DISTORTION PROPERTIES OF QUASICONFORMAL MAPS

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In this Licentiate thesis we investigate the absolute ratio δ, j, $\tilde{j}$ and hyperbolic ρ metrics and their relations with each other. Various growth estimates are given for quasiconformal mpas both in plane and space. Some Hölder constants were refined with respect $\delta, j\ \tilde{j}$ metrics. Some new results regarding the Hölder continuity of quasiconformal and quasiregular mapping of unit ball with respect to Euclidean and hyperbolic metrics are given, which were obtained by many authors in 1980's.

Applications are given to the study of metric space, quasiconformal and quasiregular maps in the plane and as well as in the space.

Keywords: quasiconformal and quasiregular maps, Hölder continuity of unit ball, distortion fucntions, comparison of constants.

Contents

1. **Introduction**

This Licentiate thesis has been written under the supervision of Prof. Matti Vuorinen at the University of Turku in the academic year 2008-2009. The topic of this thesis is geometric function theory, more precisely the theory of quasiconformal mappings in the Euclidean n-dimensional space. For an authoritative survey of the field, see F. W. Gehring [G] in the handbook of Kühnau [K] which also contains many other surveys on quasiconformal mappings and related topics. In this study, some of the main sources have been the monographs of O. Lehto and K.I. Virtanen [LV], of J. Väisälä [Va] and of M. Vuorinen [Vu1].

In the early of 1960's, F. W. Gehring and J. Väisälä originated the theory of quasiconformal mappings in the Euclidean n-space. Their work generalized the theory of quasiconformal mappings due to H. Grötzsch 1928, O. Teichmüller in the period 1938-44, and L. Bers, L. V. Ahlfors from the early 1950's in two-dimensional case.

In this thesis, which is largely motivated by [Vu3], the notion of a metric space plays a central role. In particular, the hyperbolic metric and related metrics are often used. Some useful references are the textbook [KL] and the collection of surveys [PSV]. The thesis is divided into seven sections. The results are largely based on the cited literature. We give some preliminary definitions in the second section, and also we define the j and $\tilde{j}$ metrics. In Section 3, we discuss the absolute ratio metric δ and its comparison with various other metrics, and Lipschitz mappings with respect to δ metric [S]. Section 4 starts with the modulus of a curve family and then proceeds to basic properties of quasiconformal mappings. We give here the properties of the distortion function and the Hölder constant with respect to the δ metric under quasiconformal mappings. In Section 5 we discuss the Hölder continuity of quasiconformal mappings of the proper subdomains of $\mathbf{R}^2$ and $\mathbf{R}^n$ with respect to j and $\tilde{j}$ metrics. In Section 6 we discuss the Hölder continuity of quasiconformal mappings of the unit ball with respect to the Euclidean metric. In the last section, we discuss the Hölder continuity ofquasiregular mappings of the unit ball with respect to the hyperbolic ρ metric. F. W. Gehring and K. Hag gave some results, which discussed in two-dimensional case the j and ρ metrics [GH1]. We improve their results in two-dimensional case as well as prove those results for n-dimensional case. In 1988 G. D. Anderson and M. K. Vamanamurthy gave the Hölder constant of the quasiconformal mappings of the unit ball with respect to the Euclidean metric [AV]. A sharp version of this result was proved at the same time by R. Fehlmann and M. Vuorinen [FV]. Here we give same kind of constant which is better than [AV] and [FV] in some specific range [BV]. The last theorem of Section 7 is the explicit form of the Schwarz lemma, probably a new result. This kind of results are also given in [EMM, Theorem 5.1] and [Vu1, Theorem 11.2] in implicit form.

2. **Topology of metric spaces**

This section consists of some elementary definitions, which will be used frequently later on. Most of this section is taken from the first part of the book [GG].

2.1. **Metric space.** A non-empty set X, whose elements we shall call points, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p,q)$, called the *distance* from p to q, such that

(1) $d(p,p) = 0$;
(2) $d(p,q) > 0$ if $p \neq q$;
(3) $d(p,q) = d(q,p)$;
(4) $d(p,r) \leq d(p,q) + d(q,r)$, *for all* $p, q, r \in X$.

If (1) and (4) hold, then d is *pseudometric.* If also (2) and (3) hold then d is a metric and (X,d) is a metric space.

2.2. **Open balls.** Let (X,d) be any metric space. The *open ball* $B(x,r)$ with center $x \in X$ and radius $r > 0$ is defined by

$$B(x,r) = \{y \in X : d(x,y) < r\}.$$

The balls centered x form a nested family of subsets of X that increase with r, that is, $B(x,r_1) \subset B(x,r_2)$ if $r_1 \leq r_2$. Furthermore,

$$\bigcup_{r>0} B(x,r) = X.$$

2.3. **Open and closed sets.** Let Y be a subset of X. A point $x \in X$ is an *interior point* of Y if there exists $r > 0$ such that $B(x,r) \subset Y$. The set of interior points of Y is the *interior* of Y, and it is denoted by $\mathrm{int}(Y)$. Note that every interior point of Y belongs to Y:

$$\mathrm{int}(Y) \subset Y.$$

A subset Y of X is *open* if every point of Y is an interior point of Y, that is, if int $(Y) = Y$. In particular, the empty set $\emptyset$ and the entire space X are open subsets of X.

Let Y be a subset of a metric space X. A point $x \in X$ is *adherent* to Y if for all $r > 0$,

$$B(x,r) \cap Y \neq \emptyset.$$

The *closure* of Y, denoted by $\overline{Y}$, consists of all points in X that are adherent to Y. Evidently each point of Y is adherent to Y, so that

$$Y \subset \overline{Y}.$$

A point $x \in X$ is a *boundary point* of a subset Y of X if x is adherent both to Y and to $X \setminus Y$. The *boundary* of Y, denoted by ∂Y, is the set of boundary points of Y. Then

$$\partial Y = \overline{Y} \cap (\overline{X \setminus Y}),$$

so that ∂Y is closed.

The subset Y is *closed* if $Y = \overline{Y}$. In particular, the empty set $\emptyset$ and the entire space X are closed subsets of X.

2.4. **Compact sets.** A family $\{U_\alpha\}_{\alpha\in A}$ of sets is said to be a *cover* of a set S if S is contained in the union of the sets U_α. An *open cover* of a metric space X is a family of open subsets of X that covers X. A metric space X is *compact* if every open cover has a finite subcover. In other words, X is compact if, for each open cover $\{U_\alpha\}_{\alpha\in A}$ of X, there are finitely many of the sets U_α, say $U_{\alpha_1}, \ldots, U_{\alpha_m}$, such that

$$X \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_m}.$$

A metric space X is *totally bounded* if for each $\epsilon > 0$, there exists a finite number of open balls of radius ϵ that cover X.

2.5. **Continuous mappings.** Let (X, d_1) and (Y, d_2) be metric spaces. We say that $f : X \to Y$ is continuous at $x \in X$ if whenever $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $f(x_n) \to f(x)$. The function f is *continuous* if it is continuous at each $x \in X$. Moreover, f is continuous if and only if the preimage of every open (*resp.* closed) set A of Y is open (*resp.* closed) in X. Also $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$ implies that f is continuous.

2.6. **Homeomorphism.** A function f from one metric space to another is a *homeomorphism* if f is continuous, one to one, and onto and if moreover the inverse function f^{-1} is continuous. A homeomorphism preserves all the properties of a metric space that are definable in terms of open sets only.

2.7. **Isometry.** Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$ be a homeomorphism. We call f an *isometry* if

$$d_2(f(x), f(y)) = d_1(x, y)$$

for all $x, y \in X$.

2.8. **Lipschitz mappings.** Let (X, d_1) and (Y, d_2) be metric spaces, let $f : X \to Y$ be continuous and let $L \geq 1$. We say that f is *L-Lipschitz* if

$$d_2(f(x), f(y)) \leq L d_1(x, y)$$

for all $x, y \in X$. If, in addition, f is homeomorphism and

$$d_1(x, y)/L \leq d_2(f(x), f(y)) \leq L d_1(x, y)$$

for all $x, y \in X$, we say that f is *L-bilipschitz.*

2.9. **Modulus of continuity.** Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$ be a continuous mapping. Then we say that f is uniformly continuous if there exists an increasing continuous function $w : [0, \infty) \to [0, \infty)$ with $w(0) = 0$ and $d_2(f(x), f(y)) \leq w(d_1(x, y))$ for all $x, y \in X$. We call the function w the modulus of continuity of f. If there exists $C, \alpha > 0$ such that $w(t) \leq C t^\alpha$ for all $t \in (0, t_o]$ ($t_o > 0$ is fixed), we say that f is Hölder-continuous with Hölder constant α.

2.10**. Open, closed and discrete mappings.** Let (X, d_1) and (Y, d_2) be metric spaces. Then $f : X \to Y$ is *open* if fA is open whenever $A \subset X$ is open, and *closed* if fK is closed whenever $K \subset X$ is closed. In general, these are not equivalent (but for bijective mappings they are). A function $f : X \to Y$ is *discrete* if for every $y \in Y$, $f^{-1}(y)$ consists of isolated points.

2.11**. Metric space topology.** Let X be a non-empty set. A collection τ of subsets of X is topology on X if

(1) $A_i \in \tau, i \in I$ implies $\bigcup A_i \in \tau$, for countable I
(2) $A_1, \ldots, A_m \in \tau$ implies $\bigcap_{i=1}^m A_i \in \tau$
(3) $\emptyset, X \in \tau$.

If (X, d) is a metric space, then we denote by τ_d the topology defined by the balls $B(x, r)$. This is the standard topology we use for X.

If (X, d) is a metric space and G is a subset of X with nonempty boundary, then $d(x) = d(x, \partial G)$ is the distance between a point $x \in G$ and the boundary ∂G.

2.12. **Lemma.** [S, Lemma 2.2] *Let (X, d) be a metric space and $G \subset X$ an open set with nonempty boundary. The formula*

$$j_G(x, y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right), \quad x, y \in G,$$

defines a metric on G.

Proof. We need only to prove the triangle inequality

$$j_G(x, y) + j_G(y, z) \ge j_G(x, z) \quad \text{for all } x, y, z \in G.$$

By symmetry, we may assume that $d(x) \le d(z)$. Then we need to prove that

$$\frac{|x-y|}{\min\{d(x), d(y)\}} + \frac{|y-z|}{\min\{d(y), d(z)\}} + \frac{|x-y||y-z|}{\min\{d(x), d(y)\}\min\{d(y), d(z)\}} \ge \frac{|y-z|}{d(x)}. \tag{2.13}$$

If $d(y) \le d(x)$, then (2.13) follows immediately. Next, suppose that $d(x) < d(y)$. Then the left hand side of (2.13) is equal to

$$\frac{1}{d(x)}\left(|x-y| + |y-z|\left(\frac{d(x) + |x-y|}{\min\{d(y), d(z)\}}\right)\right),$$

which is greater than or equal to $|y-z|/d(x)$, since $d(x) + |x-y| \ge d(y)$. $\square$

We shall use this metric only when X is the Euclidean space $\mathbf{R}^n$.

2.14. **Lemma.** [S, Lemma 2.3] *Let (X, d) be a metric space and $G \subset X$ be an open set with nonempty boundary. The formula*

$$\tilde{j}_G(x, y) = \log\left(1 + \frac{|x-y|}{d(x)}\right) + \log\left(1 + \frac{|x-y|}{d(y)}\right), \quad x, y \in G,$$

defines a metric on G*.*

Proof. We only need to prove the triangle inequality

$$\tilde{j}_G(x,y)+\tilde{j}_G(y,z)\geq \tilde{j}_G(x,z) \quad \text{for all } x,y,z\in G.$$

This is equivalent to the inequality

$$\begin{aligned}(2.15)\quad \log(d(x)+|x-y|)+\log(d(y)+|x-y|)+\log(d(y)+|y-z|)+\log(d(z)+|y-z|)\\ \geq \log(d(x)+|x-z|)+\log(d(z)+|x-z|)+2\log d(y).\end{aligned}$$

Now we have

$$\begin{aligned}&\log(d(x)+|x-y|)-\log(d(x)+|x-z|)\\ &\geq \log(d(x)+|x-y|)-\log(d(x)+|x-y|+|y-z|)\\ &\geq \log d(y)-\log(d(y)+|y-z|),\end{aligned}$$

and similarly,

$$\log(d(z)+|y-z|)-\log(d(z)+|x-z|)\geq \log d(y)-\log(d(y)+|x-y|).$$

We obtain (2.15) by combining these inequalities. □

2.16. **Remark.** Since

$$\frac{|x-y|}{\max\{d(x),d(y)\}}\geq \frac{|x-y|}{\min\{d(x),d(y)\}+|x-y|}=1-e^{j_G(x,y)},$$

we always have

$$j_G(x,y)+\log(2-e^{-j_G(x,y)})=\log(2e^{j_G(x,y)}-1)\leq \tilde{j}_G(x,y)\leq 2j_G(x,y).$$

3. Geometry of Euclidean and hyperbolic spaces

In this section we discuss the absolute ratio metric, and its comparison with other metrics. We have taken this section from [S]. Here the hyperbolic tangent and its inverse function are denoted by th and arth, respectively.

3.1. **Notation.**

We shall write

$$\begin{gathered}\mathbf{B}^n(x,r)=\{z\in\mathbf{R}^n : |z-x|<r\},\\ \mathbf{B}^n(r)=B^n(0,r),\ \mathbf{B}^n=B^n(0,1),\\ S^{n-1}(x,r)=\{z\in\mathbf{R}^n : |z-x|=r\},\\ S^{n-1}(r)=S^{n-1}(0,r),\ S^{n-1}=S^{n-1}(1)\end{gathered}$$

and

$$\mathbf{H}^n=\{(x_1,\ldots,x_n)\in\mathbf{R}^n : x_n>0\}.$$

The standard unit vector of $\mathbf{R}^n$ are denoted by $e_1,e_2,\ldots,e_n$. In addition, ω_{n-1} is the surface area of the unit sphere $\mathbf{S}^{n-1}$ and Ω_n is the volume of the unit ball $\mathbf{B}^n$. We always assume that $n\geq 2$. The complex plane is denoted by $\mathbf{C}$. We denote Möbius space $\mathbf{R}^n\cup\{\infty\}$ by $\overline{\mathbf{R}}^n$. If $G\subset\overline{\mathbf{R}}^n$, and the metric space (G,d) is not given explicitly, then ∂G means the boundary of G with respect to the topology of $\overline{\mathbf{R}}^n$ and $d(x)$ is the Euclidean distance between x and ∂G, when $x \in G \setminus \{\infty\}$ and $\partial G \neq \{\infty\}$. For distinct points $a, b, c, d \in \overline{\mathbf{R}}^n$, the *absolute (cross) ratio* is defined by

$$|a,b,c,d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)}, \tag{3.2}$$

where $q(x,y)$ is the *spherical (chordal) metric*, defined by the formulas

$$\begin{cases} q(x,y) = \dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \ x,y \in \mathbf{R}^n, \ x \neq \infty \neq y, \\ q(x,\infty) = \dfrac{1}{\sqrt{1+|x|^2}}, \ x \in \mathbf{R}^n. \end{cases}$$

It is not difficult to prove [Be, p.32] that a function from $\overline{\mathbf{R}}^n$ into $\overline{\mathbf{R}}^n$ is a Möbius transformation if and only if it preserves absolute ratio. Note also that if a, b, c and d are distinct points in $\mathbf{R}^n$, then

$$|a,b,c,d| = \frac{|a-c||b-d|}{|a-b||c-d|}.$$

3.3. **Definition.** A *path* in $\mathbf{R}^n$ ($\overline{\mathbf{R}}^n$) is a continuous mapping $\gamma : \Delta \to \mathbf{R}^n$ (resp. $\overline{\mathbf{R}}^n$) where $\Delta \subset \mathbf{R}$ is an interval. If $\Delta' \subset \Delta$ is an interval, we call $\gamma|\Delta'$ a sub path of γ. The path γ is called *closed* (*open*) if Δ is *closed* (resp. *open*). (Note that according to this definition, e.g. the path $\gamma : [0,1] \to \mathbf{R}^n$ is closed and that it is not required that $\gamma(0) = \gamma(1)$). The *locus* (or *trace*) of the path γ is the set $\gamma\Delta$. The locus is also denoted by $|\gamma|$ or simply by γ if there is no danger of confusion. We use the word *curve* as the synonym for the *path*. The *length* $\ell(\gamma)$ of the curve $\gamma : \Delta \to \mathbf{R}^n$ is defined in the usual way, with the help of polygonal approximations and a passage to the limit (see [Va, pp. 1-8]). The path $\gamma : \Delta \to \mathbf{R}^n$ is called *rectifiable* if $\ell(\gamma) < \infty$ and *locally rectifiable* if each closed subpath of γ is rectifiable. If $\gamma : [0,1] \to \mathbf{R}^n$ is a rectifiable path, then γ has a parametrization by means of arc length, also called the *normal representation* of γ. The normal representation of γ is denoted by $\gamma^o : [0, \ell(\gamma)] \to \mathbf{R}^n$.

Given $x, y \in G$, $\Gamma(x,y)$ stands for the collection of all the rectifiable paths $\gamma \in G$ joining x and y. Let $w : G \to (0,\infty)$ be continuous, the *w-length* is defined by

$$\ell_w(\gamma) = \int_\gamma w(z)|dz|$$

for all $\gamma \in \Gamma(x,y)$. We define the *w-metric* by

$$w_G(x,y) = \inf_{\gamma \in \Gamma(x,y)} \ell_w(\gamma)$$

for all γ. For $A \subset \mathbf{R}^n$ let $A_+ = \{x \in A : x_n > 0\}$. We define a weight function $w : \mathbf{R}^n_+ \to \mathbf{R}_+ = \{x \in \mathbf{R} : x > 0\}$ by

$$w(x) = \frac{1}{x_n}, \ x = (x_1, \ldots, x_n) \in \mathbf{R}^n_+.$$

If $\gamma : [0,1) \to \mathbf{R}^n_+$ is a continuous mapping such that $\gamma[0,1)$ is a rectifiable curve with length $s = \ell(\gamma)$, then γ has a normal representation $\gamma^o : [0,s) \to \mathbf{R}^n$ parametrized by arc length. Then the *hyperbolic length* of $\gamma[0,s)$ is defined by

$$\ell_h([0,1)) = \int_0^s |(\gamma^o)'(t)| w(\gamma^o(t)) dt = \int_\gamma \frac{|dx|}{x_n}.$$

3.4. **Hyperbolic metric.** The hyperbolic metric $\rho_{\mathbf{B}^n}$ in $\mathbf{B}^n$ is defined by

$$\rho_{\mathbf{B}^n}(x,y) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{2|dz|}{1-|z|^2}, \quad x, y \in \mathbf{B}^n,$$

and the hyperbolic metric $\rho_{\mathbf{H}^n}$ in $\mathbf{H}^n$ is defined by

$$\rho_{\mathbf{H}^n}(x,y) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{|dz|}{d(z, \partial \mathbf{H}^n)}, \quad x, y \in \mathbf{H}^n,$$

where Γ is the family of all rectifiable curves in $\mathbf{B}^n$ and $\mathbf{H}^n$, respectively, joining x and y. Note that $\rho_{\mathbf{B}^n}$ and $\rho_{\mathbf{H}^n}$ are the particular cases of the w-metric $w_G(x,y)$.

If D is a simply connected domain of the extended complex plane $\overline{\mathbf{C}}$ and card$\partial D > 1$, then by the Riemann mapping theorem there exists a conformal mapping f from D onto the unit disk $\{z \in \mathbf{C} : |z| < 1\}$. The hyperbolic metric in D is defined by the formula

$$\rho_D(x,y) = \inf_{\gamma \in \Gamma} \int_\gamma \eta_D(z)|dz|, \qquad \eta_D(z) = \frac{2|f'(z)|}{1-|f(z)|^2},$$

where Γ is the family of all rectifiable curves in D joining x and y.

The hyperbolic density is monotonic with respect to the domain. If D_1 is a simply connected subdomain of D and $z \in D_1$, then

$$\eta_D(z) \leq \eta_{D_1}(z). \tag{3.5}$$

Let f and f_1 be conformal maps of D and D_1 onto the unit disk, both vanishing at z. Then $\eta_D(z) = |f'(z)|$, $\eta_{D_1}(z) = |f_1'(z)|$, and the application of Schwarz's lemma to the function $f \circ f_1^{-1}$ yields (3.5). Similar reasoning gives an upper bound for $\eta_D(z)$ in terms of the Euclidean distance $d(z, \partial D)$ from z to the boundary of D. Now we apply Schwarz's lemma to the function $\xi \mapsto f(z + d(z, \partial A)\xi)$ and we obtain

$$\eta_D(z) \leq \frac{1}{d(z, \partial D)}.$$

For simply connected set D, not containing ∞ we also have the lower bound

$$\eta_D(z) \geq \frac{1}{4d(z, \partial D)}. \tag{3.6}$$

If f is a conformal mapping of the unit disk A with $f(0) = 0,\ f^{'}(0) = 1,\ f(\infty) \neq \infty$, then $d(0, \partial f(A)) \geq 1/4$. We apply this to the function $w \mapsto (g(w) - z)/g^{'}(0)$, where g is a conformal mapping of A onto D with $g(0) = z$. Inequality (3.6) follows, because $\eta_D(z) = 1/|g^{'}(0)|$. Hence

$$\frac{1}{4d(z, \partial D)} \leq \eta_D(z) \leq \frac{1}{d(z, \partial D)} \tag{3.7}$$

for every $z \in D$.

3.8**. Quasihyperbolic metric.** Let $D \subseteq \mathbf{R}^n$ be a domain. The quasihyperbolic metric k_D is defined by

$$k_D(x, y) = \inf_{\gamma \in \Gamma} \int_{\gamma} \frac{|dz|}{d(z, \partial D)}, \quad x, y \in D,$$

where Γ is the family of all rectifiable curves in D joining x and y.

3.9. **Definition.** A finite curve in a metric space is a continuous map of the unit interval $[0, 1]$ into a metric space; an infinite curve is a continuous map of the real line $(-\infty, \infty)$ into a metric space; a semi-infinite curve is a continuous map of the half line $[0, \infty)$ into a metric space. The word curve stands for any one of these.

We can use the third condition of the definition of the metric known as the triangle inequality to characterize straight lines in the Euclidean geometry.

3.10. **Definition.** We say a curve is a straight line or *geodesic* in the *Euclidean plane* if for every triple of points z_1, z_3, z_2 on the curve with z_3 between z_1 and z_2 we have

$$d(z_1, z_2) = d(z_1, z_3) + d(z_3, z_2).$$

If the curve satisfying this condition is finite we call it a straight line segment or *geodesic segment*; if it is semi-finite curve satisfying the condition it is called a ray or *geodesic ray* and if it is infinite it is called an infinite geodesic or an *extended line*.

3.11**. Ferrand's metric.** Let $D \subset \overline{\mathbf{R}}^n$ be a domain with card $\partial D \geq 2$. We define a density function

$$w_D(z) = \sup_{a,b \in \partial D} \frac{|a-b|}{|z-a||z-b|}, \quad z \in D \setminus \{\infty\},$$

and a metric σ in D,

$$\sigma_D(x, y) = \inf_{\gamma \in \Gamma} \int_{\gamma} w_D(z)|dz|,$$

where Γ is the family of all rectifiable curves joining x and y in D.

3.12**. Apollonian metric.** For a proper subdomain G of $\overline{\mathbf{R}}^n$, the Apollonian metric is defined by

$$\alpha_G(x, y) = \sup_{a,b \in \partial G} \log |a, x, y, b| \quad \text{for all } x, y \in G.$$

3.13. **Absolute ratio metric δ_G.** Let G be an open subset of $\overline{\mathbf{R}}^n$ with card $\partial G \geq 2$. Set

$$m_G(x,y) = \sup_{a,b \in \partial G} |a,x,b,y| \tag{3.14}$$

and

$$\delta_G(x,y) = \log(1 + m_G(x,y)) \tag{3.15}$$

for all $x, y \in G$. Here ∂G is the boundary of G, and the absolute (cross) ratio $|a,x,b,y|$ is defined in (3.2). For the proof that δ_G is a metric, see Theorem 3.19.

3.16. **Remark.**

(1) The supremum in (3.14) is also the maximum.
(2) The function m_G has the following monotonicity property [Vu1, p. 116]: if G_1 and G_2 are sets for which m_{G_1} and m_{G_2} are defined, and $G_1 \subset G_2$, then $m_{G_1} \geq m_{G_2}$.
(3) It follows from the definitions that $\delta_{\mathbf{R}^n \setminus \{a\}} = j_{\mathbf{R}^n \setminus \{a\}}$ for all $a \in \mathbf{R}^n$.

The following Bernoulli inequalities will be used frequently:

$$\log(1 + sx) \leq s \log(1 + x), \quad x > 0, \ s \geq 1, \tag{3.17}$$

$$\log(1 + sx) \geq s \log(1 + x), \quad x > 0, \ 0 \leq s \leq 1. \tag{3.18}$$

3.19. **Theorem.** *The function δ_G is a metric in G.*

Proof. Clearly $\delta_G(x,y) = 0$ if and only if $x = y$. Also δ_G is symmetric, because $|a,x,b,y| = |b,y,a,x|$. For the triangle inequality, let x, y and z be points in G. Then there exist distinct points u and v in ∂G such that

$$m_G(x,z) = |u,x,v,z|.$$

Since

$$\delta_G(x,y) + \delta_G(y,z) \geq \delta_{\overline{\mathbf{R}}^n \setminus \{u,v\}}(x,y) + \delta_{\overline{\mathbf{R}}^n \setminus \{u,v\}}(y,z)$$

and

$$\delta_G(x,z) = \delta_{\overline{\mathbf{R}}^n \setminus \{u,v\}}(x,z),$$

we need to prove the triangle inequality only when $G = \overline{\mathbf{R}}^n \setminus \{u,v\}$. In addition, since δ_G is a Möbius-invariant, we can assume that $\{u,v\} = \{0,\infty\}$. But then $\delta_G = j_G$, and the triangle inequality follows, because j_G is a metric by Lemma 2.12. $\square$

3.20. **Theorem.** *The inequalities $j_G \leq \delta_G \leq \tilde{j}_G \leq 2 j_G$ hold for every open set $G \subseteq \mathbf{R}^n$.*

Proof. Let x and y be distinct points in G, with $d(x) \leq d(y)$. Choose $a, b \in \partial G$ such that $d(x) = |x - a|$ and $|b - a| \geq |b - y|$ (possibly $b = \infty$). Then

$$m_G(x,y) \geq |a,x,b,y| \geq \frac{|x-y|}{d(x)},$$

and we obtain the first inequality.

For the second inequality, let a and b be in ∂G. If $a, b \neq \infty$, then

$$\begin{aligned} |a,x,b,y| &\leq \frac{|x-y|}{|x-a|}\left(\frac{|a-x|}{|y-b|}+\frac{|x-y|}{|y-b|}+1\right) \\ &\leq \frac{|x-y|}{d(y)}+\frac{|x-y|^2}{d(x)d(y)}+\frac{|x-y|}{d(x)} \end{aligned}$$

and we get

$$\delta_G(x,y) \leq \log\left(\left(1+\frac{|x-y|}{d(x)}\right)\left(1+\frac{|x-y|}{d(y)}\right)\right) = \tilde{j}(x,y).$$

If $b = \infty$, then $|a,x,b,y| = |x-y|/d(x)$ and similarly if $a = \infty$. $\square$

3.21. **Remark.** The equality $\delta_G(x,y) = \tilde{j}(x,y)$ holds if and only if $x = y$ or there exist points $a, b \in \partial G$ such that $d(x) = |x-y|$, $d(y) = |y-b|$, and a, x, y and b lie on the same line in this order. In particular, $\delta_{\mathbf{B}^n}(x,-x) = 2j_{\mathbf{B}^n}(x,-x)$ for all $x \in \mathbf{B}^n$. Also,

$$\delta_{\mathbf{H}^n}(re_n, se_n) = 2j_{\mathbf{H}^n}(re_n, se_n) = |\log(r/s)|$$

for all $r, s > 0$.

3.22. **Theorem.** *If G is an open subset of $\overline{\mathbf{R}}^n$ with* card $\partial G \geq 2$ *and the metric j_G^* is defined by*

$$j_G^*(x,y) = \log\left(1+\frac{q(x,y)}{q(x,\partial G)}\right) + \log\left(1+\frac{q(x,y)}{q(y,\partial G)}\right), \quad x, y \in G,$$

then $\delta_G \leq j_G^*$.

Proof. The function j_G^* is a metric by Lemma 2.14. The inequality $\delta_G \leq j_G^*$ can be proved in the same way as the second inequality in Theorem 3.20. $\square$

3.23. **Notation.** Let G be an open set for which j_G is defined. We define the open ball $B_j(x,t)$, with centre x and radius t, in this metric as the set

$$B_j(x,t) = \{z \in G : j_G(x,z) < t\},$$

for all $x \in G$ and $t > 0$. The balls $B_{\tilde{j}}(x,t)$ and $B_\delta(x,t)$ are defined similarly.

3.24. **Theorem.** *If $G \subseteq \mathbf{R}^n$ is open, $x \in G$ and $t > 0$ then*

$$B^n(x,r) \subset B_j(x,t) \subset B^n(x,R),$$

where $r = (1-e^{-t})d(x)$ and $R = (e^t - 1)d(x)$. The formulas for r and R are the best possible expressed in terms of t and $d(x)$ only.

Proof. Let $y \in G$ and write $j = j_G(x,y)$. Then

$$|x-y| = (1-e^{-j})(\min\{d(x), d(y)\} + |x-y|) \geq (1-e^{-j})d(x)$$

and the first inclusion follows. The second inclusion follows from the inequality

$$|x-y| = (e^{-j}-1)\min\{d(x), d(y)\} \leq (e^j - 1)d(x).$$

To show that r and R are the best possible, let $G = \mathbf{R}^n \setminus \{0\}$, $x \in G$, and $t > 0$. Now, choose $y = e^t x$ and $z = e^{-t} x$. Then $j_G(x,y) = j_G(x,z) = t$, $|x-y| = R$, and $|x-z| = r$. $\square$

3.25. **Theorem.** *If $G \subseteq \mathbf{R}^n$ is open, $x \in G$ and $t > 0$, then*

$$B^n(x,r) \subset B_{\tilde{j}}(x,t) \subset B^n(x,R),$$

where $r = (\operatorname{th}(t/2))d(x)$ and $R = ((e^t-1)/2)d(x)$. The formulas for r and R are the best possible expressed in terms of t and $d(x)$ only.

Proof. By Remark 2.16, $j \le \log((e^{\tilde{j}}+1)/2)$. Then

$$B_{\tilde{j}}(x,t) \subset B_j\left(x, \log\left(\frac{e^t+1}{2}\right)\right) \subset B^n\left(x, \frac{e^t-1}{2}d(x)\right)$$

by Theorem 3.24. For the first inclusion, let $y \in G$ and denote $\tilde{j} = \tilde{j}_G(x,y)$. If $|x-y| < d(x)$, then

$$\begin{aligned}\tilde{j} &\le \log\left(\left(1+\frac{|x-y|}{d(x)}\right)\left(1+\frac{|x-y|}{d(x)-|x-y|}\right)\right)\\ &= \log\frac{d(x)+|x-y|}{d(x)-|x-y|} = 2\operatorname{arth}\frac{|x-y|}{d(x)},\end{aligned}$$

and $|x-y| \ge (\operatorname{th}(\tilde{j}/2))d(x)$. The case $|x-y| \ge d(x)$ is clear.

To show that r and R are the best possible, let $G = \mathbf{R}^n \setminus \{0\}$, $x \in G$ and $t > 0$. Now, choose $y = ((e^t+1)/2)x$ and $z = (2/(e^t+1))x$. Then $\tilde{j}_G(x,y) = \tilde{j}_G(x,z) = t$, $|x-y| = R$ and $|x-z| = r$. $\square$

3.26. **Corollary.** *If $G \subseteq \mathbf{R}^n$ is open, $x \in G$ and $t > 0$, then*

$$B^n(x,r) \subset B_\delta(x,t) \subset B^n(x,R),$$

where $r = (\operatorname{th}(t/2))d(x)$ and $R = (e^t-1)d(x)$. In addition, for all $x \in \mathbf{H}^n$ there exists $y \in \partial B_\delta(x,t)$ such that $|x-y| = R$. If $\mathbf{H}^n = \mathbf{B}^n(x,d), d > 0$, then $B_\delta(x,t) = \mathbf{B}^n(x,r)$.

Proof. The inequalities $j_G \le \delta_G \le \tilde{j}_G$ imply that

$$B_{\tilde{j}}(x,t) \subset B_\delta(x,t) \subset B_j(x,t).$$

The inclusions follow now from Theorems 3.24 and 3.25. For the cases where G is a half-space or a ball, see [Vu1, p. 22,25]. $\square$

The following theorem gives a connection between the Apollonian metric α_G and the absolute ratio metric δ_G.

3.27. **Theorem.** *Let $G \subset \overline{\mathbf{R}}^n$ be an open set with* card $\partial G \ge 2$. *Then*

$$\alpha_G \le \delta_G \le \log(e^{\alpha_G}+2) \le \alpha_G + \log 3.$$

The first two inequalities are the best possible for δ_G expressed in terms of α_G only.

Proof. Let $x, y \in G$. The first two inequalities follow if

$$|a,x,b,y| \le 1 + |a,x,b,y| \le |a,x,b,y| + 2 \tag{3.28}$$

for all $a, b \in \partial G$. Since the absolute ratio is invariant under Möbius maps, we may assume that $b = \infty$. But then (3.28) takes the form

$$\frac{|a-y|}{|a-x|} \le 1 + \frac{|x-y|}{|a-x|} \le \frac{|a-y|}{|a-x|} + 2,$$

and the result follows from the triangle inequality. The third inequality is obvious. □

For curves $\gamma : [0,1] \to D$, we write $d(\gamma, \partial D) = \inf_{t \in [0,1]} d(\gamma(t), \partial D)$.

3.29. **Theorem.** *The inequality $\sigma_D \ge \delta_D$ holds for any domain $D \subset \overline{\mathbf{R}}^n$ with* card $\partial D \ge 2$.

Proof. Since both metrics are Möbius-invariant, we may assume that $D \subset \mathbf{R}^n$. Let $\gamma : [0,1] \to D$ be a rectifiable curve with end points $\gamma(0) = x$ and $\gamma(1) = y$. Set $d = d(\gamma, \partial D)$ and choose $\epsilon \in (0, d)$. Now divide $[0,1]$ into m subintervals $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$, such that the diameters of the sets $\gamma([t_i, t_{i+1}])$ are less than ϵ. Set $\gamma(t_i) = x_i$ and $\gamma(t_{i+1}) = y_i$ for all $i = 0, 1, \ldots, m-1$. Because $(1 - \epsilon/d) < 1$, we have

$$w_D(\gamma(t)) \ge \left(1 - \frac{\epsilon}{d}\right)^2 \left(\sup_{a,b \in \partial D} \frac{|a-b|}{|x_i - a||y_i - b|} \right) = \left(1 - \frac{\epsilon}{d}\right)^2 \frac{m_D(x_i, y_i)}{|x_i - y_i|}$$

for every $t \in [t_i, t_{i+1}]$, where w_D is as in 3.11 . Now

$$\begin{aligned} \int_\gamma w_D(\gamma(t))|d\gamma(t)| &\ge \sum_{i=0}^{m-1} |x_i - y_i| \left(1 - \frac{\epsilon}{d}\right)^2 \frac{m_D(x_i, y_i)}{|x_i - y_i|} \\ &\ge \left(1 - \frac{\epsilon}{d}\right)^2 \sum_{i=0}^{m-1} \delta_D(x_i, y_i) \ge \left(1 - \frac{\epsilon}{d}\right)^2 \delta_D(x, y), \end{aligned}$$

and we obtain $\sigma_D(x,y) \ge \delta_D(x,y)$ as ϵ tends to 0. □

3.30. **Theorem.** *If D is a simply connected domain of $\overline{\mathbf{C}}$ and* card $\partial D > 1$, *then* $\rho_D \le \sigma_D \le 2\rho_D$.

3.31. **Remark.** In [Be] there is an example of a domain D such that $\alpha_D(0, \infty) = 2\rho_D(0, \infty)$. This implies that the constant 2 in the second inequality of Theorem 3.30 is the best possible, because $\alpha_D \le \sigma_D$.

Next we shall study the behavior of δ_G and σ_G under Euclidean bilipschitz maps. Using the definition of the absolute ratio, one can easily prove the following lemma.

3.32. **Lemma.** *Let $G, G' \subseteq \mathbf{R}^n$ be open sets and $f : G \to G'$ an L-bilipschitz mapping. Then*

$$m_G(x,y) \le L^4 m_{G'}(f(x), f(y)) \text{ for all } x, y \in G. \tag{3.33}$$

The inequality (3.33) also holds if G and G' are open subsets of $\overline{\mathbf{R}}^n$ with at least two boundary points and $f : G \to G'$ is an L-bilipschitz map with respect to the spherical metric.

3.34. **Theorem.** *Let $G, G' \subseteq \mathbf{R}^n$ be open sets and $f : G \to G'$ an L-bilipschitz mapping. Then*

$$\frac{1}{L^4}\delta_{G'}(f(x), f(y)) \leq \delta_G(x, y) \leq L^4\delta_{G'}(f(x), f(y)) \; for \; all \; x, y \in G. \tag{3.35}$$

The inequalities (3.35) also hold if G and G' are open subsets of $\overline{\mathbf{R}}^n$ with at least two boundary points and $f : G \to G'$ is an L-bilipschitz map with respect to the spherical metric.

Proof. The proof follows from Lemma 3.32 and the Bernoulli inequality (3.17). □

3.36. **Lemma.** *Let $D \subset \overline{\mathbf{R}}^n$ be a domain with* card $\partial D \geq 2$. *For every $\varepsilon \in (0, 1/2)$, if $x, y \in D$ and $\delta_D(x, y) < \varepsilon$, then $\sigma_D(x, y) \leq c(\varepsilon)\delta_D(x, y)$, where $c(\varepsilon) = (1-\varepsilon)/(1-2\varepsilon)^2$. Also, $c(0+) = 1$.*

Proof. We can assume that $D \subset \mathbf{R}^n$ and $d(x) \leq d(y)$. Let $\varepsilon \in (0, 1/2)$ and $j_D(x, y) \leq \delta_D(x, y) < \varepsilon$, then $|x - y| < d(x)$ so that the segment $[x, y]$ is contained in D, and

$$\begin{aligned}
\sigma_D(x, y) &\leq |x - y| \sup_{a,b\in\partial D} \frac{|a - b|}{|x - a||y - b|} \frac{|x - a|}{|x - a| - |x - y|} \frac{|y - b|}{|y - b| - |x - y|} \\
&\leq m_D(x, y) \left(\frac{d(x)}{d(x) - |x - y|}\right)^2 \leq \frac{e^\varepsilon - 1}{\varepsilon}\delta_D(x, y)\frac{1}{(2 - e^\varepsilon)^2}.
\end{aligned}$$

The formula for $c(\varepsilon)$ follows now from the inequality $e^\varepsilon < 1/(1 - \varepsilon)$. □

3.37. **Theorem.** *Let $D, D' \subseteq \mathbf{R}^n$ be domains and let $f : D \to D'$ be an L-bilipschitz mapping. Then*

$$\frac{1}{L^4}\sigma_{D'}(f(x), f(y)) \leq \sigma_D(x, y) \leq L^4\sigma_{D'}(f(x), f(y)) \; for \; all \; x, y \in D. \tag{3.38}$$

The inequalities (3.38) also hold if D and D' are open subsets of $\overline{\mathbf{R}}^n$ with at least two boundary points and $f : D \to D'$ is an L-bilipschitz map with respect to the spherical metric.

Proof. Since σ_D is Möbius-invariant, we may assume that $D, D' \subset \mathbf{R}^n$. Let $\gamma : [0, 1] \to D$ be a rectifiable curve with end points $\gamma(0) = x$ and $\gamma(1) = y$. Set $d = d(\gamma, \partial D)$. For $\epsilon \in (0, 1/2)$, divide $[0, 1]$ into subintervals $[t_i, t_{i+1}]$, $0 = t_0 < t_1 < \ldots < t_m = 1$, so that $|\gamma(t_i), \gamma(t_{i+1})| < \epsilon d/2$ for all $i = 0, 1, \ldots, m-1$. Let $x_i = \gamma(t_i)$ and $y_i = \gamma(t_{i+1})$. Then $\delta_D(x_i, y_i) \leq 2j_D(x_i, y_i) < \varepsilon$, and by Lemma 3.37,

$$\sigma_D(x_i, y_i) \leq c(\varepsilon)\delta_D(x_i, y_i)$$

for all $i = 0, 1, \ldots, m - 1$. Now

$$
\begin{aligned}
\sigma_D(x,y) &\le \sum_{i=0}^{m-1} \sigma_D(x_i,y_i) \le L^4 c(\varepsilon) \sum_{i=0}^{m-1} \delta_{D'}(f(x_i),f(y_i)) \\
&\le L^4 c(\varepsilon) \sum_{i=0}^{m-1} \sigma_{D'}(f(x_i),f(y_i)) \le L^4 c(\varepsilon) \int_{f(\gamma)} w_{D'}(t)|dt|.
\end{aligned}
$$

Next, we can choose γ so that $\int_{f(\gamma)} w_{D'}(t)|dt|$ is arbitrarily close to $\sigma_{D'}(f(x),f(y))$. Then, letting ϵ tend to 0, we obtain

$$
\sigma_D(x,y) \le L^4 \sigma_{D'}(f(x),f(y)).
$$

The first inequality in (3.38) follows from the second one applied to f^{-1}. $\square$

3.39. **Notation.** Let $f : G \to \mathbf{R}^n$ be a continuous injective function, where G is a subdomain of $\mathbf{R}^n$. The linear dilatation of f at a point $x \in G$ is

$$
H(x,f) = \lim_{r \to 0+} \sup \frac{L(x,f,r)}{l(x,f,r)},
$$

where

$$
\begin{aligned}
L(x,f,r) &= \max_z \{|f(z)-f(x)| : |z-x| = r\}, \\
l(x,f,r) &= \min_z \{|f(z)-f(x)| : |z-x| = r\}
\end{aligned}
$$

for all $r \in (0, d(x))$.

3.40. **Lipschitz maps with respect to δ_G.** Next we shall study maps $f : G \to G' = fG$ that are bilipschitz with respect to the absolute ratio metric, namely there exists a constant $L \ge 1$ such that

$$
\frac{1}{L}\delta_G(x,y) \le \delta_{G'}(f(x),f(y)) \le L\delta_G(x,y) \ for\ all\ x,y \in G.
$$

It follows from Corollary 3.26 that these maps are homeomorphisms.

Now we shall find an upper bound for the linear dilatation $H(x,f)$, when f is a bilipschitz map with respect to δ.

3.41. **Lemma.** *Let $G \subseteq \mathbf{R}^n$ be an open set and $x_0 \in G$. Then*

$$
\lim_{r \to 0+} \frac{\delta_G(x,x_0)}{\delta_G(y,x_0)} = 1,
$$

where $|x-x_0| = |y-x_0| = r \in (0, d(x_0))$.

Proof. Let $a,b,c,d \in \partial G$ be points for which $m_G(x,x_0) = |a,x,b,x_0|$ and $m_G(y,x_0) = |c,y,d,x_0|$. Then

$$
m_G(x,x_0) = |a,y,b,x_0| \cdot \frac{|a-y|}{|a-x|} \le m_G(y,x_0)\left(1 + \frac{2r}{d((x_0)-r)}\right).
$$

Similarly,

$$m_G(x,x_0) \geq m_G(y,x_0)\left(1 - \frac{2r}{d((x_0)-r)}\right).$$

Using the Bernoulli inequalities (3.17) and (3.18), we have

$$1 - \frac{2r}{d(x_0)-r} \leq \frac{\delta_G(x,x_0)}{\delta_G(y,x_0)} \leq 1 + \frac{2r}{d(x_0)-r}.$$

$\square$

3.42. **Theorem.** *Let $G, G^{'} \subseteq \mathbf{R}^n$ be open sets and $f : G \to G^{'}$ an L-bilipschitz map with respect to the metric δ_G. Then $H(x,f) \leq L^2$ for all $x \in G$.*

Proof. Let $x_0 \in G$ and $r \in (0, d(x_0))$. Choose $x, y \in S^{n-1}(x_0, r)$ such that

$$m \equiv l(x_0, f, r) = |f(x) - f(x_0)|$$

and

$$M \equiv l(x_0, f, r) = |f(x) - f(x_0)|,$$

and let $a, b \in \partial G^{'}$ be points for which $m_{G'}(f(x), f(x_0)) = |a, f(x), b, f(x_0)|$. Then

$$\begin{aligned} \frac{M}{m} &= \frac{|a, f(y), b, f(x_0)|}{|a, f(x), b, f(x_0)|} \cdot \frac{|a - f(y)|}{|a - f(x)|} \\ &\leq \frac{m_{G'}(f(y), f(x_0))}{m_{G'}(f(x), f(x_0))} \cdot \left(1 + \frac{m+M}{d(f(x_0)) - m}\right), \end{aligned}$$

and

$$\lim_{r\to 0+} \frac{M}{m} \leq \lim_{r\to 0+} \sup \frac{\exp(\delta_{G'}(f(y), f(x_0))) - 1}{\exp(\delta_{G'}(f(x), f(x_0))) - 1} \leq \lim_{r\to 0+} \frac{L\delta_G(y,x_0)}{\delta_G(x,x_0)/L} = L^2$$

by Lemma 3.41. $\square$

4. **Growth estimates under quasiconformal maps**

Now we will discuss the behavior of the absolute ratio metric δ_G under quasiconformal maps of $\overline{\mathbf{R}}^n$. For this purpose, we need to define a distortion function $\eta_{K,n}$. We denote the hyperbolic sine function and its inverse by sh and arsh, respectively.

4.1. **Definition.** A domain D in $\overline{\mathbf{R}}^n$ is called a *ring*, if $\overline{\mathbf{R}}^n \setminus D$ has two components. If the components are C_0 and C_1 we write $D = R(C_0, C_1)$. The complementary components of the *Grötzsch ring* $R_{G,n}(s)$ in $\mathbf{R}^n$ are $\overline{\mathbf{B}}^n$ and $[se_1, \infty]$, $s > 1$, while those of the *Teichmüller ring* $R_{T,n}(t)$ are $[-e_1, 0]$ and $[te_1, \infty]$, $t > 0$.

For sets $E, F \subset G$, and $G \subset \overline{\mathbf{R}}^n$, let $\Delta(E, F; G)$ denote the curve family of all curves joining the sets E and F in G, and let $\Delta(E, F) = \Delta(E, F; \overline{\mathbf{R}}^n)$.

4.2**. Modulus of a curve family.** Let Γ be a family of curves in $\overline{\mathbf{R}}^n$. That is, the elements of Γ are curves in $\overline{\mathbf{R}}^n$. We denote by $\mathrm{F}(\Gamma)$ the set of all non-negative Borel functions $\rho : \overline{\mathbf{R}}^n \to \mathbf{R} \cup \{\infty\}$ such that $\int_\gamma \rho ds \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$. We define the modulus

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{R^n} \rho^n dm,$$

where m stands for the n-dimensional Lebesgue measure.
The capacity of a ring $R(E,F)$ is

$$\mathrm{cap} R(E,F) = M(\Delta(E,F)).$$

The capacities of $R_{T,n}(s)$ and $R_{G,n}(s)$ are denoted by functions $\tau_n(s)$ and $\gamma_n(s)$, respectively. The functions $\tau_n : (0,\infty) \to (0,\infty)$ and $\gamma_n : (1,\infty) \to (0,\infty)$ are decreasing homeomorphisms and they satisfy the functional identity

$$\gamma_n(t) = 2^{n-1}\tau_n(t^2-1), \quad t > 1, \tag{4.3}$$

(see [Vu1, lemma 5.53]).

4.4**. The conformal invariants λ_G and μ_G.** If G is a proper subdomain of $\overline{\mathbf{R}}^n$, then for $x, y \in G$ with $x \neq y$ we define

$$\lambda_G(x,y) = \inf_{C_x, C_y} M(\Delta(C_x, C_y; G)), \tag{4.5}$$

where $C_z = \gamma_z[0,1)$ and $\gamma_z : [0,1) \to G$ is a curve such that $z \in |\gamma_z|$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$.
For a proper subdomain G of $\overline{\mathbf{R}}^n$ and for all $x, y \in G$ define

$$\mu_G(x,y) = \inf_{C_{xy}} M(\Delta(C_{xy}, \partial G; G)),$$

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0,1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$.

4.6. **Remark.** If D_1 and D_2 are proper subdomains of $\overline{\mathbf{R}}^n$, $D_1 \subset D_2$ and $x, y \in D_1$ are distinct points, then $\mu_{D_1}(x,y) \geq \mu_{D_2}(x,y)$ and $\lambda_{D_1}(x,y) \leq \lambda_{D_2}(x,y)$.

4.7**. Special functions.** The function $\mu : (0,1] \to [0,\infty)$ is defined by the formula

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}, \ 0 < r < 1, \quad \mu(1) = 0,$$

where

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2 t}}.$$

For all $K > 0$ and $n \geq 2$, the distortion function $\varphi_{K,n} : [0,1] \to [0,1]$ is defined by

$$\varphi_{K,n}(t) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))}, \quad t \in (0,1), \tag{4.8}$$

$\varphi_{K,n}(0) = 0$ and $\varphi_{K,n}(1) = (1)$.

The function μ is a decreasing and $\varphi_{K,n}$ is an increasing homeomorphism. It can be shown that $\varphi_K(r) = \varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K)$ for all $r \in (0,1)$ and $K > 0$.

4.9**. Quasiconformal mapping.** Given domains $D, D^{'}$ in $\mathbf{R}^n$,$n \geq 2$, let $f : D \to D^{'}$ be a homeomorphism. For $x \in D,\ r \in (0, d(x, \partial D))$, let

$$L(x,r) = \max\{|f(x) - f(y)| : |x - y| = r\},$$
$$\ell(x,r) = \min\{|f(x) - f(y)| : |x - y| = r\},$$

and let $C \in [1, \infty)$. We say that f is in the class $\mathcal{F}(D, D^{'}, C)$, if for each point $x \in D$,

$$H(x,f) = \limsup_{r \to 0} \frac{L(x,r)}{\ell(x,r)} \leq C.$$

4.10**. Metric definition.** It is clear that we can regard the class $\mathcal{F}(D, D^{'}, 1)$ as consisting of the conformal maps of D onto $D^{'}$. The maps of the class $\mathcal{F}(D, D^{'}, C)$ are said to be *quasiconformal*. The quantity $H(x, f)$ is called the *linear dilatation* of f at x. Following [Va, p. 113] we give also another definition of quasiconformal mappings. As shown in [Va] these definitions are equivalent, but interrelation of the parameters C and K in these definitions is an involved question (see [Va],[Vu2]).

4.11**. Geometrical definition.** Given domains $D, D^{'}$ in $\overline{\mathbf{R}}^n$, let $f : D \to D^{'}$ be a homeomorphism. We let

$$K_I(f) = \sup \frac{M(\Gamma^{'})}{M(\Gamma)},\ K_o(f) = \sup \frac{M(\Gamma)}{M(\Gamma^{'})},\ \Gamma^{'} = f(\Gamma),$$
$$K(f) = \max\{K_I(f), K_o(f)\},$$

where the suprema is taken over all curve families Γ in D such that moduli $M(\Gamma)$ and $M(\Gamma^{'})$ are not simultaneously 0 or ∞. We call $K_I(f)$ the *inner dilatation*, $K_0(f)$ the *outer dilatation*, and $K(f)$ the *maximal dilatation* of f. We say that f is a *K-quasiconformal* if $K(f) \leq K \leq \infty$, and f is said to be *quasiconformal* if it is *K-quasiconformal* for some K, $1 \leq K \leq \infty$.

4.12. **Theorem.** (*Schwarz lemma for quasiconformal mappings*) *Let* $f : \mathbf{B}^n \to f\mathbf{B}^n$ *be* K*-quasiconformal map with* $f\mathbf{B}^n \subset \mathbf{B}^n$*. Then*

$$\mathrm{th}\frac{\rho(f(x), f(y))}{2} \leq \varphi_{K,n}\left(\mathrm{th}\frac{\rho(x,y)}{2}\right),$$

for all $x, y \in \mathbf{B}^n$*.*
In particular, if $f(0) = 0$*, then*

$$|f(x)| \leq \varphi_{K,n}(|x|) \ for\ all\ x \in \mathbf{B}^n.$$

Following Lemmas will be used frequently onward.

4.13. **Lemma.** [Vu1, Theorem 7.47] *For* $n \geq 2$*,* $K \geq 1$ *and* $0 \leq r \leq 1$ *we have*

(1) $\varphi_{K,n}(r) \le \lambda_n^{1-\alpha} r^{\alpha}, \quad \alpha = K^{1/(1-n)}$,
(2) $\varphi_{1/K,n}(r) \ge \lambda_n^{1-\beta} r^{\beta}, \quad \beta = 1/\alpha$.

4.14. **Lemma.** [AVV1, Lemma 8.75] *For $n \ge 2$, $K \ge 1$ and $\alpha = K^{1/(1-n)} = 1/\beta$, the following inequalities hold:*
(1) $\lambda_n^{1-\alpha} \le 2^{1-\alpha} K \le 2^{1-1/K} K$,
(2) $\lambda_n^{1-\beta} \ge 2^{1-\beta} K^{-\beta} \ge 2^{1-K} K^{-K}$.

4.15. **Definition.** Let D be a domain of $\mathbf{R}^n$. A mapping $f : D \to \mathbf{R}^n$ is η-quasisymmetric, if there exists an increasing homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that

$$\frac{|f(a)-f(b)|}{|f(a)-f(c)|} \le \eta\left(\frac{|a-b|}{|a-c|}\right) \tag{4.16}$$

for all $a, b, c \in D$ with $a \ne c$.

For domains D and D' in $\overline{\mathbf{R}}^n$ we let $QC_K(D, D')$ denote the class of K-quasiconformal mappings of D into D'. We also let $QC_K(D,D) = QC_K(D)$.

We introduce some necessary notation before we give the proof of Theorem 4.19. For $n \ge 2,\ 1 \le K < \infty, t \in [0,\infty)$, we let

$$\begin{cases} \eta^*_{K,n}(t) = \sup\{|f(x)| : |x| \le t, f \in QC_K(\overline{\mathbf{R}}^n), \\ \\ f(0) = 0, f(e_1) = e_1, f(\infty) = \infty\}. \end{cases} \tag{4.17}$$

for $t > 0$.

4.18. **Lemma.** [Vu2, Lemma 2.8] *For $t > 0$ we have*

$$\begin{aligned} \eta^*_{K,n}(1) &= \sup\left\{\frac{|f(x)|}{|f(y)|} : |x| = |y| > 0, f \in \mathcal{F}(n,K)\right\} \\ &= \sup\left\{\frac{|f(x)|}{|f(y)|} : |x| = |y| = t, f \in \mathcal{F}(n,K)\right\} \\ &= \inf_{0<t<1}\left(\sup\left\{\frac{|f(x)|}{|f(y)|} : |x| = |y| = t, f \in \mathcal{F}(n,K)\right\}\right) \\ &= \inf_{0<t<1} \frac{A_{K,n}(t)}{B_{K,n}(t)}, \end{aligned}$$

where

$$\begin{cases} A_{K,n}(t) = \dfrac{\varphi_{K,n}(\sqrt{t})^2}{1-\varphi_{K,n}(\sqrt{t})^2}, \\ B_{K,n}(t) = \varphi_{1/K,n}\left(\sqrt{\frac{t}{1+t}}\right)^2 = A^{-1}_{K,n}(t), \end{cases}$$

and $\mathcal{F}(n,K) = \{f \in QC_K(\overline{\mathbf{R}}^n), f(e_1) = e_1\}$.

4.19. **Theorem.** [Vu2, Theorem 1.8] *For $n \ge 2$ and $K \ge 1$,*
(1) $\eta^*_{K,n}(1) \le \exp\left(6(K+1)^2\sqrt{K-1}\right)$;

(2) $\eta^*_{K,n}(t) \le \eta_{K,n}(1)\varphi_{K,n}(t), \quad 0 \le t < 1;$
(3) $\eta^*_{K,n}(t) \le \eta_{K,n}(1)/\varphi_{1/K,n}(1/t), \quad t > 1.$

Proof. The main part of this theorem is (1), where it is crucial that the majorant of $\eta^*_{K,n}(1)$ tends to 1 as $K \to 1$. By Lemma 4.13(1) we have $1 - \varphi_{K,n}(\sqrt{t})^2 \ge 1 - \lambda_n^{2(1-\alpha)} t^\alpha$. If $1 - \lambda_n^{2(1-\alpha)} t_o^\alpha = 1/K$, then $t_o = (\lambda_n^{2(\alpha-1)}(K-1)/K)^\beta$, so $t_o \le (K-1)/K$. Thus $1 - \varphi_{K,n}(\sqrt{t})^2 \ge 1/K$ for $0 < t \le t_o$. Therefore by Lemmas 4.13 and 4.18

$$\begin{aligned}
\eta^*_{K,n}(1) &\le \frac{A_{K,n}(t_o)}{B_{K,n}(t_o)} \le K\lambda_n^{2(\beta-\alpha)} t_o^{\alpha-\beta}(1+t_o)^\beta \\
&\le K\lambda_n^{2(\beta-\alpha)} t_o^{(\alpha-\beta)} \left(2 - \frac{1}{K}\right)^\beta \\
&= K^{\beta(\beta-1)}(K-1)^{1-\beta^2}\lambda_n^{2(\beta^2-1)}(2K-1)^\beta \equiv E.
\end{aligned}$$

Since $\max_{x>0} x^{-x} = e^{1/e}$ by elementary calculus, it follows that

$$(K-1)^{1-K} \le \exp\left((2/e)\sqrt{K-1}\right),$$

where we have used $x = \sqrt{K-1}$. We will also need the estimate $\lambda_n^{\beta-1} \le 2^{K-1}K^K$ by Lemma 4.14(2). The rest of the proof is divided into two cases.

Case 1. If $K \ge 2$, then $(K-1)^{1-\beta^2} \le 1$ and

$$\begin{aligned}
E &\le K^{K(K-1)}\lambda_n^{2(\beta^2-1)}(2K-1)^K \\
&\le K^{K(K-1)}\left(2^{K-1}K^K\right)^{2(K+1)}(2K-1)^K \\
&= K^{K(K-1)}2^{2(K^2-1)}K^{2K(K+1)}(2K-1)^K \\
&\le (2K)^{3K(K+1)} \\
&\le \exp(6(K+1)^2\sqrt{K-1}).
\end{aligned}$$

Case 2. Next, if $1 < K \le 2$, we have

$$\left(\frac{1}{K-1}\right)^{\beta-1} \le \left(\frac{1}{K-1}\right)^{K-1} \le \exp\left(\frac{2}{e}\sqrt{K-1}\right),$$

so

$$\begin{aligned}
E &\le K^{K(K-1)}\left(2^{K-1}K^K\right)^{2(K+1)}(2K-1)^K\exp\left(\frac{2}{e}(K+1)\sqrt{K-1}\right)\\
&\le K^{3K(K+1)}2^{2(K-1)(K+1)}K^{2K}\exp\left(\frac{2}{e}(K+1)\sqrt{K-1}\right)\\
&\le \exp[(3K^2+5K)\log K+2(K-1)(K+1)+\frac{2}{e}(K+1)\sqrt{K-1}]\\
&\le \exp[(3K^2+5K)\sqrt{K-1}+3(K+1)\sqrt{K-1}]\\
&= \exp[(3K^2+8K+3)\sqrt{K-1}]\\
&\le \exp(6(K+1)^2\sqrt{K-1})
\end{aligned}$$

where we have used the inequality $\log x \le \sqrt{x-1},\ x>1$. $\square$

For $r\in(0,1)$

$$\varphi^*_{K,n}(r)=\sup\{|f(x)| : f\in QC_K(\mathbf{B}^n), f(0)=0, f(\mathbf{B}^n)\subset \mathbf{B}^n, |x|\le r\},$$

also $\varphi^*_{K,n}(r)\le\varphi_{K,n}(r)$ by [AVV1, (13.3)].

4.20. **Lemma.** [AVV1, Lemma 14.5] *The following equalities hold for $n\ge 2,\ K\ge 1$:*

(1) $\eta^*_{K,n}(1)=\sup\left\{\dfrac{|f(x)|}{|f(y)|} : |x|=|y|>0, f\in\mathcal{F}(n,K)\right\}$

(2) $\eta^*_{K,n}(1)=\sup\left\{\dfrac{|f(x)|}{|f(y)|} : |x|=|y|=t, f\in\mathcal{F}(n,K)\right\}$

for each $t>0$, where $\mathcal{F}(n,K)=\{f\in QC_K(\overline{\mathbf{R}}^n), f(0)=0$ and $f(\infty)=\infty\}$.

4.21. **Theorem.** [AVV1, Theorem 14.6] *For $n\ge 2$ and $K\ge 1$:*

(1) $\eta^*_{K,n}(t)\le\eta^*_{K,n}(1)\varphi^*_{K,n}(t),\ 0\le t\le 1;$

(2) $\eta^*_{K,n}(t)\le\eta^*_{K,n}(1)/\varphi_{1/K,n}(1/t), t\ge 1.$

A refined version of the proof of Theorem 4.19 gives the next theorem.

4.22. **Theorem.** [AVV1, Theorem 14.8] *For $n\ge 2$ and $K\ge 1$,*

$$\eta^*_{K,n}(1)\le\exp(4K(K+1)\sqrt{K-1}).$$

4.23. **Theorem.** [AVV1, Theorem 14.18] *Let $f:\overline{\mathbf{R}}^n\to\overline{\mathbf{R}}^n$ be K-quasiconformal mapping. Then*

$$|f(a),f(b),f(c),f(d)|\le\eta^*_{K,n}(|a,b,c,d|) \tag{4.24}$$

for all distinct points $a,b,c,d\in\overline{\mathbf{R}}^n$.

4.25. **Theorem.** *Let $f:\overline{\mathbf{R}}^n\to\overline{\mathbf{R}}^n$ be a K-quasiconformal mapping with $f(0)=0, f(e_1)=e_1$ and $f(\infty)=\infty$. Then*

$$|f(x)|\le 2^{K-1}K^K\exp(4K(K+1)\sqrt{K-1})\max\{|x|^\alpha,|x|^\beta\}$$

for $x\in\mathbf{R}^n$ and $\beta=K^{1/(n-1)}=1/\alpha$.

Proof. By the definition of $\eta^*_{K,n}$, we have

$$|f(x)| \le \eta^*_{K,n}(|x|).$$

Write $|x| = t$ and $d(K) = \exp(4K(K+1)\sqrt{K-1})$. For $t \ge 1$, we get

$$\eta^*_{K,n}(t) \le \eta^*_{K,n}(1)/\varphi_{1/K,n}(1/t) \le d(K)\lambda_n^{\beta-1}t^{\beta}$$

by Theorems 4.21(2), 4.22 and Lemma 4.13(2).

Again for $0 \le t \le 1$, we obtain

$$|f(x)| \le \eta^*_{K,n}(t) \le \eta^*_{K,n}(1)\varphi^*_{K,n}(t) \le \eta^*_{K,n}(1)\varphi_{K,n}(t) \le d(K)\lambda_n^{1-\alpha}t^{\alpha}$$

by [AVV1, (13.3)] and Theorem 4.21(1). We conclude

$$|f(x)| \le \begin{cases} d(K)\lambda_n^{\beta-1}t^{\beta}, & t \ge 1; \\ d(K)\lambda_n^{1-\alpha}t^{\alpha}, & 0 \le t \le 1. \end{cases}$$

Observing that $\max\{\lambda_n^{1-1/\beta}, \lambda_n^{\beta-1}\} = \lambda_n^{\beta-1}$ we have

$$|f(x)| \le d(K)\lambda_n^{\beta-1}\max\{t^{\alpha}, t^{\beta}\} \le d(K)2^{K-1}K^K\max\{t^{\alpha}, t^{\beta}\}$$

by Lemma 4.14(2). $\square$

4.26. **Corollary.** *Let $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping. Then*

$$|f(z_1), f(z_2), f(z_3), f(z_4)| \le c(K)\max\{(|z_1, z_2, z_3, z_4|)^{\alpha}, (|z_1, z_2, z_3, z_4|)^{\beta}\}$$

for each quadruple of points $z_1, z_2, z_3, z_4 \in \overline{\mathbf{R}}^n$, where $\beta = K^{1/(n-1)} = 1/\alpha$ and $c(K) = 2^{K-1}K^K\exp(4K(K+1)\sqrt{K-1})$.

Proof. We get

$$\begin{aligned} |f(z_1), f(z_2), f(z_3), f(z_4)| &\le \eta^*_{K,n}(|z_1, z_2, z_3, z_4|) \\ &\le c(K)\max\{(|z_1, z_2, z_3, z_4|)^{\alpha}, (|z_1, z_2, z_3, z_4|)^{\beta}\} \end{aligned}$$

by Theorems 4.23 and 4.25. $\square$

4.27. **Lemma.** [KMV, Lemma 4.13] *Let $g(t) = \max\{t^a, t^b\}$, $\quad 0 < a \le 1 \le b$. If $c > 1$ then for $t > 0$*

$$\log(1 + cg(t)) \le cb\max\{\log(1+t), \log^a(1+t)\}.$$

4.28. **Theorem.** *Let $D \subset \mathbf{R}^n$, $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping. Then*

$$\delta_{fD}(f(x), f(y)) \le c(K)\beta\max\{\delta_D(x,y), \delta_D(x,y)^{\alpha}\},$$

where $c(K) = 2^{K-1}K^K\exp(4(K+1)\sqrt{K-1})$ and $\beta = K^{1/(1-n)} = 1/\alpha$.

Proof. Fix $x, y \in D$. For $a, b \in \partial D$ by Corollary 4.26

$$|f(a), f(x), f(b), f(y)| \le c(K)\max\{(|a, x, b, y|)^{\beta}, (|a, x, b, y|)^{\alpha}\}.$$

Write

$$\begin{cases} m_{fD}(f(x), f(y)) = \sup\limits_{f(a), f(b) \in \partial(fD)} |f(a), f(x), f(b), f(y)|, \\ m_D(x,y) = \sup\limits_{a,b \in \partial D} |a, x, b, y|. \end{cases}$$

Choose $a, b \in \partial D$ such that

$$m_{fD}(f(x), f(y)) = |f(a), f(x), f(b), f(y)|.$$

Then

$$\begin{aligned} m_{fD}(f(x), f(y)) + 1 &\leq c(K) \max\{|a, x, b, y|^\beta, |a, x, b, y|^\alpha\} + 1 \\ &\leq c(K) \max\{m_D(x, y)^\beta, m_D(x, y)^\alpha\} + 1. \end{aligned}$$

This yields

$$\begin{aligned} \log(m_{fD}(f(x), f(y)) + 1) &\leq \log(c(K) \max\{m_D(x, y)^\beta, m_D(x, y)^\alpha\} + 1) \\ &\leq c(K)\beta \max\{\log(1 + m_D(x, y)), \log^\alpha(1 + m_D(x, y))\} \\ &= c(K)\beta \max\{\delta_D(x, y), \delta_D^\alpha(x, y)\} \end{aligned}$$

by Lemma 4.27. $\square$

4.29. **Theorem.** [AQV] *For $K > 1$, the function $f(t) = \eta_K(t) - \log(1 + \eta_K(e^t - 1))$ is strictly increasing on $(0, \infty)$. In particular,*

$$\eta_K(t) > \log(1 + \eta_K(e^t - 1)) \tag{4.30}$$

for all $K > 1$ and $t > 0$.

4.31. **Corollary.** [S, Corollary 5.8] *If $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is a K-quasiconformal mapping, G and $G' = fG$ open sets of $\overline{\mathbf{R}}^n$ with* card $\partial G \geq 2$, *and $x, y \in G$, then*

$$\delta_{G'}(f(x), f(y)) \leq \eta_K(\delta_G(x, y)).$$

Next we shall study the conformal invariants λ_D and μ_D, especially when D is a quasiball.

4.32. **Theorem.** [Vu1, Theorem 8.6] *For all distinct $x, y \in \mathbf{B}^n$,*

$$\mu_{\mathbf{B}^n}(x, y) = \gamma_n \left(\frac{1}{\operatorname{th}(\rho(x, y)/2)} \right), \tag{4.33}$$

$$\lambda_{\mathbf{B}^n}(x, y) = \frac{1}{2} \tau_n \left(\operatorname{sh}^2(\rho(x, y)/2) \right). \tag{4.34}$$

4.35. **Theorem.** [S, Theorem 5.10] *If $f : D \to D'$ is a K-quasiconformal map, where D and D' are proper subdomains of $\overline{\mathbf{R}}^n$, then*

$$\frac{\mu_D(x, y)}{K} \leq \mu_{D'}(f(x), f(y)) \leq K \mu_D(x, y) \ for \ all \ x, y \in D.$$

If also card $\partial D \geq 2$, then

$$\frac{\lambda_D(x, y)}{K} \leq \lambda_{D'}(f(x), f(y)) \leq K \lambda_D(x, y)$$

for all distinct $x, y \in D$.

4.36. **Theorem.** [S, Theorem 5.11] *Let $D \subset \overline{\mathbf{R}}^n$ be a K-quasiball and let $x, y \in D$ be distinct points. Then*

$$\frac{1}{K}\gamma_n\left(\frac{1}{\operatorname{th}(r/2)}\right) \le \mu_D(x,y) \le K\gamma_n\left(\frac{1}{\operatorname{th}(s/2)}\right) \tag{4.37}$$

and

$$\frac{1}{2K}\tau_n\left(\operatorname{sh}^2\frac{s}{2}\right) \le \lambda_D(x,y) \le \frac{K}{2}\tau_n\left(\operatorname{sh}^2\frac{r}{2}\right), \tag{4.38}$$

where

$$r = \log(1+\eta_{K,n}^{-1}(e^{\delta_D(x,y)}-1)),$$

$$s = \log(1+\eta_{K,n}(e^{\delta_D(x,y)}-1)).$$

If $n = 2$, then also

$$\frac{1}{K}\gamma_2\left(\frac{1}{\operatorname{th}(\eta_K^{-1}(\delta_D)/2)}\right) \le \mu_D(x,y) \le K\gamma_2\left(\frac{1}{\operatorname{th}(\eta_K(\delta_D)/2)}\right) \tag{4.39}$$

and

$$\frac{1}{2K}\tau_2\left(\operatorname{sh}^2\frac{\eta_K(\delta_D)}{2}\right) \le \lambda_D(x,y) \le \frac{K}{2}\tau_2\left(\operatorname{sh}^2\frac{\eta_K^{-1}(\delta_D)}{2}\right). \tag{4.40}$$

Proof. Let $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal map for which $f(D) = \mathbf{B}^n$. Theorem 4.23 implies that

$$\log(1+\eta_{K,n}^{-1}(e^{\delta_D(x,y)}-1)) \le \rho(f(x),f(y)) \le \log(1+\eta_{K,n}(e^{\delta_D(x,y)}-1)).$$

The inequalities (4.37) and (4.38) now follow from Theorem 4.32 and 4.35. The inequalities in the case of $n = 2$ follow from (4.37) and (4.38), since $r \ge \eta_{K,n}^{-1}(\delta_D(x,y))$ and $s \le \eta_K(\delta_D(x,y))$ by (4.30). $\square$

4.41. **Theorem.** [S, Theorem 5.12] *Let $D_i \subset \overline{\mathbf{R}}^n$ be a K_i-quasiball, $i = 1, 2$ and $f : D_1 \to fD_1$ a K-quasiconformal map such that $fD_1 \subset D_2$. Then*

$$\operatorname{th}\frac{r}{2} \le \varphi_{K',n}\left(\operatorname{th}\frac{s}{2}\right), \tag{4.42}$$

where

$$\begin{aligned} r &= \log(1+\eta_{K_2,n}^{-1}(\exp(\delta_{D_2}(f(x),f(y))-1)), \\ s &= \log(1+\eta_{K_1,n}(\exp(\delta_{D_1}(x,y))-1)), \end{aligned}$$

$K' = KK_1K_2$, and $x, y \in D_1$. In particular, if $K_1 = K_2 = 1$, this reduces to the Schwarz lemma 4.12. If $n = 2$, then also

$$\operatorname{th}\frac{\eta_{K_2}^{-1}(\delta_{D_2}(f(x),f(y)))}{2} \le \varphi_{K'}\left(\operatorname{th}\frac{\eta_{K_1}(\delta_{D_1}(x,y))}{2}\right). \tag{4.43}$$

Proof. By (4.37), Remark 4.6 and Theorem 4.35,

$$\frac{1}{K_2}\gamma_n\left(\frac{1}{\operatorname{th}(r/2)}\right) \le \mu_{D_2}(f(x),f(y)) \le \mu_{D_1}(x,y) \le KK_1\gamma_n\left(\frac{1}{\operatorname{th}(s/2)}\right).$$

Then

$$\operatorname{th}\frac{r}{2} \le \frac{1}{\gamma_n^{-1}(KK_1K_2(1/\operatorname{th}(s/2)))} \le \varphi_{K',n}\left(\operatorname{th}\frac{s}{2}\right)$$

by formula (4.8). For $n=2$, the inequality (4.43) follows, since $r \ge \eta_{K_2}^{-1}(\delta_{D_2}(f(x),f(y)))$ and $s \le \eta_{K_1}(\delta_{D_1}(x,y))$ by (4.30). □

5. **Bounds for various metrics**

F. W. Gehring and K. Hag [GH1] investigated the variation of metrics in the two-dimensional case. Here we refine their results with new constant and also we discuss the n-dimensional case of their results. We also discuss some other results in this section.

5.1. **Theorem.** *If $f:\overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}^2$ is K-quasiconformal mapping which fixes $0,1$ and ∞, then*

$$|f(x)| \le e^{\pi(K-1/K)}\max\{|x|^K, |x|^{1/K}\}$$

for $x \in \overline{\mathbf{R}}^2$.

Proof. We have

$$|f(x)| \le \eta^*_{K,2}(|x|) = \eta_{K,2}(|x|)$$

by the definition of $\eta^*_{K,n}$ and [AVV1, (14.4)]. This implies

$$\begin{aligned}|f(x)| &\le \eta_{K,2}(|x|)\\ &\le \lambda(K)\max\{|x|^K,|x|^{1/K}\}\\ &\le e^{\pi(K-1/K)}\max\{|x|^K,|x|^{1/K}\}\end{aligned}$$

by [AVV1, Theorem 10.24] and [Vu1, Remark 10.31]. □

5.2. **Corollary.** *If $f:\overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}^2$ is K-quasiconformal mapping, then*

$$|f(z_1),f(z_2),f(z_3),f(z_4)| \le e^{\pi(K-1/K)}\max\{(|z_1,z_2,z_3,z_4|)^K, (|z_1,z_2,z_3,z_4|)^{1/K}\}$$

for each quadruple of points $z_1,z_2,z_3,z_4 \in \mathbf{R}^2$.

Proof. We get

$$\begin{aligned}|f(z_1),f(z_2),f(z_3),f(z_4)| &\le \eta^*_{K,2}(|z_1,z_2,z_3,z_4|) = \eta_{K,2}(|z_1,z_2,z_3,z_4|)\\ &\le \lambda(K)\max\{(|z_1,z_2,z_3,z_4|)^\alpha,(|z_1,z_2,z_3,z_4|)^\beta\}\\ &\le e^{\pi(K-1/K)}\max\{(|z_1,z_2,z_3,z_4|)^\alpha,(|z_1,z_2,z_3,z_4|)^\beta\}\end{aligned}$$

by Theorem 4.23, [AVV1, (14.4), Theorem 10.24] and [Vu1, Remark 10.31]. □

5.3. **Theorem.** *If* $f : \overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}^2$ *is a* K*-quasiconformal mapping with* $f(\infty) = \infty$, *then for each proper subdomain* D *of* $\mathbf{R}^2$,

$$\tilde{\jmath}_{fD}(f(z_1), f(z_2)) \le e^{\pi(K-1/K)} K \max\{\tilde{\jmath}_D(z_1, z_2), \tilde{\jmath}_D^{1/K}(z_1, z_2)\}$$

for $z_1, z_2 \in D$.

Proof. Fix $z_1, z_2 \in D$ and choose $w_1, w_2 \in \partial D$ so that

$$|f(z_k) - f(w_k)| = \mathrm{dist}(f(z_k), \partial(fD))$$

for $k = 1, 2$. By Corollary 5.2 we get

$$\begin{aligned}
\frac{|f(z_1) - f(z_2)|}{\mathrm{dist}(f(z_1), \partial(fD))} + 1 &= |(f(z_2), f(\infty), f(z_1), f(w_1))| + 1 \\
&\le e^{\pi(K-1/K)} \max\{(|z_2, \infty, z_1, w_1|)^K, (|z_2, \infty, z_1, w_1|)^{1/K}\} + 1 \\
&= e^{\pi(K-1/K)} \max\left\{\left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right)^K, \left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right)^{1/K}\right\} + 1.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{|f(z_1) - f(z_2)|}{\mathrm{dist}(f(z_2), \partial(fD))} + 1 &= |(f(z_1), f(\infty), f(z_2), f(w_2))| + 1 \\
&\le e^{\pi(K-1/K)} \max\left\{\left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_2, \partial D)}\right)^K, \left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_2, \partial D)}\right)^{1/K}\right\} + 1.
\end{aligned}$$

We denote

$$a = \left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right), \quad b = \left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_2, \partial D)}\right),$$

and conclude that

$$\begin{aligned}
\tilde{\jmath}_{fD}(f(z_1), f(z_2)) &= \log\left(\frac{|f(z_1) - f(z_2)|}{\mathrm{dist}(f(z_1), \partial(fD))} + 1\right)\left(\frac{|f(z_1) - f(z_2)|}{\mathrm{dist}(f(z_2), \partial(fD))} + 1\right) \\
&\le \log\left(e^{\pi(K-1/K)} \max\{a^K, a^{1/K}\} + 1\right) + \log\left(e^{\pi(K-1/K)} \max\{b^K, b^{1/K}\} + 1\right) \\
&\le e^{\pi(K-1/K)} K \max\{\log\left((1+a)(1+b)\right), \log^{1/K}\left((1+a)(1+b)\right)\} \\
&= e^{\pi(K-1/K)} K \max\{\tilde{\jmath}_D(z_1, z_2), \tilde{\jmath}_D^{1/K}(z_1, z_2)\}
\end{aligned}$$

by Lemma 4.27. $\square$

If D is a disk or half plane then

$$\rho_D(z_1, z_2) \le \tilde{\jmath}_D(z_1, z_2) \tag{5.4}$$

(see [GH2]), where ρ is the hyperbolic metric.

5.5. **Corollary.** *If D is a domain in $\mathbf{R}^2$ and if there exists a K-quasiconformal self mapping f of $\mathbf{R}^2$ which maps D conformally onto a disk or a half plane, then*

$$\rho_D(z_1,z_2) \le e^{\pi(K-1/K)} K \max\{\tilde{j}_D(z_1,z_2), \tilde{j}_D^{1/K}(z_1,z_2)\}$$

for $z_1, z_2 \in D$.

Proof. If $z_1, z_2 \in D$, then

$$\begin{aligned} \rho_D(z_1,z_2) &= \rho_{fD}(f(z_1), f(z_2)) \\ &\le \tilde{j}_{fD}(f(z_1), f(z_2)) \\ &\le e^{\pi(K-1/K)} K \max\{\tilde{j}_D(z_1,z_2), \tilde{j}_D^{1/K}(z_1,z_2)\} \end{aligned}$$

by the conformal invariance of hyperbolic distance, (5.4) and Theorem 5.3. □

5.6. **Theorem.** *If $D \subset \mathbf{R}^2$ is a K-quasidisk, then*

$$\rho_D(z_1,z_2) \le e^{\pi(K^2-1/K^2)} K^2 \max\{\tilde{j}_D(z_1,z_2), \tilde{j}_D^{1/K^2}(z_1,z_2)\} \tag{5.7}$$

for $z_1, z_2 \in D$.

Proof. By hypothesis there exists a K-quasiconformal self mapping of $\overline{\mathbf{C}}$ which maps D onto a disk or a half place.

If D is bounded, then by composing f with a Möbius transformation we may assume that f fixes ∞ and $f(D)$ is the unit disk B. The existence theorem for the Beltrami equation implies there exists a K-quasiconformal self mapping of $g : B \to B$ which fixes 0 such that $g \circ f$ is conformal in D. Reflection in ∂B extends to a K-quasiconformal self mapping of $\mathbf{C}$. Then $h = g \circ f$ is K^2-quasiconformal and we can apply Corollary 5.5 to obtain (5.7). If D is unbounded, then D is the union of an increasing sequence of bounded K-quasidisks D_n which contains z_1, z_2 and

$$\rho_D(z_1,z_2) = \lim_{n\to\infty} \rho_{D_n}(z_1,z_2), \qquad \tilde{j}_D(z_1,z_2) = \lim_{n\to\infty} \tilde{j}_{D_n}(z_1,z_2).$$

Hence (5.7) follows from what was proved above. □

5.8. **Theorem.** *Let $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping with $f(\infty) = \infty$, then for each proper subdomain D of $\mathbf{R}^n$ with $fD \subset \mathbf{R}^n$,*

$$\tilde{j}_{fD}(f(z_1), f(z_2)) \le c(K)\beta \max\{\tilde{j}_D(z_1,z_2), \tilde{j}_D^{\alpha}(z_1,z_2)\}$$

for $z_1, z_2 \in D$, where $c(K) = 2^{K-1}K^K \exp(4K(K+1)\sqrt{K-1})$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

Proof. Fix $z_1, z_2 \in D$ and choose $w_1, w_2 \in \partial D$ so that

$$|f(z_k) - f(w_k)| = \operatorname{dist}(f(z_k), \partial(fD))$$

for $k = 1, 2$. By Corollary 4.26 we get

$$\begin{aligned}
\frac{|f(z_1) - f(z_2)|}{\mathrm{dist}(f(z_1), \partial(fD))} + 1 &= |(f(z_1), f(w_1), f(z_2), f(\infty))| + 1 \\
&\leq c(K) \max\{(|z_1, w_1, z_2, \infty|)^\beta, (|z_1, w_1, z_2, \infty|)^\alpha\} + 1 \\
&= c(K) \max\left\{\left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right)^\beta, \left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right)^\alpha\right\} + 1.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{|f(z_1) - f(z_2)|}{\mathrm{dist}(f(z_2), \partial(fD))} + 1 &= |f(z_2), f(w_2), f(z_1), \infty| + 1 \\
&\leq c(K) \max\left\{\left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_2, \partial D)}\right)^\beta, \left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_2, \partial D)}\right)^\alpha\right\} + 1.
\end{aligned}$$

Write

$$a = \left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_1, \partial D)}\right), \quad b = \left(\frac{|z_1 - z_2|}{\mathrm{dist}(z_2, \partial D)}\right).$$

Now we conclude that

$$\begin{aligned}
\tilde{j}_{fD}(f(z_1), f(z_2)) &= \log\left(\frac{|f(z_1) - f(z_2)|}{\mathrm{dist}(f(z_1), \partial(fD))} + 1\right)\left(\frac{|f(z_1) - f(z_2)|}{\mathrm{dist}(f(z_2), \partial(fD))} + 1\right) \\
&\leq \log\left(c(K)\max\{a^\beta, a^\alpha\} + 1\right) + \log\left(c(K)\max\{b^\beta, b^\alpha\} + 1\right) \\
&\leq c(K)\beta\max\{\log\left((1+a)(1+b)\right), \log^\alpha\left((1+a)(1+b)\right)\} \\
&= c(K)\beta\max\{\tilde{j}_D(z_1, z_2), \tilde{j}_D^\alpha(z_1, z_2)\}
\end{aligned}$$

by Lemma 4.27. □

5.9. **Theorem.** [KMV, Theorem 4.17] *Let $G = \mathbf{R}^n \setminus \{0\}$, $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\infty) = \infty$. There exists $a(K)$ such that for all $x, y \in G$*

$$j_G(f(x), f(y)) \leq a(K) \max\{j_G(x, y), j_G(x, y)^\alpha\},$$

where $\alpha = K^{1/(1-n)}$, $a(K) = (\exp(60\sqrt{K-1}))/\alpha$ and $a(K) \to 1$ as $K \to 1$.

5.10. **Theorem.** *Let $G = \mathbf{R}^n \setminus \{0\}$, $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\infty) = \infty$. Then for all $x, y \in G$*

$$j_{fG}(f(x), f(y)) \leq b(K) \max\{j_G(x, y), j_G(x, y)^\alpha\},$$

where $b(K) = 2^{K-1} K^K \exp(4K(K+1)\sqrt{K-1})\beta$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

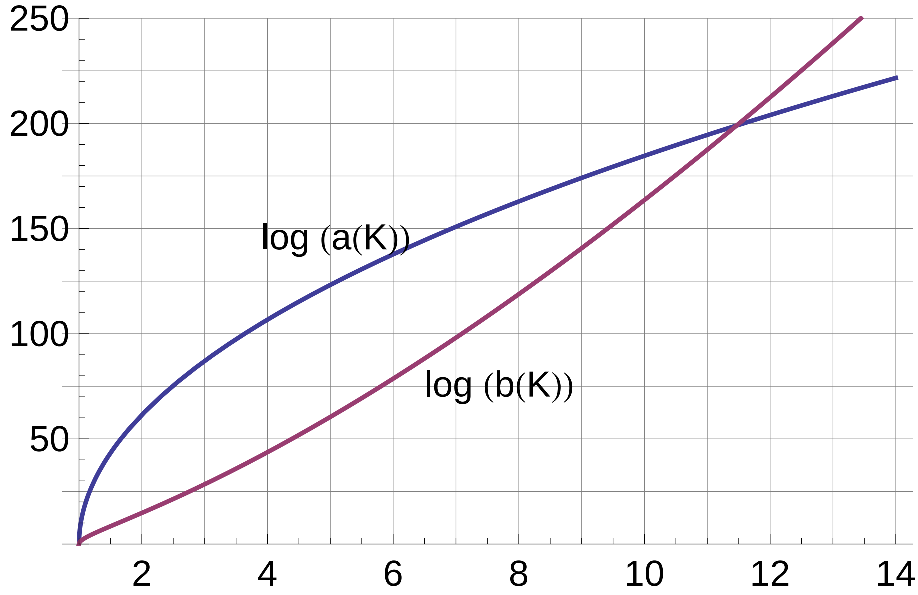

FIGURE 1. Graphical comparison of bounds given in Theorems 5.9 and 5.10.

Proof. Fix $x, y \in G$. For $a, b \in \partial G$ by Corollary 4.26

$$|f(a), f(x), f(b), f(y)| \leq c(K) \max\{(|a, x, b, y|)^{\beta}, (|a, x, b, y|)^{\alpha}\}.$$

Write

$$\begin{cases} m_{fG}(f(x), f(y)) = \sup\limits_{f(a), f(b) \in \partial(fG)} |f(a), f(x), f(b), f(y)|, \\ m_G(x, y) = \sup\limits_{a, b \in \partial G} |a, x, b, y|. \end{cases}$$

Choose $a = 0, b = \infty$ then $a, b \in \partial G$

$$m_{fG}(f(x), f(y)) = |f(a), f(x), f(b), f(y)|.$$

Then by Corollary 4.26 we have

$$\begin{aligned} m_{fG}(f(x), f(y)) + 1 &\leq (b(K)/\beta) \max\{|a, x, b, y|^{\beta}, |a, x, b, y|^{\alpha}\} + 1 \\ &\leq (b(K)/\beta) \max\{m_G(x, y)^{\beta}, m_G(x, y)^{\alpha}\} + 1. \end{aligned}$$

This yields

$$\begin{aligned} \delta_{fG}(f(x), f(y)) &= \log(m_{fG}(f(x), f(y)) + 1) = j_{fG}(f(x), f(y)) \\ &\leq \log((b(K)/\beta) \max\{m_G(x, y)^{\beta}, m_G(x, y)^{\alpha}\} + 1) \\ &\leq b(K) \max\{\log(1 + m_G(x, y)), \log^{\alpha}(1 + m_G(x, y))\} \\ &= b(K) \max\left\{\log\left(1 + \frac{|x - y|}{\min\{d(x), d(y)\}}\right), \log^{\alpha}\left(1 + \frac{|x - y|}{\min\{d(x), d(y)\}}\right)\right\} \\ &= b(K) \max\{j_G(x, y), j_G^{\alpha}(x, y)\} \end{aligned}$$

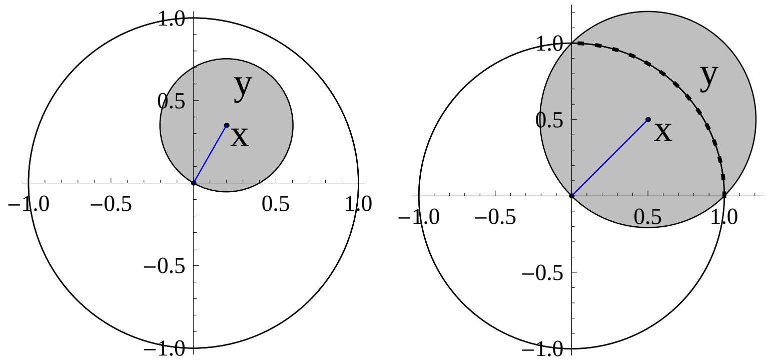

FIGURE 2. The set $\{y \in \mathbf{C} : |x-y| < |x|\}, x \in \mathbf{B}^n \setminus \{0\}$ given in (5.13) in two cases; (a) $|x| < 1/2$ (left) and (b) $|x| \in (1/2, 1)$ (right).

by Remark 3.16(3), Lemma 4.27. $\square$

Solving numerically the equation $a(K) = b(K)$ for K we obtain $K = 11.4664$. It is easy to see that $b(K)$ is better than $a(K)$ for $K \in (1, 11.7641)$. For graphical purpose, logarithmic scale is used (see Figure 1).

5.11. **Theorem.** *For $n \geq 2, K \geq 1$, let $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$, $f(e_1) = e_1$ and $f(\infty) = \infty$. Then*

$$|f(x) - f(y)| \leq c(K)^2 |x-y|^\beta \tag{5.12}$$

when $1 < |x| < |x-y|$ for $x \in \mathbf{R}^n \setminus \overline{\mathbf{B}}^n, y \in \mathbf{R}^n$, and

$$|f(x) - f(y)| \leq c(K)^2 |x-y|^\alpha \tag{5.13}$$

when $|x-y| < |x| < 1$ for $x \in \mathbf{B}^n, y \in \mathbf{R}^n$. Here $c(K) = 2^{K-1}K^K \exp(4K(K+1)\sqrt{K-1})$ and $\beta = K^{1/(n-1)} = 1/\alpha$.

Proof. By Corollary 4.26

$$|f(x), f(0), f(y), f(\infty)| \leq c(K) \max\{|x,0,y,\infty|^\alpha, |x,0,y,\infty|^\beta\}$$

this implies that

$$\frac{|f(x) - f(y)|}{|f(x)|} \leq c(K) \max\left\{ \left(\frac{|x-y|}{|x|}\right)^\alpha, \left(\frac{|x-y|}{|x|}\right)^\beta \right\}.$$

Also

$$\begin{aligned} |f(x) - f(y)| &\leq c(K)|f(x)| \max\left\{ \left(\frac{|x-y|}{|x|}\right)^\alpha, \left(\frac{|x-y|}{|x|}\right)^\beta \right\} \\ &\leq c(K) \max\{|x|^\alpha, |x|^\beta\} c(K) \max\left\{ \left(\frac{|x-y|}{|x|}\right)^\alpha, \left(\frac{|x-y|}{|x|}\right)^\beta \right\} \end{aligned}$$

by Theorem 4.25, and we can see that inequalities (5.12) and (5.13) are obvious. $\square$

5.14. **Theorem.** *Let* $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ *be a* K*-quasiconformal mapping, with* $f(0) = 0, f(e_1) = e_1$ *and* $f(\infty) = \infty$. *Then*

$$|f(x) - e_1| \le \eta^*_{K,n}(1)\lambda_n^{\beta-1} \max\{|x - e_1|^\alpha, |x - e_1|^\beta\}$$

for all $x \in \mathbf{R}^n$ *and* $\alpha = K^{1/(1-n)} = 1/\beta$.

Proof. By the definition of quasisymmetry we have

$$\frac{|f(x) - f(e_1)|}{|f(0) - f(e_1)|} \le \eta^*_{K,n}\left(\frac{|x - e_1|}{|0 - e_1|}\right) \Longrightarrow |f(x) - f(e_1)| \le \eta^*_{K,n}(|x - e_1|).$$

By Theorem 4.21 and Lemma 4.13 we get

$$\eta^*_{K,n}(|x - e_1|) \le \eta^*_{K,n}(1)\lambda_n^{1-\alpha}|x - e_1|^\alpha \quad for \quad 0 \le |x - e_1| \le 1$$

and

$$\eta^*_{K,n}(|x - e_1|) \le \eta^*_{K,n}(1)\lambda_n^{\beta-1}|x - e_1|^\beta \quad for \quad |x - e_1| \ge 1,$$

respectively. Hence

$$\begin{aligned} |f(x) - e_1| &\le \max\{\eta^*_{K,n}(1)\lambda_n^{1-\alpha}|x - e_1|^\alpha, \eta^*_{K,n}(1)\lambda_n^{\beta-1}|x - e_1|^\beta\} \\ &= \eta^*_{K,n}(1)\lambda_n^{\beta-1} \max\{|x - e_1|^\alpha, |x - e_1|^\beta\} \end{aligned}$$

here we use the fact that $\max\{\lambda_n^{1-\alpha}, \lambda_n^{\beta-1}\} = \lambda_n^{\beta-1}$. ☐

5.15. **Lemma.** [Vu1, Lemma 7.35] *Let* $R = R(E, F)$ *be a ring in* $\overline{\mathbf{R}}^n$ *and let* $a, b \in E, c, d \in F$ *be distinct points. Then*

$$\operatorname{cap} R = M(\Delta(E, F)) \ge \tau_n(|a, b, c, d|).$$

Equality holds if $b = t_1e_1, a = t_2e_1, c = t_3e_1, d = t_4e_1$ *and* $t_1 < t_2 < t_3 < t_4$.

5.16. **Theorem.** *For* $n \ge 2$, $K \ge 1$, *let* $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ *be a* K*-quasiconformal mapping with* $f(0) = 0$, $f(e_1) = e_1$ *and* $f(\infty) = \infty$. *Then*

$$|f(x)| \le K\lambda_n^{2(1-\alpha)}|x|^\alpha$$

for $0 < |x| \le (K - 1)/K$ *and for all* $x \in \mathbf{R}^n \setminus \{0\}$.

Proof. Let Γ be the family $\Delta([0, x], [e_1, \infty]; \overline{\mathbf{R}}^n)$ and $\Gamma' = f(\Gamma)$. Then by Lemma 5.15 and [AVV1, Theorem 8.47], we get

$$\tau_n(|f(0), f(x), f(e_1), f(\infty)|) \le M(\Gamma') \le KM(\Gamma) \le K\tau_n(|0, x, e_1, \infty| - 1)$$

this implies that $\tau_n(1/|f(x)|) \le K\tau_n(1/|x| - 1)$. By the identity (4.3) we get

$$\gamma_n\left(\sqrt{\frac{1}{|f(x)|} + 1}\right) \le K\gamma_n\left(\frac{1}{\sqrt{x}}\right). \tag{5.17}$$

Applying γ_n^{-1} to (5.17), we obtain

$$\sqrt{\frac{1}{|f(x)|} + 1} \ge \gamma_n^{-1}\left(K\gamma_n\left(\frac{1}{\sqrt{x}}\right)\right),$$

which implies

$$
\begin{aligned}
|f(x)| &\le \frac{1}{\left(\gamma_n^{-1}\left(K\gamma_n\left(1/\sqrt{x}\right)\right)\right)^2-1}\\
&= \frac{1}{1/\varphi_{K,n}^2(\sqrt{x})-1} = \frac{\varphi_{K,n}^2(\sqrt{x})}{1-\varphi_{K,n}^2(\sqrt{x})}\\
&\le \frac{\varphi_{K,n}^2(\sqrt{x})}{1/K} \le K\lambda_n^{2(1-\alpha)}|x|
\end{aligned}
$$

by the definition of φ_K and Lemma 4.13, also here we use $1-\varphi_{K,n}^2(\sqrt{x}) \ge 1/K$ from the proof of Theorem 4.19. $\square$

6. On the Hölder continuity of Quasiconformal maps

In this section we will discuss the Hölder continuity of quasiconformal mappings of the unit ball with respect to Euclidean metric. This section is taken from [BV]. We denote the hyperbolic cosine and its inverse function by ch and arch, respectively.

6.1. **Lemma.** *Suppose that $f : \mathbf{B}^n \to \mathbf{B}^n$ is a K-quasiconformal mapping with $f\mathbf{B}^n = \mathbf{B}^n$, $f(0)=0$, and let $h : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be the inversion $h(x) = x/|x|^2$, $h(\infty) = 0, h(0)=\infty$, and define $g : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ by $g(x) = f(x)$ for $x \in \mathbf{B}^n, g(x) = h(f(h(x)))$ for $x \in \mathbf{R}^n \setminus \overline{\mathbf{B}}^n$ and $g(x) = \lim_{z\to x} f(z)$ for $x \in \partial\mathbf{B}^n, g(\infty)=\infty$. Then g is a K-quasiconformal mapping, and we have for $x \in \mathbf{B}^n$*

$$\varphi_{1/K,n}(|x|) \le |f(x)| \le \varphi_{K,n}(|x|). \tag{6.2}$$

For $x \in \mathbf{R}^n \setminus \overline{\mathbf{B}}^n$

$$1/\varphi_{K,n}(1/|x|) \le |g(x)| \le 1/\varphi_{1/K,n}(1/|x|). \tag{6.3}$$

Proof. It is well-known that the above definition defines g as a K-quasiconformal homeomorphism. The formula (6.2) is well-known (see [AVV2, Theorem 4.2]) and (6.3) follows easily. $\square$

For $x \in \mathbf{R}^n \setminus \{0, e_1\}, n \ge 2$, we define the *Teichmüller function* by

$$p_n(x) = \inf_{E,F} M(\Delta(E,F))$$

where the infimum is taken over all the pairs of continua E and F in $\overline{\mathbf{R}}^n$ with $0, e_1 \in E$ and $x, \infty \in F$.

6.4. **Lemma.** [HV, Theorem 3.20] *For $z \in \mathbf{R}^n, |z| > 1$, the following inequalities hold:*

$$\tau_n(|z|) = p_n(-|z|e_1) \le p_n(z) \le p_n(|z|e_1) = \tau_n(|z|-1)$$

where $p_n(z)$ is the Teichmüller function. Furthermore, for $z \in \mathbf{R}^n \setminus \{0, e_1\}$, the upper bound may be refined to

$$p_n(z) \le \tau_n\left(\frac{|z|+|z-e_1|-1}{2}\right) = M(\Delta(E,F;\mathbf{R}^n)) \le \tau_n(|z|-1) \tag{6.5}$$

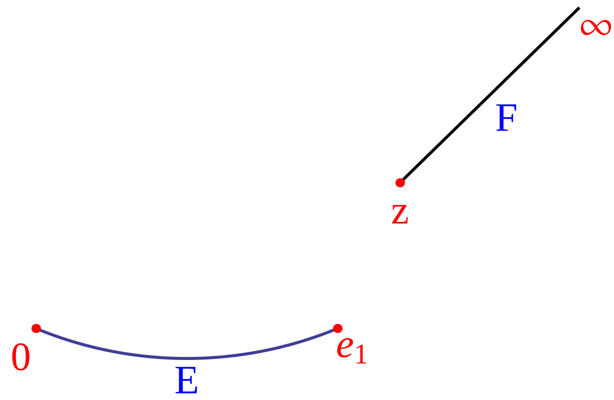

FIGURE 3. Circular arc E and ray F given in Lemma 6.4.

with equality in the first inequality both for $z = -se_1, s > 0$, $z = se_1, s > 1$, *and where* E *is a circular arc with* $0, e_1 \in E$ *and* F *is a ray with* $z, \infty \in F$.

6.6. **Theorem.** *For* $n \geq 2, K \geq 1$, *let* $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ *be a* K*-quasiconformal mapping with* $f\mathbf{B}^n \subset \mathbf{B}^n$, $f(0) = 0$ *and* $f(\infty) = \infty$. *Then*

$$\begin{aligned} |f(x) - f(y)| &\leq (3 + \varphi_{1/K,n}(1/t)^{-1})\varphi^2_{K,n}\left(\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{1/2}\right) \\ &\leq (3 + \lambda_n^{(\beta-1)}t^{\beta})\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s_1 + |x-y|}\right)^{\alpha}, \end{aligned}$$

where $\alpha = K^{1/(1-n)} = 1/\beta$ *and* $s_1 = \max\{t + |x| + |y + t\frac{x}{|x|}|, t + |y| + |x + t\frac{y}{|y|}|\}, t > 1$, *for all* $x, y \in \mathbf{B}^n$.

Proof. Let Γ be a family $\Delta(E, F; \overline{\mathbf{R}}^n)$, E and F are connected sets as in Lemma 6.4 with $x, y \in E, z, \infty \in F$, where $z = -tx/|x|$ and $\Gamma' = f(\Gamma)$. By Lemma 5.15 and (6.5), we have

$$\begin{aligned} \tau_n\left(\frac{|f(z) - f(x)|}{|f(x) - f(y)|}\right) &\leq M(\Gamma') \leq KM(\Gamma) \\ &\leq K\tau_n\left(\frac{|x-z| + |z-y| - |x-y|}{2|x-y|}\right) = K\tau_n(u-1), \end{aligned}$$

where $u = \dfrac{|x-z| + |z-y| + |x-y|}{2|x-y|}$ and by (4.3) we get

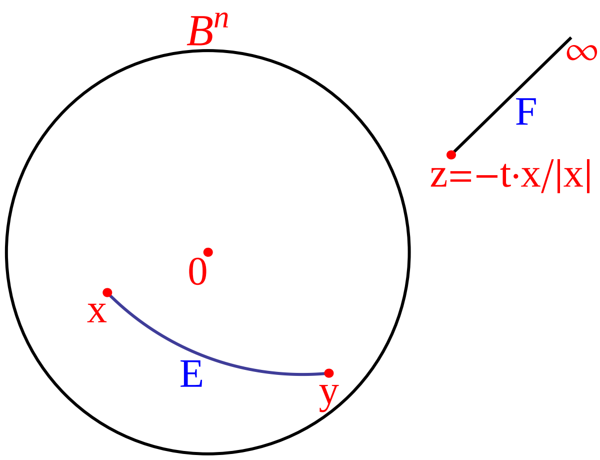

FIGURE 4. Geometrical meaning of the proof of Theorem 6.6.

$$\begin{aligned}\gamma_n\left(\left(\frac{|f(z)-f(y)|+|f(x)-f(y)|}{|f(x)-f(y)|}\right)^{1/2}\right) &\leq K\gamma_n\left((u)^{1/2}\right)\\ &= K\gamma_n\left(\left(\frac{t+|x|+|y+t\frac{x}{|x|}|+|x-y|}{2|x-y|}\right)^{1/2}\right).\end{aligned}$$

Write $a=t+|x|+|y+t\frac{x}{|x|}|$. Applying γ^{-1} to (6.7) we have

$$\frac{|f(z)-f(y)|+|f(x)-f(y)|}{|f(x)-f(y)|}\geq\left(\gamma_n^{-1}\left(K\gamma_n\left(\left(\frac{a+|x-y|}{2|x-y|}\right)^{1/2}\right)\right)\right)^2.$$

Because $f\mathbf{B}^n\subset\mathbf{B}^n$, by (6.3) and Lemma 4.13(2) we know that $|f(z)-f(y)|+|f(x)-f(y)|\leq 3+\varphi_{1/K,n}(1/t)^{-1}\leq 3+\lambda_n^{(\beta-1)}t^\beta$,

$$\begin{aligned}\frac{|f(x)-f(y)|}{3+\lambda_n^{(\beta-1)}t^\beta} &\leq \frac{|f(x)-f(y)|}{|f(z)-f(y)|+|f(x)-f(y)|}\\ &\leq \frac{1}{\left(\gamma_n^{-1}\left(K\gamma_n\left(\left(\frac{a+|x-y|}{2|x-y|}\right)^{1/2}\right)\right)\right)^2},\end{aligned}$$

also

$$\begin{aligned}|f(x)-f(y)| &\leq (3+\lambda_n^{(\beta-1)}t^\beta)\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{a+|x-y|}\right)^{1/2}\right)\\ &\leq (3+\lambda_n^{(\beta-1)}t^\beta)\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{a+|x-y|}\right)^\alpha\end{aligned}$$

by inequalities (4.3), (4.8) and Lemma 4.13(1).
Exchanging the role of x and y we see that

$$\begin{aligned}|f(x)-f(y)| &\le (3+\lambda_n^{(\beta-1)}t^\beta)\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{\max\{a,b\}+|x-y|}\right)^{1/2}\right)\\ &\le (3+\lambda_n^{(\beta-1)}t^\beta)\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{\max\{a,b\}+|x-y|}\right)^{\alpha}\end{aligned}$$

where $b=t+|y|+|x+t\frac{y}{|y|}|$. $\square$

6.7. **Corollary.** *For $n\ge 2, K\ge 1$, let $f:\overline{\mathbf{R}}^n\to\overline{\mathbf{R}}^n$ be a K-quasiconformal mapping with $f\mathbf{B}^n\subset\mathbf{B}^n$, $f(0)=0$ and $f(\infty)=\infty$. Then for all $x,y\in\mathbf{B}^n$*

$$|f(x)-f(y)|\le 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{s+|x-y|}\right)^{\alpha},$$

where $\alpha=K^{1/(1-n)}$ and $s=\max\{1+|x|+|y+\dfrac{x}{|x|}|,1+|y|+|x+\dfrac{y}{|y|}|\}$.

Proof. Proof is similar to the proof of the previous theorem. Here we consider $t=1$. Because $f\mathbf{B}^n\subset\mathbf{B}^n$, we know that $|f(z)-f(y)|+|f(x)-f(y)|\le 4$,

$$\begin{aligned}\frac{|f(x)-f(y)|}{4} &\le \frac{|f(x)-f(y)|}{|f(z)-f(y)|+|f(x)-f(y)|}\\ &\le \frac{1}{\left(\gamma_n^{-1}\left(K\gamma_n\left(\left(\frac{a+|x-y|}{2|x-y|}\right)^{1/2}\right)\right)\right)^2},\end{aligned}$$

also

$$\begin{aligned}|f(x)-f(y)| &\le 4\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{a+|x-y|}\right)^{1/2}\right)\\ &\le 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{a+|x-y|}\right)^{\alpha}\end{aligned}$$

by (4.8) and Lemma 4.13(1), where $a=1+|x|+|y+\frac{x}{|x|}|$. Exchanging the roles of x and y we get

$$|f(x)-f(y)|\le 4\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{\max\{a,b\}+|x-y|}\right)^{\alpha}$$

where $b=1+|y|+|x+\frac{y}{|y|}|$. $\square$

6.8. **Corollary.** *For $n\ge 2, K\ge 1, t\ge 1$, let f be as in Theorem 6.6. Then*

$$|f(x)-f(y)|\le (3+\lambda_n^{(\beta-1)}t^\beta)\lambda_n^{2(1-\alpha)}\left(\frac{2|x-y|}{2t+||x|-|y||+|x-y|}\right)^{\alpha}, \tag{6.9}$$

for all $x, y \in \mathbf{B}^n$,

$$|f(x) - f(y)| \le (3 + \lambda_n^{(\beta-1)} t^\beta) \lambda_n^{2(1-\alpha)} \left(\frac{|x-y|}{\max\{t+|x|, t+|y|\}} \right)^\alpha , \tag{6.10}$$

for all $x, y \in \mathbf{B}^n$, *and*

$$|f(x) - f(y)| \le (3 + \lambda_n^{(\beta-1)} t^\beta) \lambda_n^{2(1-\alpha)} \left(\frac{|x-y|}{t+|x|+(|x-y|)/2} \right)^\alpha , \tag{6.11}$$

if $|y + t\frac{x}{|x|}| > t + |x|, x, y \in \mathbf{B}^n$.

Proof. Write $B(y,x) = |y + t\frac{x}{|x|}|$. Inequality (6.9) follows because $B(y,x) > t - |y|$ and $B(x,y) > t - |x|$ for $x, y \in \mathbf{B}^n$, and hence

$$s_1 \ge \max\{2t + |x| - |y|, 2t + |y| - |x|\} = 2t + ||x| - |y|| \, .$$

It is clear that $B(y,x) \ge t + |x| - |x-y|$, and this implies that

$$s_1 \ge \max\{2(t+|x|) - |x-y|, 2(t+|y|) - |x-y|\} = 2\max\{t+|x|, t+|y|\} - |x-y|$$

and hence inequality (6.10) follows. Because $B(y,x) > t + |x|$, we see that $s > 2(t+|x|)$ and (6.11) holds. $\square$

6.12. **Definition.** Let $QC_K(\mathbf{B}^n)$, $K \ge 1$, with $f(0) = 0$. Then for all $K \ge 1$, $n \ge 2$, there exists a least constant $M(n,K) \ge 1$ such that

$$|f(x) - f(y)| \le M(n,K)|x-y|^\alpha, \quad \alpha = K^{1/(1-n)} ,$$

for all $f \in QC_K(\mathbf{B}^n), x, y \in \mathbf{B}^n$.

It is natural to expect that for all $n \ge 2$, $M(n,K) \to 1$ when $K \to 1$. This fact was pointed out by R. Fehlmann and M. Vuorinen in the following theorem.

6.13. **Theorem.** [FV, Theorem 1.3] *Let f be a K-quasiconformal mapping of $\mathbf{B}^n$ onto $\mathbf{B}^n$, $n \ge 2$, $f(0) = 0$. Then*

$$|f(x) - f(y)| \le M(n,K)|x-y|^\alpha \tag{6.14}$$

for all $x, y \in \mathbf{B}^n$ where $\alpha = K^{1/(1-n)}$ and the constant $M(n,K)$ has the following three properties:

1. *$M(n,K) \to 1$ as $K \to 1$, uniformly in n;*
2. *$M(n,K)$ remains bounded for fixed K and varying n;*
3. *$M(n,K)$ remains bounded for fixed n and varying K;*

G.D. Anderson and M. K. Vamanamurthy proved the following theorem in [AV].

6.15. **Theorem.** *For $n \ge 2, K \ge 1$,*

$$M(n,K) \le 4\lambda_n^{2(1-\alpha)} ,$$

where $\alpha = K^{1/(1-n)}$ and $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$, is the Grötzsch ring constant [AN], [Vu1, p.89].

As far as we know, the best upper bound known today for $n = 2$ is $M(2,K) \le 46^{1-1/K}$ due to S.-L. Qiu [Q] (1997)

The following theorem 6.16 refines Theorem 6.15.

6.16. **Theorem.** *(1) For $n \ge 2, K \ge 1$, $M(n,K) \le T(n,K)$ where for $t \ge 1, \alpha = K^{1/(1-n)} = 1/\beta$,*

$$T(n,K) \le \inf\{h(t) : t \ge 1\}\,, \quad h(t) = (3 + \lambda_n^{\beta-1} t^{\beta}) t^{-\alpha} \lambda_n^{2(1-\alpha)} \tag{6.17}$$

and λ_n is as in Theorem 6.15.

(2) There exists a number $K_1 > 1$ such that for all $K \in (1, K_1)$ the function h has a minimum at the point $t_1 > 1$ and

$$T(n,K) \le h(t_1) = \left[\frac{3^{1-\alpha^2}(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}} \lambda_n^{\alpha-\alpha^2} + \lambda_n^{\beta-1} \left(\frac{(3\alpha)^{\alpha}\lambda_n^{\alpha-1}}{(\beta-\alpha)^{\alpha}} \right)^{\beta-\alpha} \right] \lambda_n^{2(1-\alpha)} . \tag{6.18}$$

Moreover, for $\beta \in (1,2)$ we have

$$h(t_1) \le 3^{1-\alpha^2} 2^{5(1-\alpha)} K^5 \left(\frac{3}{2} \sqrt[4]{\beta-\alpha} + \exp(\sqrt{\beta^2-1}) \right) . \tag{6.19}$$

In particular, $h(t_1) \to 1$ when $K \to 1$.

Proof. (1) The inequality (6.17) follows easily from the inequality (6.10). (2) We see that the function h has a local minimum at $t_1 = (3\alpha)^{\alpha}\lambda_n^{\alpha-1}(\beta-\alpha)^{-\alpha}$. If $t_1 \ge 1$, then the inequality (6.10) yields the desired conclusion. The upper bound for $T(n,K)$ follows by substituting the argument t_1 in the expression of h. We next show that the value $K_1 = 4/3$ will do. Fix $K \in (1, K_1)$. Then $\alpha = K^{1/(1-n)} \ge 3/4$ and $\alpha/(1-\alpha^2) > 1$.

Because $\lambda_n^{\alpha-1} \ge 2^{1/K-1}K^{-1}$ by 4.13(1), with $d = (6/K)^{1/K}/2K$ we have

$$\begin{aligned} t_1 = (3\alpha)^{\alpha}\lambda_n^{\alpha-1}(\beta-\alpha)^{-\alpha} &\ge (3/K)^{1/K} 2^{1/K-1} K^{-1} \left(\frac{\alpha}{1-\alpha^2} \right)^{\alpha} \\ &= d \left(\frac{\alpha}{1-\alpha^2} \right)^{\alpha} \ge d \left(\frac{\alpha}{1-\alpha^2} \right)^{3/4} \\ &= \left(2r(K) \frac{\alpha}{1-\alpha^2} \right)^{3/4} \quad ; r(K) = d^{4/3}/2 . \end{aligned}$$

It suffices to observe that $t_1 > 1$ certainly holds if $2r(K)(\frac{\alpha}{1-\alpha^2}) > 1$ which holds for $\alpha > 1/(r(4/3) + \sqrt{1 + r(4/3)^2}) = 0.53...$, in particular, $t_1 > 1$ holds in the present case $\alpha > 3/4$.

For the proof of (6.19) we give following inequalities

$$\lambda_n^{\alpha-\alpha^2} \le 2^{\alpha(1-\alpha)} K^{\alpha} \le 2^{1-\alpha} K^{\alpha}, \quad K \ge 1 \tag{6.20}$$

$$\lambda_n^{\beta-\alpha} = \lambda_n^{\beta+1-1-\alpha} = \lambda_n^{\beta(1-\alpha)+1-\alpha} = \lambda_n^{(\beta+1)(1-\alpha)} \le (2^{1-\alpha}K)^3, \quad \beta \in (1,2) \tag{6.21}$$

by Lemma 4.14(1). We find upper bounds for both terms of $h(t_1)$ inequality individually,

$$\begin{aligned}
\frac{3^{1-\alpha^2}(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}\lambda_n^{\alpha-\alpha^2}\lambda_n^{2(1-\alpha)} &\leq \frac{3^{(1-\alpha)(1+\alpha)}\cdot 2^{\alpha(1-\alpha)}2^{2(1-\alpha)}K^2(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}K^\alpha \\
&\leq \frac{(9\cdot 2\cdot 4)^{1-\alpha}K^2(\beta-\alpha)^{\alpha^2}}{\alpha^{\alpha^2}}K^\alpha \\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2K^\alpha\exp(-\alpha^2\log\alpha) \\
&\leq 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2K^\alpha\exp(-\alpha\log\alpha) \\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp((\log K-\log\alpha)\alpha) \\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp\left(\left(1+\frac{1}{n-1}\log K\right)\alpha\right) \\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp\left(\frac{n}{n-1}\alpha\log K\right) \\
&\leq 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^2\exp(2\log K) \\
&= 72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^4
\end{aligned}$$

by inequality (6.20),

$$\begin{aligned}
\lambda_n^{2(1-\alpha)}\lambda_n^{\beta-1}\left(\frac{(3\alpha)^\alpha\lambda_n^{\alpha-1}}{(\beta-\alpha)^\alpha}\right)^{\beta-\alpha} &= \lambda_n^{2(1-\alpha)}\lambda_n^{\beta-1}\left((3\alpha)^\alpha\lambda_n^{\alpha-1}\right)^{\beta-\alpha}(\beta-\alpha)^{-\alpha(\beta-\alpha)} \\
&\leq (2^{1-\alpha}K)^2\lambda_n^{\beta-\alpha}\left((3\alpha)^\alpha\lambda_n^{\alpha-1}\right)^{\beta-\alpha}\left(\frac{\beta^2-1}{\beta}\right)^{-\alpha((\beta^2-1)/\beta)} \\
&\leq (2^{1-\alpha}K)^2\left(3^\alpha\lambda_n\right)^{\beta-\alpha}\beta^{\alpha^2}(\beta^2-1)^{-\alpha^2(\beta^2-1)} \\
&\leq (2^{1-\alpha}K)^2 3^{\alpha(\beta-\alpha)}\lambda_n^{(\beta+1)(1-\alpha)}\exp\left(\frac{2\alpha^2}{e}\sqrt{\beta^2-1}\right) \\
&\leq 3^{1-\alpha^2}(2^{1-\alpha}K)^2(2^{1-\alpha}K)^{(\beta+1)}\exp\left(\frac{2\alpha^2}{e}\sqrt{\beta^2-1}\right) \\
&\leq 3^{1-\alpha^2}(2^{1-\alpha}K)^5\exp(\sqrt{\beta^2-1})
\end{aligned}$$

here we assume that $\beta\in(1,2)$ this implies that $\alpha\in(1/2,1)$, also inequalities $(K-1)^{-(K-1)}\leq\exp((2/e)\sqrt{K-1})$, (6.21) are used, and we get

$$h(t_1)\leq\left[72^{1-\alpha}(\beta-\alpha)^{\alpha^2}K^4+3^{1-\alpha^2}(2^{1-\alpha}K)^5\exp(\sqrt{\beta^2-1})\right]. \tag{6.22}$$

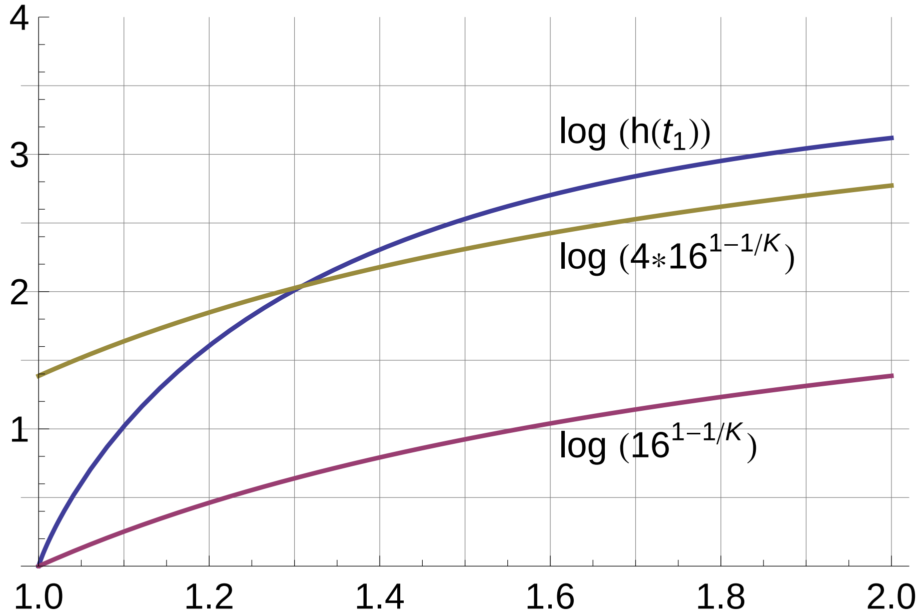

FIGURE 5. Graphical illustration of the various upper bounds for Mori's constant when $n = 2$ and $\lambda_2 = 4$ as a function of K: (a) Mori's conjectured bound $16^{1-1/K}$, (b) the Anderson-Vamanamurthy bound $4 \cdot 16^{1-1/K}$, (c) the bound from (6.18). For $K \in (1, 1.3089)$ the upper bound in (6.18) is better than the Anderson-Vamanamurthy bound.

It is easy to see that $(\beta - \alpha) \in (0, \frac{2}{3})$ this implies that $\frac{2}{3}(\beta - \alpha) \in (0, 1)$ and $\alpha^2 \in (\frac{1}{4}, 1)$ implies that $(\frac{2}{3}(\beta - \alpha))^{\alpha^2} \leq (\frac{2}{3}(\beta - \alpha))^{1/4}$, therefore

$$(\beta - \alpha)^{\alpha^2} \leq \left(\frac{2}{3}\right)^{\alpha^2} \left(\frac{2}{3}(\beta - \alpha)\right)^{1/4} \leq \left(\frac{3}{2}\right)^{1-\frac{1}{4}} \sqrt[4]{\beta - \alpha}$$
$$= \left(\frac{3}{2}\right)^{3/4} \sqrt[4]{\beta - \alpha} < \frac{3}{2} \sqrt[4]{\beta - \alpha}\,.$$

Next we prove that

$$72^{1-\alpha} \leq 3^{1-\alpha^2} 2^{5(1-\alpha)} K\,. \tag{6.23}$$

This inequality is equivalent to

$$2^{2(\alpha-1)} 3^{(1-\alpha)^2} \leq K \iff -(1-\alpha)\log 4 + (1-\alpha)^2 \log 3 \leq \log K\,.$$

This last inequality holds because the left hand side is negative. Now from (6.22) and (6.23) we get the desired inequality (6.19).

□

6.24. **Graphical and numerical comparison of various bounds.** The above bounds involve the Grötzsch ring constant λ_n, which is known only for $n = 2, \lambda_2 =$

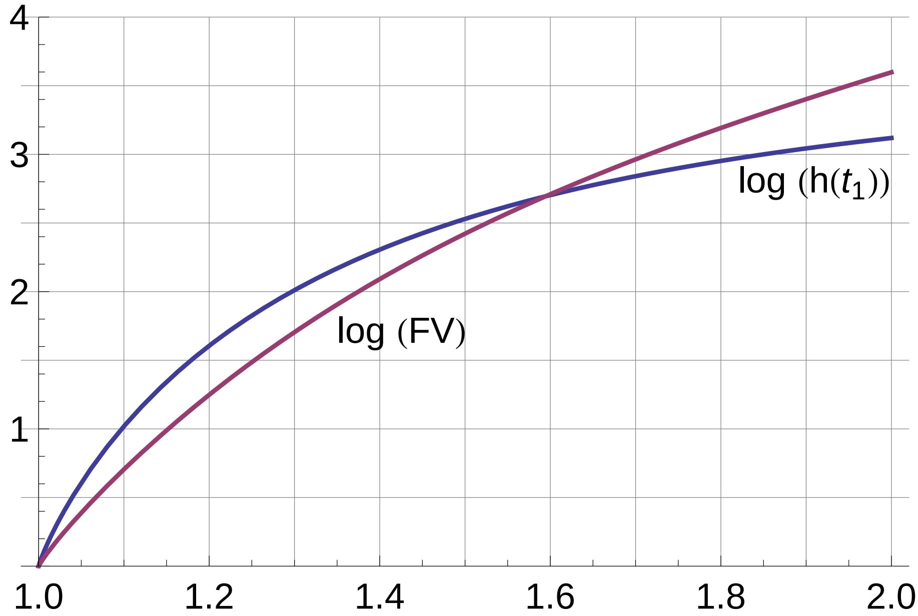

FIGURE 6. Graphical comparison of various bounds when $n = 2$ and $\lambda_2 = 4$, as a function of K: (a) the bound from (6.18), (b) the Fehlmann and Vuorinen bound [FV].

4. Therefore only for $n = 2$ we can compute the values of the bounds. Solving numerically the equation $4 \cdot 16^{1-1/K} = h(t_1)$ for K we obtain $K = 1.3089$. We give numerical and graphical comparison of the various bounds for the Mori constant.

Tabulation of the various upper bounds for Mori's constant when $n = 2$ and $\lambda_2 = 4$ as a function of K: (a) Mori's conjectured bound $16^{1-1/K}$, (b) the Anderson-Vamanamurthy bound $4 \cdot 16^{1-1/K}$, (c) the bound from (6.18). For $K \in (1, 1.3089)$ the upper bound in (6.18) is better than the Anderson-Vamanamurthy bound and for $K > 1.5946$ the upper bound in (6.18) is better than the bound of Fehlmann and Vuorinen. Numerical values of the [FV] bound given in the table were computed with the help of the algorithm for $\varphi_{K,2}(r)$ attached with [AVV1, p. 92, 439].

Formula for [FV] bound is given below:

$$M(2, K) \le \left(1 + \varphi_{K,2}\left(\frac{K^2 - 1}{K^2 + 1}\right)\right) 2^{2K-3/K} \frac{(K^2 + 1)^{(K+1/K)/2}}{(K^2 - 1)^{(K-1/K)/2}}.$$

For $K > 1.5946$ the upper bound in (6.18) is better than the Fehlmann-Vuorinen bound.

For graphing and tabulation purposes we use the logarithmic scale. Note that the upper bound for $M(2, K)$ given in [FV, Theorem 2.29] also has the desirable property that it converges to 1 when $K \to 1$, see Figure 6.

K	$\log(16^{1-1/K})$	$\log(4\cdot 16^{1-1/K})$	$\log(FV)$	$\log(h(t_1))$
1.1	0.2521	1.6384	0.7051	1.0188
1.2	0.4621	1.8484	1.2485	1.6058
1.3	0.6398	2.0261	1.7046	2.0107
1.4	0.7922	2.1785	2.0913	2.3061
1.5	0.9242	2.3105	2.4221	2.5296
1.6	1.0397	2.4260	2.7094	2.7031
1.7	1.1417	2.5280	2.9633	2.8409
1.8	1.2323	2.6186	3.1921	2.9521
1.9	1.3133	2.6996	3.4020	3.0433
2.0	1.3863	2.7726	3.5979	3.1192

TABLE 1. Numerical comparison of the bounds with $h(t_1)$.

6.25. **Comparison of estimates for the Hölder quotient.** For a K-quasiconformal mapping $f:\mathbf{B}^n\to f\mathbf{B}^n=\mathbf{B}^n,\ f(0)=0$ we call the expression

$$HQ(f)=\sup\{|f(x)-f(y)|/|x-y|^\alpha:\ x,y\in\mathbf{B}^n\}$$

the *Hölder coefficient* of f. Clearly $HQ(f)\le M(n,K)$. Theorem 6.6 yields, after dividing the both sides of the inequality by $|x-y|^\alpha$, the upper bound $HQ(f)\le HQ(K)$ for the Hölder coefficient with

$$HQ(K)=\sup\{\inf\{U(t,x,y);\ t\ge 1\},\ x,y\in\mathbf{B}^n\} \tag{6.26}$$

$$U(t,x,y)=(3+\varphi_{1/K,n}(1/t)^{-1})\varphi_{K,n}^2\left(\left(\frac{2|x-y|}{s_1+|x-y|}\right)^{1/2}\right)\frac{1}{|x-y|^\alpha}\,.$$

For $n=2$ we compare $HQ(K)$ to several other bounds (a) Mori's conjectured bound, (b) the FV bound, (c) the AV bound give the result as a table. Because the supremum and infimum in (6.26) cannot be explicitly found we use numerical methods that come with Mathematica software. For the supremum we created 100,000 random points in the unit disk because of the computational load.

7. An explicit form of Schwarz's lemma

This section is taken from [BV]. Recall that the hyperbolic metric $\rho(x,y), x,y\in\mathbf{B}^n$, of the unit ball is given by (cf. [KL], [Vu1])

$$\mathrm{th}^2\frac{\rho(x,y)}{2}=\frac{|x-y|^2}{|x-y|^2+t^2}\,,\quad t^2=(1-|x|^2)(1-|y|^2)\,. \tag{7.1}$$

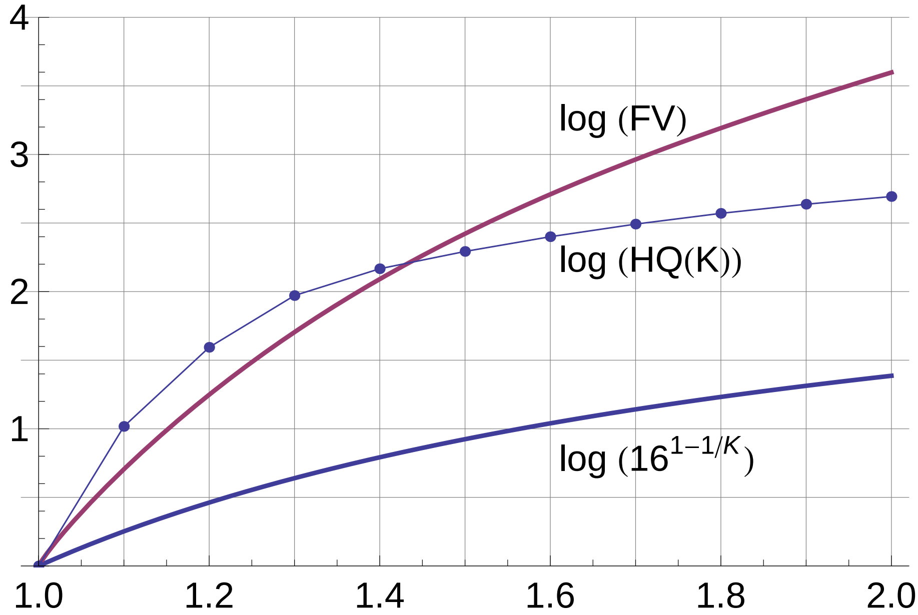

FIGURE 7. Graphical comparison of various bounds when $n = 2$ and $\lambda_2 = 4$, as a function of K: (a) the bound from (6.26), (b) the Fehlmann and Vuorinen bound [FV], (c) the bound of the Mori conjecture. Note that the bound (6.26), based on a simulation with $100,000$ random points in unit disk, gives the best estimate in the cases considered in the picture.

K	$\log(16^{1-1/K})$	$\log(4 \cdot 16^{1-1/K})$	$\log(FV)$	$\log(HQ(K))$
1.1	0.2521	1.6384	0.7051	1.0171
1.2	0.4621	1.8484	1.2485	1.5940
1.3	0.6398	2.0261	1.7046	1.9712
1.4	0.7922	2.1785	2.0913	2.1668
1.5	0.9242	2.3105	2.4221	2.2928
1.6	1.0397	2.4260	2.7094	2.4003
1.7	1.1417	2.5280	2.9633	2.4922
1.8	1.2323	2.6186	3.1921	2.5706
1.9	1.3133	2.6996	3.4020	2.6371
2.0	1.3863	2.7726	3.5979	2.6934

TABLE 2. Numerical comparison of the bounds with $HQ(K)$.

7.2. **Quasiregular mappings.** Let $G \subset \mathbf{R}^n$ be a domain. A mapping $f : G \to \mathbf{R}^n$ is said to be *quasiregular* if f is ACL^n (*absolutely continuous on almost all lines*) and there exists a constant $K \geq 1$ such that

$$|f^{'}(x)|^n \leq K J_f(x), \quad |f^{'}(x)| = \max_{|h|=1} |f^{'}(x)h|, \tag{7.3}$$

almost every where in G. Here $f^{'}(x)$ denotes the formal derivative of f at x. The smallest $K \geq 1$ for which this inequality is true is called the *outer dilatation* of f and denoted by $K_o(f)$. If f is quasiregular, then the smallest $K \geq 1$ for which the inequality

$$J_f(x) \leq K l(f^{'}(x))^n, \quad l(f^{'}(x)) = \min_{|h|=1} |f^{'}(x)h|, \tag{7.4}$$

holds almost every where in G is called the *inner dilatation* of f and denoted by $K_I(f)$. The *maximal dilatation* of f is the number $K(f) = \max\{K_o(f), K_I(f)\}$. If $K(f) \leq K$, f is said to be K*-quasiregular*. If f is not quasiregular, we set $K_o(f) = K_I(f) = K(f) = \infty$.

7.5. **Theorem.** [Vu1, 11.2] *Let $f : \mathbf{B}^n \to \mathbf{R}^n$ be a nonconstant K-quasiregular mapping with $f\mathbf{B}^n \subset \mathbf{B}^n$ and let $\alpha = K^{1/(1-n)}$. Then*

$$\mathrm{th}\frac{\rho(f(x), f(y))}{2} \leq \varphi_{K,n}\left(\mathrm{th}\frac{\rho(x,y)}{2}\right) \leq \lambda_n^{1-\alpha}\left(\mathrm{th}\frac{\rho(x,y)}{2}\right)^{\alpha}, \tag{7.6}$$

$$\rho(f(x), f(y)) \leq K(\rho(x,y) + \log 4), \tag{7.7}$$

for all $x, y \in \mathbf{B}^n$, where λ_n is the same constant as in (6.15)*. If $f(0) = 0$, then*

$$|f(x)| \leq \lambda_n^{1-\alpha}|x|^{\alpha}, \tag{7.8}$$

for all $x \in \mathbf{B}^n$.

In the case of quasiconformal mappings with $n = 2$ formulas (7.6) and (7.8) also occur in [LV, p. 65] and formula (7.7) was rediscovered in [EMM, Theorem 5.1]. Comparing Theorem 7.5 with Theorem 7.10 we see that for $n = 2$ the expression $K(\rho(x,y) + \log 4)$ may be replaced by $c(K)\max\{\rho(x,y), \rho(x,y)^{1/K}\}$, which tends to 0 when $x \to y$ and to $\rho(x,y)$ when $K \to 1$, as expected.

7.9. **Lemma.** *For $K > 1$ the function*

$$t \mapsto \frac{2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{\max\{t, t^{1/K}\}},$$

is monotone increasing on $(0,1)$ and decreasing on $(1,\infty)$.

Proof. (1) Fix $K > 1$ and consider

$$f(t) = \frac{2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{t}, \quad t > 0.$$

Let $r = \mathrm{th}\frac{t}{2}$. Then $t/2 = \mathrm{arth}\, r$, and t is an increasing function of r for $0 < r < 1$. Then

$$f(t) = \frac{\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{t/2} = \frac{\mathrm{arth}(\varphi_K(r))}{\mathrm{arth}\, r} = F(r).$$

Then by [AVV1, Theorem 10.9(3)], $F(r)$ is strictly decreasing from $(0,1)$ onto (K,∞). Hence $f(t)$ is strictly decreasing from $(0,\infty)$ onto (K,∞).

(2) Next consider

$$g(t) = \frac{2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{t^{1/K}},$$

and let $r = \mathrm{th}\frac{t}{2}$. Then $t = 2\mathrm{arth}\, r$ and

$$g(t) = \frac{2\mathrm{arth}\, s}{2^{1/K}(\mathrm{arth}\, r)^{1/K}} = \frac{2^{1-1/K}\mathrm{arth}\, s}{(\mathrm{arth}\, r)^{1/K}},$$

where $s = \varphi_K(r)$. We next apply [AVV1, Theorem 1.25]. We know $\frac{d}{dr}(\mathrm{arth} r) = 1/(1-r^2)$.

Writing $r' = \sqrt{1-r^2}$, $s' = \sqrt{1-s^2}$, we obtain the quotient of the derivatives

$$\begin{aligned}
\frac{2^{1-1/K}(1/(1-s^2))\frac{ds}{dr}}{\frac{1}{K}(\mathrm{arth}\, r)^{1/K-1}(1/(1-r^2))} &= 2^{1-1/K}\, K\, (\mathrm{arth}\, r)^{1-1/K}\frac{r'^2}{s'^2}\frac{1}{K}\frac{ss'^2\,\mathcal{K}(s)^2}{rr'^2\,\mathcal{K}(r)^2} \\
&= 2^{1-1/K}(\mathrm{arth}\, r)^{1-1/K}\frac{s\,\mathcal{K}(s)^2}{r\,\mathcal{K}(r)^2}
\end{aligned}$$

by [AVV1, appendix E(23)]. By [AVV1, Lemma 10.7(3)], $\frac{\mathcal{K}(s)^2}{\mathcal{K}(r)^2}$ is increasing, since $K > 1$, $(\mathrm{arth}\, r)^{1/K-1}$ is increasing. Finally, s/r is increasing by [AVV1, Theorem 1.25] and E(23). So $g(t)$ is increasing in t on $(0,\infty)$.

(3) Fix $K > 1$. Clearly

$$\max\{t, t^{1/K}\} = \begin{cases} t^{1/K} & \text{for} \quad 0 \le t \le 1 \\ t & \text{for} \quad 1 \le t < \infty. \end{cases}$$

Thus

$$h(t) = \frac{2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{t}{2}))}{\max\{t, t^{1/K}\}}$$

increases on $(0,1)$ and decreases on $(1,\infty)$. $\square$

7.10. **Theorem.** *If $f : \mathbf{B}^2 \to \mathbf{R}^2$ is a non-constant K-quasiregular mapping with $f\mathbf{B}^2 \subset \mathbf{B}^2$, and ρ is the hyperbolic metric of $\mathbf{B}^2$, then*

$$\rho(f(x), f(y)) \le c(K)\max\{\rho(x,y), \rho(x,y)^{1/K}\}$$

for all $x, y \in \mathbf{B}^2$ where $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$ and

$$K \le u(K-1)+1 \le \log(\mathrm{ch}(K\mathrm{arch}(e))) \le c(K) \le v(K-1)+K$$

with $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$ and $v = \log(2(1+\sqrt{1-1/e^2})) < 1.3507$. In particular, $c(1) = 1$.

Proof. The maximum value of the function considered in Lemma 7.9 is $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\frac{1}{2}))$. The inequality now follows from Lemma 7.9. $\square$

7.11**. Bounds for the constant** $c(K)$**.** In order to give upper and lower bounds for $c(K)$ given in Theorem 7.10, we observe that the identity [AVV1, Theorem 10.5(2)] yields the following formula

$$c(K) = 2\mathrm{arth}\left(\varphi_K\left(\frac{1-1/e}{1+1/e}\right)\right) = 2\mathrm{arth}\left(\frac{1-\varphi_{1/K}(1/e)}{1+\varphi_{1/K}(1/e)}\right).$$

A simplification leads to

$$c(K) = -\log\varphi_{1/K}(1/e)\,.$$

Next, from the inequality $\varphi_{1/K}(r) \geq 2^{1-K}(1+r')^{1-K}r^K$ for $K \geq 1, r \in (0,1)$ (cf. [AVV1, Corollary 8.74(2)]) we get, with $v = \log((2(1+\sqrt{1-1/e^2}))) < 1.3507$

$$\begin{aligned} c(K) &= -\log\varphi_{1/K}(1/e) \leq -\log 2^{1-K}(1+\sqrt{1-1/e^2})^{1-K}e^{-K} \\ &= v(K-1)+K < 1.3507(K-1)+K. \end{aligned}$$

In order to estimate the constant $c(K)$ from below we need an upper bound for $\varphi_{1/K,2}(r),\ K>1$, from above. For this purpose we prove the following lemma.

7.12. **Lemma.** *For every integer $n \geq 2$ and each $K > 1,\ r \in (0,1)$, there exists K-quasiconformal maps $g : \mathbf{B}^n \to \mathbf{B}^n$ and $h : \mathbf{B}^n \to \mathbf{B}^n$ with*

(*a*) $\quad g(0) = 0,\ g\mathbf{B}^n = \mathbf{B}^n,\ h(0) = 0,\ h\mathbf{B}^n = \mathbf{B}^n$

(*b*) $\quad g(re_1) = \dfrac{2r^\alpha}{(1+r')^\alpha + (1-r')^\alpha},\ h(re_1) = \dfrac{2r^\beta}{(1+r')^\beta + (1-r')^\beta}$

where $r' = \sqrt{1-r^2}$ *and* $\alpha = K^{1/(1-n)} = 1/\beta$.

In particular, for $n = 2$ and $K > 1,\ r \in (0,1)$

(*c*) $\quad \varphi_{1/K}(r) \leq \dfrac{2r^K}{(1+r')^K + (1-r')^K}\ ;\ \varphi_K(r) \geq \dfrac{2r^{1/K}}{(1+r')^{1/K} + (1-r')^{1/K}}.$

Proof. Fix $r \in (0,1)$. Let $T_a : \mathbf{B}^n \to \mathbf{B}^n$ be a Möbius automorphism with $T_a(a) = 0$ and $T_a(\mathbf{B}^n) = \mathbf{B}^n$. Choose $s \in (0,r)$ such that $T_{se_1}(0) = -T_{se_1}(re_1)$. Then $\rho(0, re_1) = 2\rho(0, se_1)$ [Vu1, (2.17)], or equivalently, $(1+r)/(1-r) = ((1+s)/(1-s))^2$ and hence $s = r/(1+r')$. Consider the K-quasiconformal mapping $f : \mathbf{B}^n \to \mathbf{B}^n$, $f(x) = |x|^{\alpha-1}x,\ \alpha = K^{1/(1-n)}$. Then $f(\pm se_1) = \pm s^\alpha e_1$. The mapping $g = T_{-s^\alpha e_1} \circ f \circ T_{se_1} : \mathbf{B}^n \to \mathbf{B}^n$ satisfies $g(0) = 0$, $g(re_1) = te_1$ where $\rho(-s^\alpha e_1, s^\alpha e_1) = \rho(0, te_1)$ and hence $t = 2r^\alpha/((1+r')^\alpha + (1-r'))^\alpha$ by [Vu1, (2.17)]. The proof for g is complete. For the map h the proof is similar except that we use the K-quasiconformal mapping $m : x \to |x|^{\beta-1}x,\ \beta = 1/\alpha$. Note that $m = f^{-1}$ and $t = 1/\mathrm{ch}(\alpha\ \mathrm{arch}(1/r))$. For the proof of (c) we apply (a), (b) together with [LV, (3.4), p. 64]. $\square$

7.13. **Lemma.** *For $K > 1$, $c(K) \geq \log(\mathrm{ch}(K\mathrm{arch}(e))) \geq u(K-1)+1$, where $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$.*

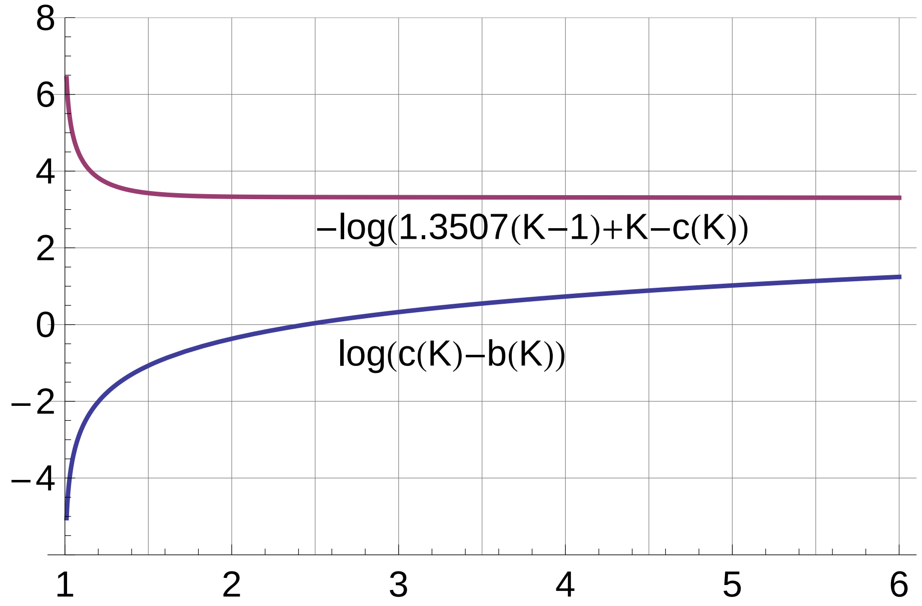

FIGURE 8. Graphical comparison of lower and upper bounds for $c(K)$ with $b(K) = \log(\mathrm{ch}(K\mathrm{arch}(e)))$.

Proof. From Lemma 7.12(c), we know that

$$\begin{aligned}\varphi_{1/K}(1/e) &\leq \frac{2/e^K}{(1+\sqrt{1-1/e^2})^K + (1-\sqrt{1-1/e^2})^K} \\ &= \frac{2}{(e+\sqrt{e^2-1})^K + (e-\sqrt{e^2-1})^K},\end{aligned}$$

hence

$$\begin{aligned}c(K) &= -\log\varphi_{1/K}(1/e) \geq -\log\left(\frac{2}{(e+\sqrt{e^2-1})^K + (e-\sqrt{e^2-1})^K}\right) \\ &= \log\left(\frac{(e+\sqrt{e^2-1})^K + (e-\sqrt{e^2-1})^K}{2}\right) \\ &= \log(\mathrm{ch}(K\mathrm{arch}(e))) \geq u(K-1)+1,\end{aligned}$$

where the last inequality follows easily from the mean value theorem, applied to the function $p(K) = \log(\mathrm{ch}(K\mathrm{arch}(e)))$. $\square$

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