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QUALITATIVE CHARACTERISTICS AND
QUANTITATIVE MEASURES OF SOLUTION'S
RELIABILITY IN DISCRETE OPTIMIZATION:
TRADITIONAL ANALYTICAL APPROACHES,
INNOVATIVE COMPUTATIONAL METHODS
AND APPLICABILITY

by

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Abstract

The purpose of this thesis is twofold. The first and major part is devoted to sensitivity analysis of various discrete optimization problems while the second part addresses methods applied for calculating measures of solution stability and solving multicriteria discrete optimization problems.

Despite numerous approaches to stability analysis of discrete optimization problems two major directions can be single out: quantitative and qualitative. Qualitative sensitivity analysis is conducted for multicriteria discrete optimization problems with minisum, minimax and minimin partial criteria. The main results obtained here are necessary and sufficient conditions for different stability types of optimal solutions (or a set of optimal solutions) of the considered problems.

Within the framework of quantitative direction various measures of solution stability are investigated. A formula for a quantitative characteristic called stability radius is obtained for the generalized equilibrium situation invariant to changes of game parameters in the case of the Hölder metric. Quality of the problem solution can also be described in terms of robustness analysis. In this work the concepts of accuracy and robustness tolerances are presented for a strategic game with a finite number of players where initial coefficients (costs) of linear payoff functions are subject to perturbations.

Investigation of stability radius also aims to devise methods for its calculation. A new metaheuristic approach is derived for calculation of stability radius of an optimal solution to the shortest path problem. The main advantage of the developed method is that it can be potentially applicable for calculating stability radii of NP-hard problems.

The last chapter of the thesis focuses on deriving innovative methods based on interactive optimization approach for solving multicriteria combinatorial optimization problems. The key idea of the proposed approach is to utilize a parameterized achievement scalarizing function for solution calculation and to direct interactive procedure by changing weighting coefficients of this function. In order to illustrate the introduced ideas a decision making process is simulated for three objective median location problem.

The concepts, models, and ideas collected and analyzed in this thesis create a good and relevant grounds for developing more complicated and integrated models of postoptimal analysis and solving the most computationally challenging problems related to it.

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This thesis is dedicated to the bright memory of my grandparents, Maria and Ivan.

Turku, November 20, 2012
Volha Karelkina

List of original publications

This thesis is based on the following articles, referred to in the text by their Roman numerals:

I. Emelichev V. and Karelkina O. (2009) Finite cooperative games: parametrisation of the concept of equilibrium (from Pareto to Nash) and stability of the efficient situation in the Hölder metric. *Discrete Mathematics and Applications*, **19** (3), pp. 229 – 236.

II. Emelichev V., Karelkina O. and Girlich E. (2009) Postoptimal analysis of multicriteria combinatorial center location problem. *Buletinul Academiei de Științe a Republicii Moldova. Matematica*, **3** (61), pp. 13 – 29.

III. Emelichev V. and Karelkina O. (2011) Postoptimal analysis of the multicriteria combinatorial median location problem. *Optimization: A Journal of Mathematical Programming and Operations Research*, **61** (9), pp. 1151 – 1167.

IV. Nikulin Y., Karelkina O. and Mäkelä M. M. (2013) On accuracy, robustness and tolerances in vector Boolean optimization. *European Journal of Operational Research*, **224** (3), pp. 449 – 457.

V. Karelkina O., Nikulin Y. and Mäkelä M. M. (2011) An adaptation of NSGA-II to the stability radius calculation for the shortest path problem. // *TUCS Technical Report* 1017. 2011.

VI. Emelichev V., Karelkina O. and Kuzmin K. (2012) Qualitative stability analysis of multicriteria combinatorial minimin problems. *Control and Cybernetics Journal*, **41** (1), pp. 57 – 79.

VII. Karelkina O. (2012) An interactive approach to solve multicriteria median location problem // *TUCS Technical Report* 1053. 2012.

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Chapter 1. Introduction

Optimization is a scientific discipline dealing with the detection of optimal solutions for a problem, among multiple (finite or infinite number of) alternatives. The optimality of solutions is based on one or several criteria being usually problem- and user-dependent. Optimization involves the study of optimality criteria for problems, the determination of algorithmic methods of solution, the study of the structure of such methods, the computer experimentation with methods both under trial conditions and on real life problems, the sensitivity and/or postoptimal analysis. There is an extremely diverse range of practical applications. Yet the subject can be studied as a branch of applied and pure mathematics.

Mathematical model of an optimization problem can be defined in several ways, depending on the underlying application. In general, any function, $f : X \rightarrow Y$, defined over a domain, X , also called *the search space*, and with range, Y , can be subjected to optimization given a total ordering relation over Y . The most common optimization problems consist of the minimization (maximization) of functions whose domain is a subset of the n -dimensional Euclidean space, $X \subset \mathbb{R}^n$, and their range is a subset of the real numbers, $Y \subset \mathbb{R}$. Moreover, the problem may have constraints in the form of equality and inequality relations, defining solution feasibility. Thus, the optimization problem can be formally described as:

$$\min_{x \in S}(\max)f(x),$$

where f is a general real or vector valued objective function of the decision parameter $x \in X$ and $S \subset X$ is *a feasible set*, that is, the set of all possible solutions, in the decision space.

1.1. General classification of optimization problems

Major optimization subfields can be distinguished based on properties of the objective function, its domain, as well as the form of the constraints. Some of the most interesting and significant subfields, with respect to the form of the objective function, are:

1. **Linear optimization (or linear programming)** studies cases where the objective function and constraints are linear.

2. **Nonlinear optimization (or nonlinear programming)** deals with cases where at least one nonlinear function is involved in the optimization problem.

3. **Convex optimization** studies problems with convex objective functions and convex feasible sets.

4. **Quadratic optimization (or quadratic programming)** involves the minimization of quadratic objective functions and linear constraints.

A categorization of optimization problems with respect to the nature of the search space can be divided into three categories:

1. **Discrete (or combinatorial) optimization**, $X \subset \mathbb{Z}^n$. In such problems we are generally looking for an object from a finite, or possibly countably infinite, set – typically an integer, set, permutation, or graph. The special cases of integer and binary variables are referred to as integer programming and 0-1 integer programming correspondingly.

2. **Continuous optimization**, $X \subset \mathbb{R}^n$. In such problems we are looking for a set of real numbers or even a function.

3. **Mixed integer optimization**, $X \subset \mathbb{Z}^n \times \mathbb{R}^n$. Both integer and real variables appear in the objective function and problem constraints.

The main target of single objective optimization is to find the best solution, which corresponds to the minimum (maximum) value of a single objective function that aggregate all different goals into one. This type of optimization is useful as a tool which should provide *decision makers* (a person or a group of persons who is supposed to express preference relations between different solutions and responsible for the final solution) with insights into the nature of the problem, but usually cannot provide a set of alternative solutions that trade different objectives against each other.

Numerous problems arising in engineering, design, planning and management inevitably involves decision making, choices and searching for compromises. Decisions, no matter if made by a group or an individual, usually involve several conflicting objectives. The observation that real-life problems have to be solved optimally in situations in which several objectives must be satisfied has led to the development of multiobjective optimization. From its first roots, which were laid by French-Italian polymathematician, economist and sociologist Pareto [86] at the end of the 19th century the discipline has prospered and grown, especially during the last three decades. Today, many decision support systems incorporate methods to deal with conflicting objectives. The foundation for such systems is a mathematical theory of optimization under multiple objectives. Recent interest of mathematicians in multicriteria discrete

optimization problems keeps very high, as confirmed by the intensive publishing activity (see, e.g. bibliography [14], which contains 234 references, and monograph [100]).

These observations lead to the following important optimization subfields:

1. **Single objective optimization**, $Y \subset \mathbb{R}$, deals with models that include a single objective function.
2. **Multiobjective (multicriteria, vector) optimization**, $Y \subset \mathbb{R}^m$, $m \geq 2$, refers to problems where two or more objective functions need to be optimized concurrently.

One thing worth mentioning is that there is no rigid difference in definitions of multiobjective, multicriteria and vector optimizations. However they can be distinguished depending on aims of investigation. In particular, multiobjective optimization considers both objective and decision spaces. Multicriteria optimization concentrated basically on studying objective functions treating the objectives as decision making criteria and corresponding objective values as vectors in a criterion space. Finally vector optimization concerns pure theoretical questions in various fields of optimization with vector valued functions and sometimes deals with infinitely many objectives. In some literature [76] another classification based on the set of feasible solutions can be found. In multicriteria optimization, the set of feasible solutions is discrete, predetermined and finite. In multiobjective optimization, the feasible alternatives are not explicitly known in advance. An infinite number of them exists and they are represented by decision variables restricted by constrain functions. These problems can be called continuous. In this thesis, all these notions are referred as the same concept of finding solutions to optimization problems with multiple objectives.

In single objective optimization problems, the main focus is on the decision variable space. In multiobjective optimization, solutions are compared by establishing some preference relations in objective space.

Without loss of generality multiobjective optimization problem can be formulated as follows:

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x)) \rightarrow \min_{x \in S},$$

where we have m (≥ 2) objective functions (conflicting partial criteria) $f_i : S \rightarrow \mathbb{R}$, S is a set of feasible solutions, which is a subset of decision space $X \subset \mathbb{R}^n$, and $y = f(x) \in Y \subset \mathbb{R}^m$ is a vector-function.

Because of the contradiction and possible incommensurability of the objective functions, it is not possible to find a single solution being optimal for all the objectives simultaneously. Multiobjective optimization problems are in

a sense ill-posed by Hadamard (see definition of problem's well-posedness by Hadamard in the next section). Anyway, some of the objective vectors can be extracted for examination. Such vectors are those where none of the components can be improved without deterioration to at least one of the other components. The definition is usually called *Pareto optimality*. A more formal definition of this concept is the following:

Definition 1.1. A decision vector $x^* \in X$ is Pareto optimal (efficient) if there exists no $x \in X$ with $f_i(x) \leq f_i(x^*)$ for all $i = 1, \dots, m$ and $f_j(x) < f_j(x^*)$ for at least one index j .

This definition introduces a componentwise order [87] on the set of feasible solutions X :

$$x \underset{P}{\succ} \hat{x} \Leftrightarrow f(x) \geq f(\hat{x}) \wedge f(x) \neq f(\hat{x}).$$

If $x \underset{P}{\succ} \hat{x}$ we say that \hat{x} dominates x and $f(\hat{x})$ dominates $f(x)$. The set of all Pareto optimal solutions x^* is denoted by $P^m(X)$. Then the set of all nondominated points $y^* = f(x^*) \in Y$, where $x^* \in P^m(X)$ is denoted by $P^m(Y)$ and called nondominated set or Pareto front. Different shapes of Pareto front are illustrated graphically in the diagrams in Figure 1.

There are usually a lot (infinite or exponential number with respect to a size of the problem) of Pareto optimal solutions. Therefore, the problem of finding all Pareto optimal solutions is quite complicated and *intractable* already for bi-criteria case (even if original single objective problems are simple, that is, polynomially solvable). Roughly speaking, intractability means that there is an algorithm that produces a solution to the problem but the algorithm does not produce results in a reasonable amount of time. The question, whether a polynomial time algorithm for multicriteria linear programming (e.g. a generalization of Karmarkars interior point algorithm [50]) is possible depends on the number of efficient extreme points. Unfortunately, it is easy to construct examples with exponentially many. For the discrete case, in terms of algorithmic complexity this problem is *NP-complete* [51], that is, it is in the set of *NP problems* so that any given solution (produced by some oracle machine) to the decision problem can be verified in polynomial time, and also in the set of *NP-hard problems* so that any NP problem can be converted into it by a transformation of the inputs in polynomial time.

As an example we consider multiobjective discrete optimization problem with linear objectives:

$$\min \sum_{i=1}^n c_i^k x_i \quad k = 1, \dots, m, \quad c^k \in \mathbb{Z}^n, \quad (1.1)$$

s.t.

$$x_i \in \{0, 1\} \quad i = 1, \dots, n.$$

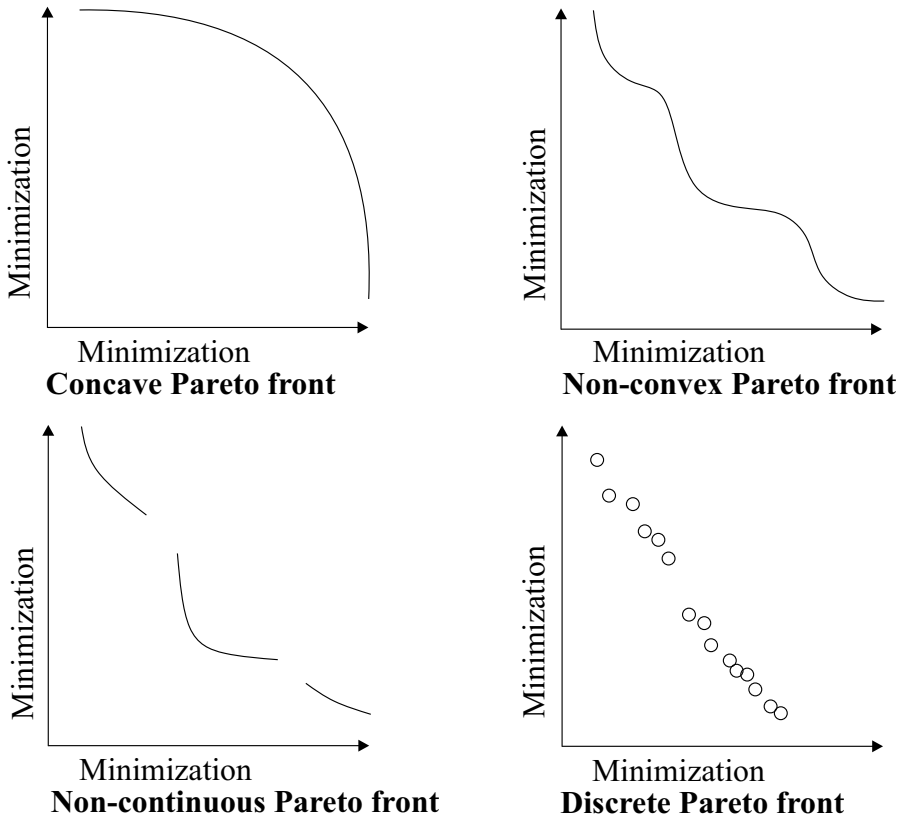


Figure 1: Different shapes of Pareto front in biobjective case

Proposition 1.1 [13] *The multiobjective combinatorial optimization problem (1.1) is NP-complete even for $m = 2$.*

This fact can be proven by reducing biobjective combinatorial optimization problem to NP-complete 0 – 1 knapsack problem [85] in polynomial time and vice verse.

In addition to Pareto optimality, other related concepts are widely used. These are weakly and strictly efficient solutions (optimal by Slater [93] and Smale [94] correspondingly).

Definition 1.2. *A feasible solution $x^* \in X$ is called weakly Pareto optimal (weakly efficient) if there exists no $x \in X$ with $f_i(x) < f_i(x^*)$ for all $i = 1, \dots, m$.*

In other words a solution is weakly optimal if there exists no solution for which all components of the vector objective function are better.

Definition 1.3. *A feasible solution $x^* \in X$ is called strictly Pareto optimal (strictly efficient) if there exists no $x \in X$, $x \neq x^*$, such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, \dots, m$.*

Analogously to Pareto optimality case the sets of weakly and strictly non-dominated points and solutions are defined by weak and strict componentwise order over domain X . The weakly and strictly efficient solutions are denoted by $Sl^m(X)$ and $Sm^m(X)$ respectively. Correspondent weakly and strictly efficient sets are denoted by $Sl^m(Y)$ and $Sm^m(Y)$. From definitions it is obvious that

$$Sm^m(X) \subset P^m(X) \subset Sl^m(X).$$

The concept of efficiency and its variants are the most important definitions of optimality in multicriteria optimization. Other choices of orders and model maps give rise to different classes of multicriteria optimization problems. Specifically if all partial criteria are ordered by importance in such a manner that each of them is more important than all the subsequent, then the principle of lexicographic optimality can be used. Lexicographic optimization problems arise naturally when conflicting objectives exist in a decision problem but for reasons outside the control of the decision maker the objectives have to be considered in a hierarchical manner.

Definition 1.4. *A feasible solution $x^* \in X$ is called lexicographically optimal if for any $x \in X$ we have $f(x^*) = f(x)$ or there exists index $l = \{1, \dots, m\}$ such that $f_l(x) > f_l(x^*)$ and $l = \min\{i \in \{1, \dots, m\} : f_i(x) \neq f_i(x^*)\}$.*

It is known (see, e.g. [13]) that the set of lexicographically optimal solutions $L^m(X)$ can be specified as a result of solving the sequence of m scalar problems

$$L_i^m(X) = \text{Arg min}\{f_i(x) : x \in L_{i-1}^m(X)\}, \quad i \in \{1, \dots, m\},$$

where $L_0^m(X) = X$, $\text{Arg min}\{\cdot\}$ is the set of all optimal solutions of a corresponding minimization problem. Hence the following inclusions

$$X \supset L_1^m(X) \supset L_2^m(X) \supset \dots \supset L_m^m(X) = L^m(X)$$

are valid.

From the above definitions it is also evident that:

$$\begin{aligned} L^m(X) &\subset P^m(X), \\ Sm^m(X) &\not\subset L^m(X), \\ Sm^m(X) &\not\supset L^m(X). \end{aligned}$$

Uncertainty and inexactness of data and outcomes pervade many aspects of most optimization problems. However, optimization models mentioned above do not take into account this factor. As it turns out, when the uncertainty in the problem is of a particular (and fairly general) form, it is relatively easy to incorporate the uncertainty into the optimization model.

Recently, interest in multicriteria decision making under uncertainty and risk with applications in, among others, game theory, mathematical economics,

optimal control, investment analysis, banking and insurance business has increased drastically. Under these conditions when solving optimization problems in practise one should take into account various factors of uncertainty and randomness such as inaccuracy of initial data, non-fit of mathematical models to real processes, rounding off, calculation errors and need of deriving algorithms for solving "close" problems etc.

Thus, the following two subfields of optimization can be single out

1. **Deterministic optimization** problems are formulated with known parameters.

2. **Optimization under uncertainty.** In this case problem parameters are known only within certain bounds.

A simple categorization of optimization problems is presented and summarized in the following table.

Table 1.1. Types of optimization problems

Classification criterion	Type of optimization problem
Form of the objective function and/or constraints	Linear
	Quadratic
	Convex
	Nonlinear
Nature of the search space	Continuous
	Discrete
	Mixed integer
Number of objectives	Single objective
	Multiobjective
Nature of the problem initial data	Deterministic
	Under uncertainty

1.2. Sensitivity analysis of discrete optimization problems

The terms "sensitivity", "stability" or "postoptimal analysis" are generally used for the phase of an algorithm at which a solution (or solutions) of the problem has been already found, and additional calculations are performed in order to investigate how this solution depends on changes in the problem data.

Recognition of the stability problem as one of the central in mathematical research goes back to Hadamard [46]. In 1923, he postulated that in order to be well-posed a problem should have three properties

- existence of a solution;

- uniqueness of the solution;
- continuous dependence of the solution on the data.

Correspondingly, ill-posed multicriteria discrete optimization problem refers to this situation that it may have multiple solutions or the feasible solution set and/or criteria functions depend on uncertain parameters.

Widespread use of discrete optimization models in the last decades stimulated many experts to investigate different aspects of stability of scalar and vector problems of discrete optimization. As a consequence, in the context of the operation research and mathematical optimization, the most closely related lines of research have been initiated:

- Postoptimal and parametric analysis of optimization problems reveals relation between behavior of found solutions and changes in initial data of a problem. Initial data is usually parameters which define an objective function [91, 92, 103] and/or a set of feasible solutions [5]. As a rule techniques and methods of this analysis are based on algorithms for solving a given optimization problem or on using properties of point-to-set mappings which associate each point of the initial data set of a problem with a certain set of optimal solutions. The notion of problem stability is usually connected with Hausdorff semi-continuity or continuity of the mappings mentioned above [92, 103]. Without going into details we note just some publications on this subject [4, 36, 103].
- Robust optimization is another traditional body of knowledge dealing with data uncertainty in optimization. The robust optimization methodology, in its simplest version, proposes to associate with an uncertain problem its robust counterpart and to use, as "real life" decisions, the associated robust optimal solutions. It aims to find a feasible/optimal solution that remains feasible/optimal under worst case realization of uncertainty in input parameters of the problem. Worst-case-oriented optimization is also known as robust optimization, and feasible/optimal solutions of worst case optimization are often referred to as robust feasible/optimal solutions (see e.g. [52, 55]).

As it was mentioned above the theoretical stability research relies on the properties of continuity of multivalued (point-to-set) mappings [58]. A solution of any optimization problem involves finding a mapping $w = \Gamma(u)$ which associates a solution w to the input data u (for instance, $u = (A, b, c) \in \mathbb{R}^{m \times n + m + n}$ for the linear programming problem), where u and w may be reviewed as elements of normed spaces. Let two normed spaces U and V be given. The mapping Γ that associates to each point of the set U some subset of the set V

is called a *multivalued (point-to-set or set-valued) mapping* of U to V and is denoted $\Gamma : U \rightarrow 2^V$.

Definition 1.5. *The multivalued mapping Γ at the point $u^0 \in U$ is called Hausdorff upper semicontinuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\Gamma(u) \supset O_\varepsilon(\Gamma(u^0))$ for every $u \in O_\delta(u^0)$, where*

$$O_\varepsilon(\Gamma(u^0)) = \{v \in V \mid \inf\{\|v - y\| < \varepsilon \mid y \in \Gamma(u^0)\}\}$$

and $O_\delta(u^0) = \{u \in U \mid \|u - u^0\| < \delta\}$.

Definition 1.6. *The multivalued mapping Γ at the point $u^0 \in U$ is called Hausdorff lower semicontinuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\Gamma(u^0) \supset O_\varepsilon(\Gamma(u)) \text{ for every } u \in O_\delta(u^0).$$

Definition 1.7. *The multivalued mapping Γ at the point $u^0 \in U$ is called Hausdorff continuous if this mapping is upper semicontinuous and lower semicontinuous at this point.*

The increased interest in stability analysis and parametric and postoptimality analysis of optimization problems is motivated by the fact that the input data, as a link between the real world and the model, are far from being unambiguously defined. Solutions to optimization problems (including discrete optimization) can exhibit remarkable sensitivity to perturbations in the parameters of the problem, thus often a computed solution is highly infeasible, suboptimal, or both (in short, potentially worthless). It is evident that an arbitrary applied problem cannot be correctly specified and solved without using the results of stability theory and related issues of parametric and postoptimality analysis at least implicitly.

Sensitivity analysis is important for several reasons:

- Stability (robustness) of the optimum solution under changes of parameters may be highly desirable. For example, using the old optimum solution point; a slight variation of a parameter in one direction may result in a large unfavorable difference in the objective function relative to the new minimum, while a large variation in the parameter in another direction may result in only a small difference. In an industrial situation where there are certain inherent variabilities in processes and materials not taken into account in the model, it may be desirable to move away from the optimum solution in order to achieve a solution less sensitive to such changes in the input parameters.
- Values of the input-output coefficients, objective function coefficients, and/or constraint constants may be to some extent controllable at some cost; in this case we want to know the effects that would result from changing these values and what the cost would be to make these changes.

- Even if the input-output and objective function coefficients and constraint constants are not controllable, their values may be only approximate; thus it is still important to know for what ranges of their values the solution is still optimal. If it turns out that the optimal solution is extremely sensitive to their values, it may become necessary to seek better estimates.

It is quite reasonable to start discussion of sensitivity analysis of combinatorial optimization problems at its boundary with continuous optimization. Linear programming plays a unique role in optimization theory; it is in one sense a continuous optimization problem, but it can also be considered combinatorial in nature and in fact is fundamental to the study of many strictly combinatorial problems.

Consider the linear programming problem in the standard form:

$$\min z = c^T x \tag{1.2}$$

$$\text{s.t. } Ax \geq b \tag{1.3}$$

$$x \geq 0, \tag{1.4}$$

where c is an n -vector, A is an $m \times n$ matrix and b is an m -vector. Then the following variations in the problem can be considered:

- change in the cost vector c
- change in the right-hand side vector b
- change in the constraints matrix A
- addition and/or deletion of a new variable
- addition and/or deletion of a new constraint

Stability of the linear programming problem has been studied in [2,36]. The definitions of stability of the linear programming problem given in [2] can be restated in the following way in terms of multivalued mappings [58].

Definition 1.8. *The problem (1.2)–(1.4) is stable with input data $u^0 = (A^0, b^0, c^0) \in \mathbb{R}^{m \times n + m + n}$ if there exists $\delta^0 > 0$ such that $\forall \delta < \delta^0$: $O_\delta(u^0) \subset U = \emptyset$.*

Definition 1.9. *The problem (1.2)–(1.4) is called functional-stable with input data u^0 if it is stable in the sense of Definition 1.8 and the function $\phi(u) = \min\{c^T x \mid Ax \leq b, x \geq 0\}$ is continuous at the point u^0 .*

Definition 1.10. *The problem (1.2)–(1.4) is called solution-stable if it is stable in the sense of Definition 1.8 and the mapping $\psi(u) = \{x' \mid c^T x' = \min\{c^T x' \mid Ax' \geq b, x' \geq 0\}\}$ is Hausdorff upper semi-continuous at the point u^0 .*

Functional-stability implies that sufficiently small perturbations in the initial data lead to sufficiently small changes in the functional value.

Solution-stability implies that, with small perturbations in the input data, for every optimal solution of the perturbed problem there exists a sufficiently close optimal solution of the original problem.

Necessary and sufficient conditions for various versions of the linear programming problem have been studied in [2]. In particular, the standard problem (1.2)-(1.4) is stable if and only if

$$\exists x^0 \in \mathbb{R}^n, p^0 \in \mathbb{R}^m : x^0 > 0, p^0 > 0, Ax^0 > b, A^T p^0 < c.$$

Moreover, if the problem is stable, then the sets of optimal solutions of the primal and dual problem are bounded.

It is well known that a major part of discrete optimization problems can be formulated as a partial case of integer linear programming problems, which in turn belong to the wide class of mathematical programming problems. Stability analysis in mathematical programming in general attempts to determine whether "small" changes in problem parameters produce a "small" changes in results.

The main difficulty while studying stability of discrete optimization problems is discrete models complexity, because even small changes of initial data make a model behave in an unpredictable manner. There is a lot of papers (see, e.g., [9, 45, 47, 56, 68, 71, 73, 97, 99]) devoted to the analysis of scalar and vector (multicriteria) discrete optimization problems sensitivity to parameters perturbations. This thesis continues investigations of different stability types of discrete optimization problems with various partial criteria and optimality principles (see, e.g., [16, 18, 23, 25–30, 34]).

Despite numerous approaches to stability analysis of discrete optimization problems (annotated bibliography [45] gives quite complete representation about various publications about stability issues) two major directions can be single out: quantitative and qualitative.

- The quantitative direction aims to derive quantitative bounds for feasible initial data changes preserving some preassigned properties of optimal solutions and create algorithms for the bounds calculation (see, e.g., [9, 16, 23, 25, 34, 47, 56, 68, 71, 73, 97, 98]). The frequently used quantitative tool of the stability theory and post-optimal analysis is so-called *stability radius* of some given optimal solution. In single objective optimization, it gives an upper bound on subset of problem parameters for which this solution remains optimal [45]. There are already similar investigations in multiobjective case, where the stability radius defines extreme level of problem parameters perturbations preserving efficiency of the given solution. For example, in [16] one can find a large survey on sensitivity

analysis of vector unconstrained integer linear programming, where the stability radius is a key object under investigation.

- The qualitative direction aims to obtain conditions under which the set of optimal solutions of the problem possesses a certain preassigned property of invariance to external influence on initial data of the problem. A number of results in this direction is connected with deriving necessary and sufficient conditions for various stability types of multicriteria integer linear and quadratic programming problems, which consist in finding of Pareto optimal, Slater optimal and Smale optimal solutions (see, e.g., [59, 60, 92]), as well as boolean and integer problems of sequential minimization of linear functions modules (see, e.g., [18, 19]), multicriteria combinatorial bottleneck problems (see, [29]) and with other nonlinear criteria (see, publications II and III).

In the stability framework, we can apply the notion of continuity to discrete optimization problems because a nontrivial, nondiscrete topology can be defined on the set $2^{\mathbb{R}^n}$ (the set of all subsets of the set \mathbb{R}^n), whose elements include both the feasible region and the set of optimal solutions [58]. This situation can be explained as follows: when analyzing the stability of any optimization problem, including discrete optimization problem, we can uniquely associate to the given problem a vector of input data u as an element of some space U in which a nontrivial nondiscrete topology can be defined.

In the current work, five stability types (see, e.g., [16, 19]) of multicriteria discrete optimization problems are considered. Suppose the initial m -criteria optimization problem with parameters defined by matrix $A \in \mathbb{R}^{m \times n}$ and the set of feasible solutions S is solved. Let $X^m(A)$ (can be $P^m(A)$, $L^m(A)$, $Sm^m(A)$, $Sl^m(A)$ or some other set of efficient solutions) be the set of optimal solutions. Perturbations of the problem are modeled by adding matrix A to matrices of the set

$$\Omega(\varepsilon) = \{A' \in \mathbb{R}^{m \times n} : \|A'\| < \varepsilon\},$$

where $\varepsilon > 0$, $\|A'\| = \max \{\|A'_i\| : i \in N_m\} = \max \{|a'_{ij}| : (i, j) \in N_m \times N_n\}$, $A' = (a'_{ij})$, $N_m = \{1, 2, \dots, m\}$ and $N_n = \{1, 2, \dots, n\}$. The set $\Omega(\varepsilon)$ is called *the set of perturbing matrices*.

A multicriteria discrete optimization problem is called *stable* if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (X^m(A + A') \subset X^m(A)).$$

In other words the problem is stable when small changes of initial problem data do not lead to appearance of new optimal solutions. Therefore stability can be interpreted as discrete analogue of the Hausdorff upper semicontinuity of the set-valued mapping

$$X : \mathbb{R}^{m \times n} \rightarrow 2^{\mathbb{R}^n} \tag{1.5}$$

at a point A [92, 103]. This mapping associates to each matrix $A \in \mathbb{R}^{m \times n}$ the set of optimal solutions $X^m(A)$.

Relaxing the requirement of non-appearance of new optimal solutions we come to the concept of strong stability. The problem is *strongly stable* if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (X^m(A) \cap X^m(A + A') \neq \emptyset).$$

Thereby this type of stability describes situation when the intersection of the set of optimal solutions to the initial problem and analogous set of any perturbed problem is nonempty. It is easy to see that the problem is strongly stable if it is stable. The problem is called *quasistable* if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (X^m(A) \subset X^m(A + A')).$$

Quasistability characterizes the case when all optimal solutions remain optimal under small perturbations of initial data of the problem. Thereby it is a discrete analogue of the Hausdorff lower semicontinuity of the set-valued mapping (1.5) at a point A .

Relaxing the requirement of the whole set of optimal solutions to remain optimal we come to the concept of strong quasistability. The problem is called *strongly quasistable* if

$$\exists \varepsilon > 0 \quad \exists x^0 \in X^m(A) \quad \forall A' \in \Omega(\varepsilon) \quad (x^0 \in X^m(A + A')).$$

This stability type is connected with existence of at least one solution that belongs to the set of optimal solutions to all perturbed problems under small perturbations of initial data. It is evident that any quasistable problem is strongly quasistable. Finally, the problem is *superstable* when the set of optimal solutions to the initial problem coincides with the set of optimal solutions to all perturbed problems, that is,

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (X^m(A) = X^m(A + A')).$$

This property of the problem is discrete analogue of the Hausdorff continuity of the set-valued mapping (1.5) at a point A that puts into correspondence any matrix A with the set of optimal solutions. It is evident that the problem is superstable if it is stable and quasistable simultaneously.

In Figure 2, in diagrams illustrating strong stability and strong quasistability matrices A' , A'' belong to the set $\Omega(\varepsilon)$. One should emphasize that, in general, for the former stability type the sets of optimal solutions corresponding to different perturbing matrices have different intersections with the initial set of optimal solutions while for the latter stability type there exists at least one common solution for the initial set of optimal solutions and the sets of optimal solutions generated by any perturbing matrix.

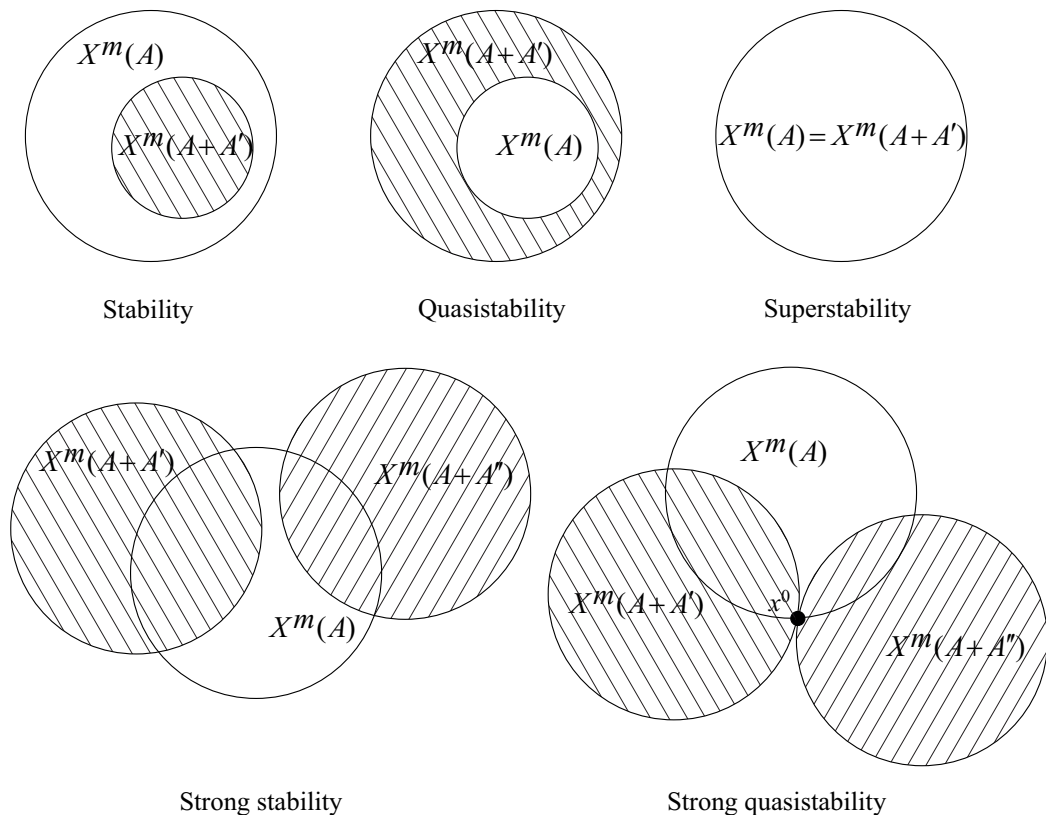


Figure 2: Venn diagrams of stability types

This thesis is motivated by enormous variety of real-life problems that are multicriteria in nature, have combinatorial structure and depend on uncertain parameters. The current work is devoted to investigation of the recent innovative approaches to multicriteria discrete optimization problems and postoptimal analysis. The major part of the thesis, which is covered in Chapters 2 and 3, focuses on stability analysis of vector discrete optimization problems with various partial criteria. It is assumed that the set of optimal solutions (or a single solution) to the problem is already known and the aim is to investigate behavioristic and invariant properties of this set (or solution). Nonetheless in order to provide complete effective tools for determining reliable solutions it is important to explore techniques for solving initial problems. Therefore, Chapter 4 of the thesis addresses the questions of interactive optimization methods that can be applied to multicriteria combinatorial optimization problems.

Chapter 2. Qualitative approach to stability analysis of vector discrete optimization problems with nonlinear partial criteria

In the works of qualitative direction authors concentrate their attention on revealing conditions for various stability types of a problem, on determining interrelations between different stability types as well as on finding and description of stability region of the problem. The group of Ukrainian mathematicians under the guidance of academic Sergienko, who introduced this approach, have studied questions of stability and parametric analysis of integer and mixed integer programming in various formulations, including solution-stability, functional-stability, constraint stability and vector-criterion stability [58].

A fairly comprehensive survey of some typical results and approaches to stability and postoptimal analysis of scalar integer programming problems is provided in [57]. Stability analysis of optimization problems with pure integer or mixed integer variables based on the theory of multivalued mappings is given in [4].

Sensitivity analysis of an integer linear programming problem is generally conducted on the basis of specific method used to solve the given problem. Postoptimal analysis of a 0-1 programming problem with changes in the objective function, in the vector of the right-hand side, and the constraint matrix, which utilizes branch and bound method is carried out in [88]. The author has demonstrated how tests performed during solution phase become a useful basis for ranging analysis on selected parameters and contribute to experimentation with a variety of parameters changes, without the necessity of resolving the entire problem for each one. The obtained results were extended on the mixed integer case in [89].

Paper [37] is devoted to modification of branch and bound method to solve a family of related integer linear programming problems, that is, the set of problems that have the same structure but differ as to the values of one or more coefficients. The approach to postoptimal analysis of integer and mixed integer programming problems with Gomory cutting-plane algorithm is presented in [54]. Utilizing this algorithm authors formulated various sufficient or necessary conditions for verifying the optimality of a solution under coefficient changes in the right hand side and the objective function, as well as conditions for introduction of a new variable.

One of the earliest papers on *the stability region*, that is, the set of all problem instances for which a given solution is optimal, is [65]. Article [73] explores the possibility of using information about the k -best solutions to the traveling salesperson problem for determining stability region of an optimal solution.

The concepts of the continuity of a point-to-set map from the parameter space into the solution or objective space are developed and used in [102, 103] to describe the stability of the solution set of multiobjective optimization problem in the case when the solution set is continuous, i.e. $X \subset \mathbb{R}^n$. In these papers a parameterized vector optimization problem with two different parameters is considered. The first one is concerned with the stability of the solution set, where the set of feasible solutions changes. The second point of view is concerned with the stability of the solution sets in the case where *the domination structure*, the positive cone in a partially ordered space representing the preference attitude, changes. Criteria for continuity, upper and lower semicontinuity of the above mentioned point-to-set map were derived.

Stability investigation in multicriteria discrete optimization has emerged not long ago. First results in this direction were obtained by Emelichev and his students (see, e.g. [15, 21]). A number of papers was devoted to stability analysis of Pareto, Slater and Smale optimal solutions under perturbations of problem parameters for different partial criteria types [16, 23–25, 34].

2.1. Multicriteria center and median location problems

Problems of finding the "best" location of facilities in networks or graphs, abound in practical situations. In particular, if a graph represents a road network with its vertices representing communities, one may have the problem of locating optimally a hospital, police station, fire station, or any other "emergency" service facility. In such cases, the criterion of optimality may justifiably be taken to be the minimization of the distance (or travel time) from the facility to the most remote vertex of the graph, that is, the optimization of the "worst-case". In a more general problem, a large number (and not just one) of such facilities may be required to be located. In this case the furthest vertex of the graph must be reachable from at least one of the facilities within a minimum distance. Such problems, involving the location of emergency facilities and whose objective is to minimize the largest travel distance to any vertex from its nearest facility, are, for obvious reasons, called minimax location problems [10]. The resulting facility locations are then called *the centers* of a graph.

In certain problems associated with the location of facilities on a graph, what is required is to locate a facility in such a way that the sum of all shortest distances from the facility to the vertices of the graph is minimized. The optimum location of the facility is then called *the median* of the graph, and because of the nature of the objective function this class of problems is referred

to as the minisum location problem [10]. The problem appears often in practice in a variety of forms: the location of switching centers in telephone networks, substations in electric power networks, supply depots in a road distribution network (where the vertices represent customers), and the location of sorting offices for letter post are some of the areas where minisum location problems occur.

In practice the situation often arises where a single facility will not be able to meet various requirements and what is then needed is to locate a number of these facilities in the best possible way. For instance, we can consider the following problem:

Find the smallest number and location of emergency centers so that no community is further away than a prespecified distance from a center, and, with that number of centers given, the distance of the furthest community from an emergency center is a minimum.

This observation leads to the generalized concepts of a multi-center and a multi-median of a graph. The situation can be formally modeled as follows. Let $N_n = \{1, 2, \dots, n\}$ be the set of possible points (centers, medians) of suppliers location, the set N_s represents consumers and parameters of the problem, that is, costs of consumer serving by each facility, are specified by a cost matrix $A \in \mathbb{R}^{n \times s}$. Let T be a family of nonempty subsets of N_n , i.e. $T \subseteq 2^{N_n} \setminus \{\emptyset\}$, $|T| \geq 2$. Then *the p -center location problem* consists in allocation of p , $1 \leq p \leq n - 1$ facilities in N_n possible points on the graph that minimizes the largest travel distance to any consumer from its nearest facility:

$$\begin{aligned} \max_{j \in N_s} \min_{i \in t} a_{ij} &\rightarrow \min \\ t \in T, |t| &= p. \end{aligned}$$

Analogously in *the p -median location problem* it is required to locate p facilities in such a way so that the sum of the shortest distances (or transport costs) to any consumer from its nearest facility to be minimized.

$$\begin{aligned} \sum_{j \in N_s} \min_{i \in t} a_{ij} &\rightarrow \min, \\ t \in T, |t| &= p. \end{aligned}$$

If the restriction that the points forming the p -center (p -median) must lie at the vertices of a graph is lifted so that points lying on the arcs are also admissible, then this more general set of p points is called *the absolute p -center (the absolute p -median)*.

It was proved in [48, 49] that general p -center and p -median problems are NP-hard. Namely, there does not exist an $O(f(n, p))$ algorithm for finding a p -

center (p -median) of a general network where $f(n, p)$ is a polynomial function in each of the variables n and p . In fact, a stronger results were formulated:

Theorem 2.1. *The problems of finding a vertex p -center and an absolute p -center are NP-hard even in the case when the network is vertex-unweighted planar graph of maximum degree 3, all whose edges are of length 1.*

Theorem 2.2. *The problem of finding a p -median is NP-hard even in the case when the network is a planar graph of maximum vertex degree 3 all whose edges are of length 1 and all whose vertices have weight 1.*

In most algorithms which have been proposed for the p -center problem, one or more of three decision factors are being employed in the search for optimum. These are

- the number of candidate points $T_p = \{i_1, i_2, \dots, i_p\}$, $p \in N_n$, actually used as facilities, $p \leq n$,
- the maximum distance λ (predefined value) within which any given facility is allowed to provide its service, and
- the coverage induced by the operating facilities T_p , defined as $Q_\lambda(T_p) = \{j \in N_s \mid a_{i_k j} \leq \lambda\}$, $i_k \in T_p$, $k = 1, \dots, p$.

Among algorithms that fit the general framework described above we can single out the earliest algorithm proposed by Minieka in [78], Kariv and Hakimi enumerative approach [48], and Christofides [10] iterative algorithm whose basic idea is to find the minimum dominating set of the graph. The differences among the p -center algorithms appear to be in three aspects. First is the choice of constraints to be imposed, namely, which among p , λ and Q_λ are restricting the search for a constrained cover. Second is the manner in which the search for a constrained cover is conducted. Third aspect is the way in which constraints are tightened or loosened.

The knowledge that the p -center problem is NP-complete suggests that if major improvements in search procedures may be developed, they need to be heuristics based on the many possible variations of the methods mentioned above.

The p -median problem can be formulated as an integer boolean programme [10]. Moreover the linear programming relaxation can be efficiently used to obtain the p -median for most problems, and if fractional values of some variables occur, then a resolution of these cases can be obtained by a tree-search procedure in which one branch fixes some fractional variable to 0 and another the same variable fixes to 1. One can then proceed to resolve the linear programming problems for each of the two resulting subproblems and so on until all variables become either 0 or 1.

Instead of an explicit formulation of the p -median problem as an integer boolean programme one could use a direct tree search approach which is better suited to exploiting the structure of the problem.

A heuristic method based on vertex substitution is described by Teitz and Bart [104]. This algorithm is one of a family of algorithms based on local optimization and on the idea of λ -optimality first introduced by Lin in [74] for the traveling salesperson problem.

In the process of locating a new facility usually more than one decision maker is involved. This is due to the fact that typically the cost connected to the decision is relatively high. Of course, different persons may have different conflicting objectives. On other occasions, different scenarios must be compared in order to be implemented, or simply uncertainty in the parameters leads to consider different replications of the objective function. Thus a natural extension of the single criterion deterministic model is to study the case when several cost criteria (not only transportation costs) have to be minimized and to investigate behavior of the objectives under changes of problem parameters.

Papers II and III focus on investigation of solution stability for multicriteria combinatorial minimax and minisum location problems with Pareto and lexicographic optimality principles.

Let us consider these problems in the following formulations.

Let N_n be the set of possible points of facilities (suppliers) location, N_s be consumers (clients) location, $A = (a_{ijk}) \in \mathbb{R}^{n \times s \times m}$ be the matrix of costs a_{ijk} . The cost is connected with client $j \in N_s$ serviced by facility $i \in N_n$ and with criterion $k \in N_m$.

Let $f : T \times \mathbb{R}^{n \times s \times m} \rightarrow \mathbb{R}^m$ be a vector-valued function, where, $f(t, A) = (f_1(t, A), f_2(t, A), \dots, f_m(t, A))$ for any $t \in T$, $A \in \mathbb{R}^{n \times s \times m}$ and partial criteria be defined as follows:

$$\begin{aligned} \text{minimax } f_k(t, A) &= \max_{j \in N_s} \min_{i \in T} a_{ijk} \rightarrow \min_{t \in T}, k \in N_m \\ \text{minisum } f_k(t, A) &= \sum_{j \in N_s} \min_{i \in T} a_{ijk} \rightarrow \min_{t \in T}, k \in N_m \end{aligned}$$

Thus, multicriteria center and median location problems consist in finding corresponding Pareto ($P^m(A)$) and lexicographic ($L^m(A)$) sets. The purpose of the research consists in finding conditions guaranteeing an optimal solution to remain optimal under "small" perturbations of vector criterion parameters. As mentioned above such perturbations are modeled by adding matrix A to matrices of the set

$$\Omega(\varepsilon) = \{A' \in \mathbb{R}^{n \times s \times m} : \|A'\| < \varepsilon\}.$$

where $\varepsilon > 0$, $\|A'\| = \max\{|a'_{ijk}| : (i, j, k) \in N_n \times N_s \times N_m\}$, $A' = (a'_{ijk})$.

Definition 2.1. A Pareto optimal solution $t \in P^m(A)$ is called stable if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t \in P^m(A + A')).$$

Definition 2.2. A lexicographically optimal solution $t \in L^m(A)$ is called stable if

$$\exists \varepsilon > 0 \quad \forall A' \in \Omega(\varepsilon) \quad (t \in L^m(A + A')).$$

Thus, the solution is stable if it remains optimal under any "small" independent perturbations of the problem parameters, that is, the elements of A .

In order to obtain necessary and sufficient conditions for an optimal solution stability it is important to determine elements of the matrix of problem parameters affecting optimality of the solution. These elements in turn depend on the nature of the partial criteria. In particular, in the case of multicriteria minimax optimization problem for any indexes $k \in N_m$, $j \in N_s$ and trajectory t we introduce the following sets:

$$N_{jk}(t, A) = \{l \in t : f_k(t, A) = g_{jk}(t, A) = a_{ljk}\},$$

$$J_k(t, A) = \{j \in N_s : f_k(t, A) = g_{jk}(t, A)\},$$

where

$$g_{jk}(t, A) = \min_{i \in t} a_{ijk}.$$

Necessary and sufficient conditions for stability of Pareto and lexicographic optimal solution to the multicriteria center location problem under initial data perturbations are formulated in the following theorems.

Theorem 2.3. [II] A solution $t^0 \in P^m(A)$ is stable if and only if for any equivalent solution t ($f(t^0, A) = f(t, A)$), and for any criteria $k \in N_m$

$$J_k(t, A) \supseteq J_k(t^0, A) \tag{2.1}$$

and

$$\forall j \in J_k(t^0, A) \quad (N_{jk}(t^0, A) \supseteq N_{jk}(t, A)). \tag{2.2}$$

Theorem 2.4. [II] A solution $t^0 \in L^m(A)$ is stable if and only if for any criteria $k \in N_m$ and for any solution $t \in L_k^m(A)$

$$J_k(t, A) \supseteq J_k(t^0, A) \tag{2.3}$$

and

$$\forall j \in J_k(t^0, A) \quad (N_{jk}(t^0, A) \supseteq N_{jk}(t, A)). \tag{2.4}$$

The proof of necessity is based on the construction of perturbing matrices whose elements are build using the sets $J_k(t, A)$, $J_k(t^0, A)$, $N_{jk}(t, A)$ and

$N_{jk}(t^0, A)$. Sufficiency is proved by utilizing continuity of the function $g_{jk}(t, A)$ in parameters space \mathbb{R}^n .

Using the same techniques analogous results are formulated and proved for the multicriteria combinatorial median location problem.

Let $N_{jk}(t, A)$ be the set of facilities from t which serve the client j with minimum costs by criterion k

$$N_{jk}(t, A) = \text{Arg min}\{a_{ijk} : i \in t\},$$

i.e.

$$N_{jk}(t, A) = \{l \in t : \min_{i \in t} a_{ijk} = a_{ljk}\}.$$

Theorem 2.5. [III] *A solution $t^0 \in P^m(A)$ is stable if and only if for any equivalent solution t , for any $k \in N_m$ and for any $j \in N_s$ the following inclusion*

$$N_{jk}(t^0, A) \supseteq N_{jk}(t, A) \quad (2.5)$$

holds.

It is easy to interpret the theorem for a scalar problem, that is, for $m = 1$. Let the distance from every client j to every facility i be known. Then the theorem states that for optimal solution t^0 to be stable it is necessary and sufficient that all optimal bindings (in the sense of a proximity by distance) of clients to facilities t^0 remain optimal bindings of all the clients to the facilities of any equivalent solution t .

Theorem 2.6. [III] *A solution $t^0 \in L^m(A)$ is stable if and only if for any criteria $k \in N_m$ and for any solution $t \in L_k^m(A)$ the following inclusion*

$$N_{jk}(t^0, A) \supseteq N_{jk}(t, A) \quad (2.6)$$

holds.

One thing worth mentioning is that in the case of $m = 1$ ($A = (a_{ik}) \in \mathbb{R}^{n \times m}$), the considered problem transforms into the m -criteria combinatorial problem with partial minimin criteria. For any $k \in N_m$, put

$$N_k(t, A) = \text{Arg min}\{a_{ik} : i \in t\}.$$

We assume $f_k(\emptyset, A_k) = +\infty$.

Next two well known results follow from Theorems 2.3, 2.4, 2.5 and 2.6.

Corollary 2.1. [24] *Let $t^0 \in P^m(A)$ be a Pareto optimal solution of the vector problem with partial criteria minimin. The next statements are equivalent:*

(i) $t^0 \in P^m(A)$ is stable;

$$(ii) \forall k \in N_m \quad \forall t \in Q^m(t^0, A) \quad (N_k(t^0, A) \supseteq N_k(t, A));$$

$$(iii) \forall k \in N_m \quad \forall t \in Q^m(t^0, A) \quad (f_k(t \setminus t^0, A) > f_k(t^0, A)).$$

Corollary 2.2. [20] *Let $t^0 \in L^m(A)$ be a lexicographically optimal solution of the vector problem with partial criteria minimin. The next statements are equivalent:*

$$(i) t^0 \in L^m(A) \text{ is stable};$$

$$(ii) \forall k \in N_m \quad \forall t \in L_k^m(t^0, A) \quad (N_k(t^0, A) \supseteq N_k(t, A));$$

$$(iii) \forall k \in N_m \quad \forall t \in L_k^m(t^0, A) \quad (f_k(t \setminus t^0, A) > f_k(t^0, A)).$$

2.2. Vector combinatorial minimin problem

This section addresses a general theoretical approach to qualitative analysis of multicriteria combinatorial minimin problems with Pareto and lexicographic principles of optimality.

Let A_i be the i -th row of matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, $m \geq 1$, $n \geq 2$, T be a family of nonempty subsets of $N_n = \{1, 2, \dots, n\}$, i.e. $T \subseteq 2^{N_n} \setminus \{\emptyset\}$, $|T| \geq 2$. Elements of the set T are called trajectories. Let $f : T \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^m$ be a vector-valued function, where, $f(t, A) = (f_1(t, A_1), f_2(t, A_2), \dots, f_m(t, A_m))$ for any $t \in T$, $A \in \mathbb{R}^{m \times n}$ and $f_i(t, A_i) = \min_{j \in t} a_{ij}$ for $i \in N_m$. The minimin problem can be written as follows:

$$\min_{t \in T} f(t, A).$$

Many extreme problems on graphs such as the traveling salesperson problem, the assignment problem, the shortest path problem etc. are included into the similar scheme of scalar combinatorial problems. In addition the multicriteria minimin problem is a special case of multicriteria variant of the well known median and center location problems. Specifically, suppose optimal locations of emergency centers (facilities) for the given communities have already been found. We want to add one new community to the network. The problem consists in finding among the existing facilities one that optimally serves the new community (consumer).

Put

$$N_i(t, A_i) = \text{Arg min}\{a_{ij} : j \in t\}, \quad i \in N_m.$$

Let us introduce a notation for Cartesian product of sets $N_i(t, A_i)$, $i \in I \subseteq N_m$:

$$\bar{N}(t, A, I) = N_{i_1}(t, A_{i_1}) \times N_{i_2}(t, A_{i_2}) \times \dots \times N_{i_k}(t, A_{i_k}),$$

where $I = \{i_1, i_2, \dots, i_k\} \subseteq N_m$, $i_1 < i_2 < \dots < i_k$.

For vector $v = (v_1, v_2, \dots, v_m) \in \mathbb{R}^m$ and set $I = \{i_1, i_2, \dots, i_k\} \subseteq N_m$, $i_1 < i_2 < \dots < i_k$, we introduce notation

$$v_I = (v_{i_1}, v_{i_2}, \dots, v_{i_k}).$$

Put

$$P^m(t, A) = \{t' \in P^m(A) : f(t, A) \geq f(t', A)\},$$

$$I(t, t') = \{i \in N_m : f_i(t, A_i) = f_i(t', A_i)\}.$$

In other words, set $P^m(t, A)$ consists of Pareto optimal trajectories t' which dominate trajectory t and $I(t, t')$ denotes the set of criteria indexes such that the values of corresponding partial criteria coincide for trajectories t and t' .

Necessary and sufficient conditions of the five stability types are formulated for Pareto principle of optimality in the following theorems.

Theorem 2.7. [VI] *The problem is stable if and only if for any solution $t \in Sl^m(A)$ the formula*

$$\forall v \in \bar{N}(t, A, N_m) \quad \exists t^* \in P^m(t, A) \quad (v_{I(t, t^*)} \in \bar{N}(t^*, A, I(t, t^*))) \quad (2.7)$$

is valid.

Formula (2.7) indicates that for any weakly efficient trajectory t there exists trajectory $t^* \in P^m(t, A)$ which is invariant to small perturbations of problem parameters.

Theorem 2.8. [VI] *The problem is strongly stable for any matrix $A \in \mathbb{R}^{m \times n}$.*

For solution $t \in P^m(A)$ we introduce a set of equivalent solutions

$$Q(t, A) = \{t' \in T : f(t, A) = f(t', A)\}.$$

Theorem 2.9. [VI] *The problem is quasistable if and only if*

$$\forall t \in P^n(A) \quad \forall t' \in Q(t, A) \quad \forall i \in N_m \quad (N_i(t, A_i) \supseteq N_i(t', A_i)). \quad (2.8)$$

Condition (2.8) indicates that for any two equivalent solutions t and t' the equality $\bar{N}(t, A, N_m) = \bar{N}(t', A, N_m)$ must hold.

The next result follows from Theorems 2.7 and 2.9 by virtue of the stability types definitions.

Theorem 2.10. [VI] *The problem is strongly quasistable if and only if both statements hold:*

$$(i) \quad \forall t \in SI^m(A) \quad \forall v \in \overline{N}(t, A, N_m) \quad \exists t^* \in P^m(t, A) \\ (v_{I(t,t^*)} \in \overline{N}(t^*, A, I(t, t^*))), \quad (2.9)$$

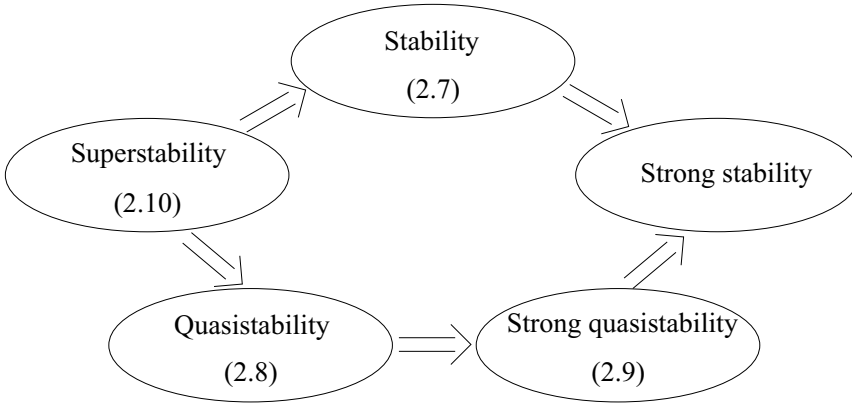
$$(ii) \quad \forall t \in P^m(A) \quad \forall t' \in Q(t, A) \quad \forall i \in N_m \quad (N_i(t, A_i) \supseteq N_i(t', A_i)).$$

Theorem 2.11. [VI] *The problem is superstable if and only if*

$$\exists t^0 \in P^m(A) \quad \forall t \in Q(t^0, A) \quad \forall i \in N_m \quad (N_i(t^0, A_i) \supseteq N_i(t, A_i)). \quad (2.10)$$

Condition (2.10) indicates the existence of efficient solution t^0 such that for all solutions t equivalent to it the inclusion $\overline{N}(t^0, A, N_m) \supseteq \overline{N}(t, A, N_m)$ holds.

Summarizing the results obtained in Theorems 2.7 – 2.11 relations between different stability types of the problem with Pareto principle of optimality are described by the following scheme:



Now let us introduce stability criteria for the multicriteria combinatorial minimin problem with lexicographic principle of optimality.

Put

$$M(t) = \{i \in N_m : t \in L_i^m(A_i)\}.$$

Theorem 2.12. [VI] *For the problem the following statements are equivalent:*

- (i) *the problem is stable,*
- (ii) *the problem is strongly stable,*
- (iii) *for any solution $t \in L_1^m(A)$*

$$\forall v \in \overline{N}(t, A, M(t)) \quad \exists t^* \in L^m(A) \quad (v \in \overline{N}(t^*, A, M(t))). \quad (2.11)$$

Statement (iii) indicates that for any non lexicographic solution $t \in L_1^m(A)$ there exists solution $t^* \in L^m(A)$ that will not allow solution t to become lexicographically optimal under small perturbations.

Theorem 2.13. [VI] For the problem the following statements are equivalent:

- (i) the problem is quasistable,
- (ii) the problem is superstable,
- (iii) the following formula holds

$$\forall t \in L^m(A) \quad \forall i \in N_n \quad \forall t' \in L_i^m(A) \quad (N_i(t, A_i) \supseteq N_i(t', A_i)). \quad (2.12)$$

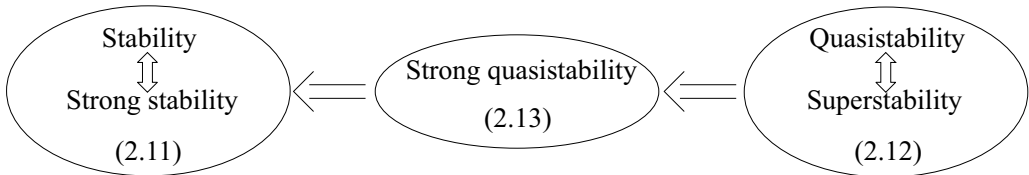
Formula (2.12) indicates that any solution $t \in L^m(A)$ must not be dominated by solutions $L_i^m(A)$, $i \in N_m$, under small perturbations of problem parameters.

Theorem 2.14. [VI] The problem is strongly quasistable if and only if

$$\exists t^0 \in L^m(A) \quad \forall i \in N_m \quad \forall t \in L_i^m(A) \quad (N_i(t^0, A_i) \supseteq N_i(t, A_i)). \quad (2.13)$$

Formula (2.13) indicates the existence of lexicographically optimal solution t^0 which must not be dominated by solutions $L_i^m(A)$, $i \in N_m$, under small perturbations of problem parameters.

Summarizing the results obtained in Theorems 2.12, 2.13 and 2.14, taking into account definitions of the stability types, one may conclude that the relations between different stability types of the multicriteria combinatorial minimin problem with lexicographic principle of optimality are described by the following scheme:



One more issue which has to be emphasized is that practical verification of conditions of Theorems 2.7 – 2.14 and their straightforward application for general case can be as hard as to solve the problem itself. Nevertheless more methodological results might be developed and implemented for special cases of the multicriteria combinatorial minimin problem with restrictions of some factors, such as structure of initial data, perturbations of particular problem parameters etc. As possible continuation of the research within this topic, it would be interesting to explore these classes of problems.

Chapter 3. Quantitative approach to stability analysis of discrete optimization problems

Within the framework of quantitative direction researchers aim to measure quality and sensitivity of an optimal solution (or a feasible solution) under possible realizations of problem parameters. Authors of most papers devoted to stability analysis and robust optimization attempt to specify analytical expressions for quantitative characteristics such as the stability radius, relative errors, accuracy functions and robust tolerances; derive algorithms for its calculation and analyze complexity of these algorithms.

In the case of a single objective function, formulae of stability radius are obtained for problems of 0–1 programming [64], problems on systems of subsets and graphs [97], scheduling problems [99] and simple assembly line balancing problem [96]. Let us discuss these results in more details.

First publications concerning quantitative direction are devoted to obtaining formula for the stability radius of the set of all possible shortest tours for the classical traveling salesperson problem (TSP) [61]. The maximum value of the stability radius for the class of matrices whose elements do not exceed in modulus some fixed number is calculated in [61]. Here a class of matrices possessing a zero minimum stability radius is described and some estimates of the stability radius are given. These estimates are easily calculated from the distance matrix for the case when the elements of the distance matrix are rational numbers. Finally some other possible methods of determining the domain of stability are discussed. Later in paper [62] the notions of strong and weak stability types were introduced for a linear scalar combinatorial problem, that is, the problem on a system of subsets of a finite set with linear partial criteria (minisum type). Bounds and formulas for the stability radii for these stability types were found.

Let us consider a mathematical formulation of a linear scalar combinatorial minisum problem. Suppose $E = \{e_1, e_2, \dots, e_n\}$ be a finite set of n elements and $T = \{t_1, t_2, \dots, t_m\}$, $m \geq 2$, be a finite family of subsets of the initial set E , $T \subset 2^E$. A nonnegative weights $w(e_1) = a_1, w(e_2) = a_2, \dots, w(e_n) = a_n$ are assigned to each element of E . For any trajectory $t_i \in T$, $i = \{1, \dots, m\}$, we define function, the length of the trajectory, which is determined by the formula:

$$f(t_i, A) = \sum_{e_j \in t_i} a_j.$$

Thus, linear trajectory problem is specified by a triplet E, T, A and function $f(t_i, A)$. Under an optimal solution to the problem we understand trajectories which have the minimal length. Let $X^m(A)$ be the set of all optimal trajectories of the problem and a norm is defined in space \mathbb{R}^m . Then the problem is stable

if there exists $\varepsilon > 0$ such that for any vector $B \in \mathbb{R}^m$, $\|B\| \leq \varepsilon$, the inclusion $X^m(B) \subseteq X^m(A)$ holds.

Definition 3.1. An open ball $O_\rho(A) \in \mathbb{R}^m$ with radius ρ and matrix A in the center is called stability ball of A if for any matrix $B \in O_\rho(A)$ the inclusion $X^m(B) \subseteq X^m(A)$ is valid.

Definition 3.2. Let M be the set of all possible real numbers ρ such that $O_\rho(A)$ is stability ball of A . Number $\rho(A) = \sup\{\rho : \rho \in M\}$ is called stability radius of $A \in \mathbb{R}^m$ if $M \neq \emptyset$ and $\rho(A) = 0$ otherwise.

In paper [62] author introduced the first analytical expression for the stability radius of the minisum problem:

$$\rho(A) = \min_{j \notin X^m(A)} \max_{i \in X^m(A)} \frac{f(t_i, A) - f(t_j, A)}{|t_i| + |t_j| - 2|t_i \cap t_j|}.$$

Analogous quantitative characteristics are found for scalar combinatorial "bottle-neck" (minimax) problem in [43]. All mentioned results are obtained under assumption that Chebyshev metric is defined in the space of problem parameters.

Note that quantitative characteristics of stability of discrete optimization problems are determined by the following three factors:

- class of the considered optimization problem
- restriction on perturbations of some problem parameters
- types of norms defined in the spaces of problem parameters

Thus deriving formula for stability radius for each class of discrete optimization problems in the case of arbitrary norm and under possible restrictions on problem parameters would be the most general result. It is evident that this problem statement is too extensive and therefore meaningful results can be obtained only if above mentioned factors are fixed or restricted. It seems quite interesting to investigate stability of a problem for different norm types because various types of normalization allows to take into account specificity of problem parameters perturbations in different ways.

Bounds for stability radius of scalar "bottle-neck" problem in the case of arbitrary norm were detected in [63]. Furthermore, in [41] a formula of stability radius was obtained for special classes of norms, in particular for monotonic norms. A situation is worse for a scalar linear combinatorial problem. Despite the fact that bounds for the stability radius were obtained for a wide class of norms [62], a formula for the stability radius is detected only for Chebyshev norm [61] and special problem types such as minimum spanning tree problem in the case of linear norm [42].

Stability analysis of an approximate solution to a scalar boolean linear programming problem was conducted in [64] for the case when all parameters of the optimization model are known approximately and the Chebyshev norm is specified in the space of problem parameters. Finalizing description of the results concerning quantitative bounds of radii for different stability types of discrete scalar problems one should mention that a general approach to obtaining formula of the stability radius for scalar combinatorial problem was suggested in [44]. It is based on reduction of the problem of finding stability radius to the mathematical programming problem of special type. In particular, let us fix the vector of parameters A in trajectory minisum problem described above and consider $B \in \mathbb{R}^m$. For an arbitrary pair of trajectories t_i and t_j , $f(t_i, A) < f(t_j, A)$, define the following optimization problem:

$$\|B\| \rightarrow \min, \quad f(t_i, A + B) \geq f(t_j, A + B), \quad B \in \mathbb{R}^m. \quad (3.1)$$

Let X_{ij} and X'_{ij} be the sets of optimal and feasible solutions to the problem (3.1). As a characteristic of this problem we introduce a parameter $r_{ij}(A)$. Suppose by definition $r_{ij}(A) = 0$ if $X_{ij} = \emptyset$ and $r_{ij}(A) = \|B\|$ if $\|B\| \in X_{ij}$. Thus, $r_{ij}(A)$ characterizes the distance from A to the set of matrices $V_{ij}(A) = \{C : C \in \mathbb{R}^m, f(t_i, C) \geq f(t_j, C)\}$ in \mathbb{R}^m .

Now we consider an arbitrary non optimal trajectory t_s and define the set $W_s(A) = \{i : f(t_i, A) < f(t_s, A)\}$. If the following set:

$$X_{is} \cap \left\{ \bigcap_{\substack{j \in W_s(A) \\ r_{js}(A) \leq r_{is}(A)}} X'_{js} \right\}$$

is non empty then for the stability radius of the problem the following relation holds [44]:

$$\rho(A) = \min_{j \notin X^m(A)} \max_{i \in X^m(A)} r_{ij}(A).$$

This relation is valid for any continuous function $f(t_i, A)$ and continuous norm defined in \mathbb{R}^m .

Now we provide a brief survey of some typical quantitative results and approaches to vector discrete optimization problems.

Stability of efficient solutions to a vector discrete optimization problem on a system of nonempty subsets of a finite set is considered in [17]. This paper addresses the vector criterion of the problem consisting of an arbitrary combination of partial criteria of the kinds minisum (linear), minimax (bottle neck) and minimin. As a result, lower bounds for stability radii of Pareto and Slater optimal solutions as well as formula for stability radius of a Smale optimal solution were obtained. Analogous bounds and formulas were derived under condition that only some elements of the matrix of initial parameters are

perturbed. Later in works devoted to quantitative direction of stability analysis formulas for radii of stability, quasistability and strong quasistability were indicated for a more general problem with criteria \sum -minimax and \sum -minimin in the case of the Chebyshev norm [38]. Attainable bounds of stability and quasistability radii for vector combinatorial problem with minimax modules criteria, that is, a modified version of bottle-neck criteria, were presented in [31], a formula for the stability radius were found a little bit later in [22].

Attainable bounds and formulas of radii were also obtained for various stability types of a vector integer linear programming problem [32, 33]. All these investigations are concerned the case when only parameters of the objective functions undergo perturbations. The case when criteria and constraints parameters are under perturbations was investigated just for the boolean programming problem. In particular, in [64] a formula for the stability radius was derived for the scalar linear boolean problem with a single optimal solution. Analogous investigations, based on the technique proposed in [64], were conducted for vector boolean problems. In [34] authors presented an approach to deriving formulae and estimations of stability radii for vector 0 – 1 programming problems with linear, linear absolute value and quadratic objective functions and linear constraints.

One thing worth mentioning is that all the problems listed above were considered for the Pareto principle of optimality. However various quantitative characteristics of stability were also detected for the vector integer linear programming problem with lexicographic principle of optimality (see, for instance, [16]).

In all the works described above the stability analysis is made in the space of problems parameters with the Chebyshev norm. Papers [24, 25] present results concerning quantitative stability analysis of the vector combinatorial optimization problems of finding the Pareto set for the case when linear norm is defined in the space of problem parameters. Here linear functions and the positive cuts of linear functions to the non negative semi-axis are considered as vector objectives.

Finally a new approach to the investigation of the stability of vector discrete optimization problems with parameterized principle of optimality is suggested in [35]. In partial cases, the parameterized principle of optimality coincides with Pareto, Slater, lexicographic or majority principle of optimality. Introduced methods allow to conduct stability analysis for problems of different nature simultaneously.

3.1. Measure of stability for finite cooperative games

The concept of stability, useful in the analysis of many kinds of optimization problems, can also be applied to some problems in game theory. Historically game theory has a lot in common with multicriteria optimization. While sensitivity analysis in vector optimization mainly concentrates on studying behavioristic and invariant properties of the Pareto set, in game theory the main object to study is a set of so-called equilibrium situations. The most common concept of equilibrium is the Nash equilibrium (named after John Nash, who proposed it in [81,82]). It is a solution concept of a game involving two or more players, in which no player has anything to gain by changing only his/her own strategy unilaterally. If each player has chosen a strategy and no player can benefit by changing his/her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium.

In the current work we consider the key object of the game theory, a finite normal-form game of m players [80], where each player i , $i \in N_m$, $m \geq 2$, chooses a corresponding strategy x_i in the set of strategies. This set is denoted by X_i , $X_i \subset \mathbb{R}$, $2 \leq |X_i| < \infty$. A realization of the game and its outcome is uniquely determined by the strategy choice by each of the participants. On the set of all situations $X = \prod_{i \in N_m} X_i$, let linear payoff functions of the players $f_i(x) = C_i x$, $i \in N_m$, be given, where C_i is the i th row of the matrix $C = [c_{ij}] \in \mathbb{R}^{m \times m}$, $x = (x_1, x_2, \dots, x_m)^T$, $x_j \in X_j$, $j \in N_m$. As the result of the game, which is referred to as a game with matrix C each player i gets, as the payoff, $f_i(x)$, which (s)he wants to maximize with the use of the preference relations.

In paper I the parametric concept of equilibrium (the generalized principle of optimality) of a finite game which induces a set of generalized equilibrium states is introduced. The parameter of this concept is the method of partitioning of the players into *coalitions*.

Definition 3.3. *For an m -person game, let N_m be the set of all players. Any nonempty subset of N_m (including N_m itself and all the one-element subsets) is called a coalition.*

We assume that the personal relations among the players inside a coalition are based on the Pareto principle. The introduced principle of optimality allows to connect two classical notions: the Pareto optimality and Nash equilibrium. In the m -person case, there are many possible coalitions and it means that, if a coalition is to form and remain for some time, the different members of the coalition must reach some sort of equilibrium. In other words we are interested in the total utility which can be attained by any one of these coalitions.

Paper I is focused on conducting quantitative sensitivity analysis of equilibrium situation invariant to changes of game parameters. The main result that was obtained here is a formula for stability radius of this situation in the case of the Hölder metric l_p , $1 \leq p \leq \infty$. Earlier in [8, 27] analogous results were obtained for radii of other stability types of optimal situations under different generalized equilibrium concepts of a finite game in the case of Chebyshev metric l_∞ , defined in the space of game parameters.

In order to present a formula for stability radius let us introduce the following notations.

A nonempty subset $I \subseteq N_m$ is referred to as a coalition of players. Let $X_I = \prod_{i \in I} X_i$. For a situation $x^0 = (x_1^0, x_2^0, \dots, x_m^0)^T$ and a coalition I , we set $x_I^0 = (x_{i_1}^0, x_{i_2}^0, \dots, x_{i_s}^0)^T$, where $I = \{i_1, i_2, \dots, i_s\} \subseteq N_m$, $i_1 < i_2 < \dots < i_s$, that is, $x_I^0 \in X_I$. For any coalition $I \subseteq N_m$ on the set of situations X of the game with matrix C we introduce a binary relation $\Omega(C, I)$ by the rule

$$x \Omega(C, I) x' \Leftrightarrow \begin{cases} C_I x \underset{P}{\prec} C_I x' \wedge x_J = x'_J, & \text{if } I \neq N_m, \\ Cx \underset{P}{\prec} Cx', & \text{if } I = N_m, \end{cases}$$

where C_I is a $|I| \times n$ matrix consisting of the rows of the matrix C whose indices belong to the coalition I , and $J = N_m \setminus I$.

Let $s \in N_m$, $N_m = \bigcup_{r \in N_s} I_r$ be a partition of the set of players N_m into s coalitions, that is, $I_r \neq \emptyset$, $r \in N_s$; $p \neq q \Rightarrow I_p \cap I_q = \emptyset$. By analogy with [79], we define the set of generalized equilibrium, or, in other words, (I_1, I_2, \dots, I_s) -efficient situations of the game with matrix C as follows:

$$Q^m(C, I_1, I_2, \dots, I_s) = \{x \in X : \forall r \in N_s \forall x' \in X (x \overline{\Omega(C, I_r)} x')\}.$$

Here and henceforth, the line above a binary relation means the negation of this relation. Thus, the relations between players inside each coalition are constructed on the base of the Pareto maximum. Therefore, the set of all N_m -efficient situations $Q^n(C, N_m)$ ($s = 1$, that is, all players from one coalition) is the Pareto set (the set of efficient situations)

$$P^m(C) = \{x \in X : \forall x' \in X (Cx \overline{\underset{P}{\succ}} Cx')\}$$

of the vector problem of maximization of the payoff functions $f_i(x)$, $i \in N_m$, on the set of situations X . The rationality of any cooperative-efficient situation $x \in P^m(C)$ consists of the fact that the increase of the payoff of any player is possible only at the sacrifice of decrease of the payoff of at least one of the other players. It is obvious that the another extreme case where the game is non-cooperative ($s = m$) any situation $x \in Q^n(C, \{1\}, \{2\}, \dots, \{m\})$ is a Nash equilibrium [82], that is, there exist no $r \in N_m$ and $x' \in X$ such that

$$C_r x < C_r x' \wedge x_{N_m \setminus \{r\}} = x'_{N_m \setminus \{r\}}.$$

Thus, the rationality of the equilibrium situation x (individually efficient situation) consists of that any deviations of a player from it (while the other players stick to it) gives him no benefit.

In the framework of our study, by the parametrization of the concept of equilibrium of a finite game with matrix C is meant the introduction of a characteristic of the binary relation of preference which allows, with the use of partitioning of players into coalitions, to link the classical concepts of optimality in the sense of Pareto and the equilibrium in the sense of Nash.

For any positive integer $k \geq 2$ in the real space \mathbb{R}^k , we introduce the Hölder metric l_p , $1 \leq p \leq \infty$, that is, the norm of $y = (y_1, y_2, \dots, y_k) \in \mathbb{R}^k$ is defined by the formulas

$$\|y\|_p = \left(\sum_{i \in N_k} |y_j|^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty,$$

$$\|y\|_\infty = \max\{|y_j| : j \in N_k\}.$$

Observe that for $p = 1$ and $p = \infty$ the Hölder metric coincides with the linear and Chebyshev metrics respectively. By the norm of a matrix, as mentioned above, is meant the norm of the vector composed of all matrix elements.

By analogy with [23, 26, 27], by the radius of stability of a situation $x \in Q^n(C, I_1, I_2, \dots, I_m)$ in the Hölder metric l_p , $1 \leq p \leq \infty$, is meant

$$\rho_p^n(x, C, I_1, I_2, \dots, I_s) = \begin{cases} \sup \Theta, & \text{if } \Theta \neq \emptyset, \\ 0, & \text{if } \Theta = \emptyset, \end{cases}$$

where

$$\Theta = \{\varepsilon > 0 : \forall B \in \Psi_p(\varepsilon) \ (x \in Q^m(C + B, I_1, I_2, \dots, I_s))\}.$$

We introduce the operator of projecting the vector $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ onto the nonnegative orthant $a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_n^+)$, where $a_i^+ = [a_i]^+ = \max\{0, a_i\}$.

The main result is formulated in the following theorem.

Theorem 3.1. [I] *For any partition (I_1, I_2, \dots, I_s) , $s \in N_m$, $m \geq 2$, of the set of players N_m into s coalitions, the radius of stability of an (I_1, I_2, \dots, I_s) -efficient situation $x^0 = (x_1^0, x_2^0, \dots, x_m^0) \in Q^n(C, I_1, I_2, \dots, I_s)$, $C \in \mathbb{R}^{m \times m}$, in the Hölder metric l_p , $1 \leq p \leq \infty$, obeys the formula*

$$\rho_p^n(x^0, C, I_1, I_2, \dots, I_s) = \min_{r \in N_s} \min_{z \in X_{I_r} \setminus \{x_{I_r}^0\}} \frac{\|[C^r(x_{I_r}^0 - z)]^+\|_p}{\|x_{I_r}^0 - z\|_q}. \quad (3.2)$$

For brevity, let ρ and φ denote the left-hand and the right-hand sides of formula (3.2) respectively. First the inequality $\rho \geq \varphi$ is proved by the reductio-ad-absurdum method. Then it is proved that $\rho \leq \varphi$. In general proofs of both

inequalities are based on the definition of φ , the outer stability of the Pareto set and the classical Hölder inequality.

The complete proof of the theorem can be found in publication I.

3.2. Stability radius and its calculation

Now let us consider questions concerning calculation of stability radii. Results of works [41, 43, 63] give evidence that straightforward calculation of the derived formulas for stability radii implies exhaustive search even in the scalar case. Therefore, a question arises: what is a benefit of founding these radii? The first principle answer to this question is following. In the case when stability radius equals zero it detects a solution not only to the initial problem but for an infinite sequence of problems with parameters from some neighborhood of the initial problem parameters. Sure the radius of this neighborhood is equal to the stability radius. Obviously even if we have a very fast algorithm for solving each of these problems, it is impossible to solve so many problems in finite time.

The second answer to the question formulated above is that for a wide class of problems it is possible to derive algorithms for calculation of stability radii exploiting and continuing the same procedures which were proposed for solving an initial problem [9]. Thereby we can improve search by considering the part of solutions whose structure may be defined by problem conditions instead of entire set of solutions. Based on this idea, some classes of scalar combinatorial problems for which algorithm for stability radius calculation has the same complexity as algorithm for solving the problem itself were revealed in [63].

Papers [9, 47] are devoted to investigation of interrelation between complexity of solving combinatorial problems and complexity of calculation stability radii. Particularly, classes of polynomially solvable problems for which there exists efficient algorithm for stability radius calculation are revealed in these works. Conditions under which stability radius of an approximate solution to scalar Boolean problem can be obtained in polynomial time, are detected in paper [95]. Nevertheless there exists a number of problems for which the calculation of stability radius is significantly more complex than solving problem itself. For instance, it was proved that the problem of stability radius calculation is NP-complete for the problem of determining the shortest path in digraph without cycles of negative weight [40]. Paper [73] discusses a method for reducing computational complexity of founding stability radius bounds for solutions to scalar combinatorial problems. This method is based on using information

about k -best solutions. One should mention that algorithms for calculation of stability radii are derived only for scalar problems. Now questions of obtaining analogous algorithms for vector discrete optimization problems are still open.

Publication V addresses the issue of deriving algorithm for calculation of stability radius of solution to the well known shortest path (SP) problem.

We can formulate SP as a linear programming problem by first defining the cost vector $C = (c_1, c_2, \dots, c_n) \in \mathbb{R}_+^n$, $c_i > 0$, n is the number of edges in the considered graph G . Denote by $X \subseteq 2^{\mathbb{E}^n}$, $\mathbb{E} = \{0, 1\}$, the set of feasible solutions, that is, the set of all valid directed paths $P = (e_{i_1}, e_{i_2}, \dots, e_{i_k})$ from source node s to sink node t . Then the decision variables specify all possible paths in graph G :

$$x_i = \begin{cases} 1, & \text{if } e_i \in P, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

Let us define by $e_i \rightarrow j$ an edge for which vertex i is a tail and vertex j is a head and by $e_i \leftarrow j$ an edge for which vertex j is a tail and vertex i is a head. Then boolean linear programming formulation of SP is given as follows:

$$\sum_{e_i \in A} c_i x_i \rightarrow \min, \quad (3.4)$$

subject to

$$\sum_{e_i: e_i \rightarrow j} x_i - \sum_{e_i: e_i \leftarrow j} x_i = \begin{cases} 1, & \text{if } j = s, \\ -1, & \text{if } j = t, \\ 0 & \text{otherwise.} \end{cases} \quad (3.5)$$

$$x_i \in \{0, 1\}. \quad (3.6)$$

Here (3.5) is classical network flow balance constraints and (3.6) is Booleanity constraints which define the set of feasible solutions (paths).

The considered problem with positive costs can be easily solved by Dijkstra's algorithm [12], which processes nodes in nondecreasing order of their actual distances from the source node. At the beginning all nodes are given an infinite distance except the source which is given a distance 0. At each step we choose the next unlabeled node which is nearest to the source and mark it, while updating the optimal distance to all its neighbors. The optimal distance of a neighbor is updated only if reaching it from the current labeling node gives a total path length that is shorter than its current distance. Doing so the algorithm constructs the so-called shortest path tree, which is a spanning tree rooted at the source node s where the shortest paths to all other nodes are determined. The shortest path to each node is then found by tracing the predecessor iteratively back to the source.

The computational complexity of the problem of determining the stability radius for a large class of 0 – 1 programming and combinatorial optimization

problems (including the traveling salesperson problem (TSP)) was considered in [47]. There it was shown that the stability radius calculation problem is as hard as the optimization problem itself. Consequently, the stability radius calculation problem for the TSP is NP-hard. Therefore, in general it is not possible to obtain this measure without spending a lot of computation time.

The methods derived in [9] and [73] have some obvious drawbacks. The exact method proposed in [9] to be efficient requires the initial problem to be polynomially solvable while the technique introduced in [73] is probably not the most effective way because it assumes that the k -best TSP has already been solved. Therefore, it is quite reasonable to apply modern heuristics in order to derive faster algorithms for the stability radius calculation.

Despite in many classical applications metaheuristic based approaches are used to tackle challenging NP-hard problems, they can also be used to attack polynomially solvable problems. One cannot hope that they outperform sophisticated problem-specific algorithms for well-studied classes of polynomially solvable combinatorial optimization problems, however the usage of metaheuristic methods could be especially beneficial on large size instances. Metaheuristic methods could also work well for the cases with higher degree of the polynomial representing complexity of some known exact method (in our case the method of [9] gives complexity $O(m^4)$ with respect to the number of vertices in the considered graph).

The choice of the shortest path problem to test the main ideas is not occasional. First of all, the shortest path problem is a well known combinatorial optimization problem, and the problem of finding the stability radius for this problem can be solved polynomially, that is, we could relatively easy get the exact solution for small and moderate size instances, and compare it with the solution obtained by our approach. Secondly, the topology of the shortest path problem is close in some sense to the topology of the traveling salesman problem, so some heuristic methods being tested and worked well for the first problem have good chance to show similar performance for the other problem and vice versa. Thus, in this work we are making a sort of pilot analysis of effectiveness and correctness of the basic ideas with potential target to apply the similar approach to more difficult optimization problems, for instance, the traveling salesperson problem, for which there no polynomial algorithms of finding the stability radius is known.

Evolutionary algorithms [11, 39, 108] are adaptive metaheuristic search algorithms based on the evolutionary ideas of natural selection and genetics. As such they represent an intelligent exploitation of a random search used to solve optimization problems. Although randomized they utilize historical information to direct the search into the region of better performance within the search space. At each generation, a new set of approximations is created by the

process of selecting individuals according to their level of fitness in the problem domain and breeding them together using the evolution of populations of individuals that are better suited to their environment than their ancestors, just as in natural selection.

In paper V, non-dominated sorting genetic algorithm (NSGA-II) [11] based approach is proposed for calculating stability radius of an optimal solution to the single criterion shortest path problem. The key idea of our method is defractionalization of the objective by means of transforming a nonlinear single objective problem into biobjective problem with linear objectives. Such transformation is performed locally (within the genetic population), that makes the problem of finding the approximation of the Pareto frontier in biobjective case realistic compare to the case if such linearization had been made globally, that is, with respect to the original set of feasible solutions. This explains and motivates our approach compare to other possible methods such as for example applying genetic algorithm directly to the single objective problem with nonlinear function, which will require some efficient nonlinear optimization tool to deal with.

Thus, in order to apply a non-dominated sorting based multiobjective evolutionary algorithm (or NSGA-II) proposed in [11] to calculation of the stability radius we suggest to treat the minimization of fraction

$$\rho(x, C) = \min_{x' \in X \setminus \{x\}} \frac{\sum_{i \in \mathbf{N}_n} c_i(x'_i - x_i)}{\|x' - x\|_1} \quad (3.7)$$

as the following biobjective discrete optimization problem:

$$f_1 = \sum_{i \in \mathbf{N}_n} c_i(x'_i - x_i) \rightarrow \min_{x' \in X \setminus \{x\}}$$

$$f_2 = \|x' - x\|_1 \rightarrow \max_{x' \in X \setminus \{x\}}$$

The Pareto set of the problem is formally defined as follows:

$$P^2(C) = \{x' \in X \mid \nexists x \in X \ (f_1(x, C) \leq f_1(x', C) \wedge$$

$$f_2(x, C) \geq f_2(x', C)) \wedge (f_1(x, C) \neq f_1(x', C) \vee f_2(x, C) \neq f_2(x', C))\}. \quad (3.8)$$

In other words, a feasible solution is Pareto efficient if there is no feasible solution which strictly dominates by one of the objectives and not worse by any other. Here the first objective function is numerator of fraction (3.7) which should be minimized and the second objective function is denominator of fraction (3.7) which should be maximized in order to obtain the minimum value of fractional ratio (3.7). Thus it is evident that the value of stability

radius corresponds to one of the points from Pareto frontier which delivers minimum to the fraction.

The details of the coding representation of a solution and developed operators of NSGA-II adaptation for stability radius calculation are presented in publication V. This work is the first attempt to derive an alternative heuristic approach to the stability radius calculation. The shortest path problem was chosen for testing based on the fact that the method proposed by Chakravarti and Wagelmans [9] runs in polynomial time if the optimization problem itself is polynomially solvable. Thus, we can estimate accuracy of results of our approach by comparison with those of the exact method. As it was shown above, theoretical time complexity of adapted NSGA-II is competitive with complexity of the algorithm proposed by Chakravarti and Wagelmans. Moreover NSGA-II complexity can be reduced by decreasing the number of generations (K), however this could affect accuracy of solutions.

The apparent computational efficiency of the proposed algorithm is explained by the following two facts: at the beginning of the solution process, breadth first search procedure provides diversity in the initial population and chosen size of the population, which is equal to the number of nodes, is enough to generate good solutions and, in addition, keeps memory and time; combining three different crossover operators lead to more exhaustive search allowing to find solutions faster.

Preliminary comparisons showed that the convergence rate of the adapted NSGA-II was good for almost all random scenarios of the shortest path problem that were tested, though the number of tested instances could have been larger. This study encourage us to believe that our approach has some real potential. In addition, the exact algorithm is not polynomial for NP-hard problems, while NSGA-II has still complexity of $O(Km^2)$ since it does not depend on complexity of the original problem. Further, our emphasis is on applying algorithm working on similar principles as adapted NSGA-II for calculating stability radius for NP-hard combinatorial optimization problems, for instance, TSP, and multi-criteria combinatorial optimization problems.

3.3. Accuracy and robustness analysis

Robust optimization is a specific and relatively novel methodology for handling optimization problems with uncertain data. Since the early 1970s there has been an increasing interest in the use of robust optimization models. By itself, the robust optimization methodology can be applied to every generic optimization problem where one can separate numerical data (that can be partly

uncertain and are only known to belong to a given uncertainty set) from problem's structure (that is known in advance and is common for all instances of the uncertain problem). Thus it appears to be important to identify classes of models and their solutions which play against the worst-case (in some sense) realization of input parameters. It is commonly accepted fact nowadays that any optimization problem arising in practice can hardly be adequately formulated and solved without the usage of results of the theory of robustness. Authors of most papers devoted to robust optimization attempt to answer the following closely related questions: How can one represent uncertainty? What is a robust solution? What could be a proper robustness measure? How to calculate robust solutions? How to interpret worst case realization under uncertainty? and many others. Different answers to these questions lead to different research approaches and investigation directions (see, e.g. [3, 6, 7, 52, 53]).

The paradigm of robust optimization can be easily explained with linear programming model (1.2) – (1.4) – the generic optimization problem that is the most frequently used in applications and whose structure and data are clear. In robust optimization, an uncertain linear programming problem is defined as a collection

$$\{\min_x \{cx : Ax = b\} : (c, A, b) \in U\}$$

of LP programs of a common structure with the data (c, A, b) varying in a given uncertainty set U . The latter summarizes all information on the "true" data that is available when solving the problem. Then the solution that remains feasible for the constraints, whatever the realization of the data within U , is called *robust feasible*. As applied to the objective, "worst-case-oriented" philosophy makes it natural to quantify the quality of a robust feasible solution x by the guaranteed value of the original objective, that is, by its largest value $\sup\{cx : (c, A, b) \in U\}$. Thus, the best possible robust feasible solution is the one that solves the optimization problem

$$\min_{x,t} \{t : cx \leq t, Ax = b \ \forall (c, A, b) \in U\}. \quad (3.9)$$

The latter problem is called the *robust counterpart* (RC) of the original uncertain problem. The feasible/optimal solutions to the RC are called *robust feasible/robust optimal* solutions to the uncertain problem. The robust optimization methodology, in its simplest version, proposes to associate with an uncertain problem its robust counterpart and to use, as "real life" decisions, the associated robust optimal solutions.

In decision making, different criteria can be used to select among robust solutions (decisions). The one described above is called the minimax criterion. It necessarily results in conservative decision, based on an anticipation that

the worst might well happen. Another possible criterion is that of minimax regret [53,55], with two variations depending on how "regret" is defined. Here the first step is to compute the "regret" associated with any possible realization of problem parameters. "Regret" can be defined as the difference between the resulting benefit (cost) to the DM and the benefit (cost) from the optimal decision for an underlying model. Alternatively, "regret" can be defined as the ratio of the previously mentioned quantities, thus serving as a measure of the percentage deviation of the robust solution (decision) from the optimal solution for any given input data scenario. The minimax criterion is then applied to the regret values, so as to choose the decision with the least maximum regret. The work of Kouvelis and Yu [55] summarizes the state-of-art in robust optimization up to 1997 and provides a comprehensive discussion of the motivation for the minimax regret approach and various aspects of applying it in practice.

Some attempts to study a quality of the problem robust solution are connected with concepts of stability and accuracy functions, which were originally proposed in [66,67] for scalar combinatorial optimization problems. Later, the results were extended for the case of multicriteria combinatorial optimization problems with Pareto and lexicographic optimality principles [72].

Paper IV continues investigations of different aspects of sensitivity analysis for different types of discrete optimization problems with various partial criteria and optimality principles [16,34,72,75]. Here we consider a strategic game in normal form with $m \geq 2$ players. We assume that $|X_i| = 2$ is a finite set of (pure) strategies of the player $i \in N_m = \{1, 2, \dots, m\}$. Thus, each player has a choice of 2 antagonistic strategies to play. A vector of payoff functions (payoff profile)

$$f(C, x) := (f_1(C, x), \dots, f_m(C, x))^T$$

consists of individual payoff functions $f_i(C, x)$ for each player $i \in N_m$, which are defined as linear functions on the set of solutions X :

$$f_i(C, x) := C_i x.$$

Here C_i is i th row of matrix $C = [c_{ij}] \in \mathbb{R}_+^{m \times m}$, $x := (x_1, x_2, \dots, x_m)^T$, $x_i \in X_i$, $i \in N_m$.

Define $\bar{x}_i = 1$ if $x_i = 0$, and $\bar{x}_i = 0$ otherwise. For any given solution $x^* \in X$, a set of solutions accessible by changing the strategy of player i only is defined as:

$$W_i(x^*) := X_i \times \prod_{j \in N_m \setminus \{i\}} x_j^* = \{(x_1^*, x_2^*, \dots, x_i^*, \dots, x_m^*), (x_1^*, x_2^*, \dots, \bar{x}_i^*, \dots, x_m^*)\}.$$

Thus, if $(x_1^*, x_2^*, \dots, \bar{x}_i^*, \dots, x_m^*)$ is feasible, then $W_i(x^*) \cap X$, the set of feasible solutions accessible by changing the strategy of player i , contains two solutions, otherwise a single solution $(x_1^*, x_2^*, \dots, x_i^*, \dots, x_m^*)$ belongs to $W_i(x^*) \cap X$ only.

To define uncertainty in the game theory model described above, we will assume that the set of game solutions X is fixed but the original payoff matrix C^0 can change or it is given with errors. Let $S(C^0)$ be a set of all possible realizations of the matrix C^0 , called the scenarios. Let us also assume that $C \in \mathbb{R}_+^{m \times m}$ for any $C \in S(C^0)$, thus we guarantee that $f_i(C, x) > 0$ for all $x \in X$ and $i \in N_n$. This is due to our assumption that at least one player always chooses strategy 1 to play, so the game solution $x = 0_{(m)} := (0, 0, \dots, 0)^T$ is not feasible. We follow the approach that define robustness measure as a maximum relative error (worst-case relative regret) of the solution considered over the set of all scenarios. Our aim is to construct a new objective that incorporates possible worst realization of uncertain parameters. In [55] one can find examples of different robustness measures and wide discussion on related complexity issues. While dealing with the multiobjective case, the definition of robustness measures must be adapted to reflect the specific of the multiple objective optimality principle chosen.

For given $x, \tilde{x} \in X$, fixed index (player) $i \in N_m$ and arbitrary $C \in \mathbb{R}_+^{m \times m}$ denote the relative deviation

$$\Delta_i(C, \tilde{x}, x) := \frac{f_i(C, \tilde{x}) - f_i(C, x)}{f_i(C, x)}. \quad (3.10)$$

Definition 3.4. For any given solution $\tilde{x} \in X$, the worst-case relative regret (or robust deviation in other terminology) of this solution on the set $S(C^0)$ is defined as follows:

in Pareto equilibrium case:

$$REG_P(S(C^0), \tilde{x}) := \max_{C \in S(C^0)} \max_{x \in X} \min_{i \in N_m} \Delta_i(C, \tilde{x}, x); \quad (3.11)$$

in Nash equilibrium case:

$$REG_N(S(C^0), \tilde{x}) := \max_{C \in S(C^0)} \max_{i \in N_m} \max_{x \in W_i(\tilde{x}) \cap X} \Delta_i(C, \tilde{x}, x). \quad (3.12)$$

For the game with matrix C , we denote $P^m(C)$ and $N^m(C)$ the set of Pareto and Nash equilibria, respectively.

The difference in $REG_N(S(C^0), \tilde{x})$ and $REG_P(S(C^0), \tilde{x})$ reflects the difference in Pareto and Nash equilibria principles. While in Pareto case, the given solution \tilde{x} must be compared with all other feasible solutions (including the solution \tilde{x} itself to guarantee that $REG_N(S(C^0), \tilde{x}) = 0$ if $\tilde{x} \in P^m(C^0)$), in the Nash case it is sufficient to compare it with solutions $x \in W_i(\tilde{x}) \cap X$ only. Both $REG_N(S(C^0), \tilde{x})$ and $REG_P(S(C^0), \tilde{x})$ give quantitative expressions to measure the relative distance how far the solution \tilde{x} from optimality under the worst case scenario, that is, the scenario which delivers maximum over the set of all possible scenarios $S(C^0)$.

In [67], [71] it was proposed to measure the quality of solutions by means of the so-called accuracy function. In this paper we introduce similar function by analogy with [83].

Definition 3.5. *In case of Pareto equilibria for $x^* \in X$ and a given matrix $C \in \mathbb{R}_+^{m \times m}$, the relative error of this solution is defined as:*

$$\varepsilon_P(C, x^*) := \max_{x \in X} \min_{i \in N_m} \Delta_i(C, x^*, x). \quad (3.13)$$

Similar in case of Nash equilibria for $x^ \in X$ and a given matrix $C \in \mathbb{R}_+^{m \times m}$, the relative error of this solution is defined as:*

$$\varepsilon_N(C, x^*) := \max_{i \in N_m} \max_{x \in W_i(x^*) \cap X} \Delta_i(C, x^*, x). \quad (3.14)$$

The difference in definitions of $\varepsilon_{P,N}(C, x^*)$, reflects the difference in the corresponding definitions of equilibria situations.

Observe that for an arbitrary $C \in \mathbb{R}_+^{m \times m}$ we have $\varepsilon_{P,N}(C, x^*) \geq 0$. If $\varepsilon_P(C, x^*) > 0$ ($\varepsilon_N(C, x^*) > 0$), then $x^* \notin P^m(C)$ ($x^* \notin N^m(C)$) and this positive value of the relative error may be treated as a measure of inefficiency of the strategy profile x^* for the game with matrix C . The equality $\varepsilon_N(C, x^*) = 0$ automatically implies that $x^* \in N^m(C)$. So, for the solution x^* to belong to $N^m(C)$ it is necessary and sufficient to have $\varepsilon_N(C, x^*) = 0$.

In the Pareto case the situation is a bit more complicated. The equality $\varepsilon_P(C, x^*) = 0$ formulates in general only necessary condition for x^* to be Pareto equilibrium in the game with matrix C , that is, $\varepsilon_P(C, x^*) = 0$ does not guarantee that $x^* \in P^m(C)$.

From now we assume that some originally specified matrix $C^0 = \{c_{ij}^0\} \in \mathbb{R}_+^{m \times m}$ defines the original problem data. In the following we are interested in the maximum value of the errors $\varepsilon_P(C, x^*)$ and $\varepsilon_N(C, x^*)$ when the matrix C belongs to some specified set, the so-called set of perturbed matrices. We are interested in relative perturbations of the elements of C^0 , and the quality of a given solution x^* is described by the so-called *accuracy function*. The value of the accuracy function for a given $\delta \in [0, 1)$ is equal to the maximum relative error of the solution x^* under the assumption that the weights of the elements are perturbed by no more than $\delta \cdot 100\%$ of their original values specified by matrix C^0 . Notice that if we compare two different equilibria for the game with matrix C from the point of view of their accuracy on data perturbation, then the smaller values of the accuracy function are more preferable. Thus, accuracy function may be used to evaluate the quality of the game solutions from the accuracy point of view.

For a given $\delta \in [0, 1)$, consider a set of perturbed matrices

$$\Theta_\delta(C^0) := \{C \in \mathbb{R}_+^{n \times n} : |c_{ij} - c_{ij}^0| \leq \delta \cdot c_{ij}^0, i \in N_m, j \in N_m\}. \quad (3.15)$$

Definition 3.6. For $x^* \in X$ and $\delta \in [0, 1)$, the value of the accuracy function in Pareto case is defined as:

$$A_P(C^0, x^*, \delta) := \max_{C \in \Theta_\delta(C^0)} \varepsilon_P(C, x^*). \quad (3.16)$$

For $x^* \in X$ and $\delta \in [0, 1)$, the value of the accuracy function in Nash case is defined as:

$$A_N(C^0, x^*, \delta) := \max_{C \in \Theta_\delta(C^0)} \varepsilon_N(C, x^*). \quad (3.17)$$

Notice that these definitions imply equivalence between accuracy functions and corresponding robust deviations, respectively. However, recall, that the robust deviation measures were used as a tool of constructing a new robust optimization counterpart problem and to find a robust solution, whereas accuracy functions are used as a tool of postoptimal analysis to express numerically the quality of the given solution under possible perturbations of initial data. Thus, we get

$$A_{P,N}(C^0, x^*, \delta) = REG_{P,N}(S(C^0), x^*),$$

if the set of scenarios $S(C^0)$ is defined as the set of perturbed matrices $\Theta_\delta(C^0)$ according to (3.15). This means that the properties of accuracy functions can be used in the solution robustness analysis.

For given $x, x^* \in X$, fixed index $i \in N_n$ and $C^0 \in \mathbb{R}_+^{n \times n}$ denote

$$\Xi_i(C^0, x^*, x, \delta) := \frac{C_i^0(x^* - x) + \delta \sum_{j \in N_m} c_{ij}^0 |x_j^* - x_j|}{(1 - \delta)C_i^0 x}. \quad (3.18)$$

The following theorem gives a formulae for calculating value of the accuracy function.

Theorem 3.2. [IV] *The following statements are true.*

(i) For $x^* \in X$ and $\delta \in [0, 1)$, the accuracy function can be expressed by the formula:

$$A_P(C^0, x^*, \delta) = \max_{x \in X} \min_{i \in N_m} \Xi_i(C^0, x^*, x, \delta). \quad (3.19)$$

(ii) For $x^* \in X$ and $\delta \in [0, 1)$, the accuracy function can be expressed by the formula:

$$A_N(C^0, x^*, \delta) = \max_{i \in N_m} \max_{x \in W_i(x^*) \cap X} \Xi_i(C^0, x^*, x, \delta). \quad (3.20)$$

Notice that analytical formula (3.20) specified in Theorem 3.2 can be computed relatively easy. At the same time analytical formula (3.19) specified in Theorem 3.2 is based on enumerating all feasible solutions, so in general it is hard to be computed. Therefore, we provide some attainable lower and upper

bounds for the Pareto accuracy function which are computationally more attractive. Next proposition gives an upper bound for the accuracy function of $x^* \in X$ in the case of Pareto optimality principle.

Proposition 3.1. [IV] For $x^* \in X$ and $\delta \in [0, 1)$,

$$A_P(C^0, x^*, \delta) \leq \frac{2\delta}{1-\delta} + \frac{1+\delta}{1-\delta} \cdot \min_{i \in N_m} A_i(C^0, x^*, 0). \quad (3.21)$$

Now it becomes clear that calculating the upper bound specified by Proposition 3.1 is as hard as calculating m times $A_i(C^0, x^*, 0)$, whose calculating turns into solving the original single objective problem.

Observe that similar upper bound were obtained in the case of single objective combinatorial optimization problem in [69].

The following corollary specifies the upper bound for the accuracy function of the originally Pareto equilibrium $x^* \in P^m(C^0)$.

Corollary 3.1. For $x^* \in P^m(C^0)$ and $\delta \in [0, 1)$,

$$A_P(C^0, x^*, \delta) \leq \frac{2\delta}{1-\delta}. \quad (3.22)$$

Corollary 3.2. For $x^* \in N^m(C^0)$ and $\delta \in [0, 1)$, the equality $A_N(C^0, x^*, \delta) = 0$ holds.

Now consider the case when x^* is an equilibrium in the original game with matrix C^0 implying $A_{P,N}(C^0, x^*, 0) = 0$. It is of special interest to know the extreme values of δ for which $A_{P,N}(C^0, x^*, \delta) = 0$, because these values determine maximum norms of perturbations which preserve the property of the given solution to be an equilibrium. These values are close analogues of the so-called stability radius introduced earlier for single and multiple objective combinatorial optimization problems (see, e.g., [16]). Formally, the accuracy radii $R_{P,N}(C^0, x^*)$ are defined in the following way:

$$R_{P,N}(C^0, x^*) = \sup \left\{ \delta \in [0, 1) : A_{P,N}(C^0, x^*, \delta) = 0 \right\}. \quad (3.23)$$

If these radii are equal to zero, then this means that there exist arbitrary small perturbations of the original game matrix C^0 such that the initial equilibrium x^* loses its property of being equilibrium under very small perturbations. Otherwise, the solution x^* remains equilibrium for any game with matrix $C \in \Theta_\delta(C^0)$, $\delta < R_{P,N}(C^0, x^*)$. The next theorem is a straightforward consequence of Theorem 3.2

Theorem 3.3. [IV] The following statements are true.

(i) For $x^* \in P^m(C^0)$, the Pareto accuracy radius can be expressed by the

formula:

$$R_P(C^0, x^*) = \min \left\{ 1, \min_{x \in X \setminus \{x^*\}} \max_{i \in N_m} \frac{C_i^0(x - x^*)}{\sum_{j \in N_m} c_{ij}^0 |x_j - x_j^*|} \right\}. \quad (3.24)$$

(ii) For $x^* \in N^m(C^0)$, the Nash accuracy radius can be expressed by the formula:

$$R_N(C^0, x^*) = \min \left\{ 1, \min_{i \in N_m} \min_{x \in W_i(x^*) \cap X \setminus \{x^*\}} \frac{C_i^0(x - x^*)}{\sum_{j \in N_m} c_{ij}^0 |x_j - x_j^*|} \right\} = 1, \quad (3.25)$$

i.e. $x^* \in N^m(C^0)$ is accurate (i.e. $R_N(C^0, x^*) \geq 0$).

Now let us consider the case when only one column in matrix C^0 is uncertain, while all the other columns are kept unchanged. It corresponds to the situation in the game, when all players are uncertain about their own costs associated with the strategy choice of a given player. Assume j be the uncertain column in the original matrix C^0 , so we denote the original matrix $C^0[j]$, where notation $[j]$ is used to indicate that column j is uncertain. Then for a fixed $\delta \in [0, 1)$ we have

$$\Theta_\delta(C^0[j]) := \left\{ C \in \mathbb{R}_+^{m \times m} : \left(|c_{ij} - c_{ij}^0| \leq \delta \cdot c_{ij}^0, i \in N_m \right) \& \left(c_{ik} = c_{ik}^0, k \in N_m \setminus \{j\}, i \in N_m \right) \right\}. \quad (3.26)$$

For $x^* \in X$ and $\delta \in [0, 1)$, the definition of the accuracy function in this case transforms into the following:

$$A_{P,N}(C^0[j], x^*, \delta) := \max_{C[j] \in \Theta_\delta(C^0[j])} \varepsilon_{P,N}(C[j], x^*), \quad (3.27)$$

where $\varepsilon_{P,N}(C[j], x^*)$ are defined according to (3.13) and (3.14).

Moreover,

$$A_{P,N}(C^0[j], x^*, \delta) = REG_{P,N}(\Theta_\delta(C^0[j]), x^*).$$

Now we are interested in the maximal level of perturbation not violating robustness of a given optimal solution.

Definition 3.7. For a given $x^* \in P^m(C^0)$ ($x^* \in N^m(C^0)$) the robustness tolerances in Pareto and Nash cases are defined as follows:

$$t_{P,N}^r(C^0[j], x^*) := \sup \left\{ \delta \in [0, 1) : A_{P,N}(C^0[j], x^*, \delta) \leq A_{P,N}(C^0[j], x, \delta) \forall x \in X \right\}.$$

Notice that the same definition can be formulated in terms of relative regrets as follows

$$t_{P,N}^r(C^0[j], x^*) := \sup \left\{ \delta \in [0, 1) : REG_{P,N}(\Theta_\delta(C^0[j]), x^*) \leq REG_{P,N}(\Theta_\delta(C^0[j]), x) \forall x \in X \right\}.$$

The superscript r is used to emphasize that we are dealing with robustness tolerances, which differ from usual tolerances.

Robustness tolerances were first mentioned in [70] for single objective linear generic combinatorial optimization problem. Our approach develops the idea of [70] by extending it to the multiobjective case under game theoretic formulation.

One of the main results obtained in IV is presented in the following theorem:

Theorem 3.4. *[IV] Assume that $P^m(C^0) = \{x^*\}$. Then the robustness tolerance can be computed according to the following expressions:*

if $x_j^* = 1$, then

$$t_P^r(C^0[j], x^*) = 1;$$

if $x_j^* = 0$ then

$$t_P^r(C^0[j], x^*) = \min \left\{ 1, \frac{\sqrt{(C_i^0 \hat{x})^2 - (C_i^0 x^*)^2}}{c_{ij}^0} \right\},$$

$$\text{where } \hat{x} := \arg \max_{x' \in X: x'_j=1} \min_{i \in N_m} C_i^0 x', \quad \hat{i} = \arg \min_{i \in N_m} C_i^0 \hat{x}.$$

Similar result can be obtained in the case of Nash optimality

Corollary 3.3. *Assume that $x^* \in N^m(C^0)$. Then*

$$t_N^r(C^0[j], x^*) = 1.$$

The results presented in this section suggest that even small changes or inaccuracies in estimating payoff function coefficients may have significant influence on the set of Pareto equilibria. Moreover, some situations being initially equilibria, cannot be considered "robust", because very small changes of data destroy their properties of being equilibria.

Chapter 4. Interactive optimization approach to multicriteria discrete optimization problems

This Chapter discusses an interactive approach to solving multicriteria variant of a p -median location problem. In interactive methods, the decision maker (DM) works together with an analyst of an interactive computer program. One can say that the analyst tries to determine the preference structure of the decision maker in an interactive way. A solution pattern is formed and repeated several times. After every iteration, some information is given to the decision maker and (s)he is asked to answer some questions or provide some other type of information. The working order in these methods is: analyst, DM, analyst, DM etc.

After a reasonable (finite) number of iterations every interactive method should yield a solution that the decision maker can be satisfied with and/or convinced that no considerably better solution exists. The basic steps in interactive algorithms can be expressed as

- find an initial feasible solution,
- interact with the decision maker, and
- obtain a new solution (or a set of new solutions). If the new solution (or one of them) or one of the previous solutions is acceptable to the decision maker, stop. Otherwise, go to the previous step.

Interactive methods differ from each other by the form in which information is given to the decision maker, by the form in which information is provided by the decision maker, and how the problem is transformed into a single objective optimization problem (see, e.g., [76, 77, 101]).

The traditional approach to solving multicriteria optimization problems with Pareto principle of optimality is by scalarization. It involves formulating a single objective problem that is related to the multicriteria problem by means of a real-valued scalarizing function typically being a function of the individualized or partial objective functions of the multicriteria problem, auxiliary scalar or vector variables, and/or scalar or vector parameters. Sometimes the feasible set of the multicriteria optimization problem is additionally restricted by new constraint functions related to the objective functions of the multicriteria problem and/or the new variables introduced. Two major requirements are set for a scalarizing function in order to provide method completeness [91]:

- every solution found by means of scalarization should be (weakly) Pareto optimal, and
- it should be able to cover the entire set of Pareto optimal solutions.

One of the widely spread approaches of dealing with multiple conflicting objectives involves constructing and optimizing a so-called achievement scalarizing function (ASF). This method was introduced in [105] and based on a reference point of aspiration level. The ASF minimizes the distance from a reference point, specified by DM, to the feasible region, if the reference point is unattainable, or maximizes the distance otherwise. The distance is defined by some appropriate metric introduced in the objective space. Sometimes the DM may want more advanced scalarization mechanisms. In [84] a parameterized version of the ASF was proposed. Authors introduced an integer parameter in order to control the degree of metric flexibility varying from L_1 to L_∞ . It was proven that the parameterized ASF is able to detect any Pareto optimal solutions. Moreover, conditions under which the Pareto optimality of each solution produced by the parameterized ASF is guaranteed was also obtained in [84].

This work investigates applicability of interactive optimization approach based on the parameterized ASF to multicriteria p -median location problem. We introduce a new way to manage an interactive process by changing weighting coefficients of scalarizing functions. The decision making procedure is simulated for three objective p -median location problem in order to illustrate how synchronous usage [77] of scalarizing functions may be potentially advantageous for interactive process.

4.1. Parameterized achievement scalarizing function

Let X be an arbitrary set of feasible solutions or a set of decision vectors. Let a vector valued function $f : X \rightarrow \mathbb{R}^m$ consisting of $m \geq 2$ partial objective functions be defined on the set of feasible solutions:

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x)).$$

Without loss of generality we assume that every objective function is subject to be minimized on the set of feasible solutions:

$$\min_{x \in X} f_i(x), \quad i \in N_m = \{1, 2, \dots, m\}. \quad (4.1)$$

We assume that

1. every objective function f_i is a lower semicontinuous function;
2. X is a nonempty compact set.

Let us denote by

$$M^i(X) = \arg \min_{x \in X} f_i(x), \quad i \in N_m$$

a set of minima of the i th objective function. Evidently, if

$$\bigcap_{i=1}^m M^i(X) \neq \emptyset,$$

then there exists at least one solution which delivers a minimum for all objectives. Such a solution can be called an *ideal solution*. An optimization problem which does not contain ideal solutions is called *non-degenerate* [84] and objectives are at least partly conflicting. Simultaneous optimization of several objectives for non-degenerate multiobjective optimization problems is not a straightforward task, and we need to define optimality for such problems. In what follows, we consider non-degenerate problems.

Here we use the traditional definitions of the Pareto and Slater principles of optimality given in introduction section. Under the assumptions 1–2 mentioned earlier in the problem formulation, we know that the set of Pareto optimal solution is non-empty, that is, there always exists at least one Pareto optimal solution [91]. Obviously, the set of Pareto optimal solutions is a subset of weakly Pareto optimal solutions.

Lower and upper bounds on objective values of all Pareto optimal solutions are given by the *ideal* and *nadir* objective vectors f^I and f^N , respectively. The components f_i of the ideal (nadir) objective vector $f^I = (f_1^I, \dots, f_m^I)$ ($f^N = (f_1^N, \dots, f_m^N)$) are obtained by minimizing (maximizing) each of the objective functions individually subject to the set of Pareto optimal solutions:

$$f_i^I = \min_{x \in P^m(X)} f_i(x), \quad i \in N_m,$$

$$f_i^N = \max_{x \in P^m(X)} f_i(x), \quad i \in N_m.$$

In reference point based methods (see, e.g., [105–107]), the DM specifies a reference point f^R consisting of desirable or reasonable aspiration levels f_i^R for each objective function f_i , $i \in N_m$. The reference point only indicates what kind of objective function values the DM prefers.

A certain class of real-valued functions $s_R : \mathbb{R}^m \rightarrow \mathbb{R}$, referred to as achievement scalarizing functions, can be used to scalarize a multiobjective optimization problem. Achievement scalarizing functions have been introduced by Wierzbicki in [105]. The scalarized problem is given by

$$\min_{x \in X} s_R(f(x)). \quad (4.2)$$

Pareto optimal solutions can be characterized by achievement scalarizing functions if the functions satisfy certain requirements.

Definition 4.1. [106] An ASF $s_R : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be

1. Increasing, if for any $y^1, y^2 \in \mathbb{R}^m$, $y_i^1 \leq y_i^2$ for all $i \in N_m$, then $s_R(y^1) \leq s_R(y^2)$.
2. Strictly increasing, if for any $y^1, y^2 \in \mathbb{R}^m$, $y_i^1 < y_i^2$ for all $i \in N_m$, then $s_R(y^1) < s_R(y^2)$.
3. Strongly increasing, if for any $y^1, y^2 \in \mathbb{R}^m$, $y_i^1 \leq y_i^2$ for all $i \in N_m$ and $y^1 \neq y^2$, then $s_R(y^1) < s_R(y^2)$.

Obviously, any strongly increasing ASF is also strictly increasing, and any strictly increasing ASF is also increasing. The following theorems define necessary and sufficient conditions for an optimal solution of (4.2) to be (weakly) Pareto optimal:

Theorem 4.1. [106]

1. Let s_R be strongly (strictly) increasing. If $x^* \in X$ is an optimal solution of problem (4.2), then x^* is (weakly) Pareto optimal.
2. If s_R is increasing and the solution of (4.2) $x^* \in X$ is unique, then x^* is Pareto optimal.

Theorem 4.2. [76] If s_R is strictly increasing and $x^* \in X$ is weakly Pareto optimal, then it is a solution of (4.2) with $f^R = f(x^*)$ and the optimal value of s_R is zero.

The advantage of ASFs is that any (weakly) Pareto optimal solution can be obtained by moving the reference point only. It was shown in [106] that the solution of an ASF depends Lipschitz continuously on the reference point. In general, ASFs are conceptually very appealing to generate Pareto optimal solutions, and they overcome most of the difficulties arising with other methods [76] in the class of methods for generating Pareto optimal solutions.

In the grate majority of cases, the ASF is based on the Chebyshev distance or L_∞ :

$$s_R^\infty(f(x), \lambda) = \max_{i \in N_m} \lambda_i (f_i(x) - f_i^R), \quad (4.3)$$

where λ is a m -vector of non-negative coefficients.

An achievement scalarizing function based on the linear distance L_1 is proposed in [90]. Given problem (4.1), a reference vector $f^R \in \mathbb{R}^m$ and a vector

of strictly positive weights λ , the additive achievement scalarizing function is defined as follows:

$$s_R^1(f(x), \lambda) = \sum_{i \in N_m} \lambda_i |f_i(x) - f_i^R|. \quad (4.4)$$

The following properties of $s_R^1(f(x), \lambda)$ were proved in [90].

Theorem 4.3. [90] *Given problem (4.2) with ASF defined by (4.4), let f^R be a reference point such that f^R is not dominated by an objective vector of any feasible solution of problem (4.2). Also assume $\lambda_i > 0$ for all $i \in N_m$. Then any optimal solution of problem (4.2) is a weakly Pareto optimal solution.*

Theorem 4.4. [90] *Given problem (4.2) with ASF defined by (4.4) and any reference point f^R , assume $\lambda_i > 0$ for all $i \in N_m$. Then among the optimal solutions of problem (4.2) there exists at least one Pareto optimal solution. If the optimal solution of problem (4.2) is unique, then it is Pareto optimal.*

In [84] authors extend ideas of [90] by introducing parameterization based on the notion of embedded subsets. Here an integer parameter $q \in N_m$ is used in order to control the degree of metric flexibility varying from L_1 to L_∞ .

Let I^q be a subset of N_m of cardinality q . A parameterized ASF is defined as follows:

$$s_R^q(f(x), \lambda) = \max_{I^q \subseteq N_m: |I^q|=q} \left\{ \sum_{i \in I^q} \max[\lambda_i(f_i(x) - f_i^R), 0] \right\}, \quad (4.5)$$

where $q \in N_m$ and $\lambda = \{\lambda_1, \dots, \lambda_m\}$, $\lambda_i > 0$, $i \in N_m$. Notice that

- for $q \in N_m$: $s_R^q(f(x), \lambda) \geq 0$;
- $q = 1$: $s_R^1(f(x), \lambda) = \max_{i \in N_m} \max[\lambda_i(f_i(x) - f_i^R), 0] \cong s_R^\infty(f(x), \lambda)$;
- $q = m$: $s_R^m(f(x), \lambda) = \sum_{i \in N_m} \max[\lambda_i(f_i(x) - f_i^R), 0] = s_R^1(f(x), \lambda)$.

Here, " \cong " means equality in the case where there exist no feasible solutions $x \in X$ which strictly dominate the reference point, that is, $f_i(x) < f_i^R$ for all $i \in N_m$.

The problem to be solved is

$$\min_{x \in X} s_R^q(f(x), \lambda). \quad (4.6)$$

It is obvious that using problem (4.6), every feasible solution of the multi-objective problem (including Pareto optimal) is supported. Indeed, given any $x \in X$, the reference point $f^R = f(x)$ and a vector of weighting coefficients $\lambda > 0$, the optimal solution to problem (4.6) is x with the optimal value of $s_R^q(f(x), \lambda)$ equals zero. Thus, the first of the two requirements, mentioned in the introduction to Chapter 4, holds.

For any $x \in X$, denote $I_x = \{i \in N_m : f_i^R \leq f_i(x)\}$. The following two results analogous to Theorems 4.3 and 4.4 describe the conditions under which the second of the two requirements mentioned in introduction the introduction to Chapter 4 holds.

Theorem 4.5. [84] *Given problem (4.6), let f^R be a reference point such that there exists no feasible solution whose image strictly dominates f^R . Also assume $\lambda_i > 0$ for all $i \in N_m$. Then any optimal solution of problem (4.6) is a weakly Pareto optimal solution.*

Theorem 4.6. [84] *Given problem (4.6), let f^R be a reference point. Also assume $\lambda_i > 0$ for all $i \in N_m$. Then among the optimal solutions of problem (4.6) there exists at least one Pareto optimal solution.*

Theorem 4.6 implies that the uniqueness of the optimal solution guarantees its Pareto optimality. Notice that the facts stated above about solutions of parameterized ASFs also implicitly follow from the results of Theorem 4.1. To show this, it is sufficient to prove that $s_R^q(f(x), \lambda)$ is increasing. Moreover, parameterized ASF is strictly increasing if there are no feasible solutions dominating f^R .

4.2. Interactive compromise programming

Key parameters in the approach utilizing achievement scalarizing functions are the reference point, which expresses desirable objective function values for the DM, and weights. According to Theorems 4.5 and 4.6 one has to keep always in mind that the reference point should not be strictly dominated by some feasible point. However, in practice sometimes it is difficult to guarantee that this condition holds. Alternatively, an ideal vector can serve as a reference point and weighting coefficients may reflect the level of penalization for "bad" deviations which DM wants to introduce into the problem. Then the goal is to find solutions as close as possible to the ideal point. This idea makes our approach close in some sense to compromise programming [13].

The main scheme of interactive techniques based on scalarizing functions is the following. Initially, an ideal vector is chosen as a reference point and weighting coefficients are inverse to corresponding components of the ideal vector to provide objective normalization. At each iteration c , objective function values calculated at the current Pareto optimal decision vector $x_c \in P^m(X)$ are presented to the DM. (S)he can then express what kind of changes would be desirable to her/him by classifying each of the objective functions into different classes [77]. Here we consider two cases:

- f_i values are desired to be improved (that is, decreased),

- f_i values may be impaired (that is, increased).

We propose a scheme to incorporate the decision maker's preference information to weighting coefficients in parameterized ASF. The most preferred solution (MPS), which reflects genuine preferences of the DM (and whose exact values the DM may even not be aware of), is supposed to be known and used to determine the stopping criterion for the entire interactive procedure. In other words, the MPS is used to simulate the behavior of the DM which has to decide to stop the process if the current solution is close enough to her/his aspiration levels with respect to a chosen distance measure.

In this paper we consider the case of three objectives, that is, $m = 3$. Then, parameterized ASF (4.5) has the following form:

$$s_R^q(f(x), \lambda) = \max_{I^q \subseteq \{1,2,3\}: |I^q|=q} \left\{ \sum_{i \in I^q} \max[\lambda_i(f_i(x) - f_i^R), 0] \right\}, \quad (4.7)$$

where $q = 1, 2, 3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i > 0$, $i \in N_3$.

In other words, taking into account that the ideal vector $f^I = (f_1^I, f_2^I, f_3^I)$ is chosen as a reference point, we have:

for $q = 1$

$$\begin{aligned} s_R^1(f(x), \lambda) &= \max \left\{ \begin{array}{l} \max[\lambda_1(f_1(x) - f_1^R), 0], \max[\lambda_2(f_2(x) - f_2^R), 0], \\ \max[\lambda_3(f_3(x) - f_3^R), 0] \end{array} \right\} \\ &= \max \left\{ \lambda_1(f_1(x) - f_1^I), \lambda_2(f_2(x) - f_2^I), \lambda_3(f_3(x) - f_3^I) \right\}; \end{aligned}$$

for $q = 2$

$$\begin{aligned} s_R^2(f(x), \lambda) &= \max \left\{ \begin{array}{l} \max[\lambda_1(f_1(x) - f_1^R), 0] + \max[\lambda_2(f_2(x) - f_2^R), 0], \\ \max[\lambda_1(f_1(x) - f_1^R), 0] + \max[\lambda_3(f_3(x) - f_3^R), 0], \\ \max[\lambda_2(f_2(x) - f_2^R), 0] + \max[\lambda_3(f_3(x) - f_3^R), 0] \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} \lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I), \\ \lambda_1(f_1(x) - f_1^I) + \lambda_3(f_3(x) - f_3^I), \\ \lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I) \end{array} \right\}; \end{aligned}$$

for $q = 3$

$$\begin{aligned} s_R^3(f(x), \lambda) &= \max \left\{ \begin{array}{l} \max[\lambda_1(f_1(x) - f_1^R), 0] + \max[\lambda_2(f_2(x) - f_2^R), 0] \\ + \max[\lambda_3(f_3(x) - f_3^R), 0] \end{array} \right\} \\ &= \lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I) \end{aligned}$$

In order to illustrate how coefficients λ_i are used to direct interactive procedure we suppose that the problem (4.6) is solved for $q = 1$, $q = 2$ and $q = 3$. Let us first consider the case $q = 1$:

$$\max\left\{\lambda_1(f_1(x) - f_1^I), \lambda_2(f_2(x) - f_2^I), \lambda_3(f_3(x) - f_3^I)\right\} = \alpha. \quad (4.8)$$

In other words, the minimal distance from the ideal point to the Pareto frontier in terms of the parameterized ASF with $q = 1$ is α . The graph of function (4.8) looks similar to what we always have for the Chebyshev type ASF, that is, of cubic shape. Now we determine the direction in which function (4.8) intersects with Pareto frontier, that is, coordinates of the corner of the cube.

Here, three cases are possible:

Case 1:

$$\{f_1(x) = \alpha/\lambda_1 + f_1^I, f_2(x) \in [f_2^I, \alpha/\lambda_2 + f_2^I], f_3(x) \in [f_3^I, \alpha/\lambda_3 + f_3^I]\};$$

Case 2:

$$\{f_1(x) \in [f_1^I, \alpha/\lambda_1 + f_1^I], f_2(x) = \alpha/\lambda_2 + f_2^I, f_3(x) \in [f_3^I, \alpha/\lambda_3 + f_3^I]\};$$

Case 3:

$$\{f_1(x) \in [f_1^I, \alpha/\lambda_1 + f_1^I], f_2(x) \in [f_2^I, \alpha/\lambda_2 + f_2^I], f_3(x) = \alpha/\lambda_3 + f_3^I\}.$$

From here we obtain that corner coordinates are $(\alpha/\lambda_1 + f_1^I, \alpha/\lambda_2 + f_2^I, \alpha/\lambda_3 + f_3^I)$. Thus, we can change coordinates of the ideal point projection onto Pareto front by moving this corner.

Now let us consider corresponding functions for $q = 2$ and $q = 3$:

$$\max\left\{\lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I), \lambda_1(f_1(x) - f_1^I) + \lambda_3(f_3(x) - f_3^I), \lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I)\right\} = \alpha. \quad (4.9)$$

$$\lambda_1(f_1(x) - f_1^I) + \lambda_2(f_2(x) - f_2^I) + \lambda_3(f_3(x) - f_3^I) = \alpha. \quad (4.10)$$

Here, α is the minimal distance from the ideal point to the Pareto frontier in terms of the parameterized ASF for $q = 2$ and $q = 3$ respectively. Detailed graphical constructions of (4.9) and (4.10) are showed in [84]. For controlling interactive process we are interested only in the case where all three sums in (4.9) and sum in (4.10) are equal to α . For (4.10), this forms a flat triangle face which is contained in a plane with normal vector $(\lambda_1, \lambda_2, \lambda_3)$. For (4.9) a flat triangle transforms into a triangle pyramid with a top vertex $(\alpha/2\lambda_1 + f_1^I, \alpha/2\lambda_2 + f_2^I, \alpha/2\lambda_3 + f_3^I)$. From here it follows that we can control

the place of potential contact of (4.9) and (4.10) with the image of the feasible set by changing the normal vector or the top vertex.

Thus the weighting coefficients vector λ indicates the relative importance of the deviations of the objectives values $f_i(x)$, $i = 1, 2, 3$, from the ideal vector (f_1^I, f_2^I, f_3^I) and can be used to reflect decision maker preferences. It follows from the above discussion that for all values of parameter q by decreasing λ_i we increase the value of $f_i(x)$ and by increasing λ_i we decrease the value of $f_i(x)$.

In order to test the introduced ideas we conducted numerical experiments for three objective median location problem. An important real life example of this problem is evacuation planning. Emergency evacuation plans are developed to ensure the safest and most efficient evacuation time of all expected residents of a structure, city, or region. Let us suppose that in a particular hazard zone there are a set of sectors S , $|S| = n$, $n \in \mathbb{N}$ that should be evacuated. Each region $i \in S$, $i = 1, \dots, n$ has a_i habitants. Let E , $|E| = l$, $l \in \mathbb{N}$ be the number of candidate shelters and $p \leq E$ is the number of shelters to be located. Each route from a sector i to a shelter j we associate with path length $d_{ij} \in \mathbb{R}$, $j = 1, \dots, l$ and risk r_{ij} , a real parameter from the real unit interval $(0, 1)$. A candidate shelter j is also associated with some risk $r_j \in (0, 1)$, capacity (number of individuals) $K_j \in \mathbb{N}$ allowed in the j th candidate shelter and integer parameter k_j which specifies minimum number of individuals required for opening the j th shelter.

Then by analogy with [1] underlying multiobjective p -median problem can be formulated as follows:

$$\min \sum_{i=1}^n \sum_{j=1}^l a_i d_{ij} x_{ij} \quad (4.11)$$

$$\min \sum_{i=1}^n \sum_{j=1}^l a_i r_{ij} x_{ij} \quad (4.12)$$

$$\min \sum_{i=1}^n \sum_{j=1}^l a_i r_j x_{ij} \quad (4.13)$$

subject to

$$\sum_{j=1}^l x_{ij} = 1, \quad i = 1, \dots, n \quad (4.14)$$

(ensures one evacuation path is chosen for each sector, with n the number of sectors; n constraints)

$$\sum_{i=1}^n a_i x_{ij} \leq K_j y_j, \quad j = 1, \dots, l \quad (4.15)$$

(ensures the maximum capacity for shelter j is not exceeded, with l the total number of candidate shelters; l constraints)

$$\sum_{i=1}^n a_i x_{ij} \geq k_j y_j, \quad j = 1, \dots, l \quad (4.16)$$

(ensures the minimum number of individuals required to open shelter j before it is opened, with E the total number of candidate shelters; l constraints)

$$\sum_{j=1}^l y_j = p \quad (4.17)$$

(ensures p of the l candidate shelters are opened)

$$x_{ij} \in \{0, 1\}, \quad i = 1, \dots, n, \quad j = 1, \dots, l \quad (4.18)$$

$$y_j \in \{0, 1\}, \quad j = 1, \dots, l. \quad (4.19)$$

Objective (4.11) minimizes the total distance required for all of the population to reach its primary evacuation shelter. Objective (4.12) minimizes the risk faced by the total population as it travels to its primary shelter. Objective (4.13) minimizes total risks associated with staying in the shelter.

Let us consider a simulation of a decision making process for solving three objective p -median location problem described above.

Example 4.1. Suppose we want to evacuate individuals from 5 sectors to 4 of 5 available shelters, that is, $n = 5$, $l = 5$ and $p = 4$.

Given the following randomly generated data set:

- Distance matrix

$$(d_{ij}) = \begin{pmatrix} 18 & 29 & 0 & 21 & 27 \\ 17 & 21 & 14 & 21 & 30 \\ 24 & 26 & 27 & 30 & 19 \\ 22 & 27 & 18 & 16 & 17 \\ 13 & 13 & 26 & 15 & 7 \end{pmatrix},$$

here length $d_{ij} = 0$ means that there is no path from sector i to shelter j ;

- Number of individuals in each sector

$$a = (5, 18, 21, 19, 29);$$

- Risk associated with a path from sector i to shelter j

$$(r_{ij}) = \begin{pmatrix} 0.7000 & 0.01190 & 0.9975 & 0.7582 & 0.9022 \\ 0.7745 & 0.2775 & 0.4996 & 0.3107 & 0.3198 \\ 0.6965 & 0.4296 & 0.4775 & 0.06067 & 0.6897 \\ 0.7466 & 0.4359 & 0.2137 & 0.1030 & 0.3448 \\ 0.6685 & 0.09851 & 0.8209 & 0.6778 & 0.01904 \end{pmatrix};$$

- Risk associated with shelter j

$$r = (0.2936, 0.1979, 0.07867, 0.9210, 0.5971);$$

- Capacity of a shelter j

$$K = (26, 25, 65, 40, 47);$$

- Minimum number of individuals required for opening shelter j

$$k = (9, 8, 6, 9, 9).$$

We can formulate the three objective 4-median location problem:

$$\min f_1(x) =$$

$$\begin{aligned} & \min\{90x_{1,1} + 145x_{1,2} + 105x_{1,4} + 135x_{1,5} + \\ & 306x_{2,1} + 378x_{2,2} + 252x_{2,3} + 378x_{2,4} + 540x_{2,5} + \\ & 504x_{3,1} + 546x_{3,2} + 567x_{3,3} + 630x_{3,4} + 399x_{3,5} + \\ & 418x_{4,1} + 513x_{4,2} + 342x_{4,3} + 304x_{4,4} + 323x_{4,5} + \\ & 377x_{5,1} + 377x_{5,2} + 754x_{5,3} + 435x_{5,4} + 203x_{5,5}\} \end{aligned}$$

$$\min f_2(x) =$$

$$\begin{aligned} & \min\{3.500x_{1,1} + 0.05951x_{1,2} + 3.791x_{1,4} + 4.511x_{1,5} + \\ & 13.94x_{2,1} + 4.994x_{2,2} + 8.992x_{2,3} + 5.592x_{2,4} + 5.756x_{2,5} + \\ & 14.63x_{3,1} + 9.022x_{3,2} + 10.03x_{3,3} + 1.274x_{3,4} + 14.48x_{3,5} + \\ & 14.19x_{4,1} + 8.281x_{4,2} + 4.060x_{4,3} + 1.956x_{4,4} + 6.551x_{4,5} + \\ & 19.39x_{5,1} + 2.857x_{5,2} + 23.80x_{5,3} + 19.66x_{5,4} + 0.5522x_{5,5}\} \end{aligned}$$

$$\min f_3(x) =$$

$$\begin{aligned} & \min\{1.468x_{1,1} + 0.9894x_{1,2} + 4.605x_{1,4} + 2.986x_{1,5} + \\ & 5.284x_{2,1} + 3.562x_{2,2} + 1.416x_{2,3} + 16.58x_{2,4} + 10.75x_{2,5} + \end{aligned}$$

$$6.165x_{3,1} + 4.155x_{3,2} + 1.652x_{3,3} + 19.34x_{3,4} + 12.54x_{3,5} + \\ 5.578x_{4,1} + 3.760x_{4,2} + 1.495x_{4,3} + 17.50x_{4,4} + 11.35x_{4,5} + \\ 8.514x_{5,1} + 5.738x_{5,2} + 2.282x_{5,3} + 26.71x_{5,4} + 17.32x_{5,5}\}$$

subject to

$$\sum_{j=1}^5 x_{ij} = 1, \quad i = 1, \dots, 5,$$

$$5x_{1,1} + 18x_{2,1} + 21x_{3,1} + 19x_{4,1} + 29x_{5,1} \leq 26y_1,$$

$$5x_{1,2} + 18x_{2,2} + 21x_{3,2} + 19x_{4,2} + 29x_{5,2} \leq 25y_2,$$

$$18x_{2,3} + 21x_{3,3} + 19x_{4,3} + 29x_{5,3} \leq 65y_3$$

$$5x_{1,4} + 18x_{2,4} + 21x_{3,4} + 19x_{4,4} + 29x_{5,4} \leq 40y_4,$$

$$5x_{1,5} + 18x_{2,5} + 21x_{3,5} + 19x_{4,5} + 29x_{5,5} \leq 47y_5,$$

$$5x_{1,1} + 18x_{2,1} + 21x_{3,1} + 19x_{4,1} + 29x_{5,1} \geq 9y_1,$$

$$5x_{1,2} + 18x_{2,2} + 21x_{3,2} + 19x_{4,2} + 29x_{5,2} \geq 8y_2,$$

$$18x_{2,3} + 21x_{3,3} + 19x_{4,3} + 29x_{5,3} \geq 6y_3,$$

$$5x_{1,4} + 18x_{2,4} + 21x_{3,4} + 19x_{4,4} + 29x_{5,4} \geq 9y_4,$$

$$5x_{1,5} + 18x_{2,5} + 21x_{3,5} + 19x_{4,5} + 29x_{5,5} \geq 9y_5,$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 4,$$

$$x_{ij} \in \{0, 1\}, \quad i = 1, \dots, 5, \quad j = 1, \dots, 5,$$

$$y_j \in \{0, 1\}, \quad j = 1, \dots, 5.$$

The ideal objective vector is $f^I = (1353, 10.9402, 23.9437)$, which is also assumed to be selected as a reference point. We define the initial aggregation weights in order to have objective functions normalized: $\lambda_1 = 1/f_1^I$, $\lambda_2 = 1/f_2^I$, $\lambda_3 = 1/f_3^I$, that is, $\lambda = (0.000739098, 0.0914061, 0.0417647)$. We also set a step of interactive process which depends on the iteration number c :

$$\lambda_i^c = \begin{cases} \lambda_i^{c-1} + 0.5\lambda_i^{c-1}/c & \text{if } f_i \text{ is desired to be decreased} \\ \lambda_i^{c-1} - 0.5\lambda_i^{c-1}/c & \text{if } f_i \text{ is desired to be increased} \end{cases}$$

$$i = 1, 2, 3, c \in \mathbb{N}.$$

Observe that the step decreases gradually with an iteration number. This somehow inline with the idea that from step to step a gap between found solution and the most preferred solution becomes smaller and big changes in λ values could affect objective values significantly. Thus, smooth changes of the weighting coefficients will lead to faster convergence of the interactive process. At every iteration of the interactive procedure the DM is allowed to increase or decrease i th component of the objective function. Then according to the rules defined above interactive process simulation is directed by changing weighting coefficients.

The most preferred solution is supposed to be given to reflect the DM's preferences. In our example the MPS (which is Pareto optimal) is calculated by using scalarizing function with Euclidean metric for $\lambda = (0.01, 0.0001, 20)$ and it is equal to $(1487, 31.0761, 29.7186)$. We simulate the interactive process for each value of q separately. At each iteration solutions (objective vectors) obtained by different ASFs are compared to the MPS. If the distance from a solution to the MPS by means of the Euclidean metric is less or equal to $\varepsilon = 0.001$ than we stop interactive process.

Decision process simulation results are presented in the following tables.

iteration	lambda	mean value	test
1	{0.000739098, 0.0914061, 0.0417647}	{1597., 17.5895, 41.0176}	111.398
2	{0.00110865, 0.0457031, 0.0626471}	{1572., 24.2921, 29.5272}	85.2705
3	{0.00138581, 0.0342773, 0.0469853}	{1487., 31.0761, 29.7186}	0.

Table 3: Parameterized ASF for $q = 1$

iteration	lambda	mean value	test
1	{0.000739098, 0.0914061, 0.0417647}	{1698., 10.9402, 42.7022}	212.356
2	{0.00110865, 0.0457031, 0.0626471}	{1487., 31.0761, 29.7186}	0.

Table 4: Parameterized ASF for $q = 2$

The column *mean value* contains values of objective functions at Pareto points found by the corresponding parameterized ASF and the column *test* shows the distance from an objective function to the MPS by means of the Euclidean metric.

In this section we have proposed to apply the interactive approach based on the parameterized ASFs for solving multicriteria combinatorial p -median location problem. We have also introduced a new way of directing interactive

iteration	lambda	mean value	test
1	{0.000739098, 0.0914061, 0.0417647}	{1698., 10.9402, 42.7022}	212.356
2	{0.00110865, 0.0457031, 0.0626471}	{1572., 24.2921, 29.5272}	85.2705
3	{0.00138581, 0.0342773, 0.0469853}	{1572., 24.2921, 29.5272}	85.2705
4	{0.00161678, 0.0285644, 0.0391544}	{1572., 24.2921, 29.5272}	85.2705
5	{0.00181887, 0.0249939, 0.0342601}	{1572., 24.2921, 29.5272}	85.2705
6	{0.00200076, 0.0224945, 0.0308341}	{1517., 27.7326, 30.0057}	30.1871
7	{0.00216749, 0.0206199, 0.0334036}	{1487., 31.0761, 29.7186}	0.

Table 5: Parameterized ASF for $q = 3$

process by changing weighting coefficients depending on the metric used in ASF.

The numerical experiment illustrates that synchronous use of different ASFs allows to detect more desirable Pareto optimal points and as a result may reduce the number of iterations needed for interactive procedure to converge.

Chapter 5. Conclusion

Presence of different conflicting goals and uncertainty are basic structural features of the technological and business environment. Due to these facts, building a suitable mathematical or computational model (that is, the formulation of the optimization problem with specifying decision variables, objectives, constraints, and variable bounds) is an important and critical task in optimization. Moreover, compared to single objective optimization problems, in multiobjective optimization, there is no single optimal solution, but a set of alternatives with different trade-offs. In practice, usually only one of these solutions is to be chosen. Thus, there are at least two equally important tasks: an optimization task for finding optimal solutions (involving a computer-based procedure) and a decision-making task for choosing a single most preferred solution. Therefore, optimal solution found by an optimization algorithm to be reliable must always be analyzed through a postoptimal analysis for their "appropriateness" in the context of the problem.

This thesis is divided into two parts. In the first part, composed of Chapters 2 and 3, a postoptimal analysis for several classes of multicriteria discrete optimization problems is conducted.

In Chapter 2 we proposed a general theoretical approach to qualitative stability analysis of the multicriteria combinatorial minimax, minisum and minimin problems with Pareto and lexicographic principles of optimality. Necessary and sufficient conditions are formulated and proved for stability of Pareto and lexicographic optimal solutions to the multicriteria center (minimax) and median (minisum) location problems. Stability criteria and relations between five stability types (stability, strong stability, quasistability, strong quasistability and superstability) are revealed for the Pareto set and lexicographic set of multicriteria combinatorial minimin problems. The proved theorems allow to analyze and predict the behavior of an optimal solution (or a set of optimal solutions) under different types of uncertainty without solving the perturbed variant of the considered problem.

One more issue which has to be emphasized is that practical verification of conditions of the proved theorems and their straightforward application for a general case can be as hard as to solve the problem itself. Nevertheless more methodological results might be developed and implemented for special cases of the multicriteria combinatorial minimax, minisum and minimin problems with restrictions of some factors, such as structure of initial data, perturbations of particular problem parameters etc. As possible continuation of the research within this topic, it would be interesting to explore these classes of problems.

Chapter 3 discusses various quantitative approaches to sensitivity analysis of combinatorial optimization problems.

Quantitative characteristic called stability radius is used to determine the limit level of perturbations of problem parameters which preserve a given property of a solution. Investigations of stability radius usually aim to derive its formulae and devise methods for its calculation or estimation.

The formula of stability radius for an optimal solution is directly connected with a given optimality principle. A common optimality principles may not fully cover all of the decision maker preferences. Sometimes, introducing a parameterized version of optimality principles may reflect the desirable preference specific much better. In publication I we consider a finite cooperative game of several players with parametric principle of optimality such that the relations between players in a coalition are based on the Pareto maximum. The introduction of this principle allows us to find a link between such classical concepts as the Pareto optimality and Nash equilibrium. In publication I, stability radius is obtained for the game situation which is optimal for the given partition method (generalized equilibrium situation) under perturbations of problem parameters in the case of the Hölder metric.

The formula of stability radius obtained in publication I implies complete enumeration of sets N_s (number of coalitions of players) and X_{I_r} (the set of all situations of coalition I_r) whose cardinality may grow exponentially with s . So far, no polynomial algorithms of calculating or estimating stability radii for multiple objective problems have been constructed. The question of whether such algorithms exist for any class of multiple objective problems of discrete optimization is still open.

Work V is the first attempt to derive an alternative heuristic approach to the stability radius calculation. Non-dominated sorting genetic algorithm based approach is proposed for calculating stability radius of an optimal solution to the single criterion shortest path problem. The shortest path problem was chosen for testing due to the fact that the method proposed by Chakravarti and Wagelmans runs in polynomial time if the original optimization problem is polynomially solvable. Thus we were able to estimate accuracy of results of our approach by comparison with those of the exact method. Despite the fact that theoretical complexity of the adapted NSGA-II is competitive with complexity of the exact method preliminary comparisons showed that NSGA-II can not outperform the last one for small and middle size problems. Nevertheless numerical experiments illustrate that the convergence rate of the adapted NSGA-II was good for almost all random instances of the shortest path problem. This study encourages us to believe that similar ideas could be efficiently applied for calculating stability radius of traveling salesperson problem, that is NP-hard problem.

Stability radius is an efficient measure of the solution reliability. But frequently, this measure is not sufficient to make a conclusion about solution

stability, among multiple optimal solutions. Therefore, it is necessary to calculate some complementary measures reflecting more information about solution behavior under uncertainty.

The accuracy function and robustness tolerances can be potentially used as an efficient tool for ranking multiple optimal solutions. A strategic game with a finite number of players where initial coefficients (costs) of linear payoff functions are subject to perturbations is considered in IV. For two different equilibria principles considered, Pareto and Nash equilibria, appropriate definitions of the worst-case relative regret are specified. Here we use the concept of robustness for dealing with uncertainty. Robust solution is defined as a feasible solution which for a given set of realizations of uncertain parameters guarantees the minimum value of the worst-case relative regret among all feasible solutions. Here we present a formula for calculating value of the accuracy function. Since this formula in general is hard to be computed, we provide some attainable lower and upper bounds for the Pareto accuracy function which are computationally more attractive. We also presented the concept of robustness tolerance of a single cost vector associated with a strategy choice of a player. In the thesis, formulae which allows calculating the robustness tolerances with respect to an equilibrium (in Pareto or Nash senses) is presented for some initial costs. The other big challenge in robust and sensitivity analysis is to construct efficient algorithms to calculate analytical expressions.

The second part of the thesis covered by Chapter 4 addresses an interactive multicriteria optimization method. The key feature of this approach is that the DM is involved into the solution process and directs this procedure by specifying preference information. As a result only such Pareto optimal solutions are generated that are interesting to the DM. Furthermore, the DM can specify and correct her/his preferences and selections during the solution procedure. The final goal of this process is to find a single most preferred solution.

Interactive methods differ from each other by the form in which information is given to the DM, the form and type of preference information the DM specifies and by the methods used for calculating an optimal solution. We proposed to utilize the parameterized achievement scalarizing function in the interactive procedure for finding Pareto optimal solutions of multicriteria p -median location problem. Here an integer parameter q is used to control the degree of metric flexibility varying from L_1 to L_∞ . This parameter can also generate different scalarizing functions for different values of q . Numerical experiments showed that various ASFs allow to detect more desirable Pareto optimal points. This means several scalarizing functions potentially can be utilized in a synchronous way [77], that is, the results of different scalarizing functions are calculated simultaneously and presented to the DM who makes

final decision.

As mentioned above, one more issue that has to be specified is how to interact with the DM. In our approach the DM's preferences are expressed in classification of the objective functions and incorporated into weighting coefficients of the scalarizing function. In classification, the DM directs the interactive solution process in the set of Pareto optimal solutions and expresses what kind of changes would be desirable to her/him by classifying each of the objective functions into different classes. Here we use only two classes in order to simulate decision making process. However in order to reflect the DM's preferences better it worth considering more different classes.

As prospective research it would be interesting to investigate applicability of the parameterized ASFs in synchronous way and to consider different classifications of the objective functions which can provide the DM with a better view of the potential compromises and give more flexible tools to detect Pareto optimal solutions.

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