-bottom degree, that is, a degree $\mathfrak{a} \in \mathbf{2}^{\mathbb{N}}/_{\equiv}$ such that $\mathbf{0} < \mathfrak{a}$ and there exists no $\mathfrak{b} \in \mathbf{2}^{\mathbb{N}}/_{\equiv}$ with $\mathbf{0} < \mathfrak{b} < \mathfrak{a}$.

4 Spiralling Words

We now consider *spiralling words* over the alphabet $\mathbf{2} = \{0, 1\}$ for which the distance of consecutive 1's in the word grows to infinity. We additionally require that the sequence of distances of consecutive 1's is ultimately periodic modulo every natural number. The class of spiralling words allows for a characterisation of their transducts in terms of weighted products.

For a function $f : \mathbb{N} \to \mathbb{N}$, we define $\langle f \rangle \in \mathbf{2}^{\mathbb{N}}$

$$\langle f \rangle = \prod_{i=0}^{\infty} 10^{f(i)} = 10^{f(0)}\, 10^{f(1)}\, 10^{f(2)} \; \cdots \quad .$$

We write $\langle f(n) \rangle$ as shorthand for $\langle n \mapsto f(n) \rangle$.

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[4] We note that finite state transducers transform infinite words to finite or infinite words. The result of the transformation is finite if the transducer outputs the empty word ε for all except a finite number of letters of the input word. We are interested in infinite words only, since the set of finite words would merely entail two spurious extra sub-bottom degrees in the hierarchy of transducer degrees.

Definition 4.1. A function $f : \mathbb{N} \to \mathbb{N}$ is called *spiralling* if

(i) $\lim_{n\to\infty} f(n) = \infty$, and
(ii) for every $m \geq 1$, the function $n \mapsto f(n) \bmod m$ is ultimately periodic.

A word $\langle f \rangle$ is called *spiralling* whenever f is spiralling.

For example, $\langle p(n) \rangle$ is spiralling for every polynomial $p(n)$ with natural numbers as coefficients. Spiralling functions are called 'cyclically ultimately periodic' in the literature [4]. For a tuple $\boldsymbol{\alpha} = \langle \alpha_0, \ldots, \alpha_m \rangle$, we define

- the *length* $|\boldsymbol{\alpha}| = m + 1$, and
- its *rotation* by $\boldsymbol{\alpha}' = \langle \alpha_1, \ldots, \alpha_m, \alpha_0 \rangle$.

Let A be a set and $f : \mathbb{N} \to A$ a function. We write $\mathcal{S}^k(f)$ for the *k-th shift* of f defined by $\mathcal{S}^k(f)(n) = f(n + k)$.

We use 'weights' to represent linear functions.

Definition 4.2. A *weight* $\boldsymbol{\alpha}$ is a tuple $\langle a_0, \ldots, a_{k-1}, b \rangle \in \mathbb{Q}^{k+1}$ of rational numbers such that $k \in \mathbb{N}$ and $a_0, \ldots, a_{k-1} \geq 0$. The weight $\boldsymbol{\alpha}$ is called

- *non-constant* if $a_i \neq 0$ for some $i < k$, else *constant*,
- *strongly non-constant* if $a_i, a_j \neq 0$ for some $i < j < k$.

Now, let us also consider a tuple of tuples. For a tuple $\boldsymbol{\alpha} = \langle \boldsymbol{\alpha_0}, \ldots, \boldsymbol{\alpha_{m-1}} \rangle$ of weights we define $||\boldsymbol{\alpha}|| = \sum_{i=0}^{m-1} (\, |\boldsymbol{\alpha_i}| - 1\,)$.

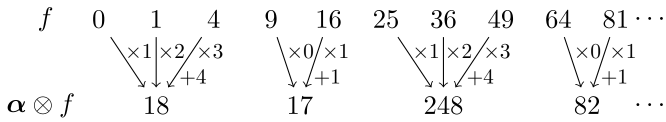
Definition 4.3. Let $f : \mathbb{N} \to \mathbb{Q}$ be a function. For a weight $\boldsymbol{\alpha} = \langle a_0, \ldots, a_{k-1}, b \rangle$ we define $\boldsymbol{\alpha} \cdot f \in \mathbb{Q}$ by $\boldsymbol{\alpha} \cdot f \;=\; a_0 f(0) + a_1 f(1) + \cdots + a_{k-1} f(k-1) + b$. For a tuple of weights $\boldsymbol{\alpha} = \langle \boldsymbol{\alpha_0}, \boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_{m-1}} \rangle$, we define the *weighted product* $\boldsymbol{\alpha} \otimes f : \mathbb{N} \to \mathbb{Q}$ by induction on n:

$$(\boldsymbol{\alpha} \otimes f)(0) = \boldsymbol{\alpha_0} \cdot f$$
$$(\boldsymbol{\alpha} \otimes f)(n+1) = (\boldsymbol{\alpha}' \otimes \mathcal{S}^{|\boldsymbol{\alpha_0}|-1}(f))(n) \qquad (n \in \mathbb{N})$$

We say that $\boldsymbol{\alpha} \otimes f$ is a *natural weighted product* if $(\boldsymbol{\alpha} \otimes f)(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$.

Weighted products are easiest understood by an example.

Example 4.4. Let $f(n) = n^2$ be a function and $\boldsymbol{\alpha} = \langle \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2} \rangle$ a tuple of weights with $\boldsymbol{\alpha_1} = \langle 1, 2, 3, 4 \rangle$, $\boldsymbol{\alpha_2} = \langle 0, 1, 1 \rangle$. Then the weighted product $\boldsymbol{\alpha} \otimes f$ can be visualised as follows



Intuitively, the weight $\boldsymbol{\alpha_1} = \langle 1, 2, 3, 4 \rangle$ means that 3 consecutive entries are added while being multiplied by 1, 2 and 3, respectively, and 4 is added to the result.

We introduce a few operations on weights. We define scalar multiplication of weights in the obvious way. We also introduce a multiplication $\odot$ that affects only the last entry of weights (the constant term).

Definition 4.5. Let $c \in \mathbb{Q}_{\geq 0}$, $\boldsymbol{\alpha} = \langle a_0, \ldots, a_{\ell-1}, b\rangle$ a weight, $\boldsymbol{\beta} = \langle \boldsymbol{\beta_0}, \ldots, \boldsymbol{\beta_{m-1}}\rangle$ a tuple of weights. We define

$$c\boldsymbol{\alpha} = \langle ca_0, \ldots, ca_{\ell-1}, cb\rangle \qquad \boldsymbol{\alpha} \odot c = \langle a_0, \ldots, a_{\ell-1}, bc\rangle$$
$$c\boldsymbol{\beta} = \langle c\boldsymbol{\beta_0}, \ldots, c\boldsymbol{\beta_{m-1}}\rangle \qquad \boldsymbol{\beta} \odot c = \langle \boldsymbol{\beta_0} \odot c, \ldots, \boldsymbol{\beta_{m-1}} \odot c\rangle$$

The following lemma follows directly from the definitions.

Lemma 4.6. *Let $c \in \mathbb{Q}_{\geq 0}$, $\boldsymbol{\alpha}$ a tuple of weights, and $f : \mathbb{N} \to \mathbb{Q}$ a function. Then $c(\boldsymbol{\alpha} \otimes f) = (c\boldsymbol{\alpha}) \otimes f = (\boldsymbol{\alpha} \odot c) \otimes (cf)$.* □

It is straightforward to define a *composition* of tuples of weights such that $\boldsymbol{\beta} \otimes (\boldsymbol{\alpha} \otimes f) = (\boldsymbol{\beta} \otimes \boldsymbol{\alpha}) \otimes f$ for every function $f : \mathbb{N} \to \mathbb{Q}$. Note that $\boldsymbol{\alpha} \otimes f$ is already defined. For the precise definition of $\boldsymbol{\beta} \otimes \boldsymbol{\alpha}$, we refer to Appendix A. It involves many details whose explicitation would not be illuminating. We will employ the following two properties of composition.

Lemma 4.7. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be tuples of weights. Then we have that $\boldsymbol{\beta} \otimes (\boldsymbol{\alpha} \otimes f) = (\boldsymbol{\beta} \otimes \boldsymbol{\alpha}) \otimes f$ for every function $f : \mathbb{N} \to \mathbb{Q}$.* □

Lemma 4.8. *Let $\boldsymbol{\alpha}$ be tuple of weights, and $\boldsymbol{\beta}$ a tuple of strongly non-constant weights. Then $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ is of the form $\langle \gamma_0, \ldots, \gamma_{k-1}\rangle$ such that for every $i \in \mathbb{N}_{<k}$, the weight γ_i is either constant or strongly non-constant.* □

We need a few results on weighted products from [5].

Lemma 4.9 ([5]). *Let $f : \mathbb{N} \to \mathbb{N}$, and $\boldsymbol{\alpha}$ a tuple of weights. If $\boldsymbol{\alpha} \otimes f$ is a natural weighted product (i.e. $\forall n \in \mathbb{N}.\ (\boldsymbol{\alpha} \otimes f)(n) \in \mathbb{N}$), then $\langle f\rangle \geq \langle \boldsymbol{\alpha} \otimes f\rangle$.* □

For the proof of Theorem 5.3, below, we use the following auxiliary lemma. The lemma gives a detailed structural analysis, elaborated and explained in [5], of the transducts of a spiralling word $\langle f\rangle$.

Lemma 4.10 ([5]). *Let $f : \mathbb{N} \to \mathbb{N}$ be a spiralling function, and let $\sigma \in \mathbf{2}^{\mathbb{N}}$ be such that $\langle f\rangle \geq \sigma$ and $\sigma \notin \mathbf{0}$. Then there exist $n_0, m \in \mathbb{N}$, a word $w \in \mathbf{2}^*$, a tuple of weights $\boldsymbol{\alpha}$, and tuples of finite words $\boldsymbol{p}$ and $\boldsymbol{c}$ with $|\boldsymbol{\alpha}| = |\boldsymbol{p}| = |\boldsymbol{c}| = m > 0$ such that $\sigma = w \cdot \prod_{i=0}^{\infty} \prod_{j=0}^{m-1} p_j\, c_j^{\varphi(i,j)}$ where $\varphi(i,j) = (\boldsymbol{\alpha} \otimes \mathcal{S}^{n_0}(f))(mi+j)$, and*

(i) $c_j^{\omega} \neq p_{j+1} c_{j+1}^{\omega}$ for every j with $0 \leq j < m-1$, and $c_{m-1}^{\omega} \neq p_0 c_0^{\omega}$, and
(ii) $c_j \neq \varepsilon$, and α_j is non-constant, for all $j \in \mathbb{N}_{<m}$. □

Example 4.11. We continue Example 4.4. We have $\alpha = \langle \alpha_0, \alpha_1\rangle$. Accordingly, we have prefixes $p_0, p_1 \in \mathbf{2}^*$ and cycles $c_0, c_1 \in \mathbf{2}^*$. Then the transduct σ in Lemma 4.10, defined by the double product, can be derived as follows:

$$\begin{array}{ccccccccccccc}
f & 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 \cdots \\
 & \searrow & \alpha_0 & \swarrow & \searrow & \alpha_1 \swarrow & \searrow & \alpha_0 & \swarrow & \searrow & \alpha_1 \swarrow \\
\boldsymbol{\alpha} \otimes f & & 18 & & & 17 & & 248 & & & 82 \quad \cdots \\
\sigma = w & \cdot & p_0\, c_0^{18} & \cdot & & p_1\, c_1^{17} & \cdot & p_0\, c_0^{248} & \cdot & & p_1\, c_1^{82} \quad \cdots
\end{array}$$

The infinite word σ is the infinite concatenation of w followed by alternating $p_0 c_0^{e_0}$ and $p_1 c_1^{e_1}$, where the exponents e_0 and e_1 are the result of applying weights α_0 and α_1, respecively.

The following theorem characterises the transducts of spiralling words up to equivalence ($\equiv$).

Theorem 4.12 ([5]). *Let $f : \mathbb{N} \to \mathbb{N}$ be spiralling, and $\sigma \in \mathbf{2}^{\mathbb{N}}$. Then $\langle f \rangle \geq \sigma$ if and only if $\sigma \equiv \langle \boldsymbol{\alpha} \otimes \mathcal{S}^{n_0}(f) \rangle$ for some $n_0 \in \mathbb{N}$, and a tuple of weights $\boldsymbol{\alpha}$.*

Roughly speaking, the next proposition states that polynomials of order k are closed under transduction.

Proposition 4.13 ([5]). *Let $p(n)$ be a polynomial of order k with non-negative integer coefficients, and let σ be a transduct of $\langle p(n) \rangle$ with $\sigma \notin \mathbf{0}$. Then $\sigma \geq \langle q(n) \rangle$ for some polynomial $q(n)$ of order k with non-negative integer coefficients.*

5 The Degree of $\langle n^k \rangle$ is Not an Atom for $k \geq 3$

We show that the degree of $\langle n^k \rangle$ is not an atom for $k \geq 3$. For this purpose, we prove a strengthening of Theorem 4.12, a lemma on weighted products of strongly non-constant weights, and we employ the power mean inequality.

First, we recall the power mean inequality [8].

Definition 5.1. For $p \in \mathbb{R}$, the *weighted power mean* $M_p(\boldsymbol{x})$ of $\boldsymbol{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n_{>0}$ with respect to $\boldsymbol{w} = \langle w_1, w_2, \ldots, w_n \rangle \in \mathbb{R}^n_{>0}$ with $\sum_{i=1}^n w_i = 1$ is

$$M_{\boldsymbol{w},0}(\boldsymbol{x}) = \prod_{i=1}^n x_i^{w_i} \qquad\qquad M_{\boldsymbol{w},p}(\boldsymbol{x}) = (\sum_{i=1}^n w_i x_i^p)^{1/p} \,.$$

Proposition 5.2 (Power mean inequality). *For all $p, q \in \mathbb{R}$, $\boldsymbol{x}, \boldsymbol{w} \in \mathbb{R}^n_{>0}$:*

$$p < q \implies M_{\boldsymbol{w},p}(\boldsymbol{x}) \leq M_{\boldsymbol{w},q}(\boldsymbol{x})$$
$$(p = q \vee x_1 = x_2 = \cdots = x_n) \iff M_{\boldsymbol{w},p}(\boldsymbol{x}) = M_{\boldsymbol{w},q}(\boldsymbol{x}) \,.$$

Theorem 4.12 characterises transducts of spiralling sequences only up to equivalence. This makes it difficult to employ the theorem for proving non-transducibility. We improve the characterisation for the case of spiralling transducts as follows.

Theorem 5.3. *Let $f, g : \mathbb{N} \to \mathbb{N}$ be spiralling functions. Then $\langle g \rangle \geq \langle f \rangle$ if and only if some shift of f is a weighted product of a shift of g, that is:*

$$\mathcal{S}^{n_0}(f) = \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g)$$

for some $n_0, m_0 \in \mathbb{N}$ and a tuple of weights $\boldsymbol{\alpha}$.

Theorem 5.3 is a strengthening of Theorem 4.12 in the sense that the characterisation uses equality (= and shifts) instead of equivalence ($\equiv$). We will employ the gained precision to show that certain spiralling transducts of $\langle n^k \rangle$ cannot be transduced back to $\langle n^k \rangle$, and conclude that $\langle n^k \rangle$ is not an atom for $k \geq 3$. See further Theorem 5.4. Note, however, that Theorem 5.3 only characterises spiralling transducts whereas Theorem 4.12 characterises all transducts.

Proof (Theorem 5.3). For the direction '$\Leftarrow$', assume that $\mathcal{S}^{n_0}(f) = \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g)$. Then we have $\langle g \rangle \equiv \langle \mathcal{S}^{m_0}(g) \rangle \geq \langle \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g) \rangle = \langle \mathcal{S}^{n_0}(f) \rangle \equiv \langle f \rangle$ by invariance under shifts and by Lemma 4.9.

For the direction '$\Rightarrow$', assume that $\langle g \rangle \geq \langle f \rangle$. Then by Lemma 4.10 there exist $m_0, m \in \mathbb{N}$, $w \in \mathbf{2}^*$, $\boldsymbol{\alpha}$, $\boldsymbol{p}$ and $\boldsymbol{c}$ with $|\boldsymbol{\alpha}| = |\boldsymbol{p}| = |\boldsymbol{c}| = m > 0$ such that:

$$\langle f \rangle = w \cdot \prod_{i=0}^{\infty} \prod_{j=0}^{m-1} p_j\, c_j^{\varphi(i,j)} \tag{1}$$

where $\varphi(i,j) = (\boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g))(mi + j)$ such that the conditions (i) and (ii) of Lemma 4.10 are fulfilled.

Note that, as $\lim_{n\to\infty} f(n) = \infty$, the distance of ones in the sequence $\langle g \rangle$ tends to infinity. For every $j \in \mathbb{N}_{<m}$, the word p_j occurs infinitely often in $\langle f \rangle$ by (1), and hence p_j can contain at most one occurrence of the symbol 1.

By condition (ii), we have for every $j \in \mathbb{N}_{<m}$ that $c_j \neq \varepsilon$, and the weight $\boldsymbol{\alpha_j}$ is not constant. As $\lim_{n\to\infty} g(n) = \infty$, it follows that c_j^2 appears infinitely often in $\langle f \rangle$ by (1). Hence c_j consists only of 0's, that is, $c_j \in \{0\}^+$ for every $j \in \mathbb{N}_{<m}$.

By condition (i) we never have $c_j^\omega = p_{j+1} c_{j+1}^\omega$ for $j \in \mathbb{N}_{<m}$ (where addition is modulo m). As $c_j^\omega = 0^\omega$ and $p_{j+1}0^\omega = p_{j+1}c_{j+1}^\omega$, we obtain that p_{j+1} must contain a 1. Hence, for every $k \in \mathbb{N}_{<m}$, the word p_j contains precisely one 1.

Finally, we apply the following transformations to ensure $p_j = 1$ and $c_j = 0$ for every $j \in \mathbb{N}_{<m}$:

(i) For every $j \in \mathbb{N}_{<m}$ such that $c_j = 0^h$ for some $h > 1$, we set $c_j = 0$ and replace the weight $\boldsymbol{\alpha_j}$ in $\boldsymbol{\alpha}$ by $h\boldsymbol{\alpha_j}$.

(ii) For every $j \in \mathbb{N}_{<m}$ such that $p_j = 0^h 1 0^\ell$ for some $h \geq 1$ or $\ell \geq 1$, we set $p_j = 1$ and replace the weight $\boldsymbol{\alpha_j}$ in $\boldsymbol{\alpha}$ by $(\boldsymbol{\alpha_j} + \ell)$ and the weight $\boldsymbol{\alpha_{j-1}}$ by $(\boldsymbol{\alpha_{j-1}} + h)$. Here, for a weight $\boldsymbol{\gamma} = \langle x_0, \ldots, x_{\ell-1}, y \rangle$ and $z \in \mathbb{Q}$, we write $\boldsymbol{\gamma} + z$ for the weight $\langle x_0, \ldots, x_{\ell-1}, y + z \rangle$. If $j = 0$, we moreover append 0^h to the word w.

Note that both transformations leave equation (1) valid, they do not change the result of the double product.

Thus we now have $p_j = 1$ and $c_j = 0$ for every $j \in \mathbb{N}_{<m}$. It follows from (1) that $\langle f \rangle = w \langle \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g) \rangle$. Hence $\mathcal{S}^{n_0}(f) = \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g)$ for some $n_0 \in \mathbb{N}$. □

Theorem 5.4. *For $k \geq 3$, the degree of $\langle n^k \rangle$ is not an atom.*

Proof. Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = n^k$. We have $\langle f \rangle \geq \langle g \rangle$ where $g : \mathbb{N} \to \mathbb{N}$ is defined by $g(n) = (2n)^k + (2n+1)^k$. Assume that we had $\langle g \rangle \geq \langle f \rangle$. Then, by Theorem 5.3 we have $\mathcal{S}^{n_0}(f) = \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(g)$ for some $n_0, m_0 \in \mathbb{N}$ and a tuple of weights $\boldsymbol{\alpha}$. Note that $g = \langle\langle 1,1,0 \rangle\rangle \otimes f$ and

$$\begin{aligned}\mathcal{S}^{n_0}(f) &= \boldsymbol{\alpha} \otimes \mathcal{S}^{m_0}(\langle\langle 1,1,0\rangle\rangle \otimes f) \\ &= \boldsymbol{\alpha} \otimes (\langle\langle 1,1,0\rangle\rangle \otimes \mathcal{S}^{2m_0}(f)) = \boldsymbol{\beta} \otimes \mathcal{S}^{2m_0}(f)\end{aligned}$$

where $\boldsymbol{\beta} = \boldsymbol{\alpha} \otimes \langle\langle 1,1,0 \rangle\rangle$. By Lemma 4.8 every weight in $\boldsymbol{\beta}$ is either constant or strongly non-constant. As $\mathcal{S}^{n_0}(f)$ is strictly increasing (and hence contains no constant subsequence), each weight in $\boldsymbol{\beta}$ must be strongly non-constant.

Let $\boldsymbol{\beta} = \langle \boldsymbol{\beta_0}, \ldots, \boldsymbol{\beta_{\ell-1}} \rangle$. For every $n \in \mathbb{N}$ we have:

$$\mathcal{S}^{n_0}(f)(\ell n) = (\boldsymbol{\beta} \otimes \mathcal{S}^{2m_0}(f))(\ell n) = \boldsymbol{\beta_0} \cdot \mathcal{S}^{2m_0 + ||\boldsymbol{\beta}|| \cdot n}(f)\ . \tag{2}$$

Then we have

$$\begin{aligned}\mathcal{S}^{n_0}(f)(\ell n) &= (n_0 + \ell n)^k = \sum_{i=0}^{k} \binom{k}{i} n_0^i \ell^{k-i} n^{k-i} \\ &= \ell^k n^k + k n_0 \ell^{k-1} n^{k-1} + \cdots + k n_0^{k-1} \ell n + n_0^k \; . \end{aligned} \tag{3}$$

Let $\boldsymbol{\beta_0} = \langle a_0, a_1, \dots, a_{h-1}, b\rangle$. We define $c_i = a_i ||\boldsymbol{\beta}||^k$ and $d_i = (2m_0 + i)/||\boldsymbol{\beta}||$. We obtain

$$\begin{aligned}\boldsymbol{\beta_0} \cdot \mathcal{S}^{2m_0 + ||\boldsymbol{\beta}|| \cdot n}(f) &= b + \sum_{i=0}^{h-1} a_i f(2m_0 + ||\boldsymbol{\beta}|| \cdot n + i) \\ &= b + \sum_{i=0}^{h-1} a_i f(||\boldsymbol{\beta}||(n + \tfrac{2m_0+i}{||\boldsymbol{\beta}||})) \\ &= b + \sum_{i=0}^{h-1} a_i ||\boldsymbol{\beta}||^k (n + d_i)^k = b + \sum_{i=0}^{h-1} c_i (n + d_i)^k \\ &= b + \sum_{i=0}^{h-1} c_i (n^k + k d_i n^{k-1} + \cdots + k d_i^{k-1} n + d_i^k) \; . \end{aligned} \tag{4}$$

Recall equation (2). Comparing the coefficients of n^k, n^{k-1} and n in (3) and (4) we obtain

$$\ell^k = \sum_{i=0}^{h-1} c_i \qquad k n_0 \ell^{k-1} = \sum_{i=0}^{h-1} c_i k d_i \qquad k n_0^{k-1} \ell = \sum_{i=0}^{h-1} c_i k d_i^{k-1} \text{, and hence}$$

$$1 = \sum_{i=0}^{h-1} \frac{c_i}{\ell^k} \qquad \frac{n_0}{\ell} = \sum_{i=0}^{h-1} \frac{c_i}{\ell^k} d_i \qquad \frac{n_0^{k-1}}{\ell^{k-1}} = \sum_{i=0}^{h-1} \frac{c_i}{\ell^k} d_i^{k-1} \; .$$

This is in contradiction with the weighted power means inequality (Proposition 5.2). Clearly all d_i are distinct, and, as a consequence of $\boldsymbol{\beta_0}$ being strongly non-constant, there are at least two $i \in \mathbb{N}_{<h}$ for which $c_i \neq 0$. Thus our assumption $\langle g\rangle \geq \langle f\rangle$ must have been wrong. Hence the degree of $\langle n^k\rangle$ is not an atom. □

6 Atoms of Every Polynomial Order

In the previous section, we have seen that $\langle n^k\rangle$ is not an atom for $k \geq 3$. In this section, we show that for every order $k \in \mathbb{N}$ there exists a polynomial $p(n)$ of order k such that the degree of the word $\langle p(n)\rangle$ is an atom. As a consequence, there are at least $\aleph_0$ atoms in the transducer degrees.

As we have seen in the proof of Theorem 5.4, whenever $k \geq 3$, we have that $\langle n^k\rangle \geq \langle g(n)\rangle$, but not $\langle g(n)\rangle \geq \langle n^k\rangle$ for $g(n) = (2n)^k + (2n+1)^k$. Thus there exist polynomials $p(n)$ of order k for which $\langle p(n)\rangle$ cannot be transduced to $\langle n^k\rangle$. However, the key observation underlying the construction in this section is the following: Although we may not be able to reach $\langle n^k\rangle$ from $\langle p(n)\rangle$, we can get arbitrarily close (Lemma 6.3, below). This enables us to employ the concept of *continuity*. We refer the reader to Appendix B, for a more elaborate, intuitive explanation of the proof of Theorem 6.9,

In order to have continuous functions over the space of polynomials to allow limit constructions, we now permit rational coefficients. For $k \in \mathbb{N}$, let $\mathfrak{Q}_k$ be the set of polynomials of order k with non-negative rational coefficients. We also use polynomials in $\mathfrak{Q}_k$ to denote spiralling sequences. However, we need to give meaning to $\langle q(n)\rangle$ for the case that the block sizes $q(n)$ are not natural numbers. For this purpose, we make use of the fact that the degree of a word $\langle f(n)\rangle$ is invariant under multiplication of the block sizes by a constant, as is easy to see. More precisely, for $f : \mathbb{N} \to \mathbb{N}$, we have $\langle f(n)\rangle \equiv \langle d \cdot f(n)\rangle$ for every $d \in \mathbb{N}$ with $d \geq 1$. So to give meaning to $\langle q(n)\rangle$, we multiply the polynomial by the least natural number $d > 0$ such that $d \cdot q(n)$ is a natural number for every $n \in \mathbb{N}$.

Definition 6.1. We call a function $f : \mathbb{N} \to \mathbb{Q}$ *naturalisable* if there exists a natural number $d \geq 1$ such that for all $n \in \mathbb{N}$ we have $(d \cdot f(n)) \in \mathbb{N}$.

For naturalisable $f : \mathbb{N} \to \mathbb{Q}$ we define $\langle f\rangle = \langle d \cdot f\rangle$ where $d \in \mathbb{N}$ is the least number such that $d \geq 1$ where for all $n \in \mathbb{N}$ we have $(d \cdot f(n)) \in \mathbb{N}$. (Note that, for $f : \mathbb{N} \to \mathbb{N}$, $\langle f(n)\rangle$ has been defined in Section 4.)

Observe that every $q(n) \in \mathfrak{Q}_k$ is naturalisable (multiply by the least common denominator of the coefficients). Also, naturalisable functions are preserved under weighted products.

Now, Lemma 4.9 can be generalised as follows. There is no longer need to require that the weighted product is natural. All weighted products of naturalisable functions can be realised by finite state transducers.

Lemma 6.2. *Let $f : \mathbb{N} \to \mathbb{Q}$ be naturalisable, and $\boldsymbol{\alpha}$ a tuple of weights. Then $\boldsymbol{\alpha} \otimes f$ is naturalisable and $\langle f\rangle \geq \langle \boldsymbol{\alpha} \otimes f\rangle$.*

Proof. Let $\boldsymbol{\alpha} = \langle \boldsymbol{\alpha_0}, \ldots, \boldsymbol{\alpha_{m-1}}\rangle$ for some $m \geq 1$. Let $c \in \mathbb{N}$ with $c \geq 1$ be minimal such that all entries of $c\boldsymbol{\alpha}$ are natural numbers. Let $d \in \mathbb{N}$ with $d \geq 1$ be the least natural number such that $\forall n \in \mathbb{N}\ (d \cdot f(n)) \in \mathbb{N}$.

Then we obtain $((dc\boldsymbol{\alpha}) \otimes f)(n) \in \mathbb{N}$ for ever $n \in \mathbb{N}$. By the definition of weighted products it follows immediately that $(dc\boldsymbol{\alpha}) \otimes f = dc(\boldsymbol{\alpha} \otimes f)$, and hence $\boldsymbol{\alpha} \otimes f$ is naturalisable. Let $e \in \mathbb{N}$ with $e \geq 1$ be the least natural number such that $\forall n \in \mathbb{N}\ (e \cdot (\boldsymbol{\alpha} \otimes f)(n)) \in \mathbb{N}$.

We have the following transduction

$$
\begin{aligned}
\langle f\rangle &= \langle df\rangle && \text{by Definition 6.1} \\
&\geq \langle ((c\boldsymbol{\alpha}) \odot d) \otimes (df)\rangle && \text{by Lemma 4.9} \\
&= \langle (dc\boldsymbol{\alpha}) \otimes f\rangle = \langle dc(\boldsymbol{\alpha} \otimes f)\rangle && \text{by Lemma 4.6} \\
&\geq \langle \langle\langle \frac{e}{dc}, 0\rangle\rangle \otimes (dc(\boldsymbol{\alpha} \otimes f))\rangle && \text{by Lemma 4.9} \\
&= \langle e(\boldsymbol{\alpha} \otimes f)\rangle = \langle \boldsymbol{\alpha} \otimes f\rangle && \text{by Definition 6.1}
\end{aligned}
$$

This concludes the proof. □

The following lemma states that every word $\langle q(n)\rangle$, for a polynomial $q(n) \in \mathfrak{Q}_k$ of order k, can be transduced arbitrarily close to $\langle n^k\rangle$.

Lemma 6.3. *Let $k \geq 1$ and let $q(n) \in \mathfrak{Q}_k$ be a polynomial of order k. For every $\varepsilon > 0$ we have $\langle q(n)\rangle \geq \langle n^k + b_{k-1}n^{k-1} + \cdots + b_1 n\rangle$ for some rational coefficients $0 \leq b_{k-1}, \ldots, b_1 < \varepsilon$.*

Proof. Let $q(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$, and let $\varepsilon > 0$ be arbitrary. Then for every $d \in \mathbb{N}$, we have

$$\begin{aligned}\langle q(n)\rangle &\geq \langle q(dn)\rangle \geq \langle \frac{q(dn)}{a_k d^k}\rangle = \langle n^k + \frac{a_{k-1}}{a_k d} n^{k-1} + \ldots + \frac{a_1}{a_k d^{k-1}} n^1 + \frac{a_0}{a_k d^k}\rangle \\ &\geq \langle n^k + \frac{a_{k-1}}{a_k d} n^{k-1} + \ldots + \frac{a_1}{a_k d^{k-1}} n^1\rangle\end{aligned}$$

The first transduction is picking a subsequence of the blocks. The second transduction is a division of the size of each block (application of Lemma 6.2 with the weight $\langle\langle 1/a_k d^k, 0\rangle\rangle$). The last transduction amounts to removing a constant number of zeros from each block (application of Lemma 6.2 with the weight $\langle\langle 1, -a_0/(a_k d^k)\rangle\rangle$). Finally, note that the last polynomial in the transduction is of the desired form if $d \in \mathbb{N}$ is chosen large enough. □

For polynomials $p(n) \in \mathfrak{Q}_k$, we want to express weighted products $\langle \boldsymbol{\alpha}\rangle \otimes p$ in terms of matrix products. For that purpose we need a couple of definitions.

Definition 6.4. For weights $\boldsymbol{\alpha} = \langle a_0, \ldots, a_{k-1}, b\rangle$ we define a column vector $U(\boldsymbol{\alpha}) = (a_0, \ldots, a_{k-1})^T$.

Definition 6.5. If $p(n) = \sum_{i=0}^{k} c_i n^i$ is a polynomial of order k, we define a column vector $V(p(n)) = (c_1, \ldots, c_k)^T$ and a square matrix

$$M(p(n)) = (V(p(kn+0)),\ \ldots,\ V(p(kn+k-1)))\,.$$

We also write $V(p)$ short for $V(p(n)$ and $M(p)$ for $M(p(n))$.

Note that we have omitted the constant term c_0 from the definition of $V(p)$. The reason is that for every $f : \mathbb{N} \to \mathbb{N}$ and $c \in \mathbb{N}$ we have $\langle f(n)\rangle \equiv \langle f(n) + c\rangle$. These words are of the same degree because a finite state transducer can add (or remove) a constant number of symbols 0 to (from) every block of 0's. For the same reason, b was omitted from the definition of $U(\boldsymbol{\alpha})$.

Example 6.6. Consider the polynomial n^3:

$$V(n^3) = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \text{and} \qquad M(n^3) = \begin{pmatrix}0 & 9 & 36\\ 0 & 27 & 54\\ 27 & 27 & 27\end{pmatrix}$$

where the columns vectors of the matrix $M(n^3)$ are given by $V((3n)^3)$, $V((3n+1)^3)$ and $V((3n+2)^3)$.

Lemma 6.7. *Let $k \geq 1$. Let $\boldsymbol{\alpha} = \langle a_0, \ldots, a_{k-1}, b\rangle$ be a weight and $p(n) \in \mathfrak{Q}_k$. Then $M(p)\,U(\boldsymbol{\alpha}) = V(\langle\boldsymbol{\alpha}\rangle \otimes p)$.*

Proof. A direct calculation shows that

$$M(p)\,U(\boldsymbol{\alpha}) = \sum_{i=0}^{k-1} a_i V(p(kn+i)) = V\big(\sum_{i=0}^{k-1} a_i p(kn+i)\big)$$

$$= V\Big(\sum_{i=0}^{k-1} a_i p(kn+i) + b\Big) = V(\langle \boldsymbol{\alpha} \rangle \otimes p) \ ,$$

which proves the lemma. □

Let us take a closer look at the matrix $M(n^k)$. The element on the ith row and jth column is $M_{i,j} = \binom{k}{i} k^i (j-1)^{k-i}$. Dividing the ith row by $\binom{k}{i} k^i$ for each i gives a Vandermonde-type matrix, which is invertible. Thus also $M(n^k)$ is invertible.

Lemma 6.8. *For $k \geq 1$, $M(n^k)$ is invertible.* □

Theorem 6.9. *Let $k \geq 1$. Let $a_0, \ldots, a_{k-1}$ be positive rational numbers, $\boldsymbol{\alpha} = \langle a_0, \ldots, a_{k-1}, 0 \rangle$, and*

$$p(n) = (\langle \boldsymbol{\alpha} \rangle \otimes n^k)(n) = \sum_{i=0}^{k-1} a_i (kn+i)^k.$$

Then $\langle q(n) \rangle \geq \langle p(n) \rangle$ for all $q(n) \in \mathfrak{Q}_k$. Moreover, the degree $\langle p(n) \rangle^{\equiv}$ is an atom. Note that the degree $\langle p(n) \rangle^{\equiv}$ is the infimum of all degrees of words $\langle q(n) \rangle$ with $q(n) \in \mathfrak{Q}_k$.

Proof. By Lemma 6.7, $M(n^k)\, U(\boldsymbol{\alpha}) = V(p)$. By Lemma 6.8, $M(n^k)$ is invertible and we can write $U(\boldsymbol{\alpha}) = M(n^k)^{-1} V(p)$. By Lemma 6.3, for every $\varepsilon > 0$ there exists $q_\varepsilon \in \mathfrak{Q}_k$ such that $\langle q(n) \rangle \geq \langle q_\varepsilon(n) \rangle$ and

$$q_\varepsilon(n) = n^k + b_{k-1} n^{k-1} + \cdots + b_1 n$$

with $0 \leq b_i \leq \varepsilon$ for every $i \in \{1, \ldots, k-1\}$. We will show that if ε is small enough, then $\langle q_\varepsilon(n) \rangle \geq \langle p(n) \rangle$.

We have $\lim_{\varepsilon \to 0} M(q_\varepsilon) = M(n^k)$. As $\det(M(n^3)) \neq 0$ and the determinant function is continuous, also $\det(M(q_\varepsilon)) \neq 0$ for all sufficiently small ε. Then $M(q_\varepsilon)$ is invertible, and we define $U_\varepsilon = M(q_\varepsilon)^{-1} V(p)$. We would like to have $U_\varepsilon = U(\gamma)$ for some weight γ. This is not always possible, because some elements of U_ε might be negative. However, by the continuity of matrix inverse and product,

$$\lim_{\varepsilon \to 0} U_\varepsilon = \lim_{\varepsilon \to 0} (M(q_\varepsilon)^{-1} V(p)) = (\lim_{\varepsilon \to 0} M(q_\varepsilon))^{-1} V(p) = M(n^k)^{-1} V(p) = U(\boldsymbol{\alpha})$$

Since every element of $U(\boldsymbol{\alpha})$ is positive, we can fix a small enough ε so that every element of U_ε is positive. Then we have $U_\varepsilon = U(\gamma)$ for some weight γ.

We have $M(q_\varepsilon)\, U(\gamma) = V(\langle \gamma \rangle \otimes q_\varepsilon)$ by Lemma 6.7, and $M(q_\varepsilon)\, U(\gamma) = V(p)$ by the definition of U_ε. As a consequence $(\langle \gamma \rangle \otimes q_\varepsilon)(n) = p(n) + c$ for some constant c. By Lemma 6.2, we obtain $\langle q_\varepsilon(n) \rangle \geq \langle p(n) \rangle$.

It remains to show that the degree $\langle p(n) \rangle^{\equiv}$ is an atom. Assume that $\langle p(n) \rangle \geq w$ and $w \notin \mathbf{0}$. By Proposition 4.13 we have $w \geq \langle q(n) \rangle$ for some $q(n) \in \mathfrak{Q}_k$. As shown above, $\langle q(n) \rangle \geq \langle p(n) \rangle$, thus $w \geq \langle p(n) \rangle$. Hence $\langle p(n) \rangle^{\equiv}$ is an atom. □

7 Future Work

Our results hint at an interesting structure of the transducer degrees of words $\langle p(n) \rangle$ for polynomials $p(n)$ of order $k \in \mathbb{N}$. Here, we have only scratched the surface of this structure. Many questions remain open, for example:

(i) What is the structure of 'polynomial spiralling' degrees (depending on $k \in \mathbb{N}$)? Is the number of degrees finite for every $k \in \mathbb{N}$?
(ii) Are there interpolant degrees between the degrees of $\langle n^k \rangle$ and $\langle p_k(n) \rangle$?
(iii) Are there continuum many atoms?
(iv) Is the degree of the Thue–Morse sequence an atom?

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Appendix

A Weighted Products

We define concatenation and unfolding of tuples of weights.

Definition A.1. Let $\boldsymbol{\alpha} = \langle \boldsymbol{\alpha_0}, \dots, \boldsymbol{\alpha_{m-1}} \rangle$, $\boldsymbol{\beta} = \langle \boldsymbol{\beta_0}, \dots, \boldsymbol{\beta_{m-1}} \rangle$ be a tuple of weights. We define *concatenation*:

$$\boldsymbol{\alpha} \,;\, \boldsymbol{\beta} = \langle \boldsymbol{\alpha_0}, \dots, \boldsymbol{\alpha_{m-1}}, \boldsymbol{\beta_0}, \dots, \boldsymbol{\beta_{m-1}} \rangle \ .$$

We define *unfolding* by induction on $n \in \mathbb{N}$ with $n > 0$:

$$\boldsymbol{\alpha}^1 = \boldsymbol{\alpha} \qquad\qquad \boldsymbol{\alpha}^{n+1} = \boldsymbol{\alpha} \,;\, \boldsymbol{\alpha}^n$$

Unfolding a tuple of weights does not change its semantics.

Lemma A.2. *Let $f : \mathbb{N} \to \mathbb{Q}$, $\boldsymbol{\alpha}$ a tuple of weights and $n \geq 1$. Then $\boldsymbol{\alpha} \otimes f = \boldsymbol{\alpha}^n \otimes f$.*

Proof. Follows immediately from the cyclic fashion in which the weights in the weighted product are applied. □

We will now define the product $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ of tuples of weights such that we have $\boldsymbol{\alpha} \otimes (\boldsymbol{\beta} \otimes f) = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \otimes f$ for every $f : \mathbb{N} \to \mathbb{Q}$. We need one auxiliary definition.

Definition A.3. For a weight $\boldsymbol{\gamma} = \langle x_0, \dots, x_{\ell-1}, y \rangle$ and a tuple of weights $\boldsymbol{\alpha} = \langle \boldsymbol{\alpha_0}, \dots, \boldsymbol{\alpha_{\ell-1}} \rangle$ with $\boldsymbol{\alpha_i} = \langle a_{i,0}, \dots, a_{i,m_i}, b_i \rangle$, we define the weight $\boldsymbol{\gamma} \cdot \boldsymbol{\alpha}$ by

$$\begin{aligned}\boldsymbol{\gamma} \cdot \boldsymbol{\alpha} = \langle & x_0 a_{0,0},\ \dots,\ x_0 a_{0,m_0}, \\ & x_1 a_{1,0},\ \dots,\ x_1 a_{1,m_1}, \\ & \qquad \vdots \\ & x_\ell a_{\ell,0},\ \dots,\ x_\ell a_{\ell,m_\ell}, \\ & x_0 b_0 + x_1 b_1 + \dots + x_\ell b_\ell + y \rangle \ .\end{aligned}$$

We now define the product of tuples of weights.

Definition A.4. For tuples of weights $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with the property $||\boldsymbol{\alpha}|| = |\boldsymbol{\beta}|$ we define $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ by induction on the tuple length:

$$\begin{aligned}\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \langle & \boldsymbol{\alpha_0} \cdot \langle \boldsymbol{\beta_0}, \dots, \boldsymbol{\beta_{|\alpha_0|-2}} \rangle \rangle \,; \\ & \big(\langle \boldsymbol{\alpha_1}, \dots, \boldsymbol{\alpha_{k-1}} \rangle \otimes \langle \boldsymbol{\beta_{|\alpha_0|-1}}, \dots, \boldsymbol{\beta_{\ell-1}} \rangle \big)\end{aligned}$$

where $\boldsymbol{\alpha} = \langle \boldsymbol{\alpha_0}, \dots, \boldsymbol{\alpha_{k-1}} \rangle$. and $\boldsymbol{\beta} = \langle \boldsymbol{\beta_0}, \dots, \boldsymbol{\beta_{\ell-1}} \rangle$. Here we stipulate that $\langle\rangle \otimes \langle\rangle = \langle\rangle$.

For $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $||\boldsymbol{\alpha}|| \neq |\boldsymbol{\beta}|$ we define $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ as follows:

$$\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \big(\boldsymbol{\alpha}^{\frac{c}{||\boldsymbol{\alpha}||}} \big) \otimes \big(\boldsymbol{\beta}^{\frac{c}{|\boldsymbol{\beta}|}} \big)$$

where $c \in \mathbb{N}$ is the least common multiple of $||\boldsymbol{\alpha}||$ and $|\boldsymbol{\beta}|$.

Example A.5. Let $\boldsymbol{\alpha} = \langle \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2} \rangle$ and $\boldsymbol{\beta} = \langle \boldsymbol{\beta_1}, \boldsymbol{\beta_2} \rangle$

$$\boldsymbol{\alpha_1} = \langle 2, 1, 3 \rangle \qquad \boldsymbol{\alpha_2} = \langle 1, 1 \rangle$$
$$\boldsymbol{\beta_1} = \langle 1, 2, 3, 4 \rangle \qquad \boldsymbol{\beta_2} = \langle 0, 1, 1 \rangle$$

Note that $\boldsymbol{\beta}$ is the tuple of weights used in Example 4.4. We compute $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$. We have $||\boldsymbol{\alpha}|| = 3$ and $|\boldsymbol{\beta}| = 2$. Thus, we have to unfold $\boldsymbol{\alpha}$ twice and $\boldsymbol{\beta}$ trice: $\boldsymbol{\alpha}^2 = \langle \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2} \rangle$ and $\boldsymbol{\beta}^3 = \langle \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2} \rangle$. Then

$$\begin{aligned}
&\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \boldsymbol{\alpha}^2 \otimes \boldsymbol{\beta}^3 \\
&= \langle \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2} \rangle \otimes \langle \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2} \rangle \\
&= \langle \boldsymbol{\alpha_1} \cdot \langle \boldsymbol{\beta_1}, \boldsymbol{\beta_2} \rangle \rangle \;;\; \langle \boldsymbol{\alpha_2}, \boldsymbol{\alpha_1}, \boldsymbol{\alpha_2} \rangle \otimes \langle \boldsymbol{\beta_1}, \boldsymbol{\beta_2}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2} \rangle \\
&\quad \vdots \\
&= \langle \boldsymbol{\alpha_1} \cdot \langle \boldsymbol{\beta_1}, \boldsymbol{\beta_2} \rangle \rangle \;;\; \langle \boldsymbol{\alpha_2} \cdot \langle \boldsymbol{\beta_1} \rangle \rangle \;;\; \langle \boldsymbol{\alpha_1} \cdot \langle \boldsymbol{\beta_2}, \boldsymbol{\beta_1} \rangle \rangle \;;\; \langle \boldsymbol{\alpha_2} \cdot \langle \boldsymbol{\beta_2} \rangle \rangle \\
&= \langle \langle 2, 4, 6, 0, 1, 12 \rangle,\ \langle 1, 2, 3, 5 \rangle,\ \langle 0, 2, 1, 2, 3, 9 \rangle,\ \langle 0, 1, 2 \rangle \rangle
\end{aligned}$$

Lemma. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ be tuples of weights. Then we have

$$\boldsymbol{\alpha} \otimes (\boldsymbol{\beta} \otimes f) = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \otimes f$$

for every function $f : \mathbb{N} \to \mathbb{Q}$. □

Lemma. Let $\boldsymbol{\alpha}$ be tuple of weights, and $\boldsymbol{\beta}$ a tuple of strongly non-constant weights. Then $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ is of the form $\langle \boldsymbol{\gamma_0}, \dots, \boldsymbol{\gamma_{k-1}} \rangle$ such that for every $i \in \mathbb{N}_{<k}$, the weight $\boldsymbol{\gamma_i}$ is either constant or strongly non-constant.

Proof. Follows directly from the definition of $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$. Every weight in $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ is a concatenation of scalar multiplications of weights in $\boldsymbol{\beta}$. □

B Additional Intuition for the Proof of Theorem 6.9

Let $k \geq 1$ and $a_0, \dots, a_{k-1} > 0$. Define

$$p_k(n) = \sum_{i=0}^{k-1} a_i (kn + i)^k \ .$$

Theorem 6.9 states that for every polynomial $q(n) \in \mathfrak{Q}_k$, we have $\langle q(n) \rangle \geq \langle p_k(n) \rangle$. Hence the degree of $\langle p(n) \rangle$ is an atom as a consequence of Proposition 4.13.

We give some more intuition for the proof of Theorem 6.9, involving a more explicit appeal to the continuity argument. For functions $f_0, \dots, f_{k-1} : \mathbb{N} \to \mathbb{Q}$, we define the function $\mathsf{zip}(f_0, \dots, f_{k-1}) : \mathbb{N} \to \mathbb{Q}$ a.k.a. *perfect shuffle* [1,11], by induction on n as follows

$$\mathsf{zip}(f_0, \dots, f_{k-1})(0) = f_0(0)$$

$$\mathsf{zip}(f_0, \dots, f_{k-1})(n+1) = \mathsf{zip}(f_1, \dots, f_{k-1}, \mathcal{S}^1(f_0))(n)$$

where $\mathcal{S}^i(f)$ is the *i-th shift* of f, defined by $m \mapsto f(m+i)$.

We have $\langle n^k \rangle \geq \langle p_k \rangle$ by the following transduction:

$$\begin{aligned}
\langle n^k \rangle &= \langle \mathsf{zip}((kn+0)^k, (kn+1)^k, \dots, (kn+k-1)^k) \rangle \\
&\geq \langle a_0(kn+0)^k + \dots + a_{k-1}(kn+k-1)^k \rangle \\
&= \langle p_k(n) \rangle
\end{aligned}$$

Thinking of n^k as an infinite word of natural numbers, then $(kn+0)^k$, $(kn+1)^k$, $\dots$, $(kn+k-1)^k$ are subsequences of n^k. Namely those subsequences picking every k-th element starting from element at index $0, 1, \dots, k-1$, respectively. Note that the transduction in the second line corresponds to the weighted product $\langle a_0, a_1, \dots, a_{k-1}, 0 \rangle$, and thus can be realised by a finite state transducer (Lemma 6.2).

However, this transduction works only for $\langle n^k \rangle$. It remains to be argued that there exists such a transduction from $\langle q(n) \rangle$ to $\langle p_k(n) \rangle$ for every polynomial $q(n) \in \mathfrak{Q}_k$,

Let us write $\sim_\varepsilon$ for the relation that relates polynomials of the same order whose coefficients differ by at most $\varepsilon > 0$. By Lemma 6.3 we can get arbitrarily close to $\langle n^k \rangle$. For every $\varepsilon > 0$, there exists $h(n) \in \mathfrak{Q}_k$ such that

$$\langle q(n) \rangle \geq \langle h(n) \rangle \qquad \text{and} \qquad h(n) \sim_\varepsilon n^k$$

Moreover, for $i \in \mathbb{N}_{<k}$, we have

$$h(kn+i) \sim_{\varepsilon'} (kn+i)^k \tag{5}$$

where ε' depends on ε (and i). If ε tends to 0, so will ε'.

The crucial observation is that $(kn+0)^k$, $(kn+1)^k$, $\dots$, $(kn+k-1)^k$ form a basis of the vector space of polynomials of order k with addition and scalar multiplication.[5] The property of 'being a basis' is continuous. Hence, for small enough ε, and using approximation (5), we conclude that $h(kn+0)$, $h(kn+1)$, $\dots$, $h(kn+k-1)$ form a basis as well. Thus, there exist $a'_0, a'_1, \dots, a'_{k-1} \in \mathbb{Q}$ such that

$$p_k(n) = a'_0 h(kn+0) + \dots + a'_{k-1} h(kn+k-1)$$

We have that

$$\begin{aligned}
\langle h(n) \rangle &= \langle \mathsf{zip}(h(kn+0), h(kn+1), \dots, h(kn+k-1)) \rangle \\
&\geq \langle a'_0 h(kn+0) + \dots + a'_{k-1} h(kn+k-1) \rangle \\
&= \langle p_k(n) \rangle
\end{aligned}$$

[5] The cautious reader will have observed that the basis consists only of k vectors while the vector space has dimension $k+1$. We tacitly ignore the constant terms of the polynomials since, in the transducer degrees, we have $\langle f \rangle \equiv \langle f + c \rangle$ for every $c \in \mathbb{Q}$.

However, for this transduction to work, we need to ensure that $a'_0, a'_1, \ldots, a'_{k-1} \geq 0$. Recall that $a_0, a_1, \ldots, a_{k-1} > 0$. Again, by continuity, a'_i approaches a_i as ε approaches 0. Hence, for small enough ε, we have $a'_i \geq 0$ for every $i \in \mathbb{N}_{<k}$. Thus we have $\langle q(n) \rangle \geq \langle h(n) \rangle \geq \langle p(n) \rangle$. Hence $p(n)$ is an atom.