TUCS

Eeva Suvitie

On Inner Products Involving Holomorphic Cusp Forms and Maass Forms

Turku Centre *for* Computer Science

TUCS Dissertations
No 123, October 2009

On Inner Products Involving Holomorphic Cusp Forms and Maass Forms

Eeva Suvitie

To be presented, with the permission of the Faculty of Mathematics and Natural Sciences of the University of Turku, for public criticism in Auditorium XXI of the University on October 30th, 2009, at 12 noon

University of Turku
Department of Mathematics
FI-20014 Turku, Finland

2009

Supervisor

Professor Emeritus Matti Jutila
Department of Mathematics
Vesilinnantie 5
University of Turku
FI-20014 Turku
Finland

Reviewers

Associate Professor Morten Skarsholm Risager
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen
Denmark

Associate Professor Andreas Strömbergsson
Department of Mathematics
University of Uppsala
Box 480
S-75106 Uppsala
Sweden

Opponent

Professor Yoichi Motohashi
Department of Mathematics
Nihon University
Surugadai
Tokyo 101-8308
Japan

ISBN 978-952-12-2353-2
ISSN 1239-1883

Acknowledgements

I wish to express my deepest gratitude to my supervisor Professor Matti Jutila for being just who he is. It has been a privilege to witness a true professional in his work. Without him this thesis truly, really would have never existed in any imaginable way.

I also want to thank Dr. Morten S. Risager, University of Copenhagen, Denmark, and Dr. Andreas Strömbergsson, University of Uppsala, Sweden, for kindly accepting the invitation to review my thesis. They provided me with many useful corrections as well as honest and friendly comments.

Special thanks are due to Professor Yoichi Motohashi, Nihon University, Japan, for giving me the honour of having him as my opponent at the public examination of my thesis. I also thank him for his valuable comments on my thesis.

I am grateful to Dr. Tom Meurman for patiently answering my every single stupid question concerning analytic number theory and helping me to get over the rocky start of my PhD studies. I thank Professor Kari Ylinen for his thorough work monitoring me as my Research Director, and teaching me the necessary analysis, as well as occasionally helping me with my non-existing French.

I would also like to thank Professor Andre Reznikov, Bar-Ilan University, Israel, for kindly answering my questions about his joint work with Professor Joseph Bernstein, Tel Aviv University, Israel, as well as for pointing out an application of my result.

I am grateful to Dr. Milla Kibble for doing awesome job in correcting my English and various misprints.

The work was funded by the Number Theory Project (Academy of Finland), the graduate school of the Turku Center for Computer Science (TUCS), the Department of Mathematics at University of Turku, Turun Suomalainen Säästöpankki scholarship fund, Finnish Cultural Foundation, Turun Suomalainen Yliopistoseura, Valto Takala's trust and Apteekkari Wäinö Edward Miettinen's scholarship fund. All this support is gratefully acknowledged.

People at the Department of Mathematics have provided a great place to work; I have been able to speak up my mind with them without any serious consequences. Especially I would like to thank Anne-Maria for being the rock I have been able to lean on, Camilla for her never-ending patience in our friendship and Tuire, Siamak, Petri, Arto, Markku, Ville, Tuomas, Henri, Otto x 2, Boa, Laura, Päivi, Irmeli, Kaisa and a couple of Rantos for just being themselves. I thank Kaisa also for her aid with the English language concerning the last minute changes in this thesis. Especially I want to thank my department family Eija, Meskanen and Jyrki. Diana, Marjo and Tommi are also special people for me as my officemates at the department.

Friends outside the university, especially Emmi, A-K, Tiina, Antti and Sini, are acknowledged for putting up with me for so long time and still carrying on. I thank both Heppo and Rokka for bringing fun and stupid movies to our evenings. My mother-in-law Noora is acknowledged for holidays in Pori and making cakes for the disputation coffee.

I thank mum for even alone providing everything to me.

Finally, I thank Roope for being the love of my life, at least apart from my beloved terriers Aaro and Lucia, and my already passed beagle Timotei.

Turku
October 2009 Eeva I. Suvitie

Contents

Introduction

According to C. G. J. Jacobi, the history of holomorphic modular forms, and thus especially holomorphic cusp forms, began in 1751 when G. C. Fagnano's work *Produzioni matematiche* was given to L. P. Euler for review by the Academy of Berlin. Fagnano's work inspired Euler to study the arc of an ellipse and ultimately to propose elliptic integrals. His work was continued by Jacobi and N. H. Abel, leading to elliptic functions as inverse functions of elliptic integrals. Research eventually led, via Jacobi's theta function and Eisenstein series, to modular forms. Generally speaking, the theory of automorphic functions formed one of the most active branches of mathematics in the late 19th century, with further developments being made by K. Weierstrass, F. Klein and H. Poincaré, among others. More recently, modular forms have appeared in various applications, most spectacularly when A. Wiles used elliptic curves in proving Fermat's last theorem in 1994.

However, as Weil stated, "Maass was needed for us to leave the ghetto of the holomorphic functions".

In 1949, H. Maass introduced the concept of real-analytic cusp forms, these being particular eigenfunctions of the non-Euclidean Laplacian Δ, which now bear his name. His work was based on theories by E. Hecke and H. Petersson. Hecke investigated the relationship between holomorphic cusp forms and the corresponding Dirichlet series, and his theory was taken forward by the metric theory created by Petersson in 1939. The line of research started by Maass was further developed by W. Roelcke and A. Selberg. Their heroic deed was to create the spectral resolution of Δ over the full modular group Γ (as a special case of a more general theory) in terms of Maass forms and Eisenstein series; see [36], Theorem 1.1.

An interesting feature of holomorphic and non-holomorphic cusp forms is the analogy between their Fourier coefficients and the classical divisor function $d(n)$. The analogy reveals itself for example in the Voronoĭ type summation formulae, which are similar in each case (see [19], Theorem 1.7, and [32], Theorem 2). One fascinating occurrence of the divisor function is in the additive divisor problem, in which one investigates the asymptotic behavior of the sum

$$D(x;m) = \sum_{n\leq x} d(n)d(n+m)$$

as x tends to infinity. Here $d(n)$ is the number of divisors of n, and m is a given positive integer. Y. Motohashi's comprehensive paper [34] gives an in-depth study of this problem along with a discussion of its history. We learn that an interesting aspect was first noticed by F. V. Atkinson in 1941, when he was looking for an estimate for the error term $E(x;m)$ in the asymptotic formula for $D(x;m)$ in order to estimate certain "non-diagonal" parts of his formula for a form of the fourth power mean of the Riemann zeta-function on the critical line. Yet another connection with the Fourier coefficients of cusp forms over the full modular group was discovered in 1982 when J.-M. Deshouillers and H. Iwaniec introduced the first application of Kuznetsov's trace formulas to $D(x;m)$, transforming sums of Kloosterman sums appearing in $E(x;m)$ into bilinear forms of these Fourier coefficients.

In his papers [21] and [22], M. Jutila considers generalizations of the additive divisor problem aiming at a unified approach to this sum along with its analogs

$$A(x;m) = \sum_{n\leq x} a(n)\overline{a(n+m)}$$

over the Fourier coefficients of a holomorphic cusp form and

$$T(x;m) = \sum_{n\leq x} t(n)t(n+m)$$

over the Hecke eigenvalues corresponding to Fourier coefficients of a Maass form. The interest in these sums does not lie purely in the formal analogy, as they are related to squares of certain L-functions in the same way as the additive divisor problem is related to the fourth moment of Riemann's zeta-function. Jutila studies these sums via the respective generating Dirichlet series $\zeta_m(s)$, $\varphi_m(s)$ and $\psi_m(s)$. In order to analytically continue these series beyond the region of absolute convergence with estimates of at most polynomial order on vertical lines, he first replaces $\zeta_m(s)$ and $\psi_m(s)$ by certain approximations, $\zeta_m^*(s)$ and $\psi_m^*(s)$, in terms of integrals over the strip $\Pi = \{z = x+iy \,|\, |x| \leq 1/2, y > 0\}$. The series $\varphi_m(s)$ can be represented precisely in such a form. He then writes the preceding functions as integrals over a fundamental domain $\mathcal{F}$ for Γ, namely in the form of Petersson inner products. Next he uses Parseval's formula for the inner products and gains meromorphic continuations of the functions $\zeta_m^*(s)$, $\varphi_m(s)$ and $\psi_m^*(s)$ to the whole complex plane with satisfactory estimates on vertical lines, the main complication being how to obtain the following estimates

$$\sum_{\kappa_j\leq K} |c_j|^2 \exp(\pi\kappa_j) \ll K^{2k+\varepsilon} \quad \text{and} \quad \sum_{\kappa_j\leq K} |\widetilde{c_j}|^2 \exp(\pi\kappa_j) \ll K^{\varepsilon}, \qquad (0.0.1)$$

where $c_j = (u_j(z), y^k|F(z)|^2)$ and $\widetilde{c}_j = (u_j(z), |u(z)|^2)$ are two inner products involving Maass wave forms u_j and the holomorphic and non-holomorphic cusp forms $F(z)$ and $u(z)$, respectively. The first of the above estimates was initially proved by A. Good in [9], the argument being specific to holomorphic cusp forms, whereas Jutila created a unified proof for both sums in (0.0.1). Analogous estimates for inner products were established independently by P. Sarnak in [39], where he considers individual inner products of a more general type than those above, gaining a result that $|\widetilde{c}_j|^2 \exp(\pi\kappa_j)$ is of polynomial order in κ_j, although the order is weaker than what follows from the estimate above.

Finally Jutila derives "almost" explicit formulae for $D(x;m)$, or rather for its most interesting part, the error term $E(x;m)$, as well as for $A(x;m)$ and $T(x;m)$ in a unified way.

In order to gain a deeper insight into the first of the κ_j-sums in (0.0.1) and into the order of a single inner product c_j, in this thesis we study this sum over a short interval $K \le \kappa_j \le K + K^{1/3}$ and achieve the expected estimate:

$$\sum_{K \le \kappa_j \le K+K^{1/3}} |c_j|^2 \exp(\pi\kappa_j) \ll K^{2k-2/3+\varepsilon}. \tag{0.0.2}$$

This result is new. Another of Jutila's papers [24] serves as a motivation for choosing the cube root of K as the length of the sum, as he investigates the sum

$$\sum_{|\kappa_j - K| \le K^{1/3}} \alpha_j H_j^4\left(\frac{1}{2}\right) \ll K^{4/3+\varepsilon} \tag{0.0.3}$$

where $H_j(s)$ is the Hecke L-function attached to the jth Maass form and $\alpha_j = |\rho_j(1)|^2/\cosh(\pi\kappa_j)$, $\rho_j(1)$ being the first Fourier coefficient of the corresponding Maass form. The reason underlying this choice goes back to a paper by A. Ivić [13], in which he proves that

$$\sum_{|\kappa_j - K| \le 1} \alpha_j H_j^3\left(\frac{1}{2}\right) \ll K^{1+\varepsilon},$$

and as a corollary it follows that

$$H_j\left(\frac{1}{2}\right) \ll \kappa_j^{1/3+\varepsilon}.$$

It is our future plan to prove in an analogous way a short interval estimate involving the inner products $\widetilde{c}_j$, namely

$$\sum_{K \le \kappa_j \le K+K^{1/3}} |\widetilde{c}_j|^2 \exp(\pi\kappa_j) \ll K^{-2/3+\varepsilon}. \tag{0.0.4}$$

This result would also be new. J. Bernstein and A. Reznikov have studied a more general case of the above in their papers [2] and [3]. They consider a product of two fixed Maass forms instead of the square of a fixed non-holomorphic cusp form to obtain a non-trivial bound $K^{-1/3+\varepsilon}$. Note that although Bernstein and Reznikov first assume, for simplicity, that the fundamental domain in question is compact, in Remark 1.4 in [2] and Remark 1.3.1 in [3] they point out that their result can be extended to the case of a general fundamental domain of finite volume. See also Corollary 1.2 in [2] or the corollary on p. 4 in [3] for how to apply their result to obtain a bound for special values of certain automorphic L-functions.

Further, if $E^*(z)$ stands for a certain non-holomorphic Eisenstein series to be defined in Section 1.1, where $d(n)$ plays the role of the Fourier coefficients, then essentially

$$H_j^2\left(\frac{1}{2}\right) = A \times (u_j(z), |E^*(z)|^2)$$

with some coefficient A. Therefore it seems plausible that our argument should give a unified approach to all three estimates (0.0.2), (0.0.3) and (0.0.4).

In the first chapter of this thesis we lay the foundations by introducing some definitions and results. In the second chapter we show that the proof for the first short sum (0.0.2) can be constructed by following the arguments in papers by Good [8] and Jutila and Motohashi [28]. In [8], Good proves that the inner product c_j grows at most polynomially with respect to κ_j. He starts with the definition of the inner product, writes the holomorphic cusp form in question as a linear combination of Poincaré series and uses the Rankin-Selberg method to arrive at a single sum over Fourier coefficients $\rho_j(m)$ and $a(m)$. The result which Good thus obtains by direct estimations is not sharp enough for our purposes, so instead we continue along the lines of the approach used by Jutila and Motohashi. In [28] they treat an analogous sum with the divisor function $d(n)$ in place of the Fourier coefficient of a holomorphic cusp form, applying a series of various transformations and approximations to spectral and arithmetic objects. The key points in Jutila's and Motohashi's proof for the sum which is analogous to our case are the use of a version of the Kloosterman-spectral sum formula of R. W. Bruggeman and N. V. Kuznetsov, the sum formula of Voronoĭ, an explicit spectral decomposition of the shifted convolution sum and the spectral large sieve. In our case, some more complications arise from the fact that Kloosterman sums appear instead of Ramanujan sums. Moreover, we have to integrate over two parameters t_1 and t_2 which appear as fixed variables in [28]. Heuristically, we lose a certain quantity when we have Kloosterman sums in place of Ramanujan sums, but with non-trivial estimation of the t_i-integrals we win back the same quantity, and hence we obtain an analogous result to that of [28].

However, our main goal is to construct a proof for the first short sum in such a manner that it can be extended to the two analogous cases mentioned above. Although holomorphic cusp forms are linear combinations of Poincaré series, there is no analogous structure known for the non-holomorphic case. Therefore we adopt a different approach in the third chapter of this work. We succeed in our goal up to a case discussed in Remark 8, which we believe can be modelled on the discussion in the Chapter 2. We follow the arguments of Jutila's paper [21], in which he estimates the long sum with this analogy in mind. The main steps are applications of a variant of the Rankin-Selberg method and the spectral large sieve along with Sobolev's lemma with additional careful modifications undertaken throughout the proof. Jutila arrives at a shifted convolution sum over the Fourier coefficients of our holomorphic cusp form with a shift of size K^{ε}. In our case, however, the original sum is taken over a short interval, leading to an oscillatory integral over this short range. The cancellation of the oscillation is thus weaker and in our case the shift is of size $K^{2/3+\varepsilon}$. While Jutila may now proceed with trivial estimations, we have to find another path to follow. We thus proceed by invoking a spectral decomposition of a shifted convolution sum and utilize Jutila's estimate for the corresponding long spectral sum. In a straightforward manner, we arrive at yet another shifted convolution sum, the shift this time being essentially of size $K^{1/3+\varepsilon}$. By use of Jutila's "almost explicit" formula for $A(x, m)$ we finally get our result, except for the above mentioned case.

Chapter 1

Preliminaries

1.1 Holomorphic and non-holomorphic cusp forms

We confine ourselves to cusp forms for the full modular group $\Gamma = SL_2(\mathbb{Z})$ operating through Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

on the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$. The standard fundamental domain $\mathcal{F}$ for Γ is the domain

$$\{z = x + iy \in \mathbb{H} \mid |z| > 1, -1/2 < x < 1/2\}$$

together with its boundary in the half plane Re $z \leq 0$.

A *holomorphic cusp form* $F(z) : \mathbb{H} \to \mathbb{C}$ of weight $k \in \mathbb{Z}$ with respect to Γ is a function satisfying the following three conditions:

(i) For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have $F(\gamma(z)) = F(z)j(\gamma, z)^k$, where

$$j(\gamma, z) = cz + d. \tag{1.1.1}$$

(ii) The function F is holomorphic in the half plane $\mathbb{H}$ and at the cusp $z = i\infty$.

(iii) The function has a zero at $z = i\infty$.

We let $S_k = S_k(\Gamma)$ denote the set of holomorphic cusp forms of weight k. In $\mathbb{H}$, $F(z)$ can be represented by its Fourier series

$$F(z) = \sum_{n=1}^{\infty} a(n)e(nz). \tag{1.1.2}$$

For k odd, $k < 12$ or $k = 14$ we have $S_k = \{0\}$. Hereafter we shall therefore always assume that k is even and $k \geq 12$. The set S_k is a finite dimensional unitary space equipped with the Petersson inner product

$$(f_1, f_2)_k = \int_{\mathcal{F}} f_1(z)\overline{f_2(z)} y^k \, d\mu(z),$$

with respect to the hyperbolic measure $d\mu(z) = \frac{dx\,dy}{y^2}$. For a proof, see Motohashi [36], p. 49. We let

$$\{\psi_{j,k} \mid 1 \leq j \leq \vartheta(k)\} \tag{1.1.3}$$

be an orthonormal basis of S_k, and write

$$\psi_{j,k}(z) = \sum_{n=1}^{\infty} \rho_{j,k}(n) n^{\frac{k-1}{2}} e(nz). \tag{1.1.4}$$

We may suppose that the basis vectors are eigenfunctions of the Hecke operators $T_k(n)$ for all positive integers n;

$$(T_k(n)f)(z) = n^{-\frac{1}{2}} \sum_{ad=n} \left(\frac{a}{d}\right)^{k/2} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right).$$

Then also $T_k(n)f \in S_k$ whenever $f \in S_k$. Moreover, the operators $T_k(n)$ are self-adjoint, and hence the eigenvalues are real. Thus, in particular, $T_k(n)\psi_{j,k} = t_{j,k}(n)\psi_{j,k}$ for certain real numbers $t_{j,k}(n)$, which we call Hecke eigenvalues. Comparing Fourier coefficients on both sides, one may verify that $\rho_{j,k}(n) = \rho_{j,k}(1)t_{j,k}(n)$ for all $n \geq 1$, $1 \leq j \leq \vartheta(k)$.

Note that we do not assume that our general cusp form $F(z)$ is an eigenfunction of all Hecke operators. Neither is this assumption needed for any of the results used in this work.

Furthermore, we let

$$H_{j,k}(s) = \sum_{n=1}^{\infty} t_{j,k}(n) n^{-s} \tag{1.1.5}$$

stand for the Hecke L-function attached to $\psi_{j,k}$. The series converges absolutely for Re $s > 5/4$; see [36], Eq. (3.1.22) p. 104. Moreover,

$$|t_{j,k}(n)| \leq d(n) \ll n^{\varepsilon}. \tag{1.1.6}$$

For a proof of (1.1.6) for n prime, see Deligne [4], Theorem 8.2, and use the Euler product of the Hecke L-function (1.1.5) and the multiplicativity of the Fourier coefficients to extend the estimate to all $n \in \mathbb{Z}_+$. As usual, $d(n)$ denotes the divisor function, and generally we use the notation

$$\sigma_{\alpha}(n) = \sum_{d|n} d^{\alpha}, \tag{1.1.7}$$

so $d(n) = \sigma_0(n)$. Eventually, (1.1.5) thus converges absolutely for Re $s > 1$.

A *non-holomorphic cusp form* or an *automorphic form* $u(z) = u(x+iy)$ is a non-constant real-analytic Γ-invariant function in the upper half-plane with the following properties:

(i) $u(z)$ is square-integrable with respect to the hyperbolic measure $d\mu(z)$ over a fundamental domain of Γ.

(ii) $u(z)$ is an eigenfunction of the non-euclidean Laplacian $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. The corresponding eigenvalue, known to exceed $1/4$, can be written as $1/4 + \kappa^2$ with $\kappa > 0$.

The Fourier series expansion for $u(z)$ is then of the form

$$u(z) = y^{1/2} \sum_{n \neq 0} \rho(n) K_{i\kappa}(2\pi|n|y) e(nx)$$

with K_ν a Bessel function of imaginary argument (see [31], p. 108).

We introduce the Hecke operator $T(n)$ which acts over the linear space of all Γ-automorphic functions

$$(T(n)f)(z) = n^{-\frac{1}{2}} \sum_{ad=n} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right).$$

Again, if f is Γ-automorphic, then so is $T(n)f$. We may suppose that our cusp forms are eigenfunctions of the Hecke operators $T(n)$ for all positive integers n and that $u(x+iy)$ is even or odd as a function of x. Thus $T(n)u = t(n)u$ for certain real numbers $t(n)$, which are again called Hecke eigenvalues, and $u(-\overline{z}) = \pm u(z)$. Comparing Fourier coefficients on both sides, one may verify that $\rho(n) = \rho(1)t(n)$ and $\rho(-n) = \pm\rho(n)$ for all $n \geq 1$. Further we may assume that the Maass forms are real, and hence $\overline{\rho(n)} = \rho(-n)$ for all n.

The Petersson inner product

$$(f, g) = \int_{\mathcal{F}} f(z)\overline{g(z)}\, d\mu(z)$$

is well-defined for two square-integrable Γ-invariant functions; note that this is analogous to the inner product of holomorphic cusp forms defined above.

Finally, *Maass (wave) forms*

$$u_j(z) = y^{1/2} \sum_{n \neq 0} \rho_j(n) K_{i\kappa_j}(2\pi|n|y) e(nx) \tag{1.1.8}$$

constitute an orthonormal set of non-holomorphic cusp forms arranged so that the corresponding parameters κ_j determined by the eigenvalues $1/4 + \kappa_j^2$ lie in an increasing order. We write $t_j(n)$ for the corresponding Hecke eigenvalues.

Further, we let

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s} \tag{1.1.9}$$

stand for the Hecke L-function attached to u_j. Now it is relatively easy to see that

$$t_j(n) \ll n^{1/4+\varepsilon},$$

where the implied constant depends only on ε (see Motohashi [36], Lemma 3.3), so the series converges absolutely for Re $s > 5/4$. In fact, this holds again for Re $s > 1$, since it is well-known that the Ramanujan-Petersson conjecture $t_j(n) \ll n^{\varepsilon}$ holds in a mean value sense.

For the number of spectral parameters κ_j in a certain range we have (see Hejhal [10], p. 511)

$$N[0 \le \kappa_j \le K] = \frac{1}{12} K^2 + \mathcal{O}(K \log K). \tag{1.1.10}$$

Finally we denote

$$c_j = (u_j(z), y^k |F(z)|^2)$$

for the inner product involving a Maass form and a holomorphic cusp form, and we are ready to formulate the main result of this thesis.

Theorem 1. *For all $K \ge 1$, $\varepsilon > 0$*

$$\sum_{K \le \kappa_j \le K + K^{1/3}} |c_j|^2 \exp(\pi \kappa_j) \ll K^{2k-2/3+\varepsilon}.$$

Here the implied constant depends on k and ε.

For future reference we also define the inner product

$$c(t) = (E(z, 1/2 + it), y^k |F(z)|^2), \tag{1.1.11}$$

involving the *non-holomorphic Eisenstein series*

$$E(z, s) = \sum_{\gamma \in \mathcal{A}} (\text{Im } \gamma(z))^s \tag{1.1.12}$$

with $z \in \mathbb{H}$ and $s \in \mathbb{C}$. Here $\mathcal{A}$ is a representative set of the right cosets $\Gamma_\infty \gamma$ in Γ, $\Gamma_\infty = \{z \mapsto z + n \,|\, n \in \mathbb{Z}\}$ being the stabilizer of the cusp in Γ. The summands are independent of the choice of the representatives, and hence $E(z,s)$ is Γ-automorphic. The sum converges absolutely for Re $s>1$, but it can be analytically continued to a meromorphic function in the complex plane (see Motohashi [36], Lemma 1.2).

Moreover we let

$$\widetilde{c_j} = (u_j(z), |u(z)|^2)$$

stand for the inner product of two non-holomorphic cusp forms. We use the notation

$$E^*(z) = \sqrt{y}(\log y - c) + 2\sqrt{y} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} d(|n|)K_0(2\pi|n|y)e(nx)$$

with $c = \log(4\pi) - \gamma$, where γ is Euler's constant. Here $E^*(z) = \lim_{s\to 1/2} E^*(z,s)$, where $E^*(z,s) = \xi(2s)E(z,s)$ with

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{1.1.13}$$

Then

$$H_j^2\left(\frac{1}{2}\right) = \frac{2\pi}{\rho_j(1)\ |\Gamma(1/4+i\kappa_j/2)|^4}(u_j(z), |E^*(z)|^2),$$

if the related Maass form $u_j(x+iy)$ is even as a function of x, that is, $\overline{\rho_j(n)} = \rho_j(-n) = \rho_j(n)$; see [42], Eqs. (4.2) and (4.8). For forms that are odd as a function of x

$$H_j\left(\frac{1}{2}\right) = 0;$$

see [36], Eq. (3.3.5).

Furthermore the following notation will be adopted. We let $\lfloor x \rfloor$ stand for the largest integer smaller or equal to $x \in \mathbb{R}$. The notation $\|f\|_1$ stands for the L_1-norm

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

We write $\alpha_j = |\rho_j|^2/\cosh(\pi\kappa_j)$ with $\rho_j = \rho_j(1)$. We use the notation $m \sim M$ when $M \le m < 2M$ and $m \asymp M$ when $AM \le m \le BM$ for some positive constants A and B. Vinogradov's relation $f(z) \ll g(z)$ is another notation for $f(z) = \mathcal{O}(g(z))$. Since the implied constant in our Theorem 1 may depend on the weight k and the arbitrarily small positive number ε, we shall often pass the forthcoming bounds without explicitly mentioning the dependence of the implied constants concerned on those variables. In the context of complex integrals, the notation $\int_{(a)}$ means integration along the vertical line, where the real part is a.

Lastly, we adopt Convention 2 from Jutila's and Motohashi's paper [28]: Let $\mathcal{X}$ be a particular object that we need to bound and $\mathcal{Y}$ some expression that comes up in the course of the proof. Let us have an approximation $\mathcal{Y} = \mathcal{Y}_0 + \mathcal{Y}_1 + \mathcal{O}(\mathcal{Z})$, in which $\mathcal{Y}_0$ is dominant, $\mathcal{Y}_1$ oscillates in the same mode as $\mathcal{Y}_0$ and $\mathcal{Z}$ contributes negligibly to $\mathcal{X}$. Then clearly it suffices to treat only $\mathcal{Y}_0$ instead of $\mathcal{Y}$ and the notation $\mathcal{Y} \sim \mathcal{Y}_0$ indicates the use of a procedure in which the treatment of $\mathcal{Y}_1$ is a repetition of that of $\mathcal{Y}_0$.

Notice that the notation $\sim$ is being used to mean different things in different places, but its meaning will be clear from the context. However, for clarity, we denote by $\approx$ the asymptotic expansions (see e.g. Olver [37], pp. 16-17).

We let ε stand generally for a small positive number, not necessarily the same at each occurrence.

1.2 Some tools

We shall now gather some auxiliary results which will be used later.

1.2.1 Special functions

In this thesis we shall need some special functions, an excellent reference for these being Lebedev's monograph [31].

The Gamma function

First we recall some basic relations for the Gamma function. Its functional equation reads as follows:

$$\Gamma(z+1) = z\Gamma(z). \tag{1.2.1}$$

The Γ-function is related to the sine function via the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{1.2.2}$$

and the duplication formula for the Gamma function is

$$\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}2^{1-2z}\Gamma(2z);$$

for proofs, see Lebedev [31], pp. 3-4.

An important role is played by the following asymptotic representation.

Lemma 1.1 (Stirling's formula). *The following asymptotic expansion holds for the* Γ*-function:*

$$\Gamma(z) \approx \sqrt{2\pi}\exp((z-1/2)\log z - z)\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots\right), \tag{1.2.3}$$

as $z \to \infty$ *in the sector* $|arg\ z| \le \pi - \varepsilon$.

In particular, in any fixed strip $b \le \sigma \le c$ *we have*

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma-1/2}e^{-|t|\pi/2}(1 + \mathcal{O}(|t|^{-1})),$$

for $|t| \to \infty$.

For a proof, see Olver [37] p. 294. In what follows, in the cases when we have not specified the estimation of the Γ-function, Stirling's formula is always used.

Moreover, we have an asymptotic expansion involving Bernoulli numbers B_{2n} for the logarithm of the Γ-function:

$$\log\Gamma(z) = (z-1/2)\log z - z + \frac{1}{2}\log(2\pi) + \sum_{n=1}^{N}\frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \mathcal{O}\left(\frac{1}{z^{2N+1}}\right), \tag{1.2.4}$$

when $|\arg z| \le \pi - \delta$, with N an arbitrary positive integer and $\delta > 0$ an arbitrary constant. See [37], pp. 293-294.

The Riemann zeta-function

The following estimate for Riemann's zeta-function holds uniformly in the half plane $\sigma \geq 0$, as $|t| \geq 1$:

$$\zeta(\sigma + it) \ll |t|^{1/2+\varepsilon}. \tag{1.2.5}$$

For a proof, see Titchmarsh [43], pp. 81-82.

Moreover we have estimate

$$\frac{1}{\zeta(1+it)} \ll \log |t|,$$

as $|t| \geq 1$. For a proof, see e.g. Titchmarsh [43], p. 114.

The Bessel functions

We have already introduced the K-Bessel function $K_\nu(z)$. Generally the Bessel functions of imaginary argument are defined by the formulas

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(\nu+k+1)}, \quad z \in \mathbb{C}, \quad |\arg z| < \pi,$$

and

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)}[I_{-\nu}(z) - I_\nu(z)], \quad z \in \mathbb{C}, \quad |\arg z| < \pi, \quad \nu \notin \mathbb{Z}.$$

With integer $\nu = n$,

$$K_n(z) = \lim_{\nu \to n} K_\nu(z), \quad \text{as } n \in \mathbb{Z}.$$

Moreover, by Lebedev [31], Eq. (5.10.23),

$$K_{ir}(y) = \int_0^\infty e^{-y\cosh u} \cos(ru)\, du$$

for all $r \in \mathbb{R}$ and $y > 0$, and therefore

$$|K_{ir}(y)| \leq \int_0^\infty e^{-y\cosh u}\, du = K_0(y).$$

For small y, the asymptotic behavior of this function is given by

$$K_0(y) = \sum_{k=0}^{\infty} \left(\frac{y}{2}\right)^{2k} \frac{\log\frac{2}{y} + \psi(k+1)}{(k!)^2} \sim \log\frac{2}{y} \quad \text{as} \quad y \to 0. \tag{1.2.6}$$

Here $\psi(z)$ is the logarithmic derivative of the Gamma function;

$$\psi(1) = -\gamma, \quad \psi(k+1) = -\gamma + 1 + \frac{1}{2} + \ldots + \frac{1}{k}, \ k = 1, 2, \ldots,$$

where again, γ is Euler's constant. For large y we have the asymptotic formula

$$K_0(y) = \sqrt{\frac{\pi}{2y}} e^{-y} \left[\sum_{k=0}^{N} (0,k)(2y)^{-k} + \mathcal{O}\left(y^{-N-1}\right) \right] \sim \sqrt{\frac{\pi}{2y}} e^{-y} \text{ as } y \to \infty. \tag{1.2.7}$$

Here

$$(0,0) = 1 \text{ and } (0,k) = \frac{(-1)(-3^2)\cdots(-(2k-1)^2)}{2^{2k}k!}.$$

For proofs, see Lebedev [31], Eqs. (5.7.11) and (5.11.9).

The hypergeometric function

The hypergeometric function is defined as the sum of the hypergeometric series

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k, \quad |z| < 1,$$

where z is a complex variable, α, β and γ are parameters which can take arbitrary real or complex values provided that $\gamma \neq 0, -1, -2, \ldots$, and the symbol $(\lambda)_k$ denotes the quantity

$$(\lambda)_0 = 1, \quad (\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+k-1), \; k = 1, 2, \ldots. \tag{1.2.8}$$

The analytic continuation of $F(\alpha, \beta; \gamma; z)$ into the z-plane cut along the segment $[1, \infty]$ is achieved through the Gaussian integral

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt,$$

$$\operatorname{Re}\gamma > \operatorname{Re}\beta > 0, \quad |\arg\,(1-z)| < \pi. \tag{1.2.9}$$

The restrictions on β and γ can be removed by the elementary recursion formula

$$\gamma(\gamma+1)F(\alpha, \beta; \gamma; z) = \gamma(\gamma - \alpha + 1)F(\alpha, \beta + 1; \gamma + 2; z)$$

$$+\alpha(\gamma - (\gamma - \beta)z)F(\alpha + 1, \beta + 1; \gamma + 2; z) \tag{1.2.10}$$

as $\gamma \neq 0, -1, -2, \ldots$ (see Lebedev [31], pp. 238-240).

We have the following transformation formulae:

$$F(\alpha+1/2, \alpha; \gamma; z) = (1+z^{1/2})^{-2\alpha} F\left(\gamma - \frac{1}{2}, 2\alpha; 2\gamma - 1; \frac{2z^{1/2}}{1+z^{1/2}}\right) \tag{1.2.11}$$

and

$$F(\alpha,\beta;\gamma;z)=z^{-\alpha}\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F\left(\alpha,1+\alpha-\gamma;1+\alpha+\beta-\gamma;\frac{z-1}{z}\right)$$

$$+z^{\alpha-\gamma}(1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F\left(\gamma-\alpha,1-\alpha;1+\gamma-\alpha-\beta;\frac{z-1}{z}\right); \tag{1.2.12}$$

for proofs, see [31], Eqs. (9.6.17) and (9.5.10). It suffices for us that both of these formulas hold on the real interval $0<z<1$, provided that $\gamma\notin\mathbb{Z}_{\le 0}$ in the former formula (1.2.11), and that $\alpha+\beta-\gamma\notin\mathbb{Z}$ in the latter formula (1.2.12).

Furthermore, we have a quadratic transformation formula

$$F(\alpha,\beta;2\beta;z)=\left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha}F\left(\alpha,\alpha-\beta+\frac{1}{2};\beta+\frac{1}{2};\left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right), \tag{1.2.13}$$

when $|\arg(1-z)|<\pi$ and $2\beta\neq-1,-3,-5,\ldots$. For a proof, see Lebedev [31], Eq. (9.6.12).

Another relation is the differentiation formula

$$\frac{\partial}{\partial z}F(\alpha,\beta;\gamma;z)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;z)$$

$$=\frac{\beta}{z}(F(\alpha,\beta+1;\gamma;z)-F(\alpha,\beta;\gamma;z)); \tag{1.2.14}$$

see Lebedev [31], Eqs. (9.2.2) and (9.2.13).

Lastly, we have an important lemma on the size of hypergeometric functions of a certain type.

Lemma 1.2. *Let $K\ge 1$, $\alpha=x+iy$ with a fixed $x\in\mathbb{R}$, $x\ge 5/2$, $x+1/2\notin\mathbb{Z}$ and $|y|\ll\log K$, $l=1,2,3$, $r\in\mathbb{R}$, $|r|\asymp K$ and $z\in\mathbb{R}_-$, $K^b\ll|z|\ll K^{1+\delta_1}$ with $\delta_1>0$ and b some fixed constants. The hypergeometric functions of the form*

$$F(\alpha,l-\alpha;1+ir;z)$$

are then of order $\ll K^{\delta_1(x-l+1/2)+\varepsilon}$.

Proof. We appeal to the Barnes integral formula

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+w)\Gamma(b+w)\Gamma(-w)}{\Gamma(c+w)}(-z)^w\,dw, \tag{1.2.15}$$

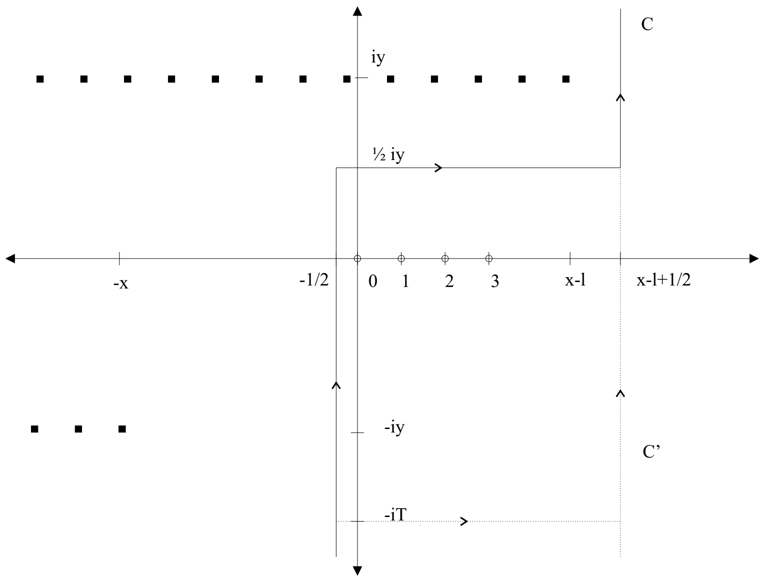
where $|\arg(-z)|<\pi$ and the contour is drawn so that the poles of $\Gamma(a+w)\Gamma(b+w)$ lie on the left and those of $\Gamma(-w)$ lie on the right of the path. For a proof, see Whittaker and Watson [45], p. 286.

Let us first suppose that Im $\alpha = y > 0$. Then

$$F(\alpha, l-\alpha; 1+ir; z) = \frac{\Gamma(1+ir)}{\Gamma(\alpha)\Gamma(l-\alpha)} \frac{1}{2\pi i}$$

$$\times \int_C \frac{\Gamma(\alpha+w)\Gamma(l-\alpha+w)\Gamma(-w)}{\Gamma(1+ir+w)} (-z)^w \, dw,$$

where C is the path joining the points $-1/2-i\infty$, $-1/2+iy/2$, $\alpha-l+1/2-iy/2 = x-l+1/2+iy/2$ and $x-l+1/2+i\infty$ by vertical or horizontal line segments, as shown in the figure below. The poles of $\Gamma(\alpha+w)\Gamma(l-\alpha+w)$ have been marked by squares and those of $\Gamma(-w)$ by circles.



By the theorem of residues,

$$\int_C f(w)\, dw = \int_{C'} f(w)\, dw + \int_{-1/2}^{x-l+1/2} f(q-iT)\, dq + \int_{-\infty}^{-T} f(-1/2+iu) i \, du$$

$$- \int_{-\infty}^{-T} f(x-l+1/2+iu) i\, du - 2\pi i \sum_{0 \le n < x-l+1/2} \mathrm{Res}(f(w), n)$$

with

$$f(w) = \frac{\Gamma(\alpha+w)\Gamma(l-\alpha+w)\Gamma(-w)}{\Gamma(1+ir+w)} (-z)^w ,$$

where C' is the vertical line Re $w = x - l + 1/2$ and T is some positive quantity. Note that by assumption $x - l + 1/2 \notin \mathbb{Z}$. Now clearly, by Stirling's formula and our assumption on z, we have

$$\frac{\Gamma(1+ir)}{\Gamma(\alpha)\Gamma(l-\alpha)} \frac{1}{2\pi i} \left(\int_{-1/2}^{x-l+1/2} f(q - iT)\, dq + \int_{-\infty}^{-T} f(-1/2 + iu) i\, du \right.$$

$$\left. - \int_{-\infty}^{-T} f(x - l + 1/2 + iu) i\, du \right) \ll 1,$$

when we fix T large enough; say $T \gg K$. Furthermore, using the same argument and dividing the path of the integral into suitable intervals, we get

$$\frac{\Gamma(1+ir)}{\Gamma(\alpha)\Gamma(l-\alpha)} \frac{1}{2\pi i} \int_{C'} f(w)\, dw \ll K^{\delta_1(x-l+1/2)+\varepsilon}.$$

Finally,

$$-2\pi i \sum_{0 \le n < x-l+1/2} \mathrm{Res}(f(w), n) = 2\pi i \sum_n \frac{\Gamma(\alpha+n)\Gamma(l-\alpha+n)}{\Gamma(1+ir+n)} (-z)^n \frac{(-1)^n}{n!}$$

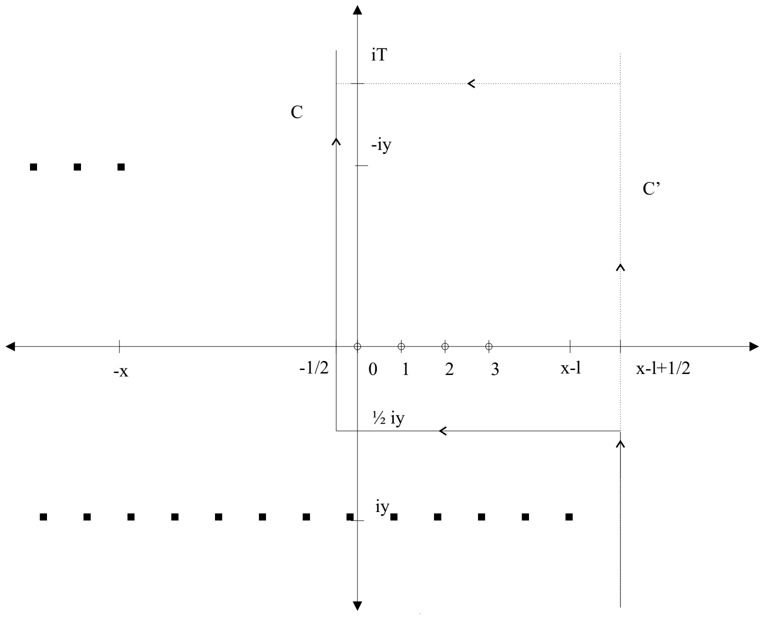
$$= 2\pi i \frac{\Gamma(\alpha)\Gamma(l-\alpha)}{\Gamma(1+ir)} \sum_n \frac{(\alpha)_n (l-\alpha)_n}{(1+ir)_n n!} (-z)^n,$$

so the sum of residues yields a partial sum of the hypergeometric series in question. By our assumption on z, the n-sum is $\ll K^{\delta_1(x-l+1/2)+\varepsilon}$ and

$$\frac{\Gamma(1+ir)}{\Gamma(\alpha)\Gamma(l-\alpha)} \frac{1}{2\pi i} (-2\pi i) \sum_{0 \le n < x-l+1/2} \mathrm{Res}(f(w), n) \ll K^{\delta_1(x-l+1/2)+\varepsilon}.$$

Hence the hypergeometric functions are in this case of order $K^{\delta_1(x-l+1/2)+\varepsilon}$, as desired.

The case $y < 0$ can be treated analogously; this time we choose C to be the path joining the points $x - l + 1/2 - i\infty$, $x - l + 1/2 + iy/2$, $-1/2 + iy/2$ and $-1/2 + i\infty$ by vertical or horizontal line segments, and C' to be the vertical line Re $w = x - l + 1/2$, as in the figure below. Again the poles of $\Gamma(\alpha + w)\Gamma(l - \alpha + w)$ have been marked by squares and the poles of $\Gamma(-w)$ by circles.



The case $y = 0$ can be handled by the continuity of the hypergeometric function; see Lebedev [31], Section 9.4. $\square$

Kloosterman sums

The Kloosterman sum is defined for the variables $m, n \in \mathbb{Z}$, $l \in \mathbb{Z}_+$ by the formula

$$S(m,n;l) = \sum_{\substack{q=1\\(q,l)=1}}^{l} e\left(\frac{mq+n\overline{q}}{l}\right), \quad q\overline{q} \equiv 1 \pmod{l}.$$

We have the Weil bound

$$S(m,n;l) \ll d(l)(m,n,l)^{1/2} l^{1/2} \tag{1.2.16}$$

which holds uniformly; see Estermann [7], p. 86.

Moreover, for the Kloosterman zeta-function we have the following spectral formula due to Kutznetsov (see [30], Lemma p. 375), originating to Selberg's paper [41].

Lemma 1.3. *For $m, n \geq 1$ and Re $s > 3/4$*

$$(2\pi\sqrt{mn})^{2s-1} \sum_{l=1}^{\infty} l^{-2s} S(m,n;l) = \frac{1}{2}\sin(\pi s) \sum_{j=1}^{\infty} \frac{\overline{\rho_j(m)}\rho_j(n)}{\cosh(\pi\kappa_j)}$$

$$\times\Gamma(s-1/2+i\kappa_j)\Gamma(s-1/2-i\kappa_j)+\frac{1}{2\pi}\sin(\pi s)\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}$$

$$\times\Gamma(s-1/2+ir)\Gamma(s-1/2-ir)\,dr+\sum_{k=1}^{\infty}(2k-1)q_{m,n}(k)\frac{\Gamma(k-1+s)}{\Gamma(k+1-s)}$$

$$-\frac{1}{2\pi}\delta_{m,n}\frac{\Gamma(s)}{\Gamma(1-s)} \tag{1.2.17}$$

with

$$q_{m,n}(k)=\sum_{l=1}^{\infty}\frac{1}{l}S(m,n;l)J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{l}\right).$$

Here

$$q_{m,n}(k)\ll(mn)^{\varepsilon}$$

uniformly in k.

For a proof of equation (1.2.17), see Lemma 2.5 in [36]. The estimate for $q_{m,n}(k)$ follows easily from Lemma 2.3 in [36] and Deligne's estimate (1.1.6).

Lastly, we have the following lemma by Iwaniec:

Lemma 1.4. *Let* $M,N,L\geq 1$ *and* $g(m,n,l)\in C^2$ *be a weight function with the properties*

$$supp\ g\subseteq[M,2M]\times[N,2N]\times[L,2L]$$

and, for $0\leq q_1,q_2,q_3\leq 2$,

$$\left|\frac{\partial^{q_1+q_2+q_3}}{\partial m^{q_1}\partial n^{q_2}\partial l^{q_3}}g(m,n,l)\right|\leq M^{-q_1}N^{-q_2}L^{-q_3}.$$

Then for $a_m,b_n\in\mathbb{C}$ *we have*

$$\sum_{m\sim M}\sum_{n\sim N}\sum_{l\sim L}a_m b_n g(m,n,l)S(m,\pm n,l)$$

$$\ll L^{1+\varepsilon}(MN)^{1/2}\left(\sum|a_m|^2\right)^{1/2}\left(\sum|b_n|^2\right)^{1/2}.$$

For a proof, see Theorem 4 in [15].

Corollary 1.5. *Let* $M,L\geq 1$ *and* n_0 *be a fixed positive integer. Let* $g(m,n,l)\in C^2$ *be a weight function with the properties*

$$supp\ g\subseteq[a_1M,b_1M]\times[a_2,b_2]\times[a_3L,b_3L]$$

for some positive constants a_i,b_i, $i=1,2,3$, *let the interval* $[a_2,b_2]$ *contain* n_0, *and, for* $0\leq q_1,q_2,q_3\leq 2$, *let*

$$\left|\frac{\partial^{q_1+q_2+q_3}}{\partial m^{q_1}\partial n^{q_2}\partial l^{q_3}}g(m,n,l)\right|\leq M^{-q_1}L^{-q_3}.$$

Finally let

$$S = \sum_{a_1 M \le m \le b_1 M} \left| \sum_{a_3 L \le l \le b_3 L} g(m, n_0, l) S(m, n_0; l) \right|^2 .$$

Then $S \ll ML^{2+\varepsilon}$.

Proof. We use the notation $a_m = \sum_{a_3 L \le l \le b_3 L} g(m, n_0, l) S(m, n_0; l)$, so that we can write S in the form

$$S = \sum_{a_1 M \le m \le b_1 M} |a_m|^2 = \sum_{\substack{a_1 M \le m \le b_1 M, \\ a_2 \le n \le b_2}} \overline{a_m} b_n \sum_{a_3 L \le l \le b_3 L} g(m, n, l) S(m, n; l)$$

with

$$b_n = \begin{cases} 1, & n = n_0, \\ 0, & n \ne n_0. \end{cases}$$

By Lemma 1.4

$$S \ll L^{1+\varepsilon} M^{1/2} S^{1/2},$$

and hence

$$S \ll ML^{2+\varepsilon}.$$

□

1.2.2 Estimations for exponential sums and integrals, Sobolev's Lemma

We shall simplify the estimation of some sums and integrals by inserting smooth weight functions. In most cases, it is enough to have a general smooth function $\psi(x)$ on the real axis with a support in some interval $[A, B]$, $A < B$, such that $\psi(x) \asymp 1$ on an interval of length $\asymp B - A$ and

$$\psi^{(\nu)}(x) \ll_\nu (B - A)^{-\nu} \tag{1.2.18}$$

for sufficiently many $\nu \ge 0$.

However, in some cases we need more information about the exact behavior of the weight function and therefore we introduce the function $\psi(x; A, B; G, H)$ with real parameters A, B, G, H and $A < A + G < B - H < B$;

$$\psi(x; A, B; G, H) = \begin{cases} 0, & x \le A \text{ or } x \ge B, \\ 1, & A + G \le x \le B - H, \end{cases}$$

and

$$\frac{d^\nu}{dx^\nu} \psi(x; A, B; G, H) \ll_\nu G^{-\nu} + H^{-\nu}.$$

We first use the notation (see Beals [1], pp. 67-68.)

$$u(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

Clearly the function u is infinitely many times differentiable. Further, we use the notation

$$g(t) = u(t)u(1-t)$$

and choose a constant c so that

$$c \int_{-\infty}^{1} g(t)\, dt = 1.$$

Then for the function

$$h(x) = c \int_{-\infty}^{x} g(t)\, dt$$

we have $0 < h(x) < 1$ when $x \in (0, 1)$,

$$h(x) = \begin{cases} 0, & x \le 0, \\ 1, & x \ge 1, \end{cases}$$

and

$$\frac{d^\nu}{dx^\nu} h(x) \ll_\nu 1.$$

Finally we can choose

$$\psi(x; A, B; G, H) = h\left(\frac{x-A}{G}\right) - h\left(\frac{x-B+H}{H}\right), \tag{1.2.19}$$

this ψ clearly satisfying the desired conditions. In the general case, we can choose $G, H \asymp B - A$ and the interval $[A + G, B - H]$ to be of the length $\asymp B - A$.

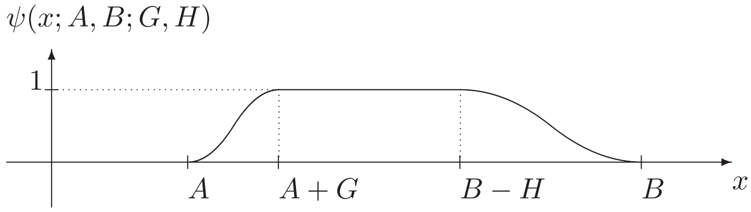


Figure 1.1: The function $\psi(x; A, B; G, H)$

Furthermore the partition of unity can now be achieved if we let

$$w_M(x) = h\left(\frac{x - M + M/4}{M/4}\right) - h\left(\frac{x - 2M + M/2}{M/2}\right)$$

for any $M \geq 1$, so that

$$U(x) = \sum_{\substack{\imath \geq 0 \\ M=2^\imath}} w_M(x) \tag{1.2.20}$$

is a continuous function such that $0 \leq U(x) \leq 1$ in the range $[3/4, 1]$, and

$$U(x) = \begin{cases} 0, & x \leq 3/4, \\ 1, & x \geq 1. \end{cases}$$

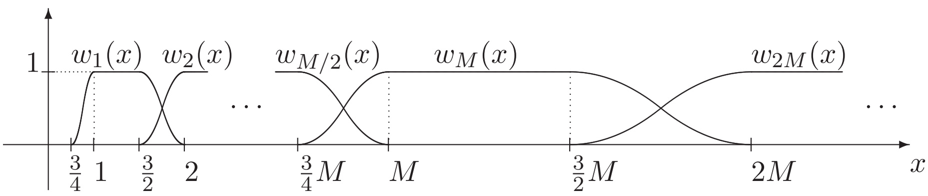


Figure 1.2: The functions $w_i(x)$

In particular, for $M_1 \geq 1$ and any complex numbers a_m we have

$$\sum_{m=1}^{M_1} a_m = \sum_{m=1}^{M_1} \sum_{\substack{0 \leq \imath < \log_2(4M_1/3), \\ M=2^\imath}} a_m w_M(m)$$

$$= \sum_{\substack{0 \leq \imath < \log_2(4M_1/3), \\ M=2^\imath}} \sum_{\frac{3}{4}M \leq m \leq \min(2M, M_1)} a_m w_M(m), \tag{1.2.21}$$

where the first sum over M now has $\ll \log M_1$ summands and the inner sum runs over a range $m \asymp M$. This device of dividing a sum into ranges of size $\asymp M$ and adding a smooth weight function will be employed frequently in the sequel.

We next introduce the basic inequality in the proof of the classical large sieve.

Lemma 1.6 (Sobolev). *Let $a \leq u \leq a + \Delta$ and let the function f be continuously differentiable on this interval. Then*

$$|f(u)|^2 \leq \Delta^{-1} \int_a^{a+\Delta} |f(x)|^2 \, dx + 2 \left(\int_a^{a+\Delta} |f(x)|^2 \, dx \right)^{1/2}$$

$$\times \left(\int_a^{a+\Delta} |f'(x)|^2 \, dx \right)^{1/2} \ll \Delta^{-1} \int_a^{a+\Delta} \left(|f(x)|^2 + \Delta^2 |f'(x)|^2 \right) dx$$

uniformly.

For a proof, see Montgomery [33], Lemma 1.1 applied to f^2.

Below we give some results concerning the estimation of exponential sums and integrals.

Lemma 1.7 (Generalization of van der Corput's Lemma). *Let $f(x)$ be a real differentiable function on the interval $[a,b]$ with $f'(x)$ monotonic, and let $|f'(x)| \leq \theta < 1$. Let $w(x)$ be a continuously differentiable function with support in $[a,b]$. Then*

$$\sum_{n=-\infty}^{\infty} e(f(n))w(n) = \int_a^b e(f(x))w(x)\,dx + \mathcal{O}\left(\left\|w'\right\|_1\right).$$

Proof. By summation by parts,

$$\sum_n e(f(n))w(n) = -\int_a^b \sum_{a\leq n\leq \xi} e(f(n))w'(\xi)\,d\xi.$$

By Titchmarsh [43], Lemma 4.8, this is equal to

$$-\int_a^b \left(\int_a^\xi e(f(x))\,dx + \mathcal{O}(1)\right) w'(\xi)\,d\xi$$

and by integration by parts we finally get

$$\int_a^b e(f(\xi))w(\xi)\,d\xi + \mathcal{O}\left(\int_a^b |w'(\xi)|\,d\xi\right).$$

□

The following important lemma is based on integration by parts.

Lemma 1.8. *Let H be a smooth function compactly supported on a finite interval $[a,b]$; and assume that there exist two quantities A_0 and A_1 such that for each integer $\nu \geq 0$ and for any t in the interval,*

$$H^{(\nu)}(t) \ll_\nu A_0 A_1^{-\nu}.$$

Also, let h be a function which is real-valued on $[a,b]$, and regular throughout the complex domain composed of all points within the distance ρ from the interval; and assume that there exists a quantity B such that

$$0 < B \ll |h'(t)|$$

for any point t in the domain. Then we have, for each fixed integer $P \geq 0$,

$$\int_a^b H(t)e^{ih(t)}\,dt \ll A_0(A_1 B)^{-P}\left(1+\frac{A_1}{\rho}\right)^P (b-a).$$

For a proof, see Jutila and Motohashi [28], Lemma 6. We will choose $\rho = a/2$ whenever we use this lemma.

Remark 1. If the regularity condition in the above lemma is replaced by the condition

$$h^{(\nu)}(t) \ll_\nu BA_1^{-\nu+1}$$

for each integer $\nu \geq 2$ and for any t in the interval $[a, b]$, then by a similar proof we get the estimate

$$\int_a^b H(t)e^{ih(t)}\,dt \ll A_0(A_1B)^{-P}(b-a).$$

In some cases it is enough to know an upper bound for the magnitude of an exponential integral with a saddle-point:

Lemma 1.9 (The second derivative test). *Let $h(t)$ be a real twice-differentiable function on a finite interval $[a, b]$ such that $h''(t) \geq m > 0$ or $h''(t) \leq -m < 0$. Further let $H(t)$ be a positive, monotonic function for $a \leq t \leq b$ such that $|H(t)| \leq G$. Then*

$$\left|\int_a^b H(t)e^{ih(t)}\,dt\right| \leq 8Gm^{-1/2}.$$

For a proof, see Ivić [12], Lemma 2.1.

When the above lemma is not sharp enough, the following lemma gives an asymptotic expansion.

Lemma 1.10 (The saddle-point method). *Let $[a, b]$ be a fixed real interval and $h : [a, b] \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{C}$ be functions that satisfy the following three conditions:*

1. *h is arbitrarily many times differentiable on the interval (a, b) and $h^{(j)}(t) \ll_j 1$ for all $t \in (a, b)$, $j \geq 1$.*

2. *$h''(t) \geq c > 0$ for some constant c and for all $t \in (a, b)$.*

3. *H is arbitrarily many times differentiable in $\mathbb{R}$, its support lies in the interval $[a, b]$ and $H^{(j)}(t) \ll_j 1$ for all $t \in \mathbb{R}$, $j \geq 0$.*

Now let $h'(t_0) = 0$ for $t_0 \in (a, b)$. Let $B > 0$ be an arbitrary constant. Then

$$I = \int_a^b H(t)e^{i\lambda h(t)}\,dt = \sqrt{2\pi}H(t_0)\lambda^{-1/2}(h''(t_0))^{-1/2}e^{i\lambda h(t_0)+\pi i/4}$$

$$+\lambda^{-1/2}(h''(t_0))^{-1/2}e^{i\lambda h(t_0)}\sum_{j=1}^{J} a_j\lambda^{-b_j} + \mathcal{O}(\lambda^{-B}) \qquad (1.2.22)$$

with $\lambda \to \infty$. Here J depends on B and for all $j = 1, 2, \ldots, J$ the exponents b_j are positive and the coefficients a_j are quotients in which the numerator is a polynomial of derivatives of the form $h^{(k)}(t_0)$ and $H^{(l)}(t_0)$ and the denominator is some power of the derivative $h''(t_0)$. If there is no saddle-point t_0 in the interval (a, b), then the terms involving t_0 on the right hand side are to be omitted.

If all of the above hypotheses hold, except that the second condition is replaced by the condition $h''(t) \leq -c < 0$ for all $t \in (a, b)$, then the right hand side of the equation (1.2.22) remains the same except that instead of $+\pi i/4$ we have $-\pi i/4$ and $(h''(t_0))^{-1/2}$ is replaced by $|h''(t_0)|^{-1/2}$.

Proof. We divide the positive real axis $\mathbb{R}_+$ into dyadic intervals and assume that $\lambda \asymp \Lambda$ for some $\Lambda \geq 1$. Then we use the notation

$$w(\tau) = \psi(\tau; -2\tau_0, 2\tau_0; \tau_0, \tau_0)$$

with $\tau_0 = \Lambda^{-1/2+\varepsilon}$ and ψ the weight function introduced in (1.2.19), and write $I = I_1 + I_2$ with

$$I_1 = \int_{-\infty}^{\infty} w(\tau) H(t_0 + \tau) e^{i\lambda h(t_0+\tau)}\, d\tau$$

and

$$I_2 = \int_{-\infty}^{\infty} (1 - w(\tau)) H(t_0 + \tau) e^{i\lambda h(t_0+\tau)}\, d\tau.$$

By integrating by parts repeatedly, we see that I_2 is $\ll \lambda^{-B}$.

As for I_1, we first assume that $w(t_0 - a) = w(b - t_0) = 0$ and write the functions $h(t_0 + \tau)$ and $H(t_0 + \tau)$ using Taylor's approximation. Then we approximate the term $\exp(\frac{i\lambda}{6} h^{(3)}(t_0)\tau^3 + \ldots)$ by a polynomial and substitute these approximations into the integral. The leading term for I_1 is

$$H(t_0)e^{i\lambda h(t_0)} \int_{-\infty}^{\infty} w(\tau) e^{\frac{i\lambda}{2} h''(t_0)\tau^2}\, d\tau = H(t_0)e^{i\lambda h(t_0)} \Bigg(\int_{-\infty}^{\infty} e^{\frac{i\lambda}{2} h''(t_0)\tau^2}\, d\tau + \int_{-\infty}^{\infty} (w(\tau) - 1) e^{\frac{i\lambda}{2} h''(t_0)\tau^2}\, d\tau \Bigg),$$

where the first term on the right hand side yields the saddle-point term

$$\sqrt{2\pi} H(t_0) \lambda^{-1/2} (h''(t_0))^{-1/2} e^{i\lambda h(t_0) + \pi i/4}$$

while the second term on the right hand side is again $\ll \lambda^{-B}$.

Next we use integration by parts suitably many times on the integral on the right hand side of the equation below

$$\int_{-\infty}^{\infty} w(\tau) e^{i\eta\tau^2}\, d\tau = \sqrt{\pi}\eta^{-1/2} e^{\pi i/4} + \int_{-\infty}^{\infty} (w(\tau) - 1) e^{i\eta\tau^2}\, d\tau.$$

Then we differentiate both sides repeatedly with respect to η, substitute $\eta = \frac{\lambda}{2}h''(t_0)$ and obtain the result

$$\int_{-\infty}^{\infty} w(\tau)\tau^{2n}e^{\frac{i\lambda}{2}h''(t_0)\tau^2}\,d\tau = \sqrt{2\pi}\left(\frac{2}{i}\right)^n e^{\pi i/4}(-1/2)(-3/2)\cdots(-1/2-n+1)$$

$$\times\lambda^{-1/2-n}(h''(t_0))^{-1/2-n} + \mathcal{O}(\lambda^{-B})$$

for all $n \geq 1$. Hence the remaining terms in I_1 are of the required type.

In case the condition $w(t_0 - a) = w(b - t_0) = 0$ does not hold, both $w(\tau)H(t_0 + \tau)$ and the right hand side of (1.2.22) are $\ll \lambda^{-B}$.

Finally, if there is no saddle-point on (a, b), we may write I_1 and I_2 as above, with a formally in place of t_0, thus obtaining the bound $I_1, I_2 \ll \lambda^{-B}$. □

Remark 2. The above lemma is also considered in Jutila's paper [26], pp. 181-183.

1.2.3 The spectral large sieve and other technical lemmas related to cusp forms and Γ-invariant functions

We introduce an important tool arising from spectral theory.

Lemma 1.11 (The spectral large sieve). *For $K \geq 1$, $1 \leq \Delta \leq K$, $M \geq 1$ and any complex numbers a_m we have*

$$\sum_{K\leq\kappa_j\leq K+\Delta} \alpha_j \left|\sum_{m\leq M} a_m t_j(m)\right|^2 \ll (K\Delta + M)(KM)^{\varepsilon} \sum_{m\leq M} |a_m|^2.$$

For a proof, see Theorem 1.1 in [25] or Theorem 3.3 in [36]. Note that the theorem in [36] gives an even sharper bound when $C^2\log^{1/2} K \leq \Delta \leq C^{-2}K\log^{-1/2} K$ and $C > 0$ is a large constant. Also, in the remaining cases $1 \leq \Delta \leq C^2\log^{1/2} K$ and $C^{-2}K\log^{-1/2} K \leq \Delta \leq K$ it clearly implies the above estimate.

For the κ_j-sum with a single m we have an exact leading term in

$$\sum_{\kappa_j\leq K} \frac{|\rho_j(m)|^2}{\cosh(\pi\kappa_j)} = \frac{K^2}{\pi^2} + \mathcal{O}(K\log K + m^{\varepsilon}K + m^{1/2+\varepsilon}), \tag{1.2.23}$$

as $\varepsilon > 0$, $K \geq 2$, $m \geq 1$. For a proof, see Kuznetsov [30], Theorem 6. As a consequence of (1.2.23) we have that for all $\varepsilon > 0$ and $m \geq 1$

$$\rho_j(m) \ll \exp\left(\frac{\pi}{2}\kappa_j\right)(\kappa_j^{1/2}(\log\kappa_j)^{1/2} + m^{\varepsilon}\kappa_j^{1/2} + m^{1/4+\varepsilon}). \tag{1.2.24}$$

By choosing $m = 1$ in (1.2.23) we have the following result: For all $K \geq 2$

$$\sum_{\kappa_j \leq K} \alpha_j \ll K^2,$$

and hence by decomposing the range $[1, K]$ into dyadic intervals of the form $\left[\frac{K}{2^l}, \frac{K}{2^{l+1}}\right]$, or using summation by parts, for all $a \in \mathbb{R}$, $\varepsilon > 0$ we have

$$\sum_{\kappa_j \leq K} \alpha_j \kappa_j^a \ll_a \begin{cases} K^{2+a+\varepsilon}, & a \geq -2, \\ 1, & a < -2. \end{cases} \tag{1.2.25}$$

Moreover, we have the following estimate for a single α_j: For all $\varepsilon > 0$

$$\alpha_j \ll \kappa_j^{\varepsilon}; \tag{1.2.26}$$

see Hoffstein and Lockhart [11], Corollary 0.3.

The next lemma is a continuous analogue of the spectral large sieve.

Lemma 1.12. *For K real, $\Delta > 0$, $M \geq 1$ and any complex numbers a_m we have*

$$\int_K^{K+\Delta} \left| \sum_{m \sim M} a_m \sigma_{2ir}(m) m^{-ir} \right|^2 dr \ll (\Delta M^{1/2} + M) M^{\varepsilon} \sum_{m \sim M} |a_m|^2$$
$$\ll (\Delta^2 + M) M^{\varepsilon} \sum_{m \sim M} |a_m|^2$$

uniformly in K. The function $\sigma_\alpha(n)$ was defined in (1.1.7).

Proof. We start from the inequality

$$\int_K^{K+\Delta} \left| \sum_{m \sim M} a_m \sigma_{2ir}(m) m^{-ir} \right|^2 dr \leq \int_{-\infty}^{\infty} \left| \sum_{m \sim M} \sum_{d | m} a_m m^{-ir} d^{2ir} \right|^2 u(r)\, dr$$

with u a suitable smooth weight function compactly supported on an interval of length $\asymp \Delta$ and $u^{(\nu)} \ll_\nu \Delta^{-\nu}$ for each $\nu \geq 0$. By writing $m = dn$ we see that

$$\left| \sum_{m \sim M} \sum_{d | m} a_m m^{-ir} d^{2ir} \right|^2 = \left| \sum_{1 \leq d < 2M} \sum_{\frac{M}{d} \leq n < \frac{2M}{d}} a_{dn} \left(\frac{d}{n}\right)^{ir} \right|^2$$

$$\leq \left(\left| \sum_{M \leq d < 2M} \sum_{n} a_{dn} \left(\frac{d}{n}\right)^{ir} \right| + \left| \sum_{\frac{M}{2} \leq d < M} \sum_{n} a_{dn} \left(\frac{d}{n}\right)^{ir} \right| + \ldots \right.$$

$$+\left|\sum_{\frac{M}{2^{j+1}}\le d<\frac{M}{2^j}}\sum_n a_{dn}\left(\frac{d}{n}\right)^{ir}\right|\Bigg)^2$$

with $\frac{M}{2^{j+1}} \le 1 < \frac{M}{2^j}$, that is, $\log_2 M - 1 \le j < \log_2 M$. Therefore, for an appropriate D in the range $\frac{M}{2^{j+1}} \le D \le M$, the integral is

$$\ll \log^2 M \int_{-\infty}^{\infty}\left|\sum_{D\le d<2D}\sum_{\frac{M}{d}\le n<\frac{2M}{d}} a_{dn}\left(\frac{d}{n}\right)^{ir}\right|^2 u(r)\,dr = \log^2 M \int_{-\infty}^{\infty}$$

$$\times \sum_{\substack{M\le d_1n_1<2M\\ D\le d_1<2D}}\sum_{\substack{M\le d_2n_2<2M\\ D\le d_2<2D}} a_{d_1n_1}\overline{a_{d_2n_2}}\left(\frac{d_1}{n_1}\right)^{ir}\left(\frac{d_2}{n_2}\right)^{-ir} u(r)\,dr$$

$$\ll \log^2 M \sum_{d_1,n_1}\sum_{d_2,n_2}\left(|a_{d_1n_1}|^2+|a_{d_2n_2}|^2\right)\left|\int_{-\infty}^{\infty}\left(\frac{d_1n_2}{n_1d_2}\right)^{ir}u(r)\,dr\right|.$$

We next study the last integral, using the abbreviation

$$A = \frac{d_1n_2}{n_1d_2}.$$

Now trivially

$$\int_{-\infty}^{\infty} A^{ir}u(r)\,dr \ll \|u\|_1 \ll \Delta$$

and by integration by parts

$$\int_{-\infty}^{\infty} A^{ir}u(r)\,dr = \frac{(-1)^\nu}{(i\log A)^\nu}\int_{-\infty}^{\infty} A^{ir}u^{(\nu)}(r)\,dr \ll (\log A)^{-\nu}\|u^{(\nu)}\|_1$$

$$\ll_\nu \Delta M^{-\nu\varepsilon},$$

if $|A-1| \gg \frac{M^\varepsilon}{\Delta}$. Then the sum involving terms for which this condition holds is under the desired bound. Therefore, by symmetry, it is sufficient to consider the sum

$$\log^2 M \sum_{\substack{d_1,n_1\\ d_2,n_2}} |a_{d_1n_1}|^2\Delta,$$

where the condition of summation is

$$\frac{d_1n_2}{n_1d_2} = 1 + \mathcal{O}\left(\frac{M^\varepsilon}{\Delta}\right),$$

that is

$$\frac{d_1}{n_1} = \frac{d_2}{n_2} + \mathcal{O}\left(\frac{DM^\varepsilon}{N\Delta}\right)$$

with $N = \frac{M}{D}$. Further, by symmetry, we may assume that $N \ll M^{1/2}$. Now if we fix d_1/n_1, then there are $\ll N$ possibilities for n_2, and $\ll 1 + \frac{DM^\varepsilon}{\Delta}$ possibilities for d_2, for each n_2. Finally we have the upper bound

$$\ll \log^2 M \sum_{M \le m < 2M} d(m)|a_m|^2 \Delta N \left(1 + \frac{DM^\varepsilon}{\Delta}\right) \ll (\Delta N + M)M^\varepsilon \sum_{m \sim M} |a_m|^2$$

$$\ll (\Delta M^{1/2} + M)M^\varepsilon \sum_{m \sim M} |a_m|^2 \ll (\Delta^2 + M)M^\varepsilon \sum_{m \sim M} |a_m|^2.$$

□

Corollary 1.13. *For K real, $M \ge 1$ and any complex numbers a_m we have*

$$\int_{-K}^{K} \left| \sum_{m \sim M} a_m \sigma_{2ir}(m) m^{-ir} \right|^2 dr \ll (K^2 + M)M^\varepsilon \sum_{m \sim M} |a_m|^2.$$

Remark 3. In his paper [15], Theorem 3, Iwaniec proved a slightly sharper result with $(K^2 + M^{1+\varepsilon})$ instead of $(K^2 + M)M^\varepsilon$.

Furthermore we mention the following well-known estimates, which we will frequently use, without specification, in estimating the order of the Fourier coefficients of holomorphic cusp forms.

Lemma 1.14. *For all Fourier coefficients of a holomorphic cusp form we have*

$$|a(n)|^2 \ll n^{k-1+\varepsilon}, \quad \sum_{n \le N} |a(n)|^2 = AN^k + \mathcal{O}(N^{k-2/5})$$

with A a positive constant.

For a proof of the first estimate, see (1.1.6) and for the second one Rankin [38], Theorem 1, or Selberg [40], p. 3. Analogously for Maass forms with the spectral parameter κ we have the following estimate due to Iwaniec [16], Lemma 1.

Lemma 1.15. *For all $\varepsilon > 0$ and $N \ge 1$ we have*

$$\sum_{n \le N} t^2(n) \ll \kappa^\varepsilon N. \tag{1.2.27}$$

Next we have a formula for the spectral decomposition of the shifted convolution sum over the Fourier coefficients of a holomorphic cusp form, due to Motohashi, [35]. For a proof see for example Jutila's and Motohashi's paper [29], Lemma 4. Note that the expansion here is independent of the fact that holomorphic cusp forms can be represented as linear combinations of the Poincaré series.

Lemma 1.16. *Let f be a positive integer and W a smooth function of compact support on $(0,\infty)$. Then*

$$\sum_{l=1}^{\infty} a(l)\overline{a(l+f)}W\left(\frac{l}{f}\right) = i\pi(4\pi)^{k-1}f^{-1/2+k}\Bigg(\sum_{j=1}^{\infty}\overline{\rho_j}t_j(f)c_j\Phi_k(\kappa_j;W)$$
$$+\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(f)c(r)}{(\pi f)^{ir}\Gamma(\frac{1}{2}-ir)\zeta(1-2ir)}\Phi_k(r;W)\,dr\Bigg) \tag{1.2.28}$$

with c_j and $c(r)$ the inner products attached to the holomorphic cusp form in question and

$$\Phi_k(r;W) = \frac{2i}{\sinh(\pi r)}\int_0^{\infty}(y(y+1))^{k-1}Im\Bigg\{\frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(k-\frac{1}{2}-ir)\Gamma(1+2ir)}$$
$$\times y^{1/2-k-ir}F\left(k-\frac{1}{2}+ir,\frac{1}{2}+ir;1+2ir;-\frac{1}{y}\right)\Bigg\}W(y)\,dy$$
$$=\frac{1}{\sinh(\pi r)}\frac{\Gamma(\frac{1}{2}+ir)}{\Gamma(k-\frac{1}{2}-ir)\Gamma(1+2ir)}\int_0^{\infty}y^{1/2-k-ir}W(y)(y(y+1))^{k-1}$$
$$\times F\left(k-\frac{1}{2}+ir,\frac{1}{2}+ir;1+2ir;-\frac{1}{y}\right)\,dy+(r\mapsto -r).$$

Here $(r\mapsto -r)$ stands for an expression similar to the preceding one, but with r replaced by $-r$. We notice that in the article, the lemma has been formulated for the eigenvalues $t(l)$ instead of the Fourier coefficients $a(l)$, under the assumption that $F(z)$ is a Hecke eigenform. This is, however, not essential. Their lemma can be formulated for Fourier coefficients, and it implies (1.2.28) if the function W is suitably modified. Note also that Jutila and Motohashi denote the weight by $2k$ instead of k.

Remark 4. We note that $a(l)\overline{a(l+f)}$ in the above lemma can be replaced by $\overline{a(l)}a(l+f)$.

Whenever we apply the lemma above, the function W always forces the variable y to be $\gg K^{\varepsilon}$ with $K\to\infty$. Therefore we transform the hypergeometric function using equation (1.2.13), and obtain the new hypergeometric function

$$F\left(-\frac{1}{2}+k+ir,-\frac{1}{2}+k;1+ir;\left(\frac{1-\sqrt{1+y^{-1}}}{1+\sqrt{1+y^{-1}}}\right)^2\right),$$

which can be expressed by its rapidly converging series. If we write $\xi = \left(\frac{1-\sqrt{1+y^{-1}}}{1+\sqrt{1+y^{-1}}}\right)^2 = \left(\sqrt{y}+\sqrt{1+y}\right)^{-4}$, then

$$F\left(-\frac{1}{2}+k+ir,-\frac{1}{2}+k;1+ir;\xi\right) = 1+\sum_{l=1}^{\infty}\frac{\left(-\frac{1}{2}+k+ir\right)_l\left(-\frac{1}{2}+k\right)_l}{(l+ir)_l\,l!}\xi^l,$$

where the symbol $(\lambda)_l$ is as in (1.2.8). Clearly the lth term is of order ξ^l, so

$$F\left(-\frac{1}{2}+k+ir,-\frac{1}{2}+k;1+ir;\xi\right)\sim 1.$$

As we mentioned in the introduction, in [21], Lemma 4, Jutila proved the estimate

$$\sum_{\kappa_j\le K}|c_j|^2\exp(\pi\kappa_j)\ll_{k,\varepsilon}K^{2k+\varepsilon}, \tag{1.2.29}$$

which corresponds to our estimate in Theorem 1, taken over a shorter interval. This estimate will be needed during the course of our proof in Chapter 3. Using a dyadic partition of the sum or summation by parts, we get the following result:

$$\sum_{\kappa_j\le K}\kappa_j^a|c_j|^2\exp(\pi\kappa_j)\ll\begin{cases}K^{2k+a+\varepsilon}, & a\ge -2k,\\ 1, & a<-2k.\end{cases} \tag{1.2.30}$$

Here the implied constant depends always on a and k. Also when $a\ge -2k$, it depends on ε, and when $a<-2k$, on the difference $-2k-a$.

In [21], Lemma 3, Jutila has also shown that for all $K\ge 1$

$$\int_{-K}^{K}|c(u)|^2\exp(\pi|u|)du\ll K^{2k+\varepsilon} \tag{1.2.31}$$

with $c(u)$ as defined in (1.1.11), the implied constant depending on k and ε. The result is not the best known, but sufficies for our purposes. Again we easily obtain the estimate

$$\int_{-K}^{K}|c(u)|^2\exp(\pi|u|)(|u|+1)^a du\ll\begin{cases}K^{2k+a+\varepsilon}, & a\ge -2k,\\ 1, & a<-2k.\end{cases} \tag{1.2.32}$$

by an appropriate decomposition of the range of integration or by integration by parts, and the dependence of the implied constant is similar to that in (1.2.30).

Lastly, we let $f(z)$ be a continuous, Γ-invariant function in the upper half-plane, integrable over the fundamental domain $\mathcal{F}$ of Γ with respect to the hyperbolic measure $d\mu(z)$. We want to express the integral

$$\int_{\mathcal{F}}f(z)\,d\mu(z) \tag{1.2.33}$$

in terms of the integral of a suitable function over the strip $\Pi=\{z=x+iy\,|\,0\le x\le 1, 0\le y\le\infty\}$.

Lemma 1.17 (A variant of the Rankin-Selberg method). *Let $f(z) : \mathbb{H} \to \mathbb{C}$ be a continuous Γ-invariant function, which is integrable over the fundamental domain $\mathcal{F}$. Let $f(z)$ further decay exponentially, as y tends to infinity. Then*

$$\int_{\mathcal{F}} f(z)\, d\mu(z) = \frac{1}{\pi i} \int_{(a)} \xi(2s)(sg(s) - (1-s)g(1-s)) \times \left(\int_0^\infty \int_0^1 f(z) y^{s-2}\, dx\, dy \right) ds,$$

with real $a > 1$, $\xi(s)$ as in (1.1.13) and

$$g(s) = \exp\left(1 - \cos\left(\frac{s-1}{B}\right)\right) \tag{1.2.34}$$

with $B > 0$ a sufficiently large constant.

For a proof, see Jutila [21], Section 2.

Chapter 2

Proof of Theorem 1 using Poincaré series

We start by presenting a proof of Theorem 1 based on papers by Good [8] and Jutila and Motohashi [28], emphasizing those points where our argument deviates from these papers. As we remarked in the Introduction, this proof cannot be extended to the analogous case of Theorem 1 involving $\widetilde{c_j}$, as it uses the fact that the holomorphic cusp forms can be represented as finite linear combinations of the Poincaré series. Therefore, in Chapter 3, we present an alternative approach to the proof of Theorem 1 which does not use Poincaré series. However, some arguments in Section 2.2 will be relevant again in this context.

2.1 Reformulation of the problem in terms of Poincaré series

In his work [8], Lemma 2, Good first represents c_j as a finite linear combination of terms of the form

$$\gamma_{n,n'}(j) = \int_{\mathcal{F}} y^{k-2} P_n(z,k) \overline{P_{n'}(z,k)} u_j(z)\, dx\, dy,$$

with $n, n' \in \mathbb{Z}_+$ and

$$P_n(z,k) = \sum_{\gamma \in \mathcal{A}} j(\gamma, z)^{-k} e(n\gamma(z)), \quad z \in \mathbb{H},\ n, k \in \mathbb{N}$$

the Poincaré series, with $j(\gamma, z)$ as in (1.1.1) and $\mathcal{A}$ as in (1.1.12). He then ends up with equation (4.9) in his paper which states that

$$\gamma_{n,n'}(j) = \left(\frac{\pi}{2}\right)^{1/2} \Gamma(k - s_j)\Gamma(k - \overline{s_j}) \sum_{\substack{m > -n, \\ m \neq 0}} \rho_j(m) \overline{a(m+n)} (2\pi|m|)^{-k+1/2}$$

$$\times \left(\frac{4n^2+4nm}{m^2} \right)^{\frac{1-k}{2}} \mathcal{P}^{1-k}_{s_j-1} \left(\frac{2n+m}{|m|} \right)$$

with $s_j = 1/2 + i\kappa_j$, $a(l)$ the lth Fourier coefficient of the Poincaré series $P_{n'}(z,k)$ and $\mathcal{P}^{\mu}_{\nu}$ the Legendre function of the first kind (see [5], p. 370).

We estimate the Γ-functions above by Stirling's formula and notice that for the proof of our Theorem 1 it suffices to prove that for all fixed $n \in \mathbb{Z}_+$

$$\sum_{K \le \kappa_j \le K+K^{1/3}} \alpha_j \Bigg| \sum_{\substack{m>-n \\ m\neq 0}} t_j(m) \overline{a(m+n)} |m|^{-1/2} \left(\sqrt{n(n+m)} \right)^{1-k}$$

$$\times \mathcal{P}^{1-k}_{s_j-1} \left(\frac{2n+m}{|m|} \right) \Bigg|^2 \ll K^{-2k+3+1/3+\varepsilon}. \tag{2.1.1}$$

Good carries on to represent the Legendre function in terms of the hypergeometric function and derives asymptotic expansions (4.18) and (4.19) which state that for $J = 1, 2, \ldots$

$$\mathcal{P}^{1-k}_{s-1}(x) = \left(\frac{x-1}{x+1} \right)^{\frac{k-1}{2}} \left\{ \sum_{j=0}^{J-1} \frac{(s)_j (1-s)_j}{\Gamma(k+j) j!} \left(\frac{1-x}{2} \right)^j \right.$$

$$\left. + \mathcal{O}\left((1+|s|)^{2J} (x-1)^J \right) \right\} \tag{2.1.2}$$

when $1 \le x \le 1 + (1+|s|)^{-2}$, and

$$\mathcal{P}^{1-k}_{s-1}(x) = \frac{x^{-s}(1+(1-x^{-2})^{1/2})^{1/2-s}\Gamma(1/2-s)}{(2\pi)^{1/2}(1-x^{-2})^{1/4}\Gamma(k-s)}$$

$$\times \left\{ \sum_{j=0}^{J} \frac{(3/2-k)_j (k-1/2)_j (1+(1-x^{-2})^{1/2})^{-j}}{(s+1/2)_j j! (-2x^2(1-x^{-2})^{1/2})^j} \right.$$

$$\left. + \mathcal{O}\left[\left((1+|s|) x^2 (1-x^{-2})^{1/2} \right)^{-J-1/2} \right] \right\}$$

$$+ \frac{x^{s-1}(1+(1-x^{-2})^{1/2})^{s-1/2}\Gamma(s-1/2)}{(2\pi)^{1/2}(1-x^{-2})^{1/4}\Gamma(s+k-1)}$$

$$\times \left\{ \sum_{j=0}^{J} \frac{(3/2-k)_j (k-1/2)_j (1+(1-x^{-2})^{1/2})^{-j}}{(3/2-s)_j j! (-2x^2(1-x^{-2})^{1/2})^j} \right.$$

$$\left. + \mathcal{O}\left[\left((1+|s|) x^2 (1-x^{-2})^{1/2} \right)^{-J-1/2} \right] \right\} \tag{2.1.3}$$

when $x \geq 1 + (1 + |s|)^{-2}$. Therefore by (2.1.2) we have

$$\mathcal{P}_{s_j-1}^{1-k}\left(\frac{2n+m}{|m|}\right) \sim \frac{1}{(k-1)!}\left(\frac{n}{n+m}\right)^{\frac{k-1}{2}}$$

if $m \gg K^{2+\varepsilon}$ and by (2.1.3)

$$\mathcal{P}_{s_j-1}^{1-k}\left(\frac{2n+m}{|m|}\right) \sim \frac{\sqrt{|m|}}{2\sqrt{\pi}(n(n+m))^{1/4}}\left\{\frac{\Gamma(-i\kappa_j)}{\Gamma(k-1/2-i\kappa_j)}\right.$$

$$\times \exp\left(-i\kappa_j \log\left(\frac{m+2n+2\sqrt{n(n+m)}}{|m|}\right)\right) + \frac{\Gamma(i\kappa_j)}{\Gamma(k-1/2+i\kappa_j)}$$

$$\left.\times \exp\left(i\kappa_j \log\left(\frac{m+2n+2\sqrt{n(n+m)}}{|m|}\right)\right)\right\}$$

if $-n+1 \leq m \ll K^{2-\varepsilon}$, $m \neq 0$.

If we are in the "transition area" $K^{2-\varepsilon} \ll m \ll K^{2+\varepsilon}$, then these asymptotic expansions do not apply, and we use the following formula

$$\mathcal{P}_{s-1}^{1-k}(x) = \frac{\pi^{-1/2}2^{1-k}(x^2-1)^{\frac{k-1}{2}}}{\Gamma(k-1/2)}\int_0^{\pi}\left(x+(x^2-1)^{1/2}\cos t\right)^{s-k}(\sin t)^{2k-2}\,dt,$$

which holds for all $k > 1/2$ and $x > 1$. For a proof, see [6], Eq. 6 on page 155. Therefore we have

$$\mathcal{P}_{s_j-1}^{1-k}\left(\frac{2n+m}{|m|}\right) = \pi^{-1/2}(\Gamma(k-1/2))^{-1}\left(\frac{n(n+m)}{m^2}\right)^{\frac{k-1}{2}}$$

$$\times \int_{-1}^{1}(1-u^2)^{k-3/2}\left(\frac{m+2n+2u\sqrt{n(n+m)}}{m}\right)^{1/2-k+i\kappa_j} du. \qquad (2.1.4)$$

Now by Taylor's approximation we are able to separate the variables κ_j and m as

$$\left(\frac{m+2n+2u\sqrt{n(n+m)}}{m}\right)^{i\kappa_j} \sim \left(\frac{m+2n+2u\sqrt{n(n+m)}}{m}\right)^{iK}.$$

Finally

$$\mathcal{P}_{s_j-1}^{1-k}\left(\frac{2n+m}{|m|}\right) \sim \left(\frac{n}{n+m}\right)^{\frac{k-1}{2}}\chi(m)$$

with χ a smooth function satisfying the condition $\chi^{(\nu)}(m) \ll_\nu (mK^{-\varepsilon})^{-\nu}$ for all $\nu \geq 0$.

Remark 5. It might be of interest to try to apply formula (2.1.4) to the intervals $m \ll K^{2-\varepsilon}$ and $m \gg K^{2+\varepsilon}$, too.

2.2 Reduction through Voronoĭ's sum formula and spectral methods

While Good now proceeds with estimations by absolute values, we transform our problem to a form similar to that treated in [28] and [29]. We first insert a smooth weight function $\phi(m)$ into the m-sum in (2.1.1) for, let us say, the range $1 \le m \le K^4$ using the method introduced in (1.2.21), and then divide the resulting double sum into five separate cases $-n+1 \le m \le -1$, $1 \le m \ll K^{4/3}$, $K^{4/3} \ll m \ll K^{2-\varepsilon}$, $K^{2-\varepsilon} \ll m \ll K^3$ and $m \gg K^3$. The first and the last case can be settled easily by estimating the m-sum trivially. For the second case we use Sobolev's Lemma and the spectral large sieve, and then estimate the m-sum trivially.

We next concentrate on proving the desired estimate for the third case $K^{4/3} \ll m \ll K^{2-\varepsilon}$, which turns out to be the most difficult one, and then comment briefly on the fourth case at the end of the chapter.

By Sobolev's Lemma 1.6 it suffices to prove that for all fixed $n \in \mathbb{Z}_+$ and $K^{4/3} \ll M \ll K^{2-\varepsilon}$

$$G^{-1}\int_K^{K+G} \sum_{K\le\kappa_j\le K+G} \alpha_j \Bigg(\Bigg| \sum_{m=1}^{\infty} \phi(m) t_j(m) \overline{a(m+n)} \left(\sqrt{n(n+m)}\right)^{1/2-k}$$

$$\times \exp(\pm ix \log A(m,n)) \Bigg|^2 + G^2 \Bigg| \sum_{m=1}^{\infty} \phi(m) t_j(m) \overline{a(m+n)} \left(\sqrt{n(n+m)}\right)^{1/2-k}$$

$$\times \exp(\pm ix \log A(m,n)) \log A(m,n) \Bigg|^2 \Bigg)\, dx \ll K^{2+1/3+\varepsilon}$$

with $G = K^{1/3+\varepsilon}$,

$$A(m,n) = \left(1 + \frac{2n}{m} + 2\frac{\sqrt{n(n+m)}}{m}\right)^{-1}$$

and ϕ a suitable smooth weight function compactly supported on the interval $[3M/4, 2M]$, and $\phi^{(\nu)} \ll_\nu M^{-\nu}$ for each $\nu \ge 0$. Note that we introduce the notation G in order to make the analogy between our case and that in the papers by Jutila and Motohashi even more visible.

In what follows, we shall concentrate on the case $+x$ and prove that for any $K \le x \le K+G$

$$\mathcal{S} = \sum_{K\le\kappa_j\le K+G} \alpha_j \Bigg(\Bigg| \sum_{m=1}^{\infty} \phi(m) t_j(m) \overline{a(m+n)} \left(\sqrt{n(n+m)}\right)^{1/2-k}$$

$$\times \exp\left(-ix \log A(m,n)^{-1}\right) \Bigg|^2 + G^2 \Bigg| \sum_{m=1}^{\infty} \phi(m) t_j(m) \overline{a(m+n)}$$

$$\times\left(\sqrt{n(n+m)}\right)^{1/2-k}\exp\left(-ix\log A(m,n)^{-1}\right)\log A(m,n)\Bigg|^2\Bigg)$$

is $\ll K^{2+1/3+\varepsilon}$ noting that the case for $-x$ follows easily by complex conjugation and some minor modifications.

We next simplify the exponential factor

$$\exp\left(-ix\log A(m,n)^{-1}\right)$$

by noticing that

$$\log\left(1+\frac{2n}{m}+2\frac{\sqrt{n(n+m)}}{m}\right)=2\left(\frac{n}{m}\right)^{1/2}-\frac{1}{3}\left(\frac{n}{m}\right)^{3/2}+\ldots+\mathcal{O}\left(K^{-P}\right)$$

with P an arbitrarily large positive constant. Hence

$$\exp\left(-ix\log A(m,n)^{-1}\right)\sim\exp\left(-2ix\sqrt{\frac{n}{m}}\right)$$

giving

$$\mathcal{S}\sim\sum_{K\le\kappa_j\le K+G}\alpha_j\left|\sum_{m=1}^{\infty}\phi(m)t_j(m)\overline{a(m+n)}\left(\sqrt{n(n+m)}\right)^{1/2-k}\right.$$
$$\left.\times\exp\left(-2ix\sqrt{\frac{n}{m}}\right)\right|^2.$$

Note that the "extra coefficients" in the additional terms left out when using the notation $\sim$ are of the form $x^\eta m^{-\rho}$ with $\eta\in\mathbb{Z}_+$ and $\rho\in\frac{1}{2}\mathbb{Z}_{\ge2}$, $\eta\le\rho$, so the variables x and m could be separated and the variable m embedded into the weight function $\phi(m)$. This embedding into various weight functions could be carried out throughout the proof in most of the cases when the notation $\sim$ emerges. See also [28], p. 76.

We now have a sum similar to Jutila's and Motohashi's sum (4.1) in [28], only there is an exponential term instead of the term m^{-it} with some fixed t. We proceed following their argument from the beginning of their Chapter 4, p. 82, and add an even function

$$h(r)=K^{-2}\left(r^2+\frac{1}{4}\right)\left[\exp\left(-\left(\frac{r-K}{G}\right)^2\right)+\exp\left(-\left(\frac{r+K}{G}\right)^2\right)\right]$$

to the κ_j-sum and multiply out the absolute values obtaining

$$\mathcal{S}\ll\sum_{j=1}^{\infty}\alpha_j\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\phi(m_1)\phi(m_2)t_j(m_1)t_j(m_2)\overline{a(m_1+n)}a(m_2+n)$$

$$\times (n+m_1)^{1/4-k/2}(n+m_2)^{1/4-k/2}\exp\left(-2ix\sqrt{\frac{n}{m_1}}\right)\exp\left(2ix\sqrt{\frac{n}{m_2}}\right)h(\kappa_j).$$

We then apply a version of the Kloosterman-spectral sum formula, Lemma 1 in [28], and arrive at the sum

$$\mathcal{S}_1 = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\phi(m_1)\phi(m_2)\overline{a(m_1+n)}a(m_2+n)$$
$$\times (n+m_1)^{1/4-k/2}(n+m_2)^{1/4-k/2}\exp\left(-2ix\sqrt{\frac{n}{m_1}}\right)\exp\left(2ix\sqrt{\frac{n}{m_2}}\right)$$
$$\times \sum_{l=1}^{\infty}\frac{\phi_1(l)}{l}S(m_1,m_2;l)\hat{h}\left(\frac{4\pi\sqrt{m_1m_2}}{l}\right), \tag{2.2.1}$$

which corresponds to formula (4.2) in [28]. Here ϕ_1 is a suitable smooth weight function compactly supported on the interval $[3L/4, 2L]$ where

$$L \ll \frac{M\log K}{KG},$$

and $\phi_1^{(\nu)} \ll_\nu L^{-\nu}$ for each $\nu \geq 0$. Again we have used the device of (1.2.21) to insert this weight into the truncated l-sum. The treatment of the first and the second term and the truncation of the l-sum in the Kloosterman-spectral sum formula can be carried out as in [28], pp. 74-76. Now it is enough to prove that

$$\mathcal{S}_1 \ll K^{2+1/3+\varepsilon}.$$

Next we find the Mellin transform $\mathcal{M}(\,\cdot\,;s)$ for the function $e^{-iy}w(y)$ with w a real-valued smooth weight function compactly supported on the interval $[T/2, 6T]$ with $T \geq 1$, $w(y) = 1$ on the interval $[T, 5T]$ and $w^{(\nu)} \ll_\nu T^{-\nu}$ for each $\nu \geq 0$. By the saddle-point method 1.10,

$$\mathcal{M}(e^{-iy}w(y);s) \sim \sqrt{2\pi}w(t)t^{\sigma-1/2+it}e^{-it-\frac{\pi i}{4}}$$

when $t \asymp T$, with $s = \sigma + it$, σ in a bounded interval, and by Lemma 1.8

$$\mathcal{M}(e^{-iy}w(y);s) \ll T^{\sigma-P}$$

otherwise, for an arbitrarily large constant P. Substituting these into the Mellin inversion formula for $e^{-iy}w(y)$ and moving the path of integration to $c = -1/2$ we get

$$e^{-iy}w(y) = \frac{1}{2\pi i}\int_{(c)}\mathcal{M}(e^{-iy}w(y);s)y^{-s}\,ds$$
$$\sim \frac{1}{\sqrt{2\pi}}e^{-\frac{\pi i}{4}}\int_{T/2}^{6T}w(t)t^{-1+it}e^{-it}y^{1/2-it}\,dt. \tag{2.2.2}$$

We use the notations $2x\sqrt{\frac{n}{m}} = y$ and

$$T = \frac{K\sqrt{n}}{\sqrt{M}},$$

and transform our exponential terms in (2.2.1) by the formula (2.2.2). We then use the same reasoning as in [28] and finally get an analogous formula to Jutila's and Motohashi's sum (4.9):

$$\mathcal{S}_2 = \sum_{m_1=1}^{\infty} \phi(m_1)\overline{a(m_1+n)}(n+m_1)^{1/4-k/2} m_1^{-1/2+it_1} \sum_{l=1}^{\infty} \frac{\phi_1(l)}{\sqrt{l}}$$

$$\times \sum_{\substack{q=1\\(q,l)=1}}^{l} e\left(\frac{qm_1}{l}\right) \sum_{m_2=1}^{\infty} \phi(m_2) a(m_2+n)(n+m_2)^{1/4-k/2} m_2^{-1/2-it_2}$$

$$\times e\left(\frac{\overline{q}(m_2+n)}{l}\right) e\left(\frac{-\overline{q}n}{l}\right) \exp\left(\delta_1 i\omega\left(r, \frac{4\pi\sqrt{m_1m_2}}{l}\right)\right)$$

and it suffices to prove that

$$\int_{T/4}^{3T}\int_{T/4}^{3T} w(2t_1)w(2t_2)(2t_1)^{-1+2it_1}(2t_2)^{-1-2it_2} e^{-2it_1+2it_2}(2x)^{-2it_1+2it_2}$$

$$\times n^{-it_1+it_2}\mathcal{S}_2 \, dt_1\, dt_2 \ll K^{\varepsilon}.$$

Here $\delta_1 = \pm 1$, $q\overline{q} \equiv 1 \pmod{l}$, $|r-K| \leq G\log K$ and

$$\omega(r,x) = x\left(1 - 2\left(\frac{r}{x}\right)^2\right).$$

Note the exception that we have t_1 and t_2 instead of one t in [28]. Notice also that we substitute t by $2t$ in (2.2.2) in order to make the analogy even more visible.

Remark 6. It is necessary to apply the Mellin transform to the exponential term

$$\exp\left(2ix\sqrt{\frac{n}{m_2}}\right)$$

at this point, in preparation for applying the Voronoĭ sum formula to the m_2-sum. The term

$$\exp\left(-2ix\sqrt{\frac{n}{m_1}}\right)$$

could be left as it stands and transformed by the Mellin inversion formula later, just before spectrally decomposing the shifted convolution sum in (2.2.4). However, we transform both at this point for the sake of symmetry.

Next we apply the sum formula of Voronoĭ, which can be found, for example, in Jutila's monograph [19], Theorem 1.7, to the inner-most sum over m_2. Note that we may write the function

$$J_{k-1}\left(\frac{4\pi\sqrt{m_2(y+n)}}{l}\right),$$

which appears in this formula, asymptotically in terms of functions

$$\frac{1}{\sqrt{2}\pi}l^{1/2}(m_2(y+n))^{-1/4}\exp\left(\delta_2 i\frac{4\pi\sqrt{m_2(y+n)}}{l}\right)\exp\left(-\delta_2 i\pi(k-1)/2\right),$$

$\delta_2=\pm 1$ (see Lebedev [31], Eq. (5.11.6)). Here y is the variable of integration arising from the sum formula. Furthermore we proceed as in the Jutila-Motohashi paper, arriving at the sum

$$\mathcal{S}_3=\sum_{f\asymp f_0}\sum_{l=1}^{\infty}\frac{\phi_1(l)S(f,n;l)}{l}\sum_{m=1}^{\infty}\phi(m)\overline{a(m+n)}a(m+f)$$

$$\times(n+m)^{1/4-k/2}m^{-1/2+it_1}(m+f)^{1/4-k/2}\int_0^{\infty}\phi(y)(y+n)^{-1/2}y^{-1/2-it_2}$$

$$\times\exp\left(\delta_1 i\omega\left(r,\frac{4\pi\sqrt{my}}{l}\right)-\delta_1 i\frac{4\pi\sqrt{(m+f)(y+n)}}{l}\right)dy \qquad (2.2.3)$$

with

$$f_0\asymp L^2T^2.$$

This corresponds to sum $\mathcal{S}_3^-$ in [28]. Now it suffices to prove that

$$\int_{T/4}^{3T}\int_{T/4}^{3T}w(2t_1)w(2t_2)(2t_1)^{-1+2it_1}(2t_2)^{-1-2it_2}e^{-2it_1+2it_2}(2x)^{-2it_1+2it_2}$$

$$\times n^{-it_1+it_2}\mathcal{S}_3\,dt_1\,dt_2\ll K^{\varepsilon}.$$

Notice that since now $t_2\asymp T$, we get the order of magnitude of f by a saddle-point analysis of the y-integral, and we find that the sum corresponding to $\mathcal{S}_3^+$ in [28] is negligibly small. Notice also that, in our case, the Kloosterman sum $S(f,n;l)$ appears instead of the Ramanujan sum $c_l(f)=S(f,0;l)$.

Now in order to make the analogy to [28] more visible we substitute $f=\tilde{f}+n$ and $m=\tilde{m}-n$ into (2.2.3) to obtain

$$\mathcal{S}_3\sim\sum_{\tilde{f}\asymp f_0}\sum_{l=1}^{\infty}\frac{\phi_1(l)S(\tilde{f}+n,n;l)}{l}\sum_{\tilde{m}=1}^{\infty}\tilde{\phi}(\tilde{m})\overline{a(\tilde{m})}a(\tilde{m}+\tilde{f})$$

$$\times \tilde{m}^{-1/4-k/2+it_1}(\tilde{m}+\tilde{f})^{1/4-k/2}\int_0^\infty \phi(y)y^{-1-it_2}$$

$$\times \exp\left(\delta_1 i\left(\frac{4\pi\sqrt{\tilde{m}y}}{l}-\frac{r^2 l}{2\pi\sqrt{\tilde{m}y}}-\frac{4\pi\sqrt{(\tilde{m}+\tilde{f})y}}{l}\right)\right)\,dy$$

with $\tilde{\phi}(\tilde{m}) = \phi(\tilde{m}-n)$ a smooth weight function similar to ϕ. For the sake of simplicity, we replace $\tilde{m}$ and $\tilde{f}$ by m and f again and continue to follow the steps in [28] arriving at

$$\mathcal{S}_4 = (4\pi)^{2it_2}\sum_{f=1}^{\infty}\frac{\phi_2(f)}{f^{k-i(t_1+t_2)}}\sum_{l=1}^{\infty}\frac{\phi_1(l)S(f+n,n;l)}{l^{1+2it_2}}$$

$$\times\sum_{m=1}^{\infty}\overline{a(m)}a(m+f)X\left(\frac{m}{f}\right). \tag{2.2.4}$$

Here ϕ_2 is a characteristic function of the interval $[F,2F]$ with

$$F \asymp L^2T^2.$$

Moreover

$$X(u) = u^{-k+i(t_1-t_2)}\int_0^\infty v^{-1-2it_2}\xi(f,l,u,v)\exp(-\delta_1 iY)\,dv$$

with

$$\xi(f,l,u,v) = \tilde{\phi}(fu)\phi\left(\frac{(lv)^2u}{16\pi^2 f}\right) \quad\text{and}\quad Y = \frac{1}{2}v+\frac{2r^2}{uv}.$$

Now

$$u \asymp \frac{M}{F} \quad\text{and}\quad v \asymp \frac{F}{L}.$$

Again it is enough to prove that

$$\int_{T/4}^{3T}\int_{T/4}^{3T} w(2t_1)w(2t_2)(2t_1)^{-1+2it_1}(2t_2)^{-1-2it_2}e^{-2it_1+2it_2}(2x)^{-2it_1+2it_2}$$

$$\times n^{-it_1+it_2}\mathcal{S}_4\,dt_1\,dt_2 \ll K^{\varepsilon}.$$

Further we use Lemma 1.16 and spectrally decompose the shifted convolution (cf. Lemma 5 in [28]) and follow the estimation of the term S_d in the lower range in [28], pp. 92-93. We concentrate in the sequel on the first term on the right hand side of Lemma 1.16, the treatment of the second one being similar but easier. We arrive at

$$\mathcal{S}_5 = (4\pi)^{2it_2}\sum_{f=1}^{\infty}\frac{\phi_2(f)}{f^{1/2-i(t_1+t_2)}}\sum_{l=1}^{\infty}\frac{\phi_1(l)S(f+n,n;l)}{l^{1+2it_2}}\sum_{\kappa_j\asymp LT^2}\overline{\rho_j}t_j(f)c_j$$

$$\times \frac{1}{\sinh(\pi\kappa_j)} \frac{\Gamma(\frac{1}{2}+i\kappa_j)}{\Gamma(k-\frac{1}{2}-i\kappa_j)\Gamma(1+2i\kappa_j)} \Xi_1(f,l,\kappa_j,\delta_1)$$

with

$$\Xi_1(f,l,\kappa_j,\delta_1) = \int_0^\infty \int_0^\infty \xi(f,l,u,v)\exp(-\delta_1 iY) u^{-3/2-i(\kappa_j-t_1+t_2)}$$
$$\times v^{-1-2it_2}\, du\, dv$$

corresponding to S_d and formulas (5.21) (without the absolute values) and (5.22) in [28]. The condition for κ_j follows by a saddle-point argument. Finally it suffices to prove that

$$\int_{T/4}^{3T} \int_{T/4}^{3T} w(2t_1)w(2t_2)(2t_1)^{-1+2it_1}(2t_2)^{-1-2it_2} e^{-2it_1+2it_2}(2x)^{-2it_1+2it_2}$$
$$\times n^{-it_1+it_2} \mathcal{S}_5\, dt_1\, dt_2 \ll K^\varepsilon.$$

Again we use the Mellin inversion to obtain

$$\Xi_1(f,l,\kappa_j,\delta_1) = \frac{\delta_1 i}{\pi^2 4^{1+it_2} r^{1+2i(\kappa_j-t_1+t_2)}} \int_{(0)} \int_{(0)} (\tilde{\phi})^*(s_1)\phi^*(s_2) f^{-s_1+s_2}$$
$$\times \left(\frac{2\pi}{l}\right)^{2s_2} r^{-2s_1-2s_2} \exp(-\delta_1 i\pi(s_1+i\kappa_j-it_1))$$
$$\times \Gamma(1/2+s_1-s_2+i(\kappa_j-t_1-t_2))\Gamma(1/2+s_1+s_2+i(\kappa_j-t_1+t_2))\, ds_1\, ds_2.$$

This corresponds to formula (5.24) in [28].

Next we represent the weight function $\phi_1(l)$ by its Mellin inversion

$$\frac{1}{2\pi i} \int_{(1/2+\varepsilon)} \phi_1^*(s_3) l^{-s_3}\, ds_3$$

and transform the l-sum spectrally by Lemma 1.3. We shall concentrate on estimating the first term on the right hand side of (1.2.17) and comment briefly on the remaining terms later. The last term obviously equals zero in our case. To avoid confusion with the previous summation over the variable κ_j, we denote the new variable of summation by $\kappa_{j'}$.

By Stirling's formula, we notice that we may assume that $\kappa_{j'} \ll TK^\varepsilon$. Further we may truncate the integrals over s_i to the intervals Im $s_i = \gamma_i \ll K^\varepsilon$, since the functions $(\tilde{\phi})^*$, ϕ^* and ϕ_1^* are of rapid decay in γ_i, $i = 1, 2, 3$, respectively, and move the s_3-integration to the line (ε). Finally, after applying Cauchy's inequality to the κ_j-sum, it remains to show that

$$\sup_{\gamma_1,\gamma_2,\gamma_3 \ll K^\varepsilon} (LT^2)^{1/2} \Bigg(\sum_{\kappa_j \asymp LT^2} \alpha_j \Bigg| \sum_f \phi_2(f) f^{-1/2-s_1+s_2}(f+n)^{-s_2-s_3/2} t_j(f)$$

$$\times \sum_{\kappa_{j'}} \frac{\overline{\rho_{j'}(f+n)}\rho_{j'}(n)}{\cosh(\pi\kappa_{j'})} \int_{T/4}^{3T}\int_{T/4}^{3T} w(2t_1)w(2t_2)(2t_1)^{-1+2it_1}(2t_2)^{-1-2it_2}$$

$$\times e^{-2it_1+2it_2}\left(\frac{2x\sqrt{f}}{r\sqrt{f+n}}\right)^{2it_2}\left(\frac{r\sqrt{f}}{2x\sqrt{n}}\right)^{2it_1}$$

$$\times\Gamma(1/2+i(\gamma_1-\gamma_2+\kappa_j-t_1-t_2))\Gamma(1/2+i(\gamma_1+\gamma_2+\kappa_j-t_1+t_2))$$

$$\times\Gamma(\varepsilon/2+i(t_2+\gamma_2+\gamma_3/2+\kappa_{j'}))\Gamma(\varepsilon/2+i(t_2+\gamma_2+\gamma_3/2-\kappa_{j'}))$$

$$\times \exp(\delta_1\pi(\gamma_1+\kappa_j-t_1))\exp(\delta_3\pi(t_2+\gamma_2+\gamma_3/2))\,dt_1\,dt_2\Bigg|^2\Bigg)^{1/2}$$

is $\ll K^{1+\varepsilon}$. The constant $\delta_3=\pm1$. Now we write $\Gamma(s)=\exp(\log\Gamma(s))$, use Taylor's approximation and derive an estimate of the νth derivative of $\log\Gamma(s)$ from the asymptotic expansion (1.2.4). Differentiation on both sides ν times can be justified by use of Cauchy's integral formula. We therefore conclude that

$$\Gamma(1/2+i(\gamma_1-\gamma_2+\kappa_j-t_1-t_2))\Gamma(1/2+i(\gamma_1+\gamma_2+\kappa_j-t_1+t_2))$$
$$\sim\Gamma^2(1/2+i(\gamma_1+\kappa_j-t_1)),$$

so we are able to separate the t_1- and t_2-integrals from each other. Further we divide the κ_j-sum into subsums of interval length LT and fix a variable κ in each of these intervals, so that we always have

$$\frac{\kappa_j-\kappa}{\kappa}\ll\frac{1}{T}.$$

Now we multiply the t_1-integral by the factor

$$\exp\left(-2i\kappa_j\log\kappa_j+2i\kappa_j\right)$$

and conclude by (1.2.3) and repeated use of Taylor's approximation that

$$\Gamma^2(1/2+i(\gamma_1+\kappa_j-t_1))e^{-2i\kappa_j\log\kappa_j+2i\kappa_j}\sim 2\pi e^{2i(\gamma_1-t_1)\log\kappa_j}e^{-\pi(\gamma_1+\kappa_j-t_1)}$$
$$\sim 2\pi e^{2i(\gamma_1-t_1)\log\kappa}e^{-\pi(\gamma_1+\kappa_j-t_1)}$$

the "extra coefficients" ($\ll K^\varepsilon$ pcs.) being polynomials of γ_1-t_1, κ_j and $(\kappa_j-\kappa)/\kappa$. Therefore we may omit the dependence of the t_1-integral on κ_j and use the spectral large sieve for the κ_j-sum. We then apply either the second derivative test or trivial estimation to the t_1- and t_2-integrals and finally end our proof.

The second term on the right hand side of (1.2.17) is treated similarly, this case being even easier. In the third term we denote the variable of summation by k' to distinguish it from the weight k. We first treat the part of the k'-sum where $k' \gg TK^{\varepsilon}$. We move the s_3-integral into the k'-sum and to the line $-P$ with P a suitably large constant, and then estimate the Γ's by Stirling's formula (1.2.3). By trivial estimations we see that this part of the k'-sum is negligibly small. For the part where $k' \ll TK^{\varepsilon}$ we move the s_3-integral to the imaginary axis and end up estimating

$$\sup_{\gamma_1,\gamma_2,\gamma_3 \ll K^{\varepsilon}} (LT^2)^{1/2} \Bigg(\sum_{\kappa_j \asymp LT^2} \alpha_j \Bigg| \sum_f \phi_2(f) f^{-1/2-s_1+s_2} (f+n)^{-s_2-s_3/2} t_j(f)$$

$$\times \sum_{k' \ll TK^{\varepsilon}} (2k'-1) q_{f+n,n}(k') \int_{T/4}^{3T} \int_{T/4}^{3T} w(2t_1) w(2t_2) (2t_1)^{-1+2it_1} (2t_2)^{-1-2it_2}$$

$$\times e^{-2it_1+2it_2} \left(\frac{2x\sqrt{f}}{r\sqrt{f+n}} \right)^{2it_2} \left(\frac{r\sqrt{f}}{2x\sqrt{n}} \right)^{2it_1}$$

$$\times \Gamma(1/2 + i(\gamma_1 - \gamma_2 + \kappa_j - t_1 - t_2)) \Gamma(1/2 + i(\gamma_1 + \gamma_2 + \kappa_j - t_1 + t_2))$$

$$\times \frac{\Gamma(k' - 1/2 + i(t_2 + \gamma_2 + \gamma_3/2))}{\Gamma(k' + 1/2 - i(t_2 + \gamma_2 + \gamma_3/2))} \exp(\delta_1 \pi (\gamma_1 + \kappa_j - t_1))\, dt_1\, dt_2 \Bigg|^2 \Bigg)^{1/2}.$$

The t_1-integral can be treated as before. For the t_2-integral, we again use either the second derivative test or trivial estimation writing $\Gamma(s) = \exp(\log \Gamma(s))$ and noticing that for $s = \sigma + it$

$$\frac{d^2}{dt^2} \operatorname{Im} \log \Gamma(s) = \operatorname{Im} \frac{d^2}{dt^2} \log \Gamma(s) = \frac{-t}{|s|^2} + \mathcal{O}(|s|^{-2})$$

by the asymptotic expansion for $\frac{d^2}{ds^2} \log \Gamma(s)$ obtained from (1.2.4).

Remark 7. The case $K^{2-\varepsilon} \ll m \ll K^3$ can be treated essentially in the same way as the case $K^{4/3} \ll m \ll K^{2-\varepsilon}$ above, without the exponential term and therefore the t_i-integrals. Now it suffices to prove that

$$\mathcal{S} = \sum_{K \le \kappa_j \le K+G} \alpha_j \Bigg| \sum_{m=1}^{\infty} \tilde{\phi}(m) t_j(m) \overline{a(m+n)} (n+m)^{1/4-k/2}$$

$$\times m^{-1/4} \left(\frac{m}{M} \right)^{-1/4} \left(\frac{n+m}{M} \right)^{3/4-k/2} \Bigg|^2 M^{1-k}$$

is $\ll K^{-2k+3+1/3+\varepsilon}$. Here $\tilde{\phi}$ is a smooth weight function compactly supported on the interval $[3M/4, 2M]$ and $\phi^{(\nu)} \ll_{\nu} (MK^{-\varepsilon})^{-\nu}$ for each $\nu \ge 0$. Further we may embed the term

$$\left(\frac{m}{M} \right)^{-1/4} \left(\frac{n+m}{M} \right)^{3/4-k/2} \ll 1$$

into $\tilde{\phi}$ having a similar situation to the case $K^{4/3} \ll M \ll K^{2-\varepsilon}$ with the exception that now $\tilde{\phi}^{(\nu)} \ll_\nu (MK^{-\varepsilon})^{-\nu}$. However, this does not cause any new difficulties. See also [28], p. 83, where Jutila and Motohashi have $\phi^{(\nu)} \ll_\nu (M \log^{-4} K)^{-\nu}$.

We proceed as above, until we reach $\mathcal{S}_3$ and estimate trivially the case $L \ll K^\varepsilon + MK^{-2+\varepsilon}$ using the bound (1.2.16). This time we insert a smooth ϕ_2 instead of a characteristic function. Finally, as we reach the point where we represented the weight function ϕ_1 by its Mellin inversion, to estimate the remaining l-sum we use this time Corollary 1.5 and conclude with the desired result.

Chapter 3

An alternative approach to proving Theorem 1

In this chapter we shall discuss an approach to proving Theorem 1 using a method which does not use any devices which are specific to holomorphic cusp forms. This proof is complete, save for the case discussed in Remark 8 on Section 3.3. As in Chapter 2, our line of argumentation combines the use of spectral theoretical methods and modifications of arithmetical problems. Repeatedly we leave a spectral theoretic portion of the proof through the use of the spectral large sieve and reencounter spectral theory once again through the spectral decomposition of a convolution sum.

We start by following the argument of Lemma 4, [21], used to estimate the respective long sum (1.2.29), with certain modifications.

3.1 A preliminary reduction of the spectral sum

We first apply a variant of the Rankin-Selberg method, Lemma 1.17, according to which we may write

$$c_j = \frac{1}{\pi i} \int_{(a)} \xi(2s)(sg(s) - (1-s)g(1-s)) I_j(s)\, ds \tag{3.1.1}$$

with $\xi(s)$ as in (1.1.13), $g(s)$ as in (1.2.34) and

$$I_j(s) = \int_0^\infty \int_0^1 |F(z)|^2 u_j(z) y^{k+s-2}\, dx\, dy. \tag{3.1.2}$$

We initially choose the constant $a > 1$ to be suitably large and the constant B so that $B > \frac{3}{\pi} a$. Hence for all $s = \sigma + it$, $0 < \sigma \le a$

$$|g(s)|, |g(1-s)| \ll \exp\left(-\frac{1}{4} e^{|t|/B}\right), \tag{3.1.3}$$

and by Stirling's formula and (1.2.5)

$$\xi(2s)(sg(s) - (1-s)g(1-s)) \ll e^{-\frac{1}{4}e^{\frac{|t|}{B}}}.$$

Next we will show that the integral with respect to s can be truncated. Let us assume that $|\mathrm{Im}\ s| = |t| > c \log K$ for some positive constant c. Now $y^k|F(z)|^2$ is exponentially decaying near the cusp $i\infty$ and bounded in $\mathbb{H}$, and by definition the Maass forms u_j are orthonormal, hence for some constant $A \geq 1$

$$\int_A^\infty \int_0^1 u_j(z) y^{k+s-2} |F(z)|^2 \, dx \, dy \ll \left(\int_A^\infty \int_0^1 |u_j(z)|^2 \, d\mu(z) \right)^{1/2}$$

$$\times \left(\int_A^\infty \int_0^1 y^{2k+2a-2} |F(z)|^4 \, dx \, dy \right)^{1/2} \ll 1.$$

Moreover

$$|u_j(z)| \ll K + K^{1/2} \left(y + \frac{1}{y} \right)^{1/2}$$

by Iwaniec [17], Proposition 7.2. Hence we deduce that

$$I_j(s) = \left(\int_0^A + \int_A^\infty \right) \int_0^1 u_j(z) y^{k+s-2} |F(z)|^2 \, dx \, dy \ll K,$$

at least when $a \geq 5/2$. Therefore, fixing c suitably, we see that the contribution of the tails $|\mathrm{Im}\ s| > c \log K$ of the s-integral to the sum in Theorem 1 is negligibly small, and hence the integral in (3.1.1) can be truncated to the interval $|t| \leq c \log K$.

Fixing s for a moment, we derive two separate expressions for $I_j(s)$. We substitute the Fourier series (1.1.2) and (1.1.8) into the integral (3.1.2), expand out the product of the series and integrate termwise to obtain

$$I_j(s) = \sum_{\substack{n_1, n_2 = 1, \\ n_1 \neq n_2}}^{\infty} a(n_1) \overline{a(n_2)} \rho_j(n_2 - n_1) \tag{3.1.4}$$

$$\times \int_0^\infty e^{-2\pi(n_1+n_2)y} K_{i\kappa_j}(2\pi |n_1 - n_2| y) y^{k+s-3/2} \, dy.$$

Notice that changing the order of summation and integration is permitted when a is large enough; see (1.2.6), (1.2.7) and (1.2.24). We split up the double sum over n_1 and n_2 into two parts $I_{j1}(s)$ and $I_{j2}(s)$ by the conditions $n_2 > n_1$ and $n_1 > n_2$, respectively. We deal with the sum I_{j1}, noticing that the case of I_{j2} is symmetric. From now on we shall write $n_1 = n$ and $n_2 = n+m$ so that n and m will run over the interval $[1,\infty)$. Thus

$$I_{j1}(s) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a(n)\overline{a(n+m)}\rho_j(m)\int_0^{\infty} e^{-2\pi(2n+m)y}K_{i\kappa_j}(2\pi my)y^{k+s-3/2}\,dy.$$

For short, we write $p = k+s-\frac{1}{2} = k+a-\frac{1}{2}+it$. To estimate the integral above, we use the Mellin transform of $e^{-\alpha y}K_{ir}(\beta y)$ for real r and positive α, β and Re p:

$$\frac{\sqrt{\pi}\beta^{ir}\Gamma(p+ir)\Gamma(p-ir)}{2^p\alpha^{p+ir}\Gamma(p+1/2)}F\left(\frac{p+ir+1}{2},\frac{p+ir}{2};p+\frac{1}{2};1-\left(\frac{\beta}{\alpha}\right)^2\right),$$

see Eq. (29) on p. 331 of [5]. We arrive at

$$I_{j1}(s) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a(n)\overline{a(n+m)}\rho_j(m)\Lambda_j(p,m,n) \tag{3.1.5}$$

with

$$\Lambda_j(p,m,n) = 4^{-p}\pi^{1/2-p}\frac{\Gamma(p+i\kappa_j)\Gamma(p-i\kappa_j)}{\Gamma(p+1/2)}\left(\frac{m}{2n+m}\right)^{i\kappa_j}(2n+m)^{-p}$$

$$\times F\left(\frac{p+i\kappa_j+1}{2},\frac{p+i\kappa_j}{2};p+\frac{1}{2};1-\left(\frac{m}{2n+m}\right)^2\right). \tag{3.1.6}$$

We obtain an alternative formula for Λ_j in the following way. Let us use the abbreviation

$$\lambda = \lambda(n,m) = \sqrt{1-\left(\frac{m}{2n+m}\right)^2} = \frac{2\sqrt{n(n+m)}}{2n+m}, \tag{3.1.7}$$

noticing that $0<\lambda<1$. We transform the hypergeometric function using the formulae (1.2.11) and (1.2.12) to obtain

$$F\left(\frac{p+i\kappa_j+1}{2},\frac{p+i\kappa_j}{2};p+\frac{1}{2};1-\left(\frac{m}{2n+m}\right)^2\right)$$

$$= \frac{\Gamma(2p)}{\Gamma(p)(2\lambda)^p}\Bigg\{(1+\lambda)^{-i\kappa_j}\frac{\Gamma(-i\kappa_j)}{\Gamma(p-i\kappa_j)}F\left(p,1-p;1+i\kappa_j;\frac{\lambda-1}{2\lambda}\right)$$

$$+(1-\lambda)^{-i\kappa_j}\frac{\Gamma(i\kappa_j)}{\Gamma(p+i\kappa_j)}F\left(p,1-p;1-i\kappa_j;\frac{\lambda-1}{2\lambda}\right)\Bigg\}.$$

Since $\frac{m}{2n+m} = ((1-\lambda)(1+\lambda))^{1/2}$, using the duplication formula for the Gamma function, we have

$$\Lambda_j(p,m,n) = 2^{-2p-1}\pi^{-p}(n(n+m))^{-p/2}\Bigg\{\Gamma(-i\kappa_j)\Gamma(p+i\kappa_j)\left(\frac{1-\lambda}{1+\lambda}\right)^{i\kappa_j/2}$$
$$\times F\left(p,1-p;1+i\kappa_j;\frac{\lambda-1}{2\lambda}\right)+\Gamma(i\kappa_j)\Gamma(p-i\kappa_j)\left(\frac{1+\lambda}{1-\lambda}\right)^{i\kappa_j/2}$$
$$\times F\left(p,1-p;1-i\kappa_j;\frac{\lambda-1}{2\lambda}\right)\Bigg\}. \qquad (3.1.8)$$

3.2 A reformulation of the problem

Next we shall restate our problem in an easier form. We start by inserting a smooth weight function $w_N(n)$ into the n-sum in (3.1.5) for, let us say, the range $1 \leq n \leq K^3$ using the method introduced in (1.2.21). Hence the beginning of the n-sum runs over ranges $n \asymp N$ with $1 \leq N \ll K^3$. We also divide the m-sum into subsums, where m runs over the ranges $M \leq m < (1+b)M$ for $M = (1+b)^{\jmath}$ with $\jmath \geq 0$ and $0 < b \leq 1$ a constant to be fixed later. Further we truncate the resulting double sum over n and m into two parts, one of which satisfies the condition

$$N(N+M) \ll K^{2+\theta}, \tag{3.2.1}$$

with $0 < \theta < 1$ some arbitrarily small positive number which will be kept fixed throughout the proof, and the other of which contains the terms with $n(n+m) \gg K^{2+\theta}$. Note that our truncation of the double sum over n and m varies slightly from that in [21], this being due to differences in our proofs later on. Combining the above steps we arrive at

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(...) = \sum_{\substack{N=2^{\imath}, M=(1+b)^{\jmath}, \\ \imath,\jmath \geq 0, \\ N(N+M) \ll K^{2+\theta}}} \sum_{3N/4 \leq n \leq 2N} \sum_{M \leq m < (1+b)M} w_N(n)(...) + Error. \tag{3.2.2}$$

Accordingly, we decompose $I_{j1}(s)$ into two parts, one of which consists of the first term on the right hand side of (3.2.2), and the other of which contains the $Error$-term. These are further substituted into the truncated version of (3.1.1) and the parameter a is chosen appropriately for each part.

For the $Error$-part, we take a sufficiently large. As in [21], we notice now that the part of c_j involving $Error$ is negligibly small. To verify this, we separate the sums within the $Error$-term into three categories, 1.) $n \gg m$, in which case $\lambda \asymp 1 \gg K^{-1}$, 2.) $n \ll m$ and $\lambda \ll K^{-1}$, 3.) $n \ll m$ and $\lambda \gg K^{-1}$. We then use the formulae (3.1.6) and (3.1.8) to represent Λ_j, respectively, depending on whether $\lambda \ll K^{-1}$ or $\lambda \gg K^{-1}$.

In the case when $\lambda \ll K^{-1}$, we use the following observation: $m \gg K^{2+\theta/2}$ by the conditions

$$n(n+m) \gg K^{2+\theta} \text{ and } \lambda \asymp \sqrt{\frac{n}{m}}. \tag{3.2.3}$$

Using equation (3.1.6), by the bound (1.2.24) the contribution of the subsums in category 2.) to $I_{j1}(s)$ is

$$\ll K^{2k+2a-2} e^{-\frac{\pi}{2}(\kappa_j - |t|)} \sum_{n,m} n^{k/2-1/2+\varepsilon} (n+m)^{-k/2-a+\varepsilon}$$

$$\times(\kappa_j^{1/2+\varepsilon} + \kappa_j^{1/2} m^{\varepsilon} + m^{1/4+\varepsilon}).$$

Here the hypergeometric function has been estimated in the following way: We write

$$F\left(\frac{p+i\kappa_j+1}{2}, \frac{p+i\kappa_j}{2}; p+1/2; \lambda^2\right) = 1 + \sum_{\nu=1}^{\infty} \frac{\left(\frac{p+i\kappa_j+1}{2}\right)_\nu \left(\frac{p+i\kappa_j}{2}\right)_\nu}{(p+1/2)_\nu \nu!} \lambda^{2\nu}$$

using the notation of (1.2.8). For $\lambda \ll K^{-1}$, with the implied constant small enough, we see that, by the standard proof of the ratio test, the above series is of order 1.

In the case when $\lambda \gg K^{-1}$, by Lemma 1.2 the hypergeometric function in question is of size $\ll K^{\varepsilon}$. Therefore, by (3.1.8), the contribution to $I_{j1}(s)$ of the subsums in either category 1.) or 3.) is

$$\ll K^{k+a-3/2+\varepsilon} e^{-\frac{\pi}{2}(\kappa_j-|t|)} \sum_{n,m} (n(n+m))^{-a/2-1/4+\varepsilon} (\kappa_j^{1/2+\varepsilon} + \kappa_j^{1/2} m^{\varepsilon} + m^{1/4+\varepsilon}).$$

In both cases we obtain the desired result if a is suitably large, and depends on both k and θ.

For the first term on the right hand side of (3.2.2), we use the formula (3.1.8) for Λ_j and move the integration close to the imaginary axis to the line $a = \delta > 0$ using the residue theorem. Note that the only residue coming from the pole $s = 1/2$ of $\xi(2s)$ is compensated by the zero of $sg(s) - (1-s)g(1-s)$ and that the integrals over the horizontal lines on the path contribute to the original sum again negligibly; see formulae (1.2.9) and (1.2.10), and estimates (1.2.5) and (3.1.3).

We once more decompose the remaining sum

$$\sum_{\substack{N=2^\imath, M=(1+b)^\jmath, \\ \imath,\jmath \geq 0, \\ N(N+M) \ll K^{2+\theta}}} \sum_{3N/4 \leq n \leq 2N} \sum_{M \leq m < (1+b)M} w_N(n)(...)$$

into two parts consisting of the sums satisfying either $M > bN/(2(1+b))$ or $M \leq bN/(2(1+b))$, respectively. After summing the latter part over M, we obtain

$$\sum_{\substack{N=2^\imath, M=(1+b)^\jmath, \\ \imath,\jmath \geq 0, M > bN/(2(1+b)) \\ NM \ll K^{2+\theta}}} \sum_{3N/4 \leq n \leq 2N} \sum_{M \leq m < (1+b)M} w_N(n)(...)$$

$$+ \sum_{\substack{N=2^\imath, \imath \geq 0, \\ N^2 \ll K^{2+\theta}}} \sum_{3N/4 \leq n \leq 2N} \sum_{1 \leq m < bN/2} w_N(n)(...). \tag{3.2.4}$$

Furthermore we let $M = N/2$ in the second of the above sum, so that m runs over the range $[1, bM)$. Hence in both sums we have $M \gg N$ and the condition (3.2.1) can be rewritten in the form $NM \ll K^{2+\theta}$.

Now $p = k - \frac{1}{2} + \delta + it$ and $|t| \le c \log K$. By (3.1.8) and the earlier considerations the estimation of the sum

$$\sum_{K \le \kappa_j \le K + K^{1/3}} |c_j|^2 \exp(\pi \kappa_j)$$

amounts to estimating

$$\sum_{K \le \kappa_j \le K + K^{1/3}} \Bigg| \int_{\delta - ic\log K}^{\delta + ic \log K} \xi(2s)(sg(s) - (1-s)g(1-s))$$

$$\times 2^{-2p-1}\pi^{-p} \sum_{N,M} \sum_{n,m} a(n)\overline{a(n+m)} \rho_j t_j(m) w_N(n)(n(n+m))^{-p/2}$$

$$\times \Bigg\{ \Gamma(-i\kappa_j)\Gamma(p + i\kappa_j) \left(\frac{1-\lambda}{1+\lambda}\right)^{i\kappa_j/2} F\left(p, 1-p; 1+i\kappa_j; \frac{\lambda-1}{2\lambda}\right)$$

$$+\Gamma(i\kappa_j)\Gamma(p - i\kappa_j) \left(\frac{1+\lambda}{1-\lambda}\right)^{i\kappa_j/2} F\left(p, 1-p; 1-i\kappa_j; \frac{\lambda-1}{2\lambda}\right) \Bigg\} ds \Bigg|^2 e^{\pi\kappa_j},$$

where the sum $\sum_{N,M} \sum_{n,m}$ stands for either of the two sums in (3.2.4). By Cauchy's inequality it suffices to consider

$$\sum_{\kappa_j} e^{\pi\kappa_j} \Bigg(\int \Bigg| \xi(2s)(sg(s) - (1-s)g(1-s)) 2^{-2p-1}\pi^{-p}\rho_j$$

$$\times \Gamma(\mp i\kappa_j)\Gamma(p \pm i\kappa_j) \Bigg|^2 |ds| \times \int \Bigg| \sum_{N,M} \sum_{n,m} a(n)\overline{a(n+m)} t_j(m) w_N(n)$$

$$\times (n(n+m))^{-p/2} \left(\frac{1 \mp \lambda}{1 \pm \lambda}\right)^{i\kappa_j/2} F\left(p, 1-p; 1 \pm i\kappa_j; \frac{\lambda-1}{2\lambda}\right) \Bigg|^2 |ds| \Bigg)$$

with either upper or lower signs. By (1.2.5), (3.1.3) and Cauchy's inequality we arrive at

$$K^{2k+2\delta-3+\varepsilon} \sum_{\kappa_j} \sum_{N,M} \int \alpha_j \Bigg| \sum_{n,m} a(n)\overline{a(n+m)} t_j(m) w_N(n)$$

$$\times (n(n+m))^{-p/2} \left(\frac{1 \mp \lambda}{1 \pm \lambda}\right)^{i\kappa_j/2} F\left(p, 1-p; 1 \pm i\kappa_j; \frac{\lambda-1}{2\lambda}\right) \Bigg|^2 |ds|.$$

When we choose δ small enough, the term 2δ in the exponent can be embeded into ε. Finally we notice that for a proof of Theorem 1 it is sufficient to show that the following estimate (with upper or lower signs) holds for all $|t| \le c \log K$ and $\operatorname{Re} s = \delta$:

$$\sum_{K \le \kappa_j \le K+K^{1/3}} \alpha_j \Bigg| \sum_{n,m} a(n)\overline{a(n+m)}t_j(m)w_N(n)(n(n+m))^{-p/2} \left(\frac{1 \mp \lambda}{1 \pm \lambda}\right)^{i\kappa_j/2}$$

$$\times F\left(p, 1-p; 1 \pm i\kappa_j; \frac{\lambda - 1}{2\lambda}\right) \Bigg|^2 \ll K^{2+1/3+\varepsilon}, \qquad (3.2.5)$$

with the summation condition over n and m either

$$\frac{3N}{4} \le n \le 2N, \quad M \le m < (1+b)M, \quad \text{(case 1)}$$

or

$$\frac{3N}{4} \le n \le 2N, \quad 1 \le m < bM, \quad \text{(case 2)}$$

and in both cases we may assume $N, M \ge 1$, $M \gg N$ and $NM \ll K^{2+\theta}$. For short, we write $w_N(n) = \phi(n)$ since N is now fixed.

Next we use summation by parts to simplify the above inequality even further. We shall present the details only for case 2, this being slightly more complicated than the first one. Using the notation

$$b(m,n) = a(n)\overline{a(n+m)}t_j(m)\phi(n)A(m,n)^{\pm i\kappa_j}$$

with

$$A(m,n) = \sqrt{\frac{1-\lambda(m,n)}{1+\lambda(m,n)}} = 1 + \frac{2n}{m} - 2\frac{\sqrt{n(n+m)}}{m}$$

$$= \left(1 + \frac{2n}{m} + 2\frac{\sqrt{n(n+m)}}{m}\right)^{-1},$$

and

$$c(m,n) = (n(n+m))^{-p/2} F\left(p, 1-p; 1 \pm i\kappa_j; \frac{\lambda-1}{2\lambda}\right),$$

by summating by parts twice and using Cauchy's inequality repeatedly, we arrive at

$$\Bigg| \sum_{3N/4 \le n \le 2N} \sum_{1 \le m < bM} a(n)\overline{a(n+m)}t_j(m)\phi(n)(n(n+m))^{-p/2} \left(\frac{1 \mp \lambda}{1 \pm \lambda}\right)^{i\kappa_j/2}$$

$$\times F\left(p, 1-p; 1 \pm i\kappa_j; \frac{\lambda-1}{2\lambda}\right) \Bigg|^2 \ll \Bigg| \sum_{3N/4 \le n \le 2N} \sum_{1 \le m < bM} b(m,n) \Bigg|^2$$

$$\times\left|c(\|bM\|,2N)\right|^2+\int_{3N/4}^{2N}\left|\sum_{3N/4\le n\le\eta}\sum_{1\le m<bM}b(m,n)\right|^2d\eta$$

$$\times\int_{3N/4}^{2N}\left|\frac{\partial}{\partial\eta}c(\|bM\|,\eta)\right|^2d\eta+\int_1^{bM}\xi^{-1}\left|\sum_{3N/4\le n\le 2N}\sum_{1\le m\le\xi}b(m,n)\right|^2d\xi$$

$$\times\int_1^{bM}\xi\left|\frac{\partial}{\partial\xi}c(\xi,2N)\right|^2d\xi+\int_{3N/4}^{2N}\int_1^{bM}\xi^{-1}\left|\sum_{3N/4\le n\le\eta}\sum_{1\le m\le\xi}b(m,n)\right|^2d\xi\,d\eta$$

$$\times\int_{3N/4}^{2N}\int_1^{bM}\xi\left|\frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi}c(\xi,\eta)\right|^2d\xi\,d\eta,$$

where we denote the largest integer smaller than $x\in\mathbb{R}$ by $\|x\|$. Notice that $\|x\|=\lfloor x\rfloor$, unless $x\in\mathbb{Z}$, when $\|x\|=\lfloor x\rfloor-1$. Now in both cases 1 and 2

$$\lambda=\lambda(m,n)\asymp\sqrt{\frac{N}{M}}\gg K^{-1-\theta/2},$$

so according to Lemma 1.2 we have

$$F\left(p,l-p;1\pm i\kappa_j;\frac{\lambda-1}{2\lambda}\right)\ll K^{(k+\delta-l)\theta/2+\varepsilon}$$

for all $l=1,2,3$. Therefore by direct calculation, using (1.2.14),

$$c(\|bM\|,2N)\ll(NM)^{-k/2+1/4}K^{(k+\delta-1)\theta/2+\varepsilon},$$

$$\frac{\partial}{\partial\eta}c(\|bM\|,\eta)\ll(NM)^{-k/2+1/4}N^{-1}K^{(k+\delta-1)\theta/2+\varepsilon},$$

$$\frac{\partial}{\partial\xi}c(\xi,2N)\ll(NM)^{-k/2+1/4}\xi^{-1}K^{(k+\delta-1)\theta/2+\varepsilon}$$

and

$$\frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi}c(\xi,\eta)\ll(NM)^{-k/2+1/4}(N\xi)^{-1}K^{(k+\delta-1)\theta/2+\varepsilon}$$

for all $\frac{3N}{4}\le\eta\le 2N$, $1\le\xi\le bM$. The assertion (3.2.5) amounts to the following: It is sufficient to show that

$$\sum_{\kappa_j}\alpha_j\left|\sum_{3N/4\le n\le 2N}\sum_{1\le m<bM}b(m,n)\right|^2+N^{-1}\int_{3N/4}^{2N}\sum_{\kappa_j}\alpha_j\left|\sum_{3N/4\le n\le\eta}\right.$$

$$\left.\times\sum_{1\le m<bM}b(m,n)\right|^2d\eta+\int_1^{bM}\xi^{-1}\sum_{\kappa_j}\alpha_j\left|\sum_{3N/4\le n\le 2N}\sum_{1\le m\le\xi}b(m,n)\right|^2d\xi$$

$$+N^{-1}\int_{3N/4}^{2N}\int_{1}^{bM}\xi^{-1}\sum_{\kappa_j}\alpha_j\Bigg|\sum_{3N/4\le n\le\eta}\ \sum_{1\le m\le\xi} b(m,n)\Bigg|^2 d\xi\, d\eta$$

$$\ll K^{2+\frac{1}{3}+\varepsilon}(NM)^{k-1/2}, \tag{3.2.6}$$

when we fix θ so that $(k+\delta-1)\theta$ is small enough.

In what follows, we shall carry out in detail the estimation of the last term on the left hand side,

$$N^{-1}\int_{3N/4}^{2N}\int_{1}^{bM}\xi^{-1}\sum_{K\le\kappa_j\le K+K^{1/3}}\alpha_j\Bigg|\sum_{3N/4\le n\le\eta}\ \sum_{1\le m\le\xi} a(n)\overline{a(n+m)}$$

$$\times t_j(m)\phi(n)A(m,n)^{\pm i\kappa_j}\Bigg|^2 d\xi\, d\eta, \tag{3.2.7}$$

the other three being similar, but easier. We also concentrate on the case $+\kappa_j$ noticing that the case for $-\kappa_j$ follows easily by complex conjugation and Remark 4 in Section 1.2.3.

Case 1 is treated in a similar way, and we end up with an analogous problem. Altogether we shall concentrate on estimating

$$N^{-1}\int_{3N/4}^{2N}\int_{I}\xi^{-1}\sum_{K\le\kappa_j\le K+K^{1/3}}\alpha_j\Bigg|\sum_{3N/4\le n\le\eta}\ \sum_{B\le m\le\xi} a(n)\overline{a(n+m)}t_j(m)$$

$$\times\phi(n)A(m,n)^{i\kappa_j}\Bigg|^2 d\xi\, d\eta, \tag{3.2.8}$$

where in case 1 we have the interval $I=[M,(1+b)M]$ and the quantity $B=M$, and in case 2, $I=[1,bM]$ and $B=1$. Note that in case 1, ξ^{-1} could be replaced by M^{-1}. However, in order to keep the notation as simple as possible, we write (3.2.8) as above.

3.3 Sobolev's lemma and the spectral large sieve

Now summation by parts does not enable us to separate the variables κ_j and m and n in the term $A(m,n)^{i\kappa_j}$, and instead we apply Sobolev's lemma 1.6 to the double sum over m and n in (3.2.8). The range $[K, K+K^{1/3}]$ for κ_j is split up into segments of length Δ in such a way that the factor $A(m,n)^{ix}$ remains essentially stationary as x runs over a segment. That is

$$\Delta\left|\frac{\partial}{\partial x}A(m,n)^{ix}\right| \ll \log K,$$

or

$$\Delta|\log A(m,n)| \ll \log K.$$

In this way, the second term in the upper bound in Lemma 1.6 will be comparable to the first, and the factors $\log A(m,n)$ can be eliminated by summation by parts as shown below. Since

$$|\log A(m,n)| \ll \begin{cases} \frac{N}{M}+\frac{\sqrt{N}}{\sqrt{M}} \ll \sqrt{\frac{N}{M}}, & \text{case 1,}\\ \log N \ll \log K, & \text{case 2,}\end{cases}$$

an appropriate choice would be

$$\Delta = \min\left(K^{1/3}, \sqrt{\frac{M}{N}}\right).$$

Now we divide the κ_j-sum in (3.2.8) into subsums of length Δ, noticing that the last subsum may be incomplete, and apply Lemma 1.6 to each subsum arriving at the bound

$$N^{-1}\int\int \xi^{-1}\sum_{l=0}^{\lfloor K^{1/3}\Delta^{-1}\rfloor}\Delta^{-1}\int_{B(l)}\sum_{\kappa_j\in B(l)}\alpha_j\Bigg(\Bigg|\sum_n\sum_m a(n)$$

$$\times\overline{a(n+m)}t_j(m)\phi(n)A(m,n)^{ix}\Bigg|^2 + \Delta^2\Bigg|\sum_n\sum_m a(n)\overline{a(n+m)}t_j(m)\phi(n)$$

$$\times A(m,n)^{ix}\log A(m,n)\Bigg|^2\Bigg)\,dx\,d\xi\,d\eta, \tag{3.3.1}$$

with the interval

$$B(l) = \begin{cases} [K+l\Delta, K+(l+1)\Delta), & 0\le l\le \lfloor K^{1/3}\Delta^{-1}\rfloor - 1,\\ [K+l\Delta, K+K^{1/3}], & l = \lfloor K^{1/3}\Delta^{-1}\rfloor.\end{cases}$$

Now the term

$$\Delta^2\Bigg|\sum_n\sum_m a(n)\overline{a(n+m)}t_j(m)\phi(n)A(m,n)^{ix}\log A(m,n)\Bigg|^2 \tag{3.3.2}$$

can be treated essentially in the same way as the simpler term

$$\left| \sum_n \sum_m a(n)\overline{a(n+m)} t_j(m) \phi(n) A(m,n)^{ix} \right|^2, \tag{3.3.3}$$

$\log A(m,n)$ being non-oscillating and "well behaved". In order to keep the notation as simple as possible, we shall concentrate on the term (3.3.3), the estimation of the term (3.3.2) being similar. Note that (3.3.2) could also be dealt with using summation by parts twice and then Cauchy's inequality as before, without any major difficulties.

We next apply the spectral large sieve to the subsum over κ_j inside the x-integrand, and finally add the results corresponding to all subsums. This leads us to the bound for the first term in (3.3.1) of

$$K^{\varepsilon} N^{-1} \Delta^{-1} (K\Delta + M) \int \int \xi^{-1} \int_K^{K+K^{1/3}} \sum_m \left| \sum_n a(n)\overline{a(n+m)} \right.$$

$$\left. \times \phi(n) A(m,n)^{ix} \right|^2 dx\, d\xi\, d\eta.$$

Further we replace $A(m,n)$ here by the simpler

$$\tilde{A}(m,n) = m^{-1} A(m,n) = (m + 2n + 2\sqrt{n(n+m)})^{-1}.$$

In case 2 we write in (3.3.2)

$$\log A(m,n) = \log m + \log \tilde{A}(m,n),$$

to give

$$\Delta^2 \sum_m \left| \sum_n (...) \log A(m,n) \right|^2 \ll K^{\varepsilon} \sum_m \left| \sum_n (...) \right|^2 + \sum_m \left| \sum_n (...) \log \tilde{A}(m,n) \right|^2.$$

In case 1, we let $\log A(m,n)$ remain as it stands.

Remark 8. Now if $M \ll K^{4/3}$, then the factor $K\Delta + M$ is dominated by $K\Delta$ up to $K^{\theta/2}$. In the case when $M \gg K^{4/3}$ the spectral large sieve is too general, and instead we need to use the information of the specific nature of the terms in our m-sum. For this purpose, we have in (3.3.1), before applying the spectral large sieve, an expression similar to that in Section 2.2, with an extra n-sum inside the absolute values. The discussion in Chapter 2 serves as a model for how this case may be treated. The details have not been examined, but will be left to future study. However, preparing to this examination, we complete the rest of the proof for all $NM \ll K^{2+\theta}$, although we bypass this point assuming that $K\Delta + M \ll K\Delta$.

Next we shall simplify the estimation of the m-sum and x-integral by inserting real-valued smooth weight functions $v(m)$ and $u(x)$. In case 1, we let v be compactly supported on $[(1-b/2)M, (1+3b/2)M]$ and $v(m) = 1$ on the interval $[M, (1+b)M]$. In case 2, for the nature of the weight function, we let v be compactly supported on the interval $[-5bM/4, 5bM/4]$ and $v(m) = 1$ on $[-bM, bM]$. In both cases, $v^{(\nu)} \ll_{\nu,b} M^{-\nu}$ for each $\nu \geq 0$. Notice that the variable m can now take the value 0 in the second case, but this does not produce any problems since we have replaced $A(m,n)$ by $\tilde{A}(m,n)$. Further we notice that in both cases, also for negative m, we always have $n + m \asymp M$. Returning to $u(x)$, we set u to be compactly supported on $[K - K^{1/3}/2, K + 3K^{1/3}/2]$, $u(x) = 1$ on $[K, K + K^{1/3}]$ and $u^{(\nu)} \ll_\nu K^{-\nu/3}$ for each $\nu \geq 0$. This gives a bound

$$N^{-1}K^{1+\varepsilon} \int\int \xi^{-1} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left| \sum_n a(n)\overline{a(n+m)}\phi(n)\tilde{A}(m,n)^{ix} \right|^2 \times v(m)u(x)\, dx\, d\xi\, d\eta. \tag{3.3.4}$$

Now there is ξ-dependence only in the term ξ^{-1}, and hence we may perform the integration over this variable, gaining a coefficient of size $\ll K^\varepsilon$. By multiplying out the squares we obtain

$$N^{-1}K^{1+\varepsilon} \int\int \sum_m \sum_{n_1} \sum_{n_2} a(n_1)\overline{a(n_2)a(n_1+m)}a(n_2+m)\phi(n_1)\phi(n_2) \times \left(\frac{\tilde{A}(m,n_1)}{\tilde{A}(m,n_2)} \right)^{ix} v(m)u(x)\, dx\, d\eta. \tag{3.3.5}$$

At this point Jutila ends up in [21] with a sum involving essentially only the diagonal terms $n_1 = n_2$. In our case, however, even certain non-diagonal terms turn out to be relevant. In particular, using the notation

$$\frac{\tilde{A}(m,n_1)}{\tilde{A}(m,n_2)} = A$$

we see by integration by parts that for all $\nu \geq 1$

$$\int_{-\infty}^{\infty} A^{ix}u(x)\, dx = (-1)^\nu \int_{-\infty}^{\infty} A^{ix} \left(\frac{1}{i \log A} \right)^\nu u^{(\nu)}(x)\, dx \ll_\nu \frac{1}{(\log A)^\nu K^{(1/3)(\nu-1)}},$$

which is

$$\ll K^{\frac{1}{3}(-\varepsilon\nu+1)}$$

if $|A-1| \gg K^{-1/3+\varepsilon}$, and by choosing ν big enough our upper bound is negligibly small. We therefore conclude that it is enough to consider the case

$$\frac{\tilde{A}(m,n_1)}{\tilde{A}(m,n_2)} = 1 + O(K^{-1/3+\varepsilon}).$$

In [21] the oscillation is more rapid with the longer range of integration, and Jutila is able to omit the terms $|A-1| \gg K^{-1+\varepsilon}$. We obtain

$$\log \frac{\tilde{A}(m,n_1)}{\tilde{A}(m,n_2)} \ll K^{-1/3+\varepsilon}$$

and straightforwardly

$$\frac{\partial}{\partial n} \log \tilde{A}(m,n) \asymp (NM)^{-1/2}$$

for all $n \asymp N$. Therefore by the mean value theorem

$$|n_1 - n_2| \le F_1 \ll \sqrt{NM} K^{-1/3+\varepsilon}, \tag{3.3.6}$$

and in view of (3.2.1) we only get the upper bound $|n_1 - n_2| \ll K^{2/3+\theta/2+\varepsilon}$, whereas in the similar case in [21] this would have been $|n_1 - n_2| \ll K^{\theta/2+\varepsilon}$. In the beginning of the next section we shall see that the trivial estimation is enough when, say, $|n_1 - n_2| \ll K^{\sqrt{\theta}}$. For the remaining case, $K^{\sqrt{\theta}} \ll |n_1 - n_2| \le F_1$, we must find another path to follow. Notice also that we may always assume that $F_1 \le 5N/4$.

After these considerations we shall fix the variable x until the end of our argument, and then integrate trivially over x. Hence, by (3.2.6), it is enough to show that the upper bound for

$$N^{-1} \int \sum_m \sum_{n_1} \sum_{n_2} a(n_1)\overline{a(n_2)a(n_1+m)}a(n_2+m)\phi(n_1)\phi(n_2) \left(\frac{\tilde{A}(m,n_1)}{\tilde{A}(m,n_2)} \right)^{ix} \times v(m)\, d\eta \tag{3.3.7}$$

under the restriction (3.3.6) is

$$\ll K^{1+\varepsilon}(NM)^{k-1/2}$$

for all $K - \frac{1}{2}K^{1/3} \le x \le K + \frac{3}{2}K^{1/3}$.

3.4 Spectral decomposition of the shifted convolution sum

We shall next follow ideas from yet another article by Jutila, [24]. In the third section of this article, Jutila considers an oscillating sum related to the additive divisor problem involving the divisor function, which, as described in the Introduction, behaves analogously to the Fourier coefficients of a holomorphic cusp form.

We start by spectrally decomposing the m-sum in (3.3.7). First we introduce the new variables $f = n_1 - n_2$ and $l = n_2 + m \geq 1$. In the case when $|f| \ll K^{\sqrt{\theta}}$, a trivial estimation is enough. Further we may assume that $f > 0$ because the case $n_2 > n_1$ is symmetric to the case $n_2 < n_1$ apart from complex conjugation. The contribution of the m-sum in (3.3.7) is therefore

$$\sum_{m=-\infty}^{\infty} \overline{a(n_1+m)} a(n_2+m) \left(\frac{\tilde{A}(m,n_1)}{\tilde{A}(m,n_2)} \right)^{ix} v(m) = \sum_{l=1}^{\infty} a(l) \overline{a(l+f)} W\left(\frac{l}{f}\right)$$

with $f \gg K^{\sqrt{\theta}}$ and with a smooth, compactly supported function

$$W(y) = \left(\frac{yf + n_2 + 2\sqrt{n_2 y f}}{yf + 2f + n_2 + 2\sqrt{(n_2+f)f(y+1)}} \right)^{ix} v(yf - n_2)$$

$$= \left(\frac{\sqrt{yf} + \sqrt{n_2}}{\sqrt{(y+1)f} + \sqrt{n_2+f}} \right)^{2ix} v(yf - n_2).$$

We use Lemma 1.16 to spectrally decompose this shifted convolution sum, arriving at

$$i\pi(4\pi)^{k-1} f^{-1/2+k} \Bigg(\sum_{j=1}^{\infty} \overline{\rho_j} t_j(f) c_j \Phi_k(\kappa_j; W)$$

$$+ \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sigma_{2ir}(f) c(r)}{(\pi f)^{ir} \Gamma(\frac{1}{2} - ir)\zeta(1-2ir)} \Phi_k(r; W)\, dr \Bigg)$$

with

$$\Phi_k(r; W) = \frac{1}{\sinh(\pi r)} \frac{\Gamma(\frac{1}{2} + ir)}{\Gamma(k - \frac{1}{2} - ir)\Gamma(1+2ir)} I(r, f, n_2) + (r \mapsto -r). \quad (3.4.1)$$

Here

$$I(r, f, n_2) = \int e(h(y, r, f, n_2)) H(y, r, f, n_2)\, dy \quad (3.4.2)$$

with

$$h(y, r, f, n_2) = -\frac{r}{2\pi} \log y + \frac{x}{\pi} \log\left(\sqrt{yf} + \sqrt{n_2}\right)$$

$$-\frac{x}{\pi}\log\left(\sqrt{(y+1)f}+\sqrt{n_2+f}\right)-\frac{r}{\pi}\log\left(1+\sqrt{1+y^{-1}}\right)+\frac{r}{\pi}\log 2,$$

and the non-oscillating part of the integrand is

$$H(y,r,f,n_2)\sim y^{-1/2}(y+1)^{k-1}v(yf-n_2)(1+\sqrt{1+y^{-1}})^{1-2k}2^{2k-1}.$$

The integral is taken over the range

$$\left[\frac{(1-b/2)M+n_2}{f},\frac{(1+3b/2)M+n_2}{f}\right]$$

in case 1 and

$$\left[\frac{-5bM/4+n_2}{f},\frac{5bM/4+n_2}{f}\right]$$

in case 2. Hence in both cases $y\asymp\frac{M}{f}$ and the length of the range of integration is $\leq\frac{5bM}{2f}$. Therefore $H(y,r,f,n_2)\asymp(M/f)^{k-3/2}$. For short, we shall write the functions h and H with only one variable y. Moreover, we follow the proof for the case $+r$, the opposite sign case being similar but easier.

Next we replace the n_1-sum in (3.3.7) by the f-sum with $K^{\sqrt{\theta}}\ll f\leq F_1$, and obtain an upper bound for (3.3.7) of

$$N^{-1}\int\sum_{K^{\sqrt{\theta}}\ll f\leq F_1}\sum_{n_2=-\infty}^{\infty}\overline{a(n_2)}a(n_2+f)\phi(n_2)\phi(n_2+f)f^{-1/2+k}\Bigg(\sum_{j=1}^{\infty}\overline{\rho_j}t_j(f)c_j$$
$$\times\Phi_k(\kappa_j;W)+\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(f)c(r)}{(\pi f)^{ir}\Gamma(\frac{1}{2}-ir)\zeta(1-2ir)}\Phi_k(r;W)\,dr\Bigg)$$
$$\times w(n_2,\eta)w(n_2+f,\eta)\,d\eta. \tag{3.4.3}$$

with

$$w(n,\eta)=\begin{cases}1, & n\leq\eta,\\ 0, & n>\eta,\end{cases} \tag{3.4.4}$$

the characteristic function of the range $(-\infty,\eta]$, truncating the f- and the n_2-sum from above. We shall follow the estimation of the first term

$$N^{-1}\int\sum_{K^{\sqrt{\theta}}\ll f\leq F_1}\sum_{n_2=-\infty}^{\infty}\overline{a(n_2)}a(n_2+f)\phi(n_2)\phi(n_2+f)f^{-1/2+k}\sum_{j=1}^{\infty}\overline{\rho_j}t_j(f)c_j$$
$$\times\Phi_k(\kappa_j;W)w(n_2,\eta)w(n_2+f,\eta)\,d\eta \tag{3.4.5}$$

in (3.4.3), the estimation of the second term being similar.

Since we assumed that $f > 0$, and there is η-dependence only in the characteristic functions, it is easy to perform the integration over η. We arrive at a smooth function of n_2 and f;

$$N^{-1}\int_{3N/4}^{2N} w(n_2,\eta)w(n_2+f,\eta)\,d\eta = 2-\frac{n_2+f}{N}.$$

Clearly this function is of size $\ll 1$ and can be embedded into the weight function $\phi(n_2+f)$, denoting it after the embedding by $\tilde{\phi}(n_2+f)$.

Moreover, by partitioning into dyadic ranges, we notice that it is enough to consider the expression

$$\sum_{f\sim F}\sum_{n_2=-\infty}^{\infty}(...)$$

for any $K^{\sqrt{\theta}} \ll F \ll \sqrt{NM}K^{-1/3+\varepsilon}$ and $F \leq 5N/8$, instead of

$$\sum_{K^{\sqrt{\theta}}\ll f\leq F_1}\sum_{n_2=-\infty}^{\infty}(...).$$

By Cauchy's inequality and estimate (1.2.30) we finally arrive at the bound for (3.4.5) of

$$K^{\varepsilon}\Bigg(\sum_{j=1}^{\infty}\alpha_j\Bigg|\sum_{f\sim F}\sum_{n_2=-\infty}^{\infty}\overline{a(n_2)}a(n_2+f)\phi(n_2)\tilde{\phi}(n_2+f)f^{-1/2+k}\kappa_j^{1/2+\delta}$$
$$\times t_j(f)I(\kappa_j,f,n_2)\Bigg|^2\Bigg)^{1/2}, \tag{3.4.6}$$

with $\delta > 0$ a small constant. For the second term in (3.4.3) we use estimate (1.2.32).

Remark 9. We recall that estimate (1.2.30), used above, follows from Jutila's estimate which corresponds to our Theorem 1 but is taken over a larger interval.

3.4.1 Estimation of the integral $I(r, f, n_2)$

Next we shall estimate the integral I above, defined in (3.4.2), by the saddle-point method and integration by parts.

Using the auxiliary function

$$\omega(t) = \frac{\sqrt{\frac{f}{y+t}}}{\sqrt{(y+t)f}+\sqrt{n_2+tf}},$$

we see that

$$h'(y) = -\frac{r}{2\pi y} + \frac{x}{2\pi}(\omega(0) - \omega(1)) + \frac{r}{2\pi y\sqrt{1+y}(\sqrt{y} + \sqrt{y+1})}. \qquad (3.4.7)$$

It can easily be seen that $\omega(t)$ may be written in the form

$$\frac{f - \sqrt{\frac{f(n_2+tf)}{y+t}}}{yf - n_2},$$

and hence

$$h'(y) = -\frac{r}{2\pi y} + \frac{x\sqrt{f}}{2\pi\left((y+1)\sqrt{n_2 y} + y\sqrt{(n_2+f)(y+1)}\right)}$$
$$+\frac{r}{2\pi y(\sqrt{y(y+1)} + y + 1)}. \qquad (3.4.8)$$

Since

$$\frac{1}{\sqrt{y(y+1)} + y + 1} < 1$$

for all $y > 0$, we notice that, in order to have $h'(y) = 0$, we at least have to have r positive. This observation is useful with the second term in (3.4.3) and with the opposite sign case $r \to -r$ in (3.4.1).

All in all, from the saddle-point condition $h'(y) = 0$ it follows that

$$r > 0 \;\text{ and }\; y \asymp \frac{x^2 f}{r^2 n_2}.$$

Therefore if the integral $I(r, f, n_2)$ has a saddle point $y_0(r, f, n_2)$, it is both of size

$$\asymp \frac{x^2 f}{r^2 n_2} \;\text{ and }\; \asymp \frac{M}{F}.$$

Hence in this case we also obtain the condition

$$r \asymp \frac{KF}{\sqrt{NM}},$$

from which it follows that

$$K^{\sqrt{\theta}-\theta/2} \ll r \ll K^{2/3+\varepsilon}.$$

For short, we use the notation

$$L = \frac{KF}{\sqrt{NM}}.$$

We obtain a formula for $h''(y_0)$ by differentiating (3.4.8) and then substituting an expression for $r/(2\pi y_0^2)$ obtained from the equation $h'(y_0) = 0$:

$$h''(y_0) = \frac{-x\sqrt{f}}{4\pi y_0}\left((y_0+1)\sqrt{n_2 y_0} + y_0\sqrt{(n_2+f)(y_0+1)}\right)^{-2}$$
$$\times\left((y_0-1)\sqrt{n_2 y_0} + y_0^2\sqrt{\frac{n_2+f}{y_0+1}}\right) - \frac{r}{4\pi y_0}\left(y_0+1+\sqrt{y_0(y_0+1)}\right)^{-2}$$
$$\times\left(2+\frac{2y_0+1}{\sqrt{y_0(y_0+1)}}\right) \asymp \frac{LF^2}{M^2}. \tag{3.4.9}$$

Hence by the mean value theorem $h''(y) \asymp \frac{LF^2}{M^2}$ for all y on the range of integration, if we now fix the constant b, introduced in the beginning of Section 3.2, suitably.

We gather relevant estimates for the derivatives of the saddle-point $y_0(r, f, n_2)$ and the function $h(y_0(r, f, n_2))$ in the following lemma.

Lemma 3.1. *For all* $\nu \geq 1$

$$\frac{\partial}{\partial r}h(y_0) \asymp \log y_0, \quad \frac{\partial^2}{\partial n_2 \partial r}h(y_0) \ll N^{-1},$$
$$\frac{\partial^2 y_0}{\partial n_2 \partial r} \ll \frac{y_0}{LN}, \quad \frac{\partial^\nu y_0}{\partial r^\nu} \ll_\nu y_0 L^{-\nu}, \quad \frac{\partial^\nu y_0}{\partial n_2^\nu} \ll_\nu y_0 N^{-\nu},$$
$$\frac{\partial^\nu y_0}{\partial f^\nu} \ll_\nu y_0 F^{-\nu} \quad \textit{and} \quad \frac{\partial y_0}{\partial n_2} \asymp \frac{y_0}{N}.$$

These estimates can be verified straightforwardly by repeated use of the equation $h'(y_0) = 0$, the formula for the differentiation of products and the following equation which is sometimes called Faá di Bruno's formula: For all integers $\nu \geq 1$ and $f,\ g \in C^\nu$

$$\frac{d^\nu}{dx^\nu}f(g(x)) = \sum \frac{\nu!}{l_1! l_2! \cdots l_\nu!}(D^l f)(g(x))\left(\frac{1}{1!}\frac{dg}{dx}\right)^{l_1}\left(\frac{1}{2!}\frac{d^2 g}{dx^2}\right)^{l_2}$$
$$\times \cdots \times \left(\frac{1}{\nu!}\frac{d^\nu g}{dx^\nu}\right)^{l_\nu}, \tag{3.4.10}$$

where the summation is taken over all $1l_1+2l_2+\ldots+\nu l_\nu = \nu$, $l_1+l_2+\ldots+l_\nu = l$.

When r is not of the same order as L, the integral does not have a saddle-point, and by repeated use of (3.4.10) and differentiation of products we have

$$\frac{\partial^\nu}{\partial y^\nu}H(y) \ll_\nu \left(\frac{M}{F}\right)^{k-3/2}\left(\frac{M}{F}\right)^{-\nu} \tag{3.4.11}$$

and

$$h'(y) \gg (|r| + L)\frac{F}{M}.$$

Therefore by Lemma 1.8 we obtain

$$I(r, f, n_2) \ll \begin{cases} K^a |r|^{-P}, & |r| \gg L, \\ K^{-P}, & |r| \ll L, \end{cases} \tag{3.4.12}$$

for an arbitrarily large constant $P \geq 0$ and $a > 0$ some fixed constant, and the contribution of these terms to the upper bound (3.4.6) is negligibly small.

In the sequel we thus assume that $r \asymp L$, and by the saddle-point method 1.10 we have that

$$I(r, f, n_2) \sim H(y_0)(-h''(y_0))^{-1/2} e(h(y_0)) e^{-i\pi/4}.$$

3.5 Sobolev's lemma and the spectral large sieve revisited

In the previous section, we obtained the spectral sum (3.4.6), which is reminiscent of the sum in (3.2.8). Again we apply both Sobolev's lemma and the spectral large sieve in similar manners as above in Section 3.3. We separate the variables f and κ_j in the term $I(\kappa_j, f, n_2)$, this time replacing κ_j by a continuous variable τ, and arrive at the following bound for (3.4.6):

$$K^{\varepsilon}F^{k-1/2}L^{1/2}\Theta^{-1/2}(L\Theta+F)^{1/2}\Bigg(\int_{\tau\asymp L}\sum_{f\sim F}\Bigg[\Bigg|\sum_{n_2=-\infty}^{\infty}\overline{a(n_2)}a(n_2+f)$$

$$\times\phi(n_2)\tilde{\phi}(n_2+f)B(y_0)e(h(y_0))\Bigg|^2+\Theta^2\Bigg|\sum_{n_2=-\infty}^{\infty}\overline{a(n_2)}a(n_2+f)$$

$$\times\phi(n_2)\tilde{\phi}(n_2+f)\frac{\partial}{\partial\tau}\Big(B(y_0)e(h(y_0))\Big)\Bigg|^2\Bigg]\,d\tau\Bigg)^{1/2}$$

with

$$B(y_0)=B(\tau,f,n_2,y_0(\tau,f,n_2))=H(y_0)(h''(y_0))^{-1/2}\asymp L^{-1/2}\left(\frac{M}{F}\right)^{k-1/2}.$$

The quantity Θ is the length of each segment in which the factor $B(y_0)e(h(y_0))$ remains essentially stationary when τ runs over it.

Remark 10. As mentioned above, the second term in (3.4.3) can be treated essentially in the same manner as the first term. Notice that instead of the spectral large sieve we use Lemma 1.12 above. This way we achieve the same upper bound for the second term as for the first term, except that $L\Theta$ is replaced by Θ^2.

In the same spirit as in Lemma 3.1, we again gather some estimates for the various derivatives of the function $B(y_0)$. By (3.4.9), formula (3.4.10), differentiation of products and Lemma 3.1, we obtain the following lemma.

Lemma 3.2. *For all* $\nu \geq 1$

$$\frac{\partial^\nu}{\partial\tau^\nu}B(y_0)\ll_\nu |B(y_0)|L^{-\nu},\quad \frac{\partial^2}{\partial n_2\partial\tau}B(y_0)\ll|B(y_0)|(NL)^{-1},$$

$$\frac{\partial^\nu}{\partial n_2^\nu}B(y_0)\ll_\nu |B(y_0)|N^{-\nu}\quad \textit{and}\quad \frac{\partial^\nu}{\partial f^\nu}B(y_0)\ll_\nu|B(y_0)|F^{-\nu}.$$

Now by Lemmas 3.1 and 3.2

$$\frac{\partial}{\partial \tau}\Big(B(y_0)e(h(y_0))\Big) \ll |B(y_0)e(h(y_0))|\, K^{\varepsilon}, \tag{3.5.1}$$

so it is enough to choose $\Theta = 1$ and both the sums $L\Theta + F$ and $\Theta^2 + F$ are always dominated by L, up to a factor $K^{\theta/2}$.

By the condition to (3.3.7) in the end of Section 3.3 we see that what we have left is to prove that

$$\int_{\tau \asymp L} \sum_{f \sim F} \Bigg[\Bigg| \sum_{n_2=-\infty}^{\infty} \overline{a(n_2)} a(n_2+f)\phi(n_2)\tilde{\phi}(n_2+f)B(y_0)e(h(y_0)) \Bigg|^2$$

$$+ \Bigg| \sum_{n_2=-\infty}^{\infty} \overline{a(n_2)} a(n_2+f)\phi(n_2)\tilde{\phi}(n_2+f)\frac{\partial}{\partial \tau}\Big(B(y_0)e(h(y_0))\Big) \Bigg|^2 \Bigg]\, d\tau$$

is of size $\ll K^{\varepsilon}(NM)^{2k}F^{-1-2k}$. Since by Lemmas 3.1 and 3.2

$$\frac{\partial}{\partial n_2}\Bigg(B(y_0)^{-1}e(-h(y_0))\frac{\partial}{\partial \tau}\Big(B(y_0)e(h(y_0))\Big)\Bigg) = -B(y_0)^{-2}\frac{\partial}{\partial n_2}B(y_0)$$

$$\times \frac{\partial}{\partial \tau}B(y_0) + B(y_0)^{-1}\frac{\partial^2}{\partial n_2 \partial \tau}B(y_0) + 2\pi i \frac{\partial^2}{\partial n_2 \partial \tau}h(y_0) \ll N^{-1},$$

by summation by parts we end up with the following condition to prove:

$$\int_{\tau \asymp L} \sum_{f \sim F} \Bigg| \sum_{3N/4 \le n_2 \le 2N} \overline{a(n_2)} a(n_2+f)\phi(n_2)\tilde{\phi}(n_2+f)B(y_0)e(h(y_0)) \Bigg|^2 d\tau$$

$$+ N^{-1}\int_{\tau \asymp L} \sum_{f \sim F} \int_{3N/4}^{2N} \Bigg| \sum_{3N/4 \le n_2 \le \eta} \overline{a(n_2)} a(n_2+f)\phi(n_2)\tilde{\phi}(n_2+f)B(y_0)$$

$$\times e(h(y_0)) \Bigg|^2 d\eta\, d\tau \ll K^{\varepsilon}(NM)^{2k}F^{-1-2k}. \tag{3.5.2}$$

We shall present the details of the proof of the estimate of the second term on the left hand side of (3.5.2), this being more involved than the estimation of the first term.

We insert a real-valued smooth weight functions $q(f)$ and $\upsilon(\tau)$ to the f-sum and τ-integral. We let q be compactly supported on $[F/2, 5F/2]$, $q(f) = 1$ on the interval $[F, 2F]$ and $q^{(\nu)} \ll_{\nu} F^{-\nu}$ for each $\nu \ge 0$. We set υ to be compactly supported on $[AL/2, 3BL/2]$, $\upsilon(\tau) = 1$ on the interval

$[AL, BL]$ for some suitable constants A and B, and $\upsilon^{(\nu)} \ll_\nu L^{-\nu}$ for each $\nu \geq 0$. Multiplying out the absolute values around the n_2-sum we obtain

$$N^{-1} \int_{-\infty}^{\infty} \sum_{f=-\infty}^{\infty} \int_{3N/4}^{2N} \sum_{m_1} \sum_{m_2} \overline{a(m_1)} a(m_2) a(m_1+f) \overline{a(m_2+f)} \phi(m_1) \tilde{\phi}(m_1+f)$$

$$\times \phi(m_2) \tilde{\phi}(m_2+f) q(f) \upsilon(\tau) B_1(y_{01}) B_2(y_{02}) e(h_1(y_{01}) - h_2(y_{02}))\, d\eta\, d\tau \quad (3.5.3)$$

with the abbreviations

$$y_{0i} = y_0(\tau, f, m_i), \quad i = 1, 2,$$

$$h_i(y_{0i}) = h(y_{0i}, \tau, f, m_i) \text{ and } B_i(y_{0i}) = H(y_{0i}, \tau, f, m_i)(h_i''(y_{0i}))^{-1/2}.$$

Finally we substitute the new variable $\mu = m_1 - m_2 \leq \frac{5N}{4}$. In the case when $\mu = 0$, the bound (3.5.2) trivially holds, and again we may assume that $\mu > 0$, the opposite sign case being symmetric apart from complex conjugation. We insert a smooth weight function $w_R(\mu)$ into the μ-sum for the range $[1, 5N/4]$ using the method introduced in (1.2.21).

Next we prepare ourselves to truncate the size of μ as for $f = n_1 - n_2$ above. We start by estimating some derivatives. By Lemma 3.1 and either an argument similar to that used in (3.4.7) and the second mean value theorem, or by writing

$$h_1(y_{01}) - h_2(y_{02}) = \int_{m_2}^{m_2+\mu} \frac{\partial}{\partial n_2} h(y_0)\, dn_2,$$

we obtain the following lemma.

Lemma 3.3. *For all $\nu \geq 1$*

$$\frac{\partial}{\partial \tau}(h_1(y_{01}) - h_2(y_{02})) \asymp \frac{\mu}{N}, \quad \frac{\partial^\nu}{\partial \tau^\nu}(h_1(y_{01}) - h_2(y_{02})) \ll_\nu \frac{\mu L}{N} L^{-\nu},$$

$$\frac{\partial^\nu}{\partial m_2^\nu}(h_1(y_{01}) - h_2(y_{02})) \ll_\nu \frac{\mu L}{N} N^{-\nu}, \quad \frac{\partial^\nu}{\partial f^\nu}(h_1(y_{01}) - h_2(y_{02})) \ll_\nu \frac{\mu L}{N} F^{-\nu},$$

$$\frac{\partial^\nu}{\partial \mu^\nu}(h_1(y_{01}) - h_2(y_{02})) = \frac{\partial^\nu}{\partial \mu^\nu} h_1(y_{01}) \ll_\nu LN^{-\nu}.$$

Now by Lemmas 3.2 and 3.3 we are able to invoke integration by parts in the spirit of Remark 1 and obtain the estimate

$$\int_{-\infty}^{\infty} \upsilon(\tau) B_1(y_{01}) B_2(y_{02}) e(h_1(y_{01}) - h_2(y_{02}))\, d\tau \ll \left(\frac{M}{F}\right)^{2k-1} \left(\frac{\mu L}{N}\right)^{-P}$$

for any $P \geq 0$. Therefore if we have, say, $\mu \gg NL^{-1}K^\theta$, then the contribution of the integral is negligibly small. We may thus assume that

$$\mu \ll \frac{N}{L} K^\theta.$$

Note that the first shift f was of size $K^{2/3+\varepsilon}$, but the shift μ has essentially been reduced to size $K^{1/3+\varepsilon}$.

We thus arrive at the bound for (3.5.3) of

$$N^{-1}\int\sum_{f}\int\sum_{m_2=-\infty}^{\infty}\sum_{\substack{0\le\imath<\log_2 5N/3\\ R=2^\imath\ll NL^{-1}K^\theta}}\sum_{3R/4\le\mu\le 2R} a(m_2)\overline{a(m_2+\mu)a(m_2+f)}$$

$$\times a(m_2+\mu+f)\phi(m_2)\tilde{\phi}(m_2+f)\phi(m_2+\mu)\tilde{\phi}(m_2+f+\mu)q(f)\upsilon(\tau)w_R(\mu)$$

$$\times B_1(y_{01})B_2(y_{02})e(h_1(y_{01})-h_2(y_{02}))w(m_2,\eta)w(m_2+\mu,\eta)\,d\eta\,d\tau$$

with the characteristic function w as in (3.4.4). Now

$$N^{-1}\int_{3N/4}^{2N} w(m_2,\eta)w(m_2+\mu,\eta)\,d\eta = 2-\frac{m_2+\mu}{N}\ll 1,$$

and this term can be embedded into the weight function $\phi(m_2+\mu)$, denoting it again by $\tilde{\phi}(m_2+\mu)$ after the embedding.

After this we again fix the variable $\tau\asymp L$ until the end of our argument, and then integrate trivially over τ. Therefore we see that finally for our Theorem 1 it is enough to prove that

$$\sum_{m_2=-\infty}^{\infty}\sum_{f=-\infty}^{\infty}\sum_{\mu=-\infty}^{\infty} a(m_2)\overline{a(m_2+\mu)a(m_2+f)}a(m_2+\mu+f)$$
$$\times\phi(m_2)\tilde{\phi}(m_2+f)\tilde{\phi}(m_2+\mu)\tilde{\phi}(m_2+f+\mu)q(f)w_R(\mu)$$
$$\times B_1(y_{01})B_2(y_{02})e(h_1(y_{01})-h_2(y_{02}))$$
$$\ll K^{-1+\varepsilon}(NM)^{2k+1/2}F^{-2-2k} \tag{3.5.4}$$

for any $1\le R\ll NL^{-1}K^\theta$. Since $NL^{-1}K^\theta\ll N$, we may assume that $R\le 5N/8$, similarly for the upper bound we have for F.

Now (3.5.4) reminds us of the bound (3.3.7). However, instead of applying Lemma 1.16 this time to the f-sum, it proves to be simpler to invoke a result by Jutila, Lemma 3.4, which rests on his formula for

$$\sum_{n\le x} a(n)\overline{a(n+m)}$$

in [21]. For this reason we also inserted the weight function into the μ-sum immediately unlike in the situation of the f-sum above. At this point we also notice that if $F\gg N^{1/2+3\sqrt{\theta}/2}$, then $R\ll N^{1/2-3\sqrt{\theta}/2}K^{3\theta/2}$ and because $N\gg F\gg K^{\sqrt{\theta}}$, we have $R\ll N^{1/2}$. Therefore either $R\ll N^{1/2}$ or $F\ll N^{1/2+\varepsilon}$, which turns out to be a crucial fact in applying Jutila's result.

Remark 11. The odd choice of $K^{\sqrt{\theta}}$ for a lower bound of the f-sum finally earns its explanation above, when we hope to gain $R\ll N^{1/2+\varepsilon}$.

3.6 Separation of variables in a double sum

We recall from the previous section that we always have either $R \ll N^{1/2}$ or $F \ll N^{1/2+\varepsilon}$. We shall first assume that $R \ll N^{1/2}$ and present the details for this case, and then comment on the case when $F \ll N^{1/2+\varepsilon}$.

We start by reformulating the left hand side of (3.5.4) to give sums containing just two Fourier coefficients of the holomorphic cusp form under consideration. First we write the left hand side of (3.5.4) as

$$\sum_{\mu=-\infty}^{\infty} S_\mu w_R(\mu), \tag{3.6.1}$$

where

$$S_\mu = \sum_{m_2=-\infty}^{\infty} \sum_{f=-\infty}^{\infty} a(m_2)\overline{a(m_2+\mu)a(m_2+f)}a(m_2+\mu+f)W_\mu(m_2,f) \tag{3.6.2}$$

with

$$W_\mu(m_2,f) = \phi(m_2)\tilde{\phi}(m_2+f)\tilde{\phi}(m_2+\mu)\tilde{\phi}(m_2+f+\mu)q(f)$$
$$\times B_1(y_{01})B_2(y_{02})e(h_1(y_{01})-h_2(y_{02})).$$

Further we introduce a new parameter $\lambda \in \mathbb{N}$ and perform the summation on both sides of (3.6.2) with respect to λ over a range, say, $[F/2, F]$. We then replace m_2 and f on the right hand side of the resulting equation by $m_2+\lambda$ and $f-\lambda$, respectively, obtaining

$$\sum_{F/2\le\lambda\le F} S_\mu = \sum_{F/2\le\lambda\le F} \sum_{m_2=-\infty}^{\infty} \sum_{f=-\infty}^{\infty} a(m_2+\lambda)\overline{a(m_2+\lambda+\mu)a(m_2+f)}$$
$$\times a(m_2+\mu+f)W_\mu(m_2+\lambda, f-\lambda).$$

Therefore

$$S_\mu \ll F^{-1} \sum_{F/2\le\lambda\le F} \sum_{N/8\le m_2\le 2N} \sum_{F\le f\le 7F/2} a(m_2+\lambda)$$
$$\times\overline{a(m_2+\lambda+\mu)a(m_2+f)}a(m_2+\mu+f)W_\mu(m_2+\lambda, f-\lambda).$$

To simplify the sum above we represent $W_\mu(m_2+\lambda, f-\lambda)$ by its Fourier inversion

$$\hat{W}_\mu(\alpha,\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\mu(x,y)e^{-i(\alpha x+\beta y)}\,dx\,dy$$

as an integral

$$W_\mu(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{W}_\mu(\alpha,\beta)e^{i(\alpha x+\beta y)}\,d\alpha\,d\beta.$$

Hence we are able to separate the variables f and λ, to give

$$S_\mu \ll F^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{W}_\mu(\alpha,\beta)| \sum_{m_2} \left| e^{i\alpha m_2} \left(\sum_f \overline{a(m_2+f)} a(m_2+\mu+f) e^{i\beta f} \right) \right.$$

$$\left. \times \left(\sum_\lambda a(m_2+\lambda) \overline{a(m_2+\lambda+\mu)} e^{i\lambda(\alpha-\beta)} \right) \right| d\alpha\, d\beta$$

$$\ll F^{-1} \int \int |\hat{W}_\mu(\alpha,\beta)| \sum_{m_2} \left(\left| \sum_f \overline{a(m_2+f)} a(m_2+\mu+f) e^{i\beta f} \right|^2 \right.$$

$$\left. + \left| \sum_\lambda a(m_2+\lambda) \overline{a(m_2+\lambda+\mu)} e^{i\lambda(\alpha-\beta)} \right|^2 \right) d\alpha\, d\beta.$$

To estimate the Fourier inversion of W_μ, we notice that trivially

$$\hat{W}_\mu(\alpha,\beta) \ll L^{-1} NF \left(\frac{M}{F} \right)^{2k-1}.$$

On the other hand, by Lemmas 3.2 and 3.3, if $|\alpha| \gg \frac{\mu L}{N^2}$ with a sufficiently large implied constant, then applying integration by parts in the spirit of Remark 1 to the x-integral and estimating the y-integral trivially, we arrive at an upper bound

$$\hat{W}_\mu(\alpha,\beta) \ll A(\alpha) = L^{-1} NF \left(\frac{M}{F} \right)^{2k-1} (N|\alpha|)^{-P}.$$

This holds, in particularly, if $|\alpha| \gg N^{-1} K^\theta$. Similarly, by estimating the y-integral with integration by parts and the x-integral trivially we have

$$\hat{W}_\mu(\alpha,\beta) \ll B(\beta) = L^{-1} NF \left(\frac{M}{F} \right)^{2k-1} (F|\beta|)^{-P},$$

in the case when $|\beta| \gg F^{-1} K^\theta$.

Therefore, if either of the conditions $|\alpha| \gg N^{-1} K^\theta$ or $|\beta| \gg F^{-1} K^\theta$ applies, the contribution of the double integral over α and β to (3.6.1) is negligibly small. Notice that if both of the above conditions hold true, then we use the observation that $\hat{W}_\mu(\alpha,\beta) \ll \min(A(\alpha), B(\beta)) \ll \sqrt{A(\alpha)B(\beta)}$.

On the other hand, when $|\alpha| \ll N^{-1} K^\theta$ and $|\beta| \ll F^{-1} K^\theta$, by summation by parts it suffices to consider

$$\left| \sum_{F \le f \le \vartheta} \overline{a(m_2+f)} a(m_2+\mu+f) \right|^2$$

for all $F \leq \vartheta \leq 7F/2$ instead of

$$\Bigg| \sum_{F \leq f \leq 7F/2} \overline{a(m_2 + f)} a(m_2 + \mu + f) e^{i\beta f} \Bigg|^2,$$

and similarly for the λ-sum. So it remains for us to prove that

$$\sum_{\mu \asymp R} \sum_{m_2 \asymp N} \Bigg| \sum_{f \asymp F} a(m_2 + f) \overline{a(m_2 + \mu + f)} \Bigg|^2 \ll K^\varepsilon N^{2k} M F^{-1}.$$

For this purpose we introduce the following lemma, based on Jutila's "almost" explicit formula for a convolution sum in [21].

Lemma 3.4 (Convolutions over short intervals in mean). *Let* $1 \leq L \leq N$ *and* $P \ll N^{1/2+\varepsilon}$. *Then*

$$S = \sum_{p \asymp P} \sum_{n \asymp N} \Bigg| \sum_{l \asymp L} a(n+l) \overline{a(n+l+p)} \Bigg|^2 \ll N^{2k+\varepsilon} P.$$

Proof. Let $P \ll N^{1/2+\delta}$ for some arbitrary $\delta > 0$. Theorem 3 in [21] with $T = N^{1/2-2\delta}$ tells us that

$$\sum_{l \asymp L} a(n+l) \overline{a(n+l+p)} = \breve{A}(n + BL, p) - \breve{A}(n + AL, p) + \mathcal{O}(N^{k-1/2+2\delta+\varepsilon}),$$

with A and B some positive constants,

$$\breve{A}(x,p) = (-1)^{k/2} (4\pi x)^{k-1/2} \mathrm{Re} \Bigg\{ - ie \left(\frac{1}{8}\right) \sum_{\kappa_j \leq T} c_j \overline{\rho_j} t_j(p) \kappa_j^{-k-1/2} \left(\frac{4x}{p}\right)^{i\kappa_j}$$

$$\times \sum_{\nu=0}^{\nu_0} \frac{(-i\kappa_j p/2x)^\nu}{\nu!} (1 + \mathcal{O}(\kappa_j^{-1})) \Bigg\}$$

and ν_0 a sufficiently large positive integer. Now $\mathcal{O}(N^{k-1/2+2\delta+\varepsilon})$ produces a term that stays under the desired bound, and the terms involving the factor $\mathcal{O}(\kappa_j^{-1})$ contribute to S by

$$N^{2k} \sum_{p \asymp P} \Bigg| \sum_{\kappa_j \leq T} |c_j \overline{\rho_j} t_j(p)| \kappa_j^{-k-3/2} \Bigg|^2.$$

By Cauchy's inequality, the estimations (1.2.25) and (1.2.30) and Lemma 1.15 the above value is $\ll N^{2k} P$.

We next introduce a real-valued smooth weight function $\varphi(n)$ to the n-sum. We set $\varphi(n)$ to be compactly supported on an interval $[CN/2, 3DN/2]$

and $\varphi(n) = 1$ on $[CN, DN]$ for some positive constants C and D. As usual, $\varphi^{(\nu)} \ll_\nu N^{-\nu}$ for each $\nu \geq 0$. For $c = A, B$ we thus obtain

$$\sum_{n \asymp N} |\breve{A}(n + cL, p)|^2 \ll \sum_{n=1}^{\infty} |\breve{A}(n + cL, p)|^2 \varphi(n)$$

$$\ll N^{2k-1} \sum_{\kappa_{j1}, \kappa_{j2} \leq T} |c_{j1} c_{j2} \rho_{j1} \rho_{j2} t_{j1}(p) t_{j2}(p)| (\kappa_{j1}\kappa_{j2})^{-k-1/2} \Bigg| \sum_n \varphi(n)$$

$$\times (n + cL)^{i(\kappa_{j1} - \kappa_{j2})} \sum_{\nu_1=0}^{\nu_0} \frac{(-i\kappa_{j1} p/2(n + cL))^{\nu_1}}{\nu_1!} \sum_{\nu_2=0}^{\nu_0} \frac{(i\kappa_{j2} p/2(n + cL))^{\nu_2}}{\nu_2!} \Bigg|. \tag{3.6.3}$$

First we let $|\kappa_{j1} - \kappa_{j2}| \geq N^\varepsilon$ and prepare ourselves to use Lemma 1.7. If we use the notation

$$w_1(n) = \varphi(n) \sum_{\nu_1=0}^{\nu_0} \frac{(-i\kappa_{j1} p/2(n + cL))^{\nu_1}}{\nu_1!} \sum_{\nu_2=0}^{\nu_0} \frac{(i\kappa_{j2} p/2(n + cL))^{\nu_2}}{\nu_2!}$$

and

$$f_1(n) = \frac{1}{2\pi}(\kappa_{j1} - \kappa_{j2}) \log(n + cL),$$

then clearly $f_1'(t)$ is monotonic, $|f_1'(t)| \ll N^{-1/2-2\delta}$ and supp $w_1 = [CN/2, 3DN/2]$. Therefore the n-sum in (3.6.3) is

$$= \int_{CN/2}^{3DN/2} w_1(t) e(f_1(t))\, dt + \mathcal{O}\left(\left\|w_1'\right\|_1\right). \tag{3.6.4}$$

Now clearly

$$\frac{\partial^\nu}{\partial t^\nu} w_1(t) \ll_\nu N^{-\nu} \text{ and } f_1'(t) \gg N^{-1+\varepsilon},$$

so by Lemma 1.8 the integral in (3.6.4) is negligibly small, and trivially

$$\left\|w_1'\right\|_1 \ll 1.$$

Therefore the contribution of these terms to S is

$$\ll N^{2k-1} \sum_{p \asymp P} \left(\sum_{\kappa_j \leq T} |c_j \rho_j t_j(p)| \kappa_j^{-k-1/2} \right)^2 \ll N^{2k-1/2+\varepsilon} P.$$

Finally, we let $|\kappa_{j1} - \kappa_{j2}| \leq N^\varepsilon$. In this case we notice first that

$$|c_{j1} c_{j2} \rho_{j1} \rho_{j2} t_{j1}(p) t_{j2}(p)| (\kappa_{j1}\kappa_{j2})^{-k-1/2} \leq \sum_{\nu=1}^{2} |c_{j\nu} \rho_{j\nu} t_{j\nu}(p)|^2 \kappa_{j\nu}^{-2k-1}.$$

Therefore, estimating the n-sum in (3.6.3) trivially and using the estimations (1.1.10) and (1.2.26) we obtain the bound

$$N^{2k}P\sum_{\kappa_j\le T}|c_j|^2\alpha_j e^{\pi\kappa_j}\kappa_j^{-2k-1+\varepsilon}\sum_{\kappa_j-N^\varepsilon\le\kappa\le\kappa_j+N^\varepsilon}1\ll N^{2k+\varepsilon}P$$

thus completing the proof of Lemma 3.4. □

Remark 12. In Lemma 6 of his paper [23], Jutila proves a similar result for Hecke eigenvalues of Maass forms: For $N, P\ge 1$ and $1\le L\le N$, we have

$$\sum_{0\le p\le P}\sum_{1\le n\le N}\left|\sum_{1\le l\le L}t(n+l)t(n+p+l)\right|^2\ll(N+P)^{1+\varepsilon}NL.$$

In order to use Lemma 3.4 we require that in the outermost p-sum $P\ll N^{1/2+\varepsilon}$. Now if we assume that $R\ll N^{1/2}$, we conclude with the desired result.

On the other hand, in the case when $R\gg N^{1/2+\varepsilon}$ and $F\ll N^{1/2+\varepsilon}$ we change the roles of the μ- and the f-sum in the beginning of this section. This time we write the left hand side of (3.5.4) as

$$\sum_{f=-\infty}^{\infty}S_f q(f), \tag{3.6.5}$$

where

$$S_f=\sum_{m_2=-\infty}^{\infty}\sum_{\mu=-\infty}^{\infty}a(m_2)\overline{a(m_2+\mu)a(m_2+f)}a(m_2+\mu+f)W_f(m_2,\mu)$$

with

$$W_f(m_2,\mu)=\phi(m_2)\tilde{\phi}(m_2+f)\tilde{\phi}(m_2+\mu)\tilde{\phi}(m_2+f+\mu)w_R(\mu)$$
$$\times B_1(y_{01})B_2(y_{02})e(h_1(y_{01})-h_2(y_{02})).$$

Following the steps for estimating S_μ we obtain in a similar way

$$S_f\ll R^{-1}\sum_{R/2\le\lambda\le R}\sum_{N/8\le m_2\le 2N}\sum_{5R/4\le\mu\le 3R}a(m_2+\lambda)\overline{a(m_2+\mu)}$$

$$\times\overline{a(m_2+\lambda+f)}a(m_2+\mu+f)W_f(m_2+\lambda,\mu-\lambda)\ll R^{-1}\int\int|\hat{W}_f(\alpha,\beta)|$$

$$\times \sum_{m_2} \left(\left| \sum_{\mu} \overline{a(m_2+\mu)} a(m_2+\mu+f) e^{i\beta\mu} \right|^2 + \left| \sum_{\lambda} a(m_2+\lambda) \overline{a(m_2+\lambda+f)} \right. \right.$$

$$\left. \left. \times e^{i\lambda(\alpha-\beta)} \right|^2 \right) d\alpha\, d\beta$$

with the Fourier inversion

$$\hat{W}_f(\alpha,\beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(x,y) e^{-i(\alpha x+\beta y)}\, dx\, dy.$$

This time, trivially

$$\hat{W}_f(\alpha,\beta) \ll L^{-1} N R \left(\frac{M}{F}\right)^{2k-1}.$$

Again, if $|\alpha| \gg N^{-1}K^{\theta} \gg \frac{\mu L}{N^2}$ with a sufficiently large implied constant, then by integration by parts we arrive at an upper bound

$$\hat{W}_f(\alpha,\beta) \ll C(\alpha) = L^{-1} N R \left(\frac{M}{F}\right)^{2k-1} (N|\alpha|)^{-P}.$$

Similarly,

$$\hat{W}_f(\alpha,\beta) \ll D(\beta) = L^{-1} N R \left(\frac{M}{F}\right)^{2k-1} (R|\beta|)^{-P},$$

if $|\beta| \gg K^{\theta} R^{-1}$. Again the contribution of the double integral over α and β to (3.6.5) is negligibly small. By summation by parts it suffices to prove that

$$\sum_{f \asymp F} \sum_{m_2 \asymp N} \left| \sum_{\mu \asymp R} a(m_2+\mu) \overline{a(m_2+\mu+f)} \right|^2 \ll K^{\varepsilon} N^{2k} M F^{-2} R.$$

This follows from Lemma 3.4 and our assumptions $R \gg N^{1/2+\varepsilon}$ and $F \ll N^{1/2+\varepsilon}$. Hence our proof of Theorem 1 is complete, up to the case discussed in Remark 8.

Chapter 4

Suggestions for further work

The first task would be to verify and write out the details for the proof of the case discussed in Remark 8.

As suggested in the Introduction, the next step is to prove estimate (0.0.3) and hypothesis (0.0.4) using a technique analogous to the one presented in Chapter 3. In both (0.0.3) and (0.0.4) $K \geq 1$ and $\varepsilon > 0$, and the implied constants depend on ε. In addition to this in (0.0.4) the implied constant also depends on the fixed form $u(z)$. As described in the Introduction, the second estimate would be completely new, although it is expected to hold in light of the result in [21], whereas the first estimate is known, the methods used in the proof being different from the ones in this thesis.

Moreover, as is pointed out in both [21] and [22], with (0.0.4) it would be of interest to make the dependence of the implied constant on $u(z)$ explicit in terms of the related parameter κ; see the remark on p. 626 of [22].

Another question arises from the fact that we have confined ourselves to the case of the full modular group. It might be interesting to extend the above results to the case of a more general group. Further, a study of the more general works by Sarnak [39] and Bernstein and Reznikov [2], [3] may provide us with some additional information on this subject. It also would be interesting to apply these results to bounds for appropriate automorphic L-functions as in Corollary 1.2 of [2] and the corollary on page 4 of [3].

Bibliography

[1] Beals R. : Advanced Mathematical Analysis, Springer-Verlag, Berlin Heidelberg New York, 1973.

[2] Bernstein J., Reznikov A. : Periods, subconvexity of L-functions and representation theory, J. Differ. Geom. **70** (2005), 129-142.

[3] Bernstein J., Reznikov A. : Subconvexity bounds for triple L-functions and representation theory, arXiv:math.NT/0608555.

[4] Deligne P. : La conjecture de Weil, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273-307.

[5] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. : Tables of Integral Transforms, vol. 1, Mc Graw-Hill, New York Toronto London, 1954.

[6] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. : Higher Transcendental Functions, vol. 1, Mc Graw-Hill, New York Toronto London, 1953.

[7] Estermann T. : On Kloosterman's sum, Mathematika **8** (1961), 83-86.

[8] Good A. : Beitraege zur Theorie der Dirichletreihen, die Spitzenformen zugeordnet sind, J. Number Theory **13** (1981), 18-65.

[9] Good A. : Cusp forms and eigenfunctions of the Laplacian, Math. Ann. **255** (1981), 523-548.

[10] Hejhal D. A. : The Selberg Trace Formula for PSL(2,$\mathbb{R}$) II, Lect. Notes in Math., Vol. 1001, Springer-Verlag, Berlin, 1983.

[11] Hoffstein J., Lockhart P. : Coefficients of Maass forms and the Siegel zero, Ann. Math. **140** (1994), 161-181.

[12] Ivić A. : The Riemann Zeta-Function, Theory and Applications, John Wiley & Sons, New York, 1985.

[13] Ivić A. : On sums of Hecke series in short intervals, J. Théor. Nombres Bordx. **13** No. 2 (2001), 453-468.

[14] Iwaniec H. : Fourier coefficients of cusp forms and the Riemann zeta-function, in: Séminaire de Théorie des Nombres 1979-1980, Exp. No. 18, Univ. Bordeaux I, Talence, 1980, pp. 1-36.

[15] Iwaniec H. : Mean values for Fourier coefficients of cusp forms and sums of Kloosterman sums, in: Journées Arithmétiques 1980, London Math. Soc. Lecture Note Ser., 56, Cambridge University Press, Cambridge, 1982, pp. 306-321.

[16] Iwaniec H. : The spectral growth of automorphic L-functions, J. Reine Angew. Math. **428** (1992), 139-159.

[17] Iwaniec H. : Introduction to the Spectral Theory of Automorphic Forms, Revista Matemtica Iberoamericana, Madrid, 1995.

[18] Iwaniec H., Kowalski E. : Analytic Number Theory, Amer. Math. Soc., Providence, Rhode Island, 2004.

[19] Jutila M. : Lectures on a Method in the Theory of Exponential Sums, Tata Institute of Fundamantal Research, Bombay (Lectures on Mathematics and Physics, vol. 80), Springer-Verlag, Berlin Heidelberg New York Tokio, 1987.

[20] Jutila M. : Mean value estimates for exponential sums with applications to L-functions, Acta Arith. **57** (1991), 93-114.

[21] Jutila M. : The additive divisor problem and its analogs for Fourier coefficients of cusp forms. I, Math. Z. **223** (1996), 435-461.

[22] Jutila M. : The additive divisor problem and its analogs for Fourier coefficients of cusp forms. II, Math. Z. **225** (1997), 625-637.

[23] Jutila M. : Convolutions of Fourier coefficients of cusp forms, Publ. Inst. Math., Nouv. Sér. **65** 79 (1999), 31-51.

[24] Jutila M. : The fourth moment of central values of Hecke series, in: Number Theory; Proceedings of the Turku Symposium on Number Theory in Memory of Kustaa Inkeri, May 31-June 4, 1999, de Gruyter, Berlin, 2001, pp. 167-177.

[25] Jutila M. : On spectral large sieve inequalities, Funct. Approximatio, Comment. Math. **28** (2000), 7-18.

[26] Jutila M. : The Mellin transform of the square of Riemann's zeta-function, Period. Math. Hung. **42** 1-2 (2001), 179-190.

[27] Jutila M. : Convolutions of Fourier coefficients of cusp forms and the circle method, in: The conference on L-functions, World Sci. Publ., Hackensack, NJ, 2007, pp. 71-87.

[28] Jutila M., Motohashi Y. : Uniform bound for Hecke L-functions, Acta Math. **195** (2005), 61-115.

[29] Jutila M., Motohashi Y. : Uniform bounds for Rankin-Selberg L-functions, in: Multiple Dirichlet series, automorphic forms, and analytic number theory, Proc. Symp. Pure Math. 75, Amer. Math. Soc., Providence, RI, 2006, pp. 243-256.

[30] Kuznetsov N. V. : Petersson's hypothesis for parabolic forms of weight zero and Linnik's hypothesis. Sums of Kloosterman sums (in Russian), Mat. Sb. **111** 153 No. 3 (1980), 334-383.

[31] Lebedev N. N. : Special Functions and Their Applications, Dover Publications, New York, 1972.

[32] Meurman T. : On exponential sums involving the Fourier coefficients of Maass wave forms, J. Reine Angew. Math. **384** (1988), 192-207.

[33] Montgomery H. L. : Topics in Multiplicative Number Theory, Lecture Notes in Mathematics, Vol. 227, Springer-Verlag, Berlin Heidelberg New York, 1971.

[34] Motohashi Y. : The binary additive divisor problem, Ann. Sci. Éc. Norm. Supér. **27** (1994), 529-572.

[35] Motohashi Y. : The mean square of Hecke L-series attached to holomorphic cusp-forms, RIMS Kyoto Univ. Kokyuroku **886** (1994), 214-227.

[36] Motohashi Y. : Spectral Theory of the Riemann Zeta-Function, Cambridge University Press, Cambridge, 1997.

[37] Olver F. W. J. : Asymptotics and Special Functions, Academic Press, New York London, 1974.

[38] Rankin R. A. : Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions II. The order of Fourier coefficients of integral modular forms, Math. Proc. Camb. Philos. Soc. **35** (1939), 357-372.

[39] Sarnak P. : Integrals of products of eigenfunctions, Int. Math. Res. Not. **6** (1994), 251-260.

[40] Selberg A. : Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist, Arch. Math. Naturvid. **43** (1940), 47-50.

[41] Selberg A. : On the estimation of Fourier coefficients of modular forms, Proc. Symp. Pure Math. **8** (1965), 1-15.

[42] Tahtadjan L. A., Vinogradov A. I. : The zeta-function of the additive divisor problem and spectral decomposition of the automorphic Laplacian (in Russian), Zap. Nauchn. Semin. LOMI **134** (1984), 84-116.

[43] Titchmarsh E. C. : The Theory of the Riemann Zeta-Function, Oxford University Press, London, 1951.

[44] Watson G. N. : A Treatise on the Theory of Bessel Functions, Second Edition, Cambridge University Press, London, 1952.

[45] Whittaker E. T., Watson G. N. : A Course of Modern Analysis, Cambridge University Press, London, 1950.

Turku Centre for Computer Science
TUCS Dissertations

88. **Elena Czeizler**, Intricacies of Word Equations
89. **Marcus Alanen**, A Metamodeling Framework for Software Engineering
90. **Filip Ginter**, Towards Information Extraction in the Biomedical Domain: Methods and Resources
91. **Jarkko Paavola**, Signature Ensembles and Receiver Structures for Oversaturated Synchronous DS-CDMA Systems
92. **Arho Virkki**, The Human Respiratory System: Modelling, Analysis and Control
93. **Olli Luoma**, Efficient Methods for Storing and Querying XML Data with Relational Databases
94. **Dubravka Ilić**, Formal Reasoning about Dependability in Model-Driven Development
95. **Kim Solin**, Abstract Algebra of Program Refinement
96. **Tomi Westerlund**, Time Aware Modelling and Analysis of Systems-on-Chip
97. **Kalle Saari**, On the Frequency and Periodicity of Infinite Words
98. **Tomi Kärki**, Similarity Relations on Words: Relational Codes and Periods
99. **Markus M. Mäkelä**, Essays on Software Product Development: A Strategic Management Viewpoint
100. **Roope Vehkalahti**, Class Field Theoretic Methods in the Design of Lattice Signal Constellations
101. **Anne-Maria Ernvall-Hytönen**, On Short Exponential Sums Involving Fourier Coefficients of Holomorphic Cusp Forms
102. **Chang Li**, Parallelism and Complexity in Gene Assembly
103. **Tapio Pahikkala**, New Kernel Functions and Learning Methods for Text and Data Mining
104. **Denis Shestakov**, Search Interfaces on the Web: Querying and Characterizing
105. **Sampo Pyysalo**, A Dependency Parsing Approach to Biomedical Text Mining
106. **Anna Sell**, Mobile Digital Calendars in Knowledge Work
107. **Dorina Marghescu**, Evaluating Multidimensional Visualization Techniques in Data Mining Tasks
108. **Tero Säntti**, A Co-Processor Approach for Efficient Java Execution in Embedded Systems
109. **Kari Salonen**, Setup Optimization in High-Mix Surface Mount PCB Assembly
110. **Pontus Boström**, Formal Design and Verification of Systems Using Domain-Specific Languages
111. **Camilla J. Hollanti**, Order-Theoretic Mehtods for Space-Time Coding: Symmetric and Asymmetric Designs
112. **Heidi Himmanen**, On Transmission System Design for Wireless Broadcasting
113. **Sébastien Lafond**, Simulation of Embedded Systems for Energy Consumption Estimation
114. **Evgeni Tsivtsivadze**, Learning Preferences with Kernel-Based Methods
115. **Petri Salmela**, On Commutation and Conjugacy of Rational Languages and the Fixed Point Method
116. **Siamak Taati**, Conservation Laws in Cellular Automata
117. **Vladimir Rogojin**, Gene Assembly in Stichotrichous Ciliates: Elementary Operations, Parallelism and Computation
118. **Alexey Dudkov**, Chip and Signature Interleaving in DS CDMA Systems
119. **Janne Savela**, Role of Selected Spectral Attributes in the Perception of Synthetic Vowels
120. **Kristian Nybom**, Low-Density Parity-Check Codes for Wireless Datacast Networks
121. **Johanna Tuominen**, Formal Power Analysis of Systems-on-Chip
122. **Teijo Lehtonen**, On Fault Tolerance Methods for Networks-on-Chip
123. **Eeva Suvitie**, On Inner Products Involving Holomorphic Cusp Forms and Maass Forms

Turku Centre *for* Computer Science

Joukahaisenkatu 3-5 B, 20520 Turku, Finland | www.tucs.fi

University of Turku

- Department of Information Technology
- Department of Mathematics

Åbo Akademi University

- Department of Information Technologies

Turku School of Economics

- Institute of Information Systems Sciences

ISBN 978-952-12-2353-2
ISSN 1239-1883

Eeva Suvitie On Inner Products Involving Holomorphic Cusp Forms and Maass Forms

Eeva Suvitie On Inner Products Involving Holomorphic Cusp Forms and Maass Forms