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On one type of stability for multiobjective integer linear programming problem with parameterized optimality

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Abstract

A multiobjective problem of integer linear programming with parametric optimality is addressed. The parameterization is introduced by dividing a set of objectives into a family of disjoint subsets, within each Pareto optimality is used to establish dominance between alternatives. The introduction of this principle allows us to connect such classical optimality sets as extreme and Pareto. The admissible perturbation in such problem is formed by a set of additive matrices, with arbitrary Hölder's norms specified in the solution and criterion spaces. The lower and upper bounds for the radius of strong stability are obtained with some important corollaries concerning previously known results.

Keywords: Multiobjective problem, integer programming, Pareto set, a set of extreme solutions, stability radius, Hölder's norms.

1 Introduction

Under certain restrictions on the type of space and the properties of the norm, it may turn out that an efficient solution of a specific optimization problem is preserved as a solution for all problems within some nonzero neighborhood in a metric space. Such conservation can be interpreted as the stability of the solution, and the non-existence of such a nonzero neighborhood can be considered as its instability. Quantitative characteristic of such a neighborhood can be called the stability radius. The widespread use of discrete optimization models has attracted the attention of many experts to the study of various aspects of stability, as well as the problems of parametric and post-optimal analysis of both scalar (single-criterion) and vector (multicriteria) discrete optimization problems (see, for example, the monograph [1], the review [2] as well as the bibliography therein).

The main purpose of works based on the qualitative approach is to obtain conditions guaranteeing the problem possesses some beforehand given property of stability to small changes of the initial data. In the framework of the qualitative direction, the authors focus on identifying various types of stability of the problem [3, 4, 5, 6, 7, 8, 9], establishing a relationship between different types of stability [10], as well as on searching and describing the stability region of the optimal solution [11]. In some recent papers [12, 13], the proximity of some approaches is analyzed at the level of both problem statements and interpreting the common results.

The quantitative direction, described in sufficient detail in [12] (see also, [2] and [13]), is associated with obtaining estimates of permissible changes in the initial data of the problem, preserving a certain predetermined property of optimal solutions. For multiobjective problems this direction is developed in series of papers of V. Emelichev et. al [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. Attempts to elaborate algorithms for calculating (and approximating) such estimates have been made in [26, 27, 28]. The need for such researches is caused by two basic reasons. First, for checking correctness of a concrete optimization model it is important to know borders of change of the input parameters, for which the solution of an optimization problem is not misrepresented. Secondly, there is an opportunity to build algorithms for solving discrete optimization problems, which are based on procedures of finding a stability radius. For example, such procedures can be useful for constructing algorithms solving a sequence of problems of similar type with initial data varying insignificantly.

The concept of stability radius was introduced and investigated for the first time by V. Leontev [29, 30] for the linear scalar trajectory problem, i.e. for the problem on a system of subsets of a finite set with the linear objective function. Obviously, the most discrete optimization problems may be formulated as a particular case of integer linear programming. Therefore, the concept of stability radius naturally arises therein. Stability radius of integer linear programming problem is defined as the limiting level of independent perturbations of the vector criterion parameters for which new efficient solutions do not appear. Relaxing the demand of nonappearance of new efficient solutions we come to the concept of the strong stability introduced earlier for some scalar and vector discrete optimization problem [2, 3, 16]. This type of stability is understood as existence of a small neighborhood of problem parameters such that for any perturbation there exists an efficient solution preserving its Pareto optimality, although appearance of new efficient optima is not prohibited.

The paper is organized as follows. In Section 2, we formulate parametric optimality and introduce basic concepts. Section 3 contains some auxiliary statements about norms and several lemmas used later for the proof of the main result. In Section 4, we formulate and prove the main result regarding the lower and upper bounds for the strong stability radius. Section 5 lists most important corollaries.

2 Main definitions and problem formulation

Consider a multicriteria integer linear programming problem (ILP) in the following formulation. Let $C=[c_{ij}]\in\mathbf{R}^{m\times n}$ be a matrix whose rows are denoted by $C_i=(c_{i1},c_{i2},...,c_{in})\in\mathbf{R}^n$, $i\in N_m=\{1,2,...,m\}$, $m\geq 1$. Let $x=(x_1,x_2,...,x_n)^T\in X\subset\mathbf{Z}^n$, $n\geq 2$, and the number of elements of the set X is finite and greater than one. On the set of (admissible) solutions X, we define a vector linear criterion

$$Cx=(C_1x,C_2x,\dots,C_mx)^T\to\min_{x\in X}. \tag{1}$$

In the space $\mathbf{R}^k$ of arbitrary dimension $k\in\mathbf{N}$, we introduce a binary relation that generates the Pareto optimality principle [31]:

$$y\succ y' \quad\Leftrightarrow\quad y\geq y' \;\;\&\;\; y\neq y',$$

where $y = (y_1, y_2, ..., y_k)^T \in \mathbf{R}^k$, $y' = ({y'}_1, {y'}_2, ..., {y'}_k)^T \in \mathbf{R}^k$.
The symbol $\overline{\succ}$, as usual, denotes the negation of the relation $\succ$.

Let $\emptyset \neq I \subseteq N_m, |I| = v$, and let C_I denote the submatrix of the matrix $C \in \mathbf{R}^{m\times n}$ consisting of rows of this matrix with the numbers of the subset I, i.e.

$$C_I = (C_{i_1}, C_{i_2}, ..., C_{i_v})^T, \quad I = \{i_1, i_2, ..., i_v\},$$

$$1 \leq i_1 < i_2 < \ ... \ < i_v \leq m, \quad C_I \in \mathbf{R}^{v\times n}.$$

Let $s \in N_m$, and let $N_m = \cup_{k\in N_s} I_k$ be a partition of the set N_m into s nonempty sets, i.e. $I_k \neq \emptyset$, $k \in N_s$, and $i \neq j \Rightarrow I_i \cap I_j = \emptyset$. For this partition, we introduce a set of $(I_1, I_2, ..., I_s)$-efficient solutions according to the formula:

$$G^m(C, I_1, I_2, ..., I_s) = \{x \in X : \ \exists k \in N_s \ \ \forall x' \in X \ \ (C_{I_k}x \ \overline{\succ} \ C_{I_k}x')\}. \tag{2}$$

Sometimes for brevity we denote this set by $G^m(C)$.

Obviously, any N_m-efficient solution $x \in G^m(C, N_m)$ $(s = 1)$ is Pareto optimal, i.e. efficient solution to problem (1). Therefore, the set $G^m(C, N_m)$ is the Pareto set [31]:

$$P^m(C) = \{x \in X : \ \forall x' \in X \ \ (Cx \ \overline{\succ} \ Cx')\}.$$

We also use the following set:

$$X(x, C) = \{x' \in X : \ Cx \succ Cx'\},$$

which is a set of solutions $x' \in X$ such that x' dominates x in Pareto sense in problem (1). Therefore,

$$P^m(C) = \{x \in X : \ X(x, C) = \emptyset\}.$$

In the other extreme case, when $s = m$, $G^m(C, \{1\}, \{2\}, ..., \{m\})$ is a set of extreme solutions (see e.g. [32]). This set is denoted by $E^m(C)$. Thereby, we have:

$$E^m(C) = \{x \in X : \ \exists k \in N_m \ \ \forall x' \in X \ \ (C_k x \ \overline{\succ} \ C_k x')\} =$$

$$\{x \in X : \ \exists k \in N_m \ \ \forall x' \in X \ \ (C_k x \leq C_k x')\}.$$

It is easy to see that the set of extreme solutions is composed of the best solutions for each of the m criteria. So, in this context, the parameterization of the optimality principle refers to the introduction of such a characteristic of the binary preference relation that allows us to connect the well-known choice functions, parameterizing them from the Pareto to the extreme.

Denoted by $Z^m(C, I_1, I_2, \ldots, I_s)$, the multicriteria ILP problem consists in finding the set $G^m(C, I_1, I_2, \ldots, I_s)$. Sometimes, for the sake of brevity, we use the notation $Z^m(C)$ for this problem.

It is easy to see that the set $P^1(C) = E^1(C)$ is the set of optimal solutions to the scalar (single-criterion) problem $Z^1(C, N_1)$, where $C \in \mathbf{R}^n$.

For any nonempty subset $I \subseteq N_m$ we introduce the notation

$$P(C_I) = \{x \in X : \ \forall x' \in X \ \ (C_I x \,\overline{\succ}\, C_I x')\},$$

$$X(x, C_I) = \{x' \in X : \ C_I x \ \succ \ C_I x')\},$$

i.e.

$$P(C_I) = \{x \in X : \ X(x, C_I) = \emptyset\}.$$

Then, by virtue of (2), we obtain

$$G^m(C, I_1, I_2, \ldots, I_s) = \{x \in X : \ \exists k \in N_s \ \ (x \in P(C_{I_k}))\}. \qquad (3)$$

Therefore, we have

$$G^m(C, I_1, I_2, \ldots, I_s) = \bigcup_{k \in N_s} P(C_{I_k}), \ \ N_m = \bigcup_{k \in N_s} I_k.$$

It is obvious that all the sets given here are nonempty for any matrix $C \in \mathbf{R}^{m \times n}$.

In the space of solutions $\mathbf{R}^n$, we define an arbitrary Hölder's norm l_p, $p \in [1,\infty]$, i.e. by the norm of a vector $a = (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$ we mean the number

$$\|a\|_p = \begin{cases} \left(\sum_{j \in N_n} |a_j|^p\right)^{1/p} & \text{if } \ 1 \leq p < \infty, \\ \max\{|a_j| : \ j \in N_n\} & \text{if } \ p = \infty. \end{cases}$$

In the space of criteria $\mathbf{R}^m$, we define an arbitrary Hölder's norm l_q, $q \in [1,\infty]$, and $l_p \neq l_q$. By the norm of the matrix $C \in \mathbf{R}^{m\times n}$ with the rows C_i, $i \in N_m$, we mean the norm of a vector whose components are the norms of the rows of the matrix. By that, we have

$$\|C\|_{pq}=\|(\|C_1\|_p,\|C_2\|_p,\ldots,\|C_m\|_p)\|_q.$$

Obviously,

$$\|C_i\|_p \leq \|C_I\|_{pq} \leq \|\mathrm{C}\|_{pq},\ i \in I \subseteq N_m. \tag{4}$$

So, it is easy to see that for any $a= (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$ with

$$|a_j| =\alpha, \quad j \in N_n,$$

the following equality holds

$$\|a\|_p=\alpha n^{1/p} \tag{5}$$

for any $p \in [1,\infty]$.

In the solution space $\mathbf{R}^n$ along with the norm l_p, $p \in [1,\infty]$, we will use the conjugate norm l_{p^*}, where the numbers p and p^* are connected, as usual, by the equality

$$\frac{1}{p}+\frac{1}{p^*}= 1,$$

assuming $p^*= 1$ if $p=\infty$, and $p^*=\infty$ if $p= 1$. Therefore, we further suppose that the range of variation of the numbers p and p^* is the closed interval $[1,\infty]$, and the numbers themselves are connected by the above conditions.

Further we use the well-known Hölder's inequality

$$|a^T b| \leq \|a\|_p\|b\|_{p^*} \tag{6}$$

that is true for any two vectors $a= (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$ and $b= (b_1, b_2, \ldots, b_n)^T \in \mathbf{R}^n$ and for any $p \in [1,\infty]$.

Perturbation of the elements of the matrix C is imposed by adding matrices C' from $\mathbf{R}^{m\times n}$ to it. Thus, the perturbed problem $Z^m(C+C')$ has the form

$$(C + C')x \to \min_{x\in X},$$

and the set of its $(I_1, I_2, ..., I_s)$-efficient solutions is $G^m(C{+}C', I_1, I_2, \dots,$ $I_s)$.
For an arbitrary number $\varepsilon > 0$, we define the set of perturbing matrices

$$\Omega_{pq}(\varepsilon) = \left\{ C' \in \mathbf{R}^{m\times n} : \ \|C'\|_{pq} < \varepsilon \right\}$$

with rows $C'_i,\ i \in N_m$.

Following [16, 34], the *strong stability radius* of the ILP problem $Z^m(C, I_1, I_2, \dots, I_s)$, $m \in \mathbf{N}$, (called T_1-stability radius in the terminology of [1, 2, 8, 9]) is the number

$$\rho = \rho_s^m(p,q) = \begin{cases} \sup\Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \left\{ \varepsilon > 0 : \ \forall C' \in \Omega_{pq}(\varepsilon) \quad \left(G^m(C) \cap G^m(C + C') \neq \emptyset \right) \right\}.$$

Thus, the strong stability radius of the problem $Z^m(C)$ determines the limit level of perturbations of the elements of the matrix C that preserve optimality of at least one (not necessarily the same) solution of the set $G^m(C)$ of the original problem. For any $C' \in \Omega_{pq}(\varepsilon)$ and $\varepsilon > 0$, it is obvious that $G^m(C) \cap G^m(C + C') \neq \emptyset$ if $G^m(C) = \mathrm{X}$. Therefore, the problem $Z^m(C)$ with $\bar{G}^m(C) = G^m(C) \backslash X = \emptyset$ is called *non-trivial*.

The problem $Z^m(C)$ is called *degenerated* if the following formula holds

$$\forall x \notin G^m(C)\ \forall a \in \mathbf{R}^n\ \exists x^0 \in G^m(C)\ (a^T(x - x^0) \geq 0).$$

If the negation of the formula above is true, i.e.

$$\exists x^0 \notin G^m(C)\ \exists a \in \mathbf{R}^n\ \forall x \in G^m(C)\ (a^T(x^0 - x) < 0), \tag{7}$$

then the problem $Z^m(C)$ is called *non-degenerated*.

It is easy to see that non-trivial problem is also non-degenerated if and only if there exists a solution $x^0 \notin G^m(C)$ such that a system containing $|G^m(C)|$ strict inequalities with n variables has a solution. In particular, as we show later (see the proof of Theorem 1), the Boolean problem $Z_B^m(C)$ has solutions.

3 Lemmas

Before formulating the main result regarding the strong stability radius bounds in the next section, we need to prove five supplementary statements presented in this section as lemmas.

Lemma 1. *A solution $x \notin G^m(C, I_1, I_2, \ldots, I_s)$ if and only if for any index $k \in N_s$ the solution $x \notin P(C_{I_k})$.*

Hereinafter, a^+ is a projection of a vector $a = (a_1, a_2, \ldots, a_k) \in \mathbf{R}^k$ on a positive orthant, i.e. $a^+ = [a]^+ = (a_1^+, a_2^+, \ldots, a_k^+)$, where superscript + implies positive cut of vector a. That is, we have

$$a_i^+ = [a_i]^+ = \max\{0, a_i\}, \ i \in N_k.$$

Lemma 2. *Given $p, q \in [1, \infty]$, $x^0 \in G^m(C, I_1, I_2, \ldots, I_s)$, $k \in N_s$ and $\varphi > 0$ such that for any $x \notin G^m(C)$, the inequality*

$$\left\| \left[C_{I_k}(x - x^0) \right]^+ \right\|_q \geq \varphi \|x - x^0\|_{p^*} > 0 \tag{8}$$

holds. Then the following formula is true:

$$\forall\ x \notin G^m(C) \quad \forall C' \in \Omega_{pq}(\varphi) \ \ \left(x \notin X(x^0, C_{I_k} + C'_{I_k}) \right). \tag{9}$$

Proof. Assume there exists a solution $\tilde{x} \notin G^m(C)$ and a perturbing matrix $\tilde{C} \in \Omega_{pq}(\varphi)$ such that

$$\tilde{x} \in X(x^0, C_{I_k} + \tilde{C}'_{I_k}).$$

Then for any index $i \in I_k$, the following inequality is true:

$$\left(C_i + \tilde{C}_i \right) x^0 \geq \left(C_i + \tilde{C}_i \right) \tilde{x}.$$

Hence, we have

$$\tilde{C}_i \left(x^0 - \tilde{x} \right) \geq C_i \left(x^0 - \tilde{x} \right), \quad i \in I_k.$$

From the above we derive

$$\left| \tilde{C}_i \left(\tilde{x} - x^0 \right) \right| \geq \left[C_i \left(\tilde{x} - x^0 \right) \right]^+, \quad i \in I_k.$$

Taking into consideration Hölder's inequality (6), we obtain

$$\|\tilde{C}_i\|_p \|\tilde{x} - x^0\|_{p^*} \geq \left[C_i\left(\tilde{x} - x^0\right)\right]^+, \quad i \in I_k.$$

Due to inequalities (4), we get a contradiction with (8):

$$\varphi\|\tilde{x} - x^0\|_{p^*} > \|\tilde{C}\|_{pq}\|\tilde{x} - x^0\|_{p^*} \geq \|\tilde{C}_{I_k}\|_{pq}\|\tilde{x} - x^0\|_{p^*} \geq$$

$$\geq \left\|\left[C_{I_k}\left(\tilde{x} - x^0\right)\right]^+\right\|_q,$$

so formula (9) is valid. □

Lemma 3. *For any non-degenerated ILP problem $Z^m(C)$, there exists a non-zero matrix $C^* \in \mathbf{R}^{m \times n}$ such that*

$$G^m(C) \cap G^m(C^*) = \emptyset.$$

Proof. According to the definition of non-degenerated problem $Z^m(C)$, the equation (7) holds, i.e. for all $x \in G^m(C)$ the inequality

$$a^T(x^0 - x) < 0 \tag{10}$$

is true for any $x^0 \notin G^m(C)$. Obviously, $a \neq \mathbf{0} = (0, 0, \ldots, 0)^T \in \mathbf{R}^m$. Let rows $C_i^*,\ i \in N_m$ of the matrix $C^* \in \mathbf{R}^{m \times n}$ be defined as:

$$C_i^* = a^T,\ i \in N_m\ .$$

Then taking into account (10), we get

$$C_i^*\left(x^0 - x\right) < 0,\ i \in N_m.$$

Thus for any index $k \in N_s$, the solution $x \notin P(C_{I_k}^*)$ if $x \in G^m(C)$. Therefore, due to Lemma 1, we have $x \notin G^m(C^*)$. The last implies

$$G^m\left(C\right) \cap G^m\left(C^*\right) = \emptyset. \quad \square$$

Lemma 4. *Let $x^0 \in G^m\left(C, I_1, I_2, \ldots, I_s\right)$. For any non-trivial ILP problem $Z^m(C)$ and perturbing matrix $C' = (C'_{I_1}, C'_{I_2}, \ldots, C'_{I_s})^T \in \mathbf{R}^{m \times n}$ such that for some index $k \in N_s$, the equality*

$$X(x^0, C_{I_k} + C'_{I_k}) \cap \bar{G}^m\left(C\right) = \emptyset \tag{11}$$

holds. Then we have

$$G^m\left(C\right) \cap G^m\left(C + C'\right) \neq \emptyset \tag{12}$$

Proof. If $x^0 \in G^m(C+C')$, the statement of lemma is obvious. Assume $x^0 \notin G^m(C+C')$. Then according to Lemma 1, for any index $k \in N_s$ we have $x^0 \notin P(C_{I_k}+C'_{I_k})$. Due to the property of external stability of the Pareto set $P(C_{I_k}+C'_{I_k})$ (see e.g., [33]), there exists a solution $x^* \in P(C_{I_k}+C'_{I_k})$ such that $x^* \in X(x^0, C_{I_k}+C'_{I_k})$, and due to (3) $x^* \in G^m(C+C')$. Using (11), we get $x^* \in G^m(C)$. Hence, (12) holds. $\square$

Lemma 5. *If $\rho_s(p,q) < \infty$, then the following formula holds:*

$$\exists a \in \mathbf{R}^n \ \forall x \in G^m(C) \ \exists x^0(x) \notin G^m(C) \ (a^T(x^0(x) - x) < 0), \quad (13)$$

$\square$

Proof. Assume that (13) does not hold. Then we have

$$\forall a \in \mathbf{R}^n \ \exists x^0 \in G^m(C, I_1, I_2, \dots, I_s)$$
$$\forall x \notin G^m(C, I_1, I_2, \dots, I_s) \ \ (a^T(x - x^0) \geq 0).$$

Let $C' = (C'_{I_1}, C'_{I_2}, \dots, C'_{I_s})^T \in \mathbf{R}^{m\times n}$ be any perturbing matrix. Then for any chosen index $k \in N_s$ there exists $x^0 \in G^m(C)$ such that for any $x \notin G^m(C)$ the inequality

$$\left(C_i + C'_i\right)(x - x^0) \geq 0, \ \ i \in I_k$$

holds.

Therefore, $x \notin X(x^0, C_{I_k} + C'_{I_k})$. Further, applying Lemma 4, we get that (12) is valid for any matrix $C' \in \mathbf{R}^{m\times n}$, i.e. $\rho_s(p,q) = \infty$. This contradiction ends the proof. $\square$

4 Main result

For the multicriteria non-trivial ILP problem $Z^m(C, I_1, I_2, \dots, I_s)$, $m \in \mathbf{N}$, for any $p, q \in [1,\infty]$ and $s \in N_m$ we define:

$$\varphi_s^m(p,q) = \max_{x' \in G^m(C)} \ \max_{k \in N_s} \ \min_{x \notin G^m(C)} \ \frac{\|[C_{I_k}(x - x')]^+\|_q}{\|x - x'\|_{p^*}},$$

$$\psi_s^m(p,q) = n^{\frac{1}{p}} m^{\frac{1}{q}} \min_{x \notin G^m(C)} \max_{x' \in G^m(C)} \max_{k \in N_s} \max_{i \in I_k} \frac{C_i(x-x')}{\|x-x'\|_1}.$$

We are now ready to formulate the main result.

Theorem 1. *For any $m \in \boldsymbol{N}$, $p,q \in [1,\infty]$ and $s \in N_m$, the strong stability radius of the multicriteria non-trivial ILP problem $Z^m(C,I_1,I_2,\ldots,I_s)$ has the following lower and upper bounds:*

$$0 < \varphi_s^m(p,q) \le \rho_s^m(p,q) \begin{cases} \le \|C\|_{pq}, & \textit{if the problem is non-degenerated;} \\ = \infty, & \textit{otherwise.} \end{cases}$$

If the problem is Boolean, i.e. $Z^m(C) = Z_B^m(C)$, then

$$0 < \varphi_s^m(p,q) \le \rho_s^m(p,q) \le \min\{\psi_s^m(p,q),\ \|C\|_{pq}\}.$$

Proof. Due to (3), the formula is true:

$$\forall\ x' \in G^m(C) \quad \exists\ k \in N_s\ \left(x' \in P\left(C_{I_k}\right)\right).$$

Therefore, due to Lemma 1, for any index $k \in N_s$, we get $x \notin P(C_{I_k})$ if $x \notin G^m(C)$. From there we conclude that the lower bound is positive, i.e. $\varphi_s^m(p,q) > 0$.

Now we prove that $\rho_s^m(p,q) \ge \varphi_s^m(p,q)$. We choose an arbitrary perturbing matrix $C' \in \mathbf{R}^{m\times n}$ such that it belongs to $\Omega_{pq}(\varphi_s^m(p,q))$. In order to prove the lower bound for strong stability radius, it suffices to demonstrate that there exists a solution $x^* \in G^m(C) \cap G^m(C+C')$. According to the definition of the number $\varphi_s^m(p,q)$, there exist a solution $x^0 \in G^m(C)$ and an index $k \in N_s$ such that for any solution $x \notin G^m(C)$ we have:

$$\|[C_{I_k}(x-x^0)]^+\|_q \ge \varphi_s^m(p,q)\,\|x-x^0\|_{p^*} > 0.$$

From the above, by Lemma 2, we get that the following formula is true:

$$\forall\ x \notin G^m(C)\ \ \forall C' \in \Omega_{pq}(\varphi_s^m(p,q))$$

$$\forall\ x \notin G^m(C) \quad \forall C' \in \Omega_{pq}(\varphi_s^m(p,q))\ \left(x \notin X(x^0, C_{I_k} + C'_{I_k})\right). \tag{14}$$

Further, we define a way of selecting a necessary solution

$$x^* \in G^m(C) \cap G^m\left(C + C'\right),$$

where $C' \in \Omega_{pq}(\varphi_s^m(p,q))$. If $x^0 \in G^m(C+C')$, then we select $x^* = x^0$. Otherwise, due to Lemma 1 we have $x^0 \notin P(C_{I_k}+C'_{I_k})$. Thus due to the property of outer stability for the Pareto set $P(C_{I_k}+C'_{I_k})$ (see e.g. [33]), we can chose a solution $x^* \in P(C_{I_k}+C'_{I_k})$ such that $x^* \in X(x^0, C_{I_k}+C'_{I_k})$. Taking into account the proven formula (14), $x^* \in G^m(C)$. Since, due to (3), we have $x^* \in G^m(C+C')$, that involves $\rho_s^m(p,q) \geq \varphi_s^m(p,q)$.

Further, we prove that inequality $\rho_s^m(p,q) \leq \|C\|_{pq}$ is valid for any non-degenerated problem $Z^m(C)$. Let $\varepsilon > \|C\|_{pq}$. According to Lemma 3, for any such problem, there exists a non-zero matrix $C^* \in \mathbf{R}^{m\times n}$ such that

$$G^m(C) \cap G^m(C^*) = \emptyset. \tag{15}$$

We consider a perturbing matrix $C^0 \in \mathbf{R}^{m\times n}$ defined as:

$$C^0 = \xi C^* - C,$$

where $0 < \xi < \frac{\varepsilon - \|C\|_{pq}}{\|C^*\|_{pq}}$. Then we easily derive

$$\left\|C^0\right\|_{pq} = \|\xi C^* - C\|_{pq} \leq \xi \|C^*\|_{pq} + \|C\|_{pq} < \varepsilon.$$

Therefore due to (15) we obtain

$$\forall \varepsilon > \|C\|_{pq} \;\; \exists C^0 \in \Omega_{pq}(\varepsilon) \quad \left(G^m(C) \cap G^m\left(C + C^0\right) = \emptyset\right).$$

Thus, $\rho_s^m(p,q) < \varepsilon$ for any $\varepsilon > \|C\|_{pq}$. Hence, $\rho_s^m(p,q) \leq \|C\|_{pq}$.

Further, we show that for degenerate problem $Z^m(C)$, the strong stability radius is equal to infinity. Assume the opposite, i.e. assume that degenerated problems have a finite strong stability radius. Then, according to Lemma 5, formula (13) is valid. Thus letting

$$x^* = arg\min\{a^T(x^0(x)) : \; x \in G^m(C)\},$$

we get that the following inequality

$$a^T(x^0(x^*) - x) < 0$$

is true for any $x \in G^m(C)$. Thus, formula (7) is true, i.e. the problem $Z^m(C)$ is non-degenerated. The obtained contradiction proves that $\rho_s^m(p,q) = \infty$.

Further, we consider non-trivial Boolean problem $Z_B^m(C, I_1, I_2, \ldots, I_s)$, $C \in \mathbf{R}^{m\times n}$, $m \in \mathbf{N}$, $s \in N_m$, $X \subseteq \mathbf{E}^n = \{0,1\}^n$, $n \geq 2$. Clearly, the lower bounds proven above for ILP problem stay valid in Boolean case.

First, we prove that $\rho_s^m(p,q) \leq \psi_s^m(p,q)$.

According to the definition of number $\psi_s^m(p,q)$, there exists a solution $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \notin G^m(C)$ such that for any solution $x \in G^m(C)$ and any index $k \in N_s$ the following inequalities hold:

$$\psi_s^m(p,q)\,\|x^0 - x\|_1 \geq n^{\frac{1}{p}} m^{\frac{1}{q}} C_i\left(x^0 - x\right), \quad i \in I_k. \tag{16}$$

Let $\varepsilon > \psi_s^m(p,q)$. We choose a perturbing matrix $C^0 = [c_{ij}^0] \in \mathbf{R}^{m\times n}$ with rows C_i^0, $i \in N_m$ and elements defined as follows:

$$c_{ij}^0 = \begin{cases} -\delta \text{ if } & i \in N_m \text{ and } x_j^0 = 1, \\ \delta \text{ if } & i \in N_m \text{ and } x_j^0 = 0, \end{cases}$$

where

$$\psi_s^m(p,q) < \delta n^{\frac{1}{p}} m^{\frac{1}{q}} < \varepsilon. \tag{17}$$

Therefore, due to (5) we have

$$\left\|C_i^0\right\|_p = \delta n^{\frac{1}{p}}, i \in N_m,$$

$$\left\|C^0\right\|_{pq} = \delta n^{\frac{1}{p}} m^{\frac{1}{q}},$$

$$C^0 \in \Omega_{pq}(\varepsilon).$$

Moreover, the following inequalities are obvious:

$$C_i\left(x^0 - x\right) = -\delta\|x^0 - x\|_1 < 0, \; i \in I_k.$$

Using (16) and (17), we conclude that the following inequalities hold for any solution $x \in G^m(C)$:

$$\left(C_i + C_i^0\right)\left(x^0 - x\right) \leq \left(\frac{\psi_s^m(p,q)}{n^{\frac{1}{p}} m^{\frac{1}{q}}} - \delta\right)\|x^0 - x\|_1 < 0, \; i \in I_k.$$

Thus for any index $k \in N_s$ we have $x \in G^m(C)$ and $x \notin P\left(C_{I_k} + C^0_{I_k}\right)$, and hence, due to Lemma 1, $x \notin G^m\left(C + C^0\right)$. Summarizing, for any $\varepsilon > \psi_s^m(p,q)$ there exists the perturbing matrix $C^0 \in \Omega_{pq}(\varepsilon)$ such that $G^m(C) \cap G^m\left(C + C^0\right) = \emptyset$, i.e. $\rho_s^m(p,q) < \varepsilon$. Thus, we have just proven that $\rho_s^m(p,q) \leq \psi_s^m(p,q)$.

Finally, we prove that $\rho_s^m(p,q) \leq \|C\|_{pq}$ is valid for any Boolean problem $Z_B^m(C, I_1, I_2, \ldots, I_s)$. In order to do this, it suffices to show that any non-trivial Boolean problem is also non-degenerated. Let $\alpha > 0$ and $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \notin G^m(C)$. We choose a vector $a = (a_1, a_2, \ldots, a_n)^T$ with elements defined as follows:

$$a_j = \begin{cases} -\alpha \text{ if } & x_j^0 = 1, \\ \alpha \text{ if } & x_j^0 = 0. \end{cases}$$

Then for any $x \in G^m(C)$ $(x \neq x^0)$, we have

$$a^T(x^0 - x) < 0.$$

Thus, (7) is true, and hence $Z_B^m(C)$ is non-degenerated. Therefore, collecting all the proven above, we get $\rho_s^m(p,q) \leq \|C\|_{pq}$. This ends the proof of the main result. □

5 Corollaries

From Theorem 1 we get the following well-known result:

Corollary 1. *[16] For any $m \in \boldsymbol{N}$ and any $p = q = \infty$, the strong stability radius of the multicriteria non-trivial ILP problem $Z^m(C, N_m)$, $C \in \mathbf{R}^{m \times n}$ of finding the Pareto set $P^m(C)$ has the following bounds:*

$$0 < \varphi_1^m(\infty,\infty) \leq \rho_1^m(\infty,\infty).$$

Moreover,

$$0 < \varphi_1^m(\infty,\infty) \leq \rho_1^m(\infty,\infty) \leq \psi_1^m(\infty,\infty),$$

if the problem $Z^m(C, N_m)$ *is Boolean, where*

$$\varphi_1^m(\infty,\infty) = \max_{x'\in P^m(C)} \quad \min_{x\notin P^m(C)} \quad \max_{i\in N_m} \frac{C_i(x-x')}{\|x-x'\|_1},$$

$$\psi_1^m(\infty,\infty) = \min_{x\notin P^m(C)} \max_{x'\in P^m(C)} \quad \max_{i\in N_m} \frac{C_i(x-x')}{\|x-x'\|_1}.$$

The *stability radius* of an efficient solution $x^0 \in P^m(C)$ of the ILP problem $Z^m(C, N_m)$, $m \in \mathbf{N}$, is called the number

$$\rho_s^m(x^0,p,q)=\begin{cases} \sup \Theta_{\mathrm{pq}} & \text{if} \quad \Theta_{\mathrm{pq}} \neq \emptyset, \\ 0 & \text{if} \quad \Theta_{\mathrm{pq}}=\emptyset, \end{cases}$$

where

$$\Theta_{pq} = \Big\{\varepsilon > 0: \quad \forall C' \in \Omega_{pq}(\varepsilon) \qquad \big(x^0 \in P^m(C+C')\big)\Big\}.$$

In [35] it was shown that for any $m \in \boldsymbol{N}$ and $p, q \in [1,\infty]$, the stability radius of an efficient solution $x^0 \in P^m(C)$ of the multicriteria non-trivial ILP problem $Z^m(C, N_m)$ is expressed by the formula:

$$\rho^m(x^0,p,q) = \min_{x\in X\setminus\{x^0\}} \frac{\|[C(x-x^0)]^+\|_q}{\|x-x^0\|_{p^*}}.$$

It is evident that $\rho_1^m(p,q) = \rho^m(x^0,p,q)$ if $P^m(C) = \{x^0\}$. Therefore, from Theorem 1 we conclude the following result.

Corollary 2. *If* $P^m(C) = \{x^0\}$*, then the strong stability radius of the multicriteria ILP problem* $Z^m(C, N_m)$*,* $C \in \mathbf{R}^{m\times n}$*, of finding the Pareto set* $P^m(C)$ *is expressed by the formula for any* $m \in \boldsymbol{N}$ *and* $p, q \in [1,\infty]$*:*

$$\rho_1^m(p,q) = \varphi_1^m(p,q) = \min_{x\in \mathrm{X}\setminus\{x^0\}} \frac{\|[C(x-x^0)]^+\|_q}{\|x-x^0\|_{p^*}}.$$

In scalar case (single criterion), we have $P^1(C) = G^1(C, N_1)$, $C \in \mathbf{R}^n$, i.e. the Pareto set constricts to a set of optimal solutions in $Z^1(C, N_1)$. It is easy to see that the problem $Z^1(C, N_1)$ with condition $P^1(C) \neq X$ is non-degenerated. Therefore, Theorem 1 transforms into the following result for $m = 1$.

Corollary 3. *Let x^0 be an optimal solution for scalar ILP problem $Z^1(C, N_1)$, $C \in \mathbf{R}^n$. Then for any $p, q \in [1, \infty]$, the strong stability radius has the following bounds:*

$$0 < \min_{x \notin P^1(C)} \frac{C^T(x - x^0)}{\|x - x^0\|_{p^*}} \leq \rho_1^1(p.q) \leq \| C \|_{pq} .$$

From Theorem 1, we get the following known results.

Corollary 4. *[21] For any $m \in \boldsymbol{N}$, $p, q \in [1, \infty]$, the strong stability radius of the multicriteria non-trivial Boolean problem $Z_B^m(C, N_m)$ consisting in finding the Pareto set $P^m(C)$ has the following lower and upper bounds:*

$$0 < \max_{x' \in P^m(C)} \min_{x \notin P^m(C)} \frac{\|[C(x - x')]^+\|_q}{\|x - x'\|_{p^*}} \leq \rho_1^m(p, q) \leq$$

$$n^{\frac{1}{p}} m^{\frac{1}{q}} \min_{x \notin P^m(C)} \max_{x' \in P^m(C)} \max_{i \in N_m} \frac{C_i(x - x')}{\|x - x'\|_1}.$$

Corollary 5. *[36] For any $m \in \boldsymbol{N}$, $p, q \in [1, \infty]$, the strong stability radius of the multicriteria non-trivial Boolean problem $Z_B^m, (C, \{1\}, \{2\}, \ldots \{n\})$, $C \in \mathbf{R}^{n \times m}$ consisting in finding the extreme set $E^m(C)$ has the following lower and upper bounds:*

$$0 < \max_{x' \in E^m(C)} \max_{i \in N_m} \min_{x \notin E^m(C)} \frac{C_i(x - x')}{\|x - x'\|_{p^*}} \leq \rho_m^m(p, q) \leq$$

$$n^{\frac{1}{p}} m^{\frac{1}{q}} \min_{x \notin E^m(C)} \max_{x' \in E^m(C)} \max_{i \in N_m} \frac{C_i(x - x')}{\|x - x'\|_1}.$$

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