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HYPERBOLIC TYPE METRICS AND DISTORTION OF QUASICONFORMAL MAPPINGS

by

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Abstract

This Ph.D. thesis consists of four original papers. The papers cover several topics from geometric function theory, more specifically, hyperbolic type metrics, conformal invariants, and the distortion properties of quasiconformal mappings.

The first paper deals mostly with the quasihyperbolic metric. The main result gives the optimal bilipschitz constant with respect to the quasihyperbolic metric for the Möbius self-mappings of the unit ball. A quasiinvariance property, sharp in a local sense, of the quasihyperbolic metric under quasiconformal mappings is also proved.

The second paper studies some distortion estimates for the class of quasiconformal self-mappings fixing the boundary values of the unit ball or convex domains. The distortion is measured by the hyperbolic metric or hyperbolic type metrics. The results provide explicit, asymptotically sharp inequalities when the maximal dilatation of quasiconformal mappings tends to 1. These explicit estimates involve special functions which have a crucial role in this study.

In the third paper, we investigate the notion of the quasihyperbolic volume and find the growth estimates for the quasihyperbolic volume of balls in a domain in terms of the radius. It turns out that in the case of domains with Ahlfors regular boundaries, the rate of growth depends not merely on the radius but also on the metric structure of the boundary.

The topic of the fourth paper is complete elliptic integrals and inequalities. We derive some functional inequalities and elementary estimates for these special functions. As applications, some functional inequalities and the growth of the exterior modulus of a rectangle are studied.

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List of original publications

This thesis consists of the following four papers/manuscripts:

[I] R. Klén, M. Vuorinen, and X.-H. Zhang: *Quasihyperbolic metric and Möbius transformations.* Proc. Amer. Math. Soc. (to appear), Available via arXiv:1108.2967 [math.CV].

[II] M. Vuorinen and X.-H. Zhang: *Distortion of quasiconformal mappings with identity boundary values.* Available via arXiv:1203.0427 [math.CV].

[III] R. Klén, V. Suomala, M. Vuorinen, and X.-H. Zhang: *Volume growth of quasihyperbolic balls.* Available via arXiv:1208.5355 [math.MG].

[IV] M. Vuorinen and X.-H. Zhang: *On exterior moduli of quadrilaterals and special functions.* J. Fixed Point Theory Appl., 2013, DOI: 10.1007/s11784-013-0115-6, Available via arXiv:1111.3812 [math.CA].

1. Introduction

The topics of this thesis lie within geometric function theory, more specifically, in the fields of hyperbolic type metrics, conformal invariants, and the distortion properties of quasiconformal mappings. One of the central problems in the theory of quasiconformal mappings is the deformation of various metrics and conformal invariants under these mappings.

It is an old idea to study function theory by using invariance with respect to rigid motions, which goes back to the works of F. Klein, H. Poincaré, H. A. Schwarz, and C. Carathéodory. The most natural notion of invariance in geometric function theory is conformal invariance under the group of conformal self-mappings of a given domain. Ahlfors [2] presents an exposition of selected topics in the geometric function theory of one complex variable, where the notion of conformal invariant and conformally invariant extremal problems have a key role. Anderson [3] gives an elementary introduction to planar hyperbolic geometry. The survey of Beardon and Minda [9] provides a clear introduction to the hyperbolic metric and a concise treatment of a few recent applications of the hyperbolic metric to geometric function theory. Beardon [8] provides a very nice presentation of many basic facts about hyperbolic 3−space and its isometries. In the context of quasiconformal mappings, conformal invariants and conformally invariant metrics have been used extensively in the pioneering works of H. Grötzsch, O. Teichmüller, L. Ahlfors and A. Beurling in plane domains, see the classical texts [1] by Ahlfors and [28] by Lehto and Virtanen. F. W. Gehring and J. Väisälä extended these ideas to the study of quasiconformal mappings in Euclidean $n-$space [15, 40]. Vuorinen [46] provides numerous estimates for conformal invariants and distortion theorems for quasiconformal and quasiregular mappings in Euclidean $n-$space. Rickman [38] introduces the modern theory of quasiregular mappings and outlines a major achievement on higher-dimensional geometric function theory. The recent handbook of Kühnau [26] collects many surveys dealing with geometric function theory, especially, quasiconformal mappings. The monograph of Anderson, Vamanamurthy and Vuorinen [7] shows the crucial role of special functions in the theory of quasiconformal and quasiregular mappings.

During the past thirty years an increasing number of papers have been published in which the geometric properties of quasiconformal mappings and relations between different metrics on a given domain are extensively studied by considering the change of these metrics and the geometric properties of the domain under quasiconformal mappings. The quasihyperbolic metric is perhaps the most well-known and frequently used of the metrics related to these topics. Since its introduction more than three decades ago, the quasihyperbolic metric has become a popular tool in many subfields of geometric function theory. For instance, in the study of quasiconformal maps of $\mathbb{R}^n$ [16, 17, 31, 34, 46] and Banach spaces [41], analysis of metric spaces [23], and hyperbolic type metrics [21]. Recently, the geometry of quasihyperbolic metric has been studied by several authors, see [22, 24, 37, 42, 43, 47].

A fundamental principle of the theory of quasiconformal mappings in $\mathbb{R}^n$, $n \geq 2$, states that when the maximal dilatation K tends to 1, K–quasiconformal mappings approach conformal mappings. Papers [I] and [II] deal with this topic. Our results provide explicit, asymptotically sharp inequalities when K tends to 1. These explicit estimates involve special functions which have a crucial role in this study. In the paper [III] the geometry of the quasihyperbolic metric is considered. We discuss the notion of quasihyperbolic volume and find growth estimates for the quasihyperbolic volume of balls in proper domains of $\mathbb{R}^n$, in terms of the radius of the balls. In the last paper [IV] we study a conformal invariant, the exterior modulus of a quadrilateral, and prove some functional inequalities and elementary estimates for this quantity.

2. Quasiinvariance of hyperbolic type metrics

The hyperbolic metric of the unit ball $\mathbb{B}^n$ is defined by

$$\rho_{\mathbb{B}^n}(x,y) = \inf_{\gamma\in\Gamma} \int_\gamma \frac{2|dz|}{1-|z|^2},\ x,y \in \mathbb{B}^n,$$

where the infimum is taken over all rectifiable curves in $\mathbb{B}^n$ joining x and y. The formula for the hyperbolic distance in $\mathbb{B}^n$ is

$$\text{sh}^2\left(\frac{1}{2}\rho_{\mathbb{B}^n}(x,y)\right) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}, \qquad x,\, y \in \mathbb{B}^n. \tag{2.1}$$

It is a basic fact that $\rho_{\mathbb{B}^n}$ is invariant under Möbius transformations of $\mathbb{B}^n$.

Several hyperbolic type metrics have been introduced as the generalizations of the hyperbolic metric to any domain and dimension $n \geq 2$. The following quasihyperbolic metric and distance ratio metric are such examples.

Let $D \subsetneq \mathbb{R}^n$ be a domain. The quasihyperbolic metric k_D is defined by

$$k_D(x,y) = \inf_{\gamma\in\Gamma} \int_\gamma \frac{1}{d(z)}|dz|, \quad x,y \in D,$$

where Γ is the family of all rectifiable curves in D joining x and y, and $d(z) = d(z,\partial D)$ is the Euclidean distance between z and the boundary of D. The explicit formula for the quasihyperbolic metric is known only in very few domains. One such domain is the punctured space $\mathbb{R}^n \setminus \{0\}$ (see [34]). The distance ratio metric is defined as

$$j_D(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x),d(y)\}}\right), \quad x,y \in D.$$

It is well known that [16, Lemma 2.1]

$$j_D(x,y) \leq k_D(x,y)$$

for all domains $D \subsetneq \mathbb{R}^n$ and $x, y \in D$. We also have the following comparison inequalities for the domain of the unit ball (see [7, Lemma 7.56] and [46, Remark

3.3]): for $x, y \in \mathbb{B}^n$,

$$\rho_{\mathbb{B}^n}/2 \le j_{\mathbb{B}^n} \le k_{\mathbb{B}^n} \le \rho_{Bn}.$$

In his dissertation [29], H. Lindén studied the shape of a geodesic for the quasihyperbolic metric in detail for simple domains and gave several sharp comparison inequalities for specific domains.

Unlike the hyperbolic metric of the unit ball, neither the quasihyperbolic metric k_D nor the distance ratio metric j_D is invariant under Möbius transformations of the unit ball onto itself. Gehring, Palka and Osgood proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [16, Corollary 2.5] and [17, proof of Theorem 4].

2.2. **Theorem.** *If D and D' are proper subdomains of $\mathbb{R}^n$ and if f is a Möbius transformation of D onto D', then, for all $x, y \in D$,*

$$\frac{1}{2} k_D(x,y) \le k_{D'}(f(x), f(y)) \le 2 k_D(x,y)$$

and

$$\frac{1}{2} j_D(x,y) \le j_{D'}(f(x), f(y)) \le 2 j_D(x,y).$$

Gehring and Osgood [17] also proved the following quasiinvariance property of the quasihyperbolic metric under quasiconformal mappings. For basic results on quasiconformal mappings and the definition of K–quasiconformality, we refer to Väisälä [40].

2.3. **Theorem.** [17, Theorem 3] *Let D and D' be proper subdomains of $\mathbb{R}^n$. There exists a constant c depending only on n and K with the following property. If f is a K-quasiconformal mapping of D onto D', then*

$$k_{D'}(f(x), f(y)) \le c \max\{k_D(x,y), k_D(x,y)^\alpha\},\ \alpha = K^{1/(1-n)},$$

for all $x, y \in D$.

In the paper [I], we first obtain an improved version of quasiinvariance property of the quasihyperbolic metric under the Möbius transformations of the unit ball $\mathbb{B}^n$, $n \ge 2$.

2.4. **Theorem.** [I, Theorem 1.4] *Let $a \in \mathbb{B}^n$ and $h : \mathbb{B}^n \to \mathbb{B}^n$ be a Möbius transformation with $h(a) = 0$. Then, for all $x, y \in \mathbb{B}^n$,*

$$\frac{1}{1+|a|} k_{\mathbb{B}^n}(x,y) \le k_{\mathbb{B}^n}(h(x), h(y)) \le (1+|a|) k_{\mathbb{B}^n}(x,y),$$

and the constants $1+|a|$ and $1/(1+|a|)$ are both sharp.

The sharpness statement in Theorem 2.4 shows that the constant c in Theorem 2.3 cannot be chosen so that it converges to 1 when $K \to 1$. We refined this result by proving that, in a local sense, we could improve the constant for quasiconformal mappings of the unit ball onto itself.

2.5. **Theorem.** [I, Theorem 1.8] *Let $f\colon \mathbb{B}^n \to \mathbb{B}^n$ be a K-quasiconformal self-mapping and $r \in (0,1)$. There exists a constant $c = c(n,K,r)$ such that, for all $x, y \in \mathbb{B}^n(r)$ with $f(x), f(y) \in \mathbb{B}^n(r)$,*

$$k_{\mathbb{B}^n}(f(x), f(y)) \le c \max\{k_{\mathbb{B}^n}(x,y), k_{\mathbb{B}^n}(x,y)^\alpha\}$$

and

$$j_{\mathbb{B}^n}(f(x), f(y)) \le c \max\{j_{\mathbb{B}^n}(x,y), j_{\mathbb{B}^n}(x,y)^\alpha\},$$

where $\alpha = K^{1/(1-n)}$ and $c \to 1$ as $(r,K) \to (0,1)$.

3. Teichmüller's problem

Teichmüller's classical mapping problem for plane domains concerns finding a lower bound for the maximal dilatation of a quasiconformal homeomorphism which holds the boundary pointwise fixed, maps the domain onto itself, and maps a given point of the domain to another given point of the domain (see [5, 25, 30, 39]). G.J. Martin [33] has recently studied the Teichmüller problem for the mean distortion. The classical problem has found applications in the theory of homogeneity of domains as introduced in [16] and more recently in the homogeneity constants of surfaces [11, 12, 27].

Let D be a proper subdomain of $\mathbb{R}^n$ $(n \ge 2)$, and let

$$\mathrm{Id}_K(\partial D) = \{f : \mathbb{R}^n \to \mathbb{R}^n \text{ is } K\text{–quasiconformal} : f(x) = x,\ \forall x \in \mathbb{R}^n \setminus D\}.$$

In his classical work [39], Teichmüller studied the class $\mathrm{Id}_K(\partial D)$ with $D = \mathbb{R}^2 \setminus \{0, e_1\}$ and proved that the following sharp inequality

$$\rho_D(x, f(x)) \le \log K$$

holds for all $x \in D$, where ρ_D is the hyperbolic metric of $D = \mathbb{R}^2 \setminus \{0, e_1\}$. This result may be regarded as a stability result since it says that $f(x)$ is contained in the closure of the hyperbolic ball $B_{\rho_D}(x, \log K)$ centered at the point x and with the radius $\log K$. In particular, the radius tends to 0 as $K \to 1$.

Krzyż [25] considered the same problem for the case of the unit disk, and Anderson and Vamanamurthy [5] found a counterpart for Krzyż' result in the case of the unit ball $\mathbb{B}^n$ $(n \ge 3)$ under an additional symmetry hypothesis. Very recently, Manojlović and Vuorinen [30] removed the extra symmetry hypothesis and proved the following theorem.

As in [46, p. 97 (7.44), p. 138, Theorem 11.2], we denote by $\varphi_{K,n} : [0,1] \to [0,1], K > 0$, the special function connected with the quasiconformal Schwarz lemma. What is important is that it is an increasing homeomorphism with $\varphi_{K,n}(r) \to r$ as $K \to 1$.

3.1. **Theorem.** [30, Theorem 1.9] *If $f \in \mathrm{Id}_K(\partial \mathbb{B}^n)$, then for all $x \in \mathbb{B}^n$,*

$$\rho_{\mathbb{B}^n}(x, f(x)) \le \log \frac{1-a}{a}, \quad a = \varphi_{1/K,n}(1/\sqrt{2})^2,$$

where $\rho_{\mathbb{B}^n}$ is the hyperbolic metric in the unit ball.

This result is asymptotically sharp since it says that $f(x)$ tends to x as $K \to 1$. Theorem 3.1 also implies that if $f(x) \neq x$ for some $x \in \mathbb{B}^n$ then $K > 1$. As pointed out in [44], it is not true for $n \geq 3$ that, for all domains D and $f \in \mathrm{Id}_K(\partial D)$, $f(x) \neq x$ for some $x \in D$ implies $K > 1$.

In the paper [II] we improve Manojlović and Vuorinen's result as follows.

3.2. **Theorem.** [II, Theorem 1.2] *If $f \in \mathrm{Id}_K(\partial \mathbb{B}^n)$, then, for all $x \in \mathbb{B}^n$,*

$$\rho_{\mathbb{B}^n}(x, f(x)) \leq \log \frac{1 - \varphi_{1/K,n}(1/2)}{\varphi_{1/K,n}(1/2)}.$$

A comparison shows that Theorem 3.2 yields a better bound than Theorem 3.1 when $n = 2$.

We also consider Teichmüller's problem for convex domains and give a distortion theorem which shows that for each $x \in D$, the requirement $f(x) \neq x$ implies that the maximal dilatation of f is greater than 1. This kind of behavior also holds for bounded domains as Theorem 3.5 shows.

3.3. **Theorem.** [II, Theorem 1.6] *Let $D \subsetneq \mathbb{R}^n$ be a convex domain and $f \in \mathrm{Id}_K(\partial D)$. Then, for all $x \in D$,*

$$j_D(x, f(x)) \leq \log \left(1 + \sqrt{\left(\frac{2\varphi_{K,n}(1/3)}{1 - \varphi_{K,n}(1/3)} \right)^2 - 1} \right).$$

For K close to 1, the inequality of Theorem 3.3 can be simplified further.

3.4. **Theorem.** [II, Theorem 1.7] *Let $D \subsetneq \mathbb{R}^n$ be a convex domain and*

$$K_n = \left(1 + \frac{\log 2}{n - 1 + \log 3} \right)^{n-1} \in [K_2, 2), \qquad K_2 \approx 1.33029.$$

If $K \in (1, K_n]$ and $f \in \mathrm{Id}_K(\partial D)$, then for all $x \in D$

$$j_D(x, f(x)) \leq 2\sqrt{1 + \log 6}(K-1)^{1/2}.$$

3.5. **Theorem.** [II, Theorem 3.6] *Let D be a bounded domain in $\mathbb{R}^n$, and $f \in \mathrm{Id}_K(\partial D)$. Then for all $x \in D$,*

$$|f(x) - x| \leq \mathrm{diam}(D)\, \mathrm{th} \left(\frac{1}{2} \log \frac{1-b}{b} \right), \quad b = \varphi_{1/K,n}(1/2)\,.$$

We also study the Hölder continuity of quasiconformal self mappings of the unit ball with identity boundary values. For the detailed history of the Hölder continuity of quasiconformal mappings, the reader is referred to the bibliographies of [10], [14], [35] and [46].

3.6. **Theorem.** [II, Theorem 1.10] *If $f \in \mathrm{Id}_K(\partial \mathbb{B}^n)$, then, for all $x, y \in \mathbb{B}^n$,*

$$|f(x) - f(y)| \leq M_1(n,K)|x-y|^\alpha, \qquad \alpha = K^{1/(1-n)}$$

where $M_1(n,K) = \lambda_n^{1-\alpha} C(\alpha)$ and $C(\alpha) = 2^{1-\alpha}\alpha^{-\alpha/2}(1-\alpha)^{(\alpha-1)/2}$, with $M_1(n,K) \to 1$ when $K \to 1$, and $\lambda_n \in [4, 2e^{n-1})$ is the Grötzsch ring constant.

For the planar case of $n = 2$, I. Prause [36] has proved that $4^{1-1/K}$ is the optimal constant under the same conditions as in Theorem 3.6.

4. Quasihyperbolic volume

Since its introduction more than three decades ago, the quasihyperbolic metric has found many applications in subfields of geometric function theory. A natural question is whether and to what extent, the results of hyperbolic geometry have counterparts for the quasihyperbolic geometry. For instance in [24] it was noticed that some facts from hyperbolic trigonometry of the plane have counterparts in the quasihyperbolic setup while some have not.

The purpose of the paper [III] is to study the notion of the quasihyperbolic volume and to find growth estimates for the quasihyperbolic volume of balls in a domain $D \subsetneq \mathbb{R}^n$, in terms of the radius. It turns out that the rate of growth depends not merely on the radius but also on the metric structure of the boundary. The quasihyperbolic volume of a Lebesgue measurable set $A \subset D$ is defined by

$$\mathrm{vol}_{k_D}(A) = \int_A \frac{dm(z)}{d(z)^n}\,,$$

where $d(z)$ is the Euclidean distance from z to the boundary of the domain D, and m refers to the Lebesgue measure.

A set $E \subset \mathbb{R}^n$ is Q–regular for $0 < Q < n$, if there is a (Borel regular, outer–) measure μ with $\mathrm{spt}(\mu) = E$ and constants $0 < \alpha \le \beta < \infty$ such that

$$\alpha r^Q \le \mu(B(x,r)) \le \beta r^Q, \text{ for all } x \in E \text{ and } 0 < r < \mathrm{diam}(E)\,.$$

Here $\mathrm{spt}(\mu)$ denotes the smallest closed set with full μ–measure.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous strictly increasing function with $\varphi(0) = 0$. A domain $D \subset \mathbb{R}^n$ is said to be φ–uniform [45] if

$$k_D(x,y) \le \varphi(|x-y|/\min\{d(x),d(y)\}) \tag{4.1}$$

for all $x, y \in G$. In particular, the domain D is C–uniform if $\varphi(t) = C\log(1+t)$ with $C > 1$. A domain is called uniform if it is a C–uniform domain for some $C > 1$.

The main results in the paper [III] are the following theorems.

4.2. **Theorem.** [III, Theorem 4.16] *Let D be a proper subdomain of $\mathbb{R}^n$ with compact and $Q-$regular ($0 < Q < n$) boundary. There exists a constant $C < \infty$ such that for each $x \in D$ and sufficiently large $r > 0$, we have*

$$\mathrm{vol}_k(B_k(x,r)) \le Ce^{Qr}.$$

4.3. **Theorem.** [III, Theorem 4.20] *Let $E \subset \mathbb{R}^n$ be a closed Q–regular set with $0 < Q < n$ such that $D = \mathbb{R}^n \setminus E$ is a ψ–uniform domain. Then for each $x \in D$ there is $c > 0$ and $r_1 > 0$ such that*

$$\mathrm{vol}_k(B_k(x,r)) \ge c(\psi^{-1}(r))^Q\,,$$

for all $r > r_1$.

By combining Theorem 4.2 and Theorem 4.3 we get the following corollary.

4.4. **Corollary.** [III, Corollary 4.21] *Let $E \subset \mathbb{R}^n$ be a compact $Q-$regular set with $0 < Q < n$ such that $D = \mathbb{R}^n \setminus E$ is a uniform domain with the uniformity constant $L > 1$. Then for each $x \in G$ and sufficiently large $r > 0$,*

$$ce^{Qr/L} \leq \mathrm{vol}_k(B_k(x,r)) \leq Ce^{Qr},$$

where $C < \infty$ only depends on the Q-regularity data and L and $c > 0$ depends only on the Q-regularity data, L, and $d(x)$.

5. Exterior moduli of quadrilaterals

For $h > 0$ consider the rectangle D with vertices $1 + ih$, ih, 0, 1 in the upper half plane $\mathbb{H}^2 = \{x + iy : y > 0\}$ and a bounded harmonic function $u : \mathbb{C} \setminus D \to \mathbb{R}$ satisfying the Dirichlet-Neumann boundary value problem $u(z) = 0$ for $z \in [0, 1]$, $u(z) = 1$ for $z \in [ih, 1 + ih]$, $\frac{\partial u}{\partial n}(z) = 0$ for $z \in [1, 1 + ih] \cup [0, ih]$ where n is the direction of the exterior normal to ∂D. The number

$$\mathcal{M}(1 + ih, ih, 0, 1) = \int_{\mathbb{C}\setminus D} |\nabla u|^2 dm$$

is called the exterior modulus of the rectangle $D(1 + ih, ih, 0, 1)$.

This quantity also has an interpretation as the modulus of the family of all curves, joining the segments $[1 + ih, ih]$ and $[0, 1]$ in the complement of the rectangle D, which also is equal to $\mathcal{M}(1 + ih, ih, 0, 1)$ (cf. [2]). In the same way, for a polygonal quadrilateral $D(a, b, 0, 1)$ with vertices $a, b \in \mathbb{H}^2$ and base $[0, 1]$, we can define the exterior modulus $\mathcal{M}(a, b, 0, 1)$.

As far as we know there is no analytic formula for $\mathcal{M}(a, b, 0, 1)$. Numerical methods for the computation of $\mathcal{M}(a, b, 0, 1)$ were recently studied by Hakula, Rasila, and Vuorinen in [20] which motivates the present study. They used numerical methods such as hp-FEM and the Schwarz-Christoffel mapping. Similar problems for the interior modulus have been studied in [18, 19].

For $0 < r < 1$, the functions

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2 t}}, \quad \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t}\, dt \tag{5.1}$$

with limiting values $\mathcal{K}(0) = \pi/2 = \mathcal{E}(0)$, $\mathcal{K}(1-) = \infty$ and $\mathcal{E}(1) = 1$ are known as Legendre's complete elliptic integrals of the first and second kind, respectively. Let $r' = \sqrt{1 - r^2}$ for $r \in (0, 1)$. We denote $\mathcal{K}'(r) = \mathcal{K}(r')$, $\mathcal{E}'(r) = \mathcal{E}(r')$. Define the function ψ as follows

$$\psi(r) = \frac{2(\mathcal{E}(r) - (1 - r)\mathcal{K}(r))}{\mathcal{E}'(r) - r\mathcal{K}'(r)}, \quad r \in (0, 1). \tag{5.2}$$

The function $\psi : (0, 1) \to (0, \infty)$ is a homeomorphism [13]. In particular, $\psi^{-1} : (0, \infty) \to (0, 1)$ is well-defined.

In [13], Duren and Pfaltzgraff studied the modulus $\mathcal{M}(\Gamma)$ of the family of curves Γ joining the opposite sides of length b of the rectangle with sides a and b, in the exterior of the rectangle, and gave the formula [13, Theorem 5]

$$\mathcal{M}(\Gamma) = \frac{\mathcal{K}'(r)}{2\mathcal{K}(r)}, \quad \text{where} \quad r = \psi^{-1}(a/b). \tag{5.3}$$

The exterior modulus $\mathcal{M}(\Gamma)$ is a conformal invariant of a quadrilateral. In [4], the authors gave a sharp comparison between the function ψ and Robin modulus of a given rectangle. Their result can be rewritten as the following inequality

$$\frac{\pi r}{(1-r)^2} < \psi(r) < \frac{16r}{\pi(1-r)^2}, \quad r \in (0,1). \tag{5.4}$$

In the paper [IV] two identities involving the function ψ are proved, and some functional inequalities and elementary estimates for the function ψ are also derived from the monotonicity and convexity of the combinations of the function ψ and some elementary functions. The main results are as follows.

5.5. **Theorem.** [IV, Theorem 1.7] *For $r \in (0,1)$, the function ψ satisfies the identities*

$$\psi(r^2)\psi\left(\left(\frac{1-r}{1+r}\right)^2\right) = 1, \quad \psi\left(\frac{1-r}{1+r}\right)\psi\left(\frac{1-r'}{1+r'}\right) = 1.$$

5.6. **Theorem.** [IV, Theorem 1.8] *The function $f(r) = (1-\sqrt{r})^2\psi(r)/r$ is strictly decreasing from $(0,1)$ onto $(4/\pi, \pi)$. In particular, for all $r \in (0,1)$,*

$$\frac{4r}{\pi(1-\sqrt{r})^2} < \psi(r) < \frac{\pi r}{(1-\sqrt{r})^2}.$$

5.7. **Theorem.** [IV, Theorem 1.9] *The function $f(x) = \psi(1/\operatorname{ch}(x))$ is decreasing and convex from $(0,\infty)$ onto $(0,\infty)$. In particular, for $r, s \in (0,1)$,*

$$2\psi\left(\frac{\sqrt{2rs}}{\sqrt{1+rs+r's'}}\right) \le \psi(r) + \psi(s) \tag{5.8}$$

with equality in the first inequality if and only if $r = s$.

We denote $R = [0,1] \times [0,b]$. Let Γ_b be the family of curves joining the opposite sides of length b of the rectangle R in the exterior of the rectangle. By the formula of Duren and Pfaltzgraff (5.3), we have

$$\mathcal{M}(\Gamma_b) = \frac{1}{\pi}\mu(\psi^{-1}(1/b)).$$

We show that the modulus $\mathcal{M}(\Gamma_b)$ has a logarithmic growth with respect to the length of side b.

5.9. **Theorem.** [IV, Theorem 4.3] *For $b \in (0,\infty)$,*

$$L(b) < \mathcal{M}(\Gamma_b) < U(b), \tag{5.10}$$

where

$$
\begin{aligned}
L(b) \; &:= \; \frac{2}{\pi}\left(1-\left(1+\sqrt{4b/\pi}\right)^{-4}\right)^{1/4} \log\left(2\left(1+\sqrt{4b/\pi}\right)\right) \\
&> \; \frac{2}{\pi}\left(1-\left(1+\sqrt{4b/\pi}\right)^{-1}\right) \log\left(2\left(1+\sqrt{4b/\pi}\right)\right), \qquad (5.11)
\end{aligned}
$$

and

$$
\begin{aligned}
U(b) \; &:= \; \frac{1}{\pi}\log\left(2\left(1+\sqrt{\pi b}\right)^2\left(1+\sqrt{1-\left(1+\sqrt{\pi b}\right)^{-4}}\right)\right) \\
&< \; \frac{2}{\pi}\log\left(2\left(1+\sqrt{\pi b}\right)\right). \qquad (5.12)
\end{aligned}
$$

6. Concluding remarks

The stability properties of K–quasiconformal mappings in paper [I] and [II] deal with the quantitative description of the behavior of these mappings when $K \to 1$. As expected, the mapping becomes more or less like a conformal mapping under this passage to the limit. The results of these papers rely on two explicit, asymptotically sharp theorems. The first is an explicit version of the Schwarz lemma for K–quasiconformal mappings of the unit ball [46, Corollary 11.3], and the second is an explicit estimate for the function of quasisymmetry of K–quasiconformal mappings of $\mathbb{R}^n, n \geq 3$ [7, Theorem 14.6, Theorem 14.8]. Hyperbolic type metrics and special functions play an important role in this approach, however, there are still many open problems that need to be solved. We list here some of them.

1. Can we find a constant $C = C(n, K)$ with $C \to 1\,(K \to 1)$ such that, for $K > 1$, $n > 2$ and $r \in (0, 1)$,

$$\operatorname{arth}\varphi_{K,n}(\operatorname{th} r) \leq C \max\{r, r^{\alpha}\}, \quad \alpha = K^{1/(1-n)}\,?$$

2. Can we compare analytically the bounds from Theorem 3.1 and Theorem 3.2 for the case of $n \geq 3$?

3. Could the exponent Qr/L in Corollary 4.4 be replaced by cr for some uniform constant $c > 0$ independent of L?

One can find more problems on the distortion theory of quasiconformal and quasiregular mappings in [46], hyperbolic type metrics and geometry in [47], and special functions related to quasiconformal analysis in [6].

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