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# Metrics of Hyperbolic Type and Moduli of Continuity of Maps

by

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## Abstract

This PhD thesis in Mathematics belongs to the field of Geometric Function Theory. The thesis consists of four original papers. The topics studied mainly deal with hyperbolic type metrics and moduli of continuity of maps.

In the first paper, we provide a simple construction of the midpoint of the hyperbolic geodesic segment joining a pair of points in the upper half plane or the unit disk.

In the second paper, we prove sharp bounds for the product and the sum of two hyperbolic distances between the opposite sides of hyperbolic Lambert quadrilaterals in the unit disk. Furthermore, we study the images of Lambert quadrilaterals under quasiconformal mappings from the unit disk onto itself and obtain sharp results in this case, too.

In the third paper, a new similarity invariant metric $v_G$ is introduced. The visual angle metric $v_G$ is defined on a domain $G \subsetneq \mathbb{R}^n$ whose boundary is not a proper subset of a line. We find sharp bounds for $v_G$ in terms of the hyperbolic metric in the particular case when the domain is either the unit ball $\mathbb{B}^n$ or the upper half space $\mathbb{H}^n$. We also obtain the sharp Lipschitz constant for a Möbius transformation $f : G \to G'$ between domains $G$ and $G'$ in $\mathbb{R}^n$ with respect to the metrics $v_G$ and $v_{G'}$. For instance, in the case $G = G' = \mathbb{B}^n$ the result is sharp.

In the fourth paper, we find sharp Lipschitz constants for the distance ratio metric under some common classes of mappings of the Euclidean space $\mathbb{R}^n, n \geq 2$.

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## List of original publications

This thesis is based on the following four papers/manuscripts:

[I] M. Vuorinen and G.-D. Wang: *Bisection of geodesic segments in hyperbolic geometry.* Complex Analysis and Dynamical Systems V, Contemp. Math. 591, Amer. Math. Soc., Providence, RI, 2013, pp. 273-290, arXiv:1108.2948 [math. MG].

[II] M. Vuorinen and G.-D. Wang: *Hyperbolic Lambert quadrilaterals and quasiconformal mappings.* Ann. Acad. Sci. Fenn. Math. 38 (2013), 433–453, arXiv:1203. 6494 [math. MG].

[III] R. Klén, H. Lindén, M. Vuorinen, and G.-D. Wang: *The visual angle metric and Möbius transformations.* arXiv:1208.2871 [math. MG].

[IV] S. Simić, M. Vuorinen, and G.-D. Wang: *Sharp Lipschitz constants for the distance ratio metric.* Math. Scand.(to appear), arXiv:1202.6565 [math. CV].

# 1. Introduction

The topics of this thesis belong to Classical Analysis, more precisely, to Geometric Function Theory (GFT). We study well-known classes of maps such as conformal maps, quasiconformal maps, Lipschitz maps, and Möbius transformations defined on subdomains of the Euclidean space $\mathbb{R}^n, n \geq 2$. This theory has its roots in the study of conformal maps and analytic functions of the complex plane.

One of the key ideas in GFT is to use tools and notions which either are fully or partially invariant under conformal maps or Möbius transformations. This is a very natural idea, because the problems studied in GFT often have this type of invariance property, too. For instance, conformal maps preserve angles between smooth curves. The notion of the modulus of a curve family, widely applied in GFT, is conformally invariant.

When we study maps defined on subdomains of $\mathbb{R}^n, n \geq 3$, or maps other than conformal maps and Möbius transformations, things change. First, the class of conformal maps is very narrow in $\mathbb{R}^n, n \geq 3$, by Liouville's theorem which says that a conformal map of a domain $D \subset \mathbb{R}^n, n \geq 3$, is a restriction to $D$ of a Möbius transformation of $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. Liouville's classical theorem requires that the maps be sufficiently smooth. However, the differentiability assumption can be replaced with the requirement that the map be 1-quasiconformal or even 1-quasiregular, see [13, 33]. Second, quasiconformal maps, which are differentiable only almost everywhere, may substantially change "the local geometry", the images of small spheres may be fractal surfaces and hence the Hausdorff dimension does not remain invariant.

For the study of the local behavior of maps, GFT uses several metrics: the Euclidean metric, the hyperbolic metric, and the chordal metric. In addition to these, it has turned out that various generalizations of the hyperbolic metric are also useful: in this work, we call such generalizations hyperbolic type metrics. For an overview of this research, see Vuorinen [40].

We now give an overview of some of the main sources of GFT. Ahlfors's book [2] provides an exposition of the topics of GFT of functions of one complex variable and the book [3] deals with the higher-dimensional Möbius transformations in particular. For a collection of several surveys of GFT, see the handbooks of Kühnau [24, 25]. Quasiconformal mappings and quasiregular mappings are studied in the monographs of Gehring and Hag [14], Väisälä [35], Vuorinen [39]. The well-known references of the hyperbolic geometry are Anderson [5], Beardon [7], Keen and Lakic [20]. The book of Mumford, Series, and Wright [29] presents the vision of F. Klein: the connections between group theory, symmetry, and geometry; and builds an approach to discrete group theory on these ideas. Conformal invariants are often expressed in terms of special functions. This connection is due to the fact that certain canonical regions can be mapped onto each other by means of such maps, e.g., the upper half plane with four prescribed points on the real axis can be conformally mapped onto a rectangle such that the prescribed points are mapped onto the vertices of the rectangle. For a treatment of the theory of special functions we refer to the monograph of Anderson, Vamanamurthy, and Vuorinen [4], handbooks of Abramowitz and Stegun [1], Olver, Lozier, Boisvert, and Clark [31].

This thesis consists of four original papers and a summary outlining the main ideas and results of each paper. Papers [I] and [II] deal with hyperbolic geometry. In paper [I], we study the problem of bisection of a segment in the hyperbolic geometry. Paper [II] concerns hyperbolic quadrilaterals of special type, so called Lambert quadrilaterals, and gives sharp bounds for the sum and the product of distances between opposite pairs of sides. Papers [III] and [IV] are related to the estimate of the moduli of continuity of maps between two metric spaces. In paper [III], a new metric in a domain $G$ of $\mathbb{R}^n$, the visual angle metric, is introduced and its basic properties are examined. Also, estimates are given in terms of hyperbolic metric in the case when the domain is either the unit ball or the upper half space. In paper [IV], we study the change of distances with respect to the distance ratio metric under some maps, e.g., conformal maps of some plane domains. For some special domains we compare in [III] the visual angle metric and the distance ratio metric. Some problems concerning the topics of papers [III, IV] are listed at the end of this summary.

## 2. Background and definitions

We introduce some basic notation, terminology, and background for this thesis.

2.1. **Euclidean geometry.** The group of Möbius transformations in $\overline{\mathbb{R}^n}$ is generated by transformations of two types:

(1) reflections in the hyperplane $P(a,t) = \{x \in \mathbb{R}^n : x \cdot a = t\} \cup \{\infty\}$

$$f_1(x) = x - 2(x \cdot a - t)\frac{a}{|a|^2},\ f_1(\infty) = \infty,$$

where $a \in \mathbb{R}^n \setminus \{0\}$ and $t \in \mathbb{R}$;

(2) inversions (reflections) in the sphere $S^{n-1}(a,r) = \{x \in \mathbb{R}^n : |x-a| = r\}$

$$f_2(x) = a + \frac{r^2(x-a)}{|x-a|^2},\ f_2(a) = \infty, f_2(\infty) = a,$$

where $a \in \mathbb{R}^n$ and $r > 0$. For $x \in \mathbb{R}^n \setminus \{0\}$, we denote by $x^* = x/|x|^2$ the inversion in the sphere $S^{n-1}(0,1) = S^{n-1}$. If $G \subset \overline{\mathbb{R}^n}$, we denote by $\mathcal{GM}(G)$ the group of all Möbius transformations which map $G$ onto itself. For an ordered quadruple $a, b, c, d$ of distinct points in $\mathbb{R}^n$, we define the absolute ratio by (for the case $a, b, c, d \in \overline{\mathbb{R}^n}$ defined below)

$$|a,b,c,d| = \frac{|a-c||b-d|}{|a-b||c-d|}.$$

The most important property of the absolute ratio is the Möbius invariance, see [7, Theorem 3.2.7], i.e., if $f$ is a Möbius transformation, then

$$|f(a), f(b), f(c), f(d)| = |a,b,c,d|,$$

for all distinct $a, b, c, d \in \mathbb{R}^n$. This Möbius invariance property also holds when the points are in $\overline{\mathbb{R}^n}$.

The notion of a metric space was introduced by M. Fréchet in his thesis in 1906. It became quickly one of the key concepts in mathematical areas such as geometry, linear algebra, and topology, see the book [12].

Let $X$ be a nonempty set. A function $d : X \times X \to [0, \infty)$ is a metric if for all $x\,, y\,, z\, \in X$
(1) $d(x, y) = d(y, x)$;
(2) $d(x, y) \le d(x, z) + d(z, y)$;
(3) $d(x, y) \ge 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.

2.2. **Spherical geometry.** The chordal metric is defined by

$$\begin{cases} q(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x\,, y \neq \infty, \\ q(x, \infty) = \frac{1}{\sqrt{1+|x|^2}}, & x \neq \infty, \end{cases}$$

for all $x\,, y \in \overline{\mathbb{R}^n}$. The triangle inequality for the chordal metric is easily proved by the relation between this metric and the stereographic projection.

For an ordered quadruple $a, b, c, d$ of distinct points in $\overline{\mathbb{R}^n}$, we define the absolute ratio by

$$|a, b, c, d| = \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)}.$$

For the points in $\mathbb{R}^n$, we easily see that this definition agrees with the above definition.

2.3. **Hyperbolic geometry.** Let $G \subsetneq \mathbb{R}^n (n \ge 2)$ be a domain and $w : G \to (0, \infty)$ be a continuous function. We define the weighted length of a rectifiable curve $\gamma \subset G$ as

$$\ell_w(\gamma) = \int_\gamma w(z)|dz|$$

and the weighted distance by

$$d_w(x, y) = \inf_\gamma \ell_w(\gamma),$$

where the infimum is taken over all rectifiable curves in $G$ joining $x$ and $y$ ($x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$). It is easy to see that $d_w$ defines a metric on $G$ and $(G, d_w)$ is a metric space. We say that a curve $\gamma : [0, 1] \to G$ is a geodesic joining $\gamma(0)$ and $\gamma(1)$ if for all $t \in (0, 1)$, we have

$$d_w(\gamma(0), \gamma(1)) = d_w(\gamma(0), \gamma(t)) + d_w(\gamma(t), \gamma(1)).$$

The hyperbolic distance in $\mathbb{H}^n$ is defined by the weight function $w_{\mathbb{H}^n}(x) = 1/x_n$ and in $\mathbb{B}^n$ by the weight function $w_{\mathbb{B}^n}(x) = 2/(1 - |x|^2)$. We also have the corresponding explicit formulas

$$\cosh \rho_{\mathbb{H}^n}(x, y) = 1 + \frac{|x - y|^2}{2x_n y_n} \tag{2.4}$$

for all $x, y \in \mathbb{H}^n$ [7, p.35], and

$$\sinh \frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}} \tag{2.5}$$

for all $x, y \in \mathbb{B}^n$ [7, p.40].

There is a third equivalent way to express the hyperbolic distance. Let $G \in \{\mathbb{H}^n, \mathbb{B}^n\}$, $x, y \in G$. Let $L$ be an arc of a circle perpendicular to the boundary $\partial G$ of the domain $G$ with $x, y \in L$ and let $\{x_*, y_*\} = L \cap \partial G$, the points being labelled so that $x_*, x, y, y_*$ occur in this order on $L$. Then by [7, (7.26)]

$$\rho_G(x,y) = \sup\{\log |a, x, y, b| : a, b \in \partial G\} = \log |x_*, x, y, y_*|. \tag{2.6}$$

The last definition (2.6) is geometric and it gives us a definition for hyperbolic lines. The hyperbolic distance is invariant under Möbius transformations of $G$ onto $G'$ for $G,\, G' \in \{\mathbb{H}^n, \mathbb{B}^n\}$.

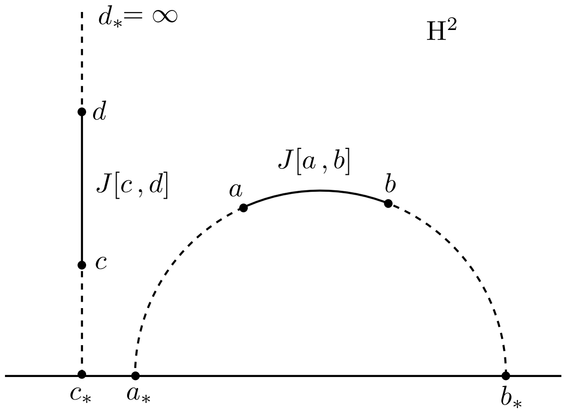


FIGURE 1.
Hyperbolic geodesics in $\mathbb{H}^2$.

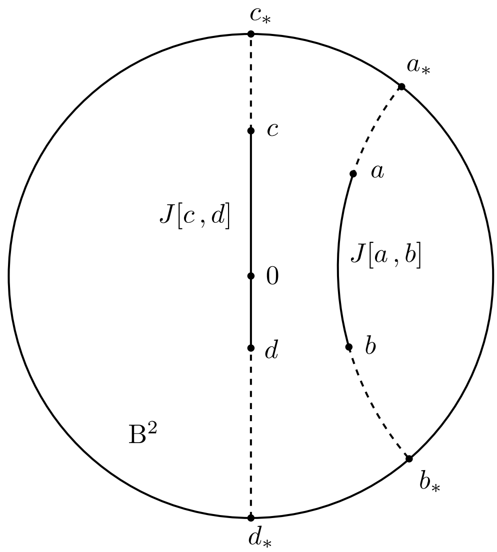


FIGURE 2.
Hyperbolic geodesics in $\mathbb{B}^2$.

Hyperbolic geodesic lines or hyperbolic lines are arcs of circles, which are orthogonal to the boundary of the domain. More precisely, for $a, b \in \mathbb{B}^n$ (or $\mathbb{H}^n$), the hyperbolic geodesic segment joining $a$ and $b$ is an arc of a circle orthogonal to $S^{n-1}$ (or $\partial\mathbb{H}^n$). In a limiting case the points $a$ and $b$ are located on a Euclidean line through 0 (or located on a normal of $\partial\mathbb{H}^n$), see [7]. Therefore, the points $x_*$ and $y_*$ are the end points of the hyperbolic line passing through the points $x\,, y$. We denote by $J[a, b]$ the hyperbolic geodesic segment or shortly hyperbolic segment joining $a$ to $b$. For any two distinct points the hyperbolic geodesic segment is unique in $\mathbb{H}^n$ and $\mathbb{B}^n$ (see Figure 1 and 2).

The hyperbolic metric has been used in complex analysis by Poincaré in his proof of the Uniformization Theorem for Riemann surfaces. The classical Schwarz-Pick Lemma shows that many results of classical function theory are more natural when expressed in terms of the hyperbolic metric than the Euclidean metric. For the history of hyperbolic geometry we refer to [5, 7, 18, 20, 28].

2.7. **F. Klein's Erlangen Program.** In his 1872 Erlangen Program, F. Klein outlined his vision of studying Geometry by use of algebraic notions. The main idea of this program is to view Geometry as the study of properties of a space invariant under the action of certain group of transformations.

The idea of Klein's Erlangen Program provides a uniform view of various geometries. The three classical geometries, the Euclidean, the hyperbolic, and the spherical geometry each has a metric, the Euclidean, the hyperbolic, and the spherical metric, respectively, that is invariant under a group of Möbius transformations. These three groups of transformations are isometric automorphisms of the respective spaces, the complex plane $\mathbb{C}$, the unit disk, and the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

2.8. **Quasihyperbolic metric.** For a proper subdomain $G$ of $\mathbb{R}^n$ and for all $x$, $y \in G$, the quasihyperbolic metric $k_G$ is defined as

$$k_G(x,y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z,\partial G)} |dz|,$$

where the infimum is taken over all rectifiable arcs $\gamma$ joining $x$ to $y$ in $G$ and $d(z,\partial G)$ denotes the Euclidean distance from the point $z$ to the boundary $\partial G$.

2.9. **Distance ratio metric.** For a proper open subset $G$ of $\mathbb{R}^n$ and for all $x, y \in G$, the distance ratio metric $j_G$ or $j$-metric is defined as

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right).$$

The distance ratio metric was introduced by Gehring and Palka [16] and in the above simplified form by Vuorinen [38]. Both definitions are frequently used in the study of hyperbolic type metrics [19], geometric theory of functions [39], and quasiconformality in Banach spaces [36].

Let $p \in G$, then for all $x, y \in G \setminus \{p\}$

$$j_{G\setminus\{p\}}(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G), |x-p|, |y-p|\}}\right).$$

This formula shows that the $j$-metric highly depends on the boundary of the domain.

It should be noted that in the case $G = \mathbb{B}^n$ or $G = \mathbb{H}^n$ the three metrics $\rho_G$, $k_G$, and $j_G$ can be compared as follows ([4, Lemma 7.56] and [39, Lemma 2.41] ):

$$\frac{1}{2}\rho_{\mathbb{B}^n}(x,y) \leq j_{\mathbb{B}^n}(x,y) \leq k_{\mathbb{B}^n}(x,y) \leq \rho_{\mathbb{B}^n}(x,y)\,, \forall\, x,y \in \mathbb{B}^n\,, \tag{2.10}$$

$$\frac{1}{2}\rho_{\mathbb{H}^n}(x,y) \leq j_{\mathbb{H}^n}(x,y) \leq k_{\mathbb{H}^n}(x,y) \equiv \rho_{\mathbb{H}^n}(x,y)\,, \forall\, x,y \in \mathbb{H}^n\,. \tag{2.11}$$

2.12. **Visual angle metric.** For the definition of the visual angle metric, we introduce some notation. Let $x, y \in \mathbb{R}^n$, $x \neq y$, and $0 < \alpha < \pi$. Let $m = (x+y)/2$ be the midpoint of the segment $[x, y]$ and $P_{xy}$ be the hyperplane orthogonal to $[x, y]$ and passing through $m$. Let $C(x, y, z)$ be the circle centered at $z \in P_{xy}$ containing $x$ and $y$. More precisely, if $z \neq m$, then $C(x, y, z) = S^{n-1}(z, r) \cap \Pi_{xyz}$, where $r = |z - x| = |z - y|$ and $\Pi_{xyz}$ stands for the plane passing through $x, y, z$; if $z = m$, then $C(x, y, z)$ is an arbitrary circle with diameter $[x, y]$.

Now denote

$$\mathcal{C}^{\alpha}_{xy} = \{C(x, y, z) \;:\; z \in P_{xy},\ 2|z - x| \sin\alpha = |x - y|\}\,.$$

Every circle $C \in \mathcal{C}^{\alpha}_{xy}$ contains the points $x$ and $y$, and therefore $C \setminus \{x, y\}$ consists of two circular arcs. We denote these two circular arcs by $\mathrm{comp}_{\alpha}(C)$ and $\mathrm{comp}_{\pi-\alpha}(C)$ and assume that the length of $\mathrm{comp}_{\alpha}(C)$ is equal to $2(\pi - \alpha)|x - z|$, see Figure 3. Then it is clear that

$$C = \{x\} \cup \{y\} \cup \mathrm{comp}_{\alpha}(C) \cup \mathrm{comp}_{\pi-\alpha}(C).$$

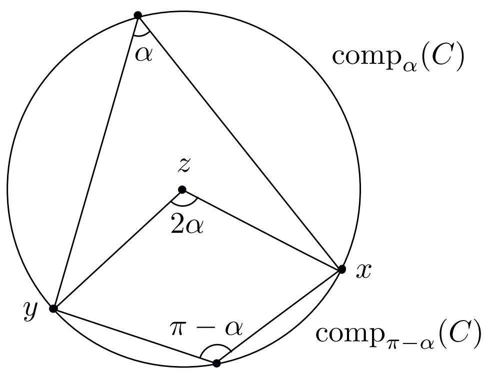


FIGURE 3. Components $\mathrm{comp}_{\alpha}(C)$ and $\mathrm{comp}_{\pi-\alpha}(C)$ of the circle $C$.

Finally, we define the $\alpha$-*envelope* of the pair $(x, y)$ to be

$$E^{\alpha}_{xy} = [x, y] \cup \left( \bigcup_{C \in \mathcal{C}^{t}_{xy},\ \alpha \le t < \pi} \mathrm{comp}_t(C) \right)$$

if $0 < \alpha < \pi$, $E^{0}_{xy} = \mathbb{R}^n$, and $E^{\pi}_{xy} = [x, y]$. For instance, in the case $n = 3$, this means that for $0 < \alpha < \pi/2$, the set $E^{\alpha}_{xy}$ is an "apple domain"; for $\alpha = \pi/2$, the closed ball $\overline{\mathbb{B}^n(m, |x - y|/2)}$ which is centered at $m$ with radius $|x - y|/2$; and for $\pi/2 < \alpha < \pi$, a "lemon domain".

It is not difficult to show that in fact

$$E^{\alpha}_{xy} = \{w \in \mathbb{R}^n \;:\; \measuredangle(x, w, y) \ge \alpha\}.$$

Now we are ready for the definition of the visual angle metric. Let $G \subsetneq \mathbb{R}^n$ be a domain and $x, y \in G$. We define a distance function $v_G$ by [III, Definition 2.21]

$$v_G(x,y) \;\;=\;\; \sup\left\{\alpha \;:\; E^{\alpha}_{xy} \cap \partial G \neq \emptyset\right\}.$$

The function $v_G \colon G \times G \to [0, \pi]$ is a similarity invariant pseudometric for every domain $G \subsetneq \mathbb{R}^n$. It is a metric unless $\partial G$ is a proper subset of a line and will be called the *visual angle metric* [III, Lemma 2.22].

2.13**. Möbius metric** (Seittenranta [34])**.** Let $G$ be an open subset of $\overline{\mathbb{R}^n}$ with card $\partial G \geq 2$. For all $x$, $y \in G$, the Möbius (or absolute ratio) metric $\delta_G$ is defined as

$$\delta_G(x,y) = \log(1 + \sup_{a,b \in \partial G} |a, x, b, y|).$$

It is a well-known basic fact that $\delta_G$ agrees with the hyperbolic metric in the case of the unit ball and in the case of the half space, see [39, Lemma 8.39].

2.14**. Hyperbolic type metrics.** As generalizations of the hyperbolic metric in general domains and in higher-dimensional spaces, many metrics have been introduced, and become popular tools in the study of modern mapping theory. For instance, the quasihyperbolic metric, the distance ratio metric, the visual angle metric, the Apollonian metric, the Möbius metric, and some weak metrics have been studied in [6, 8, 19, 21, 27, 32, 34, III, IV]. Furthermore, metrics based on conformal capacity have been studied in [34]. Väisälä's theory of quasiconformality in Banach spaces is based on the use of the quasihyperbolic metric [36]. On one hand, these metrics share several properties of the hyperbolic metric and are therefore sometimes called hyperbolic type metrics. On the other hand, these metrics differ from the classical hyperbolic metric in other respects.

It has turned out that for the hyperbolic type metrics we cannot any more expect the same invariance properties as in the classical cases, but still some type of "near-invariance" or "quasi-invariance" is a desirable feature. Therefore, it is natural to consider two problems: a) compare different hyperbolic type metrics; b) find the Lipschitz constants for these metrics under some common classes of mappings of the Euclidean space, e.g., conformal mappings or Möbius transformations. These two problems both belong to the important research theme: the estimate of the moduli of continuity of mappings between any two metric spaces, see [21, 27, 34, 38].

In this thesis, we only deal with the visual angle metric and the distance ratio metric, see [III] and [IV]. For other metrics, such as the Möbius metric, these problems are still open.

2.15**. Why do we need hyperbolic type metrics?** The basic idea of Euclidean geometry is that the space is unlimited and its local structure is similar everywhere. In GFT one often studies bounded domains and maps defined on theses domains. In this case the Euclidean geometry is no longer the natural geometry because of the boundary points. What we need is a "relative geometry" of the domain. In fact, we hope to replace the role of the point at infinity in Euclidean geometry with the boundary of the subdomain and to introduce a geometry of the subdomain that reflects this idea. This type of geometries can be constructed in several different ways, for instance by use of hyperbolic type metrics. These metrics are used in various contexts, for instance, in the characterizations of special classes of domains such as quasidisks, uniform domains, and John domains [14, 15, 26, 30, 37].

In the following four sections, we will describe the contents of each of the papers [I]–[IV].

## 3. Bisection of hyperbolic geodesic segments [I]

The bisection problem in the classical hyperbolic geometries is to find a point $z$ on the hyperbolic segment $J[x,y]$ such that $\rho_G(z,x) = \rho_G(z,y)$ ($G \in \{\mathbb{B}^2, \mathbb{H}^2\}$). It is easy to see that for $G = \mathbb{B}^2$ the hyperbolic midpoint of $J[0,x]$ ($x \in \mathbb{B}^2$) can be found by geometric construction [39, 14.1(2)]. G. Goodman-Strauss [17, Construction 3.1] gave a construction of the hyperbolic midpoint of a hyperbolic segment in a general position in $\mathbb{B}^2$ using an idea similar to the Euclidean construction, only substituting hyperbolic straightedge and compass for the Euclidean straightedge and compass. Another method was given by M. Vuorinen and R. Klén [22, 2.9], as a byproduct of studying the Apollonian circles and hyperbolic geometry. As far as we know, in hyperbolic geometry, constructions have been studied much less than in the Euclidean case, and to our surprise, we have not been able to find our results in paper [I] in the literature.

The main result of paper [I] is the following theorem.

3.1. **Theorem.** [I, Theorem 1.1, Theorem 1.2] *Given a pair of points in the upper half plane $\mathbb{H}^2$ or in the unit disk $\mathbb{B}^2$, one can bisect the hyperbolic segment joining the points by a geometric construction.*

It is a basic fact that the hyperbolic geometries of the half plane and of the unit disk are isometrically equivalent via Möbius transformations. Thus, it is natural to expect that a construction in one of these cases leads to a construction in the other case. However, our methods of construction, based on Euclidean compass and ruler, are not Möbius invariant. Because of this reason, we must treat these two cases separately.

Let $L(x,y)$ be the Euclidean line through the points $x\,,y \in \mathbb{R}^2$ and $L(z)$ be the Euclidean line through the point $z \in \mathbb{R}^2$ and orthogonal to $\partial\mathbb{H}^2$. Let $L_\rho[x,y]$ be the circular arc through the points $x\,,y \in \mathbb{H}^2$ and orthogonal to $\partial\mathbb{H}^2$.

We now describe the constructions of the bisection in the case of the upper half plane. For two distinct points $x,\ y \in \mathbb{H}^2$, let $\{x_*, y_*\} = L_\rho[x,y] \cap \partial\mathbb{H}^2$ such that $x_*, x, y, y_*$ occur in this order on $L_\rho[x,y]$. Let $S^1(a, r_a)$ be the circle through $x\,,y$ and orthogonal to $L_\rho[x,y]$. Moreover, let $\{w\} = L(x,y) \cap \partial\mathbb{H}^2$, $\{v\} = L(x,x_*) \cap L(y,y_*)$, and $\{u\} = L(x,y_*) \cap L(y,x_*)$. Then we find the hyperbolic midpoint $z$ of $J[x,y]$ by constructing the line $L(s)$ for $s \in \{u,v,a\}$ or the line through $w$ tangent to $L_\rho[x,y]$, see Figure 4. Denote $\{z_1\} = L(x,\overline{y}) \cap L(y,\overline{x})$, where $\overline{x}$, $\overline{y}$ are the complex conjugates of $x$ and $y$, respectively. Then the intersection of $J[x,y]$ and $L(z_1)$ is also

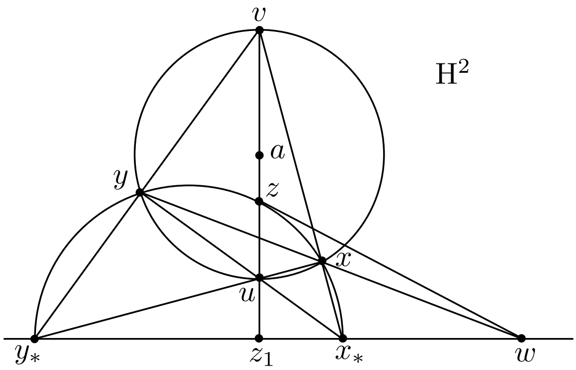


FIGURE 4. Bisection in $\mathbb{H}^2$: $z$ is the midpoint of hyperbolic segment $J[x,y]$.

the hyperbolic midpoint $z$ of $J[x,y]$. Moreover, we conclude that the four points $v\,,a\,,u\,,z_1$ are collinear and $u\,,v\in S^1(a,r_a)$.

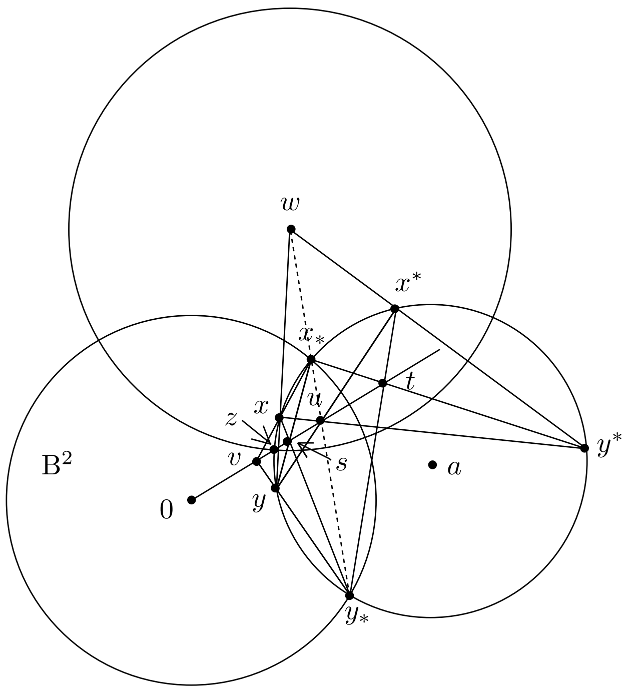


FIGURE 5. Bisection in $\mathbb{B}^2$: $z$ is the midpoint of hyperbolic segment $J[x,y]$.

We then turn to the bisection in the case of the unit disk. Let $x,y\in\mathbb{B}^2\setminus\{0\}$ such that $0,x,y$ are noncollinear. Let $S^1(a,r_a)$ be the circle through $x\,,y$ and orthogonal to the unit circle $S^1$. Let $\{x_*,y_*\}=S^1\cap S^1(a,r_a)$ such that $x_*,x,y,y_*$ occur in this order on $S^1(a,r_a)$. Let $\{w\}=L(x,y)\cap L(x^*,y^*)$, $\{u\}=L(x,y^*)\cap L(y,x^*)$, $\{v\}=L(x,x_*)\cap L(y,y_*)$, $\{s\}=L(x,y_*)\cap L(y,x_*)$, and $\{t\}=L(x_*,y^*)\cap L(y_*,x^*)$.

Then we find the hyperbolic midpoint $z$ of $J[x,y]$ by constructing the line $L(0,g)$ for $g \in \{u,v,s,t\}$ or the circle $S^1(w,r_w)$ which is orthogonal to $S^1(a,r_a)$, see Figure 5. If we let $\{k\} = L(x_*,x^*) \cap L(y_*,y^*)$, then the intersection of $J[x,y]$ and $L(0,k)$ is also the hyperbolic midpoint $z$ of $J[x,y]$. Moreover, we conclude that the five points $t\,,u\,,s\,,v\,,k$ are collinear, $u \in L(x_*,y_*)$, and $S^1(w,r_w)$ is orthogonal to $S^1$.

## 4. Hyperbolic Lambert quadrilaterals and quasiconformal mappings [II]

We first introduce the hyperbolic Lambert quadrilateral and the ideal hyperbolic quadrilateral, and then present our main results concerning these quadrilaterals.

Given a pair of points in the closure of the unit disk $\mathbb{B}^2$, there exists a unique hyperbolic geodesic line joining these two points. Hyperbolic lines are simply sets of the form $C \cap \mathbb{B}^2$, where $C$ is a circle perpendicular to the unit circle, or a Euclidean diameter of $\mathbb{B}^2$. For a quadruple of four points $\{a,b,c,d\}$ in the closure of the unit disk, we can draw these hyperbolic lines joining each of the four pairs of points $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, and $\{d,a\}$. If these hyperbolic lines bound a domain $D \subset \mathbb{B}^2$ such that the points $\{a,b,c,d\}$ are in the positive order on the boundary of the domain $D$, then we say that the quadruple of points $\{a,b,c,d\}$ determines a hyperbolic quadrilateral $Q(a,b,c,d)$ and that the points $a,b,c,d$ are its vertices. A hyperbolic quadrilateral with angles equal to $\pi/2, \pi/2, \pi/2, \phi\,(0 \le \phi < \pi/2)$, is called a hyperbolic *Lambert* quadrilateral [7, p.156], see Figure 6. Observe that one of the vertices of a Lambert quadrilateral may be on the unit circle $\partial\mathbb{B}^2$, in which case the angle at that vertex is $\phi = 0$. We say that a hyperbolic quadrilateral is an ideal hyperbolic quadrilateral if all the vertices are on the unit circle, and consequently all the angles are zero, see Figure 7. It is not difficult to observe that an ideal hyperbolic quadrilateral can be subdivided into four Lambert quadrilaterals.

Let $J^*[a,b]$ be the hyperbolic geodesic line with end points $a\,,b \in \partial\mathbb{B}^2$, and let $J[a,b]$ be the hyperbolic geodesic segment joining $a$ and $b$ when $a,b \in \mathbb{B}^2$, or the hyperbolic geodesic ray when one of the two points $a\,,b$ is on $\partial\mathbb{B}^2$ in the sequel.

Given two nonempty subsets $A,B$ of $\mathbb{B}^2$, let $d_\rho(A,B)$ denote the hyperbolic distance between $A$ and $B$, defined as

$$d_\rho(A,B) = \inf_{\substack{x\in A\\ y\in B}} \rho(x,y)\,,$$

where $\rho(x,y)$ stands for the hyperbolic distance in $\mathbb{B}^n$, see (2.5).

For the definition of quasiconformal maps we introduce the modulus of a curve family. Let $\Gamma$ be a family of curves in $\mathbb{R}^n$. By $\mathcal{F}(\Gamma)$ we denote the set of all non-negative Borel functions $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ such that

$$\int_\gamma \rho\, ds \ge 1$$

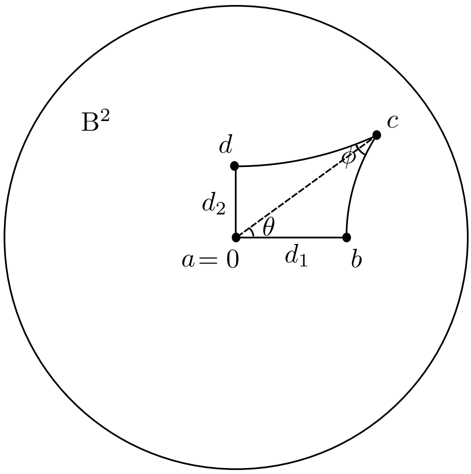


FIGURE 6. A hyperbolic Lambert quadrilateral in $\mathbb{B}^2$.

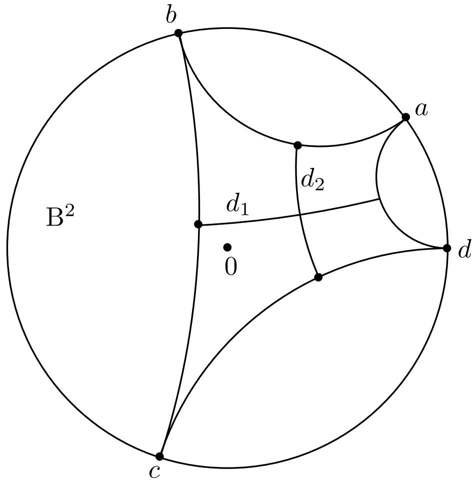


FIGURE 7. An ideal hyperbolic quadrilateral in $\mathbb{B}^2$.

for every locally rectifiable curve $\gamma \in \Gamma$. For each $p \geq 1$, we set

$$M_p(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^p \, dm,$$

where $m$ stands for the $n$-dimensional Lebesgue measure. If $\mathcal{F}(\Gamma) = \emptyset$, we set $M_p(\Gamma) = \infty$. The number $M_p(\Gamma)$ is called the $p$-modulus of $\Gamma$. If $p = n$, we denote $M_n(\Gamma)$ also by $M(\Gamma)$ and call it the modulus of $\Gamma$, see [35, 6.1].

Let $G\,, G'$ be domains in $\overline{\mathbb{R}^n}$ and let $f : G \to G'$ be a homeomorphism. Then $f$ is $K$-quasiconformal if

$$M(\Gamma)/K \leq M(f\Gamma) \leq K M(\Gamma), \quad K \geq 1,$$

for every curve family $\Gamma$ in $G$, see [35, 13.1] and [39, 10.9].

For $r \in (0,1)$ and $K \geq 1$, we define the distortion function

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K),$$

where $\mu(r)$ is the modulus of the planar Grötzsch ring, see [4, (5.1), (8.35)].

For the formulation of our results, we also need the constant [11, Theorem 1.10]

$$A(K) = 2\mathrm{arth}\left(\varphi_K\left(\mathrm{th}\frac{1}{2}\right)\right), \quad K \geq 1, \tag{4.1}$$

which satisfies

$$K \leq u(K-1) + 1 \leq \log(\mathrm{ch}(K \mathrm{arch}(e))) \leq A(K) \leq v(K-1) + K$$

with $u = \mathrm{arch}(e)\mathrm{th}(\mathrm{arch}(e)) > 1.5412$ and $v = \log(2(1 + \sqrt{1 - 1/e^2})) < 1.3507$. In particular, $A(1) = 1$.

In paper [II], we study bounds for the product and the sum of two hyperbolic distances between the opposite sides of hyperbolic Lambert quadrilaterals in the unit disk. Also, we consider the same product expression for the images of these hyperbolic Lambert quadrilaterals under quasiconformal mappings from the unit disk onto itself. In particular, we obtain similar results for ideal hyperbolic quadrilaterals. Note that the unit disk with vertices of an ideal quadrilateral on the unit circle can be conformally mapped onto a rectangle such that the vertices of the quadrilateral are mapped onto the vertices of the rectangle. The hyperbolic distance in a rectangle has been studied by A. F. Beardon [9].

The main results of paper [II] are as follows.

4.2. **Theorem.** [II, Theorem 1.1] *Let $Q(a\,,b\,,c\,,d)$ be a hyperbolic Lambert quadrilateral in $\mathbb{B}^2$ and let the quadruple of interior angles $(\frac{\pi}{2}\,,\frac{\pi}{2}\,,\phi\,,\frac{\pi}{2})$, $\phi\in[0,\pi/2)$, correspond to the quadruple $(a\,,b\,,c\,,d)$ of vertices. Let $d_1=d_\rho(J[a,d]\,,J[b,c])$, $d_2=d_\rho(J[a,b]\,,J[c,d])$ (see Figure 6), and let $L=\mathrm{th}\rho(a,c)\in(0,1]$. Then*

$$d_1 d_2 \le \left(\mathrm{arth}\left(\frac{\sqrt{2}}{2}L\right)\right)^2 .$$

*The equality holds if and only if $c$ is on the bisector of the interior angle at $a$.*

4.3. **Theorem.** [II, Theorem 1.2] *Let $Q(a\,,b\,,c\,,d)$, $d_1$, $d_2$, and $L$ be as in Theorem 4.2. Let $m=\sqrt{(2-L^2)(3L^2-2)}$, $r_0=\sqrt{\frac{1-m/L^2}{2}}$, and $r_0'=\sqrt{1-r_0^2}$.*

*(1) If $0<L\le\sqrt{\frac{2}{3}}$, then*

$$\mathrm{arth}\,L < d_1+d_2 \le \mathrm{arth}\left(\frac{2\sqrt{2}L}{2+L^2}\right).$$

*The equality holds in the right-hand side if and only if $c$ is on the bisector of the interior angle at $a$.*

*(2) If $\sqrt{\frac{2}{3}}<L<\sqrt{2(\sqrt{2}-1)}$, then*

$$\mathrm{arth}\,L < d_1+d_2 \le \mathrm{arth}\left(\frac{L(r_0+r_0')}{1+L^2 r_0 r_0'}\right).$$

*The equality holds in the right-hand side if and only if the interior angle between $J[a,b]$ and $J[a,c]$ is $\arccos r_0$ or $\arccos r_0'$.*

*(3) If $\sqrt{2(\sqrt{2}-1)}\le L<1$, then*

$$\mathrm{arth}\left(\frac{2\sqrt{2}L}{2+L^2}\right) \le d_1+d_2 \le \mathrm{arth}\left(\frac{L(r_0+r_0')}{1+L^2 r_0 r_0'}\right).$$

*The equality holds in the left-hand side if and only if $c$ is on the bisector of the interior angle at $a$. The equality holds in the right-hand side if and only if the interior angle between $J[a,b]$ and $J[a,c]$ is $\arccos r_0$ or $\arccos r_0'$.*

*(4) If* $L = 1$, *then*

$$d_1 + d_2 \geq \operatorname{arth}\left(\frac{2\sqrt{2}}{3}\right).$$

*The equality holds if and only if* $c$ *is on the bisector of the interior angle at* $a$.

In a Lambert quadrilateral, the angle $\phi$ is related to the lengths $d_1$, $d_2$ of the sides "opposite" to it as follows [7, Theorem 7.17.1]:

$$\operatorname{sh} d_1 \operatorname{sh} d_2 = \cos\phi.$$

See also the recent paper of Beardon and Minda [10, Lemma 5]. The proof of Theorem 4.2 yields the following corollary, which provides a connection between $d_1$, $d_2$ and $L = \operatorname{th}\rho(a, c)$.

4.4. **Corollary.** [II, Corollary 1.3] *Let* $L$, $d_1$, *and* $d_2$ *be as in Theorem 4.2. Then*

$$\operatorname{th}^2 d_1 + \operatorname{th}^2 d_2 = L^2.$$

By Theorem 4.2 and Theorem 4.3, we obtain the following corollary which deals with ideal hyperbolic quadrilaterals.

4.5. **Corollary.** [II, Corollary 1.4] *Let* $Q(a, b, c, d)$ *be an ideal hyperbolic quadrilateral in* $\mathbb{B}^2$. *Let* $d_1 = d_\rho(J^*[a, d], J^*[b, c])$ *and* $d_2 = d_\rho(J^*[a, b], J^*[c, d])$ *(see Figure 7). Then*

$$d_1 d_2 \leq \left(2\log(\sqrt{2}+1)\right)^2$$

*and*

$$d_1 + d_2 \geq 4\log(\sqrt{2}+1)\,.$$

*In both cases the equalities hold if and only if* $|a, b, c, d| = 2$.

The following Theorem 4.6 and Corollary 4.7 study the images of hyperbolic Lambert quadrilaterals and ideal hyperbolic quadrilaterals, respectively, under quasiconformal mappings from the unit disk onto itself.

4.6. **Theorem.** [II, Theorem 1.6] *Let* $f : \mathbb{B}^2 \to \mathbb{B}^2$ *be a* $K$*-quasiconformal mapping with* $f\mathbb{B}^2 = \mathbb{B}^2$ *and let* $Q(a, b, c, d)$, $d_1, d_2$, $L$ *be as in Theorem 4.2. Let* $A(K)$ *be as in (4.1) and* $f_L(r) = \frac{1-(Lr')^2}{r\operatorname{arth}(Lr)}$. *Denote* $D_1 = d_\rho(f(J[a, d]), f(J[b, c]))$ *and* $D_2 = d_\rho(f(J[a, b]), f(J[c, d]))$.

*(1) If* $0 < L \leq \frac{e^2-1}{e^2+1} \approx 0.761594$, *then*

$$D_1 D_2 \leq A(K)^2\left(\operatorname{arth}\left(\frac{\sqrt{2}}{2}L\right)\right)^{2/K}.$$

*(2) If* $\frac{e^2-1}{e^2+1} < L \leq 1$, *then let* $r_L = \frac{1}{L}\frac{e^2-1}{e^2+1} \approx \frac{0.761594}{L}$ *and*

$$M_L = \frac{f_L(\sqrt{1-r_L^2})}{f_L(r_L)} > 1.$$

*Let $r_L(K)$ be the unique solution $r$ to the equation $Kf_L(r) = f_L(\sqrt{1-r^2})$ with $r_L < r < 1$. Further, define*

$$T(x, L) = \operatorname{arth}(Lx)\left(\operatorname{arth}\left(L\sqrt{1-x^2}\right)\right)^{1/K},\ 0 < x < 1.$$

*Then*

$$D_1 D_2 \le A(K)^2 \max\left\{T(r_L(K), L),\ \left(\operatorname{arth}\left(\frac{\sqrt{2}}{2}L\right)\right)^{2/K}\right\}$$

*if $K > M_L$, and*

$$D_1 D_2 \le A(K)^2 \max\left\{T(r_L, L),\ \left(\operatorname{arth}\left(\frac{\sqrt{2}}{2}L\right)\right)^{2/K}\right\}$$

*if $1 \le K \le M_L$.*

4.7. **Corollary.** [II, Corollary 1.7] *Let $f : \mathbb{B}^2 \to \mathbb{B}^2$ be a $K$-quasiconformal mapping with $f\mathbb{B}^2 = \mathbb{B}^2$ and let $Q(a, b, c, d)$, $d_1, d_2$ be as in Corollary 4.5. Let $A(K)$ be as in (4.1) and $f_1(r) = \frac{1-(r')^2}{r\operatorname{arth}(r)}$. Denote $D_1 = d_\rho(f(J^*[a, d]), f(J^*[b, c]))$ and $D_2 = d_\rho(f(J^*[a, b]), f(J^*[c, d]))$. Further denote $r_1 = \frac{2\sqrt{e}}{e+1} \approx 0.886819$ and*

$$M_1 = \frac{(e-1)(\log(\sqrt{e}+1) - \log(\sqrt{e}-1))}{\sqrt{e}} \approx 1.46618\,,$$

*and define $r_1(K)$ to be the unique solution $r$ to the equation $Kf_1(r) = f_1(\sqrt{1-r^2})$ with $r_1 < r < 1$. With the notation*

$$T(x) = \operatorname{arth}(x)\left(\operatorname{arth}(\sqrt{1-x^2})\right)^{1/K},\ 0 < x < 1\,,$$

*we have*

$$D_1 D_2 \le A(K)^2 \max\left\{2^{1+1/K} T(r_1(K)),\ \left(2\log(\sqrt{2}+1)\right)^2\right\}$$

*if $K > M_1$, and*

$$D_1 D_2 \le A(K)^2 \max\left\{2^{1+1/K} T(r_1),\ \left(2\log(\sqrt{2}+1)\right)^2\right\}$$

*if $1 \le K \le M_1$.*

## 5. The visual angle metric and Möbius transformations [III]

The metrics introduced in [6] provide a way to connect the behavior of generalized angles to the behavior of quasimöbius embeddings. Classically one studies distortion of angles locally, "in the small", whereas in [6] this topic is studied "in the large". In paper [III], we give an alternative way to look at this topic by what we call the visual angle metric $v_G$. We compare the visual angle metric and the hyperbolic metric in the unit ball and the upper half space. We also obtain sharp Lipschitz constants with respect to $v_G$ under some Möbius transformations.

The main results of paper [III] are the following.

5.1. **Theorem.** [III, Theorem 1.1] *For $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$ and $x\,,y \in G$, let $\rho_G^*(x,y) = \arctan\left(\operatorname{sh}\frac{\rho_G(x,y)}{2}\right)$. Then*

$$\rho_G^*(x,y) \le v_G(x,y) \le 2\rho_G^*(x,y).$$

*The left-hand side of the inequality is sharp and the constant 2 in the right-hand side of the inequality is the best possible.*

It is not difficult to prove that $\rho_G^*(x,y)$ is a Möbius invariant metric for $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$ [III, Proposition 3.31]. Hence by Theorem 5.1, the visual angle metric, which is similarity invariant but not Möbius invariant, is not changed by more than a factor 2 under the Möbius transformations from $G$ onto $G'$ for $G\,, G' \in \{\mathbb{B}^n, \mathbb{H}^n\}$. The following theorems show the sharp Lipschitz constants for the visual angle metric under several Möbius transformations.

5.2. **Theorem.** [III, Theorem 1.2] *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a Möbius transformation. Then*

$$\sup_{\substack{f\in \mathcal{GM}(\mathbb{B}^n),\\ x\neq y\in\mathbb{B}^n}} \frac{v_{\mathbb{B}^n}(f(x),f(y))}{v_{\mathbb{B}^n}(x,y)} = 2.$$

5.3. **Theorem.** [III, Theorem 1.3] *Let $f : \mathbb{H}^2 \to \mathbb{B}^2 = f\mathbb{H}^2$ be a Möbius transformation. Then for all $x\,,y \in \mathbb{H}^2$*

$$v_{\mathbb{H}^2}(x,y)/2 \le v_{\mathbb{B}^2}(f(x),(y)) \le 2v_{\mathbb{H}^2}(x,y),$$

*and the constants $1/2$ and $2$ are both the best possible.*

5.4. **Theorem.** [III, Theorem 1.4] *Let $a\,,b\,,c\,,d \in \mathbb{R}$ and $ad-bc = 1$ and $c \neq 0$. Let $f : \mathbb{H}^2 \to \mathbb{H}^2$ be a Möbius transformation with $f(z) = \frac{az+b}{cz+d}$. Then*

$$\sup_{x\neq y\in\mathbb{H}^2} \frac{v_{\mathbb{H}^2}(f(x),f(y))}{v_{\mathbb{H}^2}(x,y)} = 2.$$

5.5. **Remark.** [III, Remark 4.6] *If $c = 0$ in Theorem 5.4, then $f(z) = a^2z + ab$. Therefore, it is clear that the Lipschitz constant under $f$ for the visual angle metric is always* 1.

## 6. Sharp Lipschitz constants of the distance ratio metric [IV]

The hyperbolic metric in the unit ball or the half space is Möbius invariant. However, neither the quasihyperbolic metric nor the distance ratio metric is invariant under Möbius transformations. Therefore, it is natural to ask what the Lipschitz constants are for these metrics under conformal mappings or Möbius transformations in higher dimension. F. W. Gehring, B. G. Osgood, and B. P. Palka proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [15, proof of Theorem 4] and [16, Corollary 2.5]:

6.1. **Theorem.** *If $D$ and $D'$ are proper subdomains of $\mathbb{R}^n$ and if $f$ is a Möbius transformation of $D$ onto $D'$, then for all $x, y \in D$*

$$\frac{1}{2}m_D(x,y) \le m_{D'}(f(x), f(y)) \le 2m_D(x,y),$$

*where $m \in \{j, k\}$.*

R. Klén, M. Vuorinen, and X.-H. Zhang studied the sharpness of the constant 2 in Theorem 6.1. They got the sharp bilipschitz constant $1+|a|$ for the quasihyperbolic metric under Möbius self-mappings of the unit ball [23, Theorem 1.4], and proposed a conjecture for the distance ratio metric.

6.2. **Conjecture.** [23, Conjecture 2.3] *Let $a \in \mathbb{B}^n$ and $h : \mathbb{B}^n \to \mathbb{B}^n = h\mathbb{B}^n$ be a Möbius transformation with $h(a) = 0$. Then*

$$\sup_{\substack{x,y\in\mathbb{B}^n \\ x\neq y}} \frac{j_{\mathbb{B}^n}(h(x), h(y))}{j_{\mathbb{B}^n}(x,y)} = 1 + |a|.$$

The positive answer to the above conjecture is due to S. Simić [IV, Theorem 1.5]. The sharp Lipschitz constants for the distance ratio metric under some common classes of mappings of the Euclidean space $\mathbb{R}^n$ $(n \ge 2)$ are also studied in paper [IV].

The set of complex numbers is also denoted by $\mathbb{C}$. We identify $\mathbb{R}^2 = \mathbb{C}$. The planar angular domain is defined as

$$S_\varphi = \{re^{i\theta} \in \mathbb{C} : 0 < \theta < \varphi\,,\, r > 0\}.$$

The following results of paper [IV] are due to G.-D. Wang.

6.3. **Theorem.** [IV, Theorem 2.4] *Let $f : \mathbb{H}^2 \to \mathbb{B}^2$ with $f(z) = \frac{z-i}{z+i}$.*

*(1) For all $x, y \in \mathbb{H}^2$*

$$j_{\mathbb{B}^2}(f(x), f(y)) \le 2j_{\mathbb{H}^2}(x,y),$$

*and the constant 2 is the best possible.*

*(2) For all $x, y \in \mathbb{H}^2 \setminus \{i\}$*

$$j_{\mathbb{B}^2\setminus\{0\}}(f(x), f(y)) \le 2j_{\mathbb{H}^2\setminus\{i\}}(x,y),$$

*and the constant 2 is the best possible.*

6.4. **Theorem.** [IV, Theorem 2.11] *Let $f : S_{\pi/m} \to \mathbb{H}^2$ with $f(z) = z^m$ $(m \in \mathbb{N})$. Then for all $x, y \in S_{\pi/m}$,*

$$j_{\mathbb{H}^2}(f(x), f(y)) \le m j_{S_{\pi/m}}(x,y),$$

*and the constant $m$ is the best possible.*

6.5. **Theorem.** [IV, Theorem 2.19] *Let $f(z) = a + r^2\frac{(z-a)}{|z-a|^2}$ be the inversion in $S^{n-1}(a,r)$ with $\operatorname{Im} a = 0$. Then $f(\mathbb{H}^n) = \mathbb{H}^n$ and for all $x, y \in \mathbb{H}^n$,*

$$j_{\mathbb{H}^n}(f(x), f(y)) \le 2j_{\mathbb{H}^n}(x,y).$$

*The constant 2 is the best possible.*

6.6. **Remark.** *In view of Theorem 6.1, we see that Theorem 6.3 and Theorem 6.5 only show the sharpness of the constant* 2*.*

## 7. Concluding remarks

As we have seen, hyperbolic type metrics play a crucial role in GFT and the quasiconformal mapping theory. There is a wide spectrum of open problems concerning the geometry of these metric spaces and homeomorphisms between two such spaces. Several open problems of this character were listed in [40]. While some of these problems have been studied and solved in [21, 23, 27, 34], the systematic study of these problems is still in its initial stages. The results of papers [III] and [IV] raise many questions about generalizations and we list here some of them.

1. Can we compare the visual angle metric to metrics other than the hyperbolic metric?
2. Describe hypotheses on the mapping $f$ and the domains $G, G'$ under which $f : (G, v_G) \to (G', v_{G'})$ is Lipschitz.
3. Very little is known about the geometry of the visual angle metric and many basic questions are open. For example, when are the balls $B_{v_G}(x,t) = \{y \in G : v_G(x,y) < t\}$ convex for all radii $t > 0$ or for small radii $t$; when are the boundaries of balls nice/smooth?
4. Can we get the same result of Theorem 6.4 for all $m > 1$?
5. Do the above Lipschitz continuity results have counterparts for other metrics such as the Möbius metric?

Also papers [III], [IV] list a few conjectures.

## References


[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1966.

[2] L. V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[3] L. V. Ahlfors, *Möbius Transformations in Several Dimensions*, Ordway Professorship Lectures in Mathematics, University of Minnesota, School of Mathematics, Minneapolis, Minn., 1981.

[4] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal Maps*, Canadian Mathematical Society Series of Monographs and Advanced Texts, a Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1997.

[5] J. W. Anderson, *Hyperbolic Geometry*, 2nd edition, Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, 2005.

[6] V. V. Aseev, A. V. Sychëv, and A. V. Tetenov, *Möbius-invariant metrics and generalized angles in Ptolemaic spaces*, Siberian Math. J. **46** (2005), 189–204.

[7] A. F. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Mathematics **91**, Springer-Verlag, New York, 1983.

[8] A. F. Beardon, *The Apollonian metric of a domain in* $\mathbb{R}^n$, Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), 91–108, Springer, New York, 1998.

[9] A. F. Beardon, *The hyperbolic metric in a rectangle II*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 143–152.

[10] A. F. Beardon and D. Minda, *Conformal images and the angle of parallelism*, Comput. Methods Funct. Theory **12** (2012), 87–100.

[11] B. A. Bhayo and M. Vuorinen, *On Mori's theorem for quasiconformal maps in the n-space*, Trans. Amer. Math. Soc. **363** (2011), 5703–5719.

[12] M.-M. Deza and E. Deza, *Encyclopedia of Distances*, 2nd edition, Springer-Verlag, Berlin, 2013.

[13] F. W. Gehring, *Rings and quasiconformal mappings in space*, Tran. Amer. Math. Soc. **103** (1962), 353–393.

[14] F. W. Gehring and K. Hag, *The Ubiquitous Quasidisk*, Mathematical Surveys and Monographs **184**, Amer. Math. Soc., Providence, RI, 2012.

[15] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** (1979), 50–74.

[16] F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains*, J. Analyse Math. **30** (1976), 172–199.

[17] C. Goodman-Strauss, *Compass and straightedge in the Poincaré disk*, Amer. Math. Monthly **108** (2001), 38–49.

[18] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries: Development and History*, W. H. Freeman and Co., San Francisco, Calif., 1973.

[19] P. Hästö, Z. Ibragimov, D. Minda, S. Ponnusamy, and S. K. Sahoo, *Isometries of some hyperbolic-type path metrics, and the hyperbolic medial axis*, In the tradition of Ahlfors-Bers. IV, 63–74, Contemp. Math. **432**, Amer. Math. Soc., Providence, RI, 2007.

[20] L. Keen and N. Lakic, *Hyperbolic Geometry from a Local Viewpoint*, London Mathematical Society Student Texts **68**, Cambridge University Press, Cambridge, 2007.

[21] R. Klén, *On hyperbolic type metrics*, Dissertation, University of Turku, Ann. Acad. Sci. Fenn. Math. Diss. No. **152** (2009), 49 pp.

[22] R. Klén and M. Vuorinen, *Apollonian circles and hyperbolic geometry*, J. Analysis **19** (2011), 41–60, arXiv:1010.2630v2 [math. MG].

[23] R. Klén, M. Vuorinen, and X.-H. Zhang, *Quasihyperbolic metric and Möbius transformations*, Proc. Amer. Math. Soc. (to appear), arXiv: 1108.2967 [math.CV].

[24] R. Kühnau, *Handbook of Complex Analysis: Geometry Function Theory. Vol 1*, North-Holland, Amsterdam, 2002.

[25] R. Kühnau, *Handbook of Complex Analysis: Geometry Function Theory. Vol 2*, Elsevier Science B. V., Amsterdam, 2005.

[26] N. Langmeyer, *The quasihyperbolic metric, growth, and John domains*, Ann. Acad. Sci. Fenn. Math. **23** (1998), 205–224.

[27] H. Lindén, *Quasihyperbolic geodesics and uniformity in elementary domains*, Dissertation, University of Helsinki, Ann. Acad. Sci. Fenn. Math. Diss. No. **146** (2005), 50 pp.

[28] J. Milnor, *Hyperbolic geometry: The first 150 years*, Bull. Amer. Math. Soc. (N. S.) **6** (1982), 9–24.

[29] D. Mumford, C. Series, and D. Wright, *Indra's Pearls: The Vision of Felix Klein*, Cambridge University Press, New York, 2002.

[30] R. Näkki and J. Väisälä, *John disks*, Exposition. Math. **9** (1991), 3–43.

[31] W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, U. S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.

[32] A. Papadopoulos and M. Troyanov, *Weak metrics on Euclidean domains*, JP J. Geom. Topol. **7** (2007), 23–44.

[33] Yu. G. Reshetnyak, *The Liouville theorem with minimal regularity conditions (Russian)*, Sibirsk. Mat. Zh. **8** (1967), 835–840.

[34] P. Seittenranta, *Möbius-invariant metrics*, Math. Proc. Cambridge Philos. Soc. **125** (1999), 511–533.

[35] J. Väisälä, *Lectures on n-dimensional Quasiconformal Mappings*, Lecture Notes in Mathematics **229**, Springer-Verlag, Berlin-New York, 1971.

[36] J. Väisälä, *The free quasiworld. Freely quasiconformal and related maps in Banach spaces*, Quasiconformal geometry and dynamics (Lublin, 1996), 55–118, Banach Center Publ. **48**, Polish Acad. Sci., Warsaw, 1999.

[37] J. Väisälä, *Uniform domains*, Tohoku Math. J. **40** (1988), 101–118.

[38] M. Vuorinen, *Conformal invariants and quasiregular mappings*, J. Analyse Math. **45** (1985), 69–115.

[39] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Mathematics **1319**, Springer-Verlag, Berlin, 1988.

[40] M. Vuorinen, *Geometry of metrics*, J. Analysis **18** (2010), 399–424.

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