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On Short Exponential Sums Involving Fourier Coefficients of Holomorphic Cusp Forms

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"Even things that are true can be proved"
-Oscar Wilde, The Picture of Dorian Gray

Abstract

By an exponential sum of the Fourier coefficients of a holomorphic cusp form we mean the sum which is formed by first taking the Fourier series of the said form, then cutting the beginning and the tail away and considering the remaining sum on the real axis. For simplicity's sake, typically the coefficients are normalized. However, this isn't so important as the normalization can be done and removed simply by using partial summation.

We improve the approximate functional equation for the exponential sums of the Fourier coefficients of the holomorphic cusp forms by giving an explicit upper bound for the error term appearing in the equation. The approximate functional equation is originally due to Jutila [9] and a crucial tool for transforming sums into shorter sums. This transformation changes the point of the real axis on which the sum is to be considered.

We also improve known upper bounds for the size estimates of the exponential sums. For very short sums we do not obtain any better estimates than the very easy estimate obtained by multiplying the upper bound estimate for a Fourier coefficient (they are bounded by the divisor function as Deligne [2] showed) by the number of coefficients. This estimate is extremely rough as no possible cancellation is taken into account. However, with small sums, it is unclear whether there happens any remarkable amounts of cancellation.

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B1 I don't believe it.

B2 I'm not impressed.

B3 This is the style in analytic number theory that I. Don't. Like.

I would also like to thank Doctor Tapani Matala-aho for very fast and useful comments on the thesis. I'm also very grateful for his understanding and kindness. Writing thesis was stressful enough as it was, so I really appreciate him for making it a tad bit easier and more fun.

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Contents

Introduction

Holomorphic cusp forms can be represented as Fourier series

$$F(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{\kappa-1}{2}} e(nz), \tag{1}$$

where the numbers $a(n)$ are called normalized Fourier coefficients, $e(nz) = e^{2\pi inz}$ and $\kappa \in \mathbb{Z}_+$ is the weight of the form. It is of interest to consider exponential sums of the normalized Fourier coefficients:

$$A(M,\Delta,\alpha) = \sum_{M \le n \le M+\Delta} a(n) e(n\alpha). \tag{2}$$

Wilton's estimate [16]

$$A(1, M-1, \alpha) \ll M^{1/2} \log M$$

from the year 1929 is a classical result. It is nearly sharp, only the logarithm can be removed and that was done by Jutila [9] in 1987. Before that it was proved by Deligne [2] that $|a(n)| \le d(n) \ll n^{\varepsilon}$. When the parameter $\alpha = h/k$ is rational with k not too large in comparison with M, there are much better estimates as Jutila showed by proving ([8], formula 1.5.20) that

$$\sum_{1 \le n \le M} a(n) n^{\frac{\kappa-1}{2}} e\left(\frac{h}{k} n\right) \ll k^{2/3} M^{\kappa/2 - 1/6 + \varepsilon},$$

which is equivalent to

$$A\left(1, M-1, \frac{h}{h}\right) \ll k^{2/3} M^{1/3+\varepsilon},$$

which in turn implies

$$A\left(M, \Delta, \frac{h}{k}\right) \ll k^{2/3} M^{1/3+\varepsilon},$$

when $1 \le \Delta \ll M$. However, in the general case it is impossible to obtain very good estimates for long sums. The situation changes drastically when one considers short sums. From now on we assume $1 \le \Delta \le M$. Jutila [9] proved the estimate

$$A(M,\Delta,\alpha) \ll M^{1/2-a_1} \left(1 + \Delta M^{-1/2}\right)$$

with a_1 positive, and he also proved that

$$A(M,\Delta,\alpha) \ll M^{1/2}. \tag{3}$$

These results are a natural starting point for this thesis. We will improve the results. We show that for, $1 \leq \Delta \ll M^{3/4}$,

$$A(M,\Delta,\alpha) \ll M^{1/2-f(\log_M \Delta)},$$

where f is positive when $\Delta \ll M^{3/4-\varepsilon}$ for any fixed $\varepsilon > 0$. As usual, Voronoi type sum formulas are very helpful in the proofs. Results for shorter sums lean heavily on good estimates for non-linear sums. For longer sums an approximate functional equation for exponential sums is crucial. We will also give an explicit formula for f. In the final chapter we will prove that

$$\left| \sum_{M \leq n \leq M+\Delta} a(n)w(n)e\left(\frac{n}{\sqrt{M}}\right) \right| \asymp \Delta M^{-1/4},$$

where w is a smooth weight function satisfying certain conditions and $M^{1/2+\varepsilon} \ll \Delta$ for some fixed $\varepsilon > 0$. This also yields the interesting corollary telling

$$\left| A\left(M, M^{3/4}, \frac{1}{\sqrt{M}}\right) \right| \asymp M^{1/2}$$

and we may hence conclude Jutila's estimate (3) to be best possible when $M^{3/4} \ll \Delta$. Another interesting corollary is the fact that the estimate to be derived is best possible for $\Delta \gg M^{3/4-1/32+\varepsilon}$.

Already in 1933 Wilton [17] proved an approximate functional equation for linear exponential sums involving the divisor function $d(n)$ in place of $a(n)$. Later Jutila [7] generalized the result by proving an approximate functional equation which used rational approximations of the parameter α.

In this thesis we will prove that the formula

$$\frac{A(M,\Delta;\alpha)}{M^{1/2}} = \frac{A\left(Mk^2\eta^2, \Delta k^2\eta^2; \beta\right)}{(Mk^2\eta^2)^{1/2}} + O\left(\left(Mk^2\eta^2\right)^{-1/12+\varepsilon}\right)$$

holds for any $\varepsilon > 0$, under the following assumptions:

- $\alpha = \frac{h}{k} + \eta$ is a Farey approximation of order $M^{1/4}$, i.e. $k \leq M^{1/4}$ and $|\eta| \leq \left(kM^{1/4}\right)^{-1}$
- $h\overline{h} \equiv 1 \pmod{k}$
- $\beta = -\frac{\overline{h}}{k} - (k^2\eta)^{-1}$
- $M \leq M_1 \leq 2M$ and $k^2\eta^2 M \gg 1$.

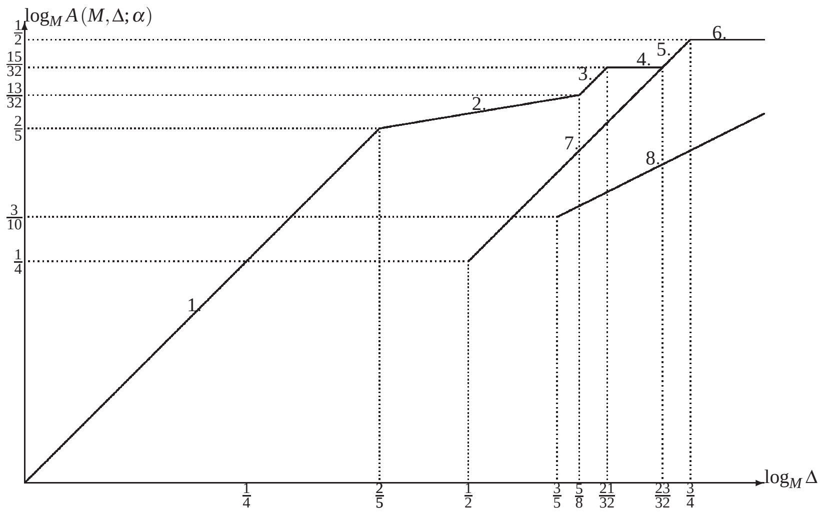
In [9] Jutila proved the approximate functional equation with an unspecified positive constant a in the place of $\frac{1}{12}$ in the error term of the above formula. He needed this result to prove the estimate (3).

Using Rankin's result [13] (the same result can also be find in Selberg's paper [14]) and partial summation, we obtain

$$\int_0^1 \left| \sum_{n\le M} a(n)e(n\alpha) \right|^2 d\alpha = \sum_{n\le M} |a(n)|^2 = BM + O\left(M^{3/5}\right).$$

Therefore, Jutila's result is sharp. However, the more precise form of the approximate functional equation is useful in attempts to obtain sharper estimates for short sums of the type $A(M,\Delta;\alpha)$.

The following figure shows the currently best estimates in a clear way.



1. The trivial estimate obtained by multiplying Deligne's estimate for a coefficient by the number of coefficients

2. The result from Theorem 24 as written in Corollary 27

3. Estimate from Theorem 30

4. Estimate from Theorem 33. However, it is not known whether this estimate may be improved.

5. Estimate from Theorem 33. It is proved in Theorem 39 that this bound is sharp.

6. Estimate derived using triangle inequality from Jutila's estimate for longer sums and proved here to be sharp

7. Sharp estimate for sum $\sum_{M \le n \le M+\Delta} a(n)w(n)e\left(\frac{n}{\sqrt{M}}\right)$

8. The average estimate derived from Rankin's formula.

Before considering the above mentioned linear sums, we will prove, using the Farey method due to Bombieri and Iwaniec [1] and Jutila [7], that

$$\sum_{M \le m \le M+\Delta} a(n)g(m)e(f(m)) \ll \left(\frac{\Delta}{M}\right)^{5/6} (G+\Delta G')M^{1/2}F^{1/3+\varepsilon}$$

for $f(z) = \eta z + Bz^{1/2}$, $F = |B|M^{1/2}$, $g \in C^1[M, M+\Delta]$, $|g(x)| \le G$ and $|g'(x)| \le G'$. For this, we will also prove certain lemmas on properties of rational numbers.

In the following $\ll$, $\gg$ and $\asymp$ are used in the standard way and the implied constants depend only on ε and κ.

A part of the results have appeared in [4]. The rest are to appear in [3]. Also, a part of the results has appeared in Karppinen's licentiate thesis [11].

Chapter 1

Notation, preliminaries and analytic lemmas

1.1 A brief introduction to holomorphic cusp forms, Voronoi formulas and Bessel functions

Definition 1. Write

$$\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}.$$

Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic. We call f a modular form of weight $\kappa \in \mathbb{Z}$ if

1. $f(z+1) = f(z)$ and $f\left(-\frac{1}{z}\right) = z^{\kappa} f(z)$, when $z \in \mathbb{H}$,
2. the function f has a Fourier expansion of the form

$$f(z) = a(0) + \sum_{m \geq 1} a(m) m^{\frac{\kappa-1}{2}} e^{2\pi i m z}.$$

In particular, f is a cusp form when $a(0) = 0$. The first condition may also be written in the following form:

1*. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with integer entries and determinant 1 (in other words, it is an element of the group $\mathbf{SL}_2(\mathbb{Z})$). Then

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{\kappa} f(\tau).$$

It is worth noting that although one may define non-holomorphic cusp forms, functions with the invariance condition with respect to another group, etc, in which case the set of possible weights κ may be fairly large, in the case of holomorphic modular forms, κ is always a positive even integer. We wrote the Fourier coefficients in the form $a(n)n^{(\kappa-1)/2}$ because we are interested in the normalized Fourier coefficients instead of the original ones. This does not really affect the results as we may change between

$$\sum_{M\leq n\leq M+\Delta} a(n)e(n\alpha) \text{ and } \sum_{M\leq n\leq M+\Delta} a(n)e(n\alpha)\, n^{(\kappa-1)2}$$

using partial summation ([**?**], Theorem 1.4):

Lemma 2 (Partial summation)**.** *Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be a sequence of real numbers such that $\lim_{n\to\infty}\lambda_n = \infty$ and let g be a continuously differentiable function on the interval $[\lambda_1,\infty]$. Let $(a_n)_{n=1}^{\infty}$ be an arbitrary sequence of complex numbers. We have*

$$\sum_{\lambda_1\leq\lambda_n\leq x} a_n g(\lambda_n) = A(x)g(x) - \int_{\lambda_1}^{\infty} a(t)g'(t)dt,$$

where

$$A(t) = \sum_{\lambda_1\leq\lambda_n\leq t} a_n.$$

Now the Ramanujan-Pettersson conjecture, proved by Deligne [2], states that

$$|a(n)| \leq d(n),$$

where

$$d(n) = \sum_{d|n, d>0} 1.$$

Using the well-known estimate for $d(n)$, we obtain $|a(n)| \ll n^{\varepsilon}$ for any fixed positive ε. Also, the normalization allows us to treat all the holomorphic cusp forms with the same calculations, without having to worry about the weights of the forms, and without having to carry along some extra terms.

It is an easy observation that a linear combination of holomorphic cusp forms of weight κ is also a holomorphic weight form of the same weight. Another easy observation is that the product of holomorphic weight forms of weights κ_1 and κ_2 is a holomorphic modular form of weight $\kappa_1+\kappa_2$. Also, if either of the modular forms is a cusp form, then their product is a cusp form as well.

Let us now devote some time to recall the Ramanujan τ-function. Define the discriminant function by the formula

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n),$$

where

$$q = e^{2\pi iz}.$$

Further, define the Eisenstein series

$$G_{2k}(z) = \sum_{\substack{c^2+d^2\neq 0\\ c,d\in\mathbb{Z}}} \frac{1}{(cz+d)^{2k}},$$

where $k \geq 2$ is an integer, and the normalized Eisenstein series

$$E_{2k}(z) = \frac{G_{2k}(z)}{2\zeta(k)}.$$

Then E_{2k} has a Fourier series of the type

$$E_{2k}(z) = 1 + \sum_{n\geq 0} a_n q^n.$$

It is an easy calculation to show that G_{2k} is a holomorphic modular form of the weight $2k$. It follows that the function

$$\Delta_1(z) = (60G_4(z))^3 - 27\,(140G_6(z))^2,$$

is a holomorphic modular form of the weight 12, and furthermore, it is a cusp form. It is not so simple (but pretty easy, anyway) to show that

$$\Delta_1(z) = \Delta(z).$$

From this we get an alternative representation for $\Delta(z)$. Typically we write the Fourier series of $\Delta(z)$ in the following form

$$\Delta(z) = \sum_{n\geq 1} \tau(n)e(nz).$$

The function $\tau(n)$, thus defined, is known as the Ramanujan τ function. This multiplicative function is of great interest due to the fact that very little is known about its properties. The Lehmer conjecture states that $\tau(n) \neq 0$. However, this has not yet been proved or disproved.

A necessary tool in this work is the Voronoi type summation formula due to Jutila, namely the formula ([8], formula (1.9.2))

$$\begin{aligned}&\sum_{a\leq n\leq b}{}' a(n)n^{(\kappa-1)/2}e_k(nh)f(n)\\ &= 2\pi k^{-1}(-1)^{\kappa/2}\sum_{n=1}^{\infty} a(n)e_k\left(-n\bar{h}\right)\int_a^b x^{(\kappa-1)/2}J_{\kappa-1}\left(\frac{4\pi\sqrt{nx}}{k}\right)f(x)dx, \quad (1.1)\end{aligned}$$

where $e_k(nh) = e\left(\frac{nh}{k}\right)$, $f \in C^1[a,b]$, $h\bar{h} \equiv 1 \pmod{k}$, $J(x)$ is the J-Bessel function, and $'$ in summation means that if a or b is an integer, then the corresponding term is to be halved, and otherwise the summation is carried out in the standard way. Typically, it is easier to consider sums when they are weighted with a smooth weight function:

Definition 3. We say that w is a *smooth weight function* on the interval $[a,b]$ if it has a support on the interval $[a,b]$ and for any $x \in [a,b]$

$$w(x) \ll 1,\ w^{(j)}(x) \ll (b-a)^{-j} \text{ for } j \in \{0,1,\dots,J\}, \tag{1.2}$$

where J is a sufficiently large integer. Assume further that $w(x)=1$ on an interval of length $\asymp b-a$.

Sometimes, to simplify the notation, we write

$$\tilde{A}(M,\Delta,\alpha) = \sum_{M \le n \le M+\Delta} a(n)e(n\alpha)w(n).$$

While working with weighted sums, a very useful version of the formula (4.2) is the following:

$$\sum_{a \le n \le b} a(n)e(n\alpha)w(n)$$
$$= 2\pi k^{-1}(-1)^{\kappa/2}\sum_{n=1}^{\infty} a(n)e_k\left(-n\bar{h}\right)\int_a^b J_{\kappa-1}\left(\frac{4\pi\sqrt{nx}}{k}\right)w(x)e(\eta x)\,dx, \tag{1.3}$$

where $w(x)$ is a sufficiently smooth weight function with support on the interval $[a,b]$, and $\alpha = \frac{h}{k}+\eta$.

Lebedev's book [12] gives several useful formulas for the J-Bessel functions. Firstly, it is possible to write it as a sum of Hankel functions $H_{\kappa-1}^{(1)}$ and $H_{\kappa-1}^{(2)}$:

$$J_{\kappa-1}(z) = \frac{1}{2}\left(H_{\kappa-1}^{(1)}(z) + H_{\kappa-1}^{(2)}(z)\right).$$

When $\Re\, z > 0$ and $|z| \ge 1$, we may use the asymptotic formula for the Hankel functions:

$$H_\nu^{(j)}(z) = \sqrt{\frac{2}{\pi z}}e\left((-1)^{j-1}\left(\frac{z}{2\pi}-\lambda\right)\right)\omega_H^{(j)}(z) + O(|z|^{-3/2-H}).$$

Here

$$\omega_H^{(j)}(z) = \sum_{h=0}^{H}(-1)^{jh}C_{h,\nu}z^{-h} = 1 + (-1)^j C_{1,\nu}z^{-1} + \cdots,$$

where $\lambda = \frac{\nu}{4}+\frac{1}{8}$ and the coefficients $C_{h,\nu}$ are certain constants ([12], formulas 5.11.4-5). Let us choose $H=1$. Then

$$H_{\kappa-1}^{(j)}\left(\frac{4\pi\sqrt{nt}}{k}\right)$$
$$= \frac{1}{\pi\sqrt{2}}k^{1/2}n^{-1/4}t^{-1/4}e\left((-1)^{j-1}\left(\frac{2\sqrt{nt}}{k}-\lambda\right)\right)\omega_1^{(j)}\left(\frac{4\pi\sqrt{nt}}{k}\right)$$
$$+ O\left(k^{5/2}n^{-5/4}t^{-5/4}\right), \tag{1.4}$$

where $\lambda = \frac{\kappa}{4} - \frac{1}{8}$. This also gives a very useful asymptotic expansion for the J-Bessel function ([12], formula (5.11.6)):

$$J_{\kappa-1}\left(4\pi\frac{\sqrt{nx}}{k}\right) = \frac{1}{\sqrt{2}\pi}\sqrt{\frac{k}{\sqrt{nx}}}\cos\left(4\pi\frac{\sqrt{nx}}{k} - \frac{1}{2}\kappa\pi + \frac{1}{4}\pi\right)$$
$$+ c_1\left(\frac{k}{\sqrt{nx}}\right)^{3/2}\sin\left(4\pi\frac{\sqrt{nx}}{k} - \frac{1}{2}\kappa\pi - \frac{1}{4}\pi\right) + O\left(\frac{k^{5/2}}{(nx)^{5/4}}\right) \quad (1.5)$$

where c_1 is a constant.

1.2 Known lemmas and enough notation to support them

As we may now easily think what kind of integrals and sums are to be estimated, it is natural to introduce a couple of derivative tests ([15] Lemmas 4.3 and 4.5).

Lemma 4 (First derivative test)**.** *Let $f(x)$ and $g(x)$ be real functions on the interval $[a,b]$ such that $\frac{g(x)}{f'(x)}$ is monotonic and*

$$\left|\frac{f'(x)}{g(x)}\right| \geq \lambda > 0.$$

Then

$$\int_a^b g(x)e(f(x))\,dx \ll \frac{1}{\lambda}.$$

Lemma 5 (Second derivative test)**.** *Let $f(x)$ and $g(x)$ be real functions on the interval $[a,b]$ such that $\frac{g(x)}{f'(x)}$ is monotonic and*

$$\left|f''(x)\right| \geq \lambda > 0,\ |g(x)| \leq G,$$

then

$$\int_a^b g(x)e(f(x))\,dx \ll \frac{G}{\sqrt{\lambda}}.$$

However, sometimes when we know estimates for the higher derivates, we get better estimates for integrals using the following lemma which is a slightly modified version of lemma ([10], Lemma 6). The proof is similar to that of the original lemma. To state the lemma, we need the following definition:

Definition 6. Given $X,Y,Z \in \mathbb{R}$ we write

$$D(X,Y,Z) = \{x \in \mathbb{C} :\ \exists y \in [X,Y] : |x-y| < Z\}.$$

The lemma reads as follows.

Lemma 7. *Let A be a function which is compactly supported in a finite interval $[M_1, M_2]$ and at least $P \geq 0$ times differentiable. Assume also that there exist two quantities A_0 and A_1 such that for any non-negative integer $\nu \leq P$ and for any $x \in [M_1, M_2]$,*

$$A^{(\nu)}(x) \ll A_0 A_1^{-\nu}.$$

Also, let B be a function which is real-valued on $[M_1, M_2]$, and regular throughout the complex domain $D(M_1, M_2, \rho)$; and assume that there exists a quantity B_1 such that

$$0 < B_1 \ll \left| B'(x) \right|$$

for any point x in the domain. Then we have

$$\int_{-\infty}^{\infty} A(x) e(B(x))\, dx \ll A_0 (A_1 B_1)^{-P} \left(1 + \frac{A_1}{\rho}\right)^P (M_2 - M_1).$$

This lemma and the derivative test are useful while estimating oscillating integrals. However, some integrals require a bit different treatment. In this case a slightly different weight function is needed.

The following definition is similar to formula (2.1.2) in [8].

Definition 8. Let $\eta_J(x)$ be the function on the interval $[a, b]$ satisfying the following equation for any integrable function h:

$$U^{-J} \int_0^U du_1 \cdots \int_0^U du_J \int_{b+u}^{b-u} h(x) dx = \int_a^b \eta_J(x) h(x) dx \tag{1.6}$$

where $u = u_1 + \cdots + u_J$ and $U < (b-a)/2J$, and define η_0 to be the characteristic function of the interval $[a, b]$.

Notice that η_J, for large J, is a smooth weight function which is equal to 1 on the interval $[a + JU, b - JU]$. A weight function of this type is very useful every now and then. For instance, the saddle-point lemma (which will be formulated soon) requires this weight function. One may easily compute the Fourier transform

$$\hat{\eta}_J(\lambda) = -(i\lambda)^{-J-1} \left(e^{-i\lambda U} - 1\right)^J \left(e^{-i\lambda b} - e^{-i\lambda a}\right).$$

This implies that $\eta_J(x)$ is $J - 1$ times differentiable.

To keep the notation short and simple, denote by D the complex domain consisting of points z satisfying the condition $|z - x| < \mu$ for some $x \in [a, b]$, where $\mu \asymp a \asymp b$. That is, $D = D(a, b; \mu)$ in the notation of Definition 6.

It is now the time to state a simplified version of the Saddle-point lemma ([8] Theorem 2.1).

Lemma 9 (Saddle-point lemma)**.** *Let $F \gg a^{\varepsilon}$ for some positive ε, G and U be positive parameters. Let f be a holomorphic function on the domain D. Assume further*

- *the function f is real on the interval $[a,b]$*
- *$f'(x) \ll \frac{F}{a}$ for $x \in D$*
- *$|f''(x)| \asymp \frac{F}{a^2}$ for $x \in [a,b]$.*

Let g be a holomorphic function with $g(z) \ll G$ in the domain D. Denote the characteristic function of

$$(a, a+JU) \bigcup (b-JU, b) \tag{1.7}$$

by $\delta(x)$. Let x_0 be the (possibly existing) zero of $f'(x)+\gamma$ in the interval (a,b), and suppose that $U \gg \delta(x_0)F^{-1/2}a + a^{\varepsilon}$. Write

$$E_J(x) = \frac{G}{\left(|f'(x)+\gamma| + F^{1/2}/a\right)^{J+1}}.$$

Then

$$\begin{aligned}\int_a^b \eta_J g(x) e\left(f(x)+\gamma x\right) dx = \xi_J(x_0) g(x_0) f''(x_0)^{-1/2} e\left(f(x_0)+\gamma x_0 + 1/8\right) \\ + O\left(\left(1+\delta(x_0)F^{1/2}\right) GaF^{-3/2}\right) \\ + O\left(U^{-J} \sum_{j=0}^{J} \left(E_J(a+jU) + E_J(b-jU)\right)\right),\end{aligned}$$

where ξ_J is a bounded function on (a,b) with the following properties:

- *$\xi_J(x) = 1$ on $(a+JU, b-JU)$*
- *$\xi_J'(x)$ is continuous and $\xi_J'(x) \ll U^{-J}$ on the set (1.7), except possibly at the points $a+jU$, $b-jU$, $j = 1, \dots, J-1$.*

If x_0 does not exist, then the terms and conditions involving x_0 are to be omitted.

This differs only slightly from Lemma 3 in [9] which was also a simplified version from the same saddle point lemma. In the original version, there is the assumption $F \gg 1$. In the version of the paper [9], this assumption is in the form $F \gg a$. This assumption allows us to omit the first error term as it may be contained in others. However, this assumption may be replaced with the assumption $F \gg a^{\varepsilon}$ for any fixed $\varepsilon \geq 0$ as is done in this version. It is still easy to see that the same error term may be omitted.

Let us now state the most technical looking lemma in this thesis. We use $\delta_1, \delta_2, \ldots$ for some fixed constants which can be chosen to be arbitrarily small. A slightly different version of the following lemma is due to Jutila ([8] Theorem 3.4). After stating the lemma, we will briefly explain the differences in the lemmas and proofs.

Lemma 10. *Assume $M_2 \leq 2M_1$ and let f and g be holomorphic functions on the open set $D(M_1, M_2, cM_1)$, where c is a positive constant. Let us assume that $f(x)$ is real when $M_1 \leq x \leq M_2$. Assume further that there are parameters F and G such that*

$$|g(z)| \ll G, \ |f'(z)| \ll \frac{F}{M_1}, \ \left|f''(x)\right| \gg \frac{F}{M_1^2}$$

when $z \in D(M_1, M_2, cM_1)$ and $M_1 \leq x \leq M_2$. Let $r = \frac{a}{k}$ be a rational number satisfying

$$1 \leq k \ll M_1^{1/2-\delta_1}, \ |r| \asymp \frac{F}{M_1}, \ f'(M(r)) = r$$

for some $M(r) \in [M_1, M_2]$. Define

$$p_{j,n}(x) = f(x) - rx + (-1)^{j-1}\left(2\frac{\sqrt{nx}}{k} - \frac{1}{8}\right)$$

and

$$n_j = k^2 M_j (r - f'(M_j))^2.$$

Let $x_{j,n}$ be the unique zero of the function $p'_{j,n}(x)$ on the interval (M_1, M_2) when $n < n_j$. Let

$$U \gg M_1^{1+\delta_2} F^{-1/2},$$

Denote

$$\begin{aligned} M_1 &= M(r) - m_1, \quad M_2 = M(r) + m_2 \\ M'_1 &= M_1 + JU = M(r) - m'_1, \quad M'_2 = M_2 - JU = M(r) + m'_2 \end{aligned}$$

assuming that J is large enough. Assume further that $m_1 \asymp m_2 \asymp m'_2 \asymp m'_1$,

$$M_1^{1+\delta_3} F^{-1/2} \ll m_j \ll M_1^{1-\delta_4}. \tag{1.8}$$

Let us define similarly to n_j

$$n'_j = k^2 M'_j (r - f'(M'_j))^2.$$

Then

$$\begin{aligned} &\sum_{M_1 \leq m \leq M_2} \eta_J(m) a(m) g(m) e(f(m)) \\ &\quad = \frac{i}{\sqrt{2k}} \sum_{j=1}^{2} \sum_{n<n_j} \Phi_j(r,n) + O\left(G(|a|k)^{1/2} M_1^{-1} m_1^{1/2} U \log M_1\right), \end{aligned} \tag{1.9}$$

where

$$\Phi_j(r,n) = v_J(r,n)a(n)e\left(-\frac{n\overline{a}}{k}\right)n^{-1/4}x_{j,n}^{-1/4}\times$$
$$g(x_{j,n})\left|p''_{j,n}(x_{j,n})\right|^{-1/2}e\left(p_{j,n}(x_{j,n})+\frac{|f''(M_1)|}{f''(M_1)}\frac{1}{8}\right),$$

and

$$\begin{cases} v_J(r,n) = 1, & \text{when } n < n'_j \\ v_J(r,n) \ll 1, & \text{when } n < n_j \\ v_J(r,n) = 0, & \text{when } n \geq n_j. \end{cases}$$

The functions $v_J(r,y)$ *and* $\frac{\partial}{\partial y}(v_J(r,y))$ *are piecewise continuous on the interval* (n'_j, n_j) *having at most* $J-1$ *discontinuities and*

$$\frac{\partial}{\partial y}(v_J(r,y)) \ll (n_j - n'_j)^{-1}$$

when $\frac{\partial}{\partial y}(v_J(r,y))$ *exists.*

Remark 11. In the original theorem $f''(x)$ is assumed to be positive, but it may be negative, as well, if absolute values and signs are introduced. Also, the condition 1.8 is weaker than the original condition (3.1.8) but it may be used in the case of the holomorphic cusp forms.

1.3 Miscellaneous lemmas

In this section we collect various technical results and simple lemmas which will be used in the proofs to come. Some of the results may look completely random, and maybe even utterly useless. However, the reader should not worry, they will all be used. We begin by fixing some notation. Remember that k and η satisfy the conditions given in the introduction.

Let J be a fixed natural number that can be chosen to be arbitrarily large. Assume $0 < \Delta \leq M$ and let $w(x)$ stand for a J times differentiable weight function on the interval $[M-JU, M+\Delta+JU]$, where $M^{1/2+\varepsilon}k \ll U$ for some fixed ε and $M-JU \asymp M$. Let

$$w(x) = 1 \quad \text{for} \quad x \in [M, M+\Delta]$$

and suppose moreover that

$$w^{(m)}(x) \ll U^{-m},\ w(M-JU) = w(M+\Delta+JU) = 0 \tag{1.10}$$

for any non-negative $m \leq J$.

Denote

$$M_{-1} = M-JU,\ M_1 = M+\Delta, \text{ and } M_2 = M+\Delta+JU.$$

Write also,

$$N_{-1} = k^2\eta^2 M_{-1},\ N = k^2\eta^2 M,\ N_1 = k^2\eta^2 M_1 \text{ and } N_2 = k^2\eta^2 M_2.$$

The proofs of the following lemmas are simple. They mostly use the derivative tests and Lemma 7 in addition to basic analysis and arithmetic. Their importance is visible while treating the terms arising from the asymptotic expansion of the J-Bessel function and the integral around it in the current work. To simplify, write $N = k^2\eta^2 M$.

Lemma 12. *Let $\eta > 0$ and $c > 1$ be a fixed constant. We have*

$$k^{-1}(-1)^{\kappa/2}\sum_{n\geq cN} a(n)e_k(-n\bar{h})n^{-1/4}\int_{M_{-1}}^{M_2} w(x)x^{-1/4}e\left(\eta x - 2\frac{\sqrt{nx}}{k}\right)dx \ll_c 1.$$

Proof. We first use Lemma 7 with $A(x) = w(x)x^{-1/4}$, $B(x) = \eta x - 2\frac{\sqrt{nx}}{k}$, $A_0 = M_{-1}^{-1/4}$, $A_1 = U$, $B_1 = \frac{\sqrt{n}}{k\sqrt{M}}$ and $\rho \asymp M_{-1}$. This gives

$$\int_{M_{-1}}^{M_2} w(x)x^{-1/4}e\left(\eta x - 2\frac{\sqrt{nx}}{k}\right)dx \ll M^{-1/4}U^{-P}k^P M^{P/2}n^{-P/2}(U+\Delta)$$

for any $P \leq J$. Substituting this estimate, the left-hand side is dominated by

$$\ll k^{-1}\sum_{n\geq cN} n^{\varepsilon-1/4-P/2}M^{-1/4+P/2}U^{-P}k^P(U+\Delta)$$

and this is $\ll 1$ when P is sufficiently large. $\square$

Lemma 13. *Let c be a (large) constant and $\eta > 0$. We have*

$$k^{1/2}\sum_{n\geq cN} a(n)e_k(-n\bar{h})n^{-3/4}\int_{M_{-1}}^{M_2} w(x)x^{-3/4}e\left(\eta x \pm \frac{\sqrt{nx}}{k}\right)dx \ll 1.$$

Lemma 14. *Let $\eta \geq 0$. Then*

$$k^{-1}(-1)^{\kappa/2}\sum_{n=1}^{\infty} a(n)e_k(-n\bar{h})n^{-1/4}\int_{M_{-1}}^{M_2} w(x)x^{-1/4}e\left(\eta x + 2\frac{\sqrt{nx}}{k}\right)dx \ll 1.$$

Lemma 15. *Let $\eta \geq 0$. Then*

$$k^{1/2}\sum_{n=1}^{\infty} a(n)e_k(-n\bar{h})n^{-3/4}\int_{M}^{M+\Delta} x^{-3/4}e\left(\eta x + 2\frac{\sqrt{nx}}{k}\right)w(x)dx \ll 1$$

The following lemma is of a very similar nature, and it is easy to prove in a slightly different way.

Lemma 16. *Let c be a constant and let $\eta > 0$. We have*

$$k^{1/2} \sum_{1 \le n \le cN} a(n) n^{-3/4} \int_{M_{-1}}^{M_2} w(x) x^{-3/4} e\left(\eta x \pm \frac{\sqrt{nx}}{k}\right) dx \ll k\left(k^2\eta^2 M\right)^{\varepsilon}.$$

Proof. The second derivative test Lemma 5 gives

$$\int_{M_{-1}}^{M_2} w(x) x^{-3/4} e\left(\eta x \pm \frac{\sqrt{nx}}{k}\right) dx \ll \frac{\sqrt{k}}{n^{1/4}}.$$

Hence

$$k^{1/2} \sum_{1 \le n \le cN} a(n) n^{-3/4} \int_{M_{-1}}^{M_2} w(x) x^{-3/4} e\left(\eta x \pm \frac{\sqrt{nx}}{k}\right) dx \ll k\left(k^2\eta^2 M\right)^{\varepsilon}.$$

□

Lemma 17. *Let c be any given constant. Let T be any number of the form $M - jU$ or $M + jU$, where $0 \le j \le J$, $U = M^{1/2}\eta^{-1/2}\left(k^2\eta^2 M\right)^d$ and d is some fixed number. Then*

$$k^{-1} \sum_{1 \le n \le cN} \frac{n^{\varepsilon}\sqrt{k}}{n^{1/4}} M^{-1/4} U^{-J} \left(\left|\eta - \frac{\sqrt{n}}{\sqrt{T}k}\right| + \frac{n^{1/4}}{\sqrt{k}M^{3/4}}\right)^{-J-1} \ll \sqrt{M}\left(k^2\eta^2 M\right)^{\varepsilon - Jd}.$$

Proof. We estimate the left-hand side by $S_1 + S_2 + S_3$, where

$$S_1 = k^{-1/2} M^{-1/4} U^{-J} \sum_{|n - k^2\eta^2 T| \le \sqrt{N}} n^{\varepsilon - 1/4} \left(\frac{n^{1/4}}{\sqrt{k}M^{3/4}}\right)^{-J-1}$$

$$S_2 = k^{-1/2} M^{-1/4} U^{-J} \sum_{1 \le n \le k^2\eta^2 T - \sqrt{N}} n^{\varepsilon - 1/4} \left|\eta - \frac{\sqrt{n}}{\sqrt{T}k}\right|^{-J-1}$$

and S_3 is like S_2 but with the condition $k^2\eta^2 T + \sqrt{N} \le n \le cN$. Then it is easy to verify the claimed upper bound for each term S_i.

□

Chapter 2

Arithmetic Lemmas

Before getting to the actual bounds, we need some more lemmas. These lemmas are arithmetic in nature, and some of the results are both new and even of independent interest.

Huxley has proved the following useful result (Chapter 1, [6]).

Lemma 18. *Let $F(Q)$ be a Farey sequence of order Q on the interval J. If the length of J is Δ, then*

$$\sum_{F(Q)\cap J} 1 \ll 1+\Delta Q^2, \quad \sum_{\frac{a}{k}\in F(Q)\cap J} \frac{1}{k} \ll 1+\Delta Q. \tag{2.1}$$

The next lemma introduces *the Magic Matrix* ([5], Lemma 7.17).

Lemma 19. *Let the rational numbers $\frac{a_1}{k_1}$ and $\frac{a_2}{k_2}$, where $(a_1,k_1)=(a_2,k_2)=1$, satisfy the condition*

$$\left|\left|\frac{\overline{a_1}}{k_1}-\frac{\overline{a_2}}{k_2}\right|\right| \le \Delta_1,\ (0<\Delta_1\le 1), \tag{2.2}$$

where $||x||$ is the distance of x from the nearest integer and $\overline{a_i}$ is the solution of the congruence $a_i\overline{a_i}\equiv 1 \pmod{k_i}$. Then there exists a matrix

$$M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},$$

with integer entries with determinant 1 such that,

$$\begin{pmatrix}a_2\\ k_2\end{pmatrix}=M\begin{pmatrix}a_1\\ k_1\end{pmatrix},$$

$$\alpha=\gamma\frac{a_2}{k_2}+\frac{k_1}{k_2},\ \beta=\frac{-\gamma a_1a_2+a_2k_2-a_1k_1}{k_1k_2},\ \delta=-\gamma\frac{a_1}{k_1}+\frac{k_2}{k_1}, \tag{2.3}$$

and

$$|\gamma|\le \Delta_1k_1k_2. \tag{2.4}$$

Remark 20. Because $\bar{a_1}$ is defined only modulo k_i, we may assume

$$\left|\frac{\bar{a_1}}{k_1}-\frac{\bar{a_2}}{k_2}\right|\leq\Delta_1.$$

As far as we know, the next result is new; to be precise, it differs from a result of Bombieri and Iwaniec ([1] Theorem 4.1) by the additional condition (2.9).

Lemma 21. *Let A and Q be natural numbers and let $\Delta_1\leq 1$ and $\Delta_2,\Delta_3\ll 1$ be positive real numbers such that $A\Delta_3\gg Q$. Let us assume that rational numbers $\frac{a_i}{k_i}$ satisfy the conditions*

$$a_i\asymp A,\ k_i\asymp Q \text{ and } (a_i,k_i)=1. \tag{2.5}$$

Let $[R,R']$ be a subinterval of $[C_1A/Q,C_2A/Q]$ such that $R'-R\asymp\Delta_3AQ^{-1}$. Assume further that $\psi(x)$ is a continuously differentiable function on $[R,R']$ such that

$$\psi(x)\asymp H \text{ and } \psi'(x)\asymp\frac{QH}{A} \tag{2.6}$$

for some parameter H. Then the number of pairs of rational numbers $\left(\frac{a_1}{k_1},\frac{a_2}{k_2}\right)$ satisfying the conditions

$$\left|\frac{\bar{a_1}}{k_1}-\frac{\bar{a_2}}{k_2}\right|\leq\Delta_1, \tag{2.7}$$

$$\left|k_1\psi\left(\frac{a_1}{k_1}\right)-k_2\psi\left(\frac{a_2}{k_2}\right)\right|\leq\Delta_2QH, \tag{2.8}$$

and

$$\frac{a_1}{k_1},\frac{a_2}{k_2}\in[R,R'] \tag{2.9}$$

is

$$\ll\Delta_3AQ+\Delta_m\Delta_3A^2+\Delta_1(\Delta_1+\Delta_2)\Delta_3^2A^2Q^2, \tag{2.10}$$

where $\Delta_m=\min(\Delta_2,\Delta_3)$.

Proof. Notice first that by lemma 18 the number of the rational numbers $\frac{a}{k}$ on the interval $[R,R']$ satisfying the conditions (2.5) is at most $\ll\Delta_3AQ$. In order to estimate the number of pairs, let us classify these by the matrices of lemma 19. Let us first consider those matrices having at least one entry equal to zero. If $\gamma=0$, then $a_2=a_1+\beta k_1$, $k_2=k_1$, and by (2.9) we have

$$|\beta|=\left|\frac{a_1}{k_1}-\frac{a_2}{k_2}\right|\ll\Delta_3AQ^{-1}.$$

However, the conditions (2.6) and (2.8) give

$$|k_1|\left|\psi\left(\frac{a_1}{k_1}\right)-\psi\left(\frac{a_1}{k_1}+\beta\right)\right|\asymp|\beta||k_1|QHA^{-1}\ll\Delta_2QH,$$

so β has at most $\ll 1+\Delta_m AQ^{-1}$ different possible values. Hence the number of pairs is

$$\ll \Delta_3 AQ(1+\Delta_m AQ^{-1}).$$

If $\beta=0$, then $a_2=a_1$, $k_2=\gamma a_1+k_1$ and

$$|\gamma| = \left|\frac{k_1}{a_1}-\frac{k_2}{a_2}\right| \ll \Delta_3 QA^{-1},$$

so the number of pairs will be $\ll \Delta_3 AQ(1+\Delta_3 QA^{-1}) \ll \Delta_3 AQ$, since $\Delta_3 QA^{-1} \ll 1$.

If $\alpha=0$, then

$$|\delta| \le \frac{k_2}{k_1}+\frac{a_1}{a_2} \le 4$$

and if $\delta=0$, then

$$|\alpha| \le \frac{a_2}{a_1}+\frac{k_1}{k_2} \le 4.$$

In both cases the number of pairs will be $\ll \Delta_3 AQ$.

Let $\alpha\beta\gamma\delta \neq 0$ and D be a constant. We will later show that the number of matrices satisfying the condition $|\gamma| \le \frac{DQ}{\Delta_3 A}$ is bounded (see formula (2.13)), and the number of corresponding pairs is $\ll \Delta_3 AQ$. Let us assume from now on that the matrix $M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is given and

$$|\gamma| > \frac{DQ}{\Delta_3 A}. \tag{2.11}$$

Now the condition (2.8) yields

$$\left|\frac{k_2}{k_1}\psi\left(\frac{a_2}{k_2}\right)-\psi\left(\frac{a_1}{k_1}\right)\right| \le \Delta_2 H.$$

Write $x=\frac{a_1}{k_1}$. Then

$$\frac{a_2}{k_2}=\frac{\alpha x+\beta}{\gamma x+\delta}, \text{ and } \frac{k_2}{k_1}=\gamma x+\delta.$$

Define

$$\varphi(x) := \frac{k_2}{k_1}\psi\left(\frac{a_2}{k_2}\right)-\psi\left(\frac{a_1}{k_1}\right) = (\gamma x+\delta)\psi\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)-\psi(x),$$

so that $|\varphi(x)| \ll \Delta_2 H$ and

$$\varphi'(x) = \gamma\psi\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)+(\gamma x+\delta)^{-1}\psi'\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)-\psi'(x)$$

if we interpret x as a continuous variable. Since

$$(\gamma x+\delta)^{-1}\psi'\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)-\psi'(x)\ll\frac{QH}{A}$$

and

$$\left|\gamma\psi\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)\right|\gg|\gamma|H>\frac{DQH}{\Delta_3 A}\gg\frac{DQH}{A},$$

we may assume that $|\varphi'(x)|\gg|\gamma|H$ (by formula (2.11), when D is large enough). Let $\left(\frac{a_1}{k_1},\frac{a_2}{k_2}\right)$ and $\left(\frac{a_1'}{k_1'},\frac{a_2'}{k_2'}\right)$ be two pairs corresponding to the matrix M and satisfying the assumptions (2.7)-(2.9) of the lemma. Let x lie between $\frac{a_1}{k_1}$ and $\frac{a_1'}{k_1'}$. Then the inequality

$$\left|\varphi'(x)\right|\gg|\gamma|H$$

holds also for this value of x, because $\frac{\alpha x+\beta}{\gamma x+\delta}$ lies between $\frac{a_2}{k_2}$ and $\frac{a_2'}{k_2'}$, and $\gamma x+\delta$ lies between $\frac{k_2}{k_1}$ and $\frac{k_2'}{k_1'}$. Hence we find upper and lower bounds for the integral

$$\int_{a_1/k_1}^{a_1'/k_1'}\varphi'(t)dt,$$

namely

$$|\gamma|H\left|\frac{a_1}{k_1}-\frac{a_1'}{k_1'}\right|\ll\left|\int_{a_1/k_1}^{a_1'/k_1'}\varphi'(t)dt\right|\ll\Delta_2 H,$$

and it follows that

$$\left|\frac{a_1}{k_1}-\frac{a_1'}{k_1'}\right|\ll\frac{\Delta_2}{|\gamma|}.$$

An interpretation of this result is that if $\frac{a_1}{k_1}$ is given, then all the other possible values of $\frac{a_1'}{k_1'}$ corresponding to the given matrix lie on an interval of length $\ll\frac{\Delta_2}{|\gamma|}$. On an interval of this length (remember that $1\ll|\gamma|^{-1}\Delta_1 Q^2$) there are

$$\ll 1+\frac{\Delta_2 Q^2}{|\gamma|}\ll\frac{(\Delta_1+\Delta_2)Q^2}{|\gamma|}$$

rational numbers satisfying the conditions (2.5). It suffices to count the number of matrices in order to estimate the number of pairs. First notice that the conditions (2.3) and (2.9) imply

$$|\alpha-\gamma R|\ll 1+|\gamma|\Delta_3 AQ^{-1} \tag{2.12}$$

$$|\delta+\gamma R|\ll 1+|\gamma|\Delta_3 AQ^{-1} \tag{2.13}$$

and therefore for any given γ we may choose α and δ in

$$1+|\gamma|\Delta_3 AQ^{-1}\ll|\gamma|\Delta_3 AQ^{-1}$$

different ways (by condition (2.11)). Moreover, $\alpha\delta \equiv 1 \pmod{\gamma}$. Then, for any given pair (γ, α), δ can be chosen only in

$$\ll 1 + \Delta_3 A Q^{-1} \ll \Delta_3 A Q^{-1}$$

different ways. Hence the number of pairs will be

$$\ll \sum_{1 \le |\gamma| \ll \Delta_1 Q^2} \Delta_3 A Q^{-1} \frac{(\Delta_1 + \Delta_2) Q^2}{|\gamma|} |\gamma| \Delta_3 A Q^{-1} \ll \Delta_1 (\Delta_1 + \Delta_2) \Delta_3^2 A^2 Q^2.$$

This also shows that when $1 \le |\gamma| < \frac{DQ}{\Delta_3 A}$, the number of suitable matrices is bounded. □

The next lemma is a useful variant of the preceding one.

Lemma 22. *Let the assumptions of Lemma 21 be satisfied with the exception of (2.7) Then the number of pairs will be*

$$\Delta_3 A Q (1 + (\Delta_2 + \Delta_3) Q)(1 + \Delta_m A). \tag{2.14}$$

Proof. Let us choose two rational numbers $\frac{a_1}{k_1}$ and $\frac{a_2}{k_2}$ satisfying the conditions of the lemma and write

$$E := k_1 \psi\left(\frac{a_1}{k_1}\right) - k_2 \psi\left(\frac{a_2}{k_2}\right).$$

Then

$$\left| E - (k_1 - k_2) \psi\left(\frac{a_1}{k_1}\right) \right| = |k_2| \left| \psi\left(\frac{a_1}{k_1}\right) - \psi\left(\frac{a_2}{k_2}\right) \right| \ll \Delta_3 Q H.$$

From the condition (3) we obtain $|E - 0| \le \Delta_2 Q H$. So if

$$|k_1 - k_2| \psi\left(\frac{a_1}{k_1}\right) \asymp |k_1 - k_2| H \gg (\Delta_2 + \Delta_3) Q H,$$

i.e. $|k_1 - k_2| \gg (\Delta_2 + \Delta_3) Q$, then the conditions (2.8) and (2.9) cannot hold simultaneously. Therefore we may suppose that

$$|k_2 - k_1| \ll (\Delta_2 + \Delta_3) Q. \tag{2.15}$$

Let $\frac{a_1}{k_1}$ be given. According to the condition (2.15) the denominator k_2 can be chosen in at most $\ll 1 + (\Delta_2 + \Delta_3) Q$ different ways. Since a_1, k_1 and k_2 will then be determined, we deduce from the assumptions (2.6) and (2.8) that a_2 can be chosen in at most $\ll 1 + \Delta_2 A$ different ways. Because this cannot exceed the number of all possibilities for a_2, the actual number of possibilities will be $\ll \min(1 + \Delta_2 A, 1 + \Delta_3 A)$. Therefore, for any given $\frac{a_1}{k_1}$, we can choose $\frac{a_2}{k_2}$ in at most

$$\ll (1 + (\Delta_2 + \Delta_3) Q)(1 + \Delta_m A) \tag{2.16}$$

different ways satisfying the conditions (2.8), (2.9) and (2.15). Because the rational number $\frac{a_1}{k_1}$ has $\ll \Delta_3 A Q$ different possibilities, we obtain (2.14) from (2.16). □

2.1 A Bound for a Non-linear Exponential Sum

Even though our main interest lies in estimating linear exponential sums, we concentrate on non-linear exponential sums for a while and prove a result that will later serve as a necessary auxiliary theorem.

Theorem 23. *Let $\eta, B \in \mathbb{R}$, $1 \le \Delta \le M$, and denote $F = |B|M^{1/2}$. We assume that*

$$M^2 \ll \Delta F. \tag{2.17}$$

Let

$$f(z) = \eta z + Bz^{1/2}. \tag{2.18}$$

Let $g \in C^1[M, M+\Delta]$ and

$$||g||_\infty \le G, \quad ||g'||_\infty \le G'. \tag{2.19}$$

Then

$$\sum_{M \le m \le M+\Delta} a(m)g(m)e(f(m)) \ll \left(\frac{\Delta}{M}\right)^{5/6}(G+\Delta G')M^{1/2}F^{1/3+\varepsilon}. \tag{2.20}$$

Proof. Even though the beginning of the proof follows closely the proof of theorem 4.6 in [8], it will be presented for the sake of completeness.

As $e^{2\pi in} = 1$ for any integer n, we may assume that $0 \le |\eta| < 1$ and that the sign of η is the same as that of B. Since the sign of B is not essential in the proof, we may choose $B > 0$. Then $f'(x)$ is decreasing on the interval $[M, M+\Delta]$. Writing $\xi = \frac{\Delta}{M}$, the condition (2.17) becomes $M \ll \xi F$ and $0 < \xi \le 1$. In the following, δ is a small fixed number.

It is enough to show that

$$S(M,\xi) = \sum_{M \le m \le M+\Delta} a(m)e(f(m)) \ll \xi^{5/6}M^{1/2}F^{1/3+\varepsilon},$$

because the original claim (2.20) can then be derived using partial summation:

$$\begin{aligned}\sum_{M \le m \le M+\Delta} & a(m)g(m)e(f(m)) \\ &= S(M,\xi)g(M+\Delta) - \int_M^{M+\Delta} S\left(M, \frac{t-M}{M}\right)g'(t)dt \\ &\ll \xi^{5/6}M^{1/2}F^{1/3+\varepsilon}G + G'\Delta\xi^{5/6}M^{1/2}F^{1/3+\varepsilon}.\end{aligned}$$

If $M \ll \xi^{-1/3}F^{2/3+\delta}$, then estimating simply by absolute values we obtain

$$S(M,\xi) \ll M^{1+\varepsilon}\xi \ll \xi^{5/6}M^{1/2}F^{1/3+\varepsilon}.$$

Therefore, the original condition (2.17) may be replaced by

$$\xi^{-1/3}F^{2/3+\delta} \le M \ll \xi F. \tag{2.21}$$

The existence of such a number M implies that

$$1 \ll \xi^{4/3}F^{1/3-\delta}. \tag{2.22}$$

When $z \in [M, M+\Delta]$, we get the easy estimates

$$|f'(z)| = \left|\eta + \frac{1}{2}Bz^{-1/2}\right| \ll \frac{F}{M}, \quad |f''(z)| = \left|-\frac{1}{4}Bz^{-3/2}\right| \asymp \frac{F}{M^2} \tag{2.23}$$

and use the Farey method. Choose $K = \left\lfloor \xi^{-1/3}M^{1/2}F^{-1/3-\delta/2} \right\rfloor$ to be the order of the Farey sequence. Note that $K \ge 1$ by (2.21). Consider now an increasing sequence of rational numbers $r = \frac{a}{k}$, with $1 \le k \le K$, on the interval

$$I = [f'(M+\Delta), f'(M)] = \left[\frac{F}{2M} + \eta - \Delta_0, \frac{F}{2M} + \eta\right],$$

with

$$\Delta_0 \asymp \xi F M^{-1},$$

then $\Delta_0 \gg 1$ by (2.21). For any given r let r_1 and r_2 be the preceding and the next member in the Farey sequence. Let us divide the sum according to Farey fractions:

$$S(M,\xi) = \sum_{r\in I} S(a/k) := \sum_{r\in I} \sum_{m\in[M,M+M\xi]\cap(M_1,M_2]} a(m)e(f(m)),$$

where $f'(M_1) = \rho_1$ is the mediant of r_1 and r and $f'(M_2) = \rho_2$ is the mediant of r and r_2. Now

$$M_j = \frac{1}{4}F^2M^{-1}(\rho_j - \eta)^{-2}. \tag{2.24}$$

This expression looks as if it were made for Lemma 10. Define $M(r)$ just as in the aforementioned lemma, to stand for the number satisfying $f'(M(r)) = r$. According to the estimate (2.23) for the second derivative of f, we have

$$\begin{aligned} m_j := |M(r) - M_j| &\asymp M^2F^{-1}\left|f'(M(r)) - f'(M_j)\right| = M^2F^{-1}|r - \rho_j| \\ &\asymp M^2F^{-1}k^{-1}K^{-1} \asymp \xi^{1/3}M^{3/2}F^{-2/3+\delta/2}k^{-1}. \end{aligned} \tag{2.25}$$

Especially this shows that $m_1 \asymp m_2$ and

$$\xi^{2/3}MF^{-1/3+\delta} \ll m_j \ll \xi^{5/6}MF^{-1/6+\varepsilon}. \tag{2.26}$$

The incomplete sums at the ends of the interval $[M, M+M\xi]$ may be estimated by absolute values:

$$\ll m_jM^{\varepsilon} \ll \xi^{5/6}MF^{-1/6+2\varepsilon} \ll \xi^{5/6}M^{1/2}F^{1/3+\varepsilon}.$$

Let us now weight the sums $S(a/k)$ with the function $w_J(x)$. Choose $U = MF^{-1/2+\varepsilon}$. Then $U \gg 1$ by (2.21). On an interval of length Δ_0 there are $\ll \Delta_0 K^2 \ll \xi^{1/3}F^{1/3+\varepsilon}$ Farey fractions of order K. Hence the error caused by the weighting is

$$\ll \Delta_0 K^2 U M^{\varepsilon} \ll \xi^{1/3} M F^{-1/6+\varepsilon} \ll \xi^{5/6} M^{1/2} F^{1/3+\varepsilon},$$

so it is sufficient to estimate the weighted sums.

First notice that

$$m_j \gg MF^{-1/2}\xi^{2/3}F^{1/6+\delta}.$$

Thus, by (2.22),

$$m_j \gg M^{1+3\delta/2}F^{-1/2}.$$

This estimate together with (2.26) shows that the condition (1.8) of Lemma 10 is satisfied. The remaining assumptions of Lemma 10 are satisfied due to the choices of the function f and the Farey sequence. It is enough to consider the case $j = 1$ because the case $j = 2$ is similar. Now

$$p_{1,n}(x) = Bx^{1/2} - (r-\eta)x + \frac{2\sqrt{nx}}{k} - \frac{1}{8}$$

and according to (2.25)

$$n_1 \asymp F^2 M^{-3} m_1^2 k^2 \asymp \xi^{2/3} F^{2/3+\delta}.$$

Define $x_{1,n}$ from the equation $p'_{1,n}(x_{1,n}) = 0$:

$$x_{1,n} = (r-\eta)^{-2}\left(\frac{B}{2} + \frac{\sqrt{n}}{k}\right)^2 \asymp (r-\eta)^{-2}B^2;$$

note that

$$r - \eta \ll FM^{-1} = |B|M^{-1/2}.$$

Let us start with the error term in (1.9). Because $|a|k \asymp k^2 FM^{-1}$, using (2.25) the error term becomes

$$F^{-1/2+\varepsilon}(|a|k)^{1/2} m_1^{1/2} \ll \xi^{1/6} M^{1/4} K^{1/2} F^{-1/3+\varepsilon} \ll M^{1/2}F^{-1/2+\varepsilon}$$

Taking now into account the total number of Farey fractions on the interval I and noting that $\xi^{2/3}F^{1/6} \gg 1$ by (2.22), the estimate becomes

$$\ll \xi M^{1/2} F^{\varepsilon} \ll \xi^{5/6} M^{1/2} F^{1/3+\varepsilon}.$$

Let us put $n \sim A$ to denote $A \le n \le 2A$. Let $K_0 \ll K$ and $N_0 \ll N$. We need to estimate $O\left(\log^2 F\right)$ sums of the type

$$S_1 := \sum_{r \in I, k \sim K_0} \sum_{n \sim N_0} v_J(r,n) a(n) Q(r,n) e(f(r,n)),$$

where

$$\begin{aligned}
Q(r,n) &= k^{-1/2}n^{-1/4}\left(\frac{B}{2}+\frac{\sqrt{n}}{k}\right)^{-1/2} x_{1,n}^{1/2},\\
f(r,n) &= -\frac{n\bar{a}}{k}+BX_{1,n}^{1/2}-(r-\eta)x_{1,n}+\frac{2\sqrt{nx_{1,n}}}{k},\\
\frac{\partial}{\partial n}f(r,n) &= -\frac{\bar{a}}{k}+p'_{1,n}(x_{1,n})\frac{\partial}{\partial n}(x_{1,n})+k^{-1}n^{-1/2}x_{1,n}^{1/2} = -\frac{\bar{a}}{k}+k^{-1}n^{-1/2}x_{1,n}^{1/2}.
\end{aligned}$$

Substituting the formula for $x_{1,n}$ we obtain

$$\begin{aligned}
Q(r,n) =& \frac{1}{\sqrt{2}}F^{1/2}M^{-1/4}k^{-1/2}n^{-1/4}(r-\eta)^{-1}\left(1+\frac{2\sqrt{nM}}{kF}\right)^{1/2}\\
\ll& F^{-1/2}K_0^{-1/2}M^{3/4}N_0^{-1/4}.
\end{aligned}$$

Now

$$\frac{\partial}{\partial n}Q(r,n) \ll N_0^{-1}Q(r,n) \tag{2.27}$$

and

$$\begin{aligned}
\frac{\partial}{\partial n}f(r,n) &= -\frac{\bar{a}}{k}+\frac{FM^{-1/2}n^{-1/2}}{2(a-k\eta)}+\frac{1}{k(a-k\eta)}\\
\frac{\partial^2}{\partial n^2}f(r,n) &= -\frac{FM^{-1/2}n^{-3/2}}{4(a-k\eta)}.
\end{aligned}$$

The sum S_1 may be estimated using Cauchy's inequality:

$$\begin{aligned}
&\sum_{n\sim N_0}a(n)\sum_{r\in I,k\sim K_0}v_J(r,n)Q(r,n)e(f(r,n))\\
&\qquad\ll N_0^{1/2}F^{\varepsilon}\left(\sum_{n\sim N_0}\left|\sum_r v_J(r,n)Q(r,n)e(f(r,n))\right|^2\right)^{1/2}\\
&\ll N_0^{1/2}F^{\varepsilon}\left(\sum_{r_1}\sum_{r_2}\left|\sum_{n\sim N_0}v_J(r_1,n)v_J(r_2,n)Q(r_1,n)Q(r_2,n)e(f(r_1,n)-f(r_2,n))\right|\right)^{1/2}.
\end{aligned}$$

Write

$$S_2=\sum_{r_1}\sum_{r_2}\max_{0<\nu\le N_0}\left|\sum_{n=N_0}^{N_0+\nu}e(F(n))\right|, \tag{2.28}$$

where

$$F(n) = f(r_1,n) - f(r_2,n) \tag{2.29}$$

$$F'(n) = \frac{\bar{a}_2}{k_2} - \frac{\bar{a}_1}{k_1} + \frac{1}{2}FM^{-1/2}n^{-1/2}\frac{(a_2-k_2\eta)-(a_1-k_1\eta)}{(a_1-k_1\eta)(a_2-k_2\eta)} + \frac{1}{k_1(a_1-k_1\eta)} \tag{2.30}$$

$$-\frac{1}{k_2(a_2-k_2\eta)} \tag{2.31}$$

$$F''(n) = -\frac{1}{4}FM^{-1/2}n^{-3/2}\frac{(a_2-k_2\eta)-(a_1-k_1\eta)}{(a_1-k_1\eta)(a_2-k_2\eta)}. \tag{2.32}$$

Because the functions $v_J(r_1,n)Q(r_1,n)$ and $v_J(r_2,n)Q(r_2,n)$ are piecewise sufficiently stationary in n, they can be eliminated by partial summation. Hence

$$S_1 \ll F^{-1/2+\varepsilon}K_0^{-1/2}M^{3/4}N_0^{1/4}|S_2|^{1/2}.$$

Assume for a while that

$$S_2 \ll \xi^{5/3}F^{5/3+\varepsilon}K_0M^{-1/2}N_0^{-1/2} \tag{2.33}$$

holds. Then

$$S_1 \ll \xi^{5/6}M^{1/2}F^{1/3+\varepsilon}.$$

This shows that it is sufficient to prove that (2.33) holds. Notice that the denominators of the derivatives (2.32) satisfy $(a_1-k_1\eta)(a_2-k_2\eta) \asymp K_0^2F^2M^{-2}$, when $r_1,r_2 \in I$ and $k_i \sim K_0$. We will use Lemmas 21 and 22. For this purpose let us choose

$$Q = K_0,\ A = K_0FM^{-1},\ \Delta_3 = \xi,\ \psi(x) = x-\eta,\ H = \frac{A}{Q},\ [R,R'] = I.$$

Then $R'-R = \Delta_0 \asymp \Delta_3AQ^{-1}$ and $\xi A \gg K_0$. The sum S_2 (defined in (2.28)) can be estimated trivially, if the n-sum is short enough. Choose $N_1 = c_0F^{-2}K_0^{-4}K^{-2}M^3$. This choice will be motivated in a moment. If $N_0 \le N_1$, we obtain

$$S_2 \ll (\xi AK_0)^2F^{-2}K_0^{-4}K^{-2}M^3 = \xi^2K^{-2}M \ll \xi^{7/2}F^{3/2+\varepsilon}K_0M^{-1/2}N_0^{-1/2}.$$

Here c_0 is a positive constant that can be chosen suitably. When $N_0 > N_1$ we may estimate the sum using first or second derivative test ([15] Lemmas 4.2, 4.7, 4.8, and 5.9), depending on the size of the derivatives, or actually, on the difference

$$(a_2-k_2\eta)-(a_1-k_1\eta) = k_2\psi\left(\frac{a_2}{k_2}\right) - k_1\psi\left(\frac{a_1}{k_1}\right).$$

Let

$$\frac{1}{2}\Delta_2A \le |(a_2-k_2\eta)-(a_1-k_1\eta)| \le \Delta_2A, \tag{2.34}$$

where Δ_2 is a variable and $0 < \Delta_2 \ll 1$. Let us write $\delta_0 = F^{-1}K_0^{-1}K^{-1}M$. Notice that $\delta_0 \leq A^{-1} \ll \xi$. Let us now assume for a while that (2.34) holds and $\Delta_2 < \delta_0$. Then the expression for the first derivative can be written as

$$F'(n) = \frac{\bar{a}_2}{k_2} - \frac{\bar{a}_1}{k_1} + O(\Delta_1) + O\left(K_0^{-2}F^{-1}M\right), \tag{2.35}$$

where

$$\Delta_1 = F^{-1}K_0^{-2}K^{-1}M^{3/2}N_0^{-1/2}.$$

Since $K = \left\lfloor \xi^{-1/3}M^{1/2}F^{-1/3-\delta/2} \right\rfloor$ and $N_0 \ll n_1 \asymp \xi^{2/3}F^{2/3+\delta}$, it follows that

$$\Delta_1 \gg F^{-1}K_0^{-2}\xi^{1/3}M^{-1/2}F^{1/3+\delta/2}M^{3/2}\xi^{-1/3}F^{-1/3-\delta/2} = F^{-1}K_0^{-2}M,$$

and the second O-term is absorbed into the Δ_1-term. Note that the choice of N_1 enables the inequality $\Delta_1 < c_0^{-1/2}$ to be satisfied when $N_0 > N_1$. Choosing c_0 large enough allows us to assume that $F'(n) < 1$. The number of the sums where

$$\left| \frac{\bar{a}_2}{k_2} - \frac{\bar{a}_1}{k_1} \right| \ll \Delta_1$$

is according to lemma 21 (remembering that $\Delta_2 < \delta_0 \ll \Delta_1$ and $\delta_0 \ll \xi$)

$$\ll \xi A K_0 + \Delta_2 \xi A^2 + \Delta_1^2 \xi^2 A^2 K_0^2.$$

Since

$$\left(\Delta_2 \xi A^2\right)(\xi A K_0)^{-1} < \delta_0 A K_0^{-1} = K_0^{-1}K^{-1} \ll 1,$$

the number of the sums will be

$$\ll \xi A K_0 + \Delta_1^2 \xi^2 A^2 K_0^2. \tag{2.36}$$

These sums may be estimated trivially (remember that we may choose δ to be an arbitrarily small positive constant, according to the desired choice of ε and that $K_0 \geq 1$):

$$\begin{aligned} &\ll \left(\xi A K_0 + \Delta_1^2 \xi^2 A^2 K_0^2\right) N_0 \\ &\qquad \ll \xi F K_0 M^{-1/2} N_0^{-1/2} \left(K M^{-1/2} N_0^{3/2} + \xi F^{-1} K^{-2} M^{3/2} N_0^{1/2}\right) \\ &\qquad\qquad \ll \xi F^{1+\varepsilon} K_0 M^{-1/2} N_0^{-1/2} \left(\xi^{2/3} F^{2/3} + \xi^{5/2} F^{1/2}\right) \\ &\qquad\qquad\qquad \ll \xi^{5/3} F^{5/3+\varepsilon} K_0 M^{-1/2} N_0^{-1/2}. \end{aligned}$$

Next let

$$\frac{1}{2}\theta \leq \left| \frac{\bar{a}_2}{k_2} - \frac{\bar{a}_1}{k_1} \right| \leq \theta$$

and $\Delta_1 \ll \theta \le \frac{1}{2}$. According to Lemmas 4.2 and 4.8 [15]

$$\max_{0<v\le N_0}\left|\sum_{n=N_0}^{N_0+v} e\left(F(n)\right)\right| \ll \frac{1}{\theta}.$$

Combining this with (2.36), when θ stands in the place of the number Δ_1, the estimate becomes

$$\begin{aligned}&\ll \xi F^{1+\varepsilon} K_0 M^{-1/2} N_0^{-1/2}\left(\theta^{-1} K_0 M^{-1/2} N_0^{1/2} + \theta \xi F K_0^3 M^{-3/2} N_0^{1/2}\right)\\ &\qquad \ll \xi F^{1+\varepsilon} K_0 M^{-1/2} N_0^{-1/2}\left(F K^4 M^{-2} N_0 + \xi F K^3 M^{-3/2} N_0^{1/2}\right)\\ &\qquad\qquad \ll \xi F^{1+\varepsilon} K_0 M^{-1/2} N_0^{-1/2}\left(\xi^{-2/3} F^{1/3} + \xi^{1/3} F^{1/3}\right).\end{aligned}$$

Since $1 \ll \xi^{4/3} F^{1/3-\delta}$ (see (2.22)), we may conclude that

$$\xi^{-2/3} F^{1/3} \ll \xi^{2/3} F^{2/3-\delta}$$

and the estimate becomes

$$\ll \xi^{5/3} F^{5/3+\varepsilon} K_0 M^{-1/2} N_0^{-1/2}.$$

When $\delta_0 \le \Delta_2 \le 1$, we can use the second derivative. Since

$$F''(x) \asymp \lambda = \Delta_2 K_0^{-1} M^{1/2} N_0^{-3/2},$$

we obtain the following estimate from the second derivative test

$$\begin{aligned}\max_{0<v\le N_0}\left|\sum_{n=N_0}^{N_0+v} e\left(F(n)\right)\right| &\ll \Delta_2^{1/2} K_0^{-1/2} M^{1/4} N_0^{1/4} + \Delta_2^{-1/2} K_0^{1/2} M^{-1/4} N_0^{3/4}\\ &\ll K_0^{-1} N_0^{-1/2}\left(\Delta_2^{-1/2} K^{1/2} M^{1/4} N_0^{3/4} + \Delta_2^{-1/2} K^{3/2} M^{-1/4} N_0^{5/4}\right) F^{\varepsilon}\\ &\ll \xi^{1/3} F^{1/3+\varepsilon} K_0^{-1} M^{1/2} N_0^{-1/2} \Delta_2^{-1/2}. \qquad (2.37)\end{aligned}$$

Because $\min(\xi, \Delta_2) \ll \xi^{1/2} \Delta_2^{1/2}$, the estimate of Lemma 22 can be simplified to

$$\begin{aligned}&\ll \xi A K_0 \left(1 + \min(\xi, \Delta_2) A + \xi K_0 + \Delta_2 K_0 + \xi \Delta_2 A K_0\right)\\ &\qquad \ll \xi A K_0 \left(1 + \xi^{1/2} \Delta_2^{1/2} A + \xi K + \Delta_2 K + \xi \Delta_2 A K\right) \qquad (2.38)\end{aligned}$$

Use now the inequality (2.22) to simplify this further to

$$\ll \xi F^{1+\varepsilon} K_0^2 M^{-1}\left(\xi^{2/3} F^{1/6} + \xi^{1/2} \Delta_2^{1/2} K_0 F M^{-1} + \Delta_2 \xi^{1/3} F^{1/3}\right).$$

Multiplication of the terms (2.37) and (2.38) gives

$$\ll \xi^{5/3} F^{3/2+\varepsilon} K_0 M^{-1/2} N_0^{-1/2}\left(\Delta_2^{-1/2} \xi^{1/3} + \xi^{1/6} K_0 F^{5/6} M^{-1} + \Delta_2^{1/2} F^{1/6}\right).$$

Use the formula (2.21) and recall that $\delta_0 \geq \xi^{2/3}F^{-1/3}$ to obtain

$$\begin{aligned}
&\Delta_2^{-1/2}\xi^{2/3}F^{1/6} + \xi^{1/2}K_0FM^{-1} + \Delta_2^{1/2}\xi^{1/3}F^{1/3} \\
&\qquad \ll \delta_0^{-1/2}\xi^{2/3}F^{1/6} + \xi^{1/2}KFM^{-1} + \xi^{1/3}F^{1/3} \\
&\qquad\qquad \ll \xi^{5/6}F^{1/3+\varepsilon} + \xi^{1/3}F^{1/3} \ll \xi^{1/3}F^{1/3+\varepsilon}.
\end{aligned}$$

This finally completes the proof of the theorem. □

Chapter 3

Bounds for Short Linear Exponential Sums

Theorem 24. *Let w be a smooth weight function on the interval $[M, M+\Delta]$, where $1 \le \Delta \le M$. Assume further that $\alpha = \frac{a}{k} + \eta$, $(a,k) = 1$, $1 \le k \le Q \asymp \Delta^{1/2-\delta/2}$, $|\eta| \le \frac{1}{kQ}$ where δ is a small fixed positive real number (which may be chosen to be arbitrarily small). If $|\eta| \le \frac{1}{3}\Delta^{-1+\delta}$, then*

$$\tilde{A}\left(M, \Delta, \frac{a}{k} + \eta\right) \ll \left(\frac{\Delta}{M}\right)^{1/6} M^{1/2+\varepsilon}. \tag{3.1}$$

If

$$\Delta \ll \frac{\sqrt{M}}{\sqrt{|\eta|}} \tag{3.2}$$

and $\eta \neq 0$, then

$$\tilde{A}(M, \Delta, \alpha) \ll \left(\frac{\Delta}{M}\right)^{1/6} M^{1/2+\varepsilon} + k^{-1}\Delta|\eta|^{-1/2}M^{-1/2+\varepsilon}. \tag{3.3}$$

Proof. Using Deligne's estimate $|a(n)| \le d(n) \ll n^{\varepsilon}$, the theorem holds for $\Delta \ll M^{2/5+\delta}$, so we may assume $\Delta \gg M^{2/5+\delta}$. First using a transformation formula of the Voronoi type ([8], Theorem 1.7) we obtain

$$\begin{aligned}
\tilde{A}\left(M, \Delta, \frac{a}{k} + \eta\right) &= \sum_{M \le m \le M+\Delta} a(m)w(m)e(\eta m)e\left(\frac{am}{k}\right) \\
&= (-1)^{\kappa/2}\frac{2\pi}{k}\sum_{n=1}^{\infty} a(n)e\left(-\frac{n\overline{a}}{k}\right)\int_M^{M+\Delta} w(t)J_{\kappa-1}\left(\frac{4\pi\sqrt{nt}}{k}\right)e(\eta t)dt.
\end{aligned} \tag{3.4}$$

Writing the J-Bessel function as a sum of Hankel functions and then substituting

(1.4) into (3.4), we obtain

$$\tilde{A}\left(M,\Delta,\frac{a}{k}+\eta\right) = \frac{(-1)^{\kappa/2}}{\sqrt{2}}k^{-1/2}\sum_{n=1}^{\infty}a(n)e\left(-\frac{n\bar{a}}{k}\right)n^{-1/4}\left(I^{(1)}(n)+I^{(2)}(n)\right) + O\left(\Delta^{7/4}M^{-5/4+\varepsilon}\right),$$

where

$$I^{(j)}(n) = \int_{M}^{M+\Delta} w(t)g_j(n,t)e(f_j(n,t))dt,$$
$$f_j(n,t) = \eta t + (-1)^{j-1}\left(\frac{2\sqrt{nt}}{k}-\lambda\right),$$
$$g_j(n,t) = t^{-1/4} + \frac{(-1)^j}{4\pi}C_{1,\kappa-1}kn^{-1/2}t^{-3/4}.$$

To make this proof a bit more reader-friendly, the rest is divided into two lemmas. The first one covers the case $\eta \leq \frac{1}{3}\Delta^{\delta-1}$.

Lemma 25. *With the assumptions of the Theorem 24 and the additional assumption* $0<\eta\leq\frac{1}{3}\Delta^{-1+\delta}$*, the following holds*

$$\tilde{A}\left(M,\Delta,\frac{a}{k}+\eta\right) \ll \left(\frac{\Delta}{M}\right)^{1/6}M^{1/2+\varepsilon}. \tag{3.5}$$

Proof. We will choose $N_0 = k^2M\Delta^{-2+2\delta}$ from the condition

$$\left|\frac{\partial}{\partial t}\left(\eta t \pm\left(\frac{2\sqrt{N_0 t}}{k}-\lambda\right)\right)\right| \asymp \frac{\sqrt{N_0}}{k\sqrt{M}} = \Delta^{-1+\delta}.$$

We split the summation into two parts

$$A+B = \frac{(-1)^{\kappa/2}}{\sqrt{2}}k^{-1/2}\left(\sum_{n\leq N_0}+\sum_{n>N_0}\right)a(n)e\left(-\frac{n\bar{a}}{k}\right)n^{-1/4}I^{(j)}(n)$$

and consider separately the cases

(1) $M^{2/5+\delta} \ll \Delta \ll M^{1/2}$ and $1\leq k\leq Q$,
(2) $M^{1/2} \ll \Delta \leq M$ and $1\leq k\leq \Delta^{1-\delta}M^{-1/2}$,
(3) $M^{1/2} \ll \Delta \leq M$ and $\Delta^{1-\delta}M^{-1/2}\leq k\leq Q$.

Let us first consider the cases (1) and (3). Now $N_0 \gg 1$. In the oscillating integrals of the sum B

$$\left|\frac{\partial}{\partial t}f_j(n,t)\right| \gg \Delta^{-1+\delta}.$$

From this we see that it is sensible to estimate by partial integration. Lemma 7 gives

$$I^{(j)}(n) \ll \Delta^{1-J}k^J M^{-1/4+J/2} n^{-J/2} \ (J \geq 2)$$

and therefore we obtain

$$\begin{aligned} B \ll & k^{-1/2+J}\Delta^{1-J}M^{-1/4+J/2}\sum_{n>N_0} n^{-1/4-J/2+\varepsilon} \\ \ll & k^{-1/2+J}\Delta^{1-J}M^{-1/4+J/2}N_0^{3/4-J/2+\varepsilon} \ll \Delta^{-J_1\delta}M^{1/2+\varepsilon}, \end{aligned}$$

where $J_1 = J-1$. Let us now choose J such that $J_1\delta \geq \frac{1}{4}$. Hereby we obtain

$$\Delta^{-J_1\delta}M^{1/2+\varepsilon} \ll \Delta^{-1/4}M^{1/2+\varepsilon} \ll \Delta^{1/6}M^{1/3+\varepsilon}.$$

Next turn to the sum A. Changing the order of the summation and integration and integrating using absolute values, we obtain

$$A \ll \Delta \max_{M\leq t\leq M+\Delta}\left| k^{-1/2}\sum_{n\leq N_0} a(n)n^{-1/4}g_j(n,t)e\left(-\frac{n\bar{a}}{k} \pm \frac{2\sqrt{nt}}{k} + \eta t\right)\right|.$$

Let $Y \leq N_0$ and write

$$S := k^{-1/2}\sum_{n\sim Y} a(n)n^{-1/4}g_j(n,t)e\left(-\frac{n\bar{a}}{k} \pm \frac{2\sqrt{nt}}{k}\right).$$

We may treat this as a long sum according to theorem 23. When we choose

$$F = \frac{2\sqrt{Yt}}{k} + \eta t,\ G = Y^{-1/4}t^{-1/4},\ G' = Y^{-1}G,\ \eta = -\frac{\bar{a}}{k},$$

the condition (2.17) will be fulfilled because

$$\frac{Y}{F} \ll 1$$

and theorem 23 gives

$$S \ll k^{-5/6}Y^{5/12}t^{-1/12+\varepsilon}.$$

Hereby we obtain

$$A \ll \Delta \max_{t,Y}\left(k^{-5/6}Y^{5/12}t^{-1/12+\varepsilon}\right) \ll \Delta^{1/6}M^{1/3+\varepsilon}.$$

Let us now consider the case (2). Now $N_0 \ll 1$. Partial integration twice gives

$$I^{(j)}(n) \ll \Delta^{-1}k^2M^{3/4}n^{-1}$$

and this yields the estimate

$$\tilde{A}\left(M,\Delta,\frac{a}{k}+\eta\right) \ll k^{3/2}\Delta^{-1}M^{3/4}\sum_{n=1}^{\infty} n^{-5/4+\varepsilon} \ll \Delta^{1/2-3\delta/2} \ll \Delta^{1/6}M^{1/3}.$$

□

The following lemma deals with bigger values of η.

Lemma 26. *In addition to the assumptions of theorem 24, assume that*

$$\frac{1}{3}\Delta^{-1+\delta} < \eta \le \frac{1}{kQ}.$$

Then

$$\tilde{A}(M,\Delta,\alpha) \ll \left(\frac{\Delta}{M}\right)^{1/6} M^{1/2+\varepsilon} + k^{-1}\Delta\eta^{-1/2}M^{-1/2+\varepsilon}. \tag{3.6}$$

Proof. Let us first consider the sum containing the integral $I^{(1)}(n)$. Now

$$\left|\frac{\partial}{\partial t}\left(\eta t + \frac{2\sqrt{nt}}{k} - \lambda\right)\right| \asymp \begin{cases} \eta, & \text{when } n \le 2n_0 \\ n^{1/2}k^{-1}M^{-1/2}, & \text{when } n > 2n_0 \end{cases} \tag{3.7}$$

where $n_0 = k^2 M\eta^2 \gg k^2 M\Delta^{-2+2\delta}$. Partial integration $J \ge 2$ times (lemma 7) gives

$$\begin{aligned} &\frac{(-1)^{\kappa/2}}{\sqrt{2}} k^{-1/2} \sum_{n=1}^{\infty} a(n) e\left(-\frac{n\overline{a}}{k}\right) n^{-1/4} I^{(1)}(n) \\ &\qquad \ll \Delta^{1-J}\eta^{-J}k^{-1/2}M^{-1/4} \sum_{n \le 2n_0} n^{-1/4+\varepsilon} \\ &\qquad + k^{-1/2+J}\Delta^{1-J}M^{-1/4+J/2} \sum_{n > 2n_0} n^{-1/4-J/2+\varepsilon} \\ &\qquad \ll k\Delta^{1-J}\eta^{3/2-J}M^{1/2+\varepsilon} \ll \Delta^{(1-J)\delta}M^{1/2+\varepsilon} \ll \Delta^{1/6}M^{1/3+\varepsilon}, \end{aligned} \tag{3.8}$$

when $J \ge 1 + \frac{1}{4\delta}$.

In the integral $I^{(2)}(n)$ the function $e(f_j(n,t))$ oscillates on the whole interval of integration $[M, M+\Delta]$ if (3.2) holds and

$$|n - n_0| > N := k^2\eta M\Delta^{-1+\delta}. \tag{3.9}$$

Then

$$\left|\frac{\partial}{\partial t}\left(\eta t - \frac{2\sqrt{nt}}{k} + \lambda\right)\right| \gg \begin{cases} \Delta^{-1+\delta} & \text{when } 1 \le n \le n_0 - N, \\ \Delta^{-1+\delta} & \text{when } n_0 + N < n < 2n_0, \\ n^{1/2}k^{-1}M^{-1/2} & \text{when } n > 2n_0, \end{cases}$$

and we may treat the case (3.9) similarly to (3.8). We have now

$$\tilde{A}(M,\Delta,\alpha) = \frac{(-1)^{\kappa/2}}{\sqrt{2}} k^{-1/2} \sum_{|n-n_0| \le N} a(n) e\left(-\frac{n\overline{a}}{k}\right) n^{-1/4} I^{(2)} + O\left(\Delta^{1/6}M^{1/3+\varepsilon}\right). \tag{3.10}$$

If $N < \frac{1}{2}$, then there is at most one term, say n', in the sum (3.10). Integration using absolute values gives

$$I^{(2)}(n') \ll \Delta M^{-1/4}.$$

Hence

$$\tilde{A}(M,\Delta,\alpha) \ll \Delta k^{-1}\eta^{-1/2}M^{-1/2+\varepsilon} + \Delta^{1/6}M^{1/3+\varepsilon}.$$

Let us now assume $N \geq \frac{1}{2}$. We change the order of the integration and summation and consider the sum

$$k^{-1/2}\sum_{|n-n_0|\leq N} a(n)n^{-1/4}g(n,t)e\left(\frac{n\bar{a}}{k} - \frac{2\sqrt{nt}}{k}\right) \tag{3.11}$$

for a fixed value of the variable t. Since

$$\frac{N}{n_0} = \frac{\Delta^{\delta}}{\Delta\eta} \ll 1,$$

the sum (3.11) is typically short, and we need the sharp result of Theorem 23. We may use it with

$$M = n_0,\ \Delta = N,\ F = \tfrac{2\sqrt{n_0 t}}{k} = 2\eta\sqrt{Mt},$$
$$G = n_0^{-1/4}t^{-1/4},\ G' = n_0^{-1}G,\ \eta = -\tfrac{\bar{a}}{k}.$$

Specifically (2.17) follows from $\eta \leq \frac{1}{kQ}$. We obtain

$$k^{-1/2}\sum_{|n-n_0|\leq N} a(n)n^{-1/4}g_j(n,t)e\left(-\frac{\bar{a}n}{k} - \frac{2\sqrt{nt}}{k}\right) \ll \Delta^{-5/6}M^{1/3+\varepsilon}.$$

Integration using absolute values completes the proof of the lemma. □

Combining the results of the above lemmas with the comment between them, proves the original theorem. □

Notice that the condition (3.2) may be considered as a condition for $|\eta|$:

$$|\eta| \ll M\Delta^{-2}.$$

If $k^{-1}Q^{-1} \ll M\Delta^{-2}$, that is, if $\Delta \ll M^{\frac{2}{3+\delta}}$, then the condition (3.2) holds for all k and the estimate (3.3) holds for all real α, so we may infer the following corollary.

Corollary 27. *Let* $1 \leq \Delta \ll M^{2/3}$. *Then*

$$\tilde{A}(M,\Delta,\alpha) \ll \Delta^{1/6}M^{1/3+\varepsilon} + \Delta^{3/2}M^{-1/2+\varepsilon}. \tag{3.12}$$

Taking the estimate obtained using absolute values into account when $1 \leq \Delta \ll M^{2/5}$, *we obtain*

$$\tilde{A}(M,\Delta,\alpha) \ll \begin{cases} \Delta M^{\varepsilon}, & \text{when } 1 \leq \Delta \ll M^{2/5} \\ \Delta^{1/6}M^{1/3+\varepsilon}, & \text{when } M^{2/5} \ll \Delta \ll M^{5/8} \\ \Delta^{3/2}M^{-1/2+\varepsilon}, & \text{when } M^{5/8} \ll \Delta \ll M^{2/3}. \end{cases}$$

The first one is the trivial estimate by absolute values. Next, we will show that quite often one is able to obtain good estimates also for non-weighted sums.

Theorem 28. *Let $M^{2/5} \le \Delta \ll M^{5/8}$. Then*

$$A(M,\Delta;\alpha) \ll \Delta^{1/6} M^{1/3+\varepsilon}.$$

Further, if $M^{5/8} \ll \Delta \ll \frac{\sqrt{M}}{\sqrt{|\eta|}}$, $k \le M^{5/16-\delta/2}$ for some positive δ and $|\eta| \le k^{-1} M^{\delta/2-5/16}$, we have

$$A(M,\Delta;\alpha) \ll \Delta^{1/6} M^{1/3+\varepsilon} + k^{-1}\Delta|\eta|^{-1/2} M^{-1/2+\varepsilon}.$$

Proof. Let $\ell > 0$. To simplify the notation, write

$$\begin{cases} M_0 = M + \frac{\Delta}{2} - \frac{\Delta}{10} \\ M_\ell = M + \frac{\Delta}{2} + \frac{\Delta}{10} + \frac{2\Delta}{5} \sum_{i=1}^{\ell} \frac{1}{2^i} \\ M_{-\ell} = M + \frac{\Delta}{2} - \frac{\Delta}{10} - \frac{\Delta}{2^{\ell+1}5} - \frac{2\Delta}{5} \sum_{i=1}^{\ell} \frac{1}{2^i} \end{cases}$$

and

$$\begin{cases} \Delta_0 = \frac{\Delta}{5} \\ \Delta_{-\ell} = \Delta_\ell = \frac{\Delta}{2^{\ell+2}} \end{cases}$$

Consider the set of weight functions $\{w_{\pm\ell} \mid 0 \le \ell \le L(x)\}$ satisfying the following conditions:

- The support of each function is bounded. To be precise,

$$w_0(x) = \begin{cases} 1, & \text{when } x \in \left[M_0 + \frac{\Delta_0}{4}, M_0 + \frac{3\Delta_0}{4}\right] \\ 0, & \text{when } x \le M_0 \\ & \text{or } x \ge M_0 + \Delta_0 \end{cases}$$

  and when $\ell > 0$,

$$w_{+\ell}(x) = \begin{cases} 1, & \text{when } x \in [M_{\ell-1} + \Delta_{\ell-1}, M_{\ell+1}] \\ 0, & \text{when } x \le M_\ell \\ & \text{or } x \ge M_\ell + \Delta_\ell. \end{cases}$$

- The derivatives are assumed to satisfy the normal conditions of a smooth weight function:

$$w_{+\ell}^{(j)}(x) \ll \left(\frac{\Delta}{2^\ell}\right)^{-j}$$

  for $0 \le j \le J$ for some suitable value of J.

- On the interval

$$\left[M+\frac{\Delta}{2}+\frac{\Delta}{10}+\frac{2\Delta}{5}\sum_{i=1}^{\ell}\frac{1}{2^i}, M+\frac{\Delta}{2}+\frac{\Delta}{10}+\frac{\Delta}{2^{\ell+1}5}+\frac{2\Delta}{5}\sum_{i=1}^{\ell}\frac{1}{2^i}\right],$$

  the functions $w_\ell(x)$ and $w_{\ell+1}$ add to one:

$$w_{\ell+1}(x) = 1 - w_\ell(x).$$

The function $w_{-\ell}(x)$ is symmetric to $w_{+\ell}(x)$ with respect to the line $x = M + \frac{\Delta}{2}$ The picture will look like the following:
Consider now the function defined as

$$W_L(x) = \sum_{\ell=-L}^{L} w_\ell(x).$$

This function is a long weight function with only short slopes:

$$W_L(x) = \begin{cases} 1, & \text{when } x \in [M_{-L-1}+\Delta_{-L-1}, M_{L+1}] \\ 0, & \text{when } x \leq M_{-L} \\ & \text{or } x \geq M_L + \Delta_L, \end{cases}.$$

This function differs from the characteristic function of the interval $[M,\Delta]$ only on the interval

$$[M, M_{-L-1}+\Delta_{-L-1}] \cup [M_{L+1}, M+\Delta].$$

The total length of these intervals is:

$$2\left(\frac{\Delta}{2}-\left(\frac{\Delta}{10}+\frac{\Delta}{2^{L+1}5}+\frac{2\Delta}{5}\sum_{i=1}^{L}\frac{1}{2^i}\right)\right) = \frac{3\Delta}{2^L 5}.$$

Notice that the length of the interval on which W_L differs from the characteristic function, is of the same magnitude as the slope of W_L. Choosing L to be sufficiently large, yields

$$\frac{3\Delta}{2^L 5} \asymp M^{2/5}. \tag{3.13}$$

Assume now $\Delta \ll M^{5/8}$. Use the result of the Corollary 27, and choose L as in (3.13). Notice that for any $\Delta' \ll M^{5/8}$

$$\tilde{A}\left(M, \Delta', \alpha\right) \ll \Delta^{1/6} M^{1/3+\varepsilon}.$$

Notice also that for this choice of L, the interval on which W_L differs from the characteristic function of the whole interval $[M, M+\Delta]$, is of length at most $M^{2/5}$.

On an interval of that length, we may use estimates by absolute values to obtain the error term of $O\left(M^{2/5+\varepsilon}\right)$. Let us now consider the sum weighted with the function W_L.

$$\sum_{n=1}^{\infty} a(n)W_L(n)e(n\alpha) = \sum_{-L\le\ell\le L}\sum_{n=1}^{\infty} A(M_\ell,\Delta_\ell;\alpha)$$
$$\ll M_0^{1/3+\varepsilon}\Delta_0^{1/6} + \sum_{0<\ell\le L} M_\ell^{1/3+\varepsilon}\Delta_\ell^{1/6} + \sum_{-L\le\ell<0} M_\ell^{1/3+\varepsilon}\Delta_\ell^{1/6}$$
$$\ll M^{1/3+\varepsilon}\left(\frac{\Delta}{5} + \sum_{1\le\ell}\frac{\Delta}{2^{\ell+2}}\right) \ll M^{1/3+\varepsilon}\Delta.$$

This proves the first part of the theorem. Let us now consider the second part. Proceed as in the earlier case. However, choose L according to the following condition instead of the condition (3.13):

$$\frac{3\Delta}{2^L 5} \asymp M^{5/8}. \tag{3.14}$$

Define now the constants Δ_ℓ and M_ℓ as above for $\ell \in \mathbb{Z}$ for the current choice of L. As $k \ll M^{5/16-\varepsilon}$, we have $k \ll \Delta_\ell^{1/2-\delta}$ for any $\ell \in [-L, L]$ and a suitable choice for δ. We may estimate the sum

$$\sum_{n=1}^{\infty} W_L(n)a(n)e(n\alpha)$$

like before and obtain

$$\sum_{n=1}^{\infty} W_L(n)a(n)e(n\eta)\,e\left(n\frac{h}{k}\right) \ll M^{1/3+\varepsilon}\Delta^{1/6} + k^{-1}\Delta|\eta|^{-1/2}M^{-1/2}M^{\varepsilon}$$

With this choice of L, the characteristic function of the interval $[M, M+\Delta]$ differs from the function W_L only on intervals of the length $\ll M^{5/8}$. We may now use partial summation in order to estimate the sums on these intervals. Write this interval in the form

$$[M, N_1] \cup [N_2, M+\Delta]$$

$$
\begin{aligned}
\sum_{M\le n\le M+\Delta} &(1-W_L(n))\,a(n)e(\alpha n) \\
&= \left(\sum_{M\le n\le N_1} + \sum_{N_2\le n\le M+\Delta}\right)(1-W_L(n))\,a(n)e(\alpha n) \\
&= \int_M^{N_1} W_L'(t) \sum_{N_1\le n\le t} a(n)e(\alpha n)\,dt + \int_{N_2}^{M+\Delta} W_L'(t) \sum_{N_2\le n\le t} a(n)e(\alpha n)\,dt \\
&\quad + \sum_{N_2\le n\le M+\Delta} a(n)e(\alpha n) \\
&\ll \max_{M\le t\le N_1}\left|\sum_{M\le n\le t} a(n)e(\alpha n)\right| + \max_{N_2\le t\le M+\Delta}\left|\sum_{N_2\le n\le t} a(n)e(\alpha n)\right| \\
&\quad + \left|\sum_{N_2\le n\le M+\Delta} a(n)e(\alpha n)\right| \\
&\ll M^{\frac{1}{6}\cdot\frac{5}{8}+\frac{1}{3}+\varepsilon} \ll M^{7/16},
\end{aligned}
$$

which finishes the proof of the theorem. □

Remark 29. Assuming similar conditions as in Theorem 24 and arguing slightly more carefully, we get

$$\tilde{A}(M,\Delta,\alpha) \ll \left(\frac{\Delta}{M}\right)^{1/6} M^{1/2+\varepsilon} + k^{-1}\Delta|\eta|^{-1/2}M^{-1/2}\left(k^2\eta^2 M\right)^{\varepsilon}. \tag{3.15}$$

With the additional assumptions of the Theorem 28, the following formula holds:

$$A(M,\Delta,\alpha) \ll \left(\frac{\Delta}{M}\right)^{1/6} M^{1/2+\varepsilon} + k^{-1}\Delta|\eta|^{-1/2}M^{-1/2}\left(k^2\eta^2 M\right)^{\varepsilon}. \tag{3.16}$$

One may easily see that the estimate (3.12) weakens too fast (it increases faster than linearly) when $M^{5/8} \ll \Delta$, and hence the following theorem is useful.

Theorem 30. *Let $M^{5/8} \ll \Delta \ll M^{11/16}$. Then*

$$A(M,\Delta,\alpha) \ll M^{-9/48+\varepsilon}\Delta$$

for any α.

Proof. By Corollary 27,

$$A(M,\Delta,\alpha) \ll \Delta^{1/6}M^{1/3+\varepsilon}$$

for any $\Delta \ll M^{5/8}$. Let us split: $\Delta = mM^{5/8} + \Delta_1$, where $\Delta_1 \leq M^{5/8}$. Using corollary 27, we obtain

$$\begin{aligned} A(M,\Delta,\alpha) &= A\left(M + mM^{5/8}, \Delta_1, \alpha\right) + \sum_{k=0}^{m-1} A\left(M + kM^{5/8}, M^{5/8}, \alpha\right) \\ &\ll \Delta_1^{1/6}(M + M^{5/8}m)^{1/3+\varepsilon} + \sum_{k=0}^{m-1} \left(M^{5/8}\right)^{1/6} \left(M + kM^{5/8}\right)^{1/3+\varepsilon} \\ &\ll \Delta M^{-9/48+\varepsilon}. \end{aligned} \tag{3.17}$$

□

Chapter 4

An approximate functional equation

Theorem 31 (Approximate functional equation)**.** *For any* $\varepsilon > 0$ *we have*

$$\frac{A(M,\Delta,\alpha)}{M^{1/2}} = \frac{A\left(Mk^2\eta^2,\Delta k^2\eta^2;\beta\right)}{\left(Mk^2\eta^2\right)^{1/2}} + O\left(\left(Mk^2\eta^2\right)^{-1/12+\varepsilon}\right), \tag{4.1}$$

whenever $\alpha = \frac{h}{k} + \eta$ *is a Farey approximation of the order* $M^{1/4}$*,* $\beta = -\frac{\overline{h}}{k} - (k^2\eta)^{-1}$ *and* $k^2\eta^2 M \gg 1$*.*

Write $U = M^{1/2}\eta^{-1/2}\left(k^2\eta^2M\right)^d$, where d is a fixed positive constant which will be chosen suitably later. Let us start by a lemma before turning to the actual proof of the theorem.

Lemma 32. *Let* $\xi(x) \ll 1$ *and* $\xi'(x) \ll (k^2\eta^2U)^{-1}$ *on the intervals* $[N_{-1},N]$ *and* $[N_1,N_2]$*. Then we have*

$$\frac{1}{k\eta}\left(\sum_{N_{-1}\le n\le N} + \sum_{N_1\le n\le N_2}\right)\xi(n)a(n)e\left(-\frac{n\overline{h}}{k} - \frac{n}{k^2\eta}\right) \ll \sqrt{M}\left(k^2\eta^2M\right)^{\varepsilon+d/6-1/12}.$$

Proof. Using partial summation we conclude that it suffices to deal with the case $\xi(n) \equiv 1$.

The length of the two sums is $N - N_{-1} = N_2 - N_1 \asymp k^2\eta^2U$. Since

$$k^2\eta^2U = \left(k^2\eta^2M\right)^{d+1/2}k\eta^{1/2} \ll \left(k^2\eta^2M\right)^{5/8} = N^{5/8},$$

for $d \le \frac{1}{8}$, Theorem 28 gives the bound

$$\frac{1}{k\eta}\left(\sum_{N_{-1}\le n\le N}+\sum_{N_1\le n\le N_2}\right)a(n)e\left(-\frac{n\bar{h}}{k}-\frac{n}{k^2\eta}\right)$$
$$\ll \frac{1}{k\eta}\left(k^2\eta^2U\right)^{1/6}\left(k^2\eta^2M\right)^{1/3+\varepsilon}+\frac{\left(k^2\eta^2M\right)^{\varepsilon}}{k\eta},$$

where the latter term corresponds the case where $k^2\eta^2U \ll 1$ but there is an integer on the interval $[N_{-1}, N]$ or $[N_1, N_2]$. By simplifying and using the fact that $k^2\eta \ll 1$, we see that this is further equal to

$$M^{1/2}(\eta M)^{-1/12}\left(k^2\eta^2M\right)^{\varepsilon+d/6}+M^{1/2}\left(k^2\eta^2M\right)^{\varepsilon-1/2}$$
$$\ll \left(k^2\eta^2M\right)^{\varepsilon+d/6-1/12}\sqrt{M}.$$

□

Now we have enough lemmas (here and in the earlier chapters) to carry out the details of the proof.

Proof of Theorem 31. Consider the weighted sum

$$\sum_{M-JU\le n\le M+\Delta+JU} a(n)w(n)e(\alpha n)$$

instead of the original sum, where $w(x)$ is the special weight function defined in (1.6) with $V = U$. Now $w(x)$ satisfies the conditions (1.10).

Using the Voronoi summation formula (1.9.2 in [8]) we obtain

$$\sum_{M_{-1}\le n\le M_2} a(n)w(n)e(\alpha n)$$
$$= 2\pi k^{-1}(-1)^{\kappa/2}\sum_{n=1}^{\infty} a(n)e_k(-n\bar{h})\int_{M_{-1}}^{M_2} J_{\kappa-1}\left(4\pi\frac{\sqrt{nx}}{k}\right)e(x\eta)\,w(x)dx, \quad (4.2)$$

where $J_{\kappa-1}$ is the Bessel J-function. Without loss of generality we may assume $\eta \ge 0$.

Substituting the asymptotic expansion (1.5) in place of the Bessel function in the formula (4.2) and writing sine and cosine in terms of exponentials, we see that it is sufficient to consider the sums

$$
\begin{aligned}
A_{\pm} &= 2\pi k^{-1}(-1)^{\kappa/2}\sum_{n=1}^{\infty} a(n)e_k(-n\bar{h})\times \\
&\quad \int_{M_{-1}}^{M_2} \frac{\sqrt{k}}{2\sqrt{2}\pi n^{1/4}} x^{-1/4} e\left(\mp 2\frac{\sqrt{nx}}{k} \pm \frac{1}{4}\kappa \mp \frac{1}{8}\right) e(x\eta)\, w(x)dx \\
B_{\pm} &= k^{-1}\sum_{n=1}^{\infty} a(n)e_k(-n\bar{h})\times \\
&\quad \int_{M_{-1}}^{M_2} \frac{k^{3/2}}{n^{3/4}} e\left(\mp 2\frac{\sqrt{nx}}{k} \pm \frac{1}{4}\kappa \mp \frac{1}{8}\right) e(x\eta)\, x^{-3/4} w(x)dx \\
C\ &= k^{-1}\sum_{n=1}^{\infty} n^{\varepsilon}\int_{M_{-1}}^{M_2} \frac{k^{5/2}}{(nx)^{5/4}} w(x)dx.
\end{aligned}
$$

The sums A_-, B_- and B_+ are handled up to constant factors in Lemmas 14, 13 and 16. The sum C is easy to estimate:

$$
\begin{aligned}
C = k^{-1}\sum_{n=1}^{\infty} n^{\varepsilon}\int_{M_{-1}}^{M_2} \frac{k^{5/2}}{(nx)^{5/4}} w(x)dx &\ll k^{3/2}\sum_{n=1}^{\infty} n^{-5/4+\varepsilon} M^{-1/4} \\
&\ll k^{3/2}M^{-1/4} \ll M^{1/8}.
\end{aligned}
$$

Now treat the sum A_+. This requires a different treatment because of the resonance between the Bessel function and the exponential term. The sum of the terms with $n \geq cN$, where c is some constant, is estimated in Lemma 12. Let us then assume $1 \leq n \leq cN$ and consider the first term of the expansion of the Bessel function in formula (1.5). Using the saddle-point Lemma 9 we obtain

$$
\begin{aligned}
&\int_{M_{-1}}^{M_2} w(x)x^{-1/4} e\left(\eta x - 2\frac{\sqrt{nx}}{k}\right) dx \\
&\qquad = \xi(n)\frac{\sqrt{2}n^{1/4}}{\sqrt{k}\eta} e\left(-\frac{n}{k^2\eta} + \frac{1}{8}\right) + O\left(\frac{k^{3/2}}{n^{3/4}} + \delta(n)\frac{M^{1/4}k}{\sqrt{n}}\right) \\
&\qquad + O\left(M^{-1/4}U^{-J}\sum_{j=0}^{J}\left(\left|\eta - \frac{\sqrt{n}}{\sqrt{M - jU}k}\right| + \frac{n^{1/4}}{\sqrt{k}M^{3/4}}\right)^{-J-1}\right) \\
&\qquad + O\left(M^{-1/4}U^{-J}\sum_{j=0}^{J}\left(\left|\eta - \frac{\sqrt{n}}{\sqrt{M + \Delta + jU}k}\right| + \frac{n^{1/4}}{\sqrt{k}M^{3/4}}\right)^{-J-1}\right), \qquad (4.3)
\end{aligned}
$$

where

$$
\begin{cases}
\xi(n) = 0 \text{ and } \delta(n) = 0, \text{ if } n \leq N_{-1} \text{ or } n \geq N_2 \\
\xi(n) = 1 \text{ and } \delta(n) = 0, \text{ if } N \leq n \leq N_1 \\
\xi(n) \ll 1,\ \xi'(n) \ll (k^2\eta^2 U)^{-1} \text{ and } \delta(n) = 1, \text{ otherwise.}
\end{cases}
$$

The sum of the main terms in (4.3) is

$$\pi k^{-1}(-1)^{\kappa/2}\sum_{1\le n\le cN} a(n)\frac{\sqrt{k}}{\pi\sqrt{2}n^{1/4}}\xi(n)\frac{\sqrt{2}n^{1/4}}{\sqrt{k}\eta}e\left(\frac{\kappa}{4}-\frac{1}{8}-\frac{n\bar{h}}{k}-\frac{n}{k^2\eta}+\frac{1}{8}\right)$$
$$=\frac{1}{k\eta}\sum_{N\le n\le N_1} a(n)e\left(-\frac{n\bar{h}}{k}-\frac{n}{k^2\eta}\right)+\frac{1}{k\eta}\sum_{N_{-1}\le n\le N}\xi(n)a(n)e\left(-\frac{n\bar{h}}{k}-\frac{n}{k^2\eta}\right)$$
$$+\frac{1}{k\eta}\sum_{N_1\le n\le N_2}\xi(n)a(n)e\left(-\frac{n\bar{h}}{k}-\frac{n}{k^2\eta}\right). \quad (4.4)$$

The first sum on the right-hand side of (4.4) is the main term of the approximate functional equation (4.1). The other two sums in (4.4) of the length $k^2\eta^2 JU$ are estimated in Lemma 32.

The sum of the first error term in formula (4.3) is equal to

$$k^{-1}\sum_{1\le n\le N_2}\frac{\sqrt{k}}{n^{1/4}}\left(\frac{k^{3/2}}{n^{3/4}}+\delta(n)\frac{M^{1/4}k}{\sqrt{n}}\right)\ll k(k^2\eta^2M)^{\varepsilon+d}.$$

The remaining error terms in (4.3) are estimated in Lemma 17.

We have now derived

$$\sum_{M_{-1}\le n\le M_2} a(n)w(n)e(\alpha n)$$
$$=\frac{1}{k\eta}\sum_{N\le n\le N_1} a(n)e(-n\beta)+O\left(M^{1/2}\left(k^2\eta^2M\right)^{\varepsilon-Jd}\right)$$
$$+O\left(M^{1/2}\left(k^2\eta^2M\right)^{\varepsilon+d/6-1/12}\right).$$

However, there is still the error caused by the introduction of the weight function. Next we will estimate its size and optimize it together with the error terms $M^{1/2}\left(k^2\eta^2M\right)^{\varepsilon-Jd}$ and $M^{1/2}\left(k^2\eta^2M\right)^{\varepsilon+d/6-1/12}$.

We use partial summation to estimate

$$\sum_{M_{-1}\le n\le M} a(n)e(\alpha n)w(n)+\sum_{M_1\le n\le M_2} a(n)w(n)e(\alpha n)$$
$$=\sum_{M_{-1}\le n\le M} a(n)e(\alpha n)-\int_{M_{-1}}^{M} w'(t)\sum_{M_{-1}\le n\le t} a(n)e(\alpha n)\,dt$$
$$-\int_{M_1}^{M_2} w'(t)\sum_{M_1\le n\le t} a(n)e(\alpha n)\,dt$$
$$\ll\left|\sum_{M_{-1}\le n\le M} a(n)e(\alpha n)\right|+\max_{M_{-1}\le t\le M}\left|\sum_{M_{-1}\le n\le t} a(n)e(\alpha n)\right|$$
$$+\max_{M_1\le n\le M_2}\left|\sum_{M_1\le n\le t} a(n)e(\alpha n)\right|.$$

We will use Theorem 24 to estimate the sums. Notice that U does not satisfy the condition (3.2), as

$$U = \frac{\sqrt{M}}{\sqrt{|\eta|}} \left(k^2\eta^2 M\right)^d.$$

Therefore, we need to divide the length of the sum by $\left(k^2\eta^2 M\right)^d$. The new length of the sum is then $M^{1/2}|\eta|^{-1/2}$. We may then apply the theorem to the shorter sums to obtain good estimates for each of them, and finally multiply the estimate by the number of sums. This is similar to the proof of the theorem **??**. As an estimate for the short sums, we obtain

$$\begin{aligned}\ll \left(\frac{M^{1/2}|\eta|^{-1/2}}{M}\right)^{1/6} M^{1/2+\varepsilon} + k^{-1}M^{1/2}|\eta|^{-1/2}\eta^{-1/2}M^{-1/2}\left(k^2\eta^2 M\right)^{\varepsilon-d} \\ = M^{1/2}\left[(M\eta)^{-1/12}M^{\varepsilon} + \left(k^2\eta^2 M\right)^{-1/2+\varepsilon}\right].\end{aligned}$$

Now we notice that

$$(M\eta)^{-1/12} = \left(k^2\eta^2 M\right)^{-1/12}\left(\frac{k^2\eta^2 M}{M}\right)^{1/24} k^{1/12},$$

and remember that

$$\left(k^2\eta^2 M\right)^{-1/2} \ll \left(k^2\eta^2 M\right)^{-1/12},$$

and therefore, the estimate for the longer sums becomes

$$\ll M^{1/2}\left(k^2\eta^2 M\right)^{d-1/12}\left[\left(\frac{k^2\eta^2 M}{M}\right)^{1/24} M^{\varepsilon}k^{1/12} + \left(k^2\eta^2 M\right)^{\varepsilon}\right].$$

Now it suffices to show that, for any given positive $\varepsilon_2 > 0$,

$$\left(\frac{k^2\eta^2 M}{M}\right)^{1/24} M^{\varepsilon}k^{1/12} \ll \left(k^2\eta^2 M\right)^{\varepsilon_2}$$

for a suitable choice of ε. Assume next $k^2\eta^2 M \gg M^{1/4}$. Now

$$\left(\frac{k^2\eta^2 M}{M}\right)^{1/24} M^{\varepsilon}k^{1/12} \ll \left(k^2\eta^2 M\right)^{4\varepsilon} k^{1/6}\eta^{1/12} \ll \left(k^2\eta^2 M\right)^{\varepsilon_2},$$

when $\varepsilon_2 = 4\varepsilon$. Let us now assume $k^2\eta^2 M \ll M^{1/4}$.

$$\begin{aligned}\left(\frac{k^2\eta^2 M}{M}\right)^{1/24} M^{\varepsilon}k^{1/12} \ll M^{-3/96+\varepsilon}k^{1/12} \ll M^{-3/96+1/48+\varepsilon} \\ \ll M^{\varepsilon-1/96} \ll \left(Mk^2\eta^2\right)^{\varepsilon_2},\end{aligned}$$

when $\varepsilon_2 = \varepsilon$. We may choose d arbitrarily small, and hence get the estimate arbitrarily close to $M^{1/2}\left(k^2\eta^2M\right)^{-1/12}$. The estimate for the error term

$$M^{1/2}\left(k^2\eta^2M\right)^{\varepsilon+d/6-1/12}$$

coincides with this. The final error term to consider is

$$M^{1/2}\left(k^2\eta^2M\right)^{-Jd}.$$

Choose $J = \frac{1}{12d}$ to complete the proof. $\square$

Chapter 5

Bounds for longer linear exponential sums

Let us next consider the case when $\Delta \gg M^{11/16}$. The bound $O\left(M^{1/2}\right)$ may be viewed as a "trivial" bound, not because of actually being trivial or simple in the standard meaning; on the contrary, the bound is by far non-trivial for long sums and only obtained by Jutila in 1987 in his article [9], but because the estimate can be obtained for short sums very simply by using the triangle inequality and estimates for longer sums. We are going to show that only when $\Delta \ll M^{3/4}$, a non-trivial upper bound is possible, and the following theorem states one such bound. After that it remains to prove that for larger values of Δ a non-trivial general upper bound does not exist.

Theorem 33. *Let $M^{5/8} \ll \Delta \ll M^{3/4}$. Then*

$$A(M,\Delta,\alpha) \ll \Delta^{1/6}M^{1/3+\varepsilon} + M^{-1/4}\Delta + M^{\frac{1}{2}-a\omega},$$

where $a < \frac{1}{2}$ is the constant in the approximate functional equation ([9]) and $\omega = \min\left\{\frac{1-\log_M \Delta}{\varepsilon+3/4}, \frac{1}{2+8a+\varepsilon}\right\}$, where ε can be chosen to be an arbitrarily small positive real number.

Currently, the best known value for a is derived in Theorem 31. It states that a can be chosen to be an arbitrary positive real number less than $\frac{1}{12}$. Before proving the actual theorem, we present several useful lemmas which will be used in the proof. In fact, the first lemma proves the theorem for certain values of k and η.

Lemma 34. *Let $Mk^2\eta^2 = M^\gamma$, where $\gamma > 0$, $k \leq M^{1/4}$ and $M^{11/16} \ll \Delta \ll M^{3/4}$. Then*

$$\sum_{M \leq n \leq M+\Delta} a(n)e\left(\alpha n\right) \ll \Delta^{1/6}M^{1/3+\varepsilon\gamma} + M^{1/2-a\gamma} \tag{5.1}$$

Proof. Since $\eta^2 \ll k^{-2}M^{-1/2}$, also

$$\left(\eta^2 k^2\right)^{3/8} \ll M^{-3/16} \ll \frac{M^{5/8}}{\Delta},$$

and therefore $k^2\eta^2\Delta \ll \left(k^2\eta^2 M\right)^{5/8}$. Now we may use the approximate functional equation (**??**) and Corollary 27 (this required the preceding inequality). We obtain

$$\begin{aligned}\sum_{M\le n\le M+\Delta} a(n)e(\alpha n) &= \frac{1}{k\eta}\sum_{k^2\eta^2 M\le n\le k^2\eta^2(M+\Delta)} a(n)e(\eta n) + O\left(M^{1/2-a\gamma}\right)\\ &\ll \frac{1}{k\eta}\left(k^2\eta^2\Delta\right)^{1/6}\left(k^2\eta^2 M\right)^{1/3+\varepsilon} + M^{1/2-a\gamma} + \left(k^2\eta^2 M\right)^{\varepsilon}(k\eta)^{-1}\\ &\ll \Delta^{1/6}M^{1/3+\varepsilon\gamma} + M^{1/2-a\gamma}.\end{aligned}$$

□

Lemma 35. *Assume $k^2\eta^2 M > \frac{1}{2}$, $\eta \ge 0$, $M^{11/16} \ll \Delta \ll M^{3/4}$, $S \ge 1$ and $S - k^2\eta^2\Delta \gg S$. Then we have*

$$\begin{aligned}&k^{-1/2}\sum_{n=1}^{\infty} a(n)n^{-1/4}e_k(-n\bar{h})\int_M^{M+\Delta} x^{-1/4}e\left(\eta x - 2\frac{\sqrt{nx}}{k}\right)w(x)dx\\ &\quad\ll M^{7/16} + M^{3/2}k^3\eta^{3/2}S^{-1}\Delta^{-1}\left(k^2\eta^2 M\right)^{\varepsilon} + \Delta M^{-1/2}k^{-1}\eta^{-1/2}S\left(k^2\eta^2 M\right)^{\varepsilon}\end{aligned}$$

Proof. To be technically correct, we should divide the proof into two cases depending on whether $S \ll k^2\eta^2 M$ or not. However, we will only handle the case $S \ll k^2\eta^2 M$ as the other one is similar and even slightly easier.

When $k^2\eta^2 M - S \le n \le k^2\eta^2(M+\Delta) + S$, we will estimate the sum by absolute values:

$$\begin{aligned}&M^{-1/4}\Delta k^{-1/2}\sum_{k^2\eta^2 M - S\le n\le k^2\eta^2(M+\Delta)+S} n^{-1/4+\varepsilon}\\ &\quad\ll M^{-1/4}\Delta k^{-1/2}\left(k^2\eta^2 M\right)^{-1/4+\varepsilon}S \ll \Delta M^{-1/2}k^{-1}\eta^{-1/2}S\left(k^2\eta^2 M\right)^{\varepsilon}.\end{aligned}$$

Otherwise, let us use lemma 7 to estimate the integral. For $n \ge k^2\eta^2(M+\Delta)$, we obtain

$$\int_M^{M+\Delta} w(x)e\left(2\frac{\sqrt{nx}}{k} - \eta x\right)x^{-1/4}dx \ll M^{-1/4}\Delta^{1-P}\left(\frac{\sqrt{n}}{k\sqrt{M+\Delta}} - \eta\right)^{P}. \quad (5.2)$$

Let now $k^2\eta^2(M+\Delta) + S \le n \le ck^2\eta^2 M$ for some suitable constant c. Substitute the result (5.2), with $P=2$, in the place of the integral in the sum:

$$
\begin{aligned}
M^{-1/4}\Delta^{-1}k^{-1/2} &\sum_{k^2\eta^2(M+\Delta)+S\le n\le ck^2\eta^2M} n^{\varepsilon-1/4}\left(\frac{\sqrt{n}}{k\sqrt{M+\Delta}}-\eta\right)^{-2} \\
&\ll M^{7/4}k^{7/2}\eta^2\Delta^{-1}\left(k^2\eta^2M\right)^{-1/4+\varepsilon}\times \\
&\sum_{k^2\eta^2(M+\Delta)+S\le m\le ck^2\eta^2M}\left(n-k^2\eta^2(M+\Delta)\right)^{-2} \\
&\ll M^{3/2}k^3\eta^{3/2}\Delta^{-1}\left(k^2\eta^2M\right)^{\varepsilon}S^{-1}.
\end{aligned}
$$

For $n\ge ck^2\eta^2M$, the calculations are similar those above, only simpler. We obtain

$$
k^{-1/2}\sum_{n\ge ck^2\eta^2M} a(n)n^{-1/4}e_k\left(-n\bar{h}\right)\int_M^{M+\Delta}x^{-1/4}e\left(\eta x-2\frac{\sqrt{nx}}{k}\right)w(x)dx \ll M^{7/16}
$$

Finally, for $n\le k^2\eta^2M-S$, the calculations are again very similar to the earlier ones except for the details requiring a bit more attention. One gets

$$
\begin{aligned}
k^{-1/2}&\sum_{n\le k^2\eta^2M-S} a(n)n^{-1/4}e_k\left(-n\bar{h}\right)\int_M^{M+\Delta}x^{-1/4}e\left(\eta x-2\frac{\sqrt{nx}}{k}\right)w(x)dx \\
&\ll M^{7/16}+M^{3/2}k^3\eta^{3/2}\Delta^{-1}S^{-1}\left(k^2\eta^2M\right)^{\varepsilon}.
\end{aligned}
$$

This completes the proof of the lemma. □

Similarly we may prove the following lemma.

Lemma 36. *Assume* $k^2\eta^2M>\frac{1}{2}$, $\eta\ge 0$ *and* $M^{11/16}\ll\Delta\ll M^{3/4}$. *Then*

$$
k^{1/2}\sum_{n=1}^{\infty}a(n)n^{-3/4}e_k(-n\bar{h})\int_M^{M+\Delta}x^{-3/4}e\left(\eta x-2\frac{\sqrt{nx}}{k}\right)w(x)dx \ll M^{3/8}\left(k^2\eta^2M\right)^{\varepsilon}
$$

Lemma 37. *Let* $k^2\eta^2M<\frac{1}{2}$ *and* $M^{11/16}\ll\Delta\ll M^{3/4}$. *Then*

$$
k^{1/2}\sum_{n=1}^{\infty}a(n)n^{-3/4}e_k(-n\bar{h})\int_M^{M+\Delta}x^{-3/4}e\left(\eta x\pm 2\frac{\sqrt{nx}}{k}\right)w(x)dx \ll M^{1/8}.
$$

Now we may finally move to the actual proof of the theorem.

Proof of Theorem 33. The details of this proof could be given more in the spirit of the proof of Theorem 24, however, we feel that the following approach supports more the proof of the next theorem.

Write $\alpha = \frac{h}{k} + \eta$, where $1 \le k \le M^{1/4}$ and $|\eta| \le \frac{1}{kM^{1/4}}$. Without loss of generality we may assume $\eta \ge 0$. Write $k^2\eta^2 M = M^\gamma$. We will estimate the sum in two different cases. The first approach works for large values of γ and the latter for the smaller values.

According to Lemma 34:

$$\sum_{M \le n \le M+\Delta} a(n)e(\alpha n) \ll \Delta^{1/6} M^{1/3+\varepsilon} + O\left(M^{1/2-a\omega}\right).$$

For an alternative approach, let us introduce a smooth sufficiently many times differentiable weight function w and consider smoothed sums. Using similar method as in the proof of the Theorem 28, we see that this does not affect the final results. Using the Voronoi summation formula ([8], Theorem 1.7) we obtain

$$\begin{aligned}\sum_{M \le n \le M+\Delta} & a(n)w(n)e(\alpha n) \\ &= 2\pi k^{-1}(-1)^{\kappa/2} \sum_{n=1}^{\infty} a(n) e_k(-n\bar{h}) \int_M^{M+\Delta} J_{\kappa-1}\left(4\pi \frac{\sqrt{nx}}{k}\right) w(x)dx\end{aligned}$$

Using the expansion for the $J-$Bessel functions (5.11.6 [12]) we obtain

$$\begin{aligned}&J_{\kappa-1}\left(4\pi\frac{\sqrt{nx}}{k}\right) \\ &= \frac{\sqrt{k}}{\sqrt{2}\pi n^{1/4}} x^{-1/4}\left(e\left(-2\frac{\sqrt{nx}}{k} + \frac{1}{4}\kappa - \frac{1}{4}\right) + e\left(2\frac{\sqrt{nx}}{k} - \frac{1}{4}\kappa + \frac{1}{4}\right)\right) \\ &+ ck^{3/2} n^{-3/4} x^{-3/4}\left(e\left(2\frac{\sqrt{nx}}{k} - \frac{1}{4}\pi + \frac{1}{8}\pi\right) - e\left(-2\frac{\sqrt{nx}}{k} + \frac{1}{4}\kappa - \frac{1}{8}\right)\right) \\ &\qquad + O\left(\frac{k^{5/2}}{(nx)^{5/4}}\right),\end{aligned}$$

where c is some constant. Substituting this, we see that the sums to be evaluated are

$$\begin{aligned}A =& c_A k^{-1/2} \sum_{n=1}^{\infty} a(n) n^{-1/4} e_k(-n\bar{h}) \int_M^{M+\Delta} x^{-1/4} e\left(\eta x - 2\frac{\sqrt{nx}}{k}\right) w(x)dx \\ B =& c_B k^{-1/2} \sum_{n=1}^{\infty} a(n) n^{-1/4} e_k(-n\bar{h}) \int_M^{M+\Delta} x^{-1/4} e\left(\eta x + 2\frac{\sqrt{nx}}{k}\right) w(x)dx \\ C =& c_C k^{1/2} \sum_{n=1}^{\infty} a(n) n^{-3/4} e_k(-n\bar{h}) \int_M^{M+\Delta} x^{-3/4} e\left(\eta x - 2\frac{\sqrt{nx}}{k}\right) w(x)dx \\ D =& c_D k^{1/2} \sum_{n=1}^{\infty} a(n) n^{-3/4} e_k(-n\bar{h}) \int_M^{M+\Delta} x^{-3/4} e\left(\eta x + 2\frac{\sqrt{nx}}{k}\right) w(x)dx \\ E =& k^{3/2} \sum_{n=1}^{\infty} n^{\varepsilon-5/4} \Delta M^{-5/4} \ll k^{3/2} \Delta M^{-5/4}.\end{aligned}$$

All the sums are estimated up to some constants in the previous lemmas, so we may conclude that

$$\sum_{M\le n\le M+\Delta} a(n)w(n)e(\alpha n) \ll M^{7/16}+M^{3/2}k^3\eta^{3/2}S^{-1}\Delta^{-1}\left(k^2\eta^2M\right)^{\varepsilon}+\Delta M^{-1/2}k^{-1}\eta^{-1/2}S\left(k^2\eta^2M\right)^{\varepsilon}. \tag{5.3}$$

In order to optimize this, we set

$$M^{3/2}k^3\eta^{3/2}S^{-1}\Delta^{-1}\left(k^2\eta^2M\right)^{\varepsilon} \asymp \Delta M^{-1/2}k^{-1}\eta^{-1/2}S\left(k^2\eta^2M\right)^{\varepsilon}.$$

Now

$$S \asymp Mk^2\eta\Delta^{-1}.$$

If $Mk^2\eta\Delta^{-1} \ll 1$ or $\left|Mk^2\eta\Delta^{-1}-k^2\eta^2\Delta\right| = o\left(Mk^2\eta\Delta^{-1}\right)$, we need to choose

$$S = \max\left(1, k^2\eta^2\Delta, Mk^2\eta\Delta^{-1}\right).$$

Assume first $S=1$ and substitute the value into (5.3). The estimate becomes

$$M^{7/16}+\Delta M^{-1/2}k^{-1}\eta^{-1/2}\left(k^2\eta^2M\right)^{\varepsilon}.$$

It suffices again to consider the latter term. Substitute $\eta = k^{-1}M^{-1/2+\gamma/2}$ and the term becomes

$$\Delta M^{-1/2}k^{-1}k^{1/2}M^{1/4-\gamma/4}\left(k^2\eta^2M\right)^{\varepsilon} = \Delta M^{-1/4-\gamma/4+\varepsilon\gamma}k^{-1/2} \ll \Delta M^{-1/4}. \tag{5.4}$$

We may now proceed to the next case. The substitution $S = k^2\eta^2\Delta$ to (5.3) gives

$$M^{7/16}+M^{3/2}k\eta^{-1/2}\Delta^{-2}\left(k^2\eta^2M\right)^{\varepsilon}+\Delta^2M^{-1/2}k\eta^{3/2}\left(k^2\eta^2M\right)^{\varepsilon}.$$

Because $Mk^2\eta\Delta^{-1} \ll k^2\eta^2\Delta$ (which is assumed whenever doing the substitution $S = k^2\eta^2\Delta$) is equivalent to $M \ll \eta\Delta^2$, we have

$$M^{3/2}k\eta^{-1/2}\Delta^{-2}\left(k^2\eta^2M\right)^{\varepsilon} \ll \Delta^2M^{-1/2}k\eta^{3/2}\left(k^2\eta^2M\right)^{\varepsilon}.$$

Now it is sufficient to consider the term

$$\Delta^2M^{-1/2}k\eta^{3/2}\left(k^2\eta^2M\right)^{\varepsilon}.$$

Substitute $\eta = M^{(\gamma-1)/2}k^{-1}$ to obtain

$$\Delta^2M^{-1/2}k\eta^{3/2}\left(k^2\eta^2M\right)^{\varepsilon} = \Delta^2M^{-5/4+3\gamma/4+\varepsilon\gamma}k^{-1/2}.$$

We may now require

$$\Delta^2M^{-5/4+3\gamma/4+\varepsilon\gamma} \ll \Delta M^{-1/4}$$

without weakening the overall estimate as it is impossible to get below this bound. This requirement obviously sets a bound for γ, and the values of γ not satisfying the condition are treated according to the other approach (Lemma 34). This approach may seem a bit risky as the obvious question regarding the harshness of the bound may rise. However, it will be seen that this bound is suitable.

We obtain

$$\gamma \leq \frac{1-\log_M \Delta}{\varepsilon + 3/4}. \tag{5.5}$$

as a bound.

The following calculations will complete the proof. Substitute now $S = Mk^2\eta\Delta^{-1} \gg 1$ into (5.3). The estimate becomes

$$M^{7/16} + M^{3/2}k^3\eta^{3/2}\Delta^{-1}\left(k^2\eta^2 M\right)^{\varepsilon}\left(\Delta^{-1}Mk^2\eta\right)^{-1} = M^{7/16} + M^{1/2}k\eta^{1/2}\left(k^2\eta^2 M\right)^{\varepsilon}.$$

Let us estimate the latter term. Remember that for large values of γ, the estimate

$$\Delta^{1/6}M^{1/3+\varepsilon} + M^{1/2-a\gamma}$$

will be used. The term $\Delta^{1/6}M^{1/3+\varepsilon}$ is relatively small and it suffices to consider the other term. To optimize, we should have

$$M^{1/2-a\gamma_0} = M^{1/2}k\eta^{1/2}\left(k^2\eta^2 M\right)^{\varepsilon} \tag{5.6}$$

when the sides attain their maxima. Further,

$$M^{1/2}k\eta^{1/2}\left(k^2\eta^2 M\right)^{\varepsilon} = \left(k^2\eta^2 M\right)^{1/4+\varepsilon}M^{1/4}k^{1/2} = M^{\gamma/4+\gamma\varepsilon}M^{1/4}k^{1/2} \ll M^{\gamma/4+\gamma\varepsilon+3/8}$$

and therefore, it suffices to consider the equation

$$M^{1/2-a\gamma} = M^{3/8+\gamma/4+\gamma\varepsilon}.$$

Thus,

$$\frac{1}{2} - a\gamma = \frac{3}{8} + \frac{\gamma}{4} + \gamma\varepsilon.$$

Now $\gamma\left(\frac{1}{4} + a + \varepsilon\right) = \frac{1}{8}$ and

$$\gamma = \frac{1}{2 + 8a + \varepsilon}.$$

Concluding, the estimate becomes

$$A\left(M, M+\Delta, \frac{a}{k} + \eta\right) \ll M^{7/16} + \Delta^{1/6}M^{1/3+\varepsilon} + \Delta M^{-1/4} + M^{\frac{1}{2}-a\omega} \ll \Delta^{1/6}M^{1/3+\varepsilon} + \Delta M^{-1/4} + M^{\frac{1}{2}-a\omega}, \tag{5.7}$$

where $\omega = \min\left(\frac{1-\log_M \Delta}{\varepsilon+3/4}, \frac{1}{2+8a+\varepsilon}\right)$. $\square$

According to the best current knowledge, derived in [3], a can be chosen to be arbitrary close to but no exceeding $\frac{1}{12}$. Notice

$$\frac{1}{2+8\cdot\frac{1}{12}} \leq \frac{1-\log_M\Delta}{3/4},$$

when $\log_M\Delta \leq \frac{23}{32}$. However,

$$M^{-1/4}\Delta \gg M^{\frac{1}{2}-\frac{1}{12}\cdot\frac{1}{2+8\cdot 112}},$$

when $\Delta \gg M^{3/4-1/32}$, and therefore, we may write the estimate of the Theorem 33 in the form

Theorem 38 (another formulation)**.** *Let* $M^{11/16} \ll \Delta \ll M^{3/4}$*. Then*

$$A(M,\Delta,\alpha) \ll M^{-1/4}\Delta + M^{\frac{1}{2}-\frac{1}{32}+\varepsilon},$$

This version is easy to use, but it is obviously less flexible than the original formulation.

Chapter 6

Sharp estimates for certain sums

In this section, it will be proved that the upper bounds can never be smaller than $M^{-1/4}\Delta$, when $M^{1/4} \ll \Delta \ll M^{3/4}$. Specifically, it will be proved that when $M^{3/4-1/32+\varepsilon} \ll \Delta \ll M$, the derived bounds are sharp. (When $M^{3/4} \ll \Delta \ll M$, the sharp estimate is $M^{1/2}$, which is the bound already derived by Jutila.)

Theorem 39. *Let $M^{1/2+\delta} < \Delta \le \lambda M^{3/4}$ where $0 < \lambda < 1$ is a constant. Let w be a smooth weight function on the interval $[M, M+\Delta]$ which equals 1 on the interval $[a,b] \subset [M, M+\Delta]$ where $a - M = M + \Delta - b = \Delta^{1-\delta}$ with δ a sufficiently small fixed positive real number. Assume further that $\alpha = M^{-\frac{1}{2}}$. Then*

$$\left| \sum_{M \le n \le M+\Delta} a(n) w(n) e(\alpha n) \right| \asymp \Delta M^{-1/4}.$$

Proof. Write $\Delta = M^{\gamma}$. We may assume thoroughout the proof that $\delta \le \frac{1}{9}$. Let us start just like in the proof of Theorem 33. Assume first $n \ge 2$. Use a sufficiently large value of P in Lemma 7 to obtain

$$\int_M^{M+\Delta} w(x) e\left(\frac{x}{\sqrt{M}}\right) e\left(2\sqrt{nx} \pm \frac{\kappa - 1}{4} - \frac{1}{8}\right) x^{-3/4} dx \ll n^{-P/2}$$

and

$$\int_M^{M+\Delta} w(x) e\left(\frac{x}{\sqrt{M}} - 2\sqrt{nx}\right) x^{-1/4} dx \ll n^{-P/2}.$$

Now, let $n = 1$. Using absolute values while integrating, we get

$$\int_M^{M+\Delta} \int_M^{M+\Delta} w(x) e\left(\frac{x}{\sqrt{M}} \pm 2\sqrt{x}\right) x^{-3/4} \mathrm{d}x \ll \int_M^{M+\Delta} x^{-3/4} \mathrm{d}x \ll 1.$$

Lemma 7, with a sufficiently large value of P, gives us

$$\int_M^{M+\Delta} w(x) e\left(\frac{x}{\sqrt{M}} + 2\sqrt{x}\right) x^{-1/4} \mathrm{d}x \ll 1.$$

We have now obtained

$$\sum_{M\le n\le M+\Delta} a(n)w(n)e\left(\frac{x}{\sqrt{M}}\right)$$
$$=\sqrt{2\pi}(-1)^{\kappa/2}\int_M^{M+\Delta} w(x)e\left(\frac{x}{\sqrt{M}}+2\sqrt{x}-\frac{\kappa-1}{4}-\frac{1}{8}\right)x^{-1/4}\mathrm{d}x+O(1).$$

Let us now treat the only lasting term. This integral

$$\int_M^{M+\Delta} e\left(\frac{x}{\sqrt{M}}-2\sqrt{x}\right)w(x)x^{-1/4}dx$$

is the most crucial one and will contribute the main term. In this case partial integration (and use of Lemma 7) fails. In order to simplify notations, write $g(x)=\frac{x}{\sqrt{M}}-2\sqrt{x}$. Taylor expansion gives the following approximation

$$g(M+\Delta)-g(M)=\frac{\Delta^2}{4M^{3/2}}+O\left(\frac{\Delta^3}{M^{5/2}}\right).$$

Write

$$y(x)=g(x)-g(M)=\sqrt{M}-2\sqrt{x}+\frac{x}{\sqrt{M}}.$$

Now y is increasing, $y(M)=0$ and

$$y(M+\Delta)=\frac{\Delta^2}{4M^{3/2}}+O\left(\frac{\Delta^3}{M^{5/2}}\right).$$

Specifically this means that $\frac{\lambda^2}{4}+O\left(M^{-1/4}\right)$ is an upper bound for $y(x)$ on the interval $[M,M+\Delta]$. Therefore, we may assume the values of cosine to be non-negative. We may rewrite the integral

$$\int_M^{M+\Delta} e\left(\frac{x}{\sqrt{M}}-2\sqrt{x}\right)w(x)x^{-1/4}dx$$
$$=e^{-2\pi i\sqrt{M}}\int_M^{M+\Delta}\left(\cos\left(2\pi y(x)\right)+i\sin\left(2\pi y(x)\right)\right)w(x)x^{-1/4}dx. \quad (6.1)$$

We immediately see that

$$\int_M^{M+\Delta} e\left(\frac{x}{\sqrt{M}}-2\sqrt{x}\right)w(x)x^{-1/4}\ll\int_M^{M+\Delta}x^{-1/4}\ll\Delta M^{-1/4}.$$

Therefore, it suffices to show that

$$\left|\int_M^{M+\Delta}\cos\left(2\pi y(x)\right)w(x)x^{-1/4}\right|\asymp M^{-1/4}\Delta.$$

First notice

$$\left|\int_M^{M+\Delta}\cos\left(2\pi y(x)\right)w(x)x^{-1/4}dx\right| \geq \left|\int_M^{M+\frac{\Delta}{2}}\cos\left(2\pi y(x)\right)w(x)x^{-1/4}dx\right|$$
$$= \int_M^{M+\frac{\Delta}{2}}\cos\left(2\pi y(x)\right)x^{-1/4}w(x)dx \geq \cos\left(2\pi y\left(M+\frac{\Delta}{2}\right)\right)\int_{M+\Delta^{1-\delta}}^{M+\frac{\Delta}{2}}x^{-1/4}dx$$
$$\gg \cos\left(2\pi y\left(M+\frac{\Delta}{2}\right)\right)\Delta M^{-1/4}.$$

We need a good approximation for y in order to estimate the cosine. First compute

$$y(M+\frac{\Delta}{2}) = \sqrt{M} - 2\sqrt{M+\frac{\Delta}{2}} + \frac{M+\frac{\Delta}{2}}{\sqrt{M}} = \frac{\Delta^2}{16M^{3/2}} + O\left(\frac{\Delta^3}{M^{5/2}}\right).$$

Notice now that for $0 < \alpha < \frac{\pi}{2}$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} \geq \sqrt{1-2\sin\alpha+\sin^2\alpha} = 1-\sin\alpha \geq 1-\alpha.$$

Therefore

$$\cos\left(2\pi y\left(M+\frac{\Delta}{2}\right)\right) \geq 1-\sin\frac{\Delta^2\pi}{M^{3/2}8} + O\left(\frac{\Delta^3}{M^{5/2}}\right),$$

which proves the theorem.

$\square$

Remark 40. This also shows that for $\Delta \gg M^{3/4}$ there is no better general upper bound than $M^{1/2}$ because then the triangle inequality would give a better bound also for $A\left(M, M^{3/4}, \frac{1}{\sqrt{M}}\right)$. Also, this shows that the estimate of Theorem 33 is sharp whenever $\Delta M^{-1/4}$ is the leading term. At the moment, the best known value for a implies that the estimate is sharp once $\Delta \geq M^{3/4-1/32+\varepsilon}$.

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