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On starlike functions related with the convex conic domain

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Abstract

In the present paper, we study a new subclass $\mathcal{M}_p(\alpha,\beta)$ of p–valent functions and obtain some inequalities concerning the coefficients for the desired class. Also, by using the Hadamard product, we define a new general operator and find a condition such that it belongs to the class $\mathcal{M}_p(\alpha,\beta)$.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad p \in \mathbb{N} := \{1,2,3,\ldots\},$$

which are analytic and p–valent in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Further, we write $\mathcal{A}_1 = \mathcal{A}$. A function $f(z) \in \mathcal{A}_p$ is said to be in the class $\mathcal{M}_p(\alpha,\beta)$, if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \left|\frac{zf'(z)}{f(z)} - p\right| + p\alpha \quad (z \in \Delta), \tag{1.1}$$

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for some $\beta \leq 0$ and $\alpha > 1$. Notice that (1.1) follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < p\alpha \quad (z \in \Delta), \tag{1.2}$$

because β is negative or 0. Moreover, if we write (1.1) in the following equivalent form

$$\frac{|F(z) - p|}{\operatorname{Re}\{p\alpha - F(z)\}} < \frac{1}{-\beta} \quad (z \in \Delta),$$

where

$$F(z) := \frac{zf'(z)}{f(z)} \quad (z \in \Delta),$$

then we see that the relation between the distance $F(z)$ from the focus p and the distance $F(z)$ from the directrix $\operatorname{Re}\{w\} = p\alpha$ depends on the eccentricity $-1/\beta$. Therefore, if $f(z) \in \mathcal{M}_p(\alpha, \beta)$, then $F(z)$, $z \in \Delta$ lies in a convex conic domain: elliptic when $\beta < -1$, parabolic when $\beta = -1$ and hyperbolic when $-1 < \beta < 0$, or $F(\Delta)$ is the half–plane (1.2), for $\beta = 0$.

The class $\mathcal{M}_p(\alpha, \beta)$ cover many subclasses considered earlier by various authors, see [2, 3, 5, 7]. We remark that the class $\mathcal{M}_1(\alpha, \beta) = \mathcal{MD}(\alpha, \beta)$ was investigated earlier by J. Nishiwaki and S. Owa [4].

In this work we shall be mainly concerned with functions $f \in \mathcal{A}_p$ of the form

$$\left(\frac{z^p}{f(z)}\right)^{\mu} = 1 - \sum_{k=p}^{\infty} b_k z^k \quad (\mu > 0, z \in \Delta \cup \{1\}), \tag{1.3}$$

where $(z^p/f(z))^{\mu}$ represents principal powers (i.e. the principal branch of $(z^p/f(z))^{\mu}$ is chosen).

This paper is organized as follows. In Section 2, we present some inequalities for the class $\mathcal{M}_p(\alpha, \beta)$. In Section 3, by using the Hadamard product and applying the generalized Bessel function and the Gaussian hypergeometric function we introduce a new operator which we denote by $\mathcal{I}^{a,b}_{c,d}(p, e, \delta)(z)$. As an application, we prove that the operator $\mathcal{I}^{a,b}_{c,d}(p, e, \delta)(z)$ belongs to the class $\mathcal{M}_p(\alpha, \beta)$.

2. Main Results

Our first result is contained in the following:

Theorem 2.1. *Let the function f be in the class $\mathcal{M}_p(\alpha, \beta)$ and let f be of the form* (1.3) *for some b_k such that*

$$b_k \geq 0 \quad for \quad k \in \{p, p+1, p+2, \ldots\} \quad and \quad \sum_{k=p}^{\infty} b_k < 1.$$

Then

$$\sum_{k=p}^{\infty} [p\mu(\alpha - 1) + k(1 - \beta)] b_k \leq p\mu(\alpha - 1), \tag{2.1}$$

where $\beta \leq 0$, $\mu > 0$ and $\alpha > 1$.

Proof . Let $f \in \mathcal{M}_p(\alpha, \beta)$. A simple calculation gives us

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^p}{f(z)}\right)^{\mu} = \mu\left[p\left(\frac{z^p}{f(z)}\right)^{\mu} - \left(\frac{z^p}{f(z)}\right)^{\mu+1}\frac{f'(z)}{z^{p-1}}\right]. \tag{2.2}$$

Thus, by use of the above relation (2.2), we have

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta\left|\frac{zf'(z)}{f(z)} - p\right| + p\alpha$$

if and only if

$$\mathrm{Re}\left(p + \frac{-\frac{z}{\mu}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^p/f(z)\right)^{\mu}}{\left(z^p/f(z)\right)^{\mu}}\right) < \beta\left|\frac{-\frac{z}{\mu}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^p/f(z)\right)^{\mu}}{\left(z^p/f(z)\right)^{\mu}}\right| + p\alpha.$$

Since f is of the form (1.3), the last inequality may be equivalently written as

$$\mathrm{Re}\left(p + \frac{1}{\mu}\frac{\sum_{k=p}^{\infty} kb_k z^k}{1 - \sum_{k=p}^{\infty} b_k z^k}\right) < \frac{\beta}{\mu}\left|\frac{\sum_{k=p}^{\infty} kb_k z^k}{1 - \sum_{k=p}^{\infty} b_k z^k}\right| + p\alpha. \tag{2.3}$$

Now suppose that $z \in \Delta$ is real and tends to 1^- through reals, then from the last inequality (2.3), we get

$$\mu p + \frac{\sum_{k=p}^{\infty} kb_k}{1 - \sum_{k=p}^{\infty} b_k} \leq \beta\left|\frac{\sum_{k=p}^{\infty} kb_k}{1 - \sum_{k=p}^{\infty} b_k}\right| + p\mu\alpha$$

or equivalently

$$\mu p + \frac{\sum_{k=p}^{\infty} kb_k}{1 - \sum_{k=p}^{\infty} b_k} \leq \beta\frac{\sum_{k=p}^{\infty} kb_k}{1 - \sum_{k=p}^{\infty} b_k} + p\mu\alpha,$$

which gives (2.1). This competes the proof. □

Remark 2.2. *Taking $p = \mu = 1$ in the above Theorem 2.1, we get the result that was recently obtained by Aghalary et al. [1, Theorem 2.1 with $n = 1$].*

Next we derive the following:

Theorem 2.3. *Let $f \in \mathcal{A}_p$ be of the form* (1.3) *with $\mu > 0$. If*

$$\sum_{k=p}^{\infty}[p\mu(\alpha - 1) + k(1 - \beta)]|b_k| < p\mu(\alpha - 1), \tag{2.4}$$

then f is in the class $\mathcal{M}_p(\alpha, \beta)$, where $\beta \leq 0$ and $\alpha > 1$.

Proof . In the proof of Theorem 2.1, we saw the following inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) - p\alpha < \beta\left|\frac{zf'(z)}{f(z)} - p\right|,$$

which is equivalent to

$$\mathrm{Re}\left(\frac{\sum_{k=p}^{\infty} kb_k z^k}{1 - \sum_{k=p}^{\infty} b_k z^k}\right) - p\mu(\alpha - 1) < \beta\left|\frac{\sum_{k=p}^{\infty} kb_k z^k}{1 - \sum_{k=p}^{\infty} b_k z^k}\right|.$$

Thus, to show that f is in the class $\mathcal{M}_p(\alpha,\beta)$, it suffices to prove that

$$\operatorname{Re}\left(\frac{\sum_{k=p}^{\infty} kb_k z^k}{1-\sum_{k=p}^{\infty} b_k z^k}\right) - \beta\left|\frac{\sum_{k=p}^{\infty} kb_k z^k}{1-\sum_{k=p}^{\infty} b_k z^k}\right| < p\mu(\alpha-1). \tag{2.5}$$

Note that from (2.4), we obtain that

$$1-\sum_{k=p}^{\infty}|b_k| > 0.$$

Therefore one can rewrite (2.4) in the following equivalent form

$$\frac{\sum_{k=p}^{\infty} k|b_k|}{1-\sum_{k=p}^{\infty}|b_k|} - \beta\frac{\sum_{k=p}^{\infty} k|b_k|}{1-\sum_{k=p}^{\infty}|b_k|} < p\mu(\alpha-1). \tag{2.6}$$

Because $\beta \le 0$, we have

$$\operatorname{Re}\left(\frac{\sum_{k=p}^{\infty} kb_k z^k}{1-\sum_{k=p}^{\infty} b_k z^k}\right) - \beta\left|\frac{\sum_{k=p}^{\infty} kb_k z^k}{1-\sum_{k=p}^{\infty} b_k z^k}\right| \le \frac{\sum_{k=p}^{\infty} k|b_k|}{1-\sum_{k=p}^{\infty}|b_k|} - \beta\frac{\sum_{k=p}^{\infty} k|b_k|}{1-\sum_{k=p}^{\infty}|b_k|}. \tag{2.7}$$

Then (2.6) and (2.7) immediately follow (2.5) and therefore, $f \in \mathcal{M}_p(\alpha,\beta)$. □

Corollary 2.4. *Assume that $f \in \mathcal{A}$ and $(z/f(z))^{\mu} = 1-\sum_{k=1}^{\infty} b_k z^k$ with $\mu > 0$. If the function f satisfies the condition*

$$\sum_{k=1}^{\infty}[\mu(\alpha-1)+k(1-\beta)]|b_k| < \mu(\alpha-1),$$

for some $\beta \le 0$ and $\alpha > 1$, then f is in the class $\mathcal{MD}(\alpha,\beta)$.

Remark 2.5. *The case $p = \mu = 1$ in Theorem 2.3 was obtained by Aghalary et al. [1, Theorem 2.2].*

Theorem 2.6. *A function f of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ is in the class $\mathcal{M}_p(\alpha,\beta)$, if*

$$\sum_{k=p+1}^{\infty}[p\alpha+\beta+k(1-\beta)]|a_k| < p(\alpha-1), \tag{2.8}$$

for some $\beta \le 0$ and $\alpha > 1$.

Proof . If the function f belong to $\mathcal{A}_p$, then by definition of the class $\mathcal{M}_p(\alpha,\beta)$, we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) - p\alpha < \beta\left|\frac{zf'(z)}{f(z)} - p\right|$$

if and only if

$$\operatorname{Re}\left(\frac{p+\sum_{k=p+1}^{\infty} ka_k z^{k-p}}{1+\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right) - p\alpha < \beta\left|\frac{\sum_{k=p+1}^{\infty}(k-1)a_k z^{k-p}}{1+\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right|.$$

Thus, it suffices to show that

$$\operatorname{Re}\left(\frac{p+\sum_{k=p+1}^{\infty} ka_k z^{k-p}}{1+\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right) - \beta\left|\frac{\sum_{k=p+1}^{\infty}(k-1)a_k z^{k-p}}{1+\sum_{k=p+1}^{\infty} a_k z^{k-p}}\right| < p\alpha. \tag{2.9}$$

Note that from (2.8), we have

$$1-\sum_{k=p+1}^{\infty}|a_k|>0,$$

thus from (2.8) we obtain that

$$\frac{p+\sum_{k=p+1}^{\infty}k|a_k|}{1-\sum_{k=p+1}^{\infty}|a_k|}-\beta\frac{\sum_{k=p+1}^{\infty}(k-1)|a_k|}{1-\sum_{k=p+1}^{\infty}|a_k|}<p\alpha. \tag{2.10}$$

Moreover, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{p+\sum_{k=p}^{\infty}ka_kz^{k-p}}{1+\sum_{k=p}^{\infty}a_kz^{k-p}}\right)-\beta\left|\frac{\sum_{k=p}^{\infty}(k-1)a_kz^{k-p}}{1+\sum_{k=p}^{\infty}a_kz^{k-p}}\right| \\ \leq \frac{p+\sum_{k=p+1}^{\infty}k|a_k|}{1-\sum_{k=p+1}^{\infty}|a_k|}-\beta\frac{\sum_{k=p+1}^{\infty}(k-1)|a_k|}{1-\sum_{k=p+1}^{\infty}|a_k|}.\end{aligned} \tag{2.11}$$

Therefore, (2.10) and (2.11) follow (2.9). This completes the proof. □

Putting $p=1$ in Theorem 2.6 we have:

Corollary 2.7. *If $f\in\mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}[\alpha+\beta+k(1-\beta)]|a_k|<\alpha-1,$$

for some $\alpha>1$ and $\beta\leq 0$, then $f\in\mathcal{MD}(\alpha,\beta)$.

At the end of this section, by Theorem 2.6 we consider an example for the class $\mathcal{M}_p(\alpha,\beta)$.

Example 2.8. *Define the function $f\in\mathcal{A}_p$ as follows*

$$f(z)=z^p+\sum_{k=p+1}^{\infty}\frac{p^2(\alpha-1)e^{i\theta}}{[p\alpha+\beta+k(1-\beta)]k(k-1)}z^k,$$

where $\alpha>1$, $\beta\leq 0$ and $\theta\in\mathbb{R}$. Then the coefficients inequality (2.8) yields

$$\sum_{k=p+1}^{\infty}[p\alpha+\beta+k(1-\beta)]|a_k|=p^2(\alpha-1)\sum_{k=p+1}^{\infty}\frac{1}{k(k-1)}=p(\alpha-1),$$

which shows $f\in\mathcal{M}_p(\alpha,\beta)$.

3. Applications

In [6] Porwal and Dixit considered the function $\mathcal{U}_{d,e,\delta}$ as follows

$$\mathcal{U}_{d,e,\delta}(z)=2^d\Gamma\left(d+\frac{e+1}{2}\right)z^{-d/2}w_{d,e,\delta}(z^{1/2}),$$

where $w_{d,e,\delta}$ is called the generalized Bessel function of the first kind of order d and has the familiar representation

$$w_{d,e,\delta}(z):=\sum_{k=0}^{\infty}\frac{(-1)^k\delta^k}{k!\Gamma(d+k+\frac{e+1}{2})}\left(\frac{z}{2}\right)^{2k+d}\qquad(z,d,e,\delta\in\mathbb{C}).$$

It is easy to see the function $\mathcal{U}_{d,e,\delta}$ has the following representation

$$\mathcal{U}_{d,e,\delta}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(\delta/4)^k}{\left(d+\frac{e+1}{2}\right)_k} \frac{z^k}{k!}, \tag{3.1}$$

where $d+\frac{e+1}{2} \neq 0, -1, -2, \ldots$ and and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n := \begin{cases} 1, & (n=0); \\ x(x+1)(x+2)\ldots(x+n-1), & (n \in \mathbb{N}). \end{cases}$$

We remark that the function $\mathcal{U}_{d,e,\delta}(z)$ is analytic on $\mathbb{C}$.

The Gaussian hypergeometric function $F(a,b;c;z)$ is given by

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(1)_k} z^k \quad (z \in \Delta), \tag{3.2}$$

where $a, b, c \in \mathbb{C}$ and $c \neq 0, -1, -2, \ldots$. We note that $F(a,b;c;1)$ converges for $\mathrm{Re}(a-b-c) > 0$ and is related to the Gamma function by

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \quad (\mathrm{Re}(c-a-b) > 0).$$

Now by using (3.1) and (3.2) we introduce a new function $\mathcal{I}_{c,d}^{a,b}(p,e,\delta)(z) : \mathcal{A}_p \to \mathcal{A}_p$ defined by

$$\mathcal{I}_{c,d}^{a,b}(p,e,\delta)(z) = z^p(F(a,b;c;z) * \mathcal{U}_{d,e,\delta}(z)),$$

where " $*$ " is the well–known Hadamard product. The function $\mathcal{I}_{c,d}^{a,b}(p,e,\delta)(z)$ has the following representation

$$\mathcal{I}_{c,d}^{a,b}(p,e,\delta)(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k-p} \frac{(a)_{k-p}(b)_{k-p}(\delta/4)^{k-p}}{(c)_{k-p}\left(d+\frac{e+1}{2}\right)_{k-p}[(1)_{k-p}]^2} z^k.$$

We set $\mathcal{I}_{c,d}^{a,b}(1,e,\delta)(z) \equiv \mathcal{I}_{c,d}^{a,b}(e,\delta)(z)$. Applying Theorem 2.6 we have the following.

Theorem 3.1. *Let $a, b \in \mathbb{C}\backslash\{0\}$ and $e, d \in \mathbb{C}$. Also, assume that $\delta, d+\frac{e+1}{2} > 0$ and c is a real number such that $c > |a|+|b|+1$. Then $\mathcal{I}_{c,d}^{a,b}(p,e,\delta)(z) \in \mathcal{M}_p(\alpha,\beta)$ if*

$$\sum_{k=p+1}^{\infty} [p\alpha+\beta+k(1-\beta)] \frac{|(a)_{k-p}(b)_{k-p}|(\delta/4)^{k-p}}{(c)_{k-p}\left(d+\frac{e+1}{2}\right)_{k-p}[(k-p)!]^2} < p(\alpha-1). \tag{3.3}$$

Proof . Let $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \in \mathcal{A}_p$. By virtue of Theorem 2.6, it suffices to show that

$$\sum_{k=p+1}^{\infty} [p\alpha+\beta+k(1-\beta)] \left| \frac{(-1)^{k-p}(a)_{k-p}(b)_{k-p}(\delta/4)^{k-p}}{(c)_{k-p}\left(d+\frac{e+1}{2}\right)_{k-p}[(1)_{k-p}]^2} \right| < p(\alpha-1).$$

Some reductions give (3.3) $\square$

Setting $p=1$ in Theorem 3.1, we have:

Corollary 3.2. *If $a, b \in \mathbb{C}\backslash\{0\}$, $e, d \in \mathbb{C}$, $\delta, d + \frac{e+1}{2} > 0$ and c is a real number such that $c > |a| + |b| + 1$, then a sufficient condition for $\mathcal{I}_{c,d}^{a,b}(e, \delta)(z) \in \mathcal{MD}(\alpha, \beta)$ is that*

$$\sum_{k=2}^{\infty}[\alpha + \beta + k(1 - \beta)]\frac{|(a)_{k-1}(b)_{k-1}|(\delta/4)^{k-1}}{(c)_{k-1}\left(d + \frac{e+1}{2}\right)_{k-1}[(k-1)!]^2} < \alpha - 1.$$

If we take $\beta = 0$ in Corollary 3.2, we have the following result:

Corollary 3.3. *If $a, b \in \mathbb{C}\backslash\{0\}$, $e, d \in \mathbb{C}$, $\delta, d + \frac{e+1}{2} > 0$ and c is a real number such that $c > |a| + |b| + 1$, then a sufficient condition for $\mathcal{I}_{c,d}^{a,b}(e, \delta)(z) \in \mathcal{M}(\alpha)$ is that*

$$\sum_{k=2}^{\infty}[\alpha + k]\frac{|(a)_{k-1}(b)_{k-1}|(\delta/4)^{k-1}}{(c)_{k-1}\left(d + \frac{e+1}{2}\right)_{k-1}[(k-1)!]^2} < \alpha - 1.$$

The class $\mathcal{M}(\alpha)$ was considered by Uralegaddi et al. [7], Nishiwaki and Owa [3], and Owa and Nishiwaki [5].

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