UNIVERSITY OF TURKU
Turku School of Economics

DYNAMIC DUOPOLY WITH DIFFERENTIATED GOODS AND SLUGGISH DEMAND

Master's thesis

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ABSTRACT

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Abstract

This thesis investigates dynamic Bertrand competition between two firms in a market where the goods are differentiated and demand is sluggish. Unlike homogeneous goods, differentiated goods are not perfect substitutes for each other. Sluggish demand means that there is a delay in the adjustment of demand after price changes. Sluggish demand is a remarkably ignored topic in the economic literature.

The competitive situation is modelled as a differential game. The dynamic model employs the demand system as in Singh and Vives 1984 and dynamics as in Wirl 2010. It is shown that the dynamic model has a unique symmetric open-loop Nash equilibrium. The long-term open-loop steady state is compared with the equilibrium point of the static model. The fundamental mathematical theory and solution methods of optimal control theory and differential games that are required in the analysis of the model are also presented in the thesis.

As the main result of the analysis of the model, it is shown that when sluggishness of demand is relatively small (i.e. the adjustment of demand after price changes is sufficiently fast), sluggishness of demand increases the market power and profits of the firms in the open-loop steady state compared to the equilibrium point of the static model. After sluggishness of demand exceeds a certain point, the profits of the firms decline below the static equilibrium profits. Moreover, it is shown that product differentiation relaxes price competition also in the presence of sluggish demand, as it does in a static model.

Keywords	industrial organisation, game theory, differential games, oligopoly, price competition, sluggish demand, dynamical systems

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Tiivistelmä

Tutkielmassa tarkastellaan kahden yrityksen välistä dynaamista Bertrand-kilpailua markkinoilla, joilla hyödykkeet ovat differoituja ja kysyntä on jäykkää. Toisin kuin homogeeniset hyödykkeet, differoidut hyödykkeet eivät ole täydellisiä korvikkeita toisilleen. Kysynnän jäykkyys tarkoittaa sitä, että kysyntä sopeutuu viiveellä hinnanmuutoksiin. Kysynnän jäykkyys on yllättävän vähälle huomiolle jäänyt aihe taloustieteellisessä kirjallisuudessa.

Kilpailutilanne mallinnetaan differentiaalisena pelinä. Dynaamisen mallin kysyntäjärjestelmä on kuten Singh ja Vivesin (1984) ja dynamiikka kuten Wirlin (2010) artikkelissa. Dynaamisella mallilla osoitetaan olevan uniikki symmetrinen open-loop Nash -tasapainostrategia. Pitkän aikavälin open-loop -tasapainopistettä vertaillaan staattisen Bertrand-mallin tasapainopisteen kanssa. Tutkielmassa esitetään lisäksi mallin tarkasteluun vaadittavan optimaalisen kontrolliteorian ja differentiaalisten pelien keskeinen matemaattinen teoria ja ratkaisumenetelmät.

Mallin analyysin päätuloksena osoitetaan, että kun kysynnän jäykkyys on verrattain pientä (ts. kysynnän sopeutuminen hinnanmuutoksiin on riittävän nopeaa), kysynnän jäykkyys lisää yritysten markkinavoimaa ja voittoja open-loop -tasapainopisteessä verrattuna staattiseen tasapainopisteeseen. Kun kysynnän jäykkyys ylittää tietyn pisteen, yritysten voitot laskevat staattisten tasapainovoittojen alle. Lisäksi osoitetaan, että tuotedifferointi pehmentää hintakilpailua myös kysynnän jäykkyyden olosuhteissa, aivan kuten staattisessa mallissa.

Avainsanat	toimialan taloustiede, peliteoria, differentiaaliset pelit, oligopoli, hintakilpailu, kysynnän jäykkyys, dynaamiset systeemit

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1 INTRODUCTION

Economists have studied firms, markets and imperfect competition extensively in industrial economics, later industrial organisation, since Cournot and Bertrand in the 1800's. Theories of monopolistic competition and their harmful effects to the society gave rise to structuralist Structure-Conduct-Performance (SCP) paradigm which focused on market structure and concentration to determine where barriers to competition existed and monopolies were likely to form. SCP models were eventually discarded because of critique posed by the Chicago School economists who placed emphasis on neoclassical price-theoretic models of free market entry and exit that, according to these models, prevented firms from forming any long-term monopolies and gaining monopoly profit. After that, the focus of much of the research in the field turned to other topics such as transaction costs to assess when market concentration and vertical arrangements between companies created market efficiencies which were socially beneficial. Since the 1970's, the development of game theory has fueled the rise of strategic approach which has been applied to study the strategic behavior of firms in markets. In addition to using game theoretic models, modern industrial organisation research now places more and more emphasis on empirical methods in which structural econometric models and economic data are used to analyze conditions in each market separately.

Theories and models in industrial organisation have produced some fundamental results on monopolies, oligopolies, mergers, vertical restraints, advertising, research and development (R&D) and competition in general in imperfect markets. However, most of these results are based on static models that – unlike dynamic models – do not account for the development of time, which is a relevant factor in many economic problems. Static models are easier to solve than dynamic models and are often assumed to capture the eventual steady state equilibria of dynamic interactions. In dynamic game-theoretic problems, however, this assumption can sometimes be detrimental and lead to incorrect conclusions about the equilibria of such interactions (Long 2010, p. 137). In such cases, properly dynamic game-theoretic models and tools are needed to address these issues (Cellini and Lambertini 2006).

In mathematics and engineering, continuously dynamical systems are modelled with differential equations and studied *inter alia* with the (optimal) control theory. Control theory studies evolution of dynamical systems with inputs and feedbacks. The question of interest is does there exist a control function with which a decision-maker could steer a system from one state to another and, if so, which control would be optimal in a sense that it maximizes (or minimizes) a certain objective functional (Weber 2011, pp. 81–82). The necessary conditions for such a control are given by the Pontryagin maximum principle (Pontryagin et al. 1962) whereas the necessary and sufficient conditions for optimality are given by the Hamilton-Jacobi-Bellman (HJB) partial differential equation, which can be obtained by using a dynamic programming method (Bellman 1957). Because of difficulty of solving partial differential equations, especially relating to nonlinear problems where analytical solutions rarely exist, the maximum principle has been popular in solving simpler optimal control problems.

In economics, the tools of the optimal control theory and dynamic programming have been employed especially in study of economic growth. In industrial organisation, however, applications of control theory have been relatively few. A reason for this is unclear, because many problems in industrial organisation involve dynamic aspects which could be fruitfully studied with optimal control theory. Even the strategic nature of interactions of firms in an oligopoly can be modelled as an optimal control problem as there is a branch of game theory, called differential game theory, which accommodates the evolution of state of the game in continuous time.

Differential games are games in which the position of the players develops continuously in time (Friedman 1971). Differential game theory builds on control theory with which it shares fundamental similarities. There are one or more state variables, which describe a dynamical system, and at least one control variable for every player. Control variables are functions of state variables. Players maximize (or minimize) their payoff functions and choose their control variables depending on what other players choose for their control variables. The selection of the control variables is analogous to strategy in conventional game theory where a strategy is a set of decisions which define the moves of a player for every possible position in a game (Isaacs 1965, p. 14). Indeed, because of close similarities between differential games and control theory, many optimal control problems in economics and other disciplines can be treated as one-player differential games. In the field of industrial organisation, differential games and control theory have been applied to subjects such as oligopolies (e.g. Simaan and Takayama 1978), R&D competition (e.g. Reinganum 1981 and 1982) and advertising (e.g. Nerlove and Arrow 1962).

In this thesis, the basic theory and methods of optimal control theory are presented and then used to solve a two-person differential game in open-loop strategies. The analysed model is a dynamic oligopoly model with differentiated goods and sluggish demand where the demand system is as in Singh and Vives 1984 and dynamics as in Wirl 2010. The existence of an open-loop Nash equilibrium is proved and the steady state of the dynamic model is then compared to that of the static Bertrand model.

First, the basic mathematics of the optimal control theory and differential games are presented in a rather rigorous manner. Second, the related economic literature is reviewed briefly after which the static and dynamic models are formulated and their equilibria solved and compared with each other. Third, concluding remarks from the analysis of the models are discussed in a larger industrial organisation context.

2 DYNAMIC OPTIMISATION AND DIFFERENTIAL GAMES

2.1 Control theory

2.1.1 A finite-horizon problem

Following the notation in Dockner et al. 2000 and Lambertini 2018, a simple optimal control problem may be formulated in a standard (or Lagrange) form[1] as

$$\max_{\mathbf{u}(t)} \quad J = \int_0^T F(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.1}$$

$$s.t. \quad \dot{\mathbf{x}}(t) = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.2}$$

$$\mathbf{x}(0) = \mathbf{x_0} \tag{2.3}$$

$$\mathbf{u}(t) \in \mathcal{U} \subseteq \mathbb{R}^m \text{ for all } t \in [0, T]. \tag{2.4}$$

Time $t \in [0, T]$ is continuous. In this formulation, the terminal time T is given while the terminal condition $\mathbf{x}(T)$ is left free. The variable $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in \mathcal{X}$ is a *state variable* which belongs to a state space $\mathcal{X} \subseteq \mathbb{R}^n$ containing all possible states.[2] The variable $\mathbf{u}(t) = (u_1(t), \ldots, u_m(t)) \in \mathcal{U}$ is a *control variable* which belongs to a set of admissible controls $\mathcal{U} \subseteq \mathbb{R}^m$. A first-order ordinary differential equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = (f_1(\mathbf{x}(t), \mathbf{u}(t), t), \ldots, f_n(\mathbf{x}(t), \mathbf{u}(t), t))$ called a *state equation* or *equation of motion* governs the evolution of the state variable and, in general, depends on the state variable $\mathbf{x}(t)$, the control variable $\mathbf{u}(t)$ and time t. The integrand function $F(\mathbf{x}(t), \mathbf{u}(t), t)$ and the equation $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ are continuous and differentiable in $\mathbf{x}(t)$, $\mathbf{u}(t)$ and t. It is not necessary that the control variable $\mathbf{u}(t)$ is continuous – it is required to be only piecewise continuous.[3] The state variable $\mathbf{x}(t)$ is assumed to be continuous but piecewise differentiable.[4] If the time argument t does not appear explicitly in $F(\mathbf{x}(t), \mathbf{u}(t))$ and $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$, the problem is called *autonomous*, and, conversely, if it does appear, the problem is called *non-autonomous*.

The above optimal control problem is to maximise an objective functional (2.1) subject to constraints (2.2)–(2.4). The constraint (2.2) determines the system dynamics, (2.3) is

[1] In addition, the functional J may depend only on the scrap value (that is, the terminal pay-off) $J = S(\mathbf{x}(T), T)$ or both the definite integral and the scrap value $J = \int_0^T F(\mathbf{x}(t), \mathbf{u}(t), t)dt + S(\mathbf{x}(T), T)$. The former type of is called the *Mayer problem* and the latter the *Bolza problem*. For the purposes of this discussion, only the standard form is considered as all the functional forms – standard, Bolza and Mayer – are convertible with each other.

[2] In this thesis, vectors are denoted by a lower case boldface letter. Vector notation is generally used where applicable. Whether a vector is a column or a row vector should be clear from the context.

[3] An example of a control with discontinuities is a switch which turns a lamp or an engine on and off. If the control has these kinds of "jump" discontinuities and it is bounded (e.g. $u(t) \in [0, 1]$), the problem may result in a *bang-bang* control where the optimal control path alternates between one boundary and another in successive time intervals (e.g. $u(t) = 0$, when $t \in [0, t_1)$, and $u(t) = 1$, when $t \in [t_1, T]$).

[4] In case of a jump control, the state variable has a finite amount of corner points where it is continuous but not differentiable.

the initial condition and (2.4) restricts the set of admissible controls so that the objective functional (2.1) is well-defined. The goal is to find an admissible control $\mathbf{u}(t)$ which is optimal in a sense that it maximises (or minimises) the value of the objective functional (2.1).[5] Feasibility and optimality of a solution may be defined as follows (Dockner et al. 2000, p. 40):

Definition 2.1 (Feasibility) A control path $\mathbf{u} : [0, T] \mapsto \mathbb{R}^n$ is *feasible* for the problem (2.1)–(2.4) if the initial value problem (2.2)–(2.3) has a unique absolutely continuous solution such that the constraints $\mathbf{x}(t) \in \mathcal{X}$ and $\mathbf{u}(t) \in \mathcal{U}$ hold for all $t \in [0, T]$ and the definite integral (2.1) is well defined.

Definition 2.2 (Optimality) The control path $\mathbf{u}^*(t)$ is *optimal* if it is feasible and if the inequality $J[\mathbf{u}^*(t)] \geq J[\mathbf{u}(t)]$ holds for all feasible control paths $\mathbf{u}(t)$.

The nature of the problem (2.1)–(2.4) bears close similarities to that of static optimisation where the task is to find an optimal value to an objective function. However, in dynamic optimisation the task is to find an optimal (time) path, a function, to an objective functional. The difference between a function and a functional is that a *function* is mapping from a set of values to another (e.g. $u : \mathbb{R} \mapsto \mathbb{R}$) while a *functional* is a mapping from a set of functions to a set of values (e.g. $J : \mathcal{U} \mapsto \mathbb{R}$). Thus, for example, the functional $J[u(t)]$, where $u \in \mathcal{U}$, is not a mapping from values of the function $u(t)$ to real numbers (as a composite function $g(u(t))$ would be) but rather a mapping from *any* admissible function u to real numbers. To underscore this point, the functional $J[u(t)]$ is often written just $J[u]$ (or simply J) without argument t.

2.1.2 The maximum principle

There are two common approaches to solving the problem (2.1)–(2.4). One of them is the *maximum principle*, which gives the necessary conditions for an optimal solution. Another is *dynamic programming*, which requires solving a Hamilton-Jacobi-Bellman partial differential equation and which gives necessary and sufficient conditions for the optimum.

The maximum principle was originally developed by Lev Pontryagin and his associates (Pontryagin et al. 1962).[6] In many cases, it provides a direct way of finding an optimal path to a problem (2.1)–(2.4) without the need to consider all possible trajectories at the same time. The maximum principle utilises a function called a *Hamiltonian function* or

[5]This thesis focuses on maximisation problems. All results considered here concerning maximisation problems apply to minimisation problems as well because any maximisation problem can be converted into a minimisation problem by simply switching the sign of the objective functional ($\max J \Leftrightarrow \min -J$).

[6]Similar work was produced independently by Magnus Hestenes who later published an extension to Pontryagin's results (Hestenes 1965).

simply a *Hamiltonian* which is defined as:

$$\begin{aligned} H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t) &= F(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\mu}(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ &= F(\mathbf{x}(t), \mathbf{u}(t), t) + \sum_{i=1}^{n} \mu_i(t) f_i(\mathbf{x}(t), \mathbf{u}(t), t). \end{aligned} \tag{2.5}$$

The variable $\boldsymbol{\mu}(t) = (\mu_1(t), \ldots, \mu_n(t)) \in \mathcal{M}$ is called a *costate* or *adjoint variable* which belongs to a set $\mathcal{M} \subseteq \mathbb{R}^n$. The Hamiltonian function (2.5) is similar to a *Lagrange function* in static optimisation, except that in the Hamiltonian function the undetermined Lagrange multiplier $\boldsymbol{\mu}$ is actually a function of time t. The costate variable $\boldsymbol{\mu}(t)$ plays a central role in the necessary conditions of the maximum principle which are the following:

$$\max_{\mathbf{u}(t)} \quad H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t) \text{ for all } t \in [0, T] \tag{2.6}$$

$$\dot{\mathbf{x}}(t) = \frac{\partial H(\cdot)}{\partial \boldsymbol{\mu}(t)} \tag{2.7}$$

$$\dot{\boldsymbol{\mu}}(t) = -\frac{\partial H(\cdot)}{\partial \mathbf{x}(t)} \tag{2.8}$$

$$\boldsymbol{\mu}(T) = 0. \tag{2.9}$$

If the Hamiltonian function is differentiable with respect to $\mathbf{u}(t)$ and its maximum $H(\mathbf{x}(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t), t) \geq H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t)$, for all $t \in [0, T]$, is an inner solution $\mathbf{u}^*(t)$, then (2.6) is simply:

$$\frac{\partial H(\cdot)}{\partial \mathbf{u}(t)} = 0 \text{ for all } t \in [0, T]. \tag{2.10}$$

The conditions (2.7)–(2.8) form a so-called *Hamiltonian system* or a *canonical system*, which describes the evolution of the system in terms of its state and costate variables.[7] The condition (2.9) in turn is called a *transversality condition* which is a condition that the optimal solution must satisfy at the terminal time T. Without the transversality condition (or some other boundary condition) it would be difficult to identify the optimal path because the terminal condition $\mathbf{x}(T)$ is a variable and not a constant as it was left free in the problem (2.1)–(2.4).[8] With alternative terminal conditions, one obtains other transversality conditions, which are not discussed further herein.[9]

Using the above tools, the maximum principle may be formulated broadly as follows:

[7]The concept of the Hamiltonian originates in Hamiltonian mechanics in physics, where energy $E(p, q)$ is a function of the position $q(t)$ and the momentum $p(t)$. The Hamiltonian function $H = T(p) + V(q) = p^2/2m + V(q)$ describes the total energy of the system (the sum of kinetic energy T and potential energy V), while the state variable q describes the position of the system and the costate variable p the momentum. Hamilton's equations $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$ describe how the system evolves over time as potential energy is converted to kinetic energy (and vice versa) while the total energy of the system remains constant.

[8]*Transverse* means "to cross" (in this context, a terminal line, for example).

[9]See e.g. Chiang 1992 for the treatment of alternative terminal conditions.

Theorem 2.1 (The maximum principle) *If there is an optimal solution* $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ *to the problem* (2.1)–(2.4)*, then there exists a nonzero continuous function* $\boldsymbol{\mu} : [0, T] \mapsto \mathbb{R}^n$ *that satisfies the conditions* (2.6)–(2.9).

A rigorous (and quite long) proof for the maximum principle can be found in Pontryagin et al. 1962. In the following, the necessary conditions (2.7)–(2.10) are derived heuristically to understand the intuition behind them and to motivate their use in this thesis. The variational method of the argument follows the reasoning in Kamien and Schwartz 1991, Léonard and Long 1992 and Chiang 1992.

Suppose that $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ is an optimal solution to the problem (2.1)–(2.4) and that the Hamiltonian function is differentiable in $\mathbf{u}(t)$ and that $\mathbf{u}^*(t)$ is an inner solution. Using the definition of the Hamiltonian (2.5) the functional (2.1) can be written as

$$\begin{aligned} J &= \int_0^T F(\mathbf{x}(t), \mathbf{u}(t), t)dt \\ &= \int_0^T [H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t) - \boldsymbol{\mu}(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)]dt \\ &= \int_0^T H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t) - \int_0^T \boldsymbol{\mu}(t)\dot{\mathbf{x}}(t)dt. \end{aligned}$$

By integration by parts, the last term becomes

$$\begin{aligned} \int_0^T \boldsymbol{\mu}(t)\dot{\mathbf{x}}(t)dt &= \Big|_0^T \boldsymbol{\mu}(t)\mathbf{x}(t) - \int_0^T \dot{\boldsymbol{\mu}}(t)\mathbf{x}(t)dt \\ &= \boldsymbol{\mu}(T)\mathbf{x}(T) - \boldsymbol{\mu}(0)\mathbf{x}(0) - \int_0^T \dot{\boldsymbol{\mu}}(t)\mathbf{x}(t)dt. \end{aligned}$$

Therefore, the objective functional (2.1) may be reformulated as

$$J = \int_0^T [H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t) + \dot{\boldsymbol{\mu}}(t)\mathbf{x}(t)]dt - \boldsymbol{\mu}(T)\mathbf{x}(T) + \boldsymbol{\mu}(0)\mathbf{x}(0). \tag{2.11}$$

Next, the optimal path $\mathbf{u}^*(t)$ is perturbed by introducing an arbitrary function of the type $\mathbf{a}(t, \epsilon) = \epsilon\boldsymbol{\alpha}(t)$, where ϵ is a parameter. This generates a trajectory

$$\mathbf{u}(t, \epsilon) = \mathbf{u}^*(t) + \epsilon\boldsymbol{\alpha}(t).$$

Obviously, $\mathbf{u}(t, 0) = \mathbf{u}^*(t)$. Through the state equation (2.2), a perturbation in the control path $\mathbf{u}(t, \epsilon)$ affects the trajectory of the state variable as well, which is denoted implicitly by $\mathbf{x}(t, \epsilon)$. Thus, (2.11) becomes

$$\begin{aligned} J = \int_0^T & [H(\mathbf{x}(t, \epsilon), \mathbf{u}^*(t) + \epsilon \boldsymbol{\alpha}(t), \boldsymbol{\mu}(t), t) + \dot{\boldsymbol{\mu}}(t)\mathbf{x}(t, \epsilon)]dt \\ & - \boldsymbol{\mu}(T)\mathbf{x}(T, \epsilon) + \boldsymbol{\mu}(0)\mathbf{x}(0, \epsilon). \end{aligned} \tag{2.12}$$

The first-order condition $\frac{dJ}{d\epsilon} = 0$ is a necessary condition for the maximum of the functional J. By applying this condition, (2.12) yields

$$\begin{aligned} \frac{dJ}{d\epsilon} = & \int_0^T \left[\frac{d}{d\epsilon} (H(\mathbf{x}(t, \epsilon), \mathbf{u}^*(t) + \epsilon \boldsymbol{\alpha}(t), \boldsymbol{\mu}(t), t)) + \frac{d}{d\epsilon} (\dot{\boldsymbol{\mu}}(t)\mathbf{x}(t, \epsilon)) \right] dt \\ & - \frac{d}{d\epsilon} (\boldsymbol{\mu}(T)\mathbf{x}(T, \epsilon)) + \frac{d}{d\epsilon} (\boldsymbol{\mu}(0) \underbrace{\mathbf{x}(0, \epsilon)}_{\overset{(2.3)}{=} \mathbf{x}_0}) = 0 \\ = & \int_0^T \left[\frac{\partial H}{\partial \mathbf{x}} \frac{d\mathbf{x}}{d\epsilon} + \frac{\partial H}{\partial \mathbf{u}} \frac{d\mathbf{u}}{d\epsilon} + \dot{\boldsymbol{\mu}} \frac{d\mathbf{x}}{d\epsilon} \right] dt - \boldsymbol{\mu}(T) \frac{d\mathbf{x}(T, \epsilon)}{d\epsilon} = 0 \\ = & \int_0^T \left[\frac{\partial H}{\partial \mathbf{u}} \boldsymbol{\alpha}(t) + \left(\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\mu}} \right) \frac{d\mathbf{x}}{d\epsilon} \right] dt - \boldsymbol{\mu}(T) \frac{d\mathbf{x}(T, \epsilon)}{d\epsilon} = 0. \end{aligned} \tag{2.13}$$

Because $\boldsymbol{\alpha}(t)$ is an arbitrary function and derivatives $\frac{d\mathbf{x}}{d\epsilon}$ and $\frac{d\mathbf{x}(T,\epsilon)}{d\epsilon}$ are not necessarily zero, it follows from (2.13) that the following conditions must hold for all $t \in [0, T]$:

$$\frac{\partial H}{\partial \mathbf{u}} = 0 \tag{2.14}$$

$$\dot{\boldsymbol{\mu}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{2.15}$$

$$\boldsymbol{\mu}(T) = 0. \tag{2.16}$$

By (2.2), it was assumed that $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ for all $t \in [0, T]$, which gives

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\mu}}. \tag{2.17}$$

The conditions (2.14)–(2.17) are the same as the conditions (2.7)–(2.10). This concludes the heuristic argument for theorem 2.1.

The Hamiltonian function and each of the necessary conditions of the maximum principle can be given an intuitive economic interpretation as demonstrated by Dorfman 1969 and Chiang 1992. A typical problem in economics is how to best allocate scarce resources to maximise a given economic objective. For example, a firm may choose a production plan that maximises its profits while being subject to constraints such as the productive capacity or the available level of technology. By investing in its productive capacity (e.g. factories or machines) or research and development activities, the firm may increase its production or lower its costs and thereby increase its profits. However, there is usually a trade-off between the current and future profits: investments decrease the firm's current profits but increase its future profits. In such a dynamic situation, the firm's overall profits then depend on both the current and future profits.

In the context of the above example, the Hamiltonian (2.5) can be interpreted to be a function that contains the prospect of current and future profits of the firm (Chiang 1992). It is a sum of two functions: the (current) profit function $F(\mathbf{x}(t), \mathbf{u}(t), t)$ and the state equation $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$, which governs the rate of change of the state variable $\mathbf{x}(t)$ that (together with the control variable $\mathbf{u}(t)$ and time t) affects the future profits. Hence, the Hamiltonian function captures the firm's overall profit prospect that is composed of the current and future profits. Maximising the Hamiltonian in the condition (2.6) means that the optimal control $\mathbf{u}^*(t)$ must maximise the profit prospect at each moment of time t. When the condition (2.10)

$$\frac{\partial H(\cdot)}{\partial \mathbf{u}(t)} = \frac{\partial F(\cdot)}{\partial \mathbf{u}(t)} + \boldsymbol{\mu}(t)\frac{\partial \mathbf{f}(\cdot)}{\partial \mathbf{u}(t)} = 0$$

is rewritten as

$$\frac{\partial F(\cdot)}{\partial \mathbf{u}(t)} = -\boldsymbol{\mu}(t)\frac{\partial \mathbf{f}(\cdot)}{\partial \mathbf{u}(t)}$$

it can be seen that a marginal increase in the current profits must be balanced by a marginal decrease in the future profits that a change in the state variable induces. Furthermore, the costate variable $\boldsymbol{\mu}(t)$ may be interpreted as a shadow price for the state variable $\mathbf{x}(t)$ at time t (just like a Lagrange multiplier $\boldsymbol{\mu}$ is a shadow price for relaxing constraints in static optimisation).[10] Likewise, the transversality condition (2.9) receives an intuitive explanation: at the terminal time T, there are no longer any prospect of future profits so the shadow price $\boldsymbol{\mu}(T)$ for the state variable $\mathbf{x}(T)$ must be zero. Lastly, when the condition (2.8)

$$\dot{\boldsymbol{\mu}}(t) = -\frac{\partial H(\cdot)}{\partial \mathbf{x}(t)} = -\frac{\partial F(\cdot)}{\partial \mathbf{x}(t)} - \boldsymbol{\mu}(t)\frac{\partial \mathbf{f}(\cdot)}{\partial \mathbf{x}(t)}$$

is reformulated as

$$-\dot{\boldsymbol{\mu}}(t) = \frac{\partial F(\cdot)}{\partial \mathbf{x}(t)} + \boldsymbol{\mu}(t)\frac{\partial \mathbf{f}(\cdot)}{\partial \mathbf{x}(t)}$$

it becomes evident that the shadow price $\boldsymbol{\mu}(t)$ of the state variable $\mathbf{x}(t)$ decreases at the rate at which the state variable $\mathbf{x}(t)$ contributes to the firm's current and future profits. The shadow value of the state variable thereby depreciates at a rate at which its potential contribution to profits turns into its past contribution (Dorfman 1969).

[10] In general, costate variables are not proper measures of shadow prices in differential games with more than one agent as shall be discussed later.

As was remarked earlier, the maximum principle gives only the necessary conditions for the optimal solution. It restricts the set of possible solutions to the problem (2.1)–(2.4) but it does not often imply which of the (possibly many) solutions satisfying its conditions is the optimal solution. There is also a confusing possibility that the necessary conditions yield a non-empty set of candidate solutions when no optimal solution actually exists. Necessary conditions might then offer false hope of finding an optimum. Therefore, the existence of an optimal solution should always be ascertained, preferably by employing a sufficient condition for optimality. Solutions that satisfy the sufficient conditions are (necessarily) optimal.[11]

Mangasarian (1966) has formulated basic sufficient conditions for the optimal control of nonlinear systems. His theorem shows that if the Hamiltonian function is concave in state and control variables, the necessary conditions of the maximum principle are also sufficient for optimality. The result is comparable to nonlinear static optimisation where the Karush-Kuhn-Tucker necessary conditions are sufficient for a global maximum if (among other things) the objective function is concave.

Theorem 2.2 (Mangasarian) *Suppose that a solution* $(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t))$ *satisfies the necessary conditions* (2.7)–(2.10). *If the Hamiltonian function* (2.5) *is differentiable and concave (strictly concave) jointly in* $(\mathbf{x}, \mathbf{u})$, *then* $(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t))$ *is an optimal solution (a unique optimal solution) to the problem* (2.1)–(2.4).

Proof. To demonstrate optimality, it must be shown that $J[\mathbf{u}^*] \geq J[\mathbf{u}]$ for all feasible control paths $\mathbf{u}$.[12] One may write the difference $J[\mathbf{u}^*] - J[\mathbf{u}]$ as

$$\begin{aligned} J[\mathbf{u}^*] - J[\mathbf{u}] &= \int_0^T F^* dt - \int_0^T F dt \\ &= \int_0^T [H^* - \boldsymbol{\mu}\mathbf{f}^* - (H - \boldsymbol{\mu}\mathbf{f})] dt \\ &= \int_0^T [H^* - H - \boldsymbol{\mu}(\dot{\mathbf{x}}^* - \dot{\mathbf{x}})] dt. \end{aligned}$$

[11] If the sufficient conditions are too strong, they may produce an empty set of solutions in which case they are not very helpful in finding the optimum.

[12] Arguments of the functions are suppressed to shorten the notation. The optimal Hamiltonian is denoted as $H^* = H^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t), t)$ while the functions $F^* = F^*(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$ and $f^* = f^*(\mathbf{x}^*(t), \mathbf{u}^*(t), t)$ are written similarly.

By integration by parts, the last term becomes

$$\begin{aligned}
\int_0^T \mu\,(\dot{\mathbf{x}}^* - \dot{\mathbf{x}})dt &= \Big|_0^T \mu\,(\mathbf{x}^* - \mathbf{x}) - \int_0^T \dot{\mu}\,(\mathbf{x}^* - \mathbf{x})dt \\
&= \underbrace{\mu(T)}_{\overset{(2.9)}{=}0}\,(\mathbf{x}^*(T) - \mathbf{x}(T)) - \mu(0)\,\underbrace{(\mathbf{x}^*(0) - \mathbf{x}(0))}_{\overset{(2.3)}{=}\mathbf{x}_0 - \mathbf{x}_0 = 0} - \int_0^T \dot{\mu}\,(\mathbf{x}^* - \mathbf{x})dt \\
&= -\int_0^T \dot{\mu}\,(\mathbf{x}^* - \mathbf{x})dt.
\end{aligned}$$

Therefore, the difference $J[\mathbf{u}^*] - J[\mathbf{u}]$ may be formulated as

$$J[\mathbf{u}^*] - J[\mathbf{u}] = \int_0^T [H^* - H + \dot{\mu}\,(\mathbf{x}^* - \mathbf{x})]dt. \tag{2.18}$$

By differentiability and concavity of the Hamiltonian (2.5), it holds that

$$\begin{aligned}
J[\mathbf{u}^*] - J[\mathbf{u}] &= \int_0^T [H^* - H + \dot{\mu}\,(\mathbf{x}^* - \mathbf{x})]dt \\
&\geq \int_0^T \left[(\mathbf{x}^* - \mathbf{x})\,H_{\mathbf{x}}^* + (\mathbf{u}^* - \mathbf{u})\,H_{\mathbf{u}}^* + \dot{\mu}\,(\mathbf{x}^* - \mathbf{x})\right]dt \\
&= \int_0^T \left[(\mathbf{x}^* - \mathbf{x})\left(H_{\mathbf{x}}^* + \dot{\mu}\right) + (\mathbf{u}^* - \mathbf{u})\,H_{\mathbf{u}}^*\right]dt \\
&= 0.
\end{aligned}$$

The last equality follows from the necessary conditions (2.8) and (2.10). If the Hamiltonian is strictly concave, then the inequality above is a strict inequality and the optimal solution is unique. □

The concavity of the Hamiltonian may be examined by decomposing it into its component functions. The Hamiltonian function is concave jointly in $(\mathbf{x}, \mathbf{u})$ when the following property of differentiable concave functions holds:

$$H^* - H \geq H_{\mathbf{x}}^*\,(\mathbf{x}^* - \mathbf{x}) + H_{\mathbf{u}}^*\,(\mathbf{u}^* - \mathbf{u})\,. \tag{2.19}$$

Decomposing the Hamiltonian in (2.19) yields

$$F^* + \mu\mathbf{f}^* - F - \mu\mathbf{f} \geq \left(F_{\mathbf{x}}^* + \mu\mathbf{f}_{\mathbf{x}}^*\right)(\mathbf{x}^* - \mathbf{x}) + \left(F_{\mathbf{u}}^* + \mu\mathbf{f}_{\mathbf{u}}^*\right)(\mathbf{u}^* - \mathbf{u}) \tag{2.20}$$

or, equivalently,

$$F^* - F + \mu\,[\mathbf{f}^* - \mathbf{f}] \geq F_{\mathbf{x}}^*\,(\mathbf{x}^* - \mathbf{x}) + F_{\mathbf{u}}^*\,(\mathbf{u}^* - \mathbf{u}) + \mu\left[\mathbf{f}_{\mathbf{x}}^*\,(\mathbf{x}^* - \mathbf{x}) + \mathbf{f}_{\mathbf{u}}^*\,(\mathbf{u}^* - \mathbf{u})\right]. \tag{2.21}$$

The following property is evident from (2.21).

Theorem 2.3 *The Hamiltonian function* (2.5) *is concave jointly in* $(\mathbf{x}, \mathbf{u})$ *if the function* $F(\mathbf{x}(t), \mathbf{u}(t), t)$ *is concave jointly in* $(\mathbf{x}, \mathbf{u})$ *and one of the following three conditions holds:*

(1) The function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ *is concave jointly in* $(\mathbf{x}, \mathbf{u})$ *and the sign of the costate variable is nonnegative (*$\boldsymbol{\mu} \geq 0$*);*

(2) The function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ *is convex jointly in* $(\mathbf{x}, \mathbf{u})$ *and the sign of the costate variable is nonpositive (*$\boldsymbol{\mu} \leq 0$*);*

(3) The function $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$ *is linear jointly in* $(\mathbf{x}, \mathbf{u})$ *(in which case the sign of the costate variable* $\boldsymbol{\mu}$ *is irrelevant).*

Arrow (1968) has observed that the Mangasarian theorem may be generalised to situations where the Hamiltonian function itself is not concave. His theorem, which was proved in Arrow and Kurz 1970 and Kamien and Schwartz 1971, provides a weaker sufficient condition for the optimality of the solution than the Mangasarian theorem, which is a special case of the Arrow theorem. The Arrow theorem is more general than the Mangasarian theorem because it requires only that a special version of the Hamiltonian function, which is obtained by substituting the optimal control into the Hamiltonian (2.5), is concave. This Hamiltonian is called a *maximised Hamiltonian* which is defined as

$$H^0(\mathbf{x}(t), \boldsymbol{\mu}(t), t) = H^0(\mathbf{x}(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t), t) \tag{2.22}$$

where $\mathbf{u}^*(t) = \mathbf{U}(\mathbf{x}(t), \boldsymbol{\mu}(t), t)$ is the control that maximises the Hamiltonian (2.5). It should be noted that (2.22) is identical to the necessary condition (2.6) formulated earlier. Here, it is only given an alternative form and a new name. The maximised Hamiltonian is a function of $\mathbf{x}(t)$, $\boldsymbol{\mu}(t)$ and t and it therefore no longer depends on $\mathbf{u}(t)$ which is fixed as $\mathbf{u}^*(t)$. It is also distinct from the optimal Hamiltonian $H^*(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t), t)$, which takes the optimal solution $(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t))$ as given and thus, in fact, is just a function of time t ($H^* = H^*(t)$).

Because the maximised Hamiltonian (2.22) is a special case of the Hamiltonian (2.5), the maximum principle applies to it as well. The necessary conditions (2.7)–(2.8) may therefore be redefined as

$$\dot{\mathbf{x}}(t) = \frac{\partial H^0(\cdot)}{\partial \boldsymbol{\mu}(t)} \tag{2.23}$$

$$\dot{\boldsymbol{\mu}}(t) = -\frac{\partial H^0(\cdot)}{\partial \mathbf{x}(t)}. \tag{2.24}$$

One may now state the following theorem.

Theorem 2.4 (Arrow) *Suppose that a solution* $(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t))$ *satisfies the necessary conditions* (2.9)–(2.10) *and* (2.23)–(2.24). *If the maximised Hamiltonian function* (2.22) *is differentiable and concave (strictly concave) in* $\mathbf{x}$*, then* $(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t))$ *is an optimal solution (a unique optimal solution) to the problem* (2.1)–(2.4).

Proof. The proof is quite similar to that of theorem 2.2. Once again, it must be shown that $J[\mathbf{u}^*] \geq J[\mathbf{u}]$ for all feasible control paths $\mathbf{u}$. Taking the difference (2.18) as a starting point, one may deduce that

$$\begin{aligned} J[\mathbf{u}^*] - J[\mathbf{u}] &= \int_0^T [H^* - H + \dot{\boldsymbol{\mu}}(\mathbf{x}^* - \mathbf{x})]dt \\ &\geq \int_0^T \left[H^* - H^0 + \dot{\boldsymbol{\mu}}(\mathbf{x}^* - \mathbf{x})\right]dt \\ &\geq \int_0^T \left[(\mathbf{x}^* - \mathbf{x})\, H^*_{\mathbf{x}} + \dot{\boldsymbol{\mu}}(\mathbf{x}^* - \mathbf{x})\right]dt \\ &= \int_0^T (\mathbf{x}^* - \mathbf{x})\left(H^*_{\mathbf{x}} + \dot{\boldsymbol{\mu}}\right)dt \\ &= 0. \end{aligned}$$

The first inequality holds because $H^0(\mathbf{x}(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t), t) \geq H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t)$ for all $\mathbf{u}(t)$. The second inequality holds by differentiability and concavity of the maximised Hamiltonian (2.22). The last equality follows from the necessary condition (2.8). Analogous to theorem 2.2, if the maximised Hamiltonian is strictly concave, then the inequality above is a strict inequality and the optimal solution is unique. □

In economic models with intertemporal choice, where consumers and firms must make decisions that affect their rewards over time, time discounting matters. Time discounting reflects the economic behaviour of consumers and firms who usually value present rewards more than those in the future. This impatience of consumers and firms is represented in dynamic economic models by a discount factor that is typically exponential in continuous time problems.[13] The purpose of the discount factor is to discount future payoffs to their present value and thereby make future and present payoffs comparable with each other.

By introducing the discount factor $e^{-\rho t}$, where a constant $\rho > 0$ represents the discount rate, into the integrand function $F(\mathbf{x}(t), \mathbf{u}(t), t) = e^{-\rho t} G(\mathbf{x}(t), \mathbf{u}(t), t)$, the problem (2.1)–(2.4) becomes

$$\max_{\mathbf{u}(t)} \quad J = \int_0^T e^{-\rho t} G(\mathbf{x}(t), \mathbf{u}(t), t)dt \tag{2.25}$$

$$s.t. \quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.26}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{2.27}$$

$$\mathbf{u}(t) \in \mathcal{U} \subseteq \mathbb{R}^m \text{ for all } t \in [0, T], \tag{2.28}$$

[13] In fact, non-exponential discount factors are generally avoided in economic models because they result in dynamic inconsistencies, see Strotz 1955–56 and Pollak 1968.

where the terminal condition $\mathbf{x}(T)$ is left free.

The Hamiltonian function for the reformulated problem (2.25)–(2.28) is

$$H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t) = e^{-\rho t} G(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\mu}(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t). \tag{2.29}$$

The Hamiltonian (2.29) can be written equivalently as

$$e^{\rho t} H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t) = G(\mathbf{x}(t), \mathbf{u}(t), t) + e^{\rho t}\boldsymbol{\mu}(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t). \tag{2.30}$$

If one defines a new costate variable

$$\boldsymbol{\lambda}(t) = e^{\rho t}\boldsymbol{\mu}(t) \tag{2.31}$$

and substitutes it into (2.30), one obtains

$$\begin{aligned}\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) &= e^{\rho t} H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\ &= G(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t).\end{aligned} \tag{2.32}$$

The function $\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t)$ is called a *current value Hamiltonian* and the variable $\boldsymbol{\lambda}(t)$ a *current value costate* (or *adjoint*) *variable*. With dynamic economic problems, it is often more convenient to work with the current value (or undiscounted) version of the Hamiltonian function $\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t)$ than the regular (discounted) Hamiltonian $H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t)$ where the discount factor appears explicitly. This is especially true of autonomous problems, where the argument t does not appear explicitly in $G(\mathbf{x}(t), \mathbf{u}(t))$ and $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$.[14] Autonomous problems are usually easier to solve than non-autonomous problems and, in addition, they have a special property that the optimal Hamiltonian is constant for all $t \in [0, T]$. This can be established by taking the time derivative of the regular Hamiltonian

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial H}{\partial \mathbf{u}}\dot{\mathbf{u}} + \frac{\partial H}{\partial \boldsymbol{\mu}}\dot{\boldsymbol{\mu}}$$

which, by the maximum principle, is at the optimum $(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\mu}(t))$

$$\frac{dH^*}{dt} = \frac{\partial H^*}{\partial t} + \frac{\partial H^*}{\partial \mathbf{x}}\frac{\partial H^*}{\partial \boldsymbol{\mu}} + 0 \cdot \dot{\mathbf{u}} + \frac{\partial H^*}{\partial \boldsymbol{\mu}}\left(-\frac{\partial H^*}{\partial \mathbf{x}}\right) = \frac{\partial H^*}{\partial t}.$$

[14] Technically speaking, with time discounting, the problem is still non-autonomous, as the argument t is present in the integrand function $F(\mathbf{x}(t), \mathbf{u}(t), t) = e^{-\rho t} G(\mathbf{x}(t), \mathbf{u}(t), t)$. However, the argument t does not show up in the current value Hamiltonian and the necessary conditions derived thereof, as will be seen, and hence the problem may be thought of as autonomous (whereas by using the regular Hamiltonian, it obviously could be not).

In autonomous problems,

$$\frac{\partial H^*}{\partial t} = 0$$

and thus

$$\frac{dH^*}{dt} = 0,$$

which implies that the Hamiltonian function is constant on the optimal path. This property applies similarly to the current value Hamiltonian.

It turns out that the necessary conditions of the maximum principle need only a slight adjustment to accommodate the use of the current value Hamiltonian. As the discount factor does not include the control variable $\mathbf{u}(t)$, it can be observed to be constant for any given time t. Therefore, the optimal control $\mathbf{u}^*(t)$ maximises both the regular and the current value Hamiltonian. The Hamiltonian in the condition (2.6) can then be substituted directly with the current value Hamiltonian:

$$\max_{\mathbf{u}(t)} \quad \mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\mu}(t), t) \text{ for all } t \in [0, T]. \tag{2.33}$$

Likewise, from the condition (2.10)

$$\frac{\partial H(\cdot)}{\partial \mathbf{u}(t)} = \frac{\partial \mathcal{H}(\cdot)}{\partial \mathbf{u}(t)} e^{\rho t} = 0$$

one can simply deduce that

$$\frac{\partial \mathcal{H}(\cdot)}{\partial \mathbf{u}(t)} = 0 \text{ for all } t \in [0, T]. \tag{2.34}$$

Regarding the condition (2.7), it is evident from (2.32) that

$$\frac{\partial H(\cdot)}{\partial \boldsymbol{\mu}(t)} = \frac{\partial \mathcal{H}(\cdot)}{\partial \boldsymbol{\lambda}(t)} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t).$$

Therefore, the condition (2.7) can be reformulated as

$$\dot{\mathbf{x}}(t) = \frac{\partial \mathcal{H}(\cdot)}{\partial \boldsymbol{\lambda}(t)}. \tag{2.35}$$

By contrast, the condition (2.8) requires a minor adjustment. Differentiating (2.31) with respect to time t yields

$$\dot{\boldsymbol{\lambda}}(t) = e^{\rho t} \dot{\boldsymbol{\mu}}(t) + \rho e^{\rho t} \boldsymbol{\mu}(t)$$

which, by (2.8) and (2.31), becomes

$$\dot{\boldsymbol{\lambda}}(t) = e^{\rho t}\left(-\frac{\partial H(\cdot)}{\partial \mathbf{x}(t)}\right) + \rho\boldsymbol{\lambda}(t)$$

and, by (2.32), further

$$\dot{\boldsymbol{\lambda}}(t) = -\frac{\partial \mathcal{H}(\cdot)}{\partial \mathbf{x}(t)} + \rho\boldsymbol{\lambda}(t). \tag{2.36}$$

Lastly, by (2.31), the costate variable $\boldsymbol{\mu}(t)$ takes the value $\boldsymbol{\mu}(T) = \boldsymbol{\lambda}(T)e^{-\rho T}$ at the terminal time T. The transversality condition (2.9) thus becomes

$$\boldsymbol{\lambda}(T)e^{-\rho T} = 0. \tag{2.37}$$

The conditions (2.33) (or (2.34)) and (2.35)–(2.37) are the revised necessary conditions of the maximum principle for the current value Hamiltonian. Theorems 2.2–2.4 likewise apply trivially with the current value Hamiltonian *mutatis mutandis*.

2.1.3 Dynamic programming

Another approach to solving control theoretic problems of the type (2.1)–(2.4) is dynamic programming which was developed by Richard Bellman in the 1950's around the same time as Pontryagin and his associates were working on the maximum principle. The main idea of dynamic programming is to reduce a multi-dimensional optimisation problem into a set of related problems with a smaller dimension by embedding a small problem into a larger problem and solving the larger problem recursively after the solution to the smaller problem has been obtained. This approach allows one to avoid what Bellman called the "curse of dimensionality" which refers to the exponentially increasing difficulty of solving mathematical problems as their dimensions increase.

In dynamic programming, the solution method for finding an optimal path (or "policy", that is, a rule for making decisions which transform the state variable) $\mathbf{u}^*$ to the problem (2.1)–(2.4) involves the application of Bellman's principle of optimality which is defined as follows (Bellman 1957, p. 83):

Definition 2.3 (Principle of optimality) An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

In accordance with the principle of optimality, the past history of the system is irrelevant for finding the optimal policy – the current value of the state variable contains all relevant information for this purpose. More formally, this means that the solution $\mathbf{u}^*(\mathbf{x}, t)$ depends on time t and the state $\mathbf{x}$ but not on the initial value $\mathbf{x}_0$ itself. By principle of optimality, dynamic programming requires that the solution is optimal for all possible trajectories starting from any initial point $\mathbf{x}$. By contrast, the necessary conditions of the maximum principle need to hold for only trajectory $\mathbf{u}^*(t)$, starting from $\mathbf{x_0}$, so, strictly speaking, the solution $\mathbf{u}^*(\mathbf{x_0}, t)$ obtained through the maximum principle depends on time t *and* the initial value $\mathbf{x_0}$. This is an important point which has major consequences for time consistency of the solutions in differential games as will be later discussed.

Using dynamic programming, it can be shown that in continuous time problems of the type (2.1)–(2.4) the sufficient condition for the solution is given by a partial differential equation called the *Hamilton-Jacobi-Bellman equation*

$$-\frac{\partial V(\mathbf{x}, t)}{\partial t} = \max_{\mathbf{u}} \left\{ F(\mathbf{x}, \mathbf{u}, t) + \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right\}, \tag{2.38}$$

where $V(\mathbf{x}, t)$ is the *optimal value function* for the problem in question.[15] By solving $V(\mathbf{x}, t)$, which is necessarily a unique function, one obtains the solution to the problem. However, due to the difficulty of solving partial differential equations, analytical solutions to the Hamilton-Jacobi-Bellman equation are very hard to find in practice even for the most simple of problems. Sometimes $V(\mathbf{x}, t)$ may be found by making an informed guess about its functional form.

The Hamilton-Jacobi-Bellman equation describes the decrease of the value function $V(\mathbf{x}, t)$ with respect to time t as the maximum of the sum of the instantaneous pay-off and the marginal value of the change of the state variable. As the value function is decreasing with respect to time (it decreases to its terminal value $V(\mathbf{x}, T)$), it is intuitive that, at the optimum, the decrease should be as fast as possible (Weber 2011).

Before proving sufficiency, it is first shown how the Hamilton-Jacobi-Bellman equation can be derived heuristically by dynamic programming. The method follows closely the heuristic arguments in Kamien and Schwartz 1991, Léonard and Long 1992 and Dockner et al. 2000.

Consider the problem (2.1)–(2.4) that is truncated and starts at some time t for which holds that $0 \leq t \leq T$. Then define $V(\mathbf{x}, t)$ as the optimal value function for the truncated problem as follows:

$$V(\mathbf{x}, t) = \max_{\mathbf{u}} \int_t^T F(\mathbf{x}(s), \mathbf{u}(s), s) ds \tag{2.39}$$

$$s.t. \quad \dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s), \mathbf{u}(s), s) \tag{2.40}$$

$$\mathbf{x}(t) = \mathbf{x} \tag{2.41}$$

$$\mathbf{u}(s) \in \mathcal{U}(\mathbf{x}(s), s) \subseteq \mathbb{R}^m \text{ for all } s \in [t, T]. \tag{2.42}$$

[15]Dynamic programming is suited to problems with discrete or continuous time. In this discussion, only latter is considered.

It can be observed that at time $t = 0$, the optimal value function is $V(\mathbf{x_0}, 0)$ and the problem (2.39)–(2.42) is identical to the problem (2.1)–(2.4). As the problem is in standard form (i.e. there is no scrap value), one can also observe that

$$V(\mathbf{x}, T) = 0. \tag{2.43}$$

The solution of the problem (2.39)–(2.42) at the terminal time $t = T$ is thus trivial. Next, instead of trying to solve the problem (2.39)–(2.42) for the whole interval $[t, T]$ at the same time, one can split the problem in two intervals $[t, t + \Delta)$ and $[t + \Delta, T]$, where $\Delta > 0$ is a small real number, and try to solve the problem separately for each interval. Thus, the integral (2.39) is broken in half

$$V(\mathbf{x}, t) = \max_{\mathbf{u}} \left\{ \int_t^{t+\Delta} F(\mathbf{x}(s), \mathbf{u}(s), s) ds + \int_{t+\Delta}^{T} F(\mathbf{x}(s), \mathbf{u}(s), s) ds \right\}. \tag{2.44}$$

By principle of optimality, the control $\mathbf{u}$ must be optimal in interval $[t + \Delta, T]$, where the initial condition is $\mathbf{x}(t + \Delta)$. Therefore, by definition of $V(\mathbf{x}, t)$, (2.44) becomes

$$V(\mathbf{x}, t) = \max_{\mathbf{u}} \left\{ \int_t^{t+\Delta} F(\mathbf{x}(s), \mathbf{u}(s), s) ds + V(\mathbf{x}(t + \Delta), t + \Delta) \right\}. \tag{2.45}$$

The problem for the interval $[t + \Delta, T]$ is now embedded into the problem for the broader interval $[t, T]$, which is possible due to the recursive nature of the problem (2.39)–(2.42). When one subtracts $V(\mathbf{x}, t)$ from both sides of equation (2.45) and multiplies them by $1/\Delta$, one obtains

$$0 = \max_{\mathbf{u}} \left\{ \frac{1}{\Delta} \int_t^{t+\Delta} F(\mathbf{x}(s), \mathbf{u}(s), s) ds + \frac{V(\mathbf{x}(t + \Delta), t + \Delta) - V(\mathbf{x}, t)}{\Delta} \right\} \tag{2.46}$$

which, when $\Delta \to 0$, becomes[16]

$$0 = \max_{\mathbf{u}} \left\{ F(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{dV(\mathbf{x}, t)}{dt} \right\} \tag{2.47}$$

and further

$$0 = \max_{\mathbf{u}} \left\{ F(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial V(\mathbf{x}, t)}{\partial t} \right\}, \tag{2.48}$$

which is the Hamilton-Jacobi-Bellman equation. This concludes the heuristic argument.

It should be noted that differentiability of the optimal value function $V(\mathbf{x}, t)$ was simply

[16]The limit $\lim_{\Delta \to 0} \frac{1}{\Delta} \int_t^{t+\Delta} F(\mathbf{x}(s), \mathbf{u}(s), s) ds = F(\mathbf{x}(t), \mathbf{u}(t), t)$ holds by the mean value theorem, which states that if a function $f(x)$ is continuous in interval $[a, b]$ and differentiable in interval (a, b), then $f'(c) = (f(b) - f(a)) / (b - a)$.

assumed in the above argument. In general, the value function may not be differentiable even when the integrand function $F(\cdot)$ and the state equation $\mathbf{f}(\cdot)$ are smooth functions. However, when the optimal value function is smooth (and its partial derivatives exist), the following sufficiency theorem applies (Dockner et al. 2000):

Theorem 2.5 *Assume that $V(\mathbf{x}, t)$ is a continuously differentiable function which satisfies the Hamilton-Jacobi-Bellman equation* (2.38) *and the terminal condition* (2.43) *for all $\mathbf{x} \in \mathcal{X}$ and $t \in [0, T]$. If $\mathbf{u}^* \in \mathcal{U}(\mathbf{x}, t)$ is a feasible control function that maximises the right-hand side of* (2.38) *for all $t \in [0, T]$, then $\mathbf{u}^*$ is an optimal control and $V(\cdot)$ is the optimal value of the problem* (2.39)–(2.42).

Proof. For optimality of the solution $\mathbf{u}^*$, it once again must be shown that $J[\mathbf{u}^*] \geq J[\mathbf{u}]$. First, it may be noted that by (2.38), the continuous differentiability of the value function $V(\mathbf{x}, t)$ and the feasibility of the control $\mathbf{u}$, it holds for $\mathbf{u}^*$ that

$$-\frac{\partial V(\mathbf{x}^*, t)}{\partial t} = F(\mathbf{x}^*, \mathbf{u}^*, t) + \frac{\partial V(\mathbf{x}^*, t)}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}^*, \mathbf{u}^*, t)$$

or equivalently

$$F(\mathbf{x}^*, \mathbf{u}^*, t) = -\frac{\partial V(\mathbf{x}^*, t)}{\partial \mathbf{x}}\dot{\mathbf{x}}^* - \frac{\partial V(\mathbf{x}^*, t)}{\partial t} = -\frac{dV(\mathbf{x}^*, t)}{dt}. \tag{2.49}$$

Similarly for $\mathbf{u}$, one can evaluate that

$$-\frac{\partial V(\mathbf{x}, t)}{\partial t} \geq F(\mathbf{x}, \mathbf{u}, t) + \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

or equivalently

$$F(\mathbf{x}, \mathbf{u}, t) \leq -\frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}}\dot{\mathbf{x}} - \frac{\partial V(\mathbf{x}, t)}{\partial t} = -\frac{dV(\mathbf{x}, t)}{dt}. \tag{2.50}$$

Therefore,

$$\begin{aligned} J[\mathbf{u}^*] - J[\mathbf{u}] &= \int_0^T F(\mathbf{x}^*(t), \mathbf{u}^*(t), t)dt - \int_0^T F(\mathbf{x}(t), \mathbf{u}(t), t)dt \\ &\geq \int_0^T \frac{dV(\mathbf{x}, t)}{dt}dt - \int_0^T \frac{dV(\mathbf{x}^*, t)}{dt}dt \\ &= V(\mathbf{x}(T), T) - V(\mathbf{x}^*(T), T) - V(\mathbf{x}(0), 0) + V(\mathbf{x}^*(0), 0) \\ &= 0. \end{aligned}$$

The above inequality holds by (2.49) and (2.50). The last equality follows from (2.43) and $\mathbf{x}^*(0) = \mathbf{x}(0) = \mathbf{x}_0$.

It has now been established that $\mathbf{u}^*$ is the optimal control path. Using $\mathbf{u}^*$ and (2.49), the value of the problem (2.39)–(2.42) over interval $[t, T]$ is thus

$$\begin{aligned} J[\mathbf{u}^*] &= \int_t^T F(\mathbf{x}^*(s), \mathbf{u}^*(s), s)ds \\ &= \int_t^T -\frac{dV(\mathbf{x}^*, s)}{ds}ds \\ &= V(\mathbf{x}^*(t), t) - \underbrace{V(\mathbf{x}^*(T), T)}_{\stackrel{(2.43)}{=}0} \\ &= V(\mathbf{x}^*(t), t), \end{aligned}$$

which is the optimal value function. This completes the proof. □

There is an interesting relationship between the maximum principle and dynamic programming. If one substitutes the partial differential equation $\frac{\partial V(\mathbf{x},t)}{\partial \mathbf{x}}$ with the costate variable $\boldsymbol{\mu}(t)$, the right-hand side of the Hamilton-Jacobi-Bellman equation (2.38) becomes the maximised Hamiltonian function (2.6).[17] Indeed, one can derive heuristically also the other necessary conditions of the maximum principle using the Hamilton-Jacobi-Bellman equation (Kamien and Schwartz 1991).

To show this, consider (2.38) where the control path $\mathbf{u}^*$ is optimal. When one differentiates such an equation with respect to state $\mathbf{x}$, one obtains

$$-\frac{\partial^2 V(\cdot)}{\partial t \partial \mathbf{x}} = \frac{\partial F(\cdot)}{\partial \mathbf{x}} + \frac{\partial^2 V(\cdot)}{\partial \mathbf{x}^2}\mathbf{f}(\cdot) + \frac{\partial V(\cdot)}{\partial \mathbf{x}}\frac{\partial \mathbf{f}(\cdot)}{\partial \mathbf{x}}. \tag{2.51}$$

Then, one may observe that the time derivative of $\frac{\partial V(\cdot)}{\partial \mathbf{x}}$ is

$$\frac{d}{dt}\left(\frac{\partial V(\cdot)}{\partial \mathbf{x}}\right) = \frac{\partial^2 V(\cdot)}{\partial \mathbf{x}\partial t} + \frac{\partial^2 V(\cdot)}{\partial \mathbf{x}^2}\mathbf{f}(\cdot). \tag{2.52}$$

By substituting $\frac{\partial V(\cdot)}{\partial \mathbf{x}} = \boldsymbol{\mu}(t)$ and using (2.51), (2.52) becomes

$$\frac{d\boldsymbol{\mu}(t)}{dt} = -\frac{\partial F(\cdot)}{\partial \mathbf{x}} - \boldsymbol{\mu}(t)\frac{\partial \mathbf{f}(\cdot)}{\partial \mathbf{x}} = -\frac{\partial H(\cdot)}{\partial \mathbf{x}} \tag{2.53}$$

which is the necessary condition (2.8).

As the Hamiltonian function, the Hamilton-Jacobi-Bellman equation can also be reformulated in a current value form that is typically used in economics. Again, one may consider the discount factor $e^{-\rho t}$ and substitute it to the integrand function $F(\mathbf{x}(t), \mathbf{u}(t), t) = e^{-\rho t}G(\mathbf{x}(t), \mathbf{u}(t), t)$. One may then define the *current value optimal value function* as

$$\mathcal{V}(\mathbf{x}, t) = e^{\rho t}V(\mathbf{x}, t).$$

[17] Clarke and Vinter 1987 prove that this relationship, in terms of generalised gradients, is true for a very large class of nonsmooth optimal control problems.

Using $\mathcal{V}(\mathbf{x}, t)$ and $e^{-\rho t} G(\mathbf{x}(t), \mathbf{u}(t), t)$ the problem (2.39)–(2.42) becomes

$$e^{-\rho t}\mathcal{V}(\mathbf{x}, t) = \max_{\mathbf{u}} \int_t^T e^{-\rho s} G(\mathbf{x}(s), \mathbf{u}(s), s) ds \tag{2.54}$$

$$s.t. \quad \dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s), \mathbf{u}(s), s) \tag{2.55}$$

$$\mathbf{x}(t) = \mathbf{x} \tag{2.56}$$

$$\mathbf{u}(s) \in \mathcal{U}(\mathbf{x}(s), s) \subseteq \mathbb{R}^m \text{ for all } s \in [t, T]. \tag{2.57}$$

The terminal condition (2.43) is now

$$e^{-\rho T}\mathcal{V}(\mathbf{x}, T) = 0. \tag{2.58}$$

Breaking up the integral (2.54) as before, one obtains

$$e^{-\rho t}\mathcal{V}(\mathbf{x}, t) = \max_{\mathbf{u}} \left\{ \int_t^{t+\Delta} e^{-\rho s} G(\mathbf{x}(s), \mathbf{u}(s), s) ds + \int_{t+\Delta}^T e^{-\rho s} G(\mathbf{x}(s), \mathbf{u}(s), s) ds \right\}. \tag{2.59}$$

By definition of $\mathcal{V}(\mathbf{x}, t)$, (2.59) becomes

$$e^{-\rho t}\mathcal{V}(\mathbf{x}, t) = \max_{\mathbf{u}} \left\{ \int_t^{t+\Delta} e^{-\rho s} G(\mathbf{x}(s), \mathbf{u}(s), s) ds + e^{-\rho(t+\Delta)}\mathcal{V}(\mathbf{x}(t+\Delta), t+\Delta) \right\}. \tag{2.60}$$

When one subtracts $e^{-\rho t}\mathcal{V}(\mathbf{x}, t)$ from both sides of equation (2.60) and multiplies them by $1/\Delta$, one obtains

$$\begin{aligned} 0 = \max_{\mathbf{u}} \Bigg\{ & \frac{1}{\Delta} \int_t^{t+\Delta} e^{-\rho s} G(\mathbf{x}(s), \mathbf{u}(s), s) ds \\ & + \frac{e^{-\rho(t+\Delta)}\mathcal{V}(\mathbf{x}(t+\Delta), t+\Delta) - e^{-\rho t}\mathcal{V}(\mathbf{x}, t)}{\Delta} \Bigg\} \end{aligned} \tag{2.61}$$

which, when $\Delta \to 0$, becomes

$$0 = \max_{\mathbf{u}} \left\{ e^{-\rho t} G(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}\left(e^{-\rho t}\mathcal{V}(\mathbf{x}, t)\right) \right\} \tag{2.62}$$

and further

$$0 = \max_{\mathbf{u}} \left\{ e^{-\rho t} G(\mathbf{x}(t), \mathbf{u}(t), t) - \rho e^{-\rho t}\mathcal{V}(\mathbf{x}, t) + e^{-\rho t} \left(\frac{\partial \mathcal{V}(\mathbf{x}, t)}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial \mathcal{V}(\mathbf{x}, t)}{\partial t} \right) \right\}. \tag{2.63}$$

Multiplying (2.63) by $e^{\rho t}$, substituting $\dot{\mathbf{x}}$ with $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ and reorganising the terms, one obtains the *current value Hamilton-Jacobi-Bellman equation*

$$\rho\mathcal{V}(\mathbf{x}, t) - \frac{\partial \mathcal{V}(\mathbf{x}, t)}{\partial t} = \max_{\mathbf{u}} \left\{ G(\mathbf{x}, \mathbf{u}, t) + \frac{\partial \mathcal{V}(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right\}. \tag{2.64}$$

Theorem 2.5 applies trivially with the current value optimal value function $\mathcal{V}(\mathbf{x}, t)$, the current value Hamilton-Jacobi-Bellman equation (2.64) and the terminal condition (2.58).

2.1.4 Infinite-horizon problems

Classic optimal control problem specifications and their solution methods discussed thus far involved a finite time horizon $t \in [0, T]$ where the terminal time T is fixed. However, in many economic problems it may be more appropriate to consider that the terminal time is very far in the future or unspecified. For example, economists are usually interested in studying the long-run equilibrium which can be approximated with an infinite time horizon $t \in [0, \infty)$. Adopting the infinite time horizon often also simplifies formulas and results as there are no need to account for a scrap value (Léonard and Long 1992). Moreover, in the context of industrial organisation, it is reasonable to assume that firms are planning to stay in business and continue their existence indefinitely. Even if firms, in practice, might plan their operations in fixed time intervals (e.g. one year at a time) they should value their assets optimally at the end of each such period and that the scrap value of their assets may be thought of as the value of the discounted future profits generated by those assets *ad infinitum* (Dockner et al. 2000).

Much of the theory that applies to finite-horizon problems carries over to infinite-horizon problems as such but not everything. Regarding the maximum principle, it turns out that all the necessary conditions with the exception of transversality conditions are valid also in case of an infinite time horizon (Halkin 1974). As transversality conditions are conditions that the optimal solution must satisfy at the terminal time T, it should be expected that they need to be modified if the time horizon is infinite. However, the task of extending standard transversality conditions to problems with an infinite time horizon has proven to be difficult.

For instance, one might consider that a condition

$$\lim_{t \to \infty} \boldsymbol{\mu}(t) = 0 \tag{2.65}$$

or, alternatively,

$$\lim_{t \to \infty} \boldsymbol{\mu}(t)\mathbf{x}^*(t) = 0 \tag{2.66}$$

would be a natural replacement for a necessary condition (2.9) when a time horizon is infinite. However, there are several counterexamples (e.g. Halkin 1974 and Shell 1969) which show that (2.65) or (2.66) are not, in general, necessary for the optimum in infinite-horizon problems.[18] Rather, they are necessary for optimal solutions only under additional (boundedness) assumptions which might be too restrictive for some economic applications (see Michel 1982 and Benveniste and Scheinkman 1982). Regardless of this, (2.65) and (2.66) can be used to establish sufficient conditions for optimal solutions (Arrow and Kurz 1970). Even then, they might be too strong for many applications which is why one may prefer to look for weaker sufficient conditions.

Another, potentially serious problem is that, when $T \to \infty$, the integral in the objective functional (2.1) becomes an improper integral which does not necessarily converge. For instance, with a bounded integrand function $G(\cdot)$ and positive time discounting $\rho > 0$, the convergence of the integral is ensured (Léonard and Long 1992). However, the boundedness of the integrand function is a strong assumption. In differential games, the integrand function of one player depends on the strategies of the other players, and it may not be reasonable to restrict the set of possible strategies too much by boundedness assumptions (Dockner et al. 2000). A better alternative is to redefine the optimality condition 2.2 in such a way that it accommodates the infinite time horizon.

One such optimality condition commonly considered in the literature is called an *overtaking criterion* (see e.g. Seierstad and Sydsaeter 1977). Before defining it, the earlier notation should be clarified a little. Let $J_T[\mathbf{u}(t)]$ denote the objective functional as in (2.1), where the terminal time is T, and $J_\infty[\mathbf{u}(t)]$ the objective functional in the infinite-horizon case, where the terminal time $T \to \infty$:[19]

$$\max_{\mathbf{u}(t)} \quad J_\infty = \int_0^\infty F(\mathbf{x}(t), \mathbf{u}(t), t)dt. \tag{2.67}$$

One may now define the overtaking criterion as follows:

Definition 2.4 (Overtaking criterion) A feasible path $\mathbf{u}^*(t)$ is *overtaking optimal* if the inequality $J_T[\mathbf{u}^*(t)] \geq J_T[\mathbf{u}(t)]$ holds for all feasible control paths $\mathbf{u}(t)$ when $T \in [\mathcal{T}, \infty)$.

Another, weaker optimality condition is called a *catching-up criterion* which may be defined as follows:

Definition 2.5 (Catching-up criterion) A feasible path $\mathbf{u}^*(t)$ is *catching-up optimal* if the inequality $\lim_{T\to\infty} \inf\, (J_T[\mathbf{u}^*(t)] \geq J_T[\mathbf{u}(t)])$ holds for all feasible control paths $\mathbf{u}(t)$ when $T \in [\mathcal{T}, \infty)$.

In other words, the overtaking criterion applies if there is a time $\mathcal{T}$ such that the inequality $J_T[\mathbf{u}^*(t)] \geq J_T[\mathbf{u}(t)]$ holds for all terminal times T thereafter. The catching-

[18]See Aseev and Kryazhimskiy 2004 and 2007 for more counterexamples and a further discussion on transversality conditions in infinite time horizon problems.

[19]If this is not explicitly specified in the notation, the terminal time in question should be clear from the context.

up criterion applies if the inequality holds at the lower limit. Obviously, any overtaking optimal solution is also catching-up optimal but not necessarily vice versa. The two criteria are equivalent if the objective functional converges for all feasible paths.

Using these definitions, the sufficiency theorems 2.2, 2.4 and 2.5 can be modified accordingly for infinite time horizon problems (Seierstad and Sydsaeter 1977 and Dockner et al. 2000).

Theorem 2.6 *Consider the problems* (2.1)–(2.4) *and* (2.39)–(2.42) *where the terminal time* $T \to \infty$. *Suppose that all the conditions of theorems 2.2, 2.4 and 2.5 hold for all* $t \in [0, \infty)$ *except for the boundary conditions* (2.9) *and* (2.43). *Then these theorems remain valid and the solutions satisfying their conditions are optimal:*

(i) according to the overtaking criterion if the condition (2.9) *is replaced by an assumption that, for all feasible control paths* $\mathbf{u}(t)$, *there exists a finite* $\mathcal{T}$ *such that*

$$\boldsymbol{\mu}(T)\,(\mathbf{x}(T) - \mathbf{x}^*(T)) \geq 0 \tag{2.68}$$

for all $T \in [\mathcal{T}, \infty)$, *and, similarly, if the condition* (2.43) *is replaced by an assumption that, for all feasible control paths* $\mathbf{u}(t)$, *there exists a finite* $\mathcal{T}$ *such that*

$$V(\mathbf{x}(T), T) - V(\mathbf{x}^*(T), T) \geq 0 \tag{2.69}$$

for all $T \in [\mathcal{T}, \infty)$*;*

(ii) according to the catching-up criterion if the condition (2.9) *is replaced by an assumption that, for all feasible control paths* $\mathbf{u}(t)$, *there exists a finite* $\mathcal{T}$ *such that*

$$\lim_{T\to\infty} \inf \boldsymbol{\mu}(T)\,(\mathbf{x}(T) - \mathbf{x}^*(T)) \geq 0 \tag{2.70}$$

for all $T \in [\mathcal{T}, \infty)$, *and, similarly, if the condition* (2.43) *is replaced by an assumption that, for all feasible control paths* $\mathbf{u}(t)$, *there exists a finite* $\mathcal{T}$ *such that*

$$\lim_{T\to\infty} \inf\,(V(\mathbf{x}(T), T) - V(\mathbf{x}^*(T), T)) \geq 0 \tag{2.71}$$

for all $T \in [\mathcal{T}, \infty)$.

Proof. As before, one must demonstrate optimality by showing that $J[\mathbf{u}^*] \geq J[\mathbf{u}]$. From the proof of theorem 2.2, when the condition (2.9) is not applicable, one obtains

$$J[\mathbf{u}^*] - J[\mathbf{u}] = \int_0^T [H^* - H + \dot{\boldsymbol{\mu}}\,(\mathbf{x}^* - \mathbf{x})]dt + \boldsymbol{\mu}(T)\,(\mathbf{x}(T) - \mathbf{x}^*(T))\,,$$

from which, along the same line of reasoning as in the proof of theorem 2.2, it can be evaluated that

$$J[\mathbf{u}^*] - J[\mathbf{u}] \geq \boldsymbol{\mu}(T)\,(\mathbf{x}(T) - \mathbf{x}^*(T))\,. \tag{2.72}$$

Similarly, from the proof of theorem 2.5, one obtains

$$J[\mathbf{u}^*] - J[\mathbf{u}] = \int_0^T F(\mathbf{x}^*(t), \mathbf{u}^*(t), t)dt - \int_0^T F(\mathbf{x}(t), \mathbf{u}(t), t)dt,$$

from which, using the same principles of deduction as in the proof of theorem 2.4 and considering that the condition (2.43) is not applicable, it may be evaluated that

$$J[\mathbf{u}^*] - J[\mathbf{u}] \geq V(\mathbf{x}(T), T) - V(\mathbf{x}^*(T), T). \tag{2.73}$$

The result follows immediately from (2.72)–(2.73) and the definitions 2.4–2.5. □

Furthermore, theorem 2.6 applies trivially with the current value formulations of the costate variable and the optimal value function.

Conditions (2.68)–(2.71) may be difficult to verify in practice but sometimes they are trivially satisfied. For example, the condition (2.70) is true automatically if both (2.65) (or $\lim_{t\to\infty} \boldsymbol{\mu}(t) \geq 0$) and (2.66) hold and all feasible state trajectories are bounded from below at 0 (i.e. the state variable is nonnegative $\mathbf{x}(t) \geq 0$ for all $t \in [0, \infty)$) (Arrow and Kurz 1970). The condition (2.70) holds similarly if $\lim_{T\to\infty} \boldsymbol{\mu}(T)\mathbf{x}(T) = 0$ (or its current value counterpart $\lim_{T\to\infty} e^{-\rho T} \boldsymbol{\lambda}(T)\mathbf{x}(T) = 0$) holds. Likewise, the current value form of the condition (2.71) $\lim_{T\to\infty} \inf e^{-\rho T} \left(\mathcal{V}(\mathbf{x}(T), T) - \mathcal{V}(\mathbf{x}^*(T), T)\right) \geq 0$ is verified if there is positive time discounting $\rho > 0$ and the value function $\mathcal{V}$ is bounded or, alternatively, $\rho > 0$, $\mathcal{V}$ is bounded from below and $\lim_{T\to\infty} \sup e^{-\rho T} \mathcal{V}(\mathbf{x}^*(T), T) \leq 0$ (Dockner et al. 2000).

Regarding dynamic programming, one additional remark should be made. If the problem (2.39)–(2.42) is autonomous and the time horizon is infinite, the current value optimal value function $\mathcal{V}(\mathbf{x})$ does not depend explicitly on time t. Therefore, $\frac{\partial \mathcal{V}(\mathbf{x})}{\partial t} = 0$ and the current value Hamilton-Jacobi-Bellman equation (2.64) simply becomes

$$\rho \mathcal{V}(\mathbf{x}) = \max_{\mathbf{u}} \left\{ G(\mathbf{x}, \mathbf{u}) + \frac{\partial \mathcal{V}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) \right\}, \tag{2.74}$$

which, obviously, is easier to solve than (2.64).

Having thus far examined the maximum principle and dynamic programming, the tools for solving (one-person) optimal control problems, their applications shall be considered next in solving a more general class of problems, where the objective functional (2.1) depends on decisions of not one but many decision-makers – that of differential games.

2.2 Differential games

2.2.1 Game-theoretic foundations

Game theory is a field of applied mathematics which studies strategic interaction of rational decision-makers in situations where the actions of these decision-makers affect each other. The decision-makers, usually denoted as *players* (or agents or persons), are assumed to be rational in a sense that they have clear preferences over different outcomes and they choose *strategies* (a set of decisions or actions or moves for every possible scenario in a game) that maximise their expected utility (usually expressed in a form of a *payoff* or *utility function*) in circumstances where they might have conflicting but also mutual interests. If the players have completely conflicting (opposite) interests, the game is a *zero-sum game*, otherwise it is a *nonzero-sum game*. Moreover, if the players of a game can make binding agreements, the game is said to be *cooperative*, and if not, the game is said to be *noncooperative*, in which case the players can only rely on each other's self-interest in enforcing the outcome of the game.[20]

Every game theoretic model needs to specify (at least) the number of players, available actions and strategies for each player, payoffs for each player and information the players receive. The moves of the players can be simultaneous or sequential. In games of *complete and perfect information*, all relevant information about the rules of the game is assumed to be common knowledge, that is, player i knows that the other players know these things and the other players know that player i knows these things and player i knows that other players know that player i knows these things *et cetera*. If the players are uncertain of each other's moves, the game is of *imperfect information*, and if they are uncertain of each other's payoffs, the game is of *incomplete information*.[21]

The objective of the game theory is to predict the outcome of the game under analysis. This requires a solution concept which specifies the outcome the players are expected to end up to. The most frequently used solution concept in noncooperative game theory is *Nash equilibrium*, originally proposed by Nash (1950). In a Nash equilibrium, the strategies of the players are best replies to each other.[22] This means that, given the strategies of other players, no single player has an incentive to deviate from the equilibrium. However, Nash equilibrium is not a very restrictive solution concept since, in general, there may be (and in applications often are) multiple Nash equilibria in a game. In such cases, where a Nash equilibrium is not unique, one may not readily predict the path of play and the outcome of the game unless the solution set is restricted further by, for example, stronger solution concepts.

[20] Only noncooperative nonzero-sum games are discussed further in this thesis.

[21] Harsanyi 1967 has shown that any game of incomplete information can be formulated as a game of complete but imperfect information.

[22] In general, the strategies may be pure strategies (decisions between different alternatives at each point of the game) or mixed strategies, which are probability distributions over sets of pure strategies.

Games may be characterised as static or dynamic based on what kind of role time plays in them. Although there is no general definition for the term, a dynamic game may be defined as a game in which the order of the decisions is important (Başar and Olsder 1998) or a game in which the players may condition their actions at any instant of time on previous actions in the game (Dockner et al. 2000). For the purposes of this discussion, it suffices to consider games, where players make decisions in only one period, as *static games* and games, where players make decisions in multiple periods, as *dynamic games* (Weber 2011).

Dynamic and sequential games can often be modelled in *extensive form*, where the structure of a game is represented by a game tree. The tree has nodes and branches – each point in a game is represented by a node which is connected to other nodes by branches and so on until the game ends at a terminal node. Branches form paths that represent different sequences of actions and alternative ways the game can be played. By contrast, in games that are modelled in *normal* or *strategic form*, the order of the moves is suppressed and all sequences of actions are played against one another. In two-person games, the structure of the game may be presented by a matrix. Strategic form is the standard form to model static games and games where moves are simultaneous. However, by suppressing the order of the moves, information is lost, and because of this the extensive form is considered to be more general than strategic form.

Differential games are dynamic games played over continuous time. More precisely, differential games are state space games in which the evolution of the state variable over time is described by a set of differential equations, which represent the system dynamics. The payoff functions of the players and the system dynamics depend on the control variables which the players choose from an admissible strategy space while taking into account the corresponding choices of the other players. The choice of the control variable is equivalent to the choice of a strategy which determines the actions the players take in each instant of time in a game. Usually, like in this thesis, it is assumed that the moves by players are simultaneous and resulting games are represented in strategic form.[23] With certain assumptions regarding the admissible strategy space, a differential game becomes an optimal control problem and can therefore be solved by using the tools of optimal control theory (Dockner et al. 2000). While differential game theory borrows much of its terminology and techniques from optimal control theory, it is more general by nature and cannot be considered to be only a branch of optimal control theory (Clemhout and Wan 1994, Başar and Olsder 1998). A more formal treatment of differential games is presented next.

[23]Differential games with hierarchical play, where one of the players moves first, are called Stackelberg games (see Dockner et al. 2000, chapter 5). It can be shown that equilibrium strategies in Stackelberg games are generally time-inconsistent and, consequently, their analysis involves their own equilibrium concepts.

2.2.2 Nash equilibrium and information structures

Consider an N-player differential game $\Gamma(\mathbf{x}_0, 0) = (\mathcal{N}, \{\mathcal{U}_i(\cdot)\}_{i\in\mathcal{N}}, \{J_i[u_i|\mathbf{u}_{-i}(\cdot)]\}_{i\in\mathcal{N}})$ of complete information played over infinite[24] time horizon $t \in [0, \infty)$, where $\mathcal{N} = \{1, \ldots, N\}$ is the set of players, $\mathcal{U}_i(\cdot)$ is the set of strategies available for each player $i \in \mathcal{N}$ and $J_i[u_i|\mathbf{u}_{-i}(\cdot)]$ is the objective functional (representing the payoff) for each player $i \in \mathcal{N}$:

$$J_i[u_i(t)|\mathbf{u}_{-i}(t)] = \int_0^\infty e^{-\rho t} G_i(\mathbf{x}(t), u_i(t), \mathbf{u}_{-i}(t), t)dt. \tag{2.75}$$

The (current value form) payoff or utility function $G_i(\cdot)$ for each player i is discounted by a discount rate $\rho > 0$, which is common to all players. It is noteworthy that the objective functional (2.75) for each player i depends not only on player i's own control $u_i(t) \in \mathcal{U}_i(\mathbf{x}, \mathbf{u}_{-i}(t), t) \subseteq \mathbb{R}^{m_i}$, $m = \sum_{i=1}^N m_i$, but also on controls of every other player $j \neq i$, which are contained in a vector $\mathbf{u}_{-i}(t) = (u_1(t), \ldots, u_{i-1}(t), u_{i+1}(t), \ldots, u_N(t))$.[25] In general, the choices of other players $j \neq i$ therefore directly affect the objective functional of player i who takes them as given when choosing his own control $u_i(t)$. The objective functional (2.75) depends also on time t and the state variable $\mathbf{x}(t) \in \mathcal{X} \subseteq \mathbb{R}^n$, which evolves according to a first-order ordinary differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u_i(t), \mathbf{u}_{-i}(t), t), \ \mathbf{x}(0) = \mathbf{x}_0. \tag{2.76}$$

The state equation $\dot{\mathbf{x}}(t)$, too, generally depends on controls of all players as well on the state variable $\mathbf{x}(t)$ and time t. As in a standard optimal control problem (2.1)–(2.4), the function $G_i(\cdot)$ and the state equation $\mathbf{f}(\cdot)$ are assumed to be continuous and differentiable in $\mathbf{x}$, $\mathbf{u}$ and t. The state variable $\mathbf{x}$ is likewise assumed to be continuous but piecewise differentiable.

Many-player differential games result in "impenetrably large and complex strategy spaces" and solving them in practice requires that the feasible strategy space is somehow restricted (Clemhout and Wan 1994, p. 803). In this respect, suppose additionally that all players $j \neq i$ restrict themselves to using *Markovian strategies* $\mathbf{u}_j(t) = \phi_j(\mathbf{x}(t), t)$.[26] Markovian strategies do not depend on the past history (other than possibly the initial condition) but only on the current state of the system – all relevant information about the system is thus contained in its current state.

With Markovian strategies, the objective functional $J_i[u_i(t)|\phi_{-i}(t)]$ for player i and

[24] Alternatively, one could consider a differential game with a finite terminal time $T < \infty$.

[25] For simplicity, each player i is associated with a single control $u_i(t)$ which is assumed to be piecewise continuous. There could be more than one control for each player but in tractable problems with analytical solutions the number of controls is almost always very limited.

[26] Strategies that depend on parameters other than $\mathbf{x}(t)$ and t are called *non-Markovian strategies*. In general, they result in more complex optimisation problems that cannot be solved by standard methods of optimal control theory (see Dockner et al. 2000, chapter 6).

the state equation $\mathbf{f}(\cdot)$ depend only on variables $\mathbf{x}(t)$, $u_i(t)$ and t. The optimisation problem for player i may therefore be formulated as a standard optimal control problem as follows:

$$\max_{u_i(t)} \quad J_i[u_i(t)|\phi_{-i}(t)] = \int_0^\infty e^{-\rho t} G_i(\mathbf{x}(t), u_i(t), \phi_{-i}(\mathbf{x}(t), t), t)dt \tag{2.77}$$

$$s.t. \quad \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u_i(t), \phi_{-i}(\mathbf{x}(t), t), t) \tag{2.78}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{2.79}$$

$$u_i(t) \in \mathcal{U}_i(\mathbf{x}, \phi_{-i}(\mathbf{x}(t), t), t) \text{ for all } t \in [0, \infty), \tag{2.80}$$

where $\phi_{-i}(\mathbf{x}(t), t) = (\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_N)$. Player i's objective is to choose a strategy that maximises his overall payoff given the fixed strategies of the other players while each player $j \neq i$ pursues to do the same. Because the strategies of the players are interdependent, solving the game requires solving a system of N optimal control problems simultaneously. This is obviously more demanding than solving a single optimal control problem. If the game permits full symmetry, the dimension of the problem may be reduced significantly, in which case a symmetric equilibrium (if one exists) might be more readily found than asymmetric equilibria (Dockner et al. 2000).

A Nash equilibrium solution for the differential game $\Gamma(\mathbf{x}_0, 0)$ with Markovian strategies considered above may be formulated as follows:

Definition 2.6 (Nash equilibrium) A strategy profile $\left(\phi_1^*, \ldots, \phi_N^*\right)$ is a *(Markovian) Nash equilibrium* of the differential game $\Gamma(\mathbf{x}_0, 0)$ if, for each $i \in \mathcal{N}$, there exists a control $u_i^*(t) = \phi_i^*(\mathbf{x}^*(t), t)$ that maximises the objective functional (2.77) subject to constraints (2.78)–(2.80).

Hence, in a Nash equilibrium of a differential game, the strategies of the players are best replies to each other and no player can do better by unilaterally changing his strategy. The definition of Nash equilibrium in differential games is thus equivalent to the Nash equilibrium concept in static games. However, in differential games, there are additional points to consider that are related to the dynamic nature of the problem. For example, as the time horizon of the differential game $\Gamma(\mathbf{x}_0, 0)$ was considered to be infinite ($t \in [0, \infty)$), the optimality of a Nash equilibrium solution for player i must be considered in terms of overtaking optimality or catching-up optimality or some other optimality condition modified for the unbounded time interval.[27]

Another point specifically related to differential games is that, in differential games of complete information with simultaneous moves, one needs to specify an *information structure* which determines what information players receive about the evolution of the state variable during the game. The observability of the state variable is not explicitly clear in a strategic form game and hence this question needs to be addressed in a model itself

[27] In this thesis, Nash equilibria in differential games with unbounded time domain are considered to be catching-up optimal unless otherwise specified.

(Dockner et al. 2000). If players observe the state variable at each instant of time, they might use this information in conditioning their strategy. Otherwise, they must commit to a strategy which does not account for the development of the state variable. Specifying an information structure affects which strategies are available to players and is thus a further restriction on a strategy space of a model.

Three different information structures are commonly considered in the differential games literature.[28] These can be defined as follows (Fershtman 1987, p. 219):

Definition 2.7 (Open-loop strategy) An *open-loop strategy* is a time path $u_i = u_i(\mathbf{x}_0, t)$ such that, given the initial state variable $\mathbf{x}_0$, it assigns for every t a control in $\mathcal{U}_i$. The set of all possible open-loop strategies for player i is denoted as ψ_i, $\Psi = \{\psi_1, \ldots, \psi_N\}$.

Definition 2.8 (Closed-loop strategy) A *closed-loop (memoryless) strategy* is a decision rule $u_i = u_i(\mathbf{x}_0, \mathbf{x}(t), t)$ such that it is continuous in t and uniformly Lipschitz in $\mathbf{x}(t)$ for each t. The set of all possible closed-loop strategies for player i is denoted as ω_i, $\Omega = \{\omega_1, \ldots, \omega_N\}$.

Definition 2.9 (Feedback strategy) A *feedback strategy* is a decision rule $u_i = u_i(\mathbf{x}(t), t)$ such that it is continuous in t and uniformly Lipschitz in $\mathbf{x}(t)$ for each t. The set of all possible feedback strategies for player i is denoted as δ_i, $\Delta = \{\delta_1, \ldots, \delta_N\}$.

It should be noted that all the above strategies are Markovian strategies (although the initial condition $\mathbf{x}_0$ was contained in the strategy space $\mathcal{U}_i(\cdot)$ only implicitly). In a sense, an open-loop strategy is a degenerate Markovian strategy because it does not depend on the state equation $\mathbf{x}(t)$. The initial condition $\mathbf{x}_0$ is typically suppressed in the formulations of open-loop strategies but, in the above definitions, it is expressed explicitly to represent that, in general, time paths of open-loop and closed-loop strategies are dependent on initial conditions whereas time paths created by feedback strategies are not. This independence of initial conditions have an important consequence of making feedback strategies (sometimes called Markov strategies) subgame perfect (or Markov perfect). In the literature, closed-loop and feedback strategies are sometimes used interchangeably, which have created some confusion of terminology. Whichever terms are used, one should make a clear distinction between them as a feedback strategy is, in fact, a special case of a closed-loop strategy (Lambertini 2018).[29]

Reinganum and Stokey (1985) describe open-loop strategies as "path strategies" and feedback strategies as "decision rule strategies" to illustrate the difference made by the state variable in strategies. By assumption, open-loop strategies require players to commit

[28] Of course, there exist other information structures as well. For various forms of closed-loop information structures, see Başar and Olsder 1998.

[29] The terminology of open-loop and closed-loop solutions originates in electrical and electronic engineering, where, in applications related to electronic amplifiers, part of the output signal is fed back to an amplifier as an input to improve the control of the signal (Lambertini 2018). This (negative) feedback creates a closed loop which is absent if the output signal is not used as an input.

to a chosen path for the entire duration of the game whereas feedback strategies require players to choose a rule to decide among available paths at any point of the game instead of committing to any single path. Open-loop strategies might therefore be appropriate in situations where the state variable is either unobservable or commitment is otherwise possible and the period of commitment extends over the entire time horizon. Feedback strategies, on the other hand, are more appropriate in situations where the state variable is observable but commitment is not possible. The choice of a strategy space of a model should not be made casually but carefully, based on the characteristics of a situation under study.

(Markovian) Nash equilibria in differential games may be further classified into *open-loop*, *closed-loop* and *feedback Nash equilibria* according to which strategy is used by the players. In general, these strategies produce different Nash equilibria. This situation arises because, in addition to player i's own control, the controls of players $j \neq i$ are also present in player i's objective functional (2.77) and the system dynamics (2.78). The optimisation problem of player i is thus directly affected by the choice of a strategy space and different information structures result in different optimisation problems for player i (Dockner et al. 2000). Different informational assumptions hence alter the model itself. By contrast, in (one-person) optimal control theory, the optimisation problem remains essentially the same with all information structures, which generate the same optimal trajectory (provided that the trajectory of the state variable is not perturbed, for example, by an unforeseen force or exogenous event afterwards).

When the strategy space is restricted to Markovian strategies, Nash equilibria of the differential game $\Gamma(\mathbf{x}_0, 0)$ can be found by employing the standard methods of optimal control theory. Open-loop and closed-loop Nash equilibrium solutions may be obtained from the necessary conditions of the maximum principle which remain valid after some appropriate changes have been made. For this purpose, define a (current value) Hamiltonian of player $i \in \mathcal{N}$ as

$$\begin{aligned}
&\mathcal{H}_i(\mathbf{x}(t), u_i(t), \phi_{-i}(\mathbf{x}(t), t), \boldsymbol{\lambda}_i(t), t) \\
&= G_i(\mathbf{x}(t), u_i(t), \phi_{-i}(\mathbf{x}(t), t), t) + \boldsymbol{\lambda}_i(t)\mathbf{f}(\mathbf{x}(t), u_i(t), \phi_{-i}(\mathbf{x}(t), t), t) \\
&= G_i(\mathbf{x}(t), u_i(t), \phi_{-i}(\mathbf{x}(t), t), t) + \sum_{j=1}^{n} \lambda_{ij}(t) f_j(\mathbf{x}(t), u_i(t), \phi_{-i}(\mathbf{x}(t), t), t).
\end{aligned} \tag{2.81}$$

Correspondingly, the maximised version of the Hamiltonian function (2.81) for player i is

$$\mathcal{H}_i^0(\mathbf{x}(t), \phi_{-i}(\mathbf{x}(t), t), \boldsymbol{\lambda}_i(t), t) = \max_{u_i(t)} \mathcal{H}_i(\mathbf{x}(t), u_i(t), \phi_{-i}(\mathbf{x}(t), t), \boldsymbol{\lambda}_i(t), t). \tag{2.82}$$

One may now formulate sufficient conditions for an open-loop Nash equilibrium as follows (Başar and Olsder 1998 and Dockner et al. 2000):

Theorem 2.7 (Open-loop solution: sufficiency) *Let $u_i^*(t) = \phi_i^*(t)$ be a feasible open-loop strategy for each player $i \in \mathcal{N}$, an N-tuple $\left(\phi_1^*, \ldots, \phi_N^*\right)$ a given strategy profile of such open-loop strategies and $\mathbf{x}^*(t)$ a corresponding state trajectory that is a unique absolutely continuous solution to the initial value problem* (2.78)–(2.79). *Assume that the Hamiltonian function* (2.81) *is differentiable and concave jointly in $(\mathbf{x}, u_i)$ for all $i \in \mathcal{N}$.*[30] *If there exist N absolutely continuous functions $\boldsymbol{\lambda}_i : [0, \infty) \mapsto \mathbb{R}^n$, $i \in \mathcal{N}$, that satisfy the conditions*

$$\max_{u_i(t)} \quad \mathcal{H}_i(\mathbf{x}^*(t), u_i^*(t), \phi_{-i}^*(t), \boldsymbol{\lambda}_i(t), t) \text{ for all } t \in [0, \infty) \tag{2.83}$$

$$\dot{\mathbf{x}}^*(t) = \frac{\partial \mathcal{H}_i(\cdot)}{\partial \boldsymbol{\lambda}_i(t)} \tag{2.84}$$

$$\dot{\boldsymbol{\lambda}}_i(t) = -\frac{\partial \mathcal{H}_i(\cdot)}{\partial \mathbf{x}^*(t)} + \rho \boldsymbol{\lambda}_i(t) \tag{2.85}$$

for all $i \in \mathcal{N}$, and, additionally, either

(i) $\lim_{t\to\infty} e^{-\rho t} \boldsymbol{\lambda}_i(t)\mathbf{x}(t) = 0$ *or*

(ii) $\lim_{t\to\infty} e^{-\rho t} \boldsymbol{\lambda}_i(t) \geq 0$, $\lim_{t\to\infty} e^{-\rho t} \boldsymbol{\lambda}_i(t)\mathbf{x}^*(t) = 0$ *and* $\mathbf{x}(t) \geq 0$ *for all* $t \in [0, \infty)$

holds for all $i \in \mathcal{N}$, then $\left(\phi_1^*, \ldots, \phi_N^*\right)$ is an open-loop Nash equilibrium of the differential game $\Gamma(\mathbf{x}_0, 0)$. *(The equilibrium is catching-up optimal.)*

Proof. The result follows directly from application of theorems 2.2 and 2.6. If the Hamiltonian function $\mathcal{H}_i(\cdot)$ is replaced with a maximised Hamiltonian $\mathcal{H}_i^0(\cdot)$, the result follows from application of theorems 2.4 and 2.6. □

Theorem 2.7 requires only minor changes to be applicable to closed-loop solutions. Notably, as closed-loop strategies depend on the state variable $\mathbf{x}(t)$, player i's adjoint equation (2.85), in general, will be affected by the state variable in the strategies of the players $j \neq i$, creating state-control loops (Başar and Olsder 1998 and Lambertini 2018):

$$\dot{\boldsymbol{\lambda}}_i(t) = -\frac{\partial \mathcal{H}_i(\cdot)}{\partial \mathbf{x}^*(t)} - \sum_{j \neq i} \frac{\partial \mathcal{H}_i(\cdot)}{\phi_j(\mathbf{x}^*(t), t)} \frac{\phi_j^*(\mathbf{x}^*(t), t)}{\partial \mathbf{x}^*(t)} + \rho \boldsymbol{\lambda}_i(t). \tag{2.86}$$

In general, open-loop and closed-loop solutions produced by the maximum principle depend on the initial conditions and, consequently, they do not satisfy Bellman's principle of optimality. Indeed, as the maximum principle yields solutions which are optimal only for a single trajectory, it is not a suitable method for finding feedback Nash equilibria, which are independent of initial conditions (exceptions are those cases, where the open-loop solution coincides with the feedback solution). Instead, feedback solutions must

[30] Alternatively, one could assume that the maximised Hamiltonian function (2.82) is differentiable and concave in $\mathbf{x}$ for all $i \in \mathcal{N}$.

usually be obtained by dynamic programming and solving the Hamilton-Jacobi-Bellman equation, which may readily be adapted to differential games. In this respect, a (current value) Hamilton-Jacobi-Bellman equation of player $i \in \mathcal{N}$ may be defined as

$$\rho \mathcal{V}_i(\mathbf{x}, t) - \frac{\partial \mathcal{V}_i(\mathbf{x}, t)}{\partial t} = \max_{u_i} \left\{ G_i(\mathbf{x}, u_i, \phi_{-i}, t) + \frac{\partial \mathcal{V}_i(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, u_i, \phi_{-i}, t) \right\}, \quad (2.87)$$

which, in case of an autonomous problem, simplifies into

$$\rho \mathcal{V}_i(\mathbf{x}) = \max_{u_i} \left\{ G_i(\mathbf{x}, u_i, \phi_{-i}) + \frac{\partial \mathcal{V}_i(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, u_i, \phi_{-i}) \right\}. \quad (2.88)$$

Similar to theorem 2.7, sufficient conditions for a feedback Nash equilibrium may be formulated as follows (Başar and Olsder 1998 and Dockner et al. 2000):

Theorem 2.8 (Feedback solution: sufficiency) *Let $u_i^*(t) = \phi_i^*(\mathbf{x}(t), t)$ be a feasible feedback strategy for each player $i \in \mathcal{N}$, an N-tuple $\left(\phi_1^*, \ldots, \phi_N^*\right)$ a given strategy profile of such feedback strategies and $\mathbf{x}^*(t)$ a corresponding state trajectory that is a unique absolutely continuous solution to the initial value problem* (2.78)–(2.79). *Assume that there exist, for all $i \in \mathcal{N}$, a value function $\mathcal{V}_i(\mathbf{x}, t)$ which is continuously differentiable and satisfies the Hamilton-Jacobi-Bellman equation*

$$\rho \mathcal{V}_i(\mathbf{x}, t) - \frac{\partial \mathcal{V}_i(\mathbf{x}, t)}{\partial t} = \max_{u_i} \left\{ G_i(\mathbf{x}, u_i, \phi_{-i}^*, t) + \frac{\partial \mathcal{V}_i(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, u_i, \phi_{-i}^*, t) \right\} \quad (2.89)$$

or, if the problem is autonomous, the Hamilton-Jacobi-Bellman equation

$$\rho \mathcal{V}_i(\mathbf{x}) = \max_{u_i} \left\{ G_i(\mathbf{x}, u_i, \phi_{-i}^*) + \frac{\partial \mathcal{V}_i(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, u_i, \phi_{-i}^*) \right\} \quad (2.90)$$

for all $\mathbf{x} \in \mathcal{X}$ and $t \in [0, \infty)$. If the control $u_i^*(t)$ maximises the right-hand side of (2.89) *(or, if the problem is autonomous,* (2.90)*) for all $t \in [0, \infty)$ and for all $i \in \mathcal{N}$, and, additionally, either*

(i) $\rho > 0$ and $\mathcal{V}_i$ is bounded or

(ii) $\rho > 0$, $\mathcal{V}_i$ is bounded from below and $\lim_{t \to \infty} \sup e^{-\rho t} \mathcal{V}_i(\mathbf{x}^*(t), t) \leq 0$

holds for all $i \in \mathcal{N}$, then $\left(\phi_1^*, \ldots, \phi_N^*\right)$ is a feedback Nash equilibrium of the differential game $\Gamma(\mathbf{x}_0, 0)$. (The equilibrium is catching-up optimal.)

Proof. The result follows directly from application of theorems 2.5 and 2.6. □

2.2.3 (Non)uniqueness and time consistency of equilibria

Like Nash equilibria in static games, open-loop, closed-loop and feedback Nash equilibria in differential games are generally not unique. In fact, it has been shown that for games with dynamic information patterns there exists infinitely many "informationally nonunique" closed-loop Nash equilibrium solutions (Başar 1977). Also with feedback strategies, where it is common to restrict the analysis to the study of linear feedback equilibria, there may still exist infinitely many nonlinear feedback solutions (Tsutsui and Mino 1990). However, sufficient conditions for the existence and uniqueness of the open-loop equilibrium have been formulated for a class of linear-quadratic games, where the function $G_i(\cdot)$ is quadratic and the state equation $\mathbf{f}(\cdot)$ is linear in control and state variables (Engwerda 1998). For feedback equilibria, similar sufficient conditions have been established for a class of scalar linear-quadratic games (Lockwood 1996).

One method to choose among possibly many Nash equilibria is to study the stability properties of the solutions. In economic applications, one is usually interested in solutions which are saddle points[31] and exhibit a property called *saddle point stability*, which is a form of conditional stability in a sense that the equilibrium is stable for some initial conditions and unstable for others (Léonard and Long 1992). In such a case, if the initial conditions can be chosen appropriately, the solution converges to a steady state (Dockner et al. 2000). However, restricting analysis to saddle point stable solutions does not, in general, guarantee a unique solution. If there exist multiple saddle point stable equilibria, other criteria to select the outcome of the play may be required.

Finally, there is an important point related to time consistency of solutions. Strategies are time consistent if no player has an incentive to change their strategy during the game as time passes (Lambertini 2018). Weak time consistency requires that strategies remain optimal "along the equilibrium path" whereas strong time consistency requires that strategies are optimal also "off the equilibrium path". Time inconsistent strategies are not credible because (at least some of) the players have an incentive to change their strategy at some point in the game. Intuitively, the requirement of time consistency is similar to the definition of Nash equilibrium. Nash equilibria of Markovian differential games are indeed (weakly) time consistent (Dockner et al. 2000). It is also easy to see that strong time consistency is equivalent to Markov or subgame perfectness, which, in extensive form games, requires an equilibrium to constitute a Nash equilibrium in every subgame.

Feedback equilibria are independent of the initial conditions and strongly time consistent (or subgame perfect) by principle of optimality. By contrast, open-loop and closed-loop equilibria depend on the initial conditions and are not, in general, subgame perfect (Fershtman 1987).[32] In addition, subgame perfect feedback Nash equilibria have another related property: the interpretation of the costate variable as a shadow price of the state variable – common in optimal control theory – is valid in differential games for feedback Nash equilibria but not, in general, for open-loop Nash equilibria (Caputo 2007).

[31] In a saddle point, the system matrix of the dynamical system has real eigenvalues with different signs.

[32] Fershtman (1987) shows that open-loop Nash equilibria are subgame perfect if and only if they are degenerate feedback Nash equilibria. Differential games where open-loop Nash equilibria are subgame perfect are called "perfect games" (Mehlmann 1988). See also Lambertini 2018 for a related discussion.

3 MODEL

3.1 Related literature

The model in this thesis combines two branches of the theoretical oligopoly literature in economics: Bertrand competition with differentiated goods and sluggish demand. Bertrand (or price) competition is one of the two classic models in oligopoly theory, the other being Cournot (or quantity) competition. Unlike homogeneous goods, differentiated goods are not perfect substitutes for each other but different in some respects. Sluggish or viscous demand means that demand does not adjust instantaneously to price changes – rather, the adjustment takes some time.

Static Bertrand models with differentiated goods are well-established in the economic literature (see e.g. Singh and Vives 1984 and Tirole 1988).[33] Dynamic Bertrand models have been considered in the literature involving differential games which have had several applications in the industrial organisation research in general.[34] However, Bertrand models with differentiated goods have received far less attention in the differential games literature than Cournot models with homogeneous goods. For example, almost all versions of the well-known sticky price model, first studied by Simaan and Takayama (1978) and afterwards extended, among others, by Fershtman and Kamien (1987 and 1990) and Tsutsui and Mino (1990), concern Cournot competition with homogeneous goods. There exist only a few works on differentiated Bertrand oligopoly models relating, among others, to capital accumulation (Cellini and Lambertini 1998), adjustment costs (Jun and Vives 2004), sticky prices (Cellini and Lambertini 2007), process and product innovation (Lambertini and Mantovani 2010) and resource extraction (Colombo and Labrecciosa 2015).[35]

Remarkably, a topic of sluggish demand, which is opposite to sticky prices, seems to be almost completely ignored in the differential games literature except for two quite recent papers; one concerning Cournot competition with homogeneous goods in energy markets (Wirl 2010) and the other Bertrand competition with differentiated goods in fast moving consumer goods markets (Wang 2019).[36]

Wirl (2010) examines, in particular in relation to energy and oil markets, an n-player differential game with homogeneous goods where both demand and supply are sluggish due to costly adjustments. Demand for energy and fuels depends on the energy efficiency of the equipment used (e.g. cars, buildings) which are long lasting and seldom replaced. Adjusting supply also requires substantial investments from firms (exploration of new oil and gas fields, construction of new production facilities and power plants etc.). In Wirl's

[33] A static Bertrand model with differentiated goods is examined in detail in the next section.

[34] Comprehensive reviews on the literature of differential games in industrial organisation may be found in e.g. Lambertini 2018, Long 2010, Dockner et al. 2000 and Clemhout and Wan 1994.

[35] All of these papers employ product differentiation as in Singh and Vives 1984.

[36] Monopoly and duopoly models with homogeneous goods and viscous demand are also considered in Radner 2003 and Radner and Richardson 2003 with dynamics that are similar to Wirl 2010.

model, the firms can affect supply through costly investments (i.e. expanding or reducing output). The cost function is assumed to be linear-quadratic and the marginal costs are increasing. Demand evolves according to a first-order differential equation:

$$\dot{D}(t) = \frac{Q(t) - D(t)}{\tau}, \tag{3.1}$$

where the parameter $\tau > 0$ determines the speed of adjustment of market demand. The instantaneous demand function or "target" $Q(t)$ is assumed to be linear $Q(t) = 1 - p(t)$. Dynamics (3.1) assumes myopic price expectations which, in energy markets, are supported by numerous empirical studies employing discrete time versions of (3.1) in estimation of demand of energy and oil products (see Wirl 2010 for references).

Focusing on symmetric strategies, Wirl derives a cooperative cartel solution and noncooperative equilibria in open-loop and (linear) Markov strategies and compares them to a Cournot-Nash equilibrium of the corresponding static game. It turns out that dynamic considerations induced by sluggish demand lower the steady state output while strategic interactions presented by Markov strategies raise the equilibrium output. Markov strategies result in a higher equilibrium output than open-loop strategies due to firms' incentive to preempt other investments; for sufficiently low levels of demand sluggishness (i.e. relatively fast demand adjustments), strategic interactions prevail over dynamic considerations and Markov output may actually be above the static equilibrium output. Open-loop strategies, in turn, result in a higher equilibrium output than a cooperative solution but lower than the static solution. Moreover, open-loop and Markov strategies replicate the static solution at the limit, when $\tau \to 0$.

Wang (2019) considers an n-player linear differential game with price competition and differentiated goods in the context of fast moving consumer goods such as dairy products. The "actual" demand $D_i(t)$ of good i, $i = 1, \ldots, n$, evolves according to a kinematic equation which is otherwise similar to (3.1) except that dynamics are directly affected by the actual demand of other goods j, $j \neq i$, as well:

$$\dot{D}_i(t) = s\,(Q_i(t) - D_i(t)) - l \sum_{j \neq i} D_j(t), \tag{3.2}$$

where s and l, $s > l > 0$, are the adjustment speeds of the actual market demand. The instantaneous or "ideal" demand function is assumed to be linear: $Q_i(t) = A_i - ap_i(t) + b\sum_{j\neq i} p_j(t)$, $(n-1)b > a > b$. The firms choose prices; the cost function is assumed to be linear and marginal costs are thus constant. Restricting the strategy space to open-loop strategies, Wang derives the steady state values at an asymmetric open-loop Nash equilibrium and shows that the equilibrium is a saddle point. Wang also proposes that product differentiation (the difference $a - b$ increases) increases the equilibrium profits of the firms and that sluggishness of demand (s decreases) increases the equilibrium prices.

3.2 Static model

3.2.1 Demand system

A static Bertrand competition model with differentiated goods will be considered first as a benchmark. Suppose there is a duopoly with two firms each producing one good. The goods are differentiated and the firms compete with each other by setting prices. Demand is linear and inverse demand functions are as in Dixit 1979 and Singh and Vives 1984:

$$\begin{aligned} p_1 &= \alpha_1 - \beta_1 q_1 - \gamma q_2 \\ p_2 &= \alpha_2 - \gamma q_1 - \beta_2 q_2 \end{aligned} \tag{3.3}$$

where p_i, α_i, $\beta_i \in \mathbb{R}_+$, for all $i = 1, 2$, and γ is the degree of substitutability between the goods of firms 1 and 2. When $\gamma > 0$, goods 1 and 2 are substitutes, and when $\gamma < 0$, the goods are complements. When $\gamma = 0$, the goods are independent of each other and each firm is a monopolist. When $\alpha_1 = \alpha_2$, product differentiation may be measured by $\gamma^2/(\beta_1\beta_2)$. In a monopoly market, when $\gamma = 0$, the degree of product differentiation is $\gamma^2/(\beta_1\beta_2) = 0$. In a homogeneous market, when $\gamma = \beta_1 = \beta_2$, the degree of product differentiation is $\gamma^2/(\beta_1\beta_2) = 1$ and the goods are called perfect substitutes.

By using some basic linear algebra, the system of equations (3.3) can be transformed into an equivalent system of (direct) demand functions:

$$\begin{aligned} q_1 &= a_1 - b_1 p_1 + c p_2 \\ q_2 &= a_2 + c p_1 - b_2 p_2 \end{aligned} \tag{3.4}$$

or

$$q_i = a_i - b_i p_i + c p_j,\ i = 1, 2 \tag{3.5}$$

where $a_i = \left(\alpha_i\beta_j - \alpha_j\gamma\right)/\delta$ and $b_i = \beta_j/\delta$ for $i \neq j$, $i = 1, 2$, and $c = \gamma/\delta$ when $\delta = \beta_1\beta_2 - \gamma^2 \neq 0$. It is further assumed that $\alpha_i\beta_j - \alpha_j\gamma > 0$ and $\beta_1\beta_2 - \gamma^2 > 0$ to restrict the analysis in the region where quantities are positive.

3.2.2 Bertrand equilibrium

The firms simultaneously choose prices to maximise their profits. The cost function $C(q_i(\mathbf{p})) = \frac{1}{2}q_i(\mathbf{p})^2$, where $\mathbf{p} = (p_1, p_2)$ is a vector of market prices, is quadratic as is typical in the differential games literature. This produces interior solutions and it means that the marginal costs of the firms are increasing. Taking the price p_j of firm j as given, the optimisation problem of firm i, $i = 1, 2$, is:

$$\max_{p_i} \quad \pi_i = q_i(\mathbf{p}) \cdot \left[p_i - \frac{1}{2} q_i(\mathbf{p}) \right]. \tag{3.6}$$

The first-order necessary condition (FOC) for the optimal solution is given by

$$\frac{\partial \pi_i}{\partial p_i} = q_i(\mathbf{p}) + p_i \cdot \frac{\partial q_i(\mathbf{p})}{\partial p_i} - q_i(\mathbf{p}) \cdot \frac{\partial q_i(\mathbf{p})}{\partial p_i} = 0 \tag{3.7}$$

which produces the best-response function

$$p_i^* = \frac{\left(a_i + c p_j\right)(b_i + 1)}{(b_i + 2)\, b_i}. \tag{3.8}$$

As the best-response function (3.8) is upward sloping ($\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} > 0$), the strategies of the firms are said to be strategic complements (Bulow et al. 1985). This means that firm i responds to a price increase (decrease) by firm j by increasing (decreasing) its own price. The second-order necessary condition (SOC) for the solution is

$$\begin{aligned} \frac{\partial^2 \pi_i}{\partial p_i^2} &= 2 \cdot \frac{\partial q_i(\mathbf{p})}{\partial p_i} + (p_i - q_i(\mathbf{p})) \cdot \frac{\partial^2 q_i(\mathbf{p})}{\partial p_i^2} - \left(\frac{\partial q_i(\mathbf{p})}{\partial p_i} \right)^2 \\ &= -\left(b_i^2 + 2b_i\right) < 0 \end{aligned} \tag{3.9}$$

for all $p_i \in \mathbb{R}_+$, which is sufficient to confirm that the solution p_i^* is a global maximum and a strategy profile (p_1^*, p_2^*) is a Nash equilibrium.

Assuming symmetry, $p_i = p_j = p$, $a_i = a_j = a$, $b_i = b_j = b$ and $q_i(\mathbf{p}) = q_j(\mathbf{p}) = q(p) = a - (b - c)\, p$, one can solve the value of p^B (where the superscript B stands for Bertrand) at the (symmetric) Nash equilibrium:

$$p^B = \frac{a\,(b + 1)}{(b + 1)(b - c) + b}. \tag{3.10}$$

The equilibrium output is

$$q^B = a - (b - c)\, p^B = \frac{ab}{(b + 1)(b - c) + b} \tag{3.11}$$

and the firms make a profit

$$\pi^B = q^B p^B - \frac{1}{2}\left(q^B\right)^2 = \frac{a^2 b\,(b + 2)}{2\,((b + 1)(b - c) + b)^2}. \tag{3.12}$$

3.2.3 Product differentiation

Substituting $a = \alpha(\beta-\gamma)/(\beta^2-\gamma^2)$, $b = \beta/(\beta^2-\gamma^2)$ and $c = \gamma/(\beta^2-\gamma^2)$ into the equations (3.10)–(3.12), the equilibrium price p^B, output q^B and profit π^B can be reformulated as:

$$p^B = \frac{\alpha\,(\beta-\gamma)\left(\beta+\beta^2-\gamma^2\right)}{(\beta+\beta^2-\gamma^2)\,(\beta-\gamma)+\beta\,(\beta^2-\gamma^2)} \tag{3.13}$$

$$q^B = \frac{\alpha\,(\beta-\gamma)\,\beta}{(\beta+\beta^2-\gamma^2)\,(\beta-\gamma)+\beta\,(\beta^2-\gamma^2)} \tag{3.14}$$

$$\pi^B = \frac{\alpha^2\,(\beta-\gamma)^2\,\beta\left(\beta+2\left(\beta^2-\gamma^2\right)\right)}{2\left((\beta+\beta^2-\gamma^2)\,(\beta-\gamma)+\beta\,(\beta^2-\gamma^2)\right)^2}. \tag{3.15}$$

From this format, the effect of product differentiation on the profits of the firms may readily be examined. The following analysis focuses on substitute goods only on the interval $[0, \beta)$. When the goods become more homogeneous ($\gamma \to \beta$), the equilibrium price p^B approaches a limit:

$$\lim_{\gamma\to\beta-} p^B = \frac{\alpha}{2\beta+1}. \tag{3.16}$$

At the limit, when $\gamma = \beta$, the firms sell homogeneous goods with identical prices. However, with homogeneous goods, the equilibrium output q^B is not well-behaved at the limit because demand functions (3.4) are no longer applicable as such. Consequently, additional assumptions about consumer behaviour must be made. Especially, an assumption called a *rationing rule* is needed to determine how the market demand is divided between the firms in a symmetric equilibrium (Shy 1995). In a duopoly with symmetric prices, it is usually assumed that demand is split evenly between the firms. Therefore, with homogeneous goods $q_i(p_i, p_j) = \frac{1}{2}q(p)$ if $p = p_i = p_j$, for all i, $i = 1, 2$. In other cases, firm i either receives no customers at all and supplies zero quantity ($q_i(p_i, p_j) = 0$ if $p_i > p_j$) or it wins all customers and supplies the entire market ($q_i(p_i, p_j) = q(p_i)$ if $p_i < p_j$).

The Bertrand game with homogeneous goods and identical unit costs results in a competitive equilibrium where price equals marginal cost and the firms receive zero profit. The reason for this result is the discontinuity of the profit functions and the intuition behind it is that the firms, in maximising their profits, have an incentive to undercut each other until their price equals their marginal cost (from which they cannot lower their price any more) and that the firms therefore choose competitive prices (Shy 1995). This is known as the *Bertrand paradox*, because it is hard to believe that competition and the number of firms would have no effect on the market equilibrium (Tirole 1988). In this regard, the Bertrand model stands in stark contrast to a Cournot oligopoly game with homogeneous goods – where the firms set quantities instead of prices – which results in an equilibrium where the market output is lower and the price higher than the competitive level.

Therefore, at the limit, when $\gamma = \beta$, there is a competitive equilibrium where the firms receive zero profit ($\pi^B = 0$) and the price equals the cost ($p^B = \frac{1}{2}q^B \Rightarrow q^B = 2p^B = \frac{2\alpha}{2\beta+1}$).

By contrast, when the goods become more differentiated ($\gamma \to 0$), the equilibrium price p^B and quantity q^B approach limits

$$\lim_{\gamma \to 0+} p^B = \frac{\alpha(\beta + 1)}{2\beta + 1} \tag{3.17}$$

$$\lim_{\gamma \to 0+} q^B = \frac{\alpha}{2\beta + 1}. \tag{3.18}$$

The equilibrium profit

$$\lim_{\gamma \to 0+} \pi^B = \frac{\alpha^2}{2(2\beta + 1)} > 0 \tag{3.19}$$

is obviously higher when the goods are fully differentiated than when the goods are homogeneous. In fact, the firms are monopolists and make a monopoly profit. This is because fully differentiated goods are independent of each other and the firms do not need to take another good's price into account when pricing their good. In accordance with a well-known monopoly pricing rule, the firms choose prices that balance their marginal revenue with marginal cost (Tirole 1988).

In addition, the partial derivative of the equilibrium profit π^B with respect to the parameter γ

$$\frac{\partial \pi^B}{\partial \gamma} = -\frac{\alpha^2 \beta (\beta - \gamma)^3 \left(2\left(\beta^3 + \gamma^3\right) + \beta^2\right)}{\left((\beta + \beta^2 - \gamma^2)(\beta - \gamma) + \beta(\beta^2 - \gamma^2)\right)^3} < 0 \tag{3.20}$$

is continuous and negative on the interval $[0, \beta)$ which means that the profits of the firms decrease when the goods become more homogeneous and increase when the goods become more differentiated.

Hence, product differentiation relaxes price competition. This is a well-known result of Bertrand models with differentiated goods that applies to Cournot models of quantity competition as well. As was noted, when the goods are homogeneous, the equilibrium price is lower and the consumer surplus higher in Bertrand than in Cournot competition. This difference however becomes smaller as the goods become more differentiated (and zero in case of fully differentiated goods) implying that the type of competition becomes less important with more distant goods (Singh and Vives 1984).

3.3 Dynamic model

3.3.1 Dynamic framework

Next, the Bertrand duopoly model is considered in a dynamic environment where there is inertia in the adjustment of demand after price changes. In general, such inertia may be caused, for example, by demand being costly to adjust (Wirl 2010), habit formation, brand loyalty or switching costs (MacKay and Remer 2022) or bounded rationality of consumers in making price comparisons (Spiegler 2011).[37] The dynamic model in this thesis is open to different interpretations. For this discussion, it suffices to think that it simply takes some time for consumers to react to price changes, for whatever reason.[38]

Assume that the demand system is the same as in (3.5) but now prices and quantities are functions of continuous time $t \in [0, \infty)$:

$$q_i(t) = a_i - b_i p_i(t) + c p_j(t), \; i = 1, 2. \tag{3.21}$$

Suppose, as in Wirl 2010, that the market demand $D_i(t)$, for a good of firm i, $i = 1, 2$, evolves according to dynamics:

$$\dot{D}_i(t) = \frac{dD_i(t)}{dt} = \frac{\widehat{D}_i(t) - D_i(t)}{\tau}, \tag{3.22}$$

where $\widehat{D}_i(t) = q_i(\mathbf{p}(t))$, for all $i = 1, 2$, is the instantaneous demand (3.21) for market prices $\mathbf{p}(t) = (p_1(t), p_2(t))$. Parameter $\tau \in (0, \infty)$ determines the speed of demand adjustment. When $\tau \to 0$, the speed of demand adjustment $\frac{1}{\tau} \to \infty$, meaning that demand adjusts instantaneously to accommodate price changes. Accordingly, when $\tau \to \infty$, the speed of demand adjustment $\frac{1}{\tau} \to 0$, meaning that demand is infinitely sluggish (i.e. constant).

Assuming Markovian strategies and taking the strategy $p_j(t)$ of firm j as given, the dynamic optimisation problem of firm i, $i = 1, 2$, can now be formulated as a standard

[37] Consumers may be biased to choose a product which they have chosen previously or which another agent, such as the government or their employer, have chosen for them for institutional reasons (e.g. health insurance). This tendency of consumers to choose a default option is called *default bias* (Spiegler 2011).

[38] One reason could be that consumers make regular purchases of the same product from the same vendor but check the price (and the prices of rivals) only occasionally (e.g. groceries, which are differentiated by brand) or they subscribe to a service provider which bills them monthly (e.g. streaming services, which are differentiated by their content, or gyms, which are differentiated by their location) and switching to another service would take some time and effort. The former is an example of bounded rationality and the latter of switching costs.

optimal control problem as follows:

$$\max_{p_i(t)} \quad \Pi_i[p_i|p_j(\cdot)] = \int_0^\infty e^{-\rho t}\left[D_i(t)p_i(t) - \frac{1}{2}\widehat{D}_i(t)^2\right]dt \tag{3.23}$$

$$s.t. \quad \dot{D}_i(t) = \frac{\widehat{D}_i(t) - D_i(t)}{\tau} \tag{3.24}$$

$$D_i(0) = D_{i,0} \tag{3.25}$$

$$p_i(t) \in \mathcal{U}_i(\mathbf{D}(t), p_j, t) \subseteq \mathbb{R}_+ \text{ for all } t \in [0, \infty), \tag{3.26}$$

where the state variable $\mathbf{D}(t) = (D_1(t), D_2(t))$ belongs to a state space $\mathcal{D} \subseteq \mathbb{R}^2$. The problem (3.23)–(3.26) is autonomous and time discounting is assumed to be strictly positive $\rho > 0$. Moreover, it is assumed that the initial state is strictly positive $D_{i,0} > 0$ for all $i = 1, 2$. It should be noted that the cost function $C(\widehat{D}_i(t)) = \frac{1}{2}\widehat{D}_i(t)^2$ is again quadratic, as it was in the static model, but in this dynamic set-up the firms' costs depend on instantaneous demand only. This assumption guarantees that the control $p_i(t)$ shows up explicitly in the first-order conditions, thereby producing interior solutions. For example, it may be understood so that the firms hold in stock enough goods to answer any short-term demand changes and that the firms, in anticipation of the future, adjust their production directly to the level of long-term (instantaneous) demand.

3.3.2 Open-loop equilibrium

The above differential game $\Gamma(\mathbf{D}(0), 0) = \{\{1,2\}, \{\mathcal{U}_i(\cdot)\}_{i=1,2}, \{\Pi_i[p_i|p_j(\cdot)]\}_{i=1,2}\}$ of complete information may be examined under different informational assumptions. In the following analysis, the strategy space is restricted to open-loop strategies. First, the existence of an open-loop Nash equilibrium is proved. This is followed by comparative statics results concerning the effect of changes in parameters on the steady state.

Proposition 3.1 *There exists a unique symmetric open-loop Nash equilibrium, where the steady state demand and market price of the goods are*

$$D^{OL} = \frac{ab}{(b+1)(b-c)(1+\tau\rho)+b}$$

$$p^{OL} = \frac{a(b+1)(1+\tau\rho)}{(b+1)(b-c)(1+\tau\rho)+b}.$$

This steady state is a saddle point.

Proof. The Hamiltonian of firm i, $i = 1, 2$, $i \neq j$, is

$$\begin{aligned}\mathcal{H}_i(\cdot) = {} & D_i(t)p_i(t) - \frac{1}{2}q_i(\mathbf{p}(t))^2 \\ & + \lambda_{ii}(t)\frac{q_i(\mathbf{p}(t)) - D_i(t)}{\tau} + \lambda_{ij}(t)\frac{q_j(\mathbf{p}(t)) - D_j(t)}{\tau}.\end{aligned} \tag{3.27}$$

From the necessary conditions of the maximum principle one obtains

$$\frac{\partial \mathcal{H}_i(\cdot)}{\partial p_i(t)} = 0 \Leftrightarrow p_i(t) = \frac{1}{b_i^2}\left(D_i(t) + a_i b_i + b_i c p_j(t) - \frac{b_i \lambda_{ii}(t) - c\lambda_{ij}(t)}{\tau}\right) \quad (3.28)$$

$$\dot{D}_i(t) = \frac{\partial \mathcal{H}_i(\cdot)}{\partial \lambda_{ii}(t)} \Leftrightarrow \dot{D}_i(t) = \frac{q_i(\mathbf{p}(t)) - D_i(t)}{\tau} \quad (3.29)$$

$$\dot{D}_j(t) = \frac{\partial \mathcal{H}_i(\cdot)}{\partial \lambda_{ij}(t)} \Leftrightarrow \dot{D}_j(t) = \frac{q_j(\mathbf{p}(t)) - D_j(t)}{\tau} \quad (3.30)$$

$$\dot{\lambda}_{ii}(t) = -\frac{\partial \mathcal{H}_i(\cdot)}{\partial D_i(t)} + \rho\lambda_{ii}(t) \Leftrightarrow \dot{\lambda}_{ii}(t) = \left(\frac{1}{\tau} + \rho\right)\lambda_{ii}(t) - p_i(t) \quad (3.31)$$

$$\dot{\lambda}_{ij}(t) = -\frac{\partial \mathcal{H}_i(\cdot)}{\partial D_j(t)} + \rho\lambda_{ij}(t) \Leftrightarrow \dot{\lambda}_{ij}(t) = \left(\frac{1}{\tau} + \rho\right)\lambda_{ij}(t). \quad (3.32)$$

Because $p_i(t)$ in (3.28) is continuous in its domain it can be differentiated with respect to time t:

$$\dot{p}_i(t) = \frac{1}{b_i^2}\left(\dot{D}_i(t) + b_i c \dot{p}_j(t) - \frac{b_i \dot{\lambda}_{ii}(t) - c\dot{\lambda}_{ij}(t)}{\tau}\right). \quad (3.33)$$

It can be observed that (3.32) is a separable differential equation which admits a solution

$$\lambda_{ij}(t) = 0 \quad (3.34)$$

for all $t \in [0, \infty)$. Using this information, one can obtain the expression of $\lambda_{ii}(t)$ from (3.28):

$$\lambda_{ii}(t) = \frac{\tau}{b_i}\left(D_i(t) + a_i b_i - b_i^2 p_i(t) + b_i c p_j(t)\right). \quad (3.35)$$

Using the state equation (3.29), costate equations (3.31)–(3.32) and costate variables (3.34)–(3.35), the control dynamics (3.33) can be rewritten as:

$$\begin{aligned}\dot{p}_i(t) = {} & \frac{1}{b_i^2}\Bigg(\frac{a_i - b_i p_i(t) + c p_j(t) - D_i(t)}{\tau} + b_i c \dot{p}_j(t) \\ & -\frac{b_i}{\tau}\left(\left(\frac{1}{\tau} + \rho\right)\frac{\tau}{b_i}\left(D_i(t) + a_i b_i - b_i^2 p_i(t) + b_i c p_j(t)\right) - p_i(t)\right) \\ & +\frac{c}{\tau}\cdot 0\Bigg) \\ = {} & \frac{1}{\tau b_i^2}\big(a_i - a_i b_i (1 + \tau\rho) - (2 + \tau\rho) D_i(t) \\ & + b_i^2 (1 + \tau\rho) p_i(t) + (c - b_i c (1 + \tau\rho)) p_j(t)\big) + \frac{c}{b_i}\dot{p}_j(t).\end{aligned} \quad (3.36)$$

Henceforth, the analysis is restricted to the symmetric duopoly game with symmetric firms and strategies. A symmetry condition is imposed whereby $p_i(t) = p_j(t) = p(t)$, $D_i(t) = D_j(t) = D(t)$ and $q_i(\mathbf{p}(t)) = q_j(\mathbf{p}(t)) = q(p(t)) = a - (b - c)p(t)$. (3.36) can then be reformulated as

$$\dot{p}(t) = \frac{a - ab\,(1+\tau\rho) - (2+\tau\rho)\,D(t) + (b\,(b-c)\,(1+\tau\rho) + c)\,p(t)}{\tau b\,(b-c)}. \tag{3.37}$$

The system dynamics can now be fully characterised in matrix form as follows:

$$\begin{pmatrix} \dot{D}(t) \\ \dot{p}(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau} & -\frac{b-c}{\tau} \\ -\frac{2+\tau\rho}{\tau b(b-c)} & \frac{b(b-c)(1+\tau\rho)+c}{\tau b(b-c)} \end{pmatrix} \begin{pmatrix} D(t) \\ p(t) \end{pmatrix} + \begin{pmatrix} \frac{a}{\tau} \\ \frac{a-ab(1+\tau\rho)}{\tau b(b-c)} \end{pmatrix} \tag{3.38}$$

where

$$J = \begin{pmatrix} -\frac{1}{\tau} & -\frac{b-c}{\tau} \\ -\frac{2+\tau\rho}{\tau b(b-c)} & \frac{b(b-c)(1+\tau\rho)+c}{\tau b(b-c)} \end{pmatrix} \tag{3.39}$$

is the Jacobian matrix. The Jacobian determinant $\det(J) = -\frac{(b+1)(b-c)(1+\tau\rho)+b}{\tau^2 b(b-c)} \neq 0$ is nonzero. The nonsingularity of (3.39) ensures that system (3.38) has a unique solution $(D(t), p(t))$ which describes the behaviour of the system for each given initial point $(D(0), p(0))$.

The solution $(D(t), p(t))$ could be solved explicitly from (3.38), which is an autonomous system of two linear ordinary differential equations. However, the properties of system (3.38) at equilibrium as $t \to \infty$ can be studied without an explicit form solution of $(D(t), p(t))$. For the purposes of this thesis, it suffices to restrict further analysis to a characterisation of the stationary point of system (3.38) at the open-loop equilibrium.

The steady state values of (D^{OL}, p^{OL}) (where the superscript OL stands for open-loop) at the open-loop equilibrium can be obtained from (3.38) by setting $\dot{D}(t) = 0$ and $\dot{p}(t) = 0$ and solving the system of equations

$$\begin{aligned} 0 &= \frac{a}{\tau} - \frac{1}{\tau} D^{OL} - \frac{b-c}{\tau} p^{OL} \\ 0 &= \frac{a - ab\,(1+\tau\rho)}{\tau b\,(b-c)} - \frac{2+\tau\rho}{\tau b\,(b-c)} D^{OL} + \frac{b\,(b-c)\,(1+\tau\rho)+c}{\tau b\,(b-c)} p^{OL}, \end{aligned} \tag{3.40}$$

from which the steady state values are obtained:

$$D^{OL} = a - (b-c)\,p^{OL} = \frac{ab}{(b+1)\,(b-c)\,(1+\tau\rho)+b} \tag{3.41}$$

$$p^{OL} = \frac{a\,(b+1)\,(1+\tau\rho)}{(b+1)\,(b-c)\,(1+\tau\rho)+b}. \tag{3.42}$$

The stability properties of the equilibrium (D^{OL}, p^{OL}) can be examined by finding the eigenvalues of the Jacobian matrix (3.39). The eigenvalues of a square matrix J are the roots of its characteristic polynomial $\det(J - \Lambda I) = 0$ which, in case of a 2×2-matrix, takes the form $\Lambda^2 - \text{tr}(J)\,\Lambda + \det(J) = 0$ or equivalently:

$$\Lambda = \frac{1}{2}\left(\text{tr}(J) \pm \sqrt{(\text{tr}(J))^2 - 4\det(J)}\right), \tag{3.43}$$

where $\text{tr}(J)$ is the trace of the Jacobian matrix J. The negativity of the Jacobian determinant is sufficient to ensure that the two eigenvalues have different signs (i.e. $\Lambda_1 < 0 < \Lambda_2$) which is the definition of a saddle point. Because $\det(J) = -\frac{(b+1)(b-c)(1+\tau\rho)+b}{\tau^2 b(b-c)} < 0$, the equilibrium $\left(D^{OL}, p^{OL}\right)$ is a saddle point. The existence of a saddle point could also be determined from the phase diagram of system (3.38) represented in Figure 3.1 where the arrows indicate the dynamics of the variables D and p.

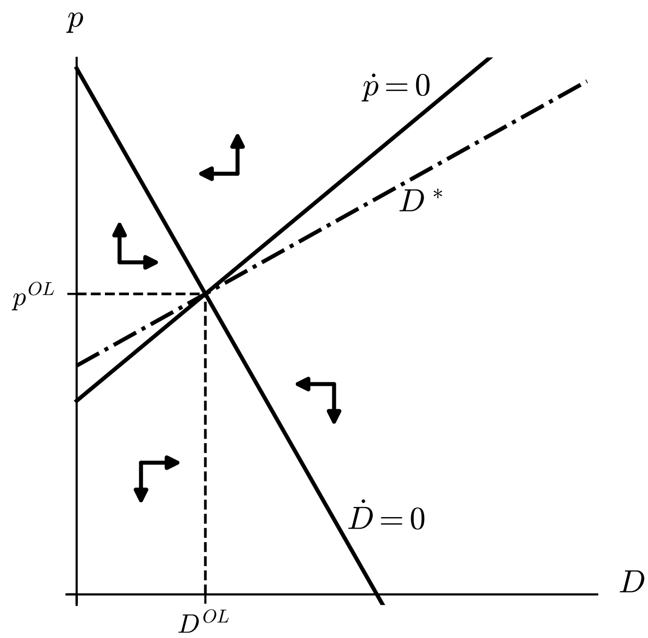

Figure 3.1: Phase diagram of system (3.38)

As the equilibrium $\left(D^{OL}, p^{OL}\right)$ is a saddle point, it is stable in one dimension (in the direction of the eigenvector associated with the eigenvalue $\Lambda_1 < 0$) and unstable in the other (in the direction of the eigenvector associated with the eigenvalue $\Lambda_2 > 0$). Consequently, only one trajectory $D^*(t)$ leads to the equilibrium while every other trajectory leads away from it. This means that, for a given initial value $D(0)$, there is a single path $p^*(t)$ which constitutes a unique optimal open-loop strategy for firms i and j. Trajectories in Figure 3.2 illustrate the saddle point property of system (3.38).

Lastly, a following reasoning verifies that a solution $p^*(t)$ is indeed an open-loop Nash equilibrium strategy for firms i and j. First, the solution $(D^*(t), p^*(t))$ trivially satisfies the conditions (2.83)–(2.85) of theorem 2.7 for all i, $i = 1, 2$, because it is the solution to system (3.38) which was derived by using these conditions.

Second, regarding the transversality condition, it may be observed that the state variable

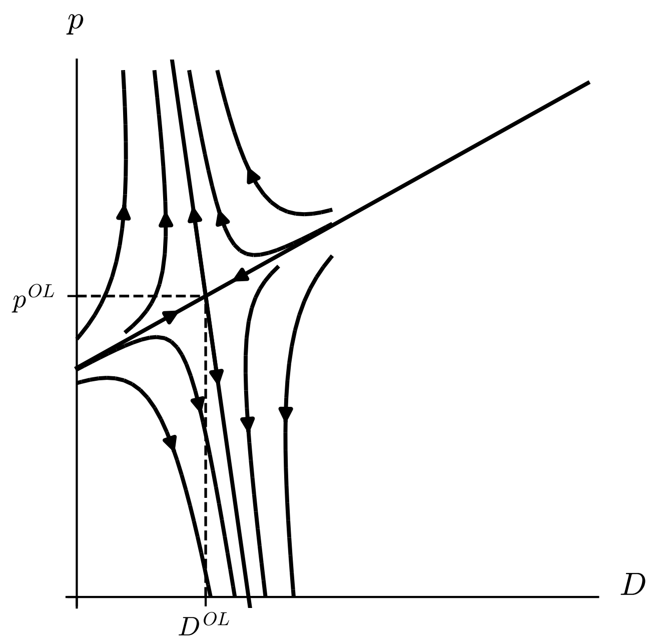

Figure 3.2: Trajectories near the steady state of system (3.38)

$D(t)$ is bounded from below at 0 for all $t \in [0, \infty)$ because $q(t)$ is nonnegative and the initial condition $D(0) = D_0$ is strictly positive by the assumptions of the model. The control $p(t)$ is also nonnegative by assumption. Furthermore, the equilibrium trajectories $(D^*(t), p^*(t))$ converge to finite values (D^{OL}, p^{OL}) as $t \to \infty$ and therefore, by (3.35), also the costate variable λ_{ii} remains bounded and nonnegative at the limit $t \to \infty$. As $D^*(t)$ and λ_{ii} remain bounded at the limit $t \to \infty$ and $\lambda_{ij} = 0$ for all $t \in [0, \infty)$, the condition (ii) of theorem 2.7 holds for all i, $i = 1, 2$.

Third, while the Hamiltonian function (3.27) itself is not concave in $(\mathbf{D}, p_i)$ jointly, the maximised Hamiltonian $\mathcal{H}_i^0(\cdot) = \max_{p_i(t)} \mathcal{H}_i(\cdot)$ can be shown to be concave in $\mathbf{D}$.[39] One may become convinced of this by observing that the optimal open-loop strategy $p^*(t)$ depends only on argument t (and the initial conditions) and when it is substituted to the Hamiltonian function (3.27), the resulting maximised Hamiltonian $\mathcal{H}_i^0(\cdot)$ is linear in the state variable and therefore concave in $\mathbf{D}$ for all i, $i = 1, 2$.

Therefore, the conditions of theorem 2.7 are satisfied and a strategy pair $(p^*(t), p^*(t))$ is an open-loop Nash equilibrium. □

3.3.3 Comparative statics

A following (trivial) observation can be made immediately by comparing the open-loop equilibrium (D^{OL}, p^{OL}) with the static equilibrium (q^B, p^B).

[39] The Hamiltonian function (3.27) is not concave because its Hessian (a matrix of second-order partial derivatives) is not negative-semidefinite for all $(\mathbf{D}, p_i)$. The Hessian matrix shows that this is caused by the presence of the state-control interaction term $D_i(t)p_i(t)$ in the instantaneous payoff $G_i(\cdot)$ of firm i.

Proposition 3.2 *The open-loop equilibrium* (D^{OL}, p^{OL}) *is 'quasi-static' in a sense that it replicates the static equilibrium* (q^B, p^B) *at the limit, when* $\tau \to 0$. *This holds also when* $\rho \to 0$.

The intuition behind proposition 3.2 is that, when the sluggishness parameter $\tau \to 0$, the speed of demand adjustment $\frac{1}{\tau} \to \infty$, which results in instantaneous adjustment of demand, just like in a static model. Correspondingly, when time discounting $\rho \to 0$, firms become indifferent between present and future profits and the dynamic nature of the game becomes irrelevant as the payoff of the firms is otherwise time-invariant.

Proposition 3.2 does not however mean that the open-loop solution should be equated with the static solution, as the open-loop solution is derived from a dynamical problem characterised by a set of differential equations (state equations) which – by definition – do not appear in a static formulation of the same problem (Lambertini 2018). This property of being quasi-static is actually typical for open-loop solutions of differential games which may be reduced to static games by omitting the state variable.

In Wirl 2010, for example, the open-loop steady state output also coincides with the static equilibrium output at the limit, when $\tau \to 0$. Furthermore, that open-loop steady state output declines monotonically when τ increases, becoming zero at the limit $\tau \to \infty$. It is easy to see from (3.41) that the open-loop equilibrium demand D^{OL} shares this same property: D^{OL} is monotonically decreasing with respect to τ ($\frac{\partial D^{OL}}{\partial \tau} < 0$ for all τ) and $D^{OL} = 0$, when $\tau \to \infty$.

Another, more important observation can be made by comparing the profits of the firms at the open-loop equilibrium and the static equilibrium.

Proposition 3.3 *The market power and profits of the firms increase with sluggishness of demand when* $\frac{c}{(b+1)(b-c)\rho} > \tau > 0$ *and decrease when* $\frac{c}{(b+1)(b-c)\rho} < \tau$. *The profits of the firms are higher at the open-loop equilibrium* (D^{OL}, p^{OL}) *than at the static equilibrium* (q^B, p^B) *when* $\bar{\tau} > \tau > 0$, *i.e. as long as demand is not too sluggish.*

Proof. The equilibrium profit π^{OL} is

$$\pi^{OL} = D^{OL} p^{OL} - \frac{1}{2} \left(D^{OL}\right)^2 = \frac{a^2 b\,(b+2) + 2a^2 b\,(b+1)\,\tau\rho}{2\left((b+1)(b-c)(1+\tau\rho) + b\right)^2}. \tag{3.44}$$

While the functional form of the open-loop equilibrium profit π^{OL} in (3.44) closely resembles the form of its static counterpart π^B in (3.12), it is not at once clear which of these profits is higher (the parameter τ is present in both the numerator and denominator of (3.44)). By proposition 3.2, the open-loop equilibrium replicates the static equilibrium at the limit, so $\pi^{OL} \to \pi^B$ as $\tau \to 0$. It is thus necessary to examine the profit equation (3.44) when $\tau > 0$ to understand its behaviour. The partial derivative of the equilibrium profit π^{OL} with respect to parameter τ

$$\frac{\partial \pi^{OL}}{\partial \tau} = \frac{a^2 b\,(b+1)\,\rho\,(c-(b+1)\,(b-c)\,\tau\rho)}{((b+1)\,(b-c)\,(1+\tau\rho)+b)^3} \tag{3.45}$$

is continuous and vanishes at

$$c-(b+1)\,(b-c)\,\tau\rho = 0 \Leftrightarrow \tau^* = \frac{c}{(b+1)\,(b-c)\,\rho}.$$

The sign of (3.45) is positive when $\frac{c}{(b+1)(b-c)\rho} > \tau$ and negative when $\frac{c}{(b+1)(b-c)\rho} < \tau$. Therefore, (3.44) is increasing on the interval $[0, \tau^*)$ and decreasing on the interval (τ^*, ∞). From (3.44) it is easy to see that when $\tau \to \infty$, the equilibrium profit $\pi^{OL} \to 0$. Because $\pi^B > 0$ is constant on the interval $\tau \in [0, \infty)$, the (continuous) profit functions π^{OL} and π^B intersect at a point $\bar{\tau} \in (\tau^*, \infty)$. (The explicit form of $\bar{\tau}$ obtained from $\pi^{OL}(\bar{\tau}) = \pi^B(\bar{\tau})$ is too complicated to be of any further analytical interest.) Therefore, the open-loop equilibrium profit is above the static equilibrium profit ($\pi^{OL} > \pi^B$) when $\bar{\tau} > \tau > 0$ and vice versa ($\pi^{OL} < \pi^B$) when $\bar{\tau} < \tau$. □

Proposition 3.3 can be understood so that the firms are, to some extent, able to take advantage of the delay in the adjustment of demand after price changes in their pricing. Because D^{OL} decreases monotonically with respect to τ, the increase in profits of the firms is entirely due to increase in market power. However, sluggishness of demand benefits the firms only as long as demand is not too sluggish. Proposition 3.3 and the effect of the parameter τ on the equilibrium profit π^{OL} is illustrated in Figure 3.3.

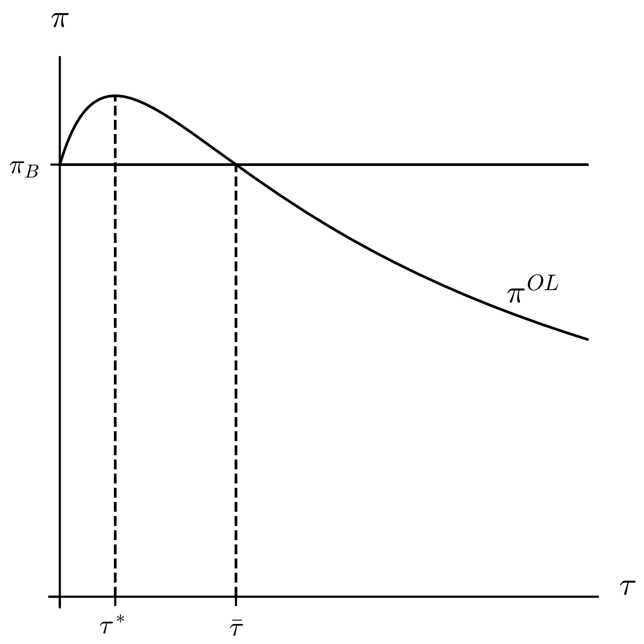

Figure 3.3: The effect of sluggishness of demand on the open-loop equilibrium profit

A following proposition, in turn, shows that sluggishness of demand does not affect the qualitative properties of product differentiation derived from the static model.

Proposition 3.4 *The profits of the firms increase when* $\gamma \to 0$ *and decrease when* $\gamma \to \beta$. *Product differentiation therefore relaxes price competition also in the presence of sluggish demand.*

Proof. Substituting $a = \alpha(\beta - \gamma)/(\beta^2 - \gamma^2)$, $b = \beta/(\beta^2 - \gamma^2)$ and $c = \gamma/(\beta^2 - \gamma^2)$ into (3.43), the equilibrium profit π^{OL} may be reformulated as:

$$\pi^{OL} = \frac{\alpha^2 (\beta - \gamma)^2 \beta \left(\beta + 2\left(\beta^2 - \gamma^2\right)\right) + 2\alpha^2 (\beta - \gamma)^2 \beta \left(\beta + \beta^2 - \gamma^2\right) \tau\rho}{2\left((\beta + \beta^2 - \gamma^2)(\beta - \gamma)(1 + \tau\rho) + \beta (\beta^2 - \gamma^2)\right)^2}. \quad (3.46)$$

As before with the static model, the analysis is restricted to substitute goods on the interval $[0, \beta)$. When the goods become more differentiated ($\gamma \to 0$), the equilibrium profit π^{OL} approaches a limit:

$$\lim_{\gamma \to 0+} \pi^{OL} = \frac{\alpha^2 (2\beta + 1 + 2(\beta + 1)\tau\rho)}{2(2\beta + 1 + (\beta + 1)\tau\rho)^2} > 0, \quad (3.47)$$

which, unsurprisingly, is strictly positive. Indeed, with fully differentiated goods, the firms make a monopoly profit. When the goods instead become more homogeneous ($\gamma \to \beta$), the equilibrium profit π^{OL} approaches a limit:

$$\lim_{\gamma \to \beta-} \pi^{OL} = \frac{\alpha^2 (1 + 2\tau\rho)}{2(2\beta + 1 + \tau\rho)^2} > 0, \quad (3.48)$$

which is likewise strictly positive. However, like in a static Bertrand model, the equilibrium demand D^{OL} is not well-behaved at the limit $\gamma = \beta$ because the demand function (3.21) is inapplicable and consequently the profit function π^{OL} is discontinuous at the limit (but continuous elsewhere). Assuming, as in the static model, that the market demand is split evenly between the firms at the symmetric equilibrium with homogeneous goods and that competition drives prices to marginal costs, the profits of the firms drop to zero. Therefore, the equilibrium profit is zero ($\pi^{OL} = 0$) when $\gamma = \beta$ and strictly positive ($\pi^{OL} > 0$) when $\beta - \epsilon < \gamma < \beta$, where $\epsilon > 0$ is an arbitrarily small positive constant.

Denote $\pi^{OL}(\gamma) = \frac{f}{g}$, where

$$f(\gamma) = \alpha^2 (\beta - \gamma)^2 \beta \left(\beta + 2\left(\beta^2 - \gamma^2\right)\right) + 2\alpha^2 (\beta - \gamma)^2 \beta \left(\beta + \beta^2 - \gamma^2\right) \tau\rho$$

and

$$g(\gamma) = 2\left(\left(\beta + \beta^2 - \gamma^2\right)(\beta - \gamma)(1 + \tau\rho) + \beta\left(\beta^2 - \gamma^2\right)\right)^2$$

are the numerator and the denominator of (3.46), respectively. Differentiating π^{OL} with respect to the parameter γ results in a partial derivative

$$\frac{\partial \pi^{OL}}{\partial \gamma} = \frac{f_\gamma g - f g_\gamma}{g^2} \tag{3.49}$$

which, when written out, becomes rather complicated and difficult to examine analytically. However, one may easily determine the signs of the functions $f > 0$ and $g > 0$ and their partial derivatives

$$\begin{aligned} f_\gamma &= -2\alpha^2\beta\left(\beta-\gamma\right)\left(\beta\left(1+2\tau\rho\right)+2\left(\gamma\left(\beta-\gamma\right)+\left(\beta^2-\gamma^2\right)\right)\left(1+\tau\rho\right)\right) < 0 \\ g_\gamma &= -4\left(\left(\beta+\beta^2-\gamma^2\right)\left(\beta-\gamma\right)\left(1+\tau\rho\right)+\beta\left(\beta^2-\gamma^2\right)\right) \\ &\quad \cdot\left(\left(2\gamma\left(\beta-\gamma\right)+\left(\beta+\beta^2-\gamma^2\right)\right)\left(1+\tau\rho\right)+2\beta\gamma\right) < 0 \end{aligned}$$

for all $\gamma \in [0, \beta)$. As a consequence of the strict positivity of the functions f and g, the equilibrium profit $\pi^{OL} > 0$ is also obviously strictly positive on the interval $[0, \beta)$. By contrast, the sign of the partial derivative (3.49) may not be determined directly from its component functions. One must resort to an alternative method to do this.

First, by (3.47) and (3.48), it may be evaluated that $0 < \pi^{OL}(\beta-\epsilon) < \pi^{OL}(0)$. Then, due to the strict positivity of the equilibrium profit π^{OL} on the interval $[0, \beta)$, one may take the logarithm

$$\ln\left(\pi^{OL}(\beta-\epsilon)\right) < \ln\left(\pi^{OL}(0)\right) \tag{3.50}$$

and decompose it to

$$\ln\left(\frac{f(\beta-\epsilon)}{g(\beta-\epsilon)}\right) < \ln\left(\frac{f(0)}{g(0)}\right). \tag{3.51}$$

From (3.51), one obtains

$$\ln\left(f(\beta-\epsilon)\right) - \ln\left(f(0)\right) < \ln\left(g(\beta-\epsilon)\right) - \ln\left(g(0)\right) \tag{3.52}$$

which is equivalently

$$\int_0^{\beta-\epsilon}\left(\frac{\partial}{\partial\gamma}\ln\left(f(\gamma)\right)\right)d\gamma < \int_0^{\beta-\epsilon}\left(\frac{\partial}{\partial\gamma}\ln\left(g(\gamma)\right)\right)d\gamma. \tag{3.53}$$

When $\epsilon \to 0$, (3.53) becomes

$$\int_0^{\beta}\left(\frac{\partial}{\partial\gamma}\ln\left(f(\gamma)\right)\right)d\gamma < \int_0^{\beta}\left(\frac{\partial}{\partial\gamma}\ln\left(g(\gamma)\right)\right)d\gamma. \tag{3.54}$$

Differentiating (3.54) (or (3.53)), one obtains

$$\frac{\partial}{\partial \gamma} \ln \left(f(\gamma) \right) < \frac{\partial}{\partial \gamma} \ln \left(g(\gamma) \right). \tag{3.55}$$

which is equivalently

$$\frac{f_\gamma}{f} < \frac{g_\gamma}{g} \tag{3.56}$$

or

$$f_\gamma g < f g_\gamma. \tag{3.57}$$

Finally, from (3.57) one obtains

$$\frac{\partial \pi^{OL}}{\partial \gamma} = \frac{f_\gamma g - f g_\gamma}{g^2} < 0 \tag{3.58}$$

which proves the negativity of the partial derivative $\frac{\partial \pi^{OL}}{\partial \gamma}$ on the interval $[0, \beta)$, and thereby, the result. □

4 DISCUSSION

This thesis has focused on the investigation of dynamic Bertrand competition between two firms in a market where the goods are differentiated and demand is sluggish. The related economic literature has been reviewed briefly and the competitive situation has been modelled as a differential game. The dynamic model employs the demand system as in Singh and Vives 1984 and dynamics as in Wirl 2010. The fundamental mathematical theory and solution methods of optimal control theory and differential games that are required in the analysis of the model have also been presented in order to serve as an introduction to differential games and also to make the analysis of the model largely self-contained.

Sluggish demand is a remarkably ignored topic in the oligopoly literature especially when one compares it to a much larger literature on sticky prices, the opposite case of sluggish demand, and considers that potentially many markets may exhibit inertia in the adjustment of demand which may be due to, for example, costly adjustment of demand (Wirl 2010), habit formation, brand loyalty or switching costs (MacKay and Remer 2022) or bounded rationality of consumers (Spiegler 2011). Static Bertrand models assume that market demand adjusts instantaneously to price changes. The analysis of the dynamic model in this thesis reveals how the equilibrium demand and price change when that assumption is relaxed.

First, in the analysis of the model, it was shown that there exists a unique symmetric open-loop Nash equilibrium and that the steady state is a saddle point. This open-loop steady state replicates the static equilibrium point at the limit, when either the sluggishness parameter $\tau \to 0$ or time discounting $\rho \to 0$. This is similar to Wirl 2010, where the open-loop steady state output also coincides with the static equilibrium output at the limit, when $\tau \to 0$. Moreover, the open-loop equilibrium demand in the dynamic model in this thesis behaves the same way as the open-loop equilibrium output in Wirl's model, declining monotonically as τ increases and becoming zero at the limit, when $\tau \to \infty$.

The main result of the analysis is that when the firms restrict themselves to open-loop strategies and sluggishness of demand is within certain parameters (i.e. sufficiently fast adjustments), sluggishness of demand increases the market power and profits of the firms compared to the static model. Because the open-loop equilibrium demand decreases monotonically as sluggishness of demand increases, the increase in profits is entirely attributable to market power that the firms gain which allows them to raise prices further above the costs. The profits of the firms are thus higher at the open-loop equilibrium point than at the static equilibrium point as long as demand is not too sluggish. Beyond that point, the open-loop equilibrium profits are lower than the static equilibrium profits, and at the limit, when $\tau \to \infty$, the open-loop profits of the firms drop to zero (as the open-loop equilibrium demand becomes zero). Lastly, it was confirmed that product differentiation relaxes price competition also in the presence of sluggish demand – sluggishness of demand does not affect the qualitative properties of product differentiation derived from the static model.

Obviously, a major limitation on the applicability of the above results to both theoretical and practical purposes is that open-loop equilibria, in general, are not subgame or Markov perfect. Open-loop strategies require players to commit to a chosen path from the beginning of the game and stick to it for the entire duration of the game, which, in the dynamic model in this thesis, is eternal. That seems like a tall order for the firms if the state variable is observable, in which case the firms might want use that information to their benefit.[40]

An immediate extension to the model in this thesis then would be to study (linear) feedback strategies which, by principle of optimality, are subgame perfect and to find feedback Nash equilibria by solving the appropriate Hamilton-Jacobi-Bellman equation. Feedback strategies do not require players to commit to any path but rather to specify a decision rule to pick the optimal path at any point of the game. Feedback strategies are robust to changes in the initial conditions and remain optimal also off the equilibrium path.

The theoretical results in this thesis also indicate that in those markets, which exhibit some degree of consumer inertia, sluggishness of demand should be taken into account in empirical research to avoid biased estimates that static models might produce. For example, MacKay and Remer (2022) show that neglecting dynamic demand can lead to significant estimation biases when predicting price increases after horizontal mergers in such markets. Similar arguments about the importance of taking the dynamic structure of the market into account when estimating econometric models have been promoted, among others, by Jun and Vives (2004).

Dynamic models increase understanding of markets and competition in situations where the market is dynamic and evolves according to some discernible rule. By using static models in such circumstances, one might overlook important economic forces that shape the market and end up with fallacious theoretical conclusions or biased empirical estimates. Further study of dynamic models may in this regard be useful to researchers, policy makers, business executives and the general public, and lead to better research, better competition policy and better economy.

[40]Nevertheless, open-loop strategies might be suitable for example in situations where there are strong commitment mechanisms or the state variable is perhaps observable only periodically at the beginning of the game (as an initial condition) and the infinite time horizon otherwise fits the competitive situation (e.g. the firms are assumed to be planning to continue their business operations indefinitely).

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