

TURUN YLIOPISTON JULKAISUJA
ANNALES UNIVERSITATIS TURKUENSIS

SARJA - SER. A / OSA - TOM. 378

ASTRONOMICA - CHEMICA - PHYSICA - MATHEMATICA

**SPINOR BOSE-EINSTEIN
CONDENSATES AND
TOPOLOGICAL DEFECTS**

by

Harri Mäkelä

TURUN YLIOPISTO
Turku 2007

From

Department of Physics
University of Turku
Finland

Supervised by

Kalle-Antti Suominen
Professor of Physics
Department of Physics
University of Turku
Finland

Reviewed by

Claus Montonen
Docent
Department of Physical Sciences
University of Helsinki
Finland

Grigory Volovik
Visiting Professor
Low Temperature Laboratory
Helsinki University of Technology
Finland

Opponent

Luis Santos
Professor of Physics
Institute of Theoretical Physics
Leibniz Universität Hannover
Germany

ISBN 978-951-29-3419-5

ISBN 978-951-29-3420-1

ISSN 0082-7002

Painosalama Oy - Turku, Finland 2007

Acknowledgments

The research presented in this thesis has been carried out at the Department of Physics in the University of Turku during the years 2002-2007. I want to thank everybody at the Laboratory of Optics and Spectroscopy for the pleasant working environment. Special thanks are due to the members of the Quantum Optics group, who have provided possibilities for a balanced combination of rigorous scientific work and recreational activities. I am especially indebted to O. Kärki for his help on computer-related questions and to Dr. S. Maniscalco for her caring attitude and plentiful advice on all conceivable topics.

I want to express my sincere gratitude to my supervisor Prof. Kalle-Antti Suominen for his guidance and support throughout the thesis work. I also warmly thank Prof. Chris Pethick for his kind hospitality during my visit to Nordita in Copenhagen and Dr. Yunbo Zhang for pleasant collaboration.

The Vilho, Yrjö and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters and the Academy of Finland are acknowledged for financial support.

I am very thankful to my friends Juha, Mika and Teiko for relaxing leisure time pursuits and intellectual discussions.

Finally, I am deeply grateful to my parents and siblings for all the support and encouragement I have received from them during my life.

Contents

Acknowledgments	3
Abstract	6
List of articles	7
1 Introduction	8
2 Energy functional and magnetic field	11
2.1 Gross-Pitaevskii energy functional	11
2.2 Atomic energy levels	13
3 Spinor condensates	17
3.1 Energy functional	17
3.1.1 Atomic spin $F = 1$	20
3.1.2 Atomic spin $F = 2$	20
3.1.3 Atomic spin $F = 3$	21
3.2 External magnetic field and magnetization	22
3.2.1 Magnetic field	22
3.2.2 Magnetization	23
4 Magnetic dipole-dipole interaction and spin-3 condensates	25
4.1 Magnetic dipole-dipole interaction	25
4.2 Ground states of spin-3 condensates	26
5 Homotopy groups and topological defects	29
5.1 Order-parameter space	29
5.2 Topological defects	30
5.3 Homotopy groups	32
5.3.1 First homotopy group	32
5.3.2 Higher homotopy groups	33

5.4	Physical interpretation	34
6	Topological defects of Bose-Einstein condensates	37
6.1	On the order-parameter spaces of BECs	37
6.2	Single component Bose-Einstein condensate	38
6.3	Mixtures of Bose-Einstein condensates	39
6.4	Spinor condensates	40
7	Decomposition of pure spin states	44
7.1	Majorana decomposition	44
7.2	Linear transformations	47
7.2.1	Spin rotation	49
7.2.2	Change of phase	50
8	Inert states	51
8.1	Definition	51
8.2	Inert states of spinor condensates	53
8.3	Monopoles	57
9	Summary	59
	Bibliography	66

Abstract

This thesis concentrates on the topological defects of spin-1 and spin-2 Bose-Einstein condensates, the ground states of spin-3 condensates, and the inert states of spinor condensates with arbitrary spin. Our work is based on the description of a spinor condensate of spin- S atoms in terms of a state vector of a spin- S particle.

The results of the homotopy theory are used to study the existence and structure of the topological defects in spinor condensates. We construct examples of defects, study their energetics, and examine how their stability is affected by the presence of an external magnetic field.

The ground states of spin-3 condensates are calculated using analytical and numerical means. Special emphasis is put on the ground states of a chromium condensate, whose dependence on the magnetic dipole-dipole interaction is studied.

A simple geometrical method for the calculation of inert states of spinor condensates is presented. This method is used to find candidates for the ground states of spin- S condensates.

List of articles

This thesis consists of an introductory review and the following five articles:

- I** H. Mäkelä, Y. Zhang, and K.-A. Suominen,
Topological defects in spinor condensates,
J. Phys. A: Math. Gen. *36*, 8555-8564 (2003).

- II** Y. Zhang, H. Mäkelä, and K.-A. Suominen,
Non-Abelian topological excitations in spinor condensates
Chin. Phys. Lett. *22*, 536-538 (2005).

- III** H. Mäkelä
Explicit expressions for the topological defects of spinor
Bose-Einstein condensates
J. Phys. A: Math. Gen. *39*, 7423-7439 (2006).

- IV** H. Mäkelä and K.-A. Suominen
Ground states of spin-3 Bose-Einstein condensates for conserved
magnetization
Phys. Rev. A *75*, 033610(8) (2007).

- V** H. Mäkelä and K.-A. Suominen
Inert states of spin- S systems
Phys. Rev. Lett. *99*, 190408(4) (2007).

Chapter 1

Introduction

The Indian physicist S. N. Bose realized in 1924 that the statistics governing photons is determined by restricting the physical Hilbert space to be a symmetric tensor product of single photon states. Soon after this A. Einstein applied this idea to massive particles and discovered the phenomenon that is now called Bose-Einstein condensation (BEC) [18, 19]. BEC is associated with the condensation of atoms in the state of lowest energy and is a consequence of quantum statistical effects. In his work Einstein considered bosonic ideal gas. If it is trapped in a box and periodic boundary conditions are used, the condensation occurs to a zero-momentum state which has zero energy. At that time this phenomenon was viewed as a mathematical curiosity with little experimental interest. It was thought that condensation disappears as soon as interactions are properly taken into account.

In 1938 two laboratories working on liquid helium simultaneously reported the discovery that below the so-called λ -temperature T_λ this liquid appeared to flow through very narrow capillaries without detectable friction. In the same year F. London suggested that this phenomenon, superfluidity, might be a manifestation of Bose-Einstein condensation [33, 32]. He believed that despite the strong interparticle interactions BEC was occurring in this system and was responsible for the superfluidity. Over the last seventy years there has been overwhelming experimental and theoretical evidence that London's hypothesis is correct, and it is now almost universally accepted by researchers in the field. However, it is extremely difficult to verify directly that BEC occurs in liquid helium. The best direct evidence comes from high-energy neutron scattering and from the evaporation of atoms from the ^4He surface. While consistent with a condensate fraction of the order of 10% at $T = 0$, it cannot be said to rigorously establish it. Because the condensate fraction in helium is so low, it was necessary to

find substances with weak particle-particle interaction. The problem was that most substances do not remain gaseous as the temperature is lowered, but form solids or liquids, and thus the effects of interaction become large.

Due to advances in the laser cooling of alkali atoms, such atoms became attractive candidates for Bose-Einstein condensation. In 1995 Bose-Einstein condensation in dilute gases was obtained using rubidium [1], sodium [15], and lithium [9, 10]. The dilute gas systems differ from liquid helium in many ways. The most important differences are that the gaseous systems are very dilute and weakly interacting, so that the perturbation theory in interatomic interaction is a very reliable method of calculating their properties. Moreover, dilute gases are extremely sensitive to manipulation of the external potentials in space and time. These differences are also illustrated by the densities of these systems. The peak density of a Bose-Einstein condensed gas cloud is of the order of $10^{13} - 10^{15} \text{ cm}^{-3}$, while the density of liquid helium is 10^{22} cm^{-3} . In a gaseous BEC the atomic velocities can be as low as 1 mm/sec. An important difference is also that, unlike ^4He , alkali atoms can have non-zero nuclear spin and therefore numerous internal hyperfine states which are stable electronic ground states. Thus, there exists the possibility of creating a quantum fluid simultaneously composed of several, distinguishable components by Bose-Einstein condensing a gas of atoms in several hyperfine states. Important examples of multi-component Bose-Einstein condensates are spinor condensates, whose creation became possible after the realization of an optical trap [57]. A spinor condensate of spin- S particles is characterized by an order-parameter with $2S + 1$ components. Due to the many-component nature of the order-parameter, several different ground states are possible in spinor condensates. One of these is chosen to be the actual ground state by the strength of the particle-particle interaction and magnetic field [13, 16, 23, 45, 63, 52]. The vectorial form of the order-parameter allows various types of topological defects. The topological defects of spinor condensates are in general considerably more complex than the topological defects of single-component condensates, which is why a systematic approach is needed in order to classify and characterize them. This is achieved by the use of homotopy groups [22]. Homotopy groups classify continuous maps which can be converted into each other continuously. This is physically relevant since topological defects correspond to order-parameters which cannot be continuously deformed into a uniform, position independent order-parameter [37].

The structure of the thesis is as follows: in Chapter 2 the Gross-

Pitaevskii energy functional and the dependence of atomic energy levels on magnetic field are discussed. The energy functional of a spinor Bose-Einstein condensate and its ground states are studied in Chapter 3. In Chapter 4 the magnetic dipole-dipole interaction and the ground states of a spin-3 condensate are examined. Chapter 5 contains a brief survey of the theory of homotopy groups and its physical applications. In Chapter 6 this theory is used to study the topological defects of Bose-Einstein condensates. Chapter 7 introduces a way to write a pure state of a spin- S particle in terms of $2S$ pure states of spin-1/2 particles. This decomposition is put into practise in Chapter 8, where the inert states of spinor condensates are calculated. The work is summarized in Chapter 9.

Chapter 2

Energy functional and magnetic field

In this chapter we briefly discuss the concept of a BEC, its energy functional, and atomic energy levels in an external magnetic field. This chapter is based mainly on [48].

2.1 Gross-Pitaevskii energy functional

As we have discussed above, Bose-Einstein condensation for non-interacting particles can be defined as a macroscopic occupation of a single particle state. The situation changes if one considers an interacting system. For a system of N particles in an external potential V and interacting via a pair potential $U(|\mathbf{r}_i - \mathbf{r}_j|)$, the Hamiltonian can be written as

$$H_N = \sum_{i=1}^N \left(-\frac{\hbar^2}{2M} \nabla_i^2 + V(\mathbf{r}_i) \right) + \sum_{1 \leq i < j \leq N} U(|\mathbf{r}_i - \mathbf{r}_j|). \quad (2.1)$$

Three and higher body interaction potentials could also be included, but they are conventionally excluded. In a dilute gas the two-body collisions are dominant and higher body scattering can usually be disregarded. Unlike in the case of a non-interacting system, even at zero temperature it is not entirely clear what is meant by a macroscopic occupation of a single particle state, because the eigenfunctions of H_N are not products of single particle states. The concept of a macroscopic occupation of a single particle state acquires a precise meaning through the one-particle density matrix. If Ψ is a normalized N -particle state vector, the one-particle state determined by

Ψ can be defined via

$$\gamma(\mathbf{r}, \mathbf{r}') = N \int \cdots \int \left(\prod_{i=2}^N d\mathbf{r}_i \right) \Psi^*(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \Psi(\mathbf{r}', \mathbf{r}_2, \dots, \mathbf{r}_N), \quad (2.2)$$

for which $\text{Tr}[\gamma] = \int d\mathbf{r} \gamma(\mathbf{r}, \mathbf{r}) = N$. BEC in the ground state of a homogeneous system means that this operator has an eigenvalue of order N in the thermodynamic limit. By thermodynamic limit we mean the limit where the particle number N and volume V approach infinity with the N/V constant. In an inhomogeneous system the thermodynamic limit has to be replaced by an appropriate scaling of the potentials involved. This definition of BEC was first presented in [47]. If more than one of the eigenvalues of γ is of the order of N , the condensate is said to be fragmented. In this work we assume in general that the condensate is not fragmented, that is, it can be described by one single-particle state only. In the orthodox approach to Bose-Einstein condensation the N -particle state of condensed particles is written as

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \prod_{i=1}^N \phi(\mathbf{r}_i), \quad (2.3)$$

where the single-particle state ϕ is normalized to one. The gas is assumed to be very dilute and cold, so that the interaction potential U can be approximated by an effective potential

$$U(\mathbf{r}_i - \mathbf{r}_j) = g\delta(\mathbf{r}_i - \mathbf{r}_j). \quad (2.4)$$

Here $g = 4\pi\hbar^2 a/M$, M is the mass of the atom, and a is the s -wave scattering length. By inserting Ψ into Eq. (2.1) and by using the contact interaction potential one obtains

$$E[\Psi] = N \int d\mathbf{r} \left[\frac{\hbar^2}{2M} |\nabla\phi(\mathbf{r})|^2 + V(\mathbf{r})|\phi(\mathbf{r})|^2 + \frac{N-1}{2} g |\phi(\mathbf{r})|^4 \right]. \quad (2.5)$$

This can be written in a different form by using the wave function of the condensate, which is defined by

$$\psi(\mathbf{r}) = \sqrt{N}\phi(\mathbf{r}). \quad (2.6)$$

If terms of order $1/N$ are neglected, one obtains the Gross-Pitaevskii energy functional

$$E[\Psi] = \int d\mathbf{r} \left[\frac{\hbar^2}{2M} |\nabla\psi(\mathbf{r})|^2 + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{1}{2} g |\psi(\mathbf{r})|^4 \right]. \quad (2.7)$$

It is experimentally possible to trap different types of atoms simultaneously. In the case of two species, the wave function (2.3) can be written as

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_{N_1}, \mathbf{r}'_1, \dots, \mathbf{r}'_{N_2}) = \prod_{i=1}^{N_1} \phi_1(\mathbf{r}_i) \prod_{j=1}^{N_2} \phi_2(\mathbf{r}'_j), \quad (2.8)$$

where $\int d\mathbf{r}_1 |\phi_1(\mathbf{r}_1)|^2 = \int d\mathbf{r}_2 |\phi_2(\mathbf{r}_2)|^2 = 1$ and we assume that the particle number of each species is conserved. The order-parameters describing the condensates are $\psi_1 = \sqrt{N_1} \phi_1$ and $\psi_2 = \sqrt{N_2} \phi_2$ and the energy functional becomes

$$E[\psi_1, \psi_2] = \int d\mathbf{r} \left[\frac{\hbar^2}{2M_1} |\nabla \psi_1|^2 + \frac{\hbar^2}{2M_2} |\nabla \psi_2|^2 + V_1(\mathbf{r}) |\psi_1|^2 + V_2(\mathbf{r}) |\psi_2|^2 + \frac{1}{2} g_{11} |\psi_1|^4 + \frac{1}{2} g_{22} |\psi_2|^4 + g_{12} |\psi_1|^2 |\psi_2|^2 \right] \quad (2.9)$$

Here, like in Eq. (2.7), terms of order $1/N_1$ and $1/N_2$ have been neglected. The masses M_i and potentials V_i may be different for the species $i = 1$ and $i = 2$. The interaction between atoms is given by $g_{ij} = g_{ji} = 2\pi\hbar^2 a_{ij}/M_{ij}$, where a_{ij} is the s -wave scattering length between atoms i and j and $M_{ij} = M_i M_j / (M_i + M_j)$ is the reduced mass for an atom i and an atom j . Two-component condensates have been produced by trapping ^{87}Rb atoms in different internal states [43].

2.2 Atomic energy levels

Next we study the energy of an atom in an external magnetic field. We denote the nuclear spin of an atom by I and the total spin of the electronic shell by S . We assume that the orbital angular momentum of the electronic shell is zero, which holds true for the alkali atoms.

In the absence of an external magnetic field the atomic levels are split by the hyperfine interaction, which is described by a Hamiltonian of the form

$$H_{hf} = A \mathbf{I} \cdot \mathbf{J} = A \mathbf{I} \cdot \mathbf{S}. \quad (2.10)$$

Here A is a constant and \mathbf{I} and $\mathbf{J} = \mathbf{L} + \mathbf{S}$ are the operators for the nuclear spin and the electronic angular momentum, respectively. The latter consists of the operator for the total orbital angular momentum of the electronic shell (\mathbf{L}) and of the operator for the total spin of the electronic shell (\mathbf{S}). Because we assume that the orbital angular momentum is zero, we can

drop the \mathbf{L} part, as has been done in the second expression of Eq. (2.10). The operator for the total spin of an atom is

$$\mathbf{F} = \mathbf{I} + \mathbf{S}. \quad (2.11)$$

From now on we assume that $\hbar = 1$ in the angular momentum operators.

To take into consideration the effect of an external magnetic field, one has to add to (2.10) the Zeeman energies arising from the interaction of the magnetic moments of the electron and the nucleus with the magnetic field. We choose the magnetic field to be parallel to the z -axis, so the total Hamiltonian becomes

$$H_{tot} = A\mathbf{I} \cdot \mathbf{S} + CS_z + DI_z. \quad (2.12)$$

Now we assume that $I = 3/2$ and $S = 1/2$, which are the relevant values for the atoms used most commonly in experiments on Bose-Einstein condensates, namely ^{87}Rb and ^{23}Na . Therefore the constants C and D are now

$$C = g\mu_B B, \quad (2.13)$$

and

$$D = -\frac{\mu}{I} B. \quad (2.14)$$

Here $\mu_B = |e|\hbar/2m_e$ is the Bohr magneton, g is the g factor of the electron and μ is the magnetic moment of the nucleus. It is of the order of the nuclear magneton $\mu_N = |e|\hbar/2m_p$, so $|D/C| \sim m_e/m_p \sim 1/2000$, which shows that in most cases D can be neglected. The basis for the total spin of the atom can be chosen to be $\{|m_I, m_S\rangle \mid m_I = \pm 3/2, \pm 1/2, m_S = \pm 1/2\}$. With the help of the raising and lowering operators $I_{\pm} = I_x \pm iI_y$ and $S_{\pm} = S_x \pm iS_y$, we can write

$$\mathbf{I} \cdot \mathbf{S} = I_z S_z + \frac{1}{2}(I_+ S_- + I_- S_+). \quad (2.15)$$

This expression commutes with $F_z = I_z + S_z$, so the Hamiltonian H_{tot} conserves the projection of the total spin \mathbf{F} in the z -direction. This means that H_{tot} couples only states with the same value of $m_F = m_I + m_S$ and hence the energy eigenvalues of H_{tot} can be calculated straightforwardly.

The results are [48]

$$\begin{aligned}
E(m_F = 2) &= \frac{3}{4}A + \frac{1}{2}C + \frac{3}{2}D, \\
E(m_F = 1) &= -\frac{1}{4}A + D \pm \sqrt{\frac{3}{4}A^2 + \frac{1}{4}(A + C - D)^2}, \\
E(m_F = 0) &= -\frac{1}{4}A \pm \sqrt{A^2 + \frac{1}{4}(C - D)^2}, \\
E(m_F = -1) &= -\frac{1}{4}A - D \pm \sqrt{\frac{3}{4}A^2 + \frac{1}{4}(A - C + D)^2}, \\
E(m_F = -2) &= \frac{3}{4}A - \frac{1}{2}C - \frac{3}{2}D.
\end{aligned} \tag{2.16}$$

Because D/C is very small, we set $D = 0$ which gives

$$\begin{aligned}
E(2)/A &= \frac{3}{4} + \frac{1}{2}b, \\
E(1)/A &= -\frac{1}{4} \pm \sqrt{\frac{3}{4} + \frac{1}{4}(1+b)^2} = -\frac{1}{4} \pm \sum_{k=0}^{\infty} \left[\sum_{l=\lceil k/2 \rceil}^k \binom{l}{k-l} \frac{c_l}{2^k} \right] b^k, \\
E(0)/A &= -\frac{1}{4} \pm \sqrt{1 + \frac{1}{4}b^2} = -\frac{1}{4} \pm \sum_{k=0}^{\infty} \frac{c_k}{4^k} b^{2k}, \\
E(-1)/A &= -\frac{1}{4} \pm \sqrt{\frac{3}{4} + \frac{1}{4}(1-b)^2} = -\frac{1}{4} \pm \sum_{k=0}^{\infty} (-1)^k \left[\sum_{l=\lceil k/2 \rceil}^k \binom{l}{k-l} \frac{c_l}{2^k} \right] b^k, \\
E(-2)/A &= \frac{3}{4} - \frac{1}{2}b,
\end{aligned} \tag{2.17}$$

where $b = C/A$ and $\lceil j \rceil$ is the smallest integer greater than or equal to j . The $+$ ($-$) sign in front of the square root refers to an $F = 2$ ($F = 1$) atom. For later use, we have expanded $E(\pm 1)$ and $E(0)$ in the Taylor series. These have been obtained using the fact that $\sqrt{1+x} = \sum_{k=0}^{\infty} c_k x^k$, where

$$c_k = \frac{(-1)^k (2k)!}{(1-2k)k!2^{4k}}. \tag{2.18}$$

The series expansion for $\sqrt{1+x}$ is valid if $|x| < 1$. In $E(1)/A$ we define $x = b/2 + b^2/4$, so this expansion of holds true if $b < -1 + \sqrt{5} \approx 1.23$.

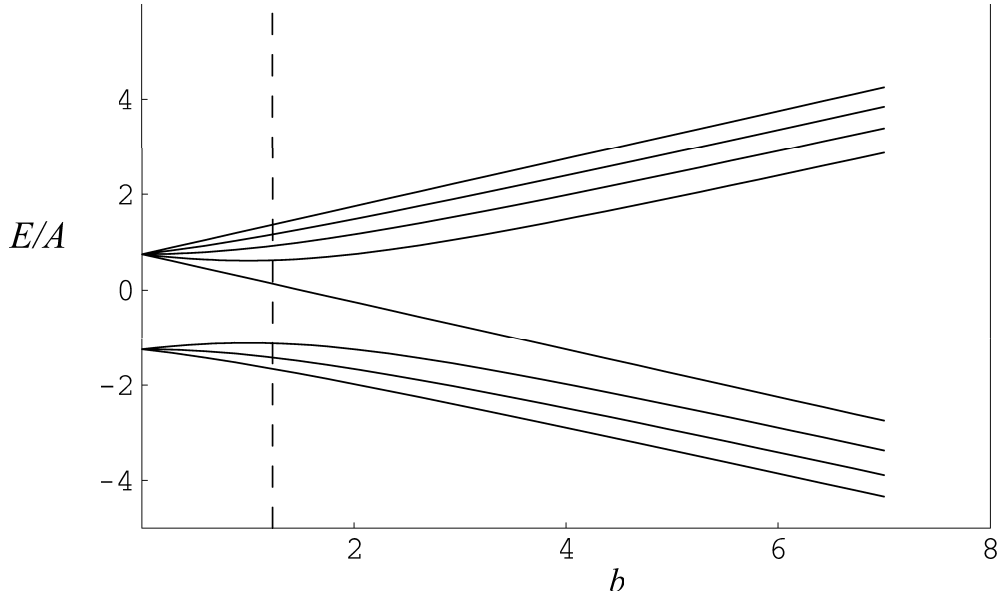


Figure 2.1: Energy of the hyperfine levels as a function of the magnetic field b . The vertical line shows the limit of validity of the series expansions of Eq. (2.17).

Under this condition the Taylor series of $E(0)/A$ is also valid. To the best of the author's knowledge, these complete forms for the series expansions of (2.17) have not been presented before, typically only the lowest order terms are given. In Fig. (2.1) we plot the energy levels of Eq. (2.17) as a function of b .

Chapter 3

Spinor condensates

By a spinor condensate we mean a Bose-Einstein condensate which is described by a state vector of a spin- F particle. As the word "spinor" is used in a different meaning in other fields of physics, a better name for a spinor condensate could simply be a spin- F condensate. However, because the expression spinor condensate is common in the literature, it shall also be used here, although interchangeably with the term spin- F condensate.

A spinor condensate can be realized experimentally by confining the condensate atoms in an optical trap [57]. If a BEC is in a magnetic trap, only particles which are in a low-field seeking state with respect to the quantization axis determined by the direction of the local magnetic field remain trapped. In an optical trap the atomic spin is liberated from the requirements of magnetic trapping and becomes a new degree of freedom. When an optical trap is used there can still be magnetic fields present. In this case the magnetic fields are not needed to trap atoms, but e.g. to diminish the effect of stray magnetic fields [12, 31, 54, 58]. Next we show how the Gross-Pitaevskii energy functional of Eq. (2.7) can be generalized for spinor Bose-Einstein condensates.

3.1 Energy functional

A spin- F particle can be described by a state vector of the form

$$\phi(\mathbf{r}) = \begin{pmatrix} \phi_F(\mathbf{r}) \\ \phi_{F-1}(\mathbf{r}) \\ \vdots \\ \phi_{-F}(\mathbf{r}) \end{pmatrix}, \quad (3.1)$$

for which $\int d\mathbf{r} \phi(\mathbf{r})^\dagger \phi(\mathbf{r}) = 1$. We again assume that all atoms are condensed in the same one-particle state, given by ϕ . Therefore, the Hartree N -particle wave function becomes

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \prod_{i=1}^N \phi(\mathbf{r}_i). \quad (3.2)$$

Because we discuss bosonic atoms, F has to be an integer. Notice that now different spin components can be converted into each other, while in the case of the two-component condensate discussed earlier the number of particles in each component was conserved. Following Eq. (2.6), we define the order-parameter describing a spinor condensate through $\psi(\mathbf{r}) = \sqrt{N} \phi(\mathbf{r})$. Usually this is written in the form

$$\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})} \xi(\mathbf{r}) = \sqrt{n(\mathbf{r})} \begin{pmatrix} \xi_F(\mathbf{r}) \\ \xi_{F-1}(\mathbf{r}) \\ \vdots \\ \xi_{-F}(\mathbf{r}) \end{pmatrix}, \quad (3.3)$$

where $\xi(\mathbf{r}) = \phi(\mathbf{r}) / \sqrt{\phi^\dagger(\mathbf{r})\phi(\mathbf{r})}$ and $n(\mathbf{r}) = N\phi^\dagger(\mathbf{r})\phi(\mathbf{r})$ is the particle density [23, 45]. When defined like this, $\xi^\dagger(\mathbf{r})\xi(\mathbf{r}) = 1$ for all \mathbf{r} for which ϕ is defined. In the rest of the book we call ξ the spinor. In some publications this term refers to ψ , so one must be careful with the terminology.

As before, we assume that only the s -wave scattering has to be taken into consideration and that the interaction can be approximated by the contact interaction. For bosons the two-particle state vector describing scattering particles has to be invariant in the interchange of the particle coordinates. The orbital part of the state vectors describing the s -wave scattering is left unchanged in this operation. Hence the spin part also has to be symmetric in the interchange of the particle coordinates, which means that only even values are possible for the total spin of two colliding particles. Under this constraint we can write the contact interaction as

$$U(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \sum_{k=0}^{2F} g_k \mathcal{P}_k, \quad (3.4)$$

where \mathcal{P}_k is the projection operator onto a two-particle state with total spin k and the summation is restricted to even values of k . The strength of the interaction is given by $g_k = \frac{4\pi\hbar^2 a_k}{M}$, where a_k is the s -wave scattering length in the total spin k scattering channel, and M is the mass of the

atom. Using this form for the interaction potential, the Gross-Pitaevskii energy functional describing a spinor condensate of spin- F particles can be written as [23, 45]

$$E^F[\psi] = \int d\mathbf{r} \left\{ \frac{1}{2M} \sum_{k=-F}^F |\nabla \psi_k(\mathbf{r})|^2 + V(\mathbf{r})n(\mathbf{r}) + \frac{1}{2} \sum_{k=0,2,\dots,2F} g_k n(\mathbf{r})^2 \langle\langle \mathcal{P}_k \rangle\rangle_\xi \right\}. \quad (3.5)$$

Here V is the external potential and $\langle\langle \mathcal{P}_i \rangle\rangle_\xi = \xi(\mathbf{r})^\dagger \otimes \xi(\mathbf{r})^\dagger \mathcal{P}_i \xi(\mathbf{r}) \otimes \xi(\mathbf{r})$. The external potential is assumed to be the same for all spin components. The energy functional of Eq. (3.5) is invariant in arbitrary spin rotations and phase changes of the spinor ξ . In this thesis the ground state spinors of (3.5) play an important role. The interaction and potential terms do not make any energetic difference between position dependent and position independent spinors, but the kinetic energy is minimized by a constant spinor. Furthermore, the term containing the external potential depends only on the density, so the spinors minimizing E^F can be found by finding all position independent spinors which minimize

$$\mathcal{E}_{int}^F[\xi] = \sum_{k=0,2,\dots,2F} g_k \langle\langle \mathcal{P}_k \rangle\rangle_\xi. \quad (3.6)$$

The interaction term can be written in a different form by using the properties of spin matrices. We consider a two-particle system and denote by \mathbf{F}_i the spin operator of particle i , that is, $\mathbf{F}_1 = \mathbf{F} \otimes I_2$, $\mathbf{F}_2 = I_1 \otimes \mathbf{F}$ and \mathbf{F} is the spin operator of a spin- F particle. The total spin operator is $\mathbf{F}_{tot} = \mathbf{F}_1 + \mathbf{F}_2$. The possible values for the total spin are $0, 2, \dots, 2F$ and therefore we get $\mathbf{F}_{tot}^2 = \sum_{k=even}^{2F} k(k+1) \mathcal{P}_k$. The projection operators can be written as

$$\mathcal{P}_k = \sum_{j=-k}^k |F_{tot} = k, m_{F_{tot}} = j\rangle \langle F_{tot} = k, m_{F_{tot}} = j|. \quad (3.7)$$

The sum of the projection operators gives the identity operator,

$$I = I_1 \otimes I_2 = \sum_{k=0,2,\dots,2F} \mathcal{P}_k. \quad (3.8)$$

Now $\mathbf{F}_1 \cdot \mathbf{F}_2 = (\mathbf{F}_{tot}^2 - \mathbf{F}_1^2 - \mathbf{F}_2^2)/2 = (\mathbf{F}_{tot}^2 - 2F(F+1)I)/2$. By using the expressions for \mathbf{F}_{tot}^2 and I we get

$$\mathbf{F}_1 \cdot \mathbf{F}_2 = \sum_{k=0,2,\dots,2F} \frac{1}{2} [k(k+1) - 2F(F+1)] \mathcal{P}_k. \quad (3.9)$$

Table 3.1: The possible zero-field ground state spinors of condensates with $F = 1$ and $F = 2$. These are defined up to a spin rotation and a phase change.

	Phase	Spinor	$ \Theta ^2$	$\langle \mathbf{F} \rangle_\xi^2$
$F = 1$	ferromagnetic	$(1, 0, 0)$	—	1
	antiferrom./polar	$(0, 1, 0)$	—	0
$F = 2$	ferromagnetic	$(1, 0, 0, 0, 0)$	0	4
		$(0, 1, 0, 0, 0)$	0	1
	cyclic	$(1, 0, \sqrt{2}, 0, -1)/2$	0	0
	antiferrom./polar	$(1, 0, 0, 0, 1)/\sqrt{2}$	1	0
		$(0, 0, 1, 0, 0)$	1	0

By elevating this to n^{th} power we obtain

$$(\mathbf{F}_1 \cdot \mathbf{F}_2)^n = \sum_{k=0,2,\dots,2F} \frac{1}{2^n} [k(k+1) - 2F(F+1)]^n \mathcal{P}_k. \quad (3.10)$$

This equation can be used to eliminate the projection operators from the interaction energy. In practise, for $F > 1$, it is favourable to leave \mathcal{P}_0 but to replace all the other projection operators with the powers of $\mathbf{F}_1 \cdot \mathbf{F}_2$. Next we consider in more detail the interaction energies and ground state spinors of spin- F condensates with $F = 1, 2$ and 3.

3.1.1 Atomic spin $F = 1$

Now we get $I = \mathcal{P}_0 + \mathcal{P}_2$ and $\mathbf{F}_1 \cdot \mathbf{F}_2 = -2\mathcal{P}_0 + \mathcal{P}_2$. Using these \mathcal{P}_0 and \mathcal{P}_2 can be eliminated and we obtain

$$\mathcal{E}_{int}^{F=1}[\xi] = \alpha_1 + \gamma_1 \langle \mathbf{F} \rangle_\xi^2, \quad (3.11)$$

where $\alpha_1 = \frac{g_0+2g_2}{3}$, $\gamma_1 = \frac{g_2-g_0}{3}$, and $\langle \mathbf{F} \rangle_\xi = \xi^\dagger(\mathbf{r})\mathbf{F}\xi(\mathbf{r})$. Depending on the sign of γ_1 , the energy is minimized either by $|\langle \mathbf{F} \rangle_\xi| = 1$ or $\langle \mathbf{F} \rangle_\xi = 0$ [23, 45]. The former is called *ferromagnetic* and the latter *antiferromagnetic* or *polar* phase, see Table (3.1).

3.1.2 Atomic spin $F = 2$

Now the use of (3.8) and (3.10) shows that the interaction term can be written as

$$\mathcal{E}_{int}^{F=2}[\xi] = \alpha_2 + \beta_2 |\Theta(\mathbf{r})|^2 + \gamma_2 \langle \mathbf{F} \rangle_\xi^2. \quad (3.12)$$

where $\alpha_2 = \frac{4g_2+3g_4}{7}$, $\beta_2 = \frac{7g_0-10g_2+3g_4}{7}$, $\gamma_2 = \frac{g_4-g_2}{7}$, and $\Theta = 2\xi_2\xi_{-2} - 2\xi_1\xi_{-1} + \xi_0^2$. The ground state spinors of this equation have been studied in [13, 56, 62, 63], and can be classified as follows: (i) If $\beta_2, \gamma_2 > 0$ the energy is minimized when $\|\langle \mathbf{F} \rangle_\xi\| = \Theta = 0$. Spinors with these properties are called *cyclic*. (ii) When $\beta_2 < 0, \gamma_2 > 0$ the minimum is obtained by making $\|\langle \mathbf{F} \rangle_\xi\| = 2, \Theta = 0$, and the ground state is *ferromagnetic*. (iii) If $\beta_2 > 0, \gamma_2 < 0$ the minimum is achieved by maximizing Θ , i.e. $|\Theta| = 1$, and $\|\langle \mathbf{F} \rangle_\xi\| = 0$. The ground state is called *polar* or *antiferromagnetic*. (iv) Finally, if α_2 and β_2 are both negative, the ground state is ferromagnetic for $4|\beta_2| > |\gamma_2|$, and polar otherwise. This is because $\|\langle \mathbf{F} \rangle_\xi\|$ and $|\Theta|$ cannot be maximized simultaneously. Explicit expressions for the ground state spinors can be found in Table (3.1). All spinors obtained from them by a spin rotation and gauge transformation are also ground state spinors. In Table (3.2) we show the ground state spinors of some alkali atom condensates.

3.1.3 Atomic spin $F = 3$

Now

$$\sum_{k=0,2,4,6} g_k \mathcal{P}_k = \alpha_3 I + 7\beta_3 \mathcal{P}_0 + \gamma_3 \mathbf{F}_1 \cdot \mathbf{F}_2 + \delta_3 (\mathbf{F}_1 \cdot \mathbf{F}_2)^2, \quad (3.13)$$

where $\alpha_3 = -\frac{1}{7}g_2 + \frac{81}{77}g_4 + \frac{1}{11}g_6$, $7\beta_3 = g_0 - \frac{5}{3}g_2 + \frac{9}{11}g_4 - \frac{5}{33}g_6$, $\gamma_3 = \frac{1}{18}(g_6 - g_2)$ and $\delta_3 = \frac{1}{126}g_2 - \frac{1}{77}g_4 + \frac{1}{198}g_6$. Eq. (3.13) gives

$$\mathcal{E}_{int}^{F=3}[\xi] = \alpha_3 + \beta_3 |\Theta(\mathbf{r})|^2 + \gamma_3 \langle \mathbf{F} \rangle_\xi^2 + \delta_3 \sum_{i,j=x,y,z} \langle F_i F_j \rangle_\xi^2, \quad (3.14)$$

where $\Theta = 2\xi_3\xi_{-3} - 2\xi_2\xi_{-2} + 2\xi_1\xi_{-1} - \xi_0^2$. Following the notation of [52] we define $O_{ij} = \langle F_i F_j \rangle_\xi$ and $O^2 = \sum_{ij} O_{ij}^2$. In the calculation of the ground state spinors the constant α_3 can be dropped. If $\gamma_3 \neq 0$, the resulting equation can be divided by $|\gamma_3|$, which gives

$$\tilde{\mathcal{E}}_{int}^{F=3}[\xi] = \frac{\beta_3}{|\gamma_3|} |\Theta(\mathbf{r})|^2 \pm \langle \mathbf{F} \rangle_\xi^2 + \frac{\delta_3}{|\gamma_3|} O^2. \quad (3.15)$$

Here the $+$ ($-$) sign refers to positive (negative) γ_3 . The analytical minimization of this energy is complicated, and so far only numerical results have been presented, see Refs. [16, 52] and Paper IV. The ground states will be discussed in more detail in Chapter 4.

Table 3.2: The zero-field ground states of some alkali atom condensates. These are based both on theoretical and experimental results.

Atom	$F = 1$	$F = 2$
^{23}Na	antiferrom./polar [14, 23, 58]	antiferrom./polar [13]
^{83}Rb	-	ferromagnetic [13]
^{85}Rb	-	antiferrom./polar [28]
^{87}Rb	ferromagnetic [23, 54, 12, 28]	antiferrom./polar [54, 31, 28]

3.2 External magnetic field and magnetization

3.2.1 Magnetic field

Thus far it has been assumed that there is no external magnetic field present. Now we consider the situation where the condensate is placed in a spatially uniform magnetic field directed parallel to the z -axis. The energy arising from the presence of a magnetic field can be calculated using the series expansions of Eq. (2.17). A straightforward calculation shows that the energy of an $F = 1$ condensate be written as

$$E_{mag}^{F=1}[\psi] = -A \int d\mathbf{r} n(\mathbf{r}) \left[\frac{1}{4} + S_2(b) + S_o(b) \langle F_z \rangle_\xi + [S_e(b) - S_2(b)] \langle F_z^2 \rangle_\xi \right]. \quad (3.16)$$

where

$$S_e(b) = \sum_{k=0}^{\infty} \left[\sum_{l=k}^{2k} \binom{l}{2k-l} \frac{c_l}{2^{2k}} \right] b^{2k}, \quad (3.17)$$

$$S_o(b) = \sum_{k=0}^{\infty} \left[\sum_{l=k+1}^{2k+1} \binom{l}{2k+1-l} \frac{c_l}{2^{2k+1}} \right] b^{2k+1}, \quad (3.18)$$

$$S_2(b) = \sum_{k=0}^{\infty} \frac{c_k}{4^k} b^{2k}, \quad (3.19)$$

and c_k is given by Eq. (2.18). The corresponding expression for an $F = 2$ condensate is

$$E_{mag}^{F=2}[\psi] = -\frac{A}{12} \int d^3x n(\mathbf{x}) \left[3 + \langle F_z^2 - F_z^4 \rangle_\xi + 2b \langle F_z - F_z^3 \rangle_\xi \right. \\ \left. - 4S_o(b) \langle 4F_z - F_z^3 \rangle_\xi - 4S_e(b) \langle 4F_z^2 - F_z^4 \rangle_\xi - 3S_2(b) \langle 4I - 5F_z^2 + F_z^4 \rangle_\xi \right]. \quad (3.20)$$

The only approximation used in the derivation of (3.16) and (3.20) is the assumption that $D = 0$. As far as the author can tell, these expressions have not been presented in the literature earlier. The only limitation for the validity of (3.16) and (3.20) is that b be smaller than ≈ 1.23 . For ^{87}Rb and ^{23}Na $A = \Delta E_{hf}/2$, where ΔE_{hf} is the hyperfine splitting between $F = 2$ and $F = 1$ states. Because $b = C/A$ and $C = g\mu_B B$, using the numerical values relevant for ^{87}Rb and ^{23}Na one finds that the largest magnetic fields for which the series expansions can be used are of the order of 1500 Gauss and 400 Gauss for rubidium and sodium, respectively. Typically, the magnetic fields used in experiments are much weaker than these, so terms which are of third or higher order in b can be neglected. The energy functionals can be simplified further by dropping the constant terms. Additionally, because Eqs. (3.16) and (3.20) are usually used to calculate the ground state spinors, all terms which do not contain expectation values of F_z^k , $k = 1, 2, 3, 4$, can be omitted. After these steps the energy becomes

$$E_{mag}[\psi] = \pm \frac{A}{32} \int d\mathbf{r} n(\mathbf{r}) [8b \langle F_z \rangle_\xi - b^2 \langle F_z^2 \rangle_\xi], \quad (3.21)$$

where the upper (lower) sign refers to $F = 2$ ($F = 1$).

3.2.2 Magnetization

From the form of the projection operators (3.7) one sees that the interaction (3.4) does not change the value of the z -component of the total spin of two scattering particles, which means that the z -component of the total spin of the condensate is a conserved quantity. The operator for this is $F_z(N) = \sum_{k=1}^N I_1 \otimes I_2 \otimes \cdots \otimes I_{k-1} \otimes F_z \otimes I_{k+1} \otimes \cdots \otimes I_N$. Evaluating the expectation value in the state (3.2) gives $\langle F_z(N) \rangle = N \int d\mathbf{r} \langle F_z \rangle_\phi$. We call this quantity the z -component of the magnetization, and denote it by M_z . Using the definition of ξ given below Eq. (3.3) one obtains

$$M_z = \int d\mathbf{r} n(\mathbf{r}) \langle F_z \rangle_\xi. \quad (3.22)$$

In general the magnetization \mathbf{M} can be defined as $\mathbf{M} = \int d\mathbf{r} n(\mathbf{r}) \langle \mathbf{F} \rangle_\xi$. The contact interaction is not capable of changing M_z or the length $\|\mathbf{M}\|$ of \mathbf{M} , but it can rotate \mathbf{M} about the z -axis.

In addition to the M_z conserving scattering described by the contact interaction, collisions (dipolar relaxation) that change the value of M_z can also occur. However, typically they are negligible over the lifetime of the

condensate. Therefore, rather than calculating the global ground state of the system, one must consider the lowest energy state under the restriction of conserved M_z . The conservation of (the z -component of) the magnetization has been experimentally verified in Refs. [12, 54]. From the definition (3.22) it follows that M_z can have any value between $-NF$ and NF , where N is the particle number. If we consider weak values of the magnetic field, it is enough to include only the first term of Eq. (3.21) in the energy. For a position independent magnetic field this term is just M_z multiplied by a constant. Thus the ground state is determined by M_z , and does not depend on the strength of the magnetic field, as long as the quadratic term is negligible.

Chapter 4

Magnetic dipole-dipole interaction and spin-3 condensates

4.1 Magnetic dipole-dipole interaction

Previously we have modeled the atom-atom interaction as a contact interaction. This simplified form for the interaction potential is justified, as it produces the same scattering as the actual interaction. The latter is the van der Waals interaction, which is caused by the electric dipole-dipole interaction between the atoms. It has approximately the form $-\alpha/r^6$, where r is the atomic separation. Although including only this interaction is in many cases sufficient, for some atoms the contribution from the magnetic dipole-dipole interaction has to be taken into account. For example in a chromium condensate [21] its effect can be almost comparable to that of the contact interaction. For a spinor condensate the dipole-dipole interaction energy is

$$E_{dd}[\psi] = \frac{\mu_0}{8\pi} \int d\mathbf{r} n(\mathbf{r}) \int d\mathbf{r}' \frac{n(\mathbf{r}') [\langle \boldsymbol{\mu} \rangle_{\xi(\mathbf{r})} \cdot \langle \boldsymbol{\mu} \rangle_{\xi(\mathbf{r}')} - 3(\hat{\mathbf{x}} \cdot \langle \boldsymbol{\mu} \rangle_{\xi(\mathbf{r})})(\hat{\mathbf{x}} \cdot \langle \boldsymbol{\mu} \rangle_{\xi(\mathbf{r}')})]}{||\mathbf{r} - \mathbf{r}'||^3}, \quad (4.1)$$

where μ_0 is the vacuum permeability, $\hat{\mathbf{x}} = (\mathbf{r} - \mathbf{r}')/(||\mathbf{r} - \mathbf{r}'||)$, $\boldsymbol{\mu} = g_F \mu_B \mathbf{F}$, and g_F is an effective g -value given by [69]

$$g_F = g \frac{F(F+1) + S(S+1) - I(I+1)}{2F(F+1)} - \frac{\mu}{I\mu_B} \frac{F(F+1) - S(S+1) + I(I+1)}{2F(F+1)}. \quad (4.2)$$

The integrand of Eq. (4.1) shows that the magnetic dipole-dipole interaction is not isotropic. It may be either attractive or repulsive, depending on the orientation of the dipoles. It decreases as $1/r^3$, thus it is a long-distance force compared to the van der Waals interaction. It also enables conversion of spin angular momentum into orbital angular momentum, which makes possible the creation of vortices via spin dynamics [24, 52]. However, the spin dynamics driven by the magnetic dipole-dipole interaction can be prevented by exposing the condensate to a strong enough external magnetic field, which suppresses the conversion of spin angular momentum into orbital angular momentum [24, 25, 52]. In this case the spinor ξ can be assumed to be position independent and the dipole-dipole interaction energy becomes

$$E_{dd}[\psi] = \frac{\mu_0(g_F\mu_B)^2\langle\mathbf{F}\rangle_\xi^2}{8\pi} \int d\mathbf{r} n(\mathbf{r}) \int d\mathbf{r}' \frac{n(\mathbf{r}') [1 - 3(\hat{\mathbf{x}} \cdot \langle\hat{\mathbf{F}}\rangle_\xi)^2]}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (4.3)$$

where $\langle\hat{\mathbf{F}}\rangle = \langle\mathbf{F}\rangle/|\langle\mathbf{F}\rangle|$. The relative strength of the dipole-dipole interaction of different atoms can be approximated using Eqs. (4.2) and (4.3). If $I = 3/2$ and $S = 1/2$, we find that $g_{F=1} \approx -1/2$ and $g_{F=2} \approx 1/2$. For ^{52}Cr $I = 0$ and $S = F = 3$, so $g_{S=3} = g \approx 2$. In our calculation we have approximated $g \approx 2$ and $\mu/\mu_B \approx 0$. Thus, the maximal expectation values for $\langle\boldsymbol{\mu}\rangle_\xi^2$ are $\mu_B^2/4$, μ_B^2 , and $36\mu_B^2$ for $F = 1$, $F = 2$, and $S = F = 3$ systems, respectively. Therefore, the magnetic dipole-dipole interaction is considerably stronger in ^{52}Cr than in ^{23}Na or ^{87}Rb and one has to add the contribution (4.3) to the energy functional of Eq. (3.5). By looking at Eqs. (3.14) and (4.3) one sees that the effect of the dipole-dipole interaction is to change the value of γ_3 and possibly the direction of $\langle\mathbf{F}\rangle_\xi$. The change $\Delta\gamma_3$ in the value of γ_3 caused by the dipole-dipole interaction is

$$\Delta\gamma_3 = 2E_{dd}[\psi] / \left(\langle\mathbf{F}\rangle_\xi^2 \int d\mathbf{r} n(\mathbf{r})^2 \right). \quad (4.4)$$

In most experiments the trap is weak in one direction and strong in orthogonal directions, which produces a cigar-shaped condensate. If $\langle\mathbf{F}\rangle_\xi$ is parallel to the long axis of the trap, the cloud remains cigar shaped after the introduction of the magnetic dipole-dipole interaction [17, 44].

4.2 Ground states of spin-3 condensates

Soon after the creation of a chromium condensate [21] the ground states of spin-3 condensates were studied independently by two groups [16, 52]. In

these papers the zero-field ground states for arbitrary scattering lengths, as well as the ground states of chromium as a function of the magnetic field, were calculated. For chromium, the energy related to the presence of an external magnetic field is easy to calculate. In ^{52}Cr the nuclear spin vanishes, so in Eq. (2.12) $A = D = 0$. If the position independent magnetic field is given by $\mathbf{B} = B\hat{\mathbf{e}}_z$, the energy becomes simply

$$\begin{aligned} E_{mag}^{Cr}[\psi] &= -\int d\mathbf{r} \langle \boldsymbol{\mu} \rangle_{\xi} \cdot \mathbf{B} = -g\mu_B B \int d\mathbf{r} n(\mathbf{r}) \langle F_z \rangle_{\xi} \\ &= -g\mu_B B M_z, \end{aligned} \quad (4.5)$$

where in the lower equation we have used Eq. (3.22). If the spinor is spatially constant, M_z and E_{mag}^{Cr} are determined by $m = \langle F_z \rangle_{\xi} = M_z/N$. Henceforth we call m the magnetization. According to Eq. (3.15), the ground state spinors of a spin-3 condensate can be obtained by minimizing

$$\mathcal{E}_{int}^3[\xi] = b|\Theta(\mathbf{r})|^2 \pm \langle \mathbf{F} \rangle_{\xi}^2 + dO^2. \quad (4.6)$$

Here $b = \beta_3/|\gamma_3|$, $d = \delta_3/|\gamma_3|$, and $+$ ($-$) refers to positive (negative) γ_3 . In the energy $0 \leq |\Theta| \leq 1$ and $46 \leq O^2 \leq 85.5$. This equation has been obtained by including only the contact interaction. However, above we showed that the contribution from the dipole-dipole interaction can be taken into account by changing the value of γ_3 , that is, by changing b and d . In the presence of a magnetic field Eq. (4.6) has to be minimized assuming that M_z is conserved. This has been done in Paper IV for arbitrary values of b, d, γ_3 , and for some values of m . More specifically, m was either allowed to vary freely or it was fixed to be equal to 0.5, 1.5, or 2.5. Examples of the ground states of (4.6) are shown in Table 4.1, whereas in Fig. 4.1 we present two examples of the ground state phase diagrams.

Table 4.1: Some of the ground states of $S = 3$ condensate, both in zero field (free magnetization) and in non-zero magnetic field (magnetization conserved). Here m gives the value of the magnetization. In I_m a is a function of b, d , and m .

Phase	Spinor	$ \Theta ^2$	$\langle \mathbf{F} \rangle_\xi^2$	O^2
A	$(1, 0, 0, 0, 0, 0, 1)$	1	0	85.5
D	$(0, 1, 0, 0, 0, 1, 0)/\sqrt{2}$	1	0	48
F	$(0, 1, 0, 0, 0, 0, 0)$	0	4	46
FF	$(1, 0, 0, 0, 0, 0, 0)$	0	9	81
A_m	$(\sqrt{\frac{3+m}{6}}, 0, 0, 0, 0, 0, \sqrt{\frac{3-m}{6}})$	$1 - \frac{1}{9}m^2$	m^2	$85.5 - \frac{1}{2}m^2$
D_m	$(0, \sqrt{2+m}, 0, 0, 0, \sqrt{2-m}, 0)$	$1 - \frac{1}{4}m^2$	m^2	$48 - \frac{1}{2}m^2$
H_m	$(0, \sqrt{\frac{3+m}{5}}, 0, 0, 0, 0, \sqrt{\frac{2-m}{5}})$ $(0, 0, \sqrt{\frac{2+m}{5}}, 0, 0, \sqrt{\frac{1-m}{3}}, 0)$	0	m^2	$45 + (3 + m)^2$
I_m	$(\sqrt{a^2 + \frac{m}{6}}, 0, 0, \sqrt{1 - 2a^2},$ $0, 0, -\sqrt{a^2 - \frac{m}{6}})$	$0 - 1$	m^2	$46 - 85.5$

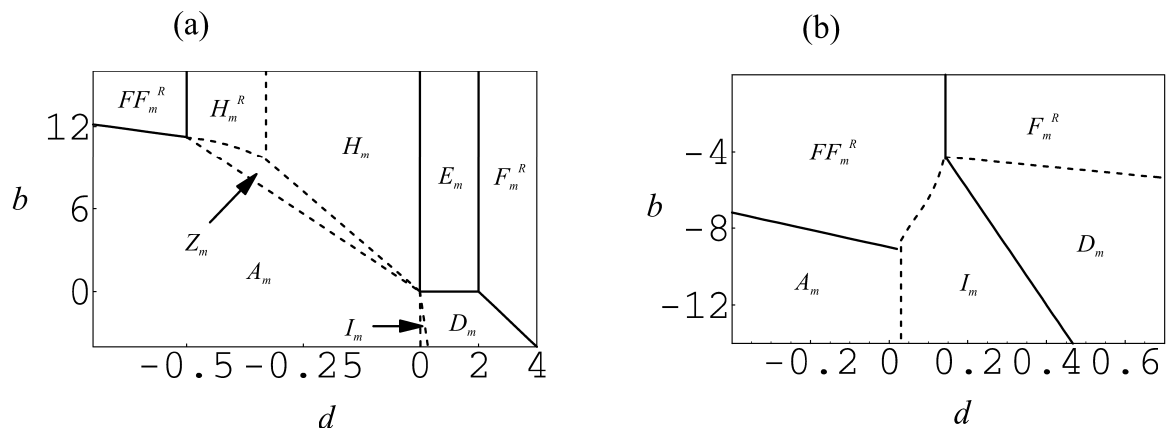


Figure 4.1: The ground state phase diagrams for $|m| = 1.5$. In (a) for $\gamma_3 > 0$ and in (b) for $\gamma_3 < 0$. In FF_m^R the superscript R means that the spinor is obtained from FF_m by a spin rotation. This holds also with F_m^R and H_m^R . In Z_m spinor all spin components are populated.

Chapter 5

Homotopy groups and topological defects

5.1 Order-parameter space

Bose-Einstein condensates of dilute gases have proved to be an excellent system to create and observe several interesting phenomena, such as topological defects. The existence of topological defects is based on the fact that a BEC can be described by an order-parameter. In general, an *order-parameter* is a continuous map $f : R \rightarrow M$, where $R \subset \mathbb{R}^3$ is some region of the physical space and M is the *order-parameter space*. Mathematically the order-parameter space M is a topological manifold which is in one-to-one correspondence with the possible values of the order-parameter. It is usually possible to associate the order-parameter space M with a Lie group G that acts on that space. The action $G \times M \rightarrow M$, $(g, x) \mapsto g \cdot x$ is denoted now simply by $g \cdot x$. If this action is transitive, i.e for every $x, x' \in M$ there exists some $g \in G$ for which $x' = g \cdot x$, we can choose some arbitrary element $x_{ref} \in M$ which we call the *reference order-parameter*¹. Every element $x \in M$ can then be obtained from x_{ref} by acting on it by a suitable element of G . Those elements of G which leave x_{ref} fixed constitute a subgroup $H_{x_{ref}}$ of G . This group is called the *isotropy group* (of x_{ref}) and it is explicitly given by

$$H_{x_{ref}} = \{g \in G \mid g \cdot x_{ref} = x_{ref}\}. \quad (5.1)$$

¹Note that the terminology is somewhat confusing. Earlier order-parameter was defined to be a map from the physical space into the order-parameter space, while here the term *reference order-parameter* refers to an element of the order-parameter space. However, because this terminology is well-established, we shall also use it here.

From the continuity of the action it follows that $H_{x_{ref}}$ is a closed subgroup of G . If G acts transitively on M the quotient space $G/H_{x_{ref}}$ can be identified with M . The elements of the quotient space are of the form $gH_{x_{ref}} = \{gh \mid h \in H_{x_{ref}}\}$, where gh denotes the element of G which is obtained by multiplying g and h . It is possible to define a multiplication $*$ in the space $G/H_{x_{ref}}$ via $(gH_{x_{ref}}) * (g'H_{x_{ref}}) = gg'H_{x_{ref}}$. The map $G/H_{x_{ref}} \rightarrow M$ identifying $G/H_{x_{ref}}$ and M is given by $gH_{x_{ref}} \mapsto g \cdot x_{ref}$. This map is a homeomorphism, that is, a continuous bijection the inverse of which is continuous. Homeomorphism is to topological spaces what isomorphism is to groups.

As an example of the above concepts we consider a system described by a normalized two-component vector with real components. We assume that the physical space is two dimensional and $R = \mathbb{R}^2$, so the order-parameter is a map $f : \mathbb{R}^2 \rightarrow S^1 = M$, where $S^1 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| = 1\} = \{\cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y \mid \theta \in [0, 2\pi)\}$. A group that acts transitively on S^1 is $U(1) = \{e^{i\tau} \mid \tau \in [0, 2\pi)\}$, the action can be defined via $e^{i\tau} \cdot (\cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y) = \cos(\theta + \tau) \mathbf{e}_x + \sin(\theta + \tau) \mathbf{e}_y$. If \mathbf{e}_x is the reference order-parameter, we find that $H_{\mathbf{e}_x} = \{1\}$ and the order-parameter space becomes $G/H_{x_{ref}} = U(1)/\{1\} = U(1)$.

5.2 Topological defects

Now we move on to discuss topological defects. We approach this topic through an example. Let $f_0, f_1 : \mathbb{R}^2 \rightarrow S^1$ be given, using polar coordinates, by $f_0(r, \varphi) = \mathbf{e}_y$ and $f_1(r, \varphi) = \cos \varphi \mathbf{e}_x + \sin \varphi \mathbf{e}_y$. These are illustrated in Fig. (5.1). The value of f_0 remains unchanged as one traverses around the dashed circular contour of Fig. 5.1 (a). However, if the same is done for the contour of Fig. 5.1 (b), the vector rotates through 2π . This holds true irrespective of the radius of the dashed circle. Because f_1 does not approach a constant value as the radius of the circle shrinks towards zero, it cannot be continuous at the origin of \mathbb{R}^2 . It is intuitively clear that the configurations of Figs. 5.1 (a) and (b) cannot be deformed into one other continuously. This is characterized by saying that f_0 and f_1 have a different winding number. In the present case the winding number indicates how many times the two-component vector rotates counterclockwise through 2π as the circular contour is traversed around counterclockwise. Thus the winding number of f_0 is 0 and that of f_1 is 1. An important property of the configurations of Fig. (5.1) is that the winding number can be determined arbitrarily far from the actual location of the possible singularity of the

order-parameter. This holds true for all singular topological defects.

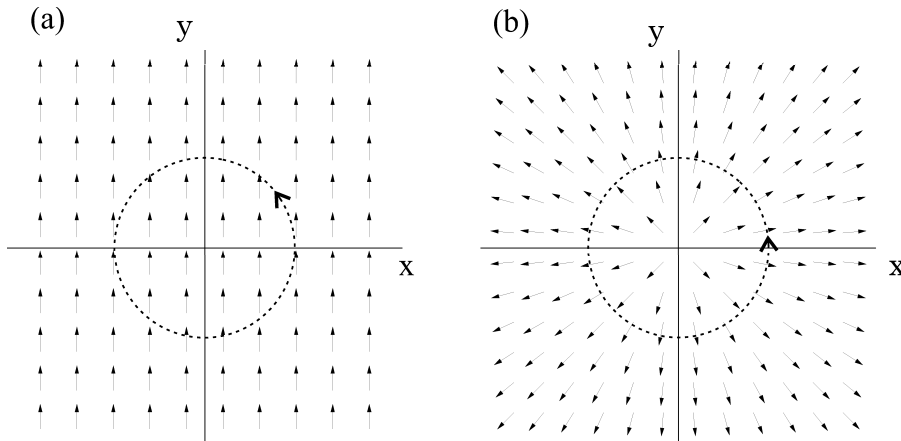


Figure 5.1: In (a) a graphical representation of f_0 and in (b) of f_1 . We show both (part of) the physical space \mathbb{R}^2 and the values of the order-parameter in the same figure. The order-parameter remains unchanged if one traverses around the dashed line of figure (a). If the same is done for (b) in a counterclockwise direction, the order-parameter rotates through 2π , also in the counterclockwise sense.

The question whether one map is or is not continuously deformable into another plays a central role in the theory of defects. Two maps of a given space into another space are said to be homotopic if one can be continuously deformed into the other. The precise definition of a homotopy is as follows. Let A, B be topological spaces and $f, g : A \rightarrow B$ be continuous maps. Then f and g are *homotopic* if there exists a continuous map $F : [0, 1] \times A \rightarrow B$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in A$. The map F is called a *homotopy*. The concept of homotopy is fundamental to the theory of topological defects, as it can be said that an order-parameter $f : R \rightarrow M$ describes a topological defect if and only if f is not homotopic to a constant map $f_{cons} : R \rightarrow M$. For the latter $f_{cons}(r) = x_0 \in M$ holds for all $r \in R$. Configurations (order-parameters) which are not homotopic to f_{cons} are called *topologically stable*. On the other hand, if a configuration is homotopic to f_{cons} , it is said to be *topologically trivial*. Physically, the deformation of a topologically stable configuration into a topologically trivial one typically requires considerably energy. In this sense topologically stable defects are also energetically stable. However, it must be stressed that topologically trivial configurations are not necessarily physically unstable. To investigate the latter it is necessary to know the energy functional of the

system. If it is impossible to get from one configuration to the other without passing through configurations of higher energy, then a topologically trivial configuration may be physically metastable.

Whether or not topologically stable defects are possible depends on the structure of the order-parameter space M . This can be analyzed by calculating the homotopy groups of M .

5.3 Homotopy groups

By examining the properties of the order-parameter space one can see what kind of, if any, topological defects are possible. This examination can be carried out with the help of the homotopy groups of the order-parameter space. The use of the theory of homotopy groups to characterize the topological defects of physical systems was first used during the late 1950s [20], but started to gain wider attention only in the 1970s, see e.g. [27, 29, 30, 37, 38, 42, 55, 59, 68]. Since then it has been applied successfully in several fields of physics, such as condensed matter physics, particle physics, and cosmology [37, 49, 51, 60, 64, 65, 66, 67]. We will now give definitions of the homotopy groups and relative homotopy groups and state some of their properties, without giving any proofs. The following discussion is very dense in order to keep this introduction compact enough. A good book discussing the mathematics of homotopy groups is [22], while a more physical insight can be found for example in [37]. The latter has become a classic text in the field. The discussion below is based on these two sources.

5.3.1 First homotopy group

As the name suggests, homotopy groups classify maps which can be continuously deformed into one another. We start by defining the first homotopy group, or the fundamental group, as it is also called. From now on we assume that all maps are continuous. Let $f, g : [0, 1] \rightarrow M$ be maps such that $f(0) = g(0) = f(1) = g(1) = x_0$. We say that f, g are *loops based at* x_0 . Because $f(0) = f(1)$, f (and g) can be considered to be a map from S^1 into M . f and g are said to be *homotopic at* x_0 if they are homotopic and the homotopy $F : [0, 1] \times [0, 1] \rightarrow M$ is such that $F(t, 0) = F(t, 1) = x_0$ for all $t \in [0, 1]$. This means that F determines a loop based at x_0 for each

fixed $t \in [0, 1]$. The multiplication of f and g can be defined through

$$\begin{aligned} f \circ g(t) &= f(2t), & 0 \leq t \leq 1/2, \\ &= g(2t - 1), & 1/2 \leq t \leq 1. \end{aligned} \tag{5.2}$$

Clearly $f \circ g$ is a loop based at x_0 , and it is obtained by traversing first f and then g . If h is a loop based at x_0 , we define $[h]$ to be the set of all loops which are homotopic at x_0 to h and call $[h]$ the *homotopy class* of h . It can be shown that homotopy defines an equivalence relation among the loops based at x_0 , thus $[h]$ is an equivalence class. The multiplication of two homotopy classes can be defined as $[f] \tilde{\circ} [g] = [f \circ g]$. With this multiplication law the set of homotopy classes of loops at x_0 acquires group structure. This group is *the first homotopy group based at x_0* and is denoted by $\pi_1(M, x_0)$. A topological space M is said to be path-connected if for all $x, y \in M$, there is a continuous map $f_{xy} : [0, 1] \rightarrow M$ such that $f_{xy}(0) = x$ and $f_{xy}(1) = y$. If M is path-connected, $\pi_1(M, x)$ and $\pi_1(M, y)$ are isomorphic for all $x, y \in M$. There is thus a single abstract group, $\pi_1(M)$, of which the based homotopy groups are isomorphic copies. If $\pi_1(M)$ contains only the identity element, all loops are homotopic to a constant map. Otherwise there can be stable topological defects characterized by $\pi_1(M)$. If the physical space is two dimensional, as in our example, these defects are point-like singularities.

5.3.2 Higher homotopy groups

We now move on to discuss the definition of the n :th homotopy group. Every M can be expressed as a union $M = M_0 \cup M_1 \cup M_2 \cup \dots$, where each M_i is such that the only path-connected subset of M containing M_i is M_i itself. M_i 's are called the path components of M . Let I^n be the n -dimensional unit cube, the product of n copies of the interval $[0, 1]$. The boundary ∂I^n of I^n is the subspace consisting of points with at least one coordinate equal to 0 or 1. We define the *n :th homotopy group of M at x_0* , $\pi_n(M, x_0)$, to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \rightarrow (M, x_0)$, where the notation means that I^n is taken to M and ∂I^n to x_0 by f . The homotopies are required to be such that the image of the boundary ∂I^n is always x_0 .² This definition extends to the case $n = 0$ by taking I^0 to be a point and ∂I^0 to be empty, so $\pi_0(M)$ is just the set of path components

²There is an alternative definition of $\pi_n(M, x_0)$ as the set of equivalence classes of maps $(D^n, S^{n-1}) \rightarrow (M, x_0)$. Here $D^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$ is the n -dimensional disk and $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\}$ is the n -sphere. Clearly S^{n-1} is the boundary of D^n .

of M . In general there is no natural group structure for such a set, but if M is a group G , then π_0 can be given the group structure of G/G_0 , where G_0 is the path component containing the identity element of G . For $n \geq 2$, a product of two maps, generalizing the composition operation in π_1 , is defined by

$$\begin{aligned} f \circ g(t_1, t_2, \dots, t_n) &= f(2t_1, t_2, \dots, t_n), & 0 \leq t_1 \leq 1/2, \\ &= f(2t_1 - 1, t_2, \dots, t_n), & 1/2 \leq t_1 \leq 1. \end{aligned} \quad (5.3)$$

The multiplication of two equivalence classes is defined as for π_1 . If M is path-connected, $\pi_n(M, x)$ and $\pi_n(M, y)$ are isomorphic for all $x, y \in M$, and the notation $\pi_n(M)$ is used to denote any $\pi_n(M, x)$. While $\pi_n(M)$ is a commuting group for $n \geq 2$, $\pi_1(M)$ may be non-commuting.

In the physical applications of the theory of homotopy groups the order-parameter space is expressed as $M = G/H$, where G is a Lie group and H is a closed subgroup of G . We denote the path component of H containing the identity element by H_0 . It is a normal subgroup of H and the quotient space H/H_0 is a group. Now we can define $\pi_0(H, e) = H/H_0$. The path components $\{H_k \mid k = 0, 1, 2, \dots\}$ of H are cosets of H in H_0 , that is $H_k = h_k H_0$ for some $h_k \in H$.

It can be shown that there is an exact sequence of homomorphism between homotopy groups. Exactness means that the image of a map is the kernel of the following map. The exact sequence is given by

$$\xrightarrow{\alpha_n} \pi_n(G_0) \xrightarrow{\beta_n} \pi_n(G/H) \xrightarrow{\gamma_n} \pi_{n-1}(H_0) \xrightarrow{\alpha_{n-1}} \pi_{n-1}(G_0) \xrightarrow{\beta_{n-1}} \dots \quad (5.4)$$

If $\pi_1(G) = \pi_2(G) = \{e\}$ and G is path-connected, i.e., $\pi_0(G) = \{e\}$, γ_1 and γ_2 become isomorphisms, and we obtain the following theorem

Theorem 1. *Let G be a Lie group with $\pi_0(G) = \pi_1(G) = \pi_2(G) = \{e\}$, let $H \subseteq G$ be a closed subgroup of G , and let $H_0 \subseteq H$ be the path component containing the identity element of H . Then there are isomorphisms*

$$\begin{aligned} \pi_1(G/H) &\cong H/H_0, \\ \pi_2(G/H) &\cong \pi_1(H_0). \end{aligned} \quad (5.5)$$

5.4 Physical interpretation

We have now given the mathematical definition of homotopy groups, but their physical meaning is yet to be clarified. Let $\psi : \mathbb{R}^d \rightarrow M$ be the order-parameter of some physical system. In the physical applications of the

homotopy groups the maps $S^n \rightarrow M$ determining the elements of $\pi_n(M)$ are often obtained as a restriction of the order-parameter to S^n . For example, if $d = 3$ and cylindrical coordinates are used, one obtains a map $\psi|_{r,z} : S^1 \rightarrow M$ from ψ by defining $\psi|_{r,z}(\varphi) = \psi(r, z, \varphi)$. Here r, z have fixed values. If two defects, characterized by $g_1, g_2 \in \pi_n(M)$, are combined, the element of $\pi_n(M)$ characterizing the resulting defect is obtained by multiplying g_1 and g_2 . Because $\pi_n(M)$ is a commuting group for $n \geq 2$, the resulting vortex does not depend on the way the vortices are combined. On the other hand, $\pi_1(M)$ can be a non-commuting group. In this case defects are characterized by the conjugacy classes of the first homotopy group. Defects can still be labelled by the elements of $\pi_1(M)$, but if these elements belong to the same conjugacy class, corresponding defects can be continuously deformed to one another. However, if they belong to different conjugacy classes this is not possible.

The type of the defects classified by $\pi_1(M)$ depends on the dimension of the physical space. In three-dimensional space it classifies one dimensional defects, that is, *vortices*. In our example of the vectorial two-component order-parameter the physical space was two dimensional and the defects characterized by the first homotopy group were point-like defects, *monopoles*. While these defects appear as singularities in the order-parameter, non-singular defects can also be characterized by $\pi_1(M)$. In these the order-parameter ψ is everywhere continuous and the topological stability follows from the boundary conditions imposed on ψ . Let the physical space be three-dimensional and $\psi : \mathbb{R}^3 \rightarrow M$ be an order-parameter. If ψ_0 is some element of M and ψ is such that $\lim_{x \rightarrow \pm\infty} \psi(x, y, z) = \psi_0$ for all $y, z, \in \mathbb{R}$, then ψ can be considered to be a map from S^1 to M . As before, ψ describes a topologically stable defect if it determines an element of $\pi_1(M)$ other than the identity element. Because ψ approaches a constant value far from a plane, these defects are called *planar solitons*. If one relaxes the boundary conditions on ψ , defects can be continuously deformed into a uniform configuration.

In three-dimensional physical space $\pi_2(M)$ classifies monopoles, but in lower dimensions it does not characterize any singular defects. It also classifies non-singular defects called *linear solitons*. These are analogous to planar solitons, except that now the order-parameter approaches a constant value far from a line.

Finally, $\pi_3(M)$ can be used in the study of *particle-like solitons*, or *skyrmions*, as they are also called. These are non-singular defects which live in three-dimensional physical space and which are characterized by an

order-parameter which becomes position independent far from a point.

The calculation of $\pi_1(M)$ and $\pi_2(M)$ can be summarized as follows.

1. Find a group G for which $\pi_0(G) = \pi_1(G) = \pi_2(G) = \{e\}$ and which acts transitively on the order-parameter space M .
2. Pick some arbitrary element $x_{ref} \in M$, the reference order-parameter.
3. Calculate the isotropy group $H_{x_{ref}}$. The order-parameter space can then be written as $M = G/H_{x_{ref}}$.
4. Determine $H_{x_{ref}0}$, the path component of the identity of $H_{x_{ref}}$.
5. Calculate $H_{x_{ref}}/H_{x_{ref}0}$ and $\pi_1(H_{x_{ref}0})$.
6. Use the isomorphisms $\pi_1(G/H_{x_{ref}}) \cong H_{x_{ref}}/H_{x_{ref}0}$ and $\pi_2(G/H_{x_{ref}}) \cong \pi_1(H_{x_{ref}0})$ to obtain the first and second homotopy groups.

Chapter 6

Topological defects of Bose-Einstein condensates

In a BEC, a vortex appears as a long-lived line-like singularity in the particle density. In a non-rotating trap, a vortex state cannot be the ground state of the system, but its decay is prevented by topological reasons. The continuous deformations of the order-parameter needed in order to reach the ground state require more energy than is available from e.g. thermal excitations. In the presence of dissipation a vortex can move to the boundary of the condensate and vanish, but even then it is stable as long as it stays in the condensate.

6.1 On the order-parameter spaces of BECs

Before embarking on the study of the topological defects of Bose-Einstein condensates, we briefly discuss the order-parameter spaces of these systems. For a single-component BEC the order-parameter can be written as $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}e^{i\theta(\mathbf{r})}$. If we assume that the density n is constant, the order-parameter space becomes $U(1) = S^1$, since it consists of the possible values of θ . In principle the order-parameter space should be written as $M' = (\mathbb{R}_+ \times M) \cup \{0\}$, where \mathbb{R}_+ is the set of real numbers larger than zero giving the possible values of the square root of the density, and M gives the order-parameter space related to the density-independent part of the order-parameter. For a single component condensate $M = U(1)$ and for a spinor condensate it consists of the possible values of the spinor. The point $\{0\}$ denotes the case where the density is zero. One sees that if M' is the order-parameter space, there are no topologically stable defects. Any

order-parameter can be converted into any other order-parameter via deformations which reduce the density to zero in an appropriate region of the physical space. From a physical point of view this is unlikely to happen, since reducing density to zero is not energetically favourable. At those points where the order-parameter is zero, the medium is in a higher energy state than at points where the order-parameter is non-zero. In principle the energy needed for making the density vanish could come, for example, from thermal excitations, but in practise this is unlikely to happen. Thus one can ignore the zero of density. Then the order-parameter space becomes $\mathbb{R}_+ \times M$. For this $\pi_n(\mathbb{R}_+ \times M) = \pi_n(\mathbb{R}_+) \times \pi_n(M) = \pi_n(M)$, since $\pi_n(\mathbb{R}_+) = \{e\}$. Thus, from the point of view of topological defects, it is enough to study the structure of M only.

6.2 Single component Bose-Einstein condensate

Now $M = S^1$ and $\pi_1(M) = \mathbb{Z}$, thus vortices with an arbitrary winding number are possible. Using the method presented in Paper III it can be shown that a vortex with winding number m has the general form

$$\psi_m(r, z, \varphi) = \sqrt{n(r, z, \varphi)} e^{i\theta_m(r, z, \varphi)}, \quad (6.1)$$

where $\theta_m(r, z, 0) = 0, \theta_m(r, z, 2\pi) = 2\pi m$. One sees that the superfluid velocity, defined by $\mathbf{v} = \frac{\hbar}{M} \nabla \theta$, cannot vanish everywhere if there is a vortex in the system. The continuity of the order-parameter requires that $n(0, z, \varphi) = 0$. If the condensate is in a trap which is cylindrically symmetric with respect to the vortex axis, the density can also be assumed to be cylindrically symmetric. In this case the angular momentum associated with the vortex is $\mathbf{L} = Nm\hbar \hat{e}_z$. Furthermore, in the minimum energy state the superfluid velocity in radial and z -direction vanishes, and thus θ is a function of φ only. Under this assumption the smallest kinetic energy is obtained when $\theta_m(r, z, \varphi) = m\varphi$. The order-parameter becomes

$$\psi_m(r, z, \varphi) = \sqrt{n(r, z)} e^{im\varphi}, \quad (6.2)$$

which is the form usually encountered in the literature.

6.3 Mixtures of Bose-Einstein condensates

Now the order-parameter can be written as

$$\begin{aligned}\psi(\mathbf{r}) &= \begin{pmatrix} \psi_1(\mathbf{r}) \\ \psi_2(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \sqrt{n_1(\mathbf{r})}e^{i\theta_1(\mathbf{r})} \\ \sqrt{n_2(\mathbf{r})}e^{i\theta_2(\mathbf{r})} \end{pmatrix} \\ &= \sqrt{n(\mathbf{r})} \begin{pmatrix} \cos \alpha(\mathbf{r})e^{i\theta_1(\mathbf{r})} \\ \sin \alpha(\mathbf{r})e^{i\theta_2(\mathbf{r})} \end{pmatrix},\end{aligned}\tag{6.3}$$

where $n = n_1 + n_2$, $\cos \alpha = \sqrt{n_1/n}$, and $\sin \alpha = \sqrt{n_2/n}$. We first assume that the total number of particles is conserved, but the number of particles in each condensate can change. Then the order-parameter space becomes $M = S^3$. For this $\pi_0(S^3) = \pi_1(S^3) = \pi_2(S^3) = \{e\}$. However, $\pi_3(S^3) = \mathbb{Z}$, which shows that skyrmions can exist. At this point it is important to note that the energy functional of Eq. (2.9) is in general invariant only in phase transformations of ψ_1 and ψ_2 and therefore the order-parameter space, consisting of the ground states, is $S^1 \times S^1$ [39]. Taking the order-parameter space to be S^3 means that also those values of ψ which cannot be ground states of the energy are allowed. Skyrmions have been studied both in systems where only the total number of particles is conserved [26, 50] and in systems where the particle number of each condensate is conserved separately [7, 53]. The existence of skyrmions in the latter system is explained by the fact that even in the presence of a skyrmion the requirement of the particle conservation of each condensate can be fulfilled, at least for some particle numbers. A similar situation is met in Paper III, where a vortex of a ferromagnetic $F = 1$ condensate is found to be possible both in the absence and presence of an external magnetic field.

If we consider only the ground states of Eq. (2.9), the order-parameter space becomes $S^1 \times S^1$. Because $\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$, vortices with an arbitrary winding number can exist. The vortex configurations are given by order-parameters where each condensate has a vortex of the form (6.1). In particular, it is possible to have a vortex in one condensate and a non-vortex state in another. In this case the total density $n = n_1 + n_2$ can be everywhere non-zero, even if there is a vortex in the system. This has also been seen in an experiment, where a vortex was created in a one spin component, while the other component was vortex free [36]. As far as the author can tell, there have been no experiments with a vortex in both spin components simultaneously.

The present analysis can straightforwardly be extended to the case of a coherent mixture of k Bose-Einstein condensates. If the particles can

convert into one another and also those values of ψ which are not necessarily ground states are allowed, the order-parameter space becomes S^{2k-1} . The only restriction for the order-parameter is $\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) = n(\mathbf{r})$. Below we show that if $k > 1/2$ topologically stable defects are not possible. If we choose the order-parameter space to be the set of ground states, the former becomes $S^1 \times S^1 \times \dots \times S^1$, where S^1 appears k times. Hence vortices can exist. Vortex configurations are those with a vortex of Eq. (6.1) in at least in one of the condensates.

6.4 Spinor condensates

For a spinor condensate the order-parameter is $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\xi(\mathbf{r})$. The order-parameter space is set of the possible values of the spinor ξ . If the normalization of ξ is the only restriction, the order-parameter space becomes S^{4F+1} . This space allows for no topological defects characterized by π_n with $n = 0, 1, 2$ or 3 . This follows from the fact that $\pi_n(S^{4F+1}) = \{e\}$ for $n = 0, 1, 2, 3$ and $F = 1, 3/2, 2, 5/2, \dots$ (for spinor condensates F is an integer, but for multicomponent condensates in general it can also be a half-integer). Instead of taking into consideration all possible values of the spinor, we study what kind of defects can exist if ξ is assumed to belong to the set of spinors minimizing the interaction energy for some given scattering lengths. For example, in a ferromagnetic condensate of $F = 1$ atoms this set can consist of all spinors that can be obtained from $|F = 1, m_F = 1\rangle$ by a spin rotation. However, defects which are stable in this set can be continuously destroyed by taking into use the whole order-parameter space S^5 . Therefore, they can be topologically stable only because of the energy barrier preventing the use of the whole order-parameter space. Thus far no-one has studied whether the decay of topological defects is energetically possible in this way in spinor condensates.

In the absence of an external magnetic field a group that acts transitively on the set of ground state spinors is $G = U(1) \times SO(3)$. However, because we want to use the Theorem 1 of Chapter 5, we use instead of $U(1) \times SO(3)$ its covering group $\mathbb{R} \times SU(2)$. This group acts on the set of spinors via equation $(x, U) \cdot \xi = e^{ix} D^{(F)}(U)\xi$, where $(x, U) \in \mathbb{R} \times SU(2)$ and $D^{(F)}$ is the $2F + 1$ dimensional irreducible representation of $SU(2)$. The elements of $SU(2)$ and corresponding $(2F + 1)$ -dimensional representation

matrices can be written either as

$$U(\alpha, \beta, \gamma) = e^{-i\frac{\alpha\sigma_z}{2}} e^{-i\frac{\beta\sigma_y}{2}} e^{-i\frac{\gamma\sigma_z}{2}} \mapsto D^{(F)}(\alpha, \beta, \gamma) = e^{-i\alpha F_z} e^{-i\beta F_y} e^{-i\gamma F_z}, \quad (6.4)$$

or as

$$\tilde{U}(\tau, \theta, \varphi) = e^{-i\tau \mathbf{n}(\theta, \varphi) \cdot \boldsymbol{\sigma}/2} \mapsto \tilde{D}^{(F)}(\tau, \theta, \varphi) = e^{-i\tau \mathbf{n}(\theta, \varphi) \cdot \mathbf{F}}, \quad (6.5)$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli σ -matrices, F_x, F_y, F_z are the spin operators of a spin- F system, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, and $\mathbf{n}(\theta, \varphi) = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$.

As an example of the calculation of topological defects we consider the ferromagnetic phase of an $F = 1$ condensate. As shown in Paper I, in this case, instead of $\mathbb{R} \times SU(2)$, we can use $SU(2)$. The reference order-parameter can be chosen to be $|1, 1\rangle = |F = 1, m_F = 1\rangle$. The isotropy group $H_{|1,1\rangle}$ is found by solving the equation $\tilde{D}^{(1)}(\tau, \theta, \varphi)|1, 1\rangle = |1, 1\rangle$ (or $D^{(1)}(\alpha, \beta, \gamma)|1, 1\rangle = |1, 1\rangle$), and it is $H_{|1,1\rangle} = \{\mathbb{I}, -\mathbb{I}\}$. Clearly the path component of the identity is $H_{|1,1\rangle,0} = \{\mathbb{I}\}$, so using Theorem 1 we obtain $\pi_1(G/H_{|1,1\rangle}) \cong \{\pm\mathbb{I}\}$ and $\pi_2(G/H_{|1,1\rangle}) \cong 0$. Thus one non-trivial vortex is possible, but topologically stable monopoles cannot exist. The order-parameter space is $G/H_{|1,1\rangle} = SU(2)/\{\pm\mathbb{I}\} = SO(3)$. This order-parameter space has also been encountered in ${}^3\text{He} - A$ [42, 65]. In Paper III it is shown that a vortex characterized by the element hH_0 of H/H_0 can be obtained by finding a map f from the physical space to G which is such that $f(r, z, \varphi = 0) = e$ and $f(r, z, \varphi = 2\pi) \in hH_0$. In the present case we can define $f(r, z, \varphi) = e^{-i\tau \mathbf{n}(\alpha, \beta) \cdot \mathbf{F}}$, where $\tau(r, z, 0) = 0$, $\tau(r, z, 2\pi) = 2\pi$, and α and β are arbitrary functions of position. The vortex becomes $\psi = \sqrt{n} e^{-i\tau \mathbf{n} \cdot \mathbf{F}} |1, 1\rangle$, which is explicitly

$$\psi(r, z, \varphi) = \sqrt{n(r, z, \varphi)} \begin{pmatrix} (\cos \frac{\tau}{2} - i \cos \beta \sin \frac{\tau}{2})^2 \\ -\sqrt{2} e^{i\alpha} \sin \beta \sin \frac{\tau}{2} (i \cos \frac{\tau}{2} + \cos \beta \sin \frac{\tau}{2}) \\ -e^{2i\alpha} \sin^2 \frac{\tau}{2} \sin^2 \beta \end{pmatrix}. \quad (6.6)$$

Now $n(r = 0, z, \varphi) = 0$ in order to keep the order-parameter well defined at $r = 0$. The superfluid velocity is defined via

$$\mathbf{v} = -i \frac{\hbar}{M} \xi^\dagger \nabla \xi. \quad (6.7)$$

When the vortex has the smallest possible energy the velocity in radial and z -directions can be assumed to vanish and thus the spinor is a function of φ only. Additionally, in a cylindrically symmetric trap the density can be

assumed to be cylindrically symmetric. Using these assumptions the Euler-Lagrange equations obtained from the energy functional show that the vortex energy is minimized when $\alpha = \text{constant}$, $\beta = \frac{\pi}{2}$ and $\tau(r, z, \varphi) = \varphi$. A simple calculation shows that when the energy is minimized the superfluid velocity vanishes. Thus, in contrast to a single-component condensate, in a spinor condensate the existence of a vortex does not have to lead to a non-zero superfluid velocity. The same phenomenon can be seen in the orbital angular momentum, which also vanishes in the ground state. The condensate does not have to contain any orbital angular momentum although there is a vortex in it.

We have briefly described the calculation of homotopy groups of a ferromagnetic $F = 1$ condensate. The calculation of the homotopy groups of other phases of $F = 1$ and $F = 2$ condensates proceeds in a similar manner. This calculation can be found in Paper I, where, among other things, it is shown that the first homotopy group of the cyclic phase of $F = 2$ condensate turns out to be a non-commuting group. Topologically stable vortices are classified by the conjugacy classes of this group and are those in which the spinor is suitably rotated and its phase changed by an integer multiple of $\pi/3$ as the defect line is encircled. In a system with non-commuting first homotopy group the vortices can be classified further using homology groups [29, 60, 61]. In the presence of other line singularities it may be possible to transform two line defects described by different conjugacy classes into one another. This is achieved by splitting a defect into two parts and combining these beyond a suitable line defect. Elements of $\pi_1(M)$ can be grouped into sets in such a way that defects described by elements in the same set can be deformed to one another either continuously or using the splitting and recombination method. Performing this grouping reveals that vortices with the same winding number can now be deformed into one another. The results of Paper I are summarized in Table 6.1.

The order-parameter space of an antiferromagnetic phase of an $F = 1$ system was found to be $[U(1) \times SO(3)]/O(2)_{G+S}$. This space can also be written as $[U(1) \times SO(3)]/Z_2$, although in Papers I and II it was stated otherwise. The latter form for the order-parameter space was first presented in [72]. The order-parameter space and defects of the polar $F = 2$ phase were not discussed in Paper I. This was a consequence of paper [63], where it was discovered that $U(1) \times SO(3)$ does not act transitively on the set of polar spinors. It seemed that the symmetry group of the energy is larger than $SO(3)$. However, recently it has been shown that due to thermal

Table 6.1: The order-parameter spaces and first and second homotopy groups of the ground states of spinor condensates with $F = 1$ and $F = 2$ (for the ground state spinors, see Table 3.1). Now the physical space is three-dimensional, so π_1 characterizes vortices and π_2 monopoles. I denotes the trivial group consisting of the identity element alone.

	Phase	M	$\pi_1(M)$	$\pi_2(M)$
$F = 1$	ferromagnetic	$SO(3)$	Z_2	I
	antiferrom./polar	$\frac{U(1) \times SO(3)}{O(2)_{G+S}} \approx \frac{U(1) \times S^2}{Z_2}$	\mathbb{Z}	\mathbb{Z}
$F = 2$	ferromagnetic	$SO(3)/Z_2$	\mathbb{Z}_4	I
		$SO(3)$	\mathbb{Z}_2	I
	cyclic	$\frac{U(1) \times SO(3)}{H_{cyclic}}$	Non-comm.	I
	antiferrom./polar	—	—	—

and quantum fluctuations this accidental continuous degeneracy is lifted [56, 62], after which the possible ground state spinors are $|2, 0\rangle$ and $(|2, 2\rangle + |2, -2\rangle)/\sqrt{2}$. Similar degeneracy has been reported also in [34, 38]. Here it should also be mentioned that in Paper III it was claimed that the order-parameter space of the cyclic phase of an $F = 2$ condensate consists of two disconnected sets. In Ref. [71] it was shown that this is not true, but the order-parameter space is a connected set. The incorrect claim resulted from an erroneous form of the spin rotation matrix which lead the author to believe that $(|2, 2\rangle + |2, -1\rangle)/\sqrt{3}$ and $(|2, 2\rangle + \sqrt{2}|2, 0\rangle - |2, -2\rangle)/2$ cannot be obtained from each other by a spin rotation.

Chapter 7

Decomposition of pure spin states

In 1932 E. Majorana showed that it is possible to express the spin state of a spin- S particle in terms of $2S$ spin states of spin-1/2 particles [35]. Therefore, because every pure spin state of spin-1/2 particle can be given in terms of a point on the Bloch sphere, every pure spin state of a spin- S particle can be characterized by $2S$ points on this sphere. Majorana used this description to study the motion of a spin- S particle in a magnetic field. In his paper the emphasis was put on the application of this decomposition, while its derivation was very short. This, together with the fact that Majorana's paper was not well known at the time, led F. Bloch and I. Rabi to derive his result in more detail in 1945 [8]. As this decomposition is needed in the next chapter, we shall now show how it is obtained. Our derivation is based on [8], although the notation and details of the calculation are different.

7.1 Majorana decomposition

In this chapter \mathbf{s} is the spin operator of a spin-1/2 particle. Let \mathcal{H} the Hilbert space related to a spin-1/2 particle. Because now we consider the spin states alone, we can exclude the orbital part and define $\mathcal{H} = \mathbb{C}^2$. For $2S$ identical spin-half particles the relevant Hilbert space is $\otimes_1^{2S} \mathcal{H} = \otimes_1^{2S} \mathbb{C}^2$. Let $\{|+\rangle, |-\rangle\}$ be the basis of \mathbb{C}^2 . Then the basis vectors of the 2^{2S} dimensional vector space $\otimes_1^{2S} \mathbb{C}^2$ are of the form

$$|\gamma_1\rangle_1 |\gamma_2\rangle_2 \cdots |\gamma_{2S}\rangle_{2S} = |\gamma_1\rangle_1 \otimes |\gamma_2\rangle_2 \otimes \cdots \otimes |\gamma_{2S}\rangle_{2S}, \quad (7.1)$$

where each γ_i is either $+$ or $-$. Our intention is to write the spin states of spin- S particles by means of $2S$ spin states of spin-1/2 particles. To achieve this, we write the spin operator of a spin- S system as

$$\mathbf{S} = \sum_{k=1}^{2S} \mathbf{S}_k, \quad (7.2)$$

where $\mathbf{S}_k = \mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_{k-1} \otimes \mathbf{s} \otimes \mathbb{I}_{k+1} \otimes \cdots \otimes \mathbb{I}_{2S}$. The operators $\mathbf{S}^2, S_z, \mathbf{S}_i^2$ are mutually commutative and hence they possess a complete set of common eigenvectors. We denote by $|S, M\rangle$ a vector for which $\mathbf{S}^2|S, M\rangle = S(S+1)|S, M\rangle$ and $S_z|S, M\rangle = M|S, M\rangle$. The vector $|S, S\rangle$ can be written in terms of the basis vectors of $\otimes_1^{2S} \mathbb{C}^2$ simply as

$$|S, S\rangle = |+\rangle_1 |+\rangle_2 \cdots |+\rangle_{2S}. \quad (7.3)$$

The lowering operator is $S_- = \sum_{k=1}^{2S} S_{k-}$, where $S_{k-} = \mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_{k-1} \otimes |-\rangle_k \langle +| \otimes \mathbb{I}_{k+1} \otimes \cdots \otimes \mathbb{I}_{2S}$. If we act on $|S, S\rangle$ with this operator we find that

$$|S, S-1\rangle = \frac{S_-}{\sqrt{2S}} |S, S\rangle = \frac{1}{(2S-1)! \sqrt{2S}} \sum_{P_{\pm}} |+\rangle_1 \cdots |+\rangle_{2S-1} |-\rangle_{2S}, \quad (7.4)$$

where $\sum_{P_{\pm}}$ denotes the sum over all different permutations of the \pm -signs. Generally we obtain

$$|S, S-r\rangle = \frac{1}{(2S-r)! r!} \sqrt{\frac{(2S-r)! r!}{(2S)!}} \sum_{P_{\pm}} \prod_{k=1}^{2S-r} |+\rangle_k \prod_{k=2S-r+1}^{2S} |-\rangle_k. \quad (7.5)$$

Let $\varphi_k = \alpha_k |+\rangle_k + \beta_k |-\rangle_k$ be the spin vector of the spin-1/2 particle labelled by k . We define

$$\begin{aligned} \xi' &= \frac{A}{(2S)!} \sum_{P_{\pm}} \bigotimes_{k=1}^{2S} \varphi_k \\ &= \frac{A}{(2S)!} \sum_{P_{\pm}} \prod_{k=1}^{2S} (\alpha_k |+\rangle_k + \beta_k |-\rangle_k), \end{aligned} \quad (7.6)$$

where A is an arbitrary non-zero complex number. The product appearing in (7.6) can be written as

$$\prod_{k=1}^{2S} (\alpha_k |+\rangle_k + \beta_k |-\rangle_k) = \sum_{r=0}^{2S} \frac{1}{(2S-r)! r!} \sum_{P_k} \prod_{k=1}^{2S-r} \alpha_k |+\rangle_k \prod_{k=2S-r+1}^{2S} \beta_k |-\rangle_k \quad (7.7)$$

where \sum_{P_k} denotes the sum over all permutations of the index k . Notice that \sum_{P_k} and $\sum_{P_{\pm}}$ are different operations. Using this Eq. (7.6) becomes

$$\begin{aligned}\xi' &= \frac{A}{(2S)!} \sum_{r=0}^{2S} \sum_{P_k} \sum_{P_{\pm}} \frac{1}{(2S-r)!r!} \prod_{k=1}^{2S-r} \alpha_k |+\rangle_k \prod_{k=2S-r+1}^{2S} \beta_k |-\rangle_k \\ &= \frac{A}{(2S)!} \sum_{r=0}^{2S} \sum_{P_k} \prod_{k=1}^{2S-r} \alpha_k \prod_{k=2S-r+1}^{2S} \beta_k \sqrt{\frac{(2S)!}{(2S-r)!r!}} |S, S-r\rangle,\end{aligned}\quad (7.8)$$

where in the lower line we have used Eq. (7.5). By defining

$$a_r = \frac{1}{(2S-r)!r!} \sum_{P_k} \prod_{k=1}^{2S-r} \alpha_k \prod_{k=2S-r+1}^{2S} \beta_k \quad (7.9)$$

we get

$$\begin{aligned}\xi' &= A \sum_{r=0}^{2S} a_r \sqrt{\frac{(2S-r)!r!}{(2S)!}} |S, S-r\rangle \\ &= A \sum_{r=0}^{2S} a_r \binom{2S}{r}^{-1/2} |S, S-r\rangle.\end{aligned}\quad (7.10)$$

The spin vector of a spin- S particle can be written as $\xi = \sum_{M=-S}^S \xi_M |S, M\rangle$. If we equate this with ξ' we get

$$\xi_{S-r} = A a_r \binom{2S}{r}^{-1/2}. \quad (7.11)$$

In conclusion, we have shown that $2S$ spin vectors $\varphi_k = \alpha_k |+\rangle_k + \beta_k |-\rangle_k$, $k = 1, 2, \dots, 2S$, determine a spin vector of a spin- S particle through Eqs. (7.9) and (7.11). As a matter of fact, because A can be chosen freely, a better formulation of this statement is that $2S$ pure states $|\varphi_k\rangle\langle\varphi_k|$ determine a unique pure state $|\xi\rangle\langle\xi|$ through these equations. The converse is also true, as can be seen by writing

$$A \prod_{k=1}^{2S} (\alpha_k x + \beta_k) = 0. \quad (7.12)$$

By expanding the product this becomes

$$\begin{aligned}
A \prod_{k=1}^{2S} (\alpha_k x + \beta_k) &= \sum_{r=0}^{2S} \frac{A}{(2S-r)! r!} \sum_{P_k} x^{2S-r} \prod_{k=1}^{2S-r} \alpha_k \prod_{k=2S-r+1}^{2S} \beta_k \\
&= A \sum_{r=0}^{2S} x^{2S-r} a_r \\
&= 0.
\end{aligned} \tag{7.13}$$

Thus, if ξ , and therefore a_r 's, is known, we get the ratios $-\beta_k/\alpha_k$ as the solutions of the equation

$$A \sum_{r=0}^{2S} x^{2S-r} a_r = \sum_{r=0}^{2S} \binom{2S}{r}^{1/2} \xi_{S-r} x^{2S-r} = 0. \tag{7.14}$$

By defining $M = S - r$ we obtain

$$\sum_{M=-S}^S \binom{2S}{S+M}^{1/2} x^{S+M} \xi_M = A \prod_{k=1}^{2S} (\alpha_k x + \beta_k) = 0. \tag{7.15}$$

This equation establishes a bijective map between the pure states of a spin- S particle and the set of $2S$ pure states of spin-1/2 particles. By choosing A properly we can define $\alpha_k = \cos \frac{\theta_k}{2} e^{-i\varphi_k/2}$ and $\beta_k = \sin \frac{\theta_k}{2} e^{i\varphi_k/2}$, where $k = 1, 2, \dots, 2S$ and (θ_k, φ_k) give the positions of the points on the Bloch sphere in spherical coordinates. The roots of (7.14) are then given by $-\beta_k/\alpha_k = -\tan \frac{\theta_k}{2} e^{i\varphi_k}$. We call the set of $2S$ points corresponding to ξ the *point distribution* or *point configuration* of ξ . The polynomial of Eq. (7.15) is called the *characteristic polynomial* of ξ . If we are given two sets of points, the characteristic polynomial of the point configuration consisting of both of these sets is obtained by multiplying the characteristic polynomials of the two sets.

It is important to notice that the point distribution is the same for ξ and $h\xi$, where h is arbitrary non-zero complex number. Therefore the point distributions are in one-to-one correspondence with pure states, not with vector representatives of these states.

7.2 Linear transformations

Now we study how ξ changes in linear transformations of the spin vectors of spin-1/2 particles. We assume that the linear transformation is the same

for all particles, that is

$$\begin{aligned}\alpha_k &\mapsto \alpha'_k = a\alpha_k + b\beta_k, \\ \beta_k &\mapsto \beta'_k = c\alpha_k + d\beta_k, \quad k = 1, 2, \dots, 2S,\end{aligned}\tag{7.16}$$

where a, b, c, d are arbitrary complex numbers. The components of the transformed spin vector are given by

$$\xi'_{S-r} = \frac{A}{(2S)!} \binom{2S}{r}^{1/2} \sum_{P_k} \prod_{k=1}^{2S-r} \alpha'_k \prod_{l=2S-r+1}^{2S} \beta'_l,\tag{7.17}$$

where

$$\begin{aligned}\prod_{k=1}^{2S-r} \alpha'_k &= \sum_{k=0}^{2S-r} \sum_{P_k} \frac{a^k b^{2S-r-k} \alpha_1 \cdots \alpha_k \beta_{k+1} \cdots \beta_{2S-r}}{k!(2S-r-k)!}, \\ \prod_{l=2S-r+1}^{2S} \beta'_l &= \sum_{l=0}^r \sum_{P_k} \frac{c^l d^{r-l} \alpha_{2S-r+1} \cdots \alpha_{2S-r+l} \beta_{2S-r+1+l} \cdots \beta_{2S}}{l!(r-l)!}.\end{aligned}\tag{7.18}$$

Combining these gives

$$\begin{aligned}\sum_{P_k} \prod_{k=1}^{2S-r} \alpha'_k \prod_{k=2S-r+1}^{2S} \beta'_k &= \sum_{P_k} \sum_{k=0}^{2S-r} \sum_{l=0}^r \frac{a^k b^{2S-r-k} c^l d^{r-l}}{k!l!(2S-r-k)!(r-l)!} \\ &\times \left(\sum_{P_k} \alpha_1 \cdots \alpha_k \beta_{k+1} \cdots \beta_{2S-r} \right) \left(\sum_{P_k} \alpha_{2S-r+1} \cdots \alpha_{2S-r+l} \beta_{2S-r+1+l} \cdots \beta_{2S} \right) \\ &= \sum_{k=0}^{2S-r} \sum_{l=0}^r \frac{a^k b^{2S-r-k} c^l d^{r-l} (2S-r)!r!}{k!l!(2S-r-k)!(r-l)!} \sum_{P_k} \alpha_1 \cdots \alpha_{k+l} \beta_{k+l+1} \cdots \beta_{2S}.\end{aligned}\tag{7.19}$$

Here there are three summations over permutations, one in each parentheses and one at the beginning. Because there are $2S-r$ terms in the first parentheses and r terms in the second, the combination of all three summations corresponds to $(2S-r)!r!$ times the first summation alone. By using this result and Eqs. (7.9), (7.11) and (7.17) we obtain

$$\begin{aligned}\xi'_{S-r} &= \frac{A}{(2S)!} \binom{2S}{r}^{1/2} \sum_{k=0}^{2S-r} \sum_{l=0}^r \frac{a^k b^{2S-r-k} c^l d^{r-l} (2S-r)!r!}{k!l!(2S-r-k)!(r-l)!} \\ &\times \frac{(k+l)!(2S-k-l)!}{A} \binom{2S}{k+l}^{1/2} \xi_{k+l-S} \\ &= \sum_{k=0}^{2S-r} \sum_{l=0}^r a^k b^{2S-r-k} c^l d^{r-l} \frac{\sqrt{(2S-r)!r!(k+l)!(2S-k-l)!}}{k!l!(2S-r-k)!(r-l)!} \xi_{k+l-S}.\end{aligned}\tag{7.20}$$

By setting $M = S - r$ and $M' = k + l - S$ we can write this as

$$\begin{aligned} \xi'_M &= \sum_{M'=-S}^S \sum_{k=\text{Max}\{0, M+M'\}}^{\text{Min}\{S+M, S+M'\}} a^k b^{S+M-k} c^{S+M'-k} d^{k-M-M'} \\ &\times \frac{\sqrt{(S+M)!(S-M)!(S+M')!(S-M')!}}{k!(S+M'-k)!(S+M-k)!(k-M-M')!} \xi_{M'}. \end{aligned} \quad (7.21)$$

7.2.1 Spin rotation

As an example of a linear map we consider a spin rotation of the spin vector of a spin-1/2 particle. The rotation matrix is given by U of Eq. (6.4), for which $a = \cos \frac{\beta}{2} e^{-i(\alpha+\gamma)/2}$, $b = -\sin \frac{\beta}{2} e^{i(\gamma-\alpha)/2}$, $c = \sin \frac{\beta}{2} e^{-i(\gamma-\alpha)/2}$, and $d = \cos \frac{\beta}{2} e^{i(\alpha+\gamma)/2}$. The elements of the rotation matrix $D^{(S)}(\alpha, \beta, \gamma)$ of (6.4) become

$$\begin{aligned} D_{MM'}^{(S)}(\alpha, \beta, \gamma) &= e^{-i(M\alpha+M'\gamma)} \sum_{k=\text{Max}\{0, M+M'\}}^{\text{Min}\{S+M, S+M'\}} (-1)^{S+M-k} \left(\cos \frac{\beta}{2} \right)^{2k-M-M'} \\ &\times \left(\sin \frac{\beta}{2} \right)^{2S-2k+M+M'} \frac{\sqrt{(S+M)!(S-M)!(S+M')!(S-M')!}}{k!(S+M'-k)!(S+M-k)!(k-M-M')!}. \end{aligned} \quad (7.22)$$

Spin rotations can also be parameterized by the rotation angle τ and the rotation axis $\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ as shown in Eq. (6.5). Using this parameterization, the 2×2 rotation matrix becomes

$$\tilde{U}(\tau, \theta, \varphi) = e^{-i\frac{\tau}{2}\mathbf{n}(\theta, \varphi) \cdot \boldsymbol{\sigma}} = \begin{pmatrix} \cos \frac{\tau}{2} - i \sin \frac{\tau}{2} \cos \theta & -ie^{-i\varphi} \sin \frac{\tau}{2} \sin \theta \\ -ie^{i\varphi} \sin \frac{\tau}{2} \sin \theta & \cos \frac{\tau}{2} + i \sin \frac{\tau}{2} \cos \theta \end{pmatrix}. \quad (7.23)$$

Using this and Eq. (7.21) we can now calculate the generating equation for the rotation matrix for arbitrary S . This form of the rotation operator is rarely presented in the literature. A straightforward calculation gives

$$\begin{aligned} \tilde{D}^{(S)}(\tau, \theta, \varphi) &= \frac{(-i \sin \frac{\tau}{2} \sin \theta)^{2S+M+M'}}{(\cos \frac{\tau}{2} + i \sin \frac{\tau}{2} \sin \theta)^{M+M'}} \sum_{k=\text{Max}\{0, M+M'\}}^{\text{Min}\{S+M, S+M'\}} \\ &\left(1 - \frac{1}{\sin^2 \frac{\tau}{2} \sin^2 \theta} \right)^k \frac{\sqrt{(S+M)!(S-M)!(S+M')!(S-M')!}}{k!(S+M'-k)!(S+M-k)!(k-M-M')!}. \end{aligned} \quad (7.24)$$

7.2.2 Change of phase

Now we show how the phase of an order-parameter ξ changes in rotations that leave the point distribution determined by this vector unchanged. These rotations leave ξ fixed up to a phase. We denote a symmetry axis of the point distribution of ξ by \mathbf{s}_ξ . Now

$$e^{-i\alpha_{rot}\mathbf{s}_\xi\cdot\mathbf{S}}\xi = e^{i\delta}\xi \quad (7.25)$$

for some rotation angle α_{rot} and some $\delta \in \mathbb{R}$. We choose a rotation $D = D^{(S)}(\alpha, \beta, \gamma)$ such that the symmetry axis of the point distribution of the rotated order-parameter, $\xi' = D\xi$, is $\hat{\mathbf{z}}$, that is, $\mathbf{s}_{\xi'} = \hat{\mathbf{z}}$. Therefore

$$e^{-i\alpha_{rot}S_z}\xi' = e^{i\delta}\xi', \quad (7.26)$$

where α_{rot} and δ are the same than in Eq. (7.25). Eq. (7.26) gives

$$e^{-iM\alpha_{rot}} = e^{i\delta} \text{ whenever } \xi'_M \neq 0. \quad (7.27)$$

Assume next that p_s of the points characterizing ξ' are at the south pole. Then p_s of the α_k 's in Eq. (7.15) are equal to zero, which is equivalent with the order of (7.15) being $2S - p_s$. This means that $\xi'_S = \xi'_{S-1} = \dots = \xi'_{S-p_s+1} = 0, \xi'_{S-p_s} \neq 0$. Similarly, if p_n of the points are at the north pole, then p_n of the β_k 's are equal to zero, which is equivalent with $\xi'_{-S} = \xi'_{-S+1} = \dots = \xi'_{-S+p_n-1} = 0, \xi'_{-S+p_n} \neq 0$. These results, combined with (7.27), give

$$\begin{aligned} \delta &= (p_s - S)\alpha_{rot} \pmod{2\pi} \\ &= (S - p_n)\alpha_{rot} \pmod{2\pi}. \end{aligned} \quad (7.28)$$

A similar expression has been given in [2, 4].

Finally, it should be stressed that the decomposition of spin states described in this chapter, as well as all the equations presented, holds for arbitrary S , including half-integer values.

Chapter 8

Inert states

8.1 Definition

States which are stationary points of the energy regardless of the exact form of the energy functional are called inert states [65]. A stationary point may change, for example, from a local maximum to a global minimum as the parameters characterizing the energy vary, but for all parameter values an inert state remains nevertheless a stationary point of energy. The independence on the details of the energy functional is related to the fact that inert states are determined by the symmetry group of the energy and the symmetry properties of the order-parameter alone. Inert states have been studied in the context of superfluid ^3He , where the analytical minimization of the energy functional is a very complicated task, but the inert states can be calculated straightforwardly [5, 6, 65], see also [46]. Inert states are always stationary points of the energy, but they do not necessarily give the global minimum of the energy. In order to see whether or not some given inert state is the global ground state of the energy functional, one has to know the exact form of the latter. All inert states are stationary points of the energy, but the reverse is not necessarily true. It is possible that there are stationary points of the energy which are not inert states. This is the case, for example, in ^3He [65, 11].

To illustrate the concept of an inert state, we consider the following simple example. Let $f_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f_\lambda(x) = \lambda x^2 + x^4$. If $\lambda \geq 0$, this function has one global minimum, located at the origin. On the other hand, if $\lambda < 0$, there are two degenerate global minima, located symmetrically at $x = \pm\sqrt{-\lambda/2}$, and local maximum at $x = 0$. These two cases are illustrated in Fig. (8.1). Because now $x = 0$ remains a stationary

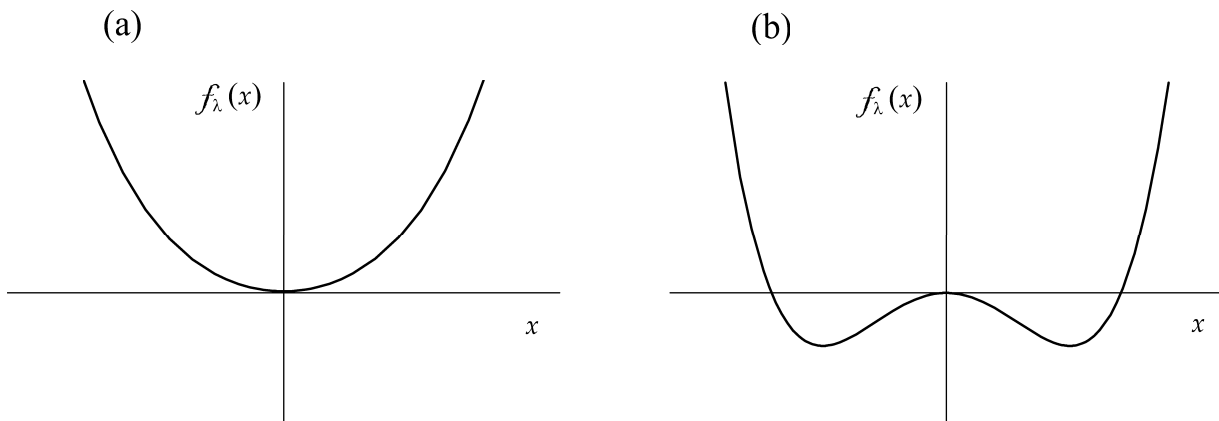


Figure 8.1: A schematic plot of f_λ for (a) positive and (b) negative λ .

point of f_λ regardless of the value of λ , one might assume that this point is an inert state. This proves to be the case, as we show below.

Before embarking on the study of inert states, we have to define some mathematical concepts. In the following we assume that G is a group that acts on a manifold M . As before, $g \cdot x$ denotes the element of M obtained by acting with $g \in G$ to $x \in M$, and the isotropy group of x is $H_x = \{g \in G \mid g \cdot x = x\}$. The *orbit* of $x \in M$ is given by $G(x) = \{g \cdot x \mid g \in G\}$. If $x, y \in M$ lie on the same orbit, i.e. $y = g \cdot x$ for some $g \in G$, then the isotropy groups H_x, H_y are conjugate, $H_y = gH_xg^{-1}$. The *stratum* of $x \in M$ is given by

$$S(x) = \{y \in M \mid H_y = gH_xg^{-1} \text{ for some } g \in G\}. \quad (8.1)$$

Thus $S(x)$ is the union of all orbits of points having isotropy groups that are conjugate to H_x . The calculation of inert states are based on the following theorem [65].

Theorem 2. *If an orbit $G(x_0)$ lies isolated in its stratum $S(x_0)$, i.e. if there exists a neighbourhood U of $G(x_0)$ such that $U \cap S(x_0) = G(x_0)$, then every G -invariant smooth real-valued function on M is stationary at all points of $G(x_0)$.*

For a complete proof, written in French, see [40]. An English sketch of the proof can be found in [41]. Here $f : M \rightarrow \mathbb{R}$ is G -invariant if $f(g \cdot x) = f(x)$ for all $g \in G, x \in M$.

In the physical applications of Theorem 2, M is the order-parameter space, G is the symmetry group of the energy, and the energy is given by a G -invariant smooth function $E : M \rightarrow \mathbb{R}$. The stationary points obtained using this theorem are called inert states.

Theorem 2 can be expressed in a different way by noting that because $G(x_0) \subset S(x_0)$, then also always $G(x_0) \subset U \cap S(x_0)$ for every neighbourhood U of $G(x_0)$. Therefore, if $U \cap S(x_0) \neq G(x_0)$, there is an element $x_U \in U \setminus G(x_0)$ such that H_{x_U} is conjugate to H_{x_0} . This result can be stated in a simpler, although at the same time more inaccurate, form by saying that x_0 is *not* an inert state if there is an element $x_1 \in M$, which is ‘infinitesimally close’ to x_0 and for which H_{x_1} and H_{x_0} are conjugate groups. Here x_1 is not allowed to be on the orbit of x_0 . On the other hand, if such an element does not exist, then x_0 is an inert state.

As a simple application of Theorem 2 we consider a symmetric function $f : \mathbb{R} \rightarrow \mathbb{R}$. Clearly f_λ is such a function. Now the symmetry group is $G = \mathbb{Z}_2 = \{1, -1\}$ and the isotropy groups are $H_{x=0} = \{1, -1\}$ and $H_{x;x \neq 0} = \{1\}$. From these and the definition (8.1) it follows that there are two possible strata, $S(x=0) = \{0\}$ and $S(x)|_{x \neq 0} = \mathbb{R} \setminus \{0\}$. The orbits are $G(x=0) = \{0\}$ and $G(x)|_{x \neq 0} = \{x, -x\}$. For any neighbourhood U of the origin $U \cap S(x=0) = S(x=0) = G(x=0)$ holds and therefore, by the above theorem, the origin is a stationary point of f . In other words, the isotropy group of a point $x \neq 0$ is not conjugate to the isotropy group of the origin, no matter how close to the origin x is.

8.2 Inert states of spinor condensates

In Paper *V* we present a geometrical method to calculate the inert states of a spin- S system. We assume that the system is described by a spin vector of a spin- S particle, denoted by ξ , and that the energy functional is invariant in rotations and phase transformations of ξ . Thus the symmetry group of the energy is $G = U(1) \times SO(3)$, where, as before, $SO(3)$ acts via its irreducible $2S+1$ dimensional representation and the action of $U(1)$ corresponds to the phase changes of ξ .

The method is based on using the spin decomposition of Chapter 7 to calculate the isotropy groups of spin vectors. The elements of the isotropy group H_ξ consists of pairs (u, r) , where $u \in U(1)$ and $r \in SO(3)$. The latter are the elements of some subgroup K of $SO(3)$ and they can be determined by finding out all rotations which leave the point configuration of ξ invariant. In the following we call K the *symmetry group* or *symmetry* of the point configuration. Because it contains only the $SO(3)$ part of H_ξ , it should not be mistaken for the isotropy group [70]. The isotropy group is the same for ξ and $e^{i\tau}\xi$, so it can depend only on the spin state, not on a particular vector representative of the state. The spin states are in one-

to-one correspondence with the point configurations on the Bloch sphere, so the isotropy groups are determined by the latter.

The inert states can be determined by finding all point distributions which cannot be modified without changing the isotropy group to a group which is not conjugate to the original isotropy group. In Paper V it is shown that this is equivalent with finding all point distributions which cannot be modified without changing the symmetry group of the point configuration.

The possible symmetry groups of the point distributions are the subgroups of $SO(3)$, that is, the continuous groups $O(2)$, $SO(2)$, and the finite groups I , C_n , D_n , T , O and Y . I is the trivial group consisting of the identity element alone, C_n (D_n) is the cyclic (dihedral) group of order n , T is the tetrahedral, O the octahedral, and Y the icosahedral group. The inert states can be calculated by first finding all configurations that have some given group as their symmetry group, and by seeing then which of these cannot be modified without changing the symmetry group.

In Paper V only those inert states with T , O , or Y symmetry which are obtained by placing points at the vertices of the convex regular polyhedra have been calculated. These polyhedra are often called the Platonic solids, and they are the tetrahedron (T), octahedron (O), cube (O), icosahedron (Y) and dodecahedron (Y), see Fig. 8.2. The symmetry group is given in parenthesis. Any change in the positions of the vertices of the Platonic solids changes the symmetry group (assuming that rotations are excluded). It is possible that there are also other ways to distribute points on the surface of a sphere with, say, octahedral symmetry than to place them at the vertices of the octahedron or cube. However, these distributions correspond to polyhedra which are not regular, and therefore the points can probably be moved without changing the symmetry group. An example of a polyhedron which is not regular can be seen in the lower figure of Fig. 8.2 (b). This object has octahedral symmetry and it is obtained from the octahedron by truncating it. The truncation of an octahedron is possible if there are at least four points at each vertex, so these kinds of point configurations do not determine inert states. Platonic solids have an even number of vertices, which means that inert states with half integer S are not possible.

The actual calculation of the inert states has been presented in Paper V. However, because the inert states with icosahedral symmetry have been discussed rather briefly there, we shall study them in more detail here. An icosahedron can be truncated if there are five or more points at each vertex, while for a dodecahedron this is possible if there are

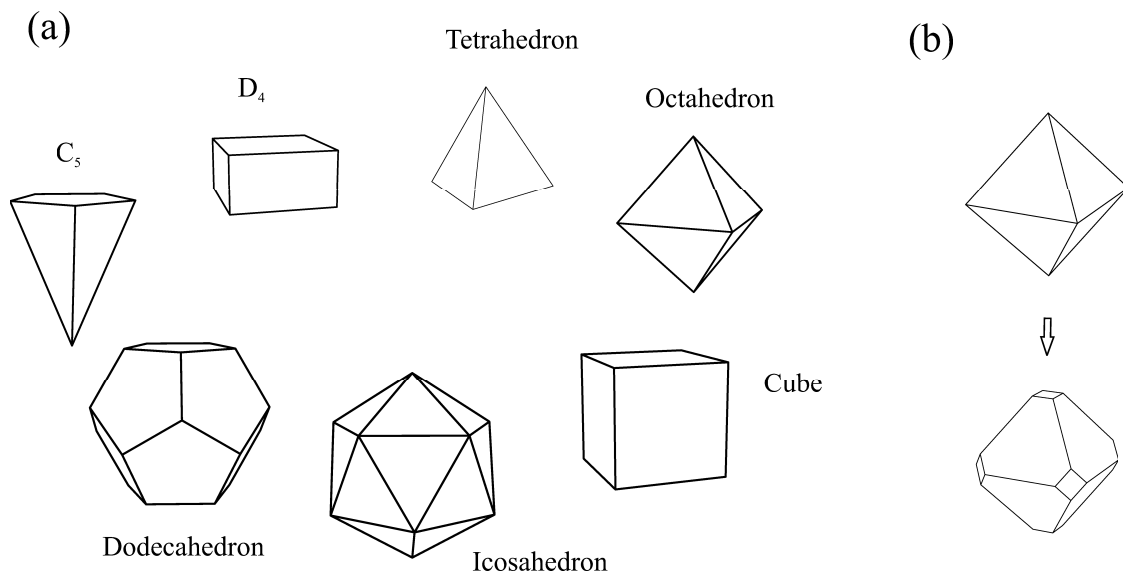


Figure 8.2: In (a) we show examples of C_5 and D_4 symmetries and the Platonic solids, while in (b) we illustrate the truncation of an octahedron.

more than two points at each vertex, see Fig. 8.2. Thus these symmetries are possible if $S = 6m + 10n$, where $m + n \geq 1$, $m \leq 4$, and $n \leq 2$. The vertices of the icosahedron can be chosen to be at $(\theta, \varphi) = (0, 0), (\pi, 0), (\theta_{Ico}, 2k\pi/5), (\pi - \theta_{Ico}, (2k+1)\pi/5)$, where $\theta_{Ico} = 2 \tan^{-1} \frac{\sqrt{5}-1}{2}$. A dodecahedron is obtained by placing the vertices at $(\theta, \varphi) = (\theta_{Dode}, (2k+1)\pi/5), (\theta'_{Dode}, (2k+1)\pi/5), (\pi - \theta_{Dode}, 2k\pi/5), (\pi - \theta'_{Dode}, 2k\pi/5)$, where $\theta_{Dode} = \sin^{-1} \sqrt{\frac{2(\sqrt{5}-1)}{3\sqrt{5}}}$ and $\theta'_{Dode} = \theta_I + 2 \sin^{-1} \left(\frac{2}{\sqrt{3}(1+\sqrt{5})} \right)$. Here k runs from one to five. Eq. (7.15) reads

$$\sum_{M=-S}^S \binom{2S}{S+M}^{1/2} x^{S+M} \xi_M = A \prod_{k=1}^{2S} \left(x \cos \frac{\theta_k}{2} e^{-i\varphi_k/2} + \sin \frac{\theta_k}{2} e^{i\varphi_k/2} \right) = 0. \quad (8.2)$$

Using the right-hand side of this equation and the angles given above it can be straightforwardly shown that the characteristic polynomial becomes $x^{11} - 11x^6 - x$ for the icosahedron and $x^{20} + 228x^{15} + 494x^{10} - 228x^5 + 1$ for the dodecahedron. Comparing these with the left-hand side shows that the inert states with icosahedral symmetry are for $S = 6$ and $S = 10$ systems

$$\xi_{Ico} = \sqrt{7}|6, 5\rangle - \sqrt{11}|6, 0\rangle - \sqrt{7}|6, -5\rangle \quad (8.3)$$

Table 8.1: The inert states for $S = 1 - 4$. The subscript of ξ gives the symmetry group of the point configuration of ξ .

Spin	Inert states
$S = 1$	$\xi_{SO(2)} = 1, 1\rangle, \xi_{O(2)} = 1, 0\rangle$
$S = 2$	$\xi_{SO(2)} = 2, 2\rangle$ and $ 2, 1\rangle, \xi_{D_4} = 2, 2\rangle + 2, -2\rangle,$ $\xi_{O(2)} = 2, 0\rangle,$ and $\xi_{Tetra} = 2, 2\rangle + i\sqrt{2} 2, 0\rangle + 2, -2\rangle$
$S = 3$	$\xi_{SO(2)} = 3, 3\rangle, 3, 2\rangle,$ and $ 3, 1\rangle, \xi_{O(2)} = 3, 0\rangle,$ $\xi_{D_6} = 3, 3\rangle + 3, -3\rangle,$ and $\xi_{Octa} = 3, 2\rangle + 3, -2\rangle$
$S = 4$	$\xi_{SO(2)} = 4, 4\rangle, 4, 3\rangle, 4, 2\rangle,$ and $ 4, 1\rangle, \xi_{O(2)} = 4, 0\rangle,$ $\xi_{D_8} = 4, 4\rangle + 4, -4\rangle, \xi_{D_6} = 4, 3\rangle + 4, -3\rangle$ $\xi_{D_4} = 4, 2\rangle + 4, -2\rangle, \xi_{Tetra} = \sqrt{7} 4, 4\rangle + 2i\sqrt{3} 4, 2\rangle$ $- \sqrt{10} 4, 0\rangle + 2i\sqrt{3} 4, -2\rangle + \sqrt{7} 4, -4\rangle.$ $\xi_{Cube} = \sqrt{5} 4, 4\rangle - \sqrt{14} 4, 0\rangle + \sqrt{5} 4, -4\rangle$

and

$$\xi_{Dode} = \sqrt{17}|10, 10\rangle + \sqrt{57}|10, 5\rangle + \sqrt{\frac{247}{11}}|10, 0\rangle - \sqrt{57}|10, -5\rangle + \sqrt{17}|10, -10\rangle. \quad (8.4)$$

In Table 8.1 we list the inert states for integer-valued spin $S = 1 - 4$. Comparing these with the ground state spinors of spinor condensates shown in Tables 3.1 and 4.1 shows that for $S = 1$ and $S = 2$ there are no other ground states than the inert states. However, there is one inert state, namely $|2, 1\rangle$, which is never a ground state. In Papers I and II the cyclic state of an $F = 2$ condensate was found to have tetrahedral symmetry. The same result was later obtained in [3] using the same graphical representation of spin states than is used here. If $S = 3$, the inert states are ground states but also numerous other ground states are possible, see Refs. [16, 52] and Paper IV. These results suggest that the inert states for $S > 3$ are ground states of spinor condensates for some values of the scattering lengths. This has interesting consequences considering the topological defects of spinor condensates. The symmetry group of ξ contains those elements of H_ξ which belong to $SO(3)$. All symmetry groups except $I, SO(2), O(2)$, and C_n are non-commuting groups, so based on Theorem 1 one can argue that the first homotopy groups of the ground states with D_n, T, O , and Y symmetries are also non-commuting groups. Therefore, vortices with non-commuting combination law can probably occur in spinor condensates for any $S \geq 1$ because then at least D_n -symmetric inert states are possible.

8.3 Monopoles

Finally we consider the monopoles of spinor condensates. This discussion contains a new result, but it was left out of Paper V due to lack of space. As before, we assume that the energy is invariant in phase changes and spin rotations of spinor ξ . In order to use Theorem 1, we again use instead of $U(1) \times SO(3)$ its covering group $\mathbb{R} \times SU(2)$, for which the first and second homotopy groups consist of the identity element alone. We denote the isotropy group of ξ by H_ξ and its connected component of the identity by H_{ξ_0} . According to Theorem 1, monopoles are possible if the first homotopy group $\pi_1(H_{\xi_0})$ is a nontrivial group, i.e. it also contains other elements than the identity element e . Assume that $H_{\xi_0} = (R, \mathbb{I})$, where R is some interval of \mathbb{R} containing zero and \mathbb{I} is the identity matrix. Now $\pi_1(H_{\xi_0}) = \pi_1(R) \times \pi_1(\{\mathbb{I}\}) = \{e\}$. Therefore, if $\pi_1(H_{\xi_0}) \neq \{e\}$, the $SU(2)$ part of H_{ξ_0} has to be a continuous set. Since the homomorphism between $SU(2)$ and $SO(3)$ is continuous, the $SO(3)$ part of the isotropy group is then a continuous set, too. Thus, by finding all those point configurations on the Bloch sphere that have a continuous part in their symmetry group, one finds all spin states that may allow monopoles. Based on the results of Paper V, these spin states are $\xi_{SO(2)}^{S,M} = |S, M\rangle$, $M \neq 0$ and $\xi_{O(2)}^S = |S, 0\rangle$. However, the isotropy group of the former is \mathbb{Z}_{2M} , so the only possibility is $\xi_{O(2)}^S$. The point distribution corresponding to this vector has S points at the north and south pole of the Bloch sphere. This configuration is invariant in arbitrary rotations about the z -axis. For these $p_n = p_s = S$ in Eq. (7.28), which gives $\delta = 2k\pi$, $k \in \mathbb{Z}$. Another symmetry axis lies in the xy -plane. For this $p_n = p_s = 0$ and $\alpha_{rot} = \pi$, so according to (7.28) $\delta = (2k + S)\pi$. Hence the isotropy group is

$$H_{|S,0\rangle} = \{(2k\pi, e^{-i\frac{\tau}{2}\sigma_z}), ((2k + S)\pi, e^{-i\frac{\pi}{2}\sigma_y} e^{-i\frac{\tau}{2}\sigma_z})\}, \quad (8.5)$$

where k is an arbitrary integer and $\tau \in [0, 4\pi)$. The connected component of the identity is $H_{|S,0\rangle_0} = \{(0, e^{-i\frac{\tau}{2}\sigma_z})\}$, which is topologically equivalent with $U(1)$ and therefore $\pi_1(H_{|S,0\rangle_0}) = \mathbb{Z}$. According to Theorem 1 this shows that monopoles with integer winding numbers are possible. Using the results of Paper III we see that a spinor representing a monopole with winding number m is given, for example, by

$$\xi(m; r, \theta, \phi) = D^{(S)}(m\varphi, \pi - \theta, 0)|S, 0\rangle, \quad (8.6)$$

where (r, ϕ, θ) are the spherical coordinates and $D^{(S)}(\alpha, \beta, \gamma)$ is given by Eq. (7.22). In conclusion, in a spinor condensate of spin- S particles monopoles

can be created, up to a spin rotation and a phase change, only from the $|S, 0\rangle$ vector, which is also an inert state.

Chapter 9

Summary

In this thesis the topological defects, ground state spinors, and inert states of spinor Bose-Einstein condensates have been studied. Their complexity, and in most cases also their existence, arises from the multicomponent nature of the order-parameter. This was discovered to enable numerous ground states, inert states with various symmetries, and topological defects characterized by complex spin textures.

The defects of spinor condensates were found to differ from the defects of single-component condensates in many ways. For example, in a ferromagnetic $F = 1$ condensate the presence of a vortex does not have to imply non-zero orbital angular momentum or superfluid velocity, unlike in a single-component condensate. Furthermore, vortices with fractional winding number and non-commuting combination law can exist. These type of vortices were explicitly studied in the case of a cyclic spin-2 condensate, but in view of the results of Paper V, these defects can occur in all spin- F condensates if F is larger than one. It has also been shown that in spinor condensates with $U(1) \times SO(3)$ symmetric energy monopoles can be generated only, up to a spin rotation and a phase change, from the $|F, m_F = 0\rangle$ vector.

In addition to the topological defects, we have discussed the ground states of a spin-3 condensate assuming that the magnetization is conserved. The ground state spinors for arbitrary and ground state phase diagrams for some values of the magnetization were calculated. The expected ground state spinor of a ^{52}Cr condensate has only two spin components populated, and its form does not seem to depend on whether or not the magnetic dipole-dipole interaction is taken into consideration. However, due to the rapid incoherent dipolar relaxation taking place in ^{52}Cr , the condensate density needs to be decreased in order to make the condensate lifetime

long enough for experiments.

We have also developed a simple geometrical method to calculate the inert states of a spin- F condensate. This technique is based on representing the spin vectors of a spin- F particle in terms of $2F$ points on the surface of a sphere and it is considerably easier to apply than the method presented earlier in Ref. [71], which becomes increasingly cumbersome as F increases. The motivation for calculating inert states is that they are good candidates for the ground state spinors of a spin- F condensate. For $F = 1$ and $F = 2$ condensates they were found to correspond to the possible ground state spinors. For $F = 3$ condensates they are also ground states, although in this case many other ground state spinors can exist as well.

During the writing of Paper V the decomposition of spin states discussed in Chapter 7 was discovered to provide a simple way to obtain the results of the first three publications. In Papers I, II, and III the explicit forms for the spin rotation matrices were used in the calculation of the isotropy groups, which made the computation cumbersome and prone to errors. The use of the representation of spin states in terms of points on the Bloch sphere makes this calculation faster and easier, as has been seen in Refs. [3, 4] and in Paper V. One future application of this description of spin states might be the determination of the isotropy groups and topological defects of the inert states of spin- F condensates for arbitrary spin.

Bibliography

- [1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell. “Observation of Bose-Einstein condensation in a dilute atomic vapor.” *Science* **269**, 198 (1995).
- [2] R. Barnett, A. Turner, and E. Demler. “Reply to comment by S. K. Yip cond-mat/0611426.” ArXiv:cond-mat/0612099v2.
- [3] R. Barnett, A. Turner, and E. Demler. “Classifying novel phases of spinor atoms.” *Phys. Rev. Lett.* **97**, 180412 (2006).
- [4] R. Barnett, A. Turner, and E. Demler. “Classifying vortices in $S = 3$ Bose-Einstein condensates.” *Phys. Rev. A* **76**, 013605 (2007).
- [5] G. Barton and M. A. Moore. “Some p-wave phases of superfluid helium-3 in strong-coupling theory.” *J. Phys. C: Solid State Phys.* **7**, 4220 (1974).
- [6] G. Barton and M. A. Moore. “The likelihood of f-wave pairing in superfluid ^3He .” *J. Phys. C: Solid State Phys.* **8**, 970 (1975).
- [7] R. A. Battye, N. R. Cooper, and P. M. Sutcliffe. “Stable skyrmions in two-component Bose-Einstein condensates.” *Phys. Rev. Lett.* **88**, 080401 (2002).
- [8] F. Bloch and I. I. Rabi. “Atoms in variable magnetic fields.” *Rev. Mod. Phys.* **17**, 237 (1945).
- [9] C. C. Bradley, C. A. Sackett, and R. G. Hulet. “Bose-Einstein condensation of lithium: Observation of limited condensate number.” *Phys. Rev. Lett.* **78**, 985 (1997).
- [10] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet. “Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions.” *Phys. Rev. Lett.* **75**, 1687 (1995).

- [11] C. Bruder and D. Vollhardt. “Symmetry and stationary points of a free energy: The case of superfluid ^3He .” *Phys. Rev. B* **34**, 131 (1986).
- [12] M.-S. Chang, C. D. Hamley, M. D. Barrett, J. A. Sauer, K. M. Fortier, W. Zhang, L. You, and M. S. Chapman. “Observation of spinor dynamics in optically trapped Rb Bose-Einstein condensates.” *Phys. Rev. Lett.* **92**, 140403 (2004).
- [13] C. V. Ciobanu, S.-K. Yip, and T.-L. Ho. “Phase diagrams of F=2 spinor Bose-Einstein condensates.” *Phys. Rev. A* **61**, 033607 (2000).
- [14] A. Crubellier, O. Dulieu, F. Masnou-Seeuws, M. Elbs, H. Knöckel, and E. Tiemann. “Simple determination of Na_2 scattering lengths using observed bound levels at the ground state asymptote.” *Eur. Phys. J. D* **6**, 211 (1999).
- [15] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle. “Bose-Einstein condensation in a gas of sodium atoms.” *Phys. Rev. Lett.* **75**, 3969 (1995).
- [16] R. B. Diener and T.-L. Ho. “ ^{52}Cr spinor condensate: A biaxial or uniaxial spin nematic.” *Phys. Rev. Lett.* **96**, 190405 (2006).
- [17] C. Eberlein, S. Giovanazzi, and D. H. J. O’Dell. “Exact solution of the Thomas-Fermi equation for a trapped Bose-Einstein condensate with dipole-dipole interactions.” *Phys. Rev. A* **71**, 033618 (2005).
- [18] A. Einstein. “Quantentheorie des einatomigen idealen Gases.” *Sitzber. Kgl. Preuss. Akad. Wiss.* page 261 (1924).
- [19] A. Einstein. “Quantentheorie des einatomigen idealen Gases. Zweite Abhandlung.” *Sitzber. Kgl. Preuss. Akad. Wiss.* page 3 (1925).
- [20] D. Finkelstein and C. W. Misner. “Some new conservation laws.” *Ann. Phys.* **6**, 230 (1959).
- [21] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau. “Bose-Einstein condensation of chromium.” *Phys. Rev. Lett.* **94**, 160401 (2005).
- [22] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge (2002).

- [23] T.-L. Ho. “Spinor Bose Condensates in optical traps.” *Phys. Rev. Lett.* **81**, 742 (1998).
- [24] Y. Kawaguchi, H. Saito, and M. Ueda. “Einstein de Haas effect in dipolar Bose-Einstein condensates.” *Phys. Rev. Lett.* **96**, 080405 (2006).
- [25] Y. Kawaguchi, H. Saito, and M. Ueda. “Can spinor dipolar effects be observed in Bose-Einstein condensates?” *Phys. Rev. Lett.* **98**, 110406 (2007).
- [26] U. A. Khawaja and H. T. C. Stoof. “Skyrmion physics in Bose-Einstein ferromagnets.” *Phys. Rev. A* **64**, 043612 (2001).
- [27] T. W. B. Kibble. “Topology of cosmic domains and strings.” *J. Phys. A: Math. Gen.* **9**, 1387 (1976).
- [28] N. N. Klausen, J. L. Bohn, and C. H. Greene. “Nature of spinor Bose-Einstein condensates in rubidium.” *Phys. Rev. A* **64**, 053602 (2001).
- [29] M. Kléman. “Relationship between Burgers circuit, Volterra process and homotopy groups.” *J. Phys. Lett. (Paris)* **38**, L-199 (1977).
- [30] M. Kléman and L. Michel. “Spontaneous breaking of Euclidean invariance and classification of topologically stable defects and configurations of crystals and liquid crystals.” *Phys. Rev. Lett.* **40**, 1387 (1978).
- [31] T. Kuwamoto, K. Araki, T. Eno, and T. Hirano. “Magnetic field dependence of the dynamics of ^{87}Rb spin-2 Bose-Einstein condensates.” *Phys. Rev. A* **69**, 063604 (2004).
- [32] F. London. “The λ -phenomenon of liquid helium and the Bose-Einstein degeneracy.” *Nature* **141**, 643 (1938).
- [33] F. London. “On the Bose-Einstein condensation.” *Phys. Rev.* **54**, 947 (1938).
- [34] I. F. Lyuksyutov. “Topological instability of singularities at small distances in nematics.” *Sov. Phys. JETP* **48**, 178 (1978). [*Zh. Eksp. Teor. Fiz.* **75**, 358 (1978)].
- [35] E. Majorana. “Atomi orientati in campo magnetico variabile.” *Nuovo Cimento* **9**, 43 (1932).

- [36] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, M. J. Holland, J. E. Williams, C. E. Wieman, and E. A. Cornell. “Watching a superfluid untwist itself: Recurrence of Rabi oscillations in a Bose-Einstein condensate.” *Phys. Rev. Lett.* **83**, 3358 (1999).
- [37] N. D. Mermin. “The topological theory of defects in ordered media.” *Rev. Mod. Phys.* **51**, 591 (1979).
- [38] N. D. Mermin, V. P. Mineyev, and G. E. Volovik. “Topological analysis of the cores of singularities in $^3\text{He-A}$.” *J. Low. Temp. Phys.* **33**, 117 (1978).
- [39] M. A. Metlitski and A. R. Zhitnitsky. “Vortons in two component Bose-Einstein condensates.” *J. High Energy Phys.* **0406**, 017 (2004).
- [40] L. Michel. “Points critiques des fonctions invariantes sur une G-variété.” *C. R. Acad. Sc. Paris* **272**, 433 (1971).
- [41] L. Michel. “Symmetry defects and broken symmetry. Configurations hidden symmetry.” *Rev. Mod. Phys.* **52**, 617 (1980).
- [42] V. P. Mineyev and G. E. Volovik. “Planar and linear solitons in superfluid ^3He .” *Phys. Rev. B* **18**, 3197 (1978).
- [43] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman. “Production of two overlapping Bose-Einstein condensates by sympathetic cooling.” *Phys. Rev. Lett.* **78**, 586 (1997).
- [44] D. H. J. O’Dell, S. Giovanazzi, and C. Eberlein. “Exact hydrodynamics of a trapped dipolar Bose-Einstein condensate.” *Phys. Rev. Lett.* **92**, 250401 (2004).
- [45] T. Ohmi and K. Machida. “Bose-Einstein condensation with internal degrees of freedom in alkali atom gases.” *J. Phys. Soc. Jpn.* **67**, 1822 (1998).
- [46] M. Ozaki, K. Machida, and T. Ohmi. “On p-wave pairing superconductivity under cubic symmetry.” *Prog. Theor. Phys.* **74**, 221 (1985).
- [47] O. Penrose and L. Onsager. “Bose-Einstein condensation and liquid helium.” *Phys. Rev.* **104**, 576 (1956).
- [48] C. Pethick and H. Smith. *Bose-Einstein condensation in dilute gases*. Cambridge University Press, Cambridge (2002).

- [49] R. Rajaraman. *Solitons and Instantons*. North-Holland, Amsterdam (1982).
- [50] J. Ruostekoski and J. R. Anglin. “Creating vortex rings and three-dimensional skyrmions in Bose-Einstein condensates.” *Phys. Rev. Lett.* **86**, 3934 (2001).
- [51] M. M. Salomaa and G. E. Volovik. “Quantized vortices in superfluid ^3He .” *Rev. Mod. Phys.* **59**, 533 (1987). [Erratum: *Rev. Mod. Phys.* **60**, 573 (1988)].
- [52] L. Santos and T. Pfau. “Spin-3 chromium Bose-Einstein condensates.” *Phys. Rev. Lett.* **96**, 190404 (2006).
- [53] C. M. Savage and J. Ruostekoski. “Energetically stable particlelike skyrmions in a trapped Bose-Einstein condensate.” *Phys. Rev. Lett.* **91**, 010403 (2003).
- [54] H. Schmaljohann, M. Erhard, J. Kronjager, M. Kottke, S. van Staa, L. Cacciapuoti, J. J. Arlt, K. Bongs, and K. Sengstock. “Dynamics of $F = 2$ spinor Bose-Einstein condensates.” *Phys. Rev. Lett.* **92**, 040402 (2004).
- [55] R. Shankar. “Applications of topology to the study of ordered systems.” *J. Phys. (Paris)* **38**, 1405 (1977).
- [56] J. L. Song, G. W. Semenoff, and F. Zhou. “Uniaxial and biaxial spin nematic phases induced by quantum fluctuations.” *Phys. Rev. Lett.* **98**, 160408 (2007).
- [57] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H.-J. Miesner, J. Stenger, and W. Ketterle. “Optical confinement of a Bose-Einstein condensate.” *Phys. Rev. Lett.* **80**, 2027 (1998).
- [58] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, and W. Ketterle. “Spin domains in ground-state Bose-Einstein condensates.” *Nature* **396**, 345 (1998).
- [59] G. Toulouse and M. Kléman. “Principles of a classification of defects in ordered media.” *J. Phys. Lett. (Paris)* **37**, L-149 (1976).
- [60] H.-R. Trebin. “The topology of non-uniform media in condensed matter physics.” *Adv. Phys.* **31**, 195 (1982).

- [61] H. R. Trebin. “Defects in frustrated media: Classification by homology groups.” *Phys. Rev. B* **30**, 4338 (1984).
- [62] A. M. Turner, R. Barnett, E. Demler, and A. Vishwanath. “Nematic order by disorder in spin-2 Bose-Einstein condensates.” *Phys. Rev. Lett.* **98**, 190404 (2007).
- [63] M. Ueda and M. Koashi. “Theory of spin-2 Bose-Einstein condensates: Spin correlations, magnetic response, and excitation spectra.” *Phys. Rev. A* **65**, 063602 (2002).
- [64] A. Vilenkin and E. P. S. Shellard. *Cosmic Strings and Other Topological Defects*. Cambridge University Press, Cambridge (1994).
- [65] D. Vollhardt and P. Wölfle. *The superfluid phases of Helium 3*. Taylor and Francis, London (1990).
- [66] G. E. Volovik. *Exotic properties of superfluid ^3He* . World Scientific, Singapore (1992).
- [67] G. E. Volovik. *The Universe in a Helium Droplet*. Clarendon Press, Oxford (2003).
- [68] G. E. Volovik and V. P. Mineev. “Line and point singularities in superfluid He^3 .” *JETP Lett.* **24**, 561 (1976). [*Pis'ma Zh. Eksp. Teor. Fiz.* **24**, 605 (1976)].
- [69] G. K. Woodgate. *Elementary atomic structure*. Clarendon Press, Oxford (1980).
- [70] S.-K. Yip. “Comment on "Classifying novel phases of spinor atoms".” ArXiv:cond-mat/0611426v1.
- [71] S.-K. Yip. “Symmetry and inert states of spin Bose-Einstein condensates.” *Phys. Rev. A* **75**, 023625 (2007).
- [72] F. Zhou. “Spin correlation and discrete symmetry in spinor Bose-Einstein condensates.” *Phys. Rev. Lett.* **87**, 080401 (2001).