THE INTERTEMPORAL CHOICE OF HOUSEHOLDS UNDER INCOMPLETE MARKETS IN GENERAL EQUILIBRIUM

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Contents

Part I

Introduction to the intertemporal choice of households under incomplete markets in general equilibrium

1 An overview: Quantitative macroeconomics with incomplete markets and heterogeneous agents

This study addresses the intertemporal choice of households in dynamic stochastic general equilibrium (DSGE) models under incomplete markets. The markets' incompleteness means that there is a lack of insurance against idiosyncratic risk, for example against the risk of unemployment, or/and there exists a limit for borrowing. Then, the idiosyncratic risk, combined with households' responses to the expected and prevailing circumstances, generates different wealth holdings among households. Hence, there exists an endogenously determined distribution of wealth which is at the heart of the analysis. These models are often also called heterogeneous agent models, since different shocks and different levels of wealth make households heterogeneous. In addition, households can differ from one another by their initial tastes and abilities. Besides the definition, the two main questions are as follows:

1. How, and under what circumstances, do these models work?
2. What are the consequences of incomplete markets – or heterogeneity among consumers – in DSGE models? That is, why do we need these types of models?

Let's find out.

In the 1970s macroeconomics focused on dynamic stochastic general equilibrium models, which were based on the optimization behavior of individuals, from an as yet prevailing agenda of estimating ad hoc aggregate macro models. Moreover, the guidelines for quantitative macroeconomics were set by the seminal work of Kydland and Prescott (1982): the parameters of the model must be consistent with micro estimates and then the predictions of the DSGE model should be compared against the actual data. The first generation of these models was based on the representative agent (or household) and the representative firm framework. There, the aggregate economy can be described *as if* one representative household makes the aggregate consumption, saving and labor supply decisions and, respectively, one firm creates the aggregate production. Under certain circumstances – which are discussed in detail by Acemoglu (2009, chap. 5) – heterogeneity among firms and households does not matter, but one can actually prove the "as if" part for the representative agent and firm. In this thesis I focus on households' intertemporal choice: hence, I will hold onto the assumption of representative firms.

There are two main assumptions that are needed for representing the "demand side" of the economy, as if it were based on the optimization behavior of a representative household/agent. The first one is that consumers must have "Gorman preferences": that is to say, preferences can be represented by the special indirect utility function (see Acemoglu, 2009, p. 151). Second, all idiosyncratic risks can be insured away in the style of Arrow-Debrau and this arrangement is known as the assumption of complete markets. That is, there is a full set of state-contingent claims which are exchanged among the agents such that all idiosyncratic risks can be insured. In the thesis, I am not interested in the demand of different types of commodities, but, rather, the focus is on the intertemporal choice of households. I will focus on the dynamics of variables which results from the decisions of economic actors under stochastic environment. Hence, I assume that households consume a single good or that they demand a basket of goods when utility function is not specifying preferences ordering over all commodities. Here, the utility function specifies preferences ordering over consumption levels at all dates.[1]

I relax the second assumption on the theory of representative agent, i.e. there is no risk sharing in the style of Arrow-Debrau. Discussion in Section 2 provides theoretical and empirical evidence for rejecting the complete markets assumption. The implications of complete markets assumption are extensive, as Lucas (1992, p. 246) puts it: "If the children of Noah had been able and willing to pool risks, Arrow-Debreu style, among themselves and their descendants, then the vast inequality we see today, within and across societies, would not exist." Further, there exists a large degree micro-econometric evidence about the heterogeneity of life, which is important for economic decisions. As Heckman (2001, p. 674) argues: "The most important discovery was the evidence on the pervasiveness of heterogeneity and diversity in economic life. When a full analysis of heterogeneity in responses was made, a variety of candidate averages emerged to describe the "average" person, and the long-standing edifice of the representative consumer was shown to lack empirical support." Moreover, the statistics provided by Budria, Diaz-Gimménez, Quadrini, and Ríos-Rull (2002) indicate a large dispersion of earnings and wealth, which needs to be explained somehow, and when this

[1]This is actually the reason why the utility function in this case is sometimes called the "felicity function".

heterogeneity is taken into consideration for the evaluations of different economic policies, it potentially has huge effects for policy recommendations. Thus, the incompleteness of markets and the heterogeneity among consumers cannot be ignored by economic theory. These facts provided the strong need for a heterogeneous agent model with incomplete markets.

The standard incomplete markets model consists of a typical consumer who faces idiosyncratic shocks to her wage rate (in addition to the aggregate shocks). The economy consists of a large number of these typical consumers who make their choices for consumption, saving and labor supply. The aggregate variables in the economy are determined as a function of these decisions. When these measures are combined with the decisions of the representative firm, the equilibrium prices can be found. As noted above, the production side of the economy is described by the standard neoclassical theory, but the problem of consumers have two essential features. First, the idiosyncratic uncertainty cannot be insured away, i.e. markets are incomplete. Second, there is some self-insurance mechanism. In the baseline case this mechanism is a precautionary saving motive. These two features will cause the standard incomplete markets model to differ significantly from its complete markets counterpart in many dimensions.

The earliest version of this setup was provided by Bewley (1983) and, therefore, sometimes these models are called as Bewley-models. More recently, Deaton (1991) and Carroll (1992, 1997) have provided a partial equilibrium model in the standard incomplete markets framework. There, the focus was on creating a theory that could explain the flaws of the permanent income hypothesis with respect to consumption behavior. Moreover, with heterogeneous agents one can provide solid micro foundations, which were revealed by micro-econometric studies for the model based on optimization. The partial equilibrium framework is discussed in Section 3. A second approach is to take a standard macro model (growth or business cycle) and add the features of the incomplete markets model into it, as done by Huggett (1993); Aiyagari (1994); Krusell and Smith (1998). A detailed discussion on these models is given in Section 4. This approach tries to find out how heterogeneity and market incompleteness matter for the aggregate economy and economic policy. In the early stages, these two standards of literature seemed to be quite separate, but these days these two somewhat different approaches have converged. As a result, we have a solid micro-founded theory for the aggregate economy.

There are two ways to model the incomplete markets economy. Heathcote, Storesletten, and Violante (2009b) call them "model what you can micro found" and "model what you can see". The first one is based on models introduced in Section 2, when the market incompleteness is micro-founded and endogenously determined. The second approach is utilized in the thesis and in the models in Section 3 and 4 – as well as in Parts II,III and IV. In this approach, the borrowing constraint and imperfect insurance are given exogenously and the focus is on the consequences of these features. However, recent models have unified these two approaches, as did Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007).

To conclude, the introduction to the first question, which was asked at the beginning of this section, I emphasize that quantitative macro economics is based on computational methods since closed form solutions are not available. Moreover, parameters for these numerically solved models must be derived from actual data: one can use calibration or estimation or both. These important issues are discussed in Section 5. Now, I turn to the second question.

The general principles for constructing a theoretical model for macroeconomic level are as follows:[2]

- Rationality and micro foundations. The model is based on optimization behavior and standard microeconomic behavior. The preferences and technology are given.
- Dynamics. Saving and investment decisions are determined by intertemporal decisions.
- Stochasticity. There is stochastic aggregate shocks that generate – at least partly – the business cycles of the economy.
- General equilibrium. The model has a general equilibrium setup, so that the factor prices and interest rates are endogenous.

These items basically provide foundations for any DSGE model. Furthermore, any model that aims to evaluate the desirability and effects of economic policy must meet these requirements as argued by Lucas (1976). However, as motivated by the discussion above, a new argument needs to be included on the list:

- Heterogeneity and market incompleteness. The model has a heterogeneous

[2] This list follows a similar list by Krusell and Smith (2006).

population structure where consumers differ in wealth and face idiosyncratic shocks against which they cannot fully insure themselves.

The last item on the list is important for many reasons. Here are some reasons that come to my mind for why incomplete markets and heterogeneous agents should be considered in a DSGE setup:

1. The dynamics and levels of aggregate variables. For example, Chang and Kim (2007) found that the dynamics of labor supply can be explained by a standard incomplete markets model. Heathcote (2005) found that a tax cut that does not matter in the representative agent model causes large real effects, i.e. Ricardian equivalence does not hold, in a model with heterogeneous agents and incomplete markets. Further, the incomplete markets and heterogeneous agents model explain a large part of the equity premium puzzle (see Storesletten, Telmer, and Yaron, 2007). Hence, these types of models have significant implications for asset pricing. Moreover, the well known excess sensitivity and smoothness puzzles related to consumption can be partly explained by the incomplete markets model. Ludvigson and Michaelides (2001) provides a partial equilibrium approach and Part II in the thesis provides a discussion using a general equilibrium framework. Moreover, the study in Part III shows that a precautionary saving motive may matter significantly for the level of aggregate capital stock.

2. Welfare questions. Introducing inequality into macro models enables a much more realistic analysis of welfare questions, since there then exist both rich and poor households. For example, the cost of business cycles can now be studied carefully, see for example (see, for example, Krusell, Mukoyama, Sahin, and Smith, 2009). Moreover, now the forces behind inequality can be modeled and they may matter for aggregate dynamics as well (see, for example, Castañeda, Díaz-Giménez, and Ríos-Rull, 2003).

3. Social security and taxation. This is related to the previous item. Questions which are related to social security and taxation are usually related to inequality. Hence, the incomplete markets model with heterogeneous agents provides a great tool for studying these questions. Some recent examples can be found from Krueger (2006), Golosov, Tsyvinski, and Werning (2006), Kocherlakota (2010) and in Part IV.

It can be concluded that many relevant questions in economic theory/policy cannot be addressed without the heterogeneous agent/households model. The essays in this thesis give more cases where market incompleteness and the heterogeneity among agents significantly affect the behavior of the aggregate economy. Hence, these essays do their own part in answering the second question on the list presented at the beginning of this section. In Section 6, I provide concluding remarks for Part I and Section 7 provides a short summary for the essays included in this thesis.

Generally, in this first part I discuss the unifying theme of the thesis. The unifying theme is the underlying model, which is applied in each of the essays in this thesis. Put differently, the essays in this thesis are closely interrelated with each other via a common model framework, that is to say, each of the essays in the thesis is based on the incomplete markets model in a general equilibrium framework. I will focus here on the general principles that are present in all of the essays, but I do not explicitly connect the essays to one another in this part. On the contrary, here I study in detail issues that are not discussed in the essays, but they are the most important matters to understand in order to know how the standard incomplete markets model works. Furthermore, this first part also introduces the most central part of the literature associated with the thesis. So, this part gives "the big picture of the thesis" which underlies the model framework and – as mentioned – which connects the different essays to one another. As I will show, the incomplete markets model provides a rich environment in which the most fundamental aspects of the intertemporal choice of households can be studied and these aspects have important consequences for the behavior of the aggregate economy.

2 Market structure and risk sharing: Micro foundations for the incomplete markets model

In this Section, I study exchange economies in which state contingent claims are available.[3] A benchmark case is the Arrow Debrau economy where a perfect insurance against idiosyncratic shocks can be provided by using state contingent claims. However, if some frictions – commitment problems or asymmetric information – are allowed in the market environment, a perfect insurance cannot be provided anymore. Then aggregate measures cannot be used for finding prices and one cannot use the representative agent framework to model aggregate consumption behavior. Hence, these frictions give us micro foundations for the incomplete markets model, which are considered in Sections 3 and 4.

2.1 Complete markets

Here, I describe a competitive equilibria for a pure exchange economy and show that with state contingent claims, it is sufficient to study the representative agent. The Arrow-Debrau market structure is used in which a complete set of dated contingent claims are traded at time 0. However, it can be shown that the consumption allocations are the same if a sequential-trading structure is used.

The economy is populate by N individuals indexed by $i \in \{1, 2, \dots, N\}$. The planning horizon for individuals is T and $T = \infty$ is allowed. Every individual i has a stochastic process $\{y_t^i\}$ which delivers the endowments of the consumption good. Let $y^{i,t} = (y_0^i, y_1^i, \dots, y_t^i)$ denote a history of endowment shocks. This process depend on s_t, which is a stochastic event for the whole economy. Hence, it can be written $s_t = (y_t^i, \dots, y_t^N)$, when the certain aggregate state implies different incomes for individuals. Or, to refer to individual income, I write $y_t^i(s^t)$ where $s^t = (s_0, \dots, s_t)$. Moreover, by $\pi_t(y^{i,t})$ I denote the objective probability of the event history of an individual and $\pi(s_t)$ represents the probability of aggregate events. Conditional probabilities are denoted by $\pi_t(s^t \mid s^\tau)$, which is the probability of observing s^t conditional upon the realization of s^τ. Let $y_t^i \in Y$ be a finite set of cardinality M and $s_t \in S$ be a finite set of cardinality K. Then $S^t = S \times S \times \dots \times S$ is a $t+1$-fold Cartesian product of S and Y^t is defined analogously, hence, $s^t \in S^t$ and $y^t \in Y^t$ for all i. Finally, assume the event s_0 as given or $\pi_0(s_0) = 1$.

[3] The main sources for this section are Ljungquist and Sargent (2004) and Krueger (2007).

The agents have identical preferences: an additively time-separable von Neumann Morgenstern expected utility function with a discount factor $\beta \in (0,1)$. That is,

$$u^i(c^i) = \sum_{t=0}^{T} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) U^i \left[c_t^i(s^t)\right] = E_0 \sum_{t=0}^{T} \beta^t U^i(c_t^i), \tag{1}$$

where the utility of every i depends upon the aggregate event history-dependent consumption plan $c^i = \{c_t^i(s^t)\}_{t=0,s^t \in S^t}^T$. Moreover, E_0 is the mathematical expectation operator conditioned on s_0. Assume now that the periodic utility functions U^i are twice continuously differentiable, strictly increasing, strictly concave and satisfy Inada conditions

$$\lim_{c \to 0} U_c^i(c) = \infty, \tag{2}$$

$$\lim_{c \to \infty} U_c^i(c) = 0, \tag{3}$$

where U_c^i is the derivate of U respect to c.

Finally, note that the consumption allocation $\{(c_t^i(s^t)_{i \in I})\}_{t=0,s^t \in S^t}^T$ is feasible if

$$c_t^i(s^t) \geq 0 \quad \forall \quad i, t, s^t, \tag{4}$$

$$\sum_{i=1}^{N} c_t^i(s^t) = \sum_{i=1}^{N} y_t^i(s^t) \quad \forall \quad t, s^t. \tag{5}$$

2.1.1 Pareto problem

I may find Pareto optimal allocations by posing a Pareto problem for a fictitious social planner. These allocations operate as a benchmark against allocations provided by a market economy. The consumption allocation $\{(c_t^i(s^t)_{i \in I})\}_{t=0,s^t \in S^t}^T$ is Pareto efficient if it is feasible and if there is no other feasible allocation $\{(\hat{c}_t^i(s^t)_{i \in I})\}_{t=0,s^t \in S^t}^T$ such that

$$u^i(\hat{c}^i) \geq u^i(c^i) \quad \text{for all} \quad i \in I, \tag{6}$$

$$u^i(\hat{c}^i) > u^i(c^i) \quad \text{for some} \quad i \in I. \tag{7}$$

Pareto efficient allocations are the solution to the planners' problem

$$\max_{\{(c_t^i(s^t)_{i\in I})\}_{t=0,s^t\in S^t}^T} \sum_{i=1}^N \lambda^i u^i(c^i) \tag{8}$$

subject to (4) and (5), for some Pareto weights $(\lambda^i)_{i=1}^N$ satisfying $\lambda^i \geq 0$ and $\sum_{i=1}^N \lambda^i = 1$. By attaching Lagrangian multipliers to the resource constraint and by ignoring the non-negativity constraints the Lagrangian for this problem is

$$L = \sum_{t=0}^T \sum_{s^t\in S^t} \left\{ \sum_{i=1}^N \lambda^i \beta^t \pi_t(s^t) U^i\left[c_t^i(s^t)\right] + \theta_t(s^t) \sum_{i=1}^N \left[c_t^i(s^t) - y_t^i(s^t)\right] \right\} \tag{9}$$

and the first order condition for an optimum is

$$\lambda^i \beta^t \pi_t(s^t) U_c^i\left[c_t^i(s^t)\right] \quad = \quad \theta_t(s^t) \quad \forall \quad i \in I \tag{10}$$

hence, for $i, j \in I$

$$\frac{U_c^i\left[c_t^i(s^t)\right]}{U_c^j\left[c_t^j(s^t)\right]} \quad = \quad \frac{\lambda^j}{\lambda^i} \quad \forall \quad t, s^t. \tag{11}$$

That is to say, the marginal utilities of consumption for any two agents are constant across state and time, which shows that there is a perfect consumption insurance.

Rewriting (11) gives

$$c_t^i(s^t) \quad = \quad U_c^{i^{-1}}\left[\frac{\lambda^j}{\lambda^i} U_c^j\left[c_t^j(s^t)\right]\right] \quad \forall \quad t, s^t. \tag{12}$$

By using (5) and assuming the CRRA utility $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$ I can define a household's j consumption as follows:

$$c_t^j(s^t) \quad = \quad \alpha^j \sum_{i\in I} y_t^i(s^t) \equiv \alpha^j y_t(s^t) = \alpha^j c_t(s^t) \quad \forall \quad t, s^t \tag{13}$$

$$\text{where} \quad \alpha^j \quad = \quad \frac{\lambda^{j^{\frac{1}{\sigma}}}}{\sum_{i=1}^N \lambda^{i^{\frac{1}{\sigma}}}} \geq 0. \tag{14}$$

$y_t(s^t)$ and $c_t(s^t)$ are aggregate income and consumption and if $\sigma = 1$ then $\alpha^j = \lambda^j$. Moreover, one can find λ's by using Negishi's method (see, Ljungquist and Sargent

(2004, p. 216)). Finally, (13) gives the growth rate of consumption between any state and time as follows:

$$\log\left(\frac{c_{t+1}^{j}(s^{t+1})}{c_{t}^{j}(s^{t})}\right) = \log\left(\frac{c_{t+1}(s^{t+1})}{c_{t}(s^{t})}\right), \tag{15}$$

i.e. individual consumption growth and aggregate consumption growth are perfectly correlated under these assumptions.

2.1.2 The competitive equilibrium with Arrow-Debrau securities

Now, households can trade dated history-contingent claims to consumption. With complete markets, I suppose that there is a complete set of these securities. All trades with these claims are done at $t = 0$ when s_0 has been realized. Households can trade claims at time t consumption contingent on history s^t at price $p_t(s^t)$. That is, the price $p_t(s^t)$ is the period 0 price of one unit of period t consumption which is delivered if s^t has been realized. Hence, the household budget constraint is as follows:

$$\sum_{t=0}^{T}\sum_{s^t\in S^t} p_t(s^t)c_t^i(s^t) \leq \sum_{t=0}^{T}\sum_{s^t\in S^t} p_t(s^t)y_t^i(s^t) \tag{16}$$

The households' problem is to maximize (1) subject to (16) and (4). A competitive Arrow Debrau equilibrium is founded if the allocations $\{(c_t^i(s^t)_{i\in I})\}_{t=0,s^t\in S^t}^{T}$ and the prices $\{p_t(s^t)\}_{t=0,s^t\in S^t}^{T}$ are such that

1. Given $\{p_t(s^t)\}_{t=0,s^t\in S^t}^{T}$ for each $i \in I$, $\{(c_t^i(s^t)_{i\in I})\}_{t=0,s^t\in S^t}^{T}$ maximizes (1) subject to (16) and (4).
2. $\{(c_t^i(s^t)_{i\in I})\}_{t=0,s^t\in S^t}^{T}$ satisfies (5) $\forall\ t, s^t$.

The first order condition for the households' problem gives for $i, j \in I$

$$\frac{U_c^i\left[c_t^i(s^t)\right]}{U_c^j\left[c_t^j(s^t)\right]} = \frac{\mu^j}{\mu^i} \quad \forall \quad t, s^t, \tag{17}$$

where μ's are the Lagrange multipliers associated with budget constraint (16) and it can be shown that an equilibrium allocation solves this equation. By setting $\mu^{i,j^{-1}} = \lambda^{i,j}$, one can see that the allocation is a particular Pareto allocation and one can solve prices by Neghishi method. Hence, the allocations for the Pareto problem and the competitive equilibrium reflect the two fundamental theorems of welfare (see Mas-Colell, Whinston, and Green, 1995, chap. 16).

2.1.3 The theory of representative agent

To derive the theory of representative agent, I rewrite (17) for individual i as follows

$$\frac{p_t(s^{t+1})}{p_t(s^t)} = \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \frac{U_c^i\left[c_{t+1}^i(s^{t+1})\right]}{U_c^i\left[c_t^i(s^t)\right]}. \tag{18}$$

Equation (15), together with the CRRA utility function, then implies that (18) can be rewritten as

$$\frac{p_t(s^{t+1})}{p_t(s^t)} = \beta \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \left(\frac{c_{t+1}(s^{t+1})}{c_t(s^t)}\right)^{-\sigma}. \tag{19}$$

The equilibrium Arrow-Debrau prices can be written as a function of aggregate consumption only. Hence, this shows that with complete markets, it is sufficient to study the representative agent economy only – one only has to care about aggregate measures.

The implications of perfect insurance can be tested. Equation (15) implies that individual consumption growth should be perfectly correlated with aggregate consumption growth, or one should not be able to forecast individual consumption growth with individual income growth when the aggregate consumption (or income) growth is controlled. However, tests by Mace (1991),[4] Cochrane (1991), Townsend (1994) and Attanasio and Davis (1996) reject the complete market hypothesis. Hence, there must be some frictions within the market environment that prevent the existence of a complete market solution.

2.2 The complete markets model with frictions

Here, it is assumed that there is a full set of contingent claims, as there was in Section 2.1. However, now there are some frictions that prevent the perfect

[4] Mace argues that with exponential utility complete markets are a good first order approximation of reality, but with CRRA-utility the complete markets hypothesis is soundly rejected. Hence, Mace partially accepts the complete market hypothesis. However, generally her results are altered by measurement errors in the data.

insurance against idiosyncratic shocks. I consider two types of imperfections: first, financial contracts are only imperfectly enforceable since there is a possibility of bankruptcy and second, the verification of the state of each individual is her private information when the payments cannot be directly conditioned on self-reported values, since individuals might lie. The effect of these frictions is that perfect risk sharing is not feasible anymore, and therefore, there is only a partial insurance against idiosyncratic risk.

2.2.1 Enforcement problems

Now the environment is the same as before, but $\{1, 2\} \in I$. Hence, there are two agents that are seeking an optimal insurance system, but there is limited enforcement when one of the agents can go to autarky and consume her stochastic endowment y_t^i forever. Moreover, the stochastic process is assumed to be a Markov process. I can approach this type of problem from a game-theoretic approach as Kocherlakota (1996) did, but here I choose a social planner approach for the problem.

The social planner's problem is the same Pareto problem as in Section 2.1.1, but now the maximization of (8) is subject to additional constraint, i.e. the participation constraint:

$$E\left[\sum_{d=t}^{\infty} \beta^{d-t} U^i(c_d^i) \mid s_t\right] \geq E\left[\sum_{d=t}^{\infty} \beta^{d-t} U^i(y_d^i) \mid s_t\right] \quad \forall \quad i, t, s_t. \tag{20}$$

Both of the agents must get a higher expected utility for participating in the exchange arrangement than from autarky for every date and state. Moreover, assume that there exists functions ω^i such that

$$\omega^i(s_t) = E\left[\sum_{d=t}^{\infty} \beta^{d-t} U^i(y^i(s_d)) \mid s_t\right] \quad \forall \quad i, t, s_t. \tag{21}$$

Hence, the problem for the social planner is as follows:

$$\max_{\left\{c_t^1(s^t),c_t^2(s^t)\right\}_{t=0,s^t\in S}^{\infty}} = E_t \sum_{t=0}^{\infty} \beta^t \left(\lambda^1 U^1(c_t^1) + \lambda^2 U^2(c_t^2)\right) \tag{22}$$

$$\text{s.t.} \quad E\left[\sum_{d=t}^{\infty} \beta^{d-t} U^i(c_d^i) \mid s_t\right] \geq \omega^i(s_t) \quad \forall \quad i,t,s_t \tag{23}$$

$$c_t^1 + c_t^2 = y^1(s_t) + y^2(s_t) = y_t \quad \forall \quad t. \tag{24}$$

The problem is non-standard from the point of view of stochastic dynamic optimization since constraint (23) makes the consumption for each day depend on the history of the entire state, hence, it is not a time-invariant function of s_t. However, Marcet and Marimon (1992, 1998) show how to form a Lagrangian for this type of problem, where a sequence of stochastic Lagrangian multipliers can be thought of as a state. Another way to formulate the problem is to use the promised utility as a state variable.

The Lagrangian for the problem is

$$\begin{aligned} L \;=\; & E_t \left[\sum_{t=0}^{\infty} \beta^t \left(\lambda^1 U^1(c_t^1) + \lambda^2 U^2(y_t - c_t^1)\right) + \sum_{t=0}^{\infty} \beta^t \theta_t^1 \left(\sum_{d=0}^{\infty} \beta^d U^1(c_{t+d}^1) - \omega^1(s_t) \right) \right. \\ & \left. + \sum_{t=0}^{\infty} \beta^t \theta_t^2 \left(\sum_{d=0}^{\infty} \beta^d U^2(y_{t+d} - c_{t+d}^1) - \omega^2(s_t) \right) \right], \end{aligned} \tag{25}$$

where $\left\{(\theta_t^i)_{i\in I}\right\}_{t=0}^{\infty}$ is a stochastic process of nonnegative Lagrange multipliers on participation constraints. I can rewrite this Lagrangian by using the equality of two square summable sequences of real numbers,[5] as follows:

[5] This proof for the equality was originally given by Paul Klein in his lecture notes. Let $\{\phi_i\}_{i=0}^{\infty}$ and $\{\psi_i\}_{i=0}^{\infty}$ be two square summable sequences of real numbers. Then I can write

$$\begin{aligned} \sum_{i=0}^{\infty} \psi_i \left(\sum_{j=0}^{\infty} \phi_{i+j} \right) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \phi_{i+j} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} I_{i\leq j} \psi_i \phi_j = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} I_{i\leq j} \psi_i \phi_j \\ &= \sum_{j=0}^{\infty} \phi_j \sum_{i=0}^{\infty} I_{i\leq j} \psi_i \sum_{j=0}^{\infty} \phi_j \left(\sum_{i=0}^{j} \psi_i \right) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \psi_j \right) \phi_i. \end{aligned}$$

where $I_{i\leq j} = 1$ if $i \leq j$ otherwise $I_{i\leq j} = 0$.

$$
\begin{aligned}
L &= E_t\left[\sum_{t=0}^{\infty}\beta^t\left(\lambda^1U^1(c_t^1)+\lambda^2U^2(y_t-c_t^1)\right)+\sum_{t=0}^{\infty}\beta^t\eta_t^1U^1(c_t^1)-\sum_{t=0}^{\infty}\beta^t\theta_t^1\omega^1(s_t)\right.\\
&\quad\left.+\sum_{t=0}^{\infty}\beta^t\eta_t^2U^2(y_t-c_t^1)-\sum_{t=0}^{\infty}\beta^t\theta_t^2\omega^2(s_t)\right] \qquad (26)\\
&\quad \text{where}\quad \eta_t^i=\sum_{d=0}^{t}\theta_d^i\quad i=1,2.
\end{aligned}
$$

One can think of η_{t-1}^i as state variables in period t, i.e. $\eta_t^i = n_{t-1}^i + \theta_t^i$.

The first order conditions with respect to c_t^1 are as follows:

$$
\frac{U_c^1(c_t^1)}{U_c^2(y_t-c_t^1)} = \frac{\lambda^2+\eta_t^2}{\lambda^1+\eta_t^1} \quad \forall \quad t, s_t. \qquad (27)
$$

Hence, the marginal utilities of agents do not depend just on the Pareto weights λ^i, but also on the aggregate state or the individual's endowments, if η_t^i gets a positive value. That is, marginal utilities are not constant anymore across state and time, which implies that there is no perfect insurance against idiosyncratic uncertainty. A special case is a situation in which the participation constraint never binds for all i, t and s_t, when equation (27) equals to equations (11) and (17). Then there is a perfect insurance against uncertainty and one has the theory of representative agent. Generally, the binding frequency of the participation constraint depends on the parameterization of the model.

The effects of limited commitment is also studied within a general equilibrium framework. Kehoe and Levine (1993, 2001) considered the Arrow Debrau market structure and Alvarez and Jermann (2000) studied a case of sequential trading in which there is a continuum of agents. For a recent application of this setup see, for example, Krueger and Perri (2006, 2009).

2.2.2 Asymmetric information

The second friction in insurance markets comes from the asymmetric information. Assume that agents are fully committed to enduring and binding contracts but that those same agents have private information about their own income. Hence, the problem for the social planner is to set up an incentive-compatible contract that makes agents report their incomes truthfully. An analysis for this type of problem is given by Thomas and Worral (1990) by using promised utility as a state variable. Due to constraints on space, I do not introduce another framework, but, rather, I skip directly to the results. It can be shown that this type of problem leads to a declining consumption path (see Ljungquist and Sargent, 2004, chap. 19.5). That is, the optimal time-invariant path for consumption is not possible since the incentive problems are present.

The effects of moral hazard in a general equilibrium framework is studied by Phelan (1995) and Atkeson and Lucas (1992). Moreover, there are applications in public finance for this type of setup (see, for example, Golosov, Tsyvinski, and Werning, 2006; Kocherlakota, 2010). This strand of the literature is referred to as the "(new) dynamic Mirrlees" approach to public finance.

2.3 Relationship between market structure and incomplete market models

Given some frictions on the market structure, it is possible to show – as was done above – that a perfect insurance against idiosyncratic risk cannot be provided. The form of the optimal contract will depend on the parametrization of the model and some insurance can be provided. However, empirical studies rejected the hypothesis of complete markets as a reasonable description of reality. That is, the enforcement problem and the problems caused by the asymmetric information imply a need to model the consumption behavior of households without the representative agent.

However, the models referred to in this section are quite complicated and I need a simplifying assumption for studying the consequences of imperfect insurance against idiosyncratic shocks within a typical macro framework. To do that, I assume that there is no insurance provided by markets, but that agents must insure themselves. In the following discussion, I assume that agents can only self-insure themselves by holding some uncontingent asset. In some models – with a particular parametrization – an optimal contract closely resembles the one-period uncontingent bonds which agents trade in typical macro models. In any case, this section provided micro foundations for models with incomplete markets, where a precautionary saving (or self-insurance) motive occurs, since there is a lack of insurance against idiosyncratic risk. Even if there is a perfect set of state contingent assets, the consequences of frictions in these markets destroy the possibility of perfect insurance. Empirical tests also favor this conclusion.

The discussion in this section provided the motivation to model the intertemporal (saving and consumption) choice of households with fluctuating income within a framework in which there is no insurance against these fluctuations. Admittedly, it is a simplifying assumption in which the lack of insurance is taken to an extreme, but at the same time it gives a reasonable approximation of reality.

3 The saving problem of consumers under incomplete markets

In this section, I assume that consumers are completely cut off from all insurance markets, but that there is a single risk-free asset which they can borrow and lend with a given interest rate.[6] Hence, models in this section are partial equilibrium models. The absence of insurance against idiosyncratic shocks causes consumers to hold assets as a means of self-insurance against income shocks. That is, under certain conditions consumers have a precautionary saving motive. In this section, I will study in detail why and when the precautionary saving motive occurs for consumers. Second, market imperfection – or market incompleteness – can also mean that borrowing is somehow limited. I will consider the effects of borrowing constraints on consumers' behavior and combine it with a precautionary saving motive. Finally, the main theoretical results for the saving problem of consumers under incomplete markets in a partial equilibrium setup are summarized.

Consider now a similar environment as in Section 2: consumers maximize their expected utility, which is discounted with $\beta = \frac{1}{1+\rho}$, by choosing the consumption of a single good c. The saving problem of consumers is

$$\max_{\{c_t\}_{t=0}^{T}} E_0 U = E_0 \sum_{t=0}^{T} \beta^t U(c_t) \tag{28}$$

$$\text{s.t.} \quad a_{t+1} = (1+r)(a_t + y_t - c_t) \quad \forall \quad t, \tag{29}$$

where a is their asset holdings and the rate of return is r. Consumers are endowed with a random sequence $\{y_t\}_{t=0}^{T}$ of the consumption good. Obviously, the problem is also constrained by a proper no-Ponzi-scheme condition and $T = \infty$ is allowed.

3.1 Incomplete markets, utility function and precautionary saving

Incomplete markets – or an uncertainty associated with the sequence of endowments (labor income) – creates a precautionary saving motive for consumers. However, the appearance of a precautionary saving motive depends on the choice of the utility function. Moreover, intuition says (probably for most economists)

[6] The main sources for this section are Deaton (1992), Ljungquist and Sargent (2004) and Krueger (2007).

that the marginal propensity to consume out of wealth declines as wealth increases. Here, I discuss which type of time-separable, von-Neumann-Morgenstern-expected utility function provides a result according to intuition. A theoretical discussion for this section is provided by Carroll and Kimball (1996). Here, the discussion is done via some simple examples.

Generally the first order conditions, i.e. Euler equations, for problem given equations (28) and (29) are

$$U_c(c_t) = \left(\frac{1+r}{1+\rho}\right) E_t\left[U_c(c_{t+1})\right] \quad \forall \quad t \tag{30}$$

and the characteristics of U_c define the intertemporal choice – or the saving and consumption behavior – of households under incomplete markets.

3.1.1 Quadratic utility

I begin with the famous implication of the permanent income hypothesis, as shown by Hall (1978): consumption follows a martingale process. However, this occurs only under very special circumstances: expectations are formed rationally, $r = \rho$, $T = \infty$ and U is a quadratic function, for example $U = -\frac{1}{2}(c_t - \bar{c})^2$, where $\bar{c}$ represents a bliss level of consumption so large that consumers cannot afford $c_t = \bar{c}$.[7] Then equation (30) becomes

$$c_t = E_t\left[c_{t+1}\right] \Leftrightarrow c_{t+1} = c_t + u_{t+1}, \tag{31}$$

where u_{t+1} is an innovation or martingale difference. If it is assumed that shocks have an i.i.d. structure, then consumption follows a random-walk process. But, generally, it can be stated that consumption follows a martingale process. However, given these assumptions, the time path of consumption *under certainty* is $c_{t+1} = c_t$ and it is identical to the time path given by (31).[8] Hence, consumption obeys the certainty equivalence.

The certainty equivalence result implies that the periods' $t+1$ consumption, c_{t+1}, is perfectly predicted by c_t, when the realization of y_{t+1} does not directly matter for c_{t+1}, but only for the unexpected part of it, i.e. the part that is described by u_{t+1}. When u_{t+1} is realized for the consumer, it will affect her

[7] Assume that no-Ponzi scheme condition binds before the consumption get at the level $\bar{c}$.
[8] Obviously, the realized consumption depends on innovation u_{t+1} when income is stochastic.

consumption, but this information cannot be anticipated with the period t information. That is, the changes in current consumption are orthogonal with lagged changes in income if these changes in income are known for the consumer before the current period.

There is a vast amount of literature that tests the implications of certainty equivalence from micro and macro data. Once again, due to space limitations, a detailed discussion is not possible. However, this evidence is summarized by Deaton (1992), who concludes that the certainty equivalence hypothesis should be rejected. Two well-known reflections from the hypothesis arise: the excess sensitivity and the excess smoothness of aggregate consumption contradict the certainty equivalence.[9] Generally, micro evidence also rejects the certainty equivalence hypothesis, since the same contradictions present in the aggregated data are also present in the micro data.

The implications of certainty equivalence are only possible in a very limited environment when these assumptions rule out potentially important factors that affect consumption. Next, I turn to a situation in which marginal utility is not linear. Beginning with Leland (1968), Sandmo (1970) and Sibley (1975), it has been known that the convexity of marginal utility causes a precautionary saving motive for households.

3.1.2 The case of CARA utility function

Here, I follow Caballero (1990, 1991) where he assumes that the utility function is given by $U(c_t) = -\frac{1}{\theta}e^{-\theta c_t}$, where θ is the coefficient for consumers' (constant) absolute risk aversion (CARA). Moreover, assume now that $\rho = r = 0$, horizon is finite and the income process follows a random-walk with normally distributed innovations whose variance is σ^2.

Then equation (30) becomes

$$c_{t+1} = c_t + \frac{\theta\sigma^2}{2} + u_{t+1}. \tag{32}$$

[9] On the one hand, the changes in consumption should be orthogonal to predictable income changes, but the correlation between consumption growth and lagged income growth has been found to be one of the most robust features of aggregate data. This feature of consumption is referred to as the excess sensitivity of consumption. On the other hand, consumption growth should be more volatile than income growth if the aggregate income growth has a positive serial correlation, which it does, but aggregate consumption growth is in fact much smoother than aggregate income growth. This feature is called the excess smoothness of consumption.

Now consumption follows a random-walk with a drift term $\frac{\theta\sigma^2}{2}$, when a higher variance or higher absolute risk aversion pushes tomorrow's consumption upward. Hence, consumers have a precautionary saving motive since uncertainty causes consumers to postpone consumption. That is, uncertainty creates additional consumption growth when compared to the case of certainty (equivalence).

However, the result given by the CARA utility function is only suggestive since there is nothing to prevent consumption being negative. To see this, I solve the consumption function, which gives the level of consumption as

$$c_t = \frac{a_t}{T+1-t} + y_t - \frac{\theta(T-t)\sigma^2}{4}. \tag{33}$$

Now it is obvious that consumption can be negative if assets and income are low and the variance is high. Moreover, the precautionary saving motive does not depend on the level of assets, but only on the uncertainty associated with labor income, which goes against intuition. Uncertainty is associated with y_t, but it has the same effect on consumption regardless of the level of assets, even if consumption could be financed completely with certain capital income. Hence, I need preferences which rule out the negative consumption and make the precautionary saving motive depend on the level of wealth.[10] The type of utility functions which obey these demands are utility functions with the property of constant relative risk aversion (CRRA). Unfortunately, very few analytical solutions are then available, but approximations or numerical methods must be used to find a solution.

3.1.3 CRRA utility and prudence

The most widely used utility functions for the saving problem of consumers under incomplete markets are utility functions which are of a CRRA-type, i.e. there is a constant relative risk aversion. For example, $U = \frac{c_t^{1-\theta}}{1-\theta}$ where θ is the coefficient for relative risk aversion. As I noted before, very few analytical solutions are available,[11] but I use a second-order Taylor approximation of $E_t U_c(c_{t+1})$ at c_t to

[10] However, an interesting result is given by Wang (2003) in a general equilibrium framework with a CARA-utility. He shows that in equilibrium consumption follows the same rule given by the permanent income hypothesis.

[11] However, by assuming an appropriate stochastic process some qualitative results could be derived.

show the properties of equation (30) with CRRA preferences.[12] This yields

$$E_t\left[\frac{c_{t+1}-c_t}{c_t}\right] = \frac{1}{\xi}\left(\frac{r-\rho}{1+r}\right)+\frac{\zeta}{2}E_t\left[\left(\frac{c_{t+1}-c_t}{c_t}\right)^2\right] \quad \text{or} \tag{34}$$

$$E_t\left[\frac{c_{t+1}-c_t}{c_t}\right] = \frac{1}{\theta}\left(\frac{r-\rho}{1+r}\right)+\frac{\theta+1}{2}\mathrm{VAR}_t\left[\frac{c_{t+1}-c_t}{c_t}\right]+u_{t+1} \tag{35}$$

where $\xi = \frac{-c_t U_{cc}}{U_c}$ and $\zeta = \frac{-c_t U_{ccc}}{U_{cc}}$.

Let me now compare this result with the previous ones. If $r = \rho$, then the first term vanishes and the expected consumption growth is determined by the coefficient of relative risk aversion and the variance of consumption growth. That is, the higher the relative risk aversion or the variance of consumption growth, the higher the expected growth of consumption.The precautionary saving motive, captured by the second term in the right hand side of equations (34) and (35), causes consumers to postpone consumption. Moreover, now the precautionary saving motive depends on the variance of consumption and not income when, with the high values of assets, the importance of the variance term diminishes.

Note that the variance term disappears if $U_{ccc} = 0$. Hence, the variance of consumption matters for the consumption decisions of consumers if and only if $U_{ccc} > 0$. Thus, a sufficient and necessary condition for precautionary saving is strictly convex marginal utility. It was Kimball (1990) who gave a measure of prudence to denote the "propensity to prepare and forearm oneself in the face of uncertainty", i.e. prudence defines the intensity of the precautionary saving motive. More precisely, the Arrow-Pratt measure of relative risk aversion is given in equation (34) by ξ and the relative prudence is given by ζ. The risk aversion depends upon the concavity of the utility function, but the prudence depends upon the convexity of the marginal utility function.

Both CARA- and CRRA-type utility functions produce a precautionary saving motive since $U_{ccc} > 0$. It should be clear that, by prudence, I refer to preferences and that precautionary saving refers to consumers' behavior. Furthermore, it is clear that CRRA-utility functions produce consumption behavior under incom-

[12] For a discussion on using Taylor approximation to describe the Euler equations see Carroll (2001a), Carroll (1997) and Dynan (1993). The Taylor approximation in this case gives

$$\frac{1+r}{1+\rho}E_t U_c(c_{t+1}) = \frac{1+r}{1+\rho}E_t\left[U_c + U_{cc}\left(c_{t+1}-c_t\right)+\frac{1}{2}U_{ccc}\left(c_{t+1}-c_t\right)^2\right].$$

plete markets which seems to be the most natural way to think about consumption: when keeping uncertainty fixed, an increase in assets reduces the effects of uncertainty for consumption. However, precautionary savings can occur due to the presence of other market imperfections. One potential candidate is the existence of borrowing constraints.

3.2 Precautionary saving and borrowing constraint

Until now, it has been assumed that consumers can borrow up to some arbitrarily large amount, which does not violate the no-Ponzi scheme condition. The first market imperfection was the lack of insurance against idiosyncratic risk. Now I introduce the second imperfection, which is a borrowing limit. For example, evidence provided by Zeldes (1989) and Jappelli (1990) shows that the presence of borrowing constraint matters for consumers' behavior.[13]

A positive probability for defaults, which may arise from commitment problems, provides a micro foundation for borrowing constraint. More precisely, Stiglitz and Weiss (1981) give a strong reason for why credit markets may not clear. High interest rates are most tempting for borrowers who will most likely default on their loans, which increases the overall default risk in the lenders' portfolio. Hence, the interest rate is not a price which clears the loan market, but, rather, a lender ratio loans by another means. Then there will be borrowers who are willing to loan, but who cannot. Second, the consumption behavior implied by certainty equivalence implies such a behavior of assets which raises questions about the model itself or shows that borrowing restrictions are likely to be an issue eventually. Appendix A shows that, under the environment given in Section 3.1.1 assets follow a random-walk when assets will eventually exceed any borrowing limit. This may not violate the no-Ponzi scheme condition if it is set carefully. However, this result requires the assumption that borrowing constraints are never binding.

Hence, there is some theoretical and empirical grounds for considering the effects of borrowing constraint when consumers solve their saving problem. The problem is given by equations (28) and (29). Now I add a second constraint, i.e.

[13]Often the borrowing constraint is also referred to as a liquidity constraint. For clarity, I just use borrowing constraint here.

the borrowing constraint, which states that

$$a_{t+1} \geq \underline{a} \quad \forall \quad t. \tag{36}$$

That is, assets must be equal to or higher than some arbitrary level $\underline{a}$ for every t.

There is no simple way to set $\underline{a}$, but often the ad-hoc borrowing constraint is such that consumers cannot borrow at all, i.e. $a_{t+1} \geq \underline{a} = 0$. Second, one may use another constraint, $c_t \geq 0$ for all t, when the budget constraint implies the so-called natural debt limit. This was given by Aiyagari (1994). The natural debt limit can be derived from the budget constraint by assuming that defaults are not allowed, i.e. consumers must always be able to pay the interest rates on their debts and still $c_t \geq 0$. Let $a_{t+1} = a_t = a_{min} \leq 0$, which is the maximum value of debt that consumers can be holding, and assume that $c_t \simeq 0$. Then, with the lowest possible income realization y_{low}, the budget constraint (29) implies that the natural borrowing limit is

$$a_{t+1} \geq a_{min} = -\left(1 + \frac{1}{r}\right) y_{low}. \tag{37}$$

So, all of the income is then used for the interest payments on the debt.

3.2.1 Precautionary saving with quadratic utility

The Euler equation for the problem given by equations (28), (29) and (36) is as follows:

$$\begin{aligned} U_c &\geq \frac{1+r}{1+\rho} E_t\left[U_c(c_{t+1})\right] \\ &= \frac{1+r}{1+\rho} E_t\left[U_c(c_{t+1})\right] \quad \text{if} \quad a_{t+1} > \underline{a}. \end{aligned} \tag{38}$$

Hence, the Euler equation holds with the equivalence sign if the liquidity constraint is not binding. Otherwise, the Euler equation holds with the inequality sign. To analyze Euler equations a bit further under quadratic preferences, define a variable "cash in hand", x_t, which defines the maximum amount of consumption that a consumer can afford at the moment.[14]

[14] We must now assume that income shocks are i.i.d.

The budget constraint then gives

$$
\begin{aligned}
x_t &= y_t + a_t - \frac{a_{t+1}}{1+r} && (39)\\
&\le y_t + a_t - \frac{\underline{a}}{1+r}. && (40)
\end{aligned}
$$

Hence, consumption cannot be greater than x_t and the marginal utility of consumption cannot be less than $U_c(x_t)$. If the borrowing constraint is binding then

$$
\begin{aligned}
U_c(c_t) &= U_c(x_t), \quad \text{but} && (41)\\
U_c(x_t) &< U_c(c_t) = \frac{1+r}{1+\rho} E_t U_c(c_{t+1}) \quad \text{if} \quad a_{t+1} > \underline{a}, && (42)
\end{aligned}
$$

where the strict concavity of utility function is used in the last line. Thus, the Euler equation (38) can be written as

$$
\begin{aligned}
U_c(c_t) &= \max\left\{U_c(x_t), \frac{1+r}{1+\rho} E_t U_c(c_{t+1})\right\} && (43)\\
&\quad \text{and with quadratic utility} \\
-(c_t - \bar{c}) &= \max\left\{-(x_t - \bar{c}), -\frac{1+r}{1+\rho}(E_t c_{t+1} - \bar{c})\right\} \quad \text{and if} \quad r = \rho && (44)\\
c_t &= \min\left\{x_t, E_t c_{t+1}\right\}. && (45)
\end{aligned}
$$

Several implications can be observed. First, the certainty equivalence solution given by equation (31) holds if with a probability 1 asset holdings satisfy $a_t \ge \underline{a}$ for every t. This, however, depends critically upon the income process and, given that income shocks are i.i.d., this condition is violated, as shown in Appendix A. Second, if there exists a period $t+s$, where consumer is borrowing constrained with a positive probability when $x_{t+s} < E_{t+s} c_{t+s}$, this implies, by using the law of iterated expectations, that $c_t < E_t c_{t+s}$. Hence, potentially binding borrowing constraints in the future matter for today's consumption. That is, it does not matter if the constraint is binding at the moment: the mere existence of the constraint already matters if it is binding with a positive probability, and that depends upon the stochastic process of the shocks. Moreover, Flodén (2008) and Park (2006) showed that even with deterministic income current consumption is lower if the borrowing constraint is binding in the future. Third, if the variance of future income increases, it makes more low and high realizations of y_{t+1} more likely. This in turn increases the probability of the borrowing constraint being binding in the future, which lowers today's level of consumption. Hence, the increased in current saving is a consequence of the increased uncertainty associated with future income. Thus, with quadratic preferences, a precautionary saving motive occurred, which originated with the existence of borrowing constraint.

Here it was shown that the existence of borrowing constraint, combined with a concave utility function, creates a precautionary saving motive when income is uncertain, even if U_c is linear. Hence, prudence is not necessarily needed to obtain a precautionary saving motive. The interpretation of a precautionary saving motive may be based on utility function, i.e. prudence, or it may depend on institutions, that is, the function of the credit markets prevent borrowing up to a certain limit. However, it is likely that both of these features create a precautionary saving motive. Next I consider both of them together.

3.3 Combining prudence and borrowing constraints: The intertemporal choice of households under uncertainty

Here, I summarize some main theoretical findings which are known regarding the saving problem of households/consumers with borrowing constraints and non-quadratic preferences. First, I focus on the so-called buffer-stock saving behavior, in which assets are used as a buffer against negative income shocks. This behavior occurs in a partial equilibrium setup and the critical assumption is that $r < \rho$, i.e. consumers' optimal path of consumption would, under complete markets, be decreasing and that, potentially, they would want to borrow. Second, I turn to a more general case and show that the solution to consumers' saving problem crucially depends on the relative magnitudes of interest rate, r, and the subjective time preference, ρ.

3.3.1 The buffer-stock saving behavior

Assume that consumers' preferences are CRRA-type and that the problem of consumers is given by equations (28), (29) and (36). Deaton (1991) showed, given $\rho > r$, that the consumption function over cash in hand, x_t, is concave when under the certainty equivalence it is linear. Moreover, the effects of bad income shocks can be set off by running assets holdings down. However, if consumer faces several poor income draws, and if assets are already exhausted, there is nothing consumers can do but let consumption decrease. Hence, assets holdings only provide a partial insurance against income shocks.

The second approach to similar results, as provided by Deaton (1991), was first suggested by Carroll (1992, 1997). The difference between these two approaches is that Carroll does not assume a borrowing constraint, but there is a well-defined lower bound for income, y_{low}. The difference between these models is that, with Deaton's model, the borrowing constraint may be binding, but with Carroll's model it is not. However, with CRRA-utility, a lower boundary for income implies a natural-borrowing limit and, under certain circumstances, the results are equivalent (see, Carroll, 2001b; Michaelides, 2003).[15]

To characterize buffer-stock behavior, I follow Carroll's approach when I do not have an explicit borrowing constraint. Moreover, I now assume that g is the growth rate of permanent income P_t and everything is divided by the level of permanent income when I can define the cash in hand to permanent income ratio as $\frac{X_t}{P_t} = x_t$. The approximated Euler equation for this type of model is given in equation (34). The expected growth rate of consumption is given by $E_t\left[\frac{C_{t+1}-C_t}{C_t}\right]$, which is now plotted in Figure 1 as a function of the cash in hand to permanent income ratio, x. Many important aspects of buffer-stock saving behavior can be understood by considering Figure 1. the theoretical proof for Figure 1 and its implications are given by Carroll (2004).

In Figure 1 the expected consumption growth is negatively associated with cash in hand: the higher the income or asset, the lower the expected growth rate of consumption. Under certainty equivalence, when the variance term in the Euler equation is zero, the expected consumption growth rate is $\frac{1}{\xi}\left(\frac{r-\rho}{1+r}\right)$, and it is negative, since $\rho > r$. With non-quadratic preferences, the expected growth of consumption will approach this growth rate when $x \to \infty$. Hence, at some point consumers are dissaving, which in turn causes x to decrease. However, with small values for x, i.e. if assets and income are low, the expected growth rate of consumption is high and consumers are saving. Hence, this type of saving behavior leads to a constant gross wealth ratio $x^\star$ where $E_t x_{t+1} = x_t$. That is, consumers have a target level of assets $x^\star$ where the consumption growth rate is equal (almost) to the growth rate of income.[16]

[15] If $y_{low} = 0$, then the natural borrowing limit is $\underline{a} = 0$, which is the one used by Deaton (1991).

[16] A concavity of the consumption function implies that the expected growth rate of consumption is strictly less than the growth rate of income g at $x^\star$ for every individual. However, if the economy is

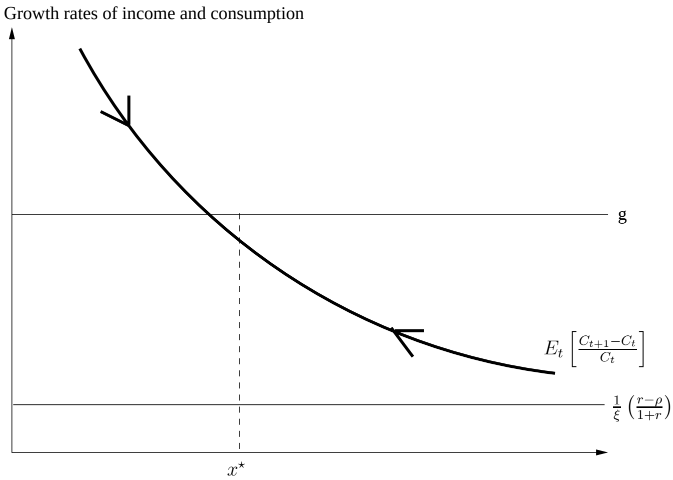

Figure 1: The expected growth rate of consumption as a function of the cash in hand to permanent income ratio, x. Source: Carroll (1997).

To summarize, it may be said that the variance of consumption growth declines when cash in hand increases, that is to say, when consumers have a better opportunity to insure themselves against income shocks. A lower level of variance in turn implies a lower expected consumption growth such that current consumption is high and consumers are dissaving (when $x_t > x^\star$). Hence, the impatience ($\rho > r$) is pulling the level of asset downwards. However, with the low levels of cash in hand ($x_t < x^\star$), the precautionary saving motive increases asset holdings, which in turn increases the expected consumption growth rate. That is, the precautionary saving motive pushes the assets upward. As a result, consumers have a target wealth ratio $x^\star$, which operates as a buffer against income shocks. Finally, it should be noticed from micro and macro data that consumption growth should be predicted with variables which affect the variance of the growth of consumption – it is not constant or zero, as assumed in many empirical studies.

The buffer-stock saving behavior explains some features that are detected in aggregated consumption data (see, Ludvigson and Michaelides, 2001) and, in general it describes the evolution of consumption in the early state of the life-cycle, as shown by Gourinchas and Parker (2002) and Cagetti (2003). Next, I turn to a general case and show that the assumption $\rho > r$ is indeed needed in order to find a well-defined solution for the saving problem of consumers in

populated by consumers who follow the buffer-stock saving behavior, then the *aggregate* consumption growth is equal to the aggregate income growth.

circumstances of this type.

3.3.2 A General case

Now, I consider in general a saving problem of consumers which is given by equations (28), (29) and (36). As shown by Chamberline and Wilson (2000), among others, the assumption $\rho > r$ is needed to find a well-defined solution for this type of problem. An important feature of this solution is that $T = \infty$, if T is finite, I do not encounter any problems that might occur with an infinite horizon, but c_t and a_t remains bounded and I have a well-defined solution. However, if the horizon is infinite and if $\rho \leq r$, then c_t and a_t will grow without any boundaries, even if the income process is stationary.

To see this, assume that for every period an endowment, y_t, is realized for the consumer from a finite set S and that the elements for the sequence of endowments are i.i.d. Moreover, assume that in state $s \in S$, an endowment y_s is realized for the consumer with $\Pr(y = y_s) = \Pi_s \geq 0$ and $\sum_{s \in S} \Pi_s = 1$. Let me now consider the Euler equation for this problem

$$\begin{aligned} U_c(c_t) &\geq \frac{1+r}{1+\rho} E_t\left[U_c(c_{t+1})\right] \quad \text{and when} \quad r > \rho && (46) \\ &> \frac{1+r}{1+\rho} E_t\left[U_c(c_{t+1})\right]. && (47) \end{aligned}$$

The marginal utility is strictly positive and follows a supermartignale. By martingale convergence theorem (see, Ljungquist and Sargent, 2004, p. 560) it follows that a sequence of random variables $\{U_c(c_t)\}$ converges almost surely to limit the random variable $U_c(c)$. Further, equation (46) implies that

$$E_t\left[U_c(c_{t+1})\right] \leq \left(\frac{1+\rho}{1+r}\right)^t U_c(c_0) \quad \forall \quad t, \tag{48}$$

and since $U_c(c_0)$ is strictly positive, it must be the case that $U_c(c)$ convergence almost surely to $U_c(c) = 0$. That is,

$$\begin{aligned} \lim_{t\to\infty} c_t &= \infty \quad \text{almost surely and} && (49) \\ \lim_{t\to\infty} a_t &= \infty \quad \text{almost surely.} && (50) \end{aligned}$$

Since y_t is stationary, then to finance infinite consumption, the asset must also be infinite.

The same result applies if $r = \rho$. I follow Aiyagari (1994) or Ljungquist and Sargent (2004) to derive the result. Assume that cash in hand, defined as $x_{t+1} = (1+r)(x_t - c_t) + y_t$ or $x_t = a_{t+1} + c_t$, gets its highest value in $\bar{x}$ when $y_t = y_{max}$, which is the maximum value of income from set S. Then I may write a first order condition (46) using the value function, V, and $a_{t+1}(x)$, which is an optimal decision rule, as follows:

$$V_x(\bar{x}) \geq E_t \left[V_x \left((1+r) a_{t+1}(\bar{x}) + y_{t+1} \right) \right] \tag{51}$$

$$> V_x \left((1+r) a_{t+1}(\bar{x}) + y_{max} \right) = V_x(\bar{x}) \tag{52}$$

which is a contradiction. The strict concavity of V implies strict inequality and the equality comes from the definition for cash in hand. Hence, a limit does not exist, but assets and consumption would here as well diverge to infinity.

Now, assume that $\rho > r$, when equation (48) implies that $U_c(c)$ does not need to converge to zero, but it does leave open the possibility that consumption fluctuates around some finite mean. However, it is quite difficult to prove that the state space is bounded above, even when $\rho > r$. But, this proof is needed since I want to use recursive methods to solve the consumers' saving problem within a general equilibrium setup. I show here in a quite abstract way that the state space has an upper boundary by following Krueger (2007).[17] However, Appendix B derives more intuitively a sufficient condition for the bounded state space.

Now suppose that the marginal utility function has such a property that there exists a finite e_{U_c} such that

$$\lim_{c \to \infty} \left(\log_c U_c(c) \right) = e_{U_c}, \tag{53}$$

where e_{U_c} is a number which is referred to as the asymptotic exponent of U_c. If e_{U_c} exists, then there also exists an $\tilde{x}$ such that $x_{t+1} = a_{t+1}(x) + y_{max} \leq x$ for all $x \geq \tilde{x}$. To apply this theorem, assume a CRRA-utility when equation (53) gives

$$\log_c c^{-\theta} = -\theta \log_c c = -\theta. \tag{54}$$

[17] Schechtman and Escudero (1977) provided the original proofs. See also Aiyagari (1993) for proofs.

But, with a CARA-utility, we get

$$\log_c e^{-c} = -c \log_c e = -\frac{c}{\ln(c)} \quad \text{when} \tag{55}$$

$$\lim_{c \to \infty} -\frac{c}{\ln(c)} = -\infty \tag{56}$$

and the previous statement does not apply.

Now, it may be concluded that there exists a well-defined solution for the saving problem of consumers when the horizon is infinite, and when income shocks follow a finite state Markov with an i.i.d. structure and when there is a *decreasing* absolute risk aversion. The borrowing constraint limits the state space from below and $\rho > r$, combined with CRRA-utility, bound the state space from above. Given these assumptions, cash in hand – or assets and consumption – stays in a bounded set. Thus, I may use the familiar tools of dynamic programming to solve the saving problem of consumers in a general equilibrium framework.

4 Incomplete markets and general equilibrium

In this section, I extend the partial equilibrium models which were discussed previously in Section 3 to the general equilibrium setup.[18] Basically, this means that prices – interest rate and wage rate – are now determined endogenously within the model. Moreover, here I have a large number of agents which are ex ante identical but ex post heterogeneous, since they receive idiosyncratic shocks, from which it follows that consumers have different levels of wealth. The presence of incomplete markets implies a precautionary saving motive when the differences in the levels of wealth cause different marginal propensities to consume. Furthermore, the existence of a precautionary saving motive gives an additional reason for consumers to save – besides the intertemporal reason – which affects prices in the economy. Hence, there are many reasons to believe that incomplete markets matter greatly in a general equilibrium setup.

The following three reasons are the most obvious ones (following Krueger (2007)):

1. As shown in Section 3.3.2, the relative size of r and ρ matters greatly for the behavior of consumption and saving over time. Both were given exogenously in partial equilibrium models and chosen by the model builder. When I let another crucial element, r, be endogenously determined, I really can show that I have a solution for the consumers' problem under incomplete markets. Hence, I impose some theoretical discipline on the problem.

2. Endogenously determined wealth distribution gives a theory of inequality which matters for the prices observed in economy. I can control the income distribution via idiosyncratic shocks, but wealth distribution is endogenous and it should reproduce some stylized facts of empirical wealth distribution. Thus, incomplete markets – at least potentially – have significant effects on business cycle dynamics and long-run growth, and these matters can only be studied within a general equilibrium framework.

3. To observe the costs and benefits of any policy experiment, I need a general equilibrium model. Many policy experiments change the relative prices within the economy, which cause additional costs or benefits, but those changes cannot be seen in a partial equilibrium model since the prices are

[18] The main source for this section is Krueger (2007).

fixed. Moreover, policy experiments may affect wealth distribution, as is the case with social security, taxes and so on. Obviously, one needs a model in which wealth distribution is endogenously determined to evaluate the pros and cons for these policies.

In this section, I first focus on a stochastic growth model when the supply of labor is fixed.[19] There is one asset – physical capital – which can be held as a store of value (intertemporal reason to save), or it may be held as a means of self-insurance against income shocks (precautionary reason to save). If only idiosyncratic shocks are considered, it is possible to find a stationary equilibrium, but when aggregate shocks are added into the model there is no hope for a similar equilibrium and the results are purely numerical. Finally, I discuss two important extensions of baseline models. First, I consider the different approaches to model labor supply. Second, I discuss different ways to generate a realistic wealth distribution.

4.1 A model without aggregate uncertainty

Here, the same type of environment – or income fluctuation problem – is studied as in Section 3, but the prices, or factors of production, are now endogenously determined. I could assume a pure credit economy, as suggested by Huggett (1993), but I choose instead to follow the model suggested by Aiyagari (1994), in which there is one asset: productive capital. However, there is no aggregate risk, but only idiosyncratic shocks, which cannot be insured. I focus only on the steady state, but Huggett (1997) provides a discussion on the transition dynamics within the model.

4.1.1 The model

Let there be a continuum of infinitely-lived consumers of measure one. Every consumer faces the same stochastic process of productivity $\{s_t\}_{t=0}^{\infty}$, where $s_t \in S = \{s_1, \ldots, s_N\}$. Consumers' labor income is given by $s_t w_t \bar{l}$, where w_t is the real wage that agents receive for a unit of labor supply $\bar{l}$. The productivity process follows a stationary Markov process in which the transition of the agent from

[19]Brock and Mirman (1972) provided the first optimizing growth model with unpredictable stochastic shocks.

state s to state s' is given by $\pi(s' \mid s) = \Pr(s_{t+1} \mid s_t)$. Not only π describes an individual's movement from s to s', but a law of large numbers holds, when π describes a deterministic fraction of population which moves from s to s'.[20] The consumers' problem is expressed as follows:

$$\begin{aligned} \max_{\{c_t\}_{t=0}^{\infty}} E_t U &= E_t \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta} && (57) \\ \text{s.t.} \quad a_{t+1} &= (1+r_t)a_t + s_t w_t \bar{l} - c_t, && (58) \\ a_{t+1} &\geq \underline{a} \quad \text{and} && (59) \\ c_t &\geq 0 \quad \forall t. && (60) \end{aligned}$$

Hence, the problem is already familiar from Sections 2 and 3, and assets holdings are restricted from below by a borrowing limit $\underline{a}$. I want to rule out any Ponzi schemes when $a \in A = [\underline{a}, \infty)$.

The production side of the economy is described using a standard neoclassical theory. Output, Y_t, is produced as follows:

$$Y_t = F(K_t, L_t). \tag{61}$$

Where F features constant return to scale, both factors of production have positive but diminishing marginal products and I assume that Inada conditions hold. Firms take prices w_t and r_t as given and they maximize their profits. Then, assumptions provided above with perfect competition implies that production can be described with a single representative firm. Further, K_t equals to the sum of a_t, i.e. it is the capital holdings of consumers, and L_t is the sum of the labor supply of consumers. Moreover, capital depreciates with the rate δ, $0 < \delta < 1$. Hence, the economy's resource constraint is

$$K_{t+1} = Y_t - C_t + (1-\delta)K_t, \tag{62}$$

where C_t is the aggregate consumption, i.e. it is the sum of the consumption by consumers.

Now, that I have defined the demand and the production side of economy, I can basically solve these problems, but I need something that describes the aggregate state of the economy. Note that this not a problem in a representative agent

[20] See Uhlig (1996) for a detailed discussion of circumstances in which the law of large numbers holds.

model, since the representative agent chooses directly C_t and K_{t+1}. Here, every consumer makes her consumption and saving decision only by observing prices (r_t, w_t) and her current state (s_t, a_t). Hence, I need to aggregate these actions somehow and keep tracking the evolution of aggregate economy as well. That is, I need a type distribution, Φ, over the state space $S \times A$. The consumer's type is described by her current asset holdings, a_t, and productivity shock, s_t, which are individual's state variables. But, the aggregate state is described using the cross-sectional distribution over individual state space, i.e. $\Phi_t(a_t, s_t)$.

The evolution of the individual state is given by the Markov process and equation (58) (given an optimal decision rule). Now, however, I must define the aggregate "law of motion". First, I need a measurable space where I can measure Φ. Define $Z = A \times S$ where a typical element of Z is $z = (a, s)$. A Borel σ-algebra on Z is given by $\mathcal{Z} = \mathcal{B}(Z) = \mathcal{S} \times \mathcal{A}$, where $\mathcal{S} = \mathcal{P}(S)$ is the power set of S and $\mathcal{A} = \mathcal{B}(A)$ is the Borel σ-algebra of A. Now I can formally define the type distribution as a probability measure $\Phi : \mathcal{Z} \to [0, 1]$. Hence, I have a probability space $(Z, \mathcal{Z}, \Phi)$. Finally, define that $\Phi \in \mathcal{M}$, where $\mathcal{M}$ is a set of all probability measures Φ on the measurable space $(Z, \mathcal{Z})$. This allows us to define the last element: the aggregate "law of motion", which is a function $H : \mathcal{M} \to \mathcal{M}$. That is,

$$\Phi_{t+1} = H(\Phi_t). \tag{63}$$

Now I can define the problem of consumers – given by equations (57)-(60) – recursively:

$$\begin{aligned} V(z, \Phi) &= \max_{c \in \mathbb{R}^+, a' \in A} \left\{ U(c) + \beta \sum_{s' \in S} \pi(s' \mid s) V(z', \Phi') \right\} && (64) \\ \text{s.t.} \quad a' + c &= [1 + r(\Phi)]\, a + s w(\Phi) \bar{l} && (65) \\ \Phi' &= H(\Phi). && (66) \end{aligned}$$

The solution to the problem is a policy function which gives the next period's asset holdings, $a' = g^a(z, \Phi)$, and current period consumption, $c = g^c(z, \Phi)$. Note that Φ will be a redundant state in equilibrium since it will be a constant. But here, the dependence is made explicit for the sake of clarity.

Obviously H will depend on g^a and π, since Φ summarizes individuals' behavior, i.e. it gives the distribution of assets and income, and hence, its movement depends on consumers' saving decisions and income shocks. But, this behavior can be captured by using a transition function. So, let me define a transition function $Q : Z \times \mathcal{Z} \to [0,1]$ by

$$Q\left(z, \mathcal{S} \times \mathcal{A}\right) = \sum_{s' \in \mathcal{S}} \begin{cases} \pi(s' \mid s) & \text{if} \quad g^a(z, \Phi) \in \mathcal{A} \\ 0 & \text{else} \end{cases} \tag{67}$$

for all $z \in Z$ and $\mathcal{S} \times \mathcal{A} \in \mathcal{Z}$. Hence, $Q\left(z, \mathcal{S} \times \mathcal{A}\right)$ delivers the mass of probability which ends up tomorrow with assets $a' \in \mathcal{A}$ and state $s' \in \mathcal{S}$, given that the current asset and state are a and s. Note that, since a' is non-stochastic, it is given by g^a, when a' falls into $\mathcal{A}$ or not. Thus, I can write

$$\Phi'(\mathcal{S} \times \mathcal{A}) = H\left(\Phi\right)\left(\mathcal{S} \times \mathcal{A}\right) = \int_Z Q\left(z, \mathcal{S} \times \mathcal{A}\right) d\Phi. \tag{68}$$

That is, a fraction of people measured with Φ will transit to some interval in $\mathcal{S} \times \mathcal{A}$, which is given by Q. So, a consumer having a state vector z today will have a state vector lying in $\mathcal{S} \times \mathcal{A}$ tomorrow with a probability given by Q.

4.1.2 The definition of equilibrium, its existence and uniqueness

Here, I derive some theoretical results concerning the model described above, but I only consider the economy in a steady state (or in a stationary equilibrium). In the steady state $r_{t+1} = r_t = r$ and $w_{t+1} = w_t = w$, i.e. prices are constant and when these prices are functions of the type distribution, Φ, it also must be constant, i.e.

$$\Phi = \int_Z Q\left(z, \mathcal{S} \times \mathcal{A}\right) d\Phi, \tag{69}$$

where the type distribution Φ maps into itself. Note a crucial difference to the representative agent model, in which in the steady state aggregate economy behaves similarly as described here. However, in the steady state of this model consumers change their places in the type distribution, since they receive idiosyncratic shocks. Thus, there is a significant dynamic taking place at the individual level, and that is the feature which makes this model extremely exciting. So, "social mobility" is meaningfully present in this model, in contrast to the complete markets case, where initial rankings persist forever (see, Chatterjee, 1994). Before proceeding to questions of uniqueness and the existence of this equilibrium, let me define a recursive competitive equilibrium.

Definition 1. *A stationary recursive competitive equilibrium consists of a value function $V : Z \to \mathbb{R}$, policy functions for consumers $g^a : Z \to \mathbb{R}$ and $g^c : Z \to \mathbb{R}^+$, policies for the firms K and L, pricing functions r and w and a measure $\Phi \in \mathcal{M}$ such that:*

1. ***Consumers maximize**: given r and w V satisfies consumers' problem (64)-(66). Moreover, g^a and g^c are associated decision rules.*

2. ***The firm maximize**: Given r and w firm's policies K and L satisfies following conditions*

$$r = F_K(K, L) - \delta \tag{70}$$

$$w = F_L(K, L) \tag{71}$$

3. ***The stationary of the type distribution**: For all $\mathcal{Z}$*

$$\Phi = \int_Z Q\left(z, \mathcal{S} \times \mathcal{A}\right) d\Phi, \tag{72}$$

where Q is generated by using g^a and π as given by equation (67).

4. ***Markets clear**:*

   - *Asset market clears:*

$$K = \int_Z g^a(z) d\Phi. \tag{73}$$

   - *Goods market clears:*

$$\int_Z g^c(z) d\Phi + \int_Z g^a(z) d\Phi = F(K, L) + (1 - \delta)K \tag{74}$$

   - *Labor market clears: $L = \int_Z s\bar{l} d\Phi$.*

Hence, the steady state is not anymore described by a number (as the value of capital stock), but by a complicated infinite-dimensonal object, namely a measure. However, according to the definition of equilibrium, the measure Φ is not an argument of decision rules anymore, since it is constant.

To study the existence and uniqueness of equilibrium defined above, I may focus on labor and asset markets. Since Walras's law implies that I can ignore the equilibrium concept for one market, and let that be the goods market. Moreover, the labor market's condition holds, since it is purely given exogenously. Hence, I am left with the asset market's condition (73) which can be rewritten as a function of r in such a way that the left-hand side defines a demand for capital and the right-hand side gives the supply of capital:

$$K(r) = \int_Z g^a(z) d\Phi \equiv Ea(r). \tag{75}$$

To characterize the existence and uniqueness of equilibrium, I must be able to show – as in any general equilibrium model – that $K(r)$ and $Ea(r)$ are both continuous and strictly monotone and that they intersect when $r \in (-\delta, \infty)$.

Let me start with the demand side. From equation (70) it is clear that the demand for capital only depends on r, since L is given exogenously. The assumption about the production function implies that $K(r)$ is a strictly decreasing and continuous function when $r \in (-\delta, \infty)$. On the supply side, it is obvious that $Ea(r) \in [\underline{a}, \infty]$ for all $r \in (-\delta, \infty)$.[21] Moreover, *if* $Ea(r)$ is continuous, $\lim_{r \to -\delta} Ea(r) < \infty$ and $\lim_{r \to \infty} Ea(r) > 0$ then a stationary recursive competitive equilibrium exists. Furthermore, *if* $Ea(r)$ is strictly increasing in r then the equilibrium is unique, since $K(r)$ is strictly decreasing.

Let me first focus on the existence of equilibrium. Given that $K(r)$ is continuous and strictly increasing, I must show that $Ea(r)$ is continuous when the existence of equilibrium is defined if $Ea(r)$ will cross $K(r)$ when $r \in (-\delta, \infty)$. Equation (73) shows that $Ea(r)$ depends on $g^a(z; r)$ and $\Phi(z; r)$ when the continuous of $Ea(r)$ can be established by showing the continuous of $g^a(z; r)$ and $\Phi(z; r)$.

If the preferences are well behaving (as they are), the continuous of $g^a(z; r)$ is given by the Theorem of Maximum (see, Stokey and Lucas, 1989, Theorem 3.6), which also states that $g^a(z; r)$ is strictly increasing in r. Hence, I must first discuss the circumstances under which I have obtained a solution for the consumers' problem, but this discussion was already provided in Section 3.3.2.

[21]Note that households may hold capital, even if r is negative, due to precautionary saving motive.

The state space is bounded below by a borrowing constraint, but the key question is whether it is bounded above. This is needed in order for any contradiction mapping theorem to hold (see, for example, Stokey and Lucas, 1989, Theorem 3.3) given that the preferences are well behaving. To show the boundedness of the state space, Aiyagari (1994) assumes an i.i.d. structure for shocks and Huggett (1993) gives a proof with serial correlated shocks. But, some limitations for the Markov process are then needed. Section 3.3.2 used an i.i.d. case to show that $\rho > r$ and that a CRRA-utility is needed when an $\bar{a}$ exists for which $a' = g^a(z) \leq \bar{a}$ for all $s \in S$ and all $a \in [\underline{a}, \bar{a}] = A$. Now I can conclude that $g^a(z; r)$ is continuous and strictly increasing in r.

The second step is to show that $\Phi(z; r)$ is continuous in r, which can be done by using Stokey and Lucas (1989, Theorem 12.13). However, the theorem requires the existence and uniqueness of an invariant distribution Φ for a given r. This can be shown if there is a contradiction mapping in $\mathcal{M}$ where Φs are indexed by r. Then, I have a unique stationary distribution Φ. Hence, I need

$$T_r^\star \Phi = \int_Z Q_r(z, \mathcal{S} \times \mathcal{A}) d\Phi, \tag{76}$$

where the operator $T_r^\star : \mathcal{M} \to \mathcal{M}$ maps $\mathcal{M}$ into itself and has unique fixed point, as discussed in Stokey and Lucas (1989, Theorem 8.2). Aiyagari used the theorem provided by Stokey and Lucas, but Huggett used a similar theorem provided by Hopenhayn and Prescott (1992, Theorem 2). In both theorems, a key condition is a *monotone mixing condition*, which is also called *the American dream and the American nightmare* condition by Ríos-Rull (1999).

In more detail,[22] the compactness of state space – which was shown above – and Q_r with a Feller property implies the existence of Φ. Q_r has a Feller property since $g^a(z)$ is continuous and bounded, as discussed above, when T_r maps continuous and bounded functions into themselves. Moreover, if Q_r satisfies monotonicity and the monotone mixing condition, then Φ is unique. The monotonicity of Q_r requires that for any increasing function $f : Z \to \mathbb{R}$, the function Tf is also increasing. It is easy to see that Q_r is monotone since g^a is increasing in both of its arguments, i.e. the higher pair (a, s) also increases the probability of being $(a', s') > (a, s)$. Note that this requires that the exogenous Markov chain does

[22]The actual proofs are hard to give and these conditions are suggested just to give some sketch for actual proofs.

not feature a negative autocorrelation. Finally, the monotone mixing condition requires (roughly speaking) that, no matter how rich (poor) the consumer is, there is a positive probability that the consumer becomes the poorest (richest) consumer in the economy in N periods. To verify this, assume that one gets a sequence of highest (lowest) productivity shocks s_N (s_1). If the process for s is stationary – that is, mean reverting – it causes assets to convergence $\bar{a}$ ($\underline{a}$) since g^a is increasing in both of its arguments. That is to say, one realizes that current income is higher (lower) than permanent income, which dictates consumption, when, by saving (dis-saving), one tries to smooth consumption. When these four requirements are met, an invariant unique measure for Φ_r can be found by iterating $T_r^\star$ and I may now conclude that Φ_r is well-defined and continuous $r \in [-\delta, r)$ by using Stokey and Lucas (1989, Theorem 12.13).

Now I know that $K(r)$ and $Ea(r)$ are both continuous in $r \in [-\delta\ , \infty)$, but still I must show that there is an intersection between $K(r)$ and $Ea(r)$ when $r \in [-\delta\ , \infty)$. The features of the production function imply that $\lim_{r \to -\delta} K(r) = \infty$ and $\lim_{r \to \infty} K(r) = 0$. Moreover, it can be verified that $\lim_{r \to -\delta} Ea(r) < \infty$, which is obvious when I have $\bar{a}$. Further, it must be shown that $\lim_{r \to \rho} Ea(r) > K(r)$. In Section 3.3.2 it was shown that when $\rho = r$, all consumers accumulate an infinite amount of assets when (loosely speaking) $Ea(\rho) = \infty$.[23] Hence, it has been shown that $K(r)$ and $Ea(r)$ are continuous $r \in [\ -\delta, \rho)$, and for the low values of r, I have $K(r) > Ea(r)$. However, for the high values the situation reversed, i.e. $K(r) < Ea(r)$. These argument guarantee that there exists $r^\star$, which solves equation (75):

$$K(r^\star) = Ea(r^\star). \tag{77}$$

That is to say, there exists a stationary recursive competitive equilibrium.

The uniqueness of this equilibrium cannot be shown since the monotonicity of $Ea(r)$ cannot be established. There are offsetting income and substitution effects directly via r and indirectly via w which makes it difficult to give the proof. Moreover, the stability of this equilibrium is not generally studied: rather, one must rely on numerical results. Hence, it can only be shown that there exists a stationary recursive competitive equilibrium given by Definition 1.

[23] Krueger (2007) points out that, actually, $Ea(\rho)$ is not well defined when it is difficult to to give the proof for this argument (and actually it is not done). However, Aiyagari asserts that when $r \to \rho$ from below, then $Ea(r)$ goes to ∞.

4.1.3 How much does idiosyncratic risk matter for the equilibrium interest rate?

The main question addressed by Huggett (1993) and Aiyagari (1994) was as follows: How much does idiosyncratic risk matter for the equilibrium interest rate? Moreover, Huggett and Ospina (2001) showed that differences in the equilibrium interest rates given by a model with perfect foresight (or under the complete market) and the interest generated by the standard incomplete markets model define aggregate precautionary savings. However, Flodén (2008) shows that the existence of borrowing constraint increases saving if the income stream fluctuates, even if this is foreseen perfectly. This saving is not due to the precautionary saving motive, but, rather, it is generated by intertemporal reasons for smoothing consumption. To avoid a question about the right definition for precautionary savings, I define precautionary savings generally as savings which occurs due to market incompleteness. That is, there exists a borrowing constraint and markets do not provide a full set of state contingent claims, and both these factors increase saving, which can be seen to be generated by a precautionary saving motive.

The results provided by Huggett (1993) and Aiyagari (1994) showed that with a reasonable calibration (I will discuss calibration in detail in Section 5) idiosyncratic risk does not significantly decrease the interest rate: hence, it seems that the contribution of precautionary savings to capital accumulation is insignificant. Thus, consumers can provide insurance for themselves against idiosyncratic shocks by accumulating and decumulating capital or trading uncontingent assets with each other. However, these results significantly depend on the risk aversion, the persistence of shocks and the variance of shocks. Increasing the values of these variables also increases the saving rate and lowers the equilibrium interest rate. Aiyagari reports that the maximum increase in the saving rate was 14 percentage points, but this result comes with quite extreme calibration.

The magnitude of precautionary savings – i.e. wealth holdings which result from precautionary saving reasons – had received substantial interest in empirical studies as well. The findings from these studies can be divided into three different groups:

1. Precautionary saving motive generates about 50-60% of capital accumulation. See Skinner (1988) and Caballero (1991), and more recently , Cagetti (2003) and Gourinchas and Parker (2002) provided a similar conclusion.

2. Precautionary saving motive generates about 20-50% of capital accumulation. See Carroll and Samwick (1998), Engen and Gruber (2001) and Carroll, Dynan, and Krane (2003) reports that precautioanry saving the motive is important.

3. Precautionary saving motive does not matter for capital accumulation. See Guiso, Jappelli, and Terlizzese (1992), Dynan (1993) and Lusardi (1998).

Hence, there is no consensus on how much the precautionary saving motive matters for capital accumulation and the equilibrium interest rate. Probably one problem is that researchers use different metrics to measure risk and there are potentially many different types of risk which should be considered. The metric used for assessing risk matters even for theoretical results, as discussed by Eeckhoudt and Schlesinger (2008). One risk which was ignored by Huggett (1993) and Aiyagari (1994) is the aggregate risk, and hence, a natural – and potentially important – extension for the model introduced here is to include aggregate risk within the model.

4.2 A model with aggregate uncertainty

Including aggregate uncertainty in addition to idiosyncratic uncertainty makes the model more feasible in many respects: most notably, the business cycles, and generally, non-equilibrium allocation can be studied. Moreover, now there is an additional risk which was missing from the previous model and which could be of substantial importance for equilibrium behavior as well. This extension was shown by Krusell and Smith (1998).

4.2.1 The model and the recursive equilibrium

Now I add the famous aggregate productivity shock into the model in the spirit of real business cycle theory. Hence, the production is now given by

$$Y_t = \gamma_t F(K_t, L_t), \tag{78}$$

where $\{\gamma_t\}$ is a sequence of random variables $\gamma_t \in \{\gamma_1, \dots, \gamma_N\} = \Gamma$ that follow a finite state Markov chain and these shocks are called aggregate productivity shocks. Everything else on the production sector is the same.

It is reasonable to think that when the aggregate productivity gets a high value, it is also more likely that the consumer will face a high productivity shock. So, there is a dependency between idiosyncratic and aggregate shocks when transition probabilities for consumers are given by $\pi(\gamma', s' \,|\gamma, s\,) \geq 0$. A conditional probability for the consumer state being s' and the aggregate state being γ' tomorrow is now defined by π, given that today's states are s and γ. Moreover, idiosyncratic shocks are identically distributed across consumers. Hence, the number of consumers in states $s \in S$ is determined by the conditional probability of aggregate shocks, $\pi(\gamma' \mid \gamma)$, since it is assumed that the law of large numbers still holds. More precisely

$$\sum_{s' \in S} \pi(\gamma', s' \,|\gamma, s\,) = \pi(\gamma' \mid \gamma) \quad \forall \quad s \in S \quad \text{and} \quad \forall \quad \gamma, \gamma' \in \Gamma. \tag{79}$$

Thus, in equilibrium the mass of consumers in each state s depend on the aggregate state γ, which implies that the equilibrium provided by the definition 1 above is not possible since Φ now varies with γ.

The consumers' problem does not change significantly when it is still given by equations (57)-(60), and it can be defined recursively. The difference with respect to the case introduced in Section 4.1 is that the consumers' problem also now depends on the aggregate state γ. The consumer's state is still given by $z = (a, s)$ i.e. it depends on asset holdings and the state of productivity, but now the aggregate state is relevant since it is given by (γ, Φ), where Φ is not constant, but depends on γ, as noted above. Moreover, productivity shocks will directly matter for prices. Hence, the consumers' problem is

$$\begin{aligned} V(z, \gamma, \Phi) &= \max_{c \in \mathbb{R}^+, a' \in A} \left\{ U(c) + \beta \sum_{s' \in S} \sum_{\gamma' \in \Gamma} \pi(\gamma', s' \,|\gamma, s) V(z', \gamma', \Phi') \right\} && (80) \\ \text{s.t.} \quad a' + c &= [1 + r(\gamma, \Phi)]\, a + s w(\gamma, \Phi) \bar{l} && (81) \\ \Phi' &= H(\gamma, \Phi, \gamma'). && (82) \end{aligned}$$

The definition of a recursive competitive equilibrium is as follows:

Definition 2. *A recursive competitive equilibrium with aggregate uncertainty consist of value function* $V : Z \times \Gamma \times \mathcal{M} \to \mathbb{R}$*, policy functions for consumers* $g^a : Z \times \Gamma \times \mathcal{M} \to \mathbb{R}$ *and* $g^c : Z \times \Gamma \times \mathcal{M} \to \mathbb{R}^+$*, policies for the firms*

$K : \Gamma \times \mathcal{M} \to \mathbb{R}$ and $L : \Gamma \times \mathcal{M} \to \mathbb{R}$, pricing functions $r : \Gamma \times \mathcal{M} \to \mathbb{R}$, $w : \Gamma \times \mathcal{M} \to \mathbb{R}$ and a law of motion $H : \Gamma \times \mathcal{M} \times \Gamma \to \mathcal{M}$ for the type distribution Φ such that:

1. ***Consumers maximize**: given r and w V satisfies consumers' problem (80)-(82). Moreover, g^a and g^c are associated decision rules.*

2. ***The firm maximize**: Given r and w firm's policies K and L satisfies following conditions*

$$r(\gamma, \Phi) = F_K \left(K(\gamma, \Phi), L(\gamma, \Phi)\right) - \delta \quad (83)$$

$$w(\gamma, \Phi) = F_L \left(K(\gamma, \Phi), L(\gamma, \Phi)\right) \quad (84)$$

3. ***Consistency condition between aggregate and individual behavior**: The law of motion for Φ, H, is generated by exogenous probabilities π from the Markov chain and policy function g^a as follows:*

$$\Phi'(\mathcal{S} \times \mathcal{A}) = H(\gamma, \Phi, \gamma')(\mathcal{S} \times \mathcal{A}) = \int_Z Q_{\gamma,\gamma'}\left(z, \mathcal{S} \times \mathcal{A}\right) d\Phi \quad (85)$$

where

$$Q_{\gamma,\gamma'}\left(z, \mathcal{S} \times \mathcal{A}\right) = \sum_{s' \in \mathcal{S}} \begin{cases} \pi(\gamma', s' \mid \gamma, s) & \textit{if} \quad g^a(z, \gamma, \Phi) \in \mathcal{A} \\ 0 & \textit{else.} \end{cases} \quad (86)$$

4. ***Markets clear**:*

   - *Asset market clears:*

$$K(\gamma, \Phi) = \int_Z a d\Phi. \quad (87)$$

   - *Goods market clears:*

$$\int_Z g^c(z, \gamma, \Phi) d\Phi + \int_Z g^a(z, \gamma, \Phi) d\Phi = \gamma F(K(\gamma, \Phi), L(\gamma, \Phi)) + (1-\delta) K(\gamma, \Phi) \quad (88)$$

   - *Labor market clears: $L(\gamma, \Phi) = \int_Z s\bar{l} d\Phi$*

for all $\Phi \in \mathcal{M}$ and all $\gamma \in \Gamma$.

Miao (2006) shows that the equilibrium given by Definition 2 generates a sequential competitive equilibrium. However, it has not been shown that a recursive equilibrium, as suggested by Definition 2, exists. To do that, one should show that the sufficient aggregate state only contains the current aggregate shock and the current wealth distribution.[24] Anyhow, Miao (2006) shows that, generally, there exists a sequential markets equilibrium for the model described in this section which can be characterized recursively when the cross-sectional distribution of expected payoffs is added as a state variable. However, the uniqueness of this sequential equilibrium is not shown.

Hence, the analysis of this type of model is based purely on the computational results. The existence, uniqueness or stability of the equilibrium cannot be theoretically established. Furthermore, the consumers' problem depends on Φ, which is basically an infinite object, when there is little hope for directly computing a solution for this type of problem. The state space for this problem is just too large – no matter how fast computer one has. Hence, I need a way to reduce the state space so that I am even able to compute these types of models.

4.2.2 The approximate aggregation

The computational methods cannot solve the problem that I processed here, but the features of the model itself enable us to reduce the state space. More precisely, it will be shown that the consumers' problem will depend on Φ to a limited extend. Φ is not a constant when I cannot use a similar approach as in Section 4.1, but now Φ changes in time as consumers need to keep tracking it in order to figure out today's and tomorrow's interest and wage rates, which in turn are needed for the recursive characterization of the problem of consumers.

Krusell and Smith (1998) showed that the movement of Φ can be approximated by using a finite set of moments of Φ. Φ is a type distribution over pairs (a, s), where s can only take a finite number of values, when, basically, I only have to deal with the first argument. Let me now describe the marginal distribution of $\Phi(\bullet, s)$ with vector m, where the first n-moment of $\Phi(\bullet, s)$ exists. Then, consumers use

[24] Kubler and Schmedders (2002) showed by using the asset-pricing model by Lucas (1978), extended with heterogeneous agents and incomplete markets, that there exists a sequential recursive equilibrium under the strong condition that the competitive equilibrium is globally unique for all possible initial values.

an approximate law of motion

$$m' = H_n(\gamma, m) \tag{89}$$

where the first n-moments given by m are mapped by H_n into the n-moments of tomorrow given by m'.[25]

The next step is to choose the appropriate n, i.e. how many moments I need to describe $\Phi(\bullet, s)$. Note that in the rational expectations equilibrium consumers use all available information when they form their expectations for future prices, but here they only use the first n-moments. Hence, this feature is sometimes interpreted as if consumers are boundedly rational. However, I hope (and will show) that there is not a large forecasting error in prices, even if only a limited amount of information is used.[26] Since the averages are equal to the aggregates in the model, I choose to use only the first moment in the forecasting equation.[27] That is, equation (89) can now be rewritten as follows:

$$\log(K') = h_{1,\gamma} + h_{2,\gamma} \log(K), \tag{90}$$

where parameters $h_{1,\gamma}$ and $h_{2,\gamma}$ depend on the aggregate state γ and are left to be determined.

The problem of consumers given by equations (80)-(82) then becomes

$$\begin{aligned} V(z,\gamma,K) &= \max_{c\in\mathbb{R}^{+},a'\in A} \left\{ U(c) + \beta \sum_{s'\in S} \sum_{\gamma'\in\Gamma} \pi(\gamma', s' \,|\gamma, s) V(z', \gamma', K') \right\} && (91) \\ \text{s.t.} \quad a' + c &= [1 + r(\gamma, K)]\, a + sw(\gamma, K)\bar{l} && (92) \\ \log(K') &= h_{1,\gamma} + h_{2,\gamma} \log(K). && (93) \end{aligned}$$

The infinite state space in the previous version of this problem is now defined as $(z, \gamma, K) \in \mathbb{R} \times S \times \Gamma \times \mathbb{R}$.

The problem of consumers can be solved for given parameters $h_{1,\gamma}$ and $h_{2,\gamma}$. Using the decision rule g^a, I can simulate the economy and construct a sequence

[25] The recently introduced method by den Haan and Rendahl (2010) avoids approximate aggregation. They explicitly generate the aggregate law of motion from individual policy rules by integrating over the individual policy rules. More details can be found from den Haan and Rendahl (2010).

[26] Moreover, Young (2010) point out that consumers are not boundedly rational, but extra information is just not useful for forecasting the prices.

[27] Note that there is a continuum of consumers of measure one which makes aggregates equal to averages.

for K_t. Finally, I can update parameters $h_{1,\gamma}$ and $h_{2,\gamma}$ by using the simulated data. If the forecast is not accurate, I will need to do all of this again.[28] When the parameters $h_{1,\gamma}$ and $h_{2,\gamma}$ have converged, the model is solved. With baseline calibration, Krusell and Smith (1998) found that $R^2 \geq 0.99999$ for the equations defined in (90). Hence, consumers practically do not make any forecasting errors, even if they use a quite simple forecasting function. For instance, a forecasting error for interest rate is 0.1% with a 25-years' horizon. Thus, te deviations are minimal – at least computationally – when compared to the rational expectations equilibrium where consumers do not make any forecasting errors with the aggergate law of motion. This result is called approximate aggregation. An interesting question is, why does the approximate aggregation even exist? The reason can be found from Figure 2.

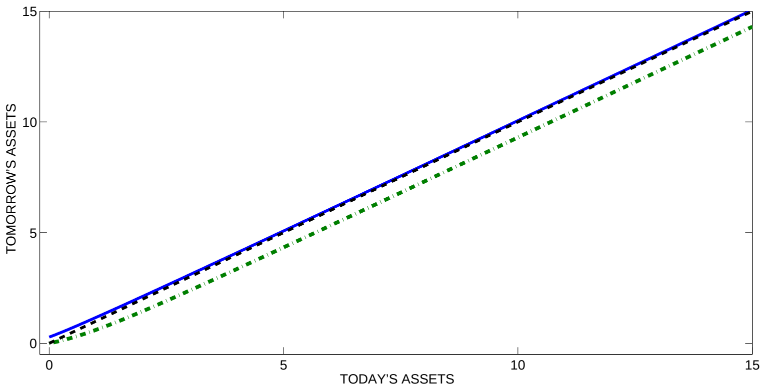

Figure 2: A sample of decision rules for given aggregate state with the baseline calibration of Krusell and Smith (1998). Solid gives a decision rule when $s = s_N$, dash-dotted line gives a decision rule when $s = s_1$ and dashed line is 45 degree line.

From Figure 2 it is evident that the non-linearity of decision rules only occurs for the lowest levels of wealth. With a linear decision rule, the propensity to save out of current assets and income is constant and the aggregation is close to one in the case of the representative agent. Moreover, consumers in the left tail of wealth distribution have very little wealth since they are poor. Hence, even if their marginal propensities to save differ from the average, they do not matter significantly for the accumulation of wealth. Finally, it should be noticed that the fluctuations of Φ are quite small when there is not much redistribution, which

[28] A more accurate description of the algorithm that solves the model can be found in Section 5.2.

could change the aggregate variables. These features enable a very good forecast for the aggregate variables and prices just by using a very simple forecasting function.

Now it seems that, for the most part of the state space consumption and saving are determined by permanent-income considerations when the marginal propensity to consume is low and very close to one that occurs with the representative agent. Hence, precautionary saving does not matter for capital accumulation, even if aggregate shocks are included within the model, but the magnitude of precautionary saving is more or less the same as reported by Huggett (1993) and Aiyagari (1994). Moreover, Krusell and Smith (1998) also argue that approximate aggregation is a robust result over a different type of specifications. Does all of this mean that the representative agent model is a superior choice?

The answer is: definitely not. First, even if capital accumulation is not affected by the heterogeneity of consumers, the behavior of many other economically meaningful variables' may differ from the behavior of the economy where only the representative exists. One candidate is consumption. Second, a model in which the wealth distribution is endogenously determined provides new opportunities for evaluating the effects of shocks and policies. Generally, one may now discuss how the distribution of wealth or income will change - basically, how welfare will change - when the economy faces a shock or a policy change. The shock or policy change will potentially affect rich and poor people differently. Obviously, the representative agent model is silent on these matters. Third, the models presented here were quite simple. I did not focus on aggregation, that is to say, on the shape of wealth distribution, and the labor supply was fixed. Thus, by extending these models, one may discuss issues which are essential for economic behavior and where heterogeneity among consumers matters.

4.3 Some extensions to baseline models

I consider two important extensions to the models presented in Sections 4.1 and 4.2. First, I let the supply of labor be endogenous. This enables an incomplete market extension for the baseline real business cycle model. One can assume that the adjustment of total hours takes place on the extensive margin (to participate or not) or on the intensive margin (how much to supply labor). Since individual and aggregate behavior are not the same, these models give a new and interesting viewpoint for the elasticity of the aggregate labor supply. Second, one of the main applications for the incomplete markets models is to explain the shape of wealth distribution. It is not only an interesting and important matter itself, but the wealth distribution also provides an aggregation for these models. When the income distribution is exogenously given, the wealth distribution endogenously determines the heterogeneity of the agents. To evaluate realistically the implications from the models, the aggregation must be realistic, i.e. one needs a realistic wealth distribution.

4.3.1 Valued leisure

There are (at least) three reasons why I should consider the labor supply decision when the incomplete markets are studied:

1. Marcet, Obiols-Homs, and Weil (2007) noted that precautionary saving increases the value of capital stock, which in turn implies a higher output *when labor supply is inelastic*. Thus, the less developed financial markets would lead to a higher output, which is somewhat paradoxical, but this result does not hold when labor supply is elastic.

2. Pijoan-Mas (2006) pointed out that labor supply does provide an additional method for consumers to insure themselves against idiosyncratic productivity shocks (see also Heathcote, Storesletten, and Violante, 2008, 2009a). Hence, by ignoring the labor supply decisions, the magnitude of precautionary savings may be overestimated.

3. Chang and Kim (2006, 2007) showed that aggregation matters for labor supply when markets are incomplete and consumers make the labor supply decisions in an extensive margin. Individual and aggregate behavior differs from each other since the elasticities of the individual labor supply and the aggregate labor supply are not the same.

Aiyagari's model were used in the first two items and Krussel and Smith's model were utilized in the last item. Now, I will discuss each of these results individually.[29]

[29] Due to space constraints, I have left out some interesting applications where the incomplete markets model and the labor market interact. Most notably, labor market behavior can be modeled by using search and matching (see Nakajima, 2008; Krusell, Mukoyama, Rogerson, and Sahin, 2008).

Marcet, Obiols-Homs, and Weil (2007) noted that in production economies, a higher precautionary saving motive also makes the capital stock higher, which in turn increases the output when the supply of labor is inelastic. They called this effect the *Aiyagari-Huggett effect*. In their model, idiosyncratic shocks are given by $s_t \in S = \{0, 1\}$ and utility is now given by $U(c_t, l_t)$, where labor supply, l_t yields dis-utility for consumers. Hence, a consumer can adjust her labor supply if she has a chance to do that, i.e. if $s_t = 1$. If leisure is normal good, then higher wealth holdings cause a wealth effect, which reduces the supply of labor. Thus, a riskier environment causes a higher precautionary saving motive – and higher capital stock – but now the wealth effect reduces the supply of labor. Marcet et al. reported that output would increase by almost 20% by completing the markets.[30] Hence, the wealth effect could be very significant and the cost of market incompleteness is high.

At the same time, Pijoan-Mas (2006) use the same setup. However, with his setup consumers face productive shocks and they can, in any case, adjust the supply of labor. The first order condition for the supply of labor is

$$U_c(c_t, l_t)w_t s_t = U_l(c_t, l_t). \tag{94}$$

Under complete markets, idiosyncratic shocks, s_t, do not have any wealth effect since they do not matter for consumption. There is only a substitution effect when l_t adjust fully for the changes in s_t. However, with incomplete markets, the low values of s_t imply a lower consumption (higher marginal utility), which in turn causes consumers to supply more labor, i.e. consumers try to compensate lower income – which results from lower productivity – by working more. That is, the supply of labor is used to smooth consumption across the states, which lowers the capital stock when compared against the case of inelastic labor supply, since precautionary savings are lower. Moreover, the market incompleteness causes consumers to work more when the productivity is low. Then, the aggregate (or average) productivity of labor is lower than in the complete markets case. Pijoan-Mas reports that labor productivity is 11.5% lower due to market incompleteness. With persistent shocks to the wage process, the lower labor productivity cause a lower output, even if capital stock is higher due to precautionary savings. Thus, the result is the same as in Marcet, Obiols-Homs, and Weil (2007), but the effects

[30]This result applies when the relative risk aversion is equal to 5.

on the behavior of households are somewhat different.

It is not obvious – at least to me – how I should model the intensive margin choice of labor supply. Both of these approaches provide a way to escape the counter-intuitive conclusion provided by the incomplete markets model with inelastic labor where the market incompleteness, which is something that should be a bad thing, leads to higher output. However, when the incomplete markets model is compared against its complete markets counterpart, it is not clear, whether the supply of labor is lower or the labor supply is higher but its productivity is lower. Probably both approaches contain elements of truth. However, a well-known fact is that the dominant source of fluctuation in total hours comes from the variation in the number of employees. So, I will turn now to the extensive margin of labor supply.

Due to empirical evidence, it seems that labor is indivisible in the sense that either a consumer works or does not work and this is the only adjustment available. This has led Hansen (1985) and Rogerson (1988) to include a lottery mechanism to model the labor supply choice in the extensive margin. However, the lottery mechanism provides a complete insurance against unemployment risk, which is questionable (and actually the unemployed consumer's utility is higher than the employed consumer's utility). Chang and Kim (2006, 2007) used the incomplete markets model of Krusell and Smith (1998) with an indivisible labor supply. In their models the labor supply decision is characterized by

$$V(z,\gamma,\Phi) = \max_{l \in \{0,\bar{l}\}} \left\{ V^E(z,\gamma,\Phi), V^N(z,\gamma,\Phi) \right\} \tag{95}$$

where V^E is the value function of employed consumer when $l = \bar{l}$ and V^U is the value function of unemployed consumer when $l = 0$.

Chang and Kim found that indivisible labor combined with incomplete markets produces a very similar behavior between the hours worked and productivity than one can observe from actual data. Moreover, they found that the elasticity of aggregate labor depends on the shape of distribution of reservation wages and that the elasticity of aggregate labor is much higher than individuals' elasticity for hours. This type of behavior cannot be generated within the representative agent model. Thus, market incompleteness seems to matter for the business cycle dynamics as well as long-run economic growth (at least) via labor supply decisions.

4.3.2 Matching the wealth distribution

One of the main reasons for the popularity of the heterogeneous agent model is that the inequality can now be studied. The endogenously determined wealth distribution (a marginal distribution of $\Phi(\bullet, s)$) provides the chance to explain why wealth is distributed so unequally in the data. The Gini coefficient for wealth in the U.S. is 0.8 (see Budria, Diaz-Gimménez, Quadrini, and Ríos-Rull, 2002), but the baseline calibration by Aiyagari (1994) gave a Gini coefficient of 0.3. Since then, different versions of incomplete markets models (or heterogeneous agent models) have been provided to explain the unequal distribution of wealth.

This matter is also important since the marginal propensities to consume and save depend on the level of wealth. This fact among the idiosyncratic shocks makes consumers ex post heterogeneous. The shocks are usually controlled by the model builder when the interesting part of heterogeneity comes from the wealth distribution. To evaluate the responses of aggregate variables on shocks or policy changes, one must be able to generate a feasible aggregation, i.e. the wealth distribution must be an approximation of its empirical counterpart. Hence, not only is the explanation for inequality in itself an interesting matter, but a realistic wealth distribution is needed to get plausible results from the model. Thus, a realistic wealth distribution is needed for any application of the incomplete markets model within a general equilibrium framework.

The problem with closely matching the wealth distribution provided by the model to the distribution observed in the data boils down to getting enough poor and rich people. Fixing the left tail of the wealth distribution is quite straightforward as shown by Hubbard, Skinner, and Zeldes (1995) and Huggett (1996) in the life-cycle framework: just introduce social security and taxes into the model. Then there are a large enough disincentives for poor people to save and the inadequate saving makes them poor. However, the real problem (in terms of modeling) is in generating a large concentration of wealth, i.e. the right tail of the wealth distribution is hard to generate. There are (at least) four different approaches to account for the stark inequality in wealth.

First, Krusell and Smith (1998) introduced a model where discount factors vary among consumers. More precisely, they showed that a small amount of heterogeneity in discount factors is enough to generate a realistic wealth distribution. Then, the discount factor follows a Markov process and one interpretation for this heterogeneity is that the discount factor changes between the generations of dynasty. These small variations are sufficient to generate enough poor and rich people.

Second, Quadrini (2000) and Cagetti and De Nardi (2006) showed that if the rate of return for savings differs between the consumers, then the wealth inequality can be generated through entrepreneurial choices. Entrepreneurs accumulate a large amount of wealth since the entrepreneur's wealth acts as collateral when she wants to borrow. Further reasons for the high wealth accumulation of entrepreneurs are that external financing could be costly and entrepreneurs face a higher risk for their income than does a typical worker. However, these models are also based on the explanation that there is heterogeneity among consumers when some consumer have better entrepreneurial ideas or abilities and, therefore, get a high return on their savings.

Third, Castañeda, Díaz-Giménez, and Ríos-Rull (2003) pursued another approach where they used identical and standard preferences for explaining the income and wealth distribution. Moreover, they showed that idiosyncratic earnings risk, retirement, altruism, and government transfers to retired households are essential parts of the model when one tries to replicate the observed earnings to wealth ratios of both the rich and the poor households simultaneously. The key to generating this result is an assumption that there is a small probability of entering a state with enormous wages. That is, the wage process must have a drastic dispersion.

Fourth, De Nardi (2004) showed that, when bequest are modeled as a luxury good, intergenerational links between parents and children causes rich households keep large amounts of wealth in old age in order to leave bequest to their children. Accidental bequest do not generate enough wealth concentration, but voluntary bequest makes possible the emergence of large estates which leads to concentrated wealth holdings. In particular, saving to leave bequests significantly affects the shape of the upper tail of the wealth distribution in the model. However, some unrealistic information restrictions must be made on children's knowledge of their parent's wealth and income.

These different approaches try to answer the question: why are the rich so rich and the poor so poor? Obviously, this question is one of the most important ones in economics and implies significant policy implications. If the theory of preference heterogeneity is used as an explanation for this question, then policies which are aimed at the redistribution of resources are hard to defend on the grounds of efficiency: the poor are poor due to their preferences, i.e. they have "chosen to be poor". However, if inequality originates from a large dispersion in the wage process or wealth accumulation, then poor people are just unlucky and there is a case for redistribution even on the grounds of efficiency. Once again, I have to note that the final answer is not yet available.

5 The implementation of general equilibrium models with incomplete markets

The discussion in Section 4 clearly showed that the analysis and results of the general equilibrium models with incomplete markets are numerical/quantitative in nature.[31] Hence, two questions arise: 1) How are the values for the parameters chosen? 2) In which way can one solve these models? Basically, these questions must be answered in any quantitative study and in this section I try to provide a short introduction to these issues.

5.1 Working with data

There are many sources of uncertainty, but the early contribution of the incomplete market models (for example, Aiyagari, 1994) only considered uninsurable idiosyncratic uncertainty for earnings. Nowadays, the sources of uncertainty have been expanded to include, for instance, family shocks and health shocks. Moreover, it is not credible to assume that consumers are completely cut off from all insurance markets, but there are many ways in which the family or government can provide some insurance against idiosyncratic shocks. Hence, an estimate is needed to give the magnitude of idiosyncratic uncertainty for earnings, but, at the same time, we should estimate how much insurance consumers can find against idiosyncratic uncertainty. First, I will discuss the dispersion of wage processes, which is the main source of uncertainty. In these estimations, researchers are specially interested in finding out how persistent the idiosyncratic shocks are. Second, different sources of uncertainty and insurance are discussed in brief.

5.1.1 The dispersion of wage processes

The most important input for the incomplete markets model is to get a parametrization for idiosyncratic productivity s. Since I have a state space system, I need to approximate the productivity process – which is often described with a simple AR-process – with a finite state Markov chain. Details for this method are given in Section C of the Appendix.

[31]The main sources for this Section are Heathcote, Storesletten, and Violante (2009b) and Heer and Maussner (2009).

Typically, PSID or equivalent data set is used, which gives a panel data about individual wage rates $\omega_{i,t}$. This wage rate, which is described in the incomplete markets model with the term $s_t w_t$. w_t, is an equilibrium price that depends on the aggregates (K_t,L_t and γ_t), when s_t is now a idiosyncratic productivity shock for a consumer i.

For instance, Floden and Lindé (2001); Pijoan-Mas (2006); Krueger and Perri (2006); Blundell, Pistaferri, and Preston (2008) used the following decomposition for a log of hourly wage rate

$$\omega_{i,t} \quad = \quad \psi_i + z_{i,t} + \epsilon_{i,t} \quad \text{with} \quad \epsilon_{i,t} \sim N(0, \sigma_\epsilon^2), \tag{96}$$

where ψ_i captures all of the effects of the factors that can be observed from the data (for instance, age or education), and $\epsilon_{i,t}$ is a measurement error, which is i.i.d. over time and across individuals. $z_{i,t}$ is a stochastic time changing individual specific component which corresponds to (log of) s_t in the models. Moreover, this shock evolves according to

$$z_{i,t} \quad = \quad \eta z_{i,t-1} + \nu_{i,t} \quad \text{with} \quad \nu_{i,t} \sim N(0, \sigma_\nu^2). \tag{97}$$

The key parameter is the η, which gives the persistence of shock $\nu_{i,t}$, since $VAR(z) = \frac{\sigma_\nu^2}{1-\eta^2}$, given that $|\eta| < 1$. That is, the higher the persistence of shock, the higher the magnitude of idiosyncratic risk. The estimates for η varies from $\eta \approx 1$ to $\eta \approx 0.9$. Hence, it seems that the effects of the idiosyncratic shocks are very persistent (perhaps permanent). Meghir and Pistaferri (2010) provide a comprehensive review for the estimation of earning dynamics.

The importance of idiosyncratic uncertainty was questioned by Keane and Wolpin (1997), who argued that 90% of lifetime earnings dispersion is explained by factors that are predetermined before an individual enters into the labor market. Thus, the initial conditions of individuals explain almost all dispersion in the wage rates. However, Storesletten, Telmer, and Yaron (2004) estimated from the panel data that roughly 50% of earnings inequality is explained by the initial conditions and the rest by very persistent idiosyncratic shocks.

Distinguishing between the effects of the initial conditions and idiosyncratic risk for earnings dispersion is important since they matter significantly for policies which try to reduce the inequality between households. If the initial conditions matter the most, then the policy intervention should be targeted at childhood (like schooling), but if idiosyncratic shocks matter, policies should help unlucky workers to effectively smooth consumption. Moreover, as discussed in Section 4.1, the variance and the persistence of shocks are important for pinning down the magnitude of precautionary savings, and quite small differences in η have significant effects on the precautionary saving motive. Further, it should be noted that aggregate uncertainty does not matter greatly for wage uncertainty, as suggested by Heaton and Lucas (1996), but idiosyncratic uncertainty is all that matters (see also Part III).

Finally, it should be recognized that recent literature emphasizes that many components of wage dispersion are perhaps not generated by some exogenous shocks, but, rather, these shocks reflect endogenous choices. For instance, Huggett, Ventura, and Yaron (2006) models earnings dynamics via endogenous human capital accumulation and Schulhofer-Wohl (2008), Cagetti and De Nardi (2006) and Quadrini (2000) assume a difference in preferences which leads to different choices and these choices in turn lead to different sizes of risk in earnings. Low, Meghir, and Pistaferri (2010) have made an important contribution to this literature, by showing that allowing for job mobility has a large effect on the estimate of productivity risk.

5.1.2 Sources of uncertainty and channels of insurance

It is not only the idiosyncratic productivity risk that should be considered when one tries to perceive risk toward consumer's/household's earnings: there are many other sources of uncertainty as well. Moreover, typically in general equilibrium models the only insurance available is a some type of asset (a bond or productive capital), but there are a range of other channels that allow consumers to insure themselves against idiosyncratic risk.

Let me start by discussing the risks. The first significant risk – in addition to productivity shocks to wage rate – are health shocks. A decline in health status leads to a higher mortality risk and potentially causes large medical expenses. De Nardi, French, and Jones (2010) and Palumbo (1999) showed that a significant part of elderly saving is done for precautionary saving reasons to mitigate the effects of increasing medical expenses. Moreover, Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007) noted that many poor households have confronted a health shock which has caused large medical expenses, which is one reason why they are poor. Hence, uncertainty related to health status definitely matters for households' consumption and saving decisions. The second important source of uncertainty are changes in the composition of the family. Marital status or the birth of children affects individual or household consumption and saving decisions. They involve some risk, but some choices as well. For example, Cubeddu and Ríos-Rull (2003) argued that changes in marital status are important determinants for economic variables. Finally, it should be noted that the shocks that households face can be potentially correlated with one another. Then, a multivariate system is needed to capture various sources of idiosyncratic risk faced by the household/consumer.

The second crucial feature in the identification of idiosyncratic risk from the data – or when building a model – is to note that idiosyncratic risk is at least partly insured. Empirical evidence has rejected the complete market benchmark (see, Section 2) and I replaced that assumption with an assumption that there are only assets which can be used as a vehicle of self-insurance. However, the truth lies between these two polarised views. I follow Heathcote, Storesletten, and Violante (2009b) and discuss the following sources of insurance:

- Financial markets. In the standard model (introduced in Section 4) the borrowing constraint is fixed and no one was allowed to default on their loans. However, in the real world defaults do occur and borrowing limits vary depending on the circumstances of the aggregate economy and the attributes of the household. Hence, the option to default loans and the possibility to affect the level of the borrowing constraint provides households an extra source of insurance. For example, Athreya (2002); Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007); Atherya, Tam, and Young (2009) studied endogenously determined default behavior which is combined with the standard incomplete markets model.

- Insurance from labor supply. As discussed in Section 4.3.1 the labor supply decisions provide a source of insurance against idiosyncratic shocks.

- Insurance within the family. In the baseline model, a household consists of one breadwinner or it is like a "bachelor household". However, the family can find many ways to be insured against idiosyncratic shocks. Attanasio, Low, and Sánchez-Marcos (2005) argue that wives' ability to adjust their labor market participation provides valuable insurance. Moreover, Kaplan (2010) noted that young people's ability to live with their parents provides insurance against labor market risk.

- Public provision of insurance. The government provides insurance via redistributive taxation and social security. The effects of social security have been, for example, examined by Imrohoroglu, Imrohoroglu, and Joines (1995), Huggett and Parra (2010) and reviewed by Krueger (2006). Taxation has been considered, for instance, by Conesa, Kitao, and Krueger (2008), Heathcote (2005) and Domeij and Heathcote (2004). An interesting observation is that the provision of public insurance will reduce the need for the provision of private insurance (see, for example, Krueger and Perri, 2009; Attanasio and Ríos-Rull, 2000). Generally, an interesting question is that the beneficial effect of progressive income taxes and social security must be traded against the adverse effect on the incentives to supply labor, to accumulate capital and to provide insurance privately.

All in all, it can be concluded that there are many sources of uncertainty and many channels of insurance as well. Currently, the baseline model takes into account one or several sources of uncertainty and considers the most important source of insurance against uncertainty. The discussion above, however, implies that the risks probably correlate with one another and the different sources of insurance also depend on one another. These connections seem to offer interesting research questions. Questions related to the sources of risk and channels of insurance are definitely important, but it is too early to give any comprehensive conclusion.

5.2 Computation

Assume that one has pinned down all of the parameters needed for the model. Then, the next question is how to solve the model. It is obvious that very few analytical solutions are available – a notable exception is Toche (2005). But, numerical methods are needed to solve the model.[32] Generally, there are many options for solving general equilibrium models numerically:[33]

1. The Bellman equation and value function iteration. Discretize the state

[32]For more analytical solutions, see also Carroll and Toche (2009); Carroll and Jeanne (2009).
[33]Details can be found from Heer and Maussner (2009).

space and then iterate the value function or policy function until they have converged.

2. Linear quadratic programming. When the transition function (budget constraint) is linear and the return function (utility function) is quadratic, the Bellman equation can be solved just by using linear algebra. However, the solution is now based on the certainty equivalence. This method can be seen as a special case of the next method.

3. Approximation methods for a system of difference equations. Derive the first order conditions of the model. Then the budget constraint, the first order conditions and an evolution equation for stochasticity create a system of stochastic difference equations. These equations can be approximated with a Taylor series around their steady state with the following choices:

   - Linear (first order) approximation. A solution, which is unique and stable, for the linear system of difference equations can be derived, as suggested by Blanchard and Kahn (1980), who applied Jordan canonical form, by Uhlig (1999) who applied the method of undetermined coefficients, by Klein (2000), who applied a general Schur method and by King and Watson (2002), who applied a system reduction.

   - Second order approximation. A solution with a quadratic approximation can be derived by applying tensors (see Schmitt-Grohé and Uribe, 2004) or standard matrix algebra (see Gomme and Klein, 2009).

4. Parametrized expectations. den Haan and Marcet (1990) proposed a parameterized expectations approach whereby a solution for the model can be derivated by approximating a time invariant function, which agents use to predict tomorrow's economic environment. The Monte Carlo techniques are used to determine the unknown parameters of the expectation function.

5. Projection methods. This method generalizes the previous one. Here, one makes a polynomial approximation (for example, using Chebyshev polynomial) for the policy function and uses some method (for instance, least squares) to update the parameters of the polynomial.

In the representative agent model the solution for the consumer's problem solves the model, since the economy consists of one agent, i.e. the economy seems like Robinson Crusoe living on a desert island. In that case, the consumer's consumption decision gives the next period's prices since the next period's capital stock is determined within the choice of consumption. However, with heterogeneous agents it is clear that the computation of the solution is more complicated, since solving the consumers' problem leaves open the question: How do you compute or handle the type distribution, or equally, how do you ensure the consistency between aggregate and individual behavior? Further, there are two additional features in the models introduced in Section 4 which deviate from the standard problem of the representative agent: 1) the problem of consumers consists of an occasionally binding borrowing constraint, 2) the solution for the consumers' problem needs to be accurate for the whole (bounded) state space since, potentially, consumers are not concentrated around the steady state.

In the following subsections, I will first, discuss how to solve the representative consumer's problem in the models described in Section 4. The two conditions given above will rule out the use of Taylor approximation, i.e. the methods described in item 3 in the list provided at the beginning of this section.[34] Approximation around a steady state does not work any more since I need to know consumers' decisions within every point in the state space. For example, I need to know whether the borrowing constraint is binding or not. Second, some numerical tools are needed for finding a stationary distribution in Aiyagari's model and somehow one must handle the dynamics of the type distribution in Krusell and Smith's model.

5.2.1 Solving the problem of consumers

There are several ways to solve the problem of consumers given in Section 4 by equations (57)-(60). The standard linear approximation is ruled out since we need an accurate solution for the whole state space. Hence, the use of Bellman equations is a suitable choice. One can solve the Bellman equations by iterating – maybe by using Howard's improvement algorithm or projection methods to make the convergence faster – but, generally, a better idea is to iterate on policy functions since their convergence is faster than the convergence of value function.

The standard method is to fix a grid for a policy function over the state variables, for example today's asset holdings a_t, but then there is a time-consuming

[34] However, recent computational methods by Preston and Roca (2007) and Kim, Kollmann, and Kim (2010) showed how Taylor approximation can be used in these types of problems.

numerical root-finding procedure at every point of the grid. The endogenous grid point method proposed by Carroll (2006) avoids this time-consuming step. The trick is that the grid is fixed over tomorrow's assets a_{t+1} and interpolation is used to update the policy function.

Consider the Euler equation for the problem given in equations (57)-(60) and assume that the type distribution has now reached its ergodic distribution when $r_{t+1} = r_t = r$ and $w_{t+1} = w_t = w$. Then the Euler equation is as follows:

$$U_c(c_t) \geq \beta(1+r) \sum_{s_{t+1} \in S} \pi(s_{t+1} \mid s_t) U_c(c_{t+1}) \quad \forall \quad s_t \in S, \tag{98}$$

and replace $c_t = (1+r)a_t + y_t - a_{t+1}$ and $c_{t+1} = (1+r)a_{t+1} + y_{t+1} - a_{t+2}$ by utilizing the budget constraint where $y = sw\bar{l}$. The algorithm that gives a policy function, which solves (98), involves the following steps:

Algorithm 1. *The endogenous gridpoint method.*

1. *Begin with fixing a grid on (a_{t+1}, s_t) where $a_{t+1} \in A_G = \{\underline{a}, \ldots, \bar{a}\}$ and $s_t \in S$. Moreover, guess a policy function $g^{a_{t+2}}(a_{t+1}, s_{t+1})$, a linear policy function $a_{t+2} = a_{t+1}$ for all s_{t+1} is a good guess.*

2. *Construct the right hand side of (98) (at the grid points) and call it $E(a_{t+1}, s_t)$, i.e.*

$$\begin{aligned} E(a_{t+1}, s_t) &= \beta(1+r) \sum_{s_{t+1} \in S} \pi(s_{t+1} \mid s_t) U_c\left([1+r]\, a_{t+1}\right. \\ &\quad \left. + y_{t+1} - g^{a_{t+2}}(a_{t+1}, s_{t+1})\right). \end{aligned} \tag{99}$$

   Note that this can be done since the grid is now set for a_{t+1} and πs are known.

3. *Use the Euler equation (98) and solve a_t. For example, if $U = \frac{c^{1-\theta}}{1-\theta}$, then*

$$[(1+r)a_t + y_t - a_{t+1}]^{-\theta} = E(a_{t+1}, s_t) \tag{100}$$

$$a_t = \frac{E(a_{t+1}, s_t)^{-\frac{1}{\theta}} - y_t + a_{t+1}}{1+r}. \tag{101}$$

   Hence, you have now defined the endogenous grid points, a_t.

4. *Use these new endogenously defined grid points to update the policy func-*

tion $g^{a_{t+2}}(a_{t+1}, s_{t+1})$ by interpolation and take into account the borrowing constraint when it is binding.

5. *Check the convergence of the policy function and stop if*

$$\max\left\{\left|g^{a_{t+2}}_{n+1} - g^{a_{t+2}}_{n}\right|\right\} < \epsilon, \tag{102}$$

where ϵ is some predetermined error tolerance. Otherwise go back to item 2.

The interpolation is the most fastest way to update the policy function and that makes this method superior to any other method for these types of problems. It should be noted that the idea of the endogenous grid point method – i.e. it is up to the researcher to decide which variable to define as the exogenous grid – can be applied to many different types of questions as shown by Barillas and Fernández-Villaverde (2007).

5.2.2 Approximating the type distribution

Now, I have an optimal decision rule $g^a(a_t, s_t)$ which delivers consumers' next period asset holdings a_{t+1}. However, I still need to find a stationary type distribution $\Phi(a, s)$ which is an infinite object and I need to somehow approximate it. Here, I consider methods which rely on the discretization of the density function. Note that the second dimension of density function, Φ, is discrete and can take the values $s \in \{s_1, \ldots, s_N\}$, so I only need to discretize the first dimension.

As discussed in Section 4.1.2, even though a unique and invariant type distribution exists (if sufficient conditions hold), the problem is how to compute it. The invariant distribution is a density function, $\Phi(a_{t+1}, s_{t+1})$, which satisfies

$$\Phi(a_{t+1}, s_{t+1}) = \sum_{s_t \in S} \pi(s_{t+1} \mid s_t)\Phi\left(g^{a^{-1}}(a_{t+1}, s_t), s_t\right) \quad \forall \quad (a_{t+1}, s_{t+1}) \in \mathcal{A} \times S$$

where $g^{a^{-1}}$ is the inverse of g^a. Note that, the density is a continuous object, when I cannot use the same methods to find an invariant distribution as in the case of a finite state Markov chain. Still, I need to approximate this continuous distribution somehow.

To compute that distribution I need to discretize the asset space by the grid

$A_f = \{\underline{a}, \ldots, \bar{a}\}$ and the grid must be more dense than the one used to compute the optimal decision rules. The problem is that the next period asset holdings $a_{t+1} = g^a(a_t, s_t)$ will be on the grid with probability zero. Hence, I must introduce a lottery: if a_{t+1} lies between the two points of grid a^j and a^{j-1} on the grid, then the next period capital stock is a^j with a probability $\frac{a_{t+1}-a^{j-1}}{a^j-a^{j-1}}$ and it is a^{j-1} with a complementary probability $\frac{a^j-a_{t+1}}{a^j-a^{j-1}}$. I define the approximate density also with Φ for notational reasons and the invariant density can be computed with the help of the following algorithm:

Algorithm 2. *Heer and Maussner (2009, Algorithm 7.2.3): Computation of the invariant density* Φ

1. *Set a grid A_f over $A \in [\underline{a}, \bar{a}]$. It should be finer than the one used to compute the decision rule.*

2. *Choose the initial $\Phi_{i=0}$ (uniform distribution) over $A_f \in [\underline{a}, \bar{a}]$ for each $s_t \in S$.*

3. *For every $a_t \in A_f$ and $s_t \in S$ compute the optimal next period wealth $a_{t+1} = g^a(a_t, s_t)$. Then for all $a_{t+1} \in A$ and $s_{t+1} \in S$ calculate the following sums:*

$$\Phi_{i+1}(a^{j-1}, s_{t+1}) = \sum_{s_t \in S} \sum_{\substack{a_t \in A_f \\ a^{j-1} \leq a_{t+1} < a^j}} \pi(s_{t+1} \mid s_t) \frac{a^j - a_{t+1}}{a^j - a^{j-1}} \Phi_i(a_t, s_t) \tag{103}$$

$$\Phi_{i+1}(a^j, s_{t+1}) = \sum_{s_t \in S} \sum_{\substack{a_t \in A_f \\ a^{j-1} < a_{t+1} < a^j}} \pi(s_{t+1} \mid s_t) \frac{a_{t+1} - a^{j-1}}{a^j - a^{j-1}} \Phi_i(a_t, s_t) \tag{104}$$

4. *Iterate until Φ converges.*

The same type of algorithm can also be constructed for distribution function, see Heer and Maussner (2009, Algorithm 7.2.2) and Ríos-Rull (1999). However, the computation time for the density function is much shorter. Another way to compute the stationary distribution is to use a Monte Carlo simulation. This technique is easy to implement and useful when considering Krusell and Smith's model. However, this method has bad convergence properties, i.e. Algorithm 2 should always be applied, if it is feasible to implement. The algorithm is as follows:

Algorithm 3. *Computation of invariant distribution function via Monte Carlo simulation*

1. *Choose the number of agents/consumers, N (for example $N = 10000$), to follow through time.*

2. *For every agent $i = 1, \ldots, N$ give a initial state $(a^i_{t=0}, s^i_{t=0})$.*

3. *Compute for every i $a^i_{t+1} = g^a(a^i_t, s^i_t)$ and, by using a random number generator and π, simulate a new s^i_{t+1} for every i. If the number of agents is higher in some state s than implied by the ergodic distribution of π, randomly change the state of agents such that the number of agents in the states is equal to the ergodic distribution of π.*

4. *Compute a set of statistics (1. and 2. moments) from the sample $\left\{a^i_{t+1}, s^i_{t+1}\right\}_{i=1}^N$.*

5. *Simulate until the statistics (moments) converge.*

Now I have all I need for an algorithm that solves the model by Aiyagari (1994) and delivers the stationary equilibrium:

Algorithm 4. *Computing the stationary equilibrium*

1. *Guess the value of aggregate capital stock K^0. This implies the prices r and w (use the first order conditions for the firm).*

2. *Given r and w, solve the problem of consumers using Algorithm 1. Obtain the optimal decision rule $g^a(a, s)$.*

3. *Using $g^a(a, s)$ and π, find a stationary distribution, for example, by using Algorithm 2.*

4. *Compute the aggregate capital supply K^1:*

$$K^1 = \int_{\mathcal{A}\times S} ad\Phi \tag{105}$$

5. *If $\max\{|K^1 - K^0|\} < \epsilon$, then stop (the model is solved). Otherwise, update the guess for the steady state capital stock K^0_{new} using:*

$$K^0_{new} = \lambda K^1 + (1-\lambda)K^0, \quad where \quad \lambda \in (0, 1]\,. \tag{106}$$

   Calculate the new r and w and go back to item 2.

5.2.3 The dynamics of the type distribution

To be able to solve Krusell and Smith (1998), I need some methods to handle the dynamics of the type distribution, Φ, since the aggregate productivity shocks cause the type distribution to evolve over time. As discussed in Section 4.2, the aggregate state can be described with (aggregate) productivity shock γ and with some moment of marginal distribution of $\Phi(\bullet, s)$: for example, I could use the first moment, K_t. Moreover, the transition function for the type distribution could just be a log-linear function where the arguments are K_t, γ_t and γ_{t+1}.

The main difference between Aiyagari's and Krusell and Smith's model is that, in the latter, the value function consists K_t and γ_t whereas in the former these same terms were constant and could be fixed before solving the problem of consumers. Hence, now I need to also discretize the state space on the dimensions of K and γ since consumers' decisions depend on the aggregate state as well. With a grid over K and γ I can calculate r_t and w_t (in the grid points), but I also need r_{t+1} and w_{t+1}. Thus, I need to fix the parameters of the forecasting function (see equation (90)), which delivers K_{t+1}, and hence, tomorrow's prices. More precisely, the following algorithm solves a heterogeneous agent model with aggregate uncertainty:

Algorithm 5. *Solution algorithm for a heterogeneous agent model with aggregate uncertainty.*

1. *Choose a statistic to describe the marginal distribution of the type distribution over the individual's asset holdings, for example the mean, i.e.* K_t. *Set a grid for this statistic, for example* $K = \{K_1, \ldots, K_{n_k}\}$. *The number of grid points can be low (for instace, Krusell and Smith used 25 points).*

2. *Parametrize the forecasting function for the statistic that you previously chose. For example, guess a log-linear form*

$$\log K_{t+1} = h_{1,\gamma} + h_{2,\gamma} \log K_t \quad \textit{for every} \quad \gamma \in \Gamma. \tag{107}$$

   Set an initial guess for the parameters $h_{1,\gamma}$ *and* $h_{2,\gamma}$. *A good guess would be* $h_{1,\gamma} = 0$ *and* $h_{2,\gamma} = 1$.

3. *For all points in* $K = \{K_1, \ldots, K_{n_k}\}$ *and for all* $\gamma \in \Gamma$ *solve today's and tomorrow's prices* r_t, w_t, r_{t+1} *and* w_{t+1}.

4. *Given the prices, solve the problem of consumers, for example using Algorithm 1. Your value function is now* $V(a_t, s_t, K_t, \gamma_t)$*. Note that to compute the value function or decision rule in the next period, you need to interpolate also in the dimension of* K*.*

5. *Given the optimal decision rule* $g^a(a_t, s_t, K_t, \gamma_t) = a_{t+1}$ *from the previous step, implement a simulation which simulates the dynamics of the type distribution. Specifically, do the following:*

   - *Set the simulation period* T *and decide how many periods of observations are discarded from the star of simulations,* T_0*.*

   - *Using a random number generator and* π*, draw a sequence of* γ_t *and* s_t *for all* T*.*

   - *Set the initial distribution for asset holdings, for example the uniform distribution ,and calculate* K_0*.*

   - *Apply a Monte Carlo simulation for the dynamics of the type distribution (see Algorithm 3) using* γ_t*,* s_t *and* $g^a(a_t, s_t, K_t, \gamma_t)$*. Every round of simulations gives you a new distribution of assets from which you can calculate* K_{t+1}*.*

   - *Keep simulating until* $t = T$*. Then you have a sequence* $\{K_t\}_{t=0}^{T}$*.*

6. *Use the simulated data,* $\{\gamma_t, K_t\}_{t=T_0+1}^{T}$*, and OLS regression to update your coefficients,* $h_{1,\gamma}$ *and* $h_{2,\gamma}$*, in the forecasting function. Note that the first* T_0 *observations are dropped to eliminate the influence of the choice of the initial distribution.*

7. *Compare* $h_{1,\gamma}^i$*,* $h_{2,\gamma}^i$ *and* $h_{1,\gamma}^{i+1}$*,* $h_{2,\gamma}^{i+1}$ *between simulation rounds* i *and* $i+1$*. If the distance between the values of coefficients is less than some predetermined tolerance level the model is solved. Otherwise, update the values of coefficients utilizing the following formulas:*

$$h_{1,\gamma} = \lambda h_{1,\gamma}^{i+1} + (1-\lambda) h_{1,\gamma}^{i} \tag{108}$$

$$h_{2,\gamma} = \lambda h_{2,\gamma}^{i+1} + (1-\lambda) h_{2,\gamma}^{i} \quad \textit{where} \quad \lambda \in (0,1] \tag{109}$$

   and return to item 3.

8. *If the values of coefficients have converged, check the fit of forecasting func-*

tion, for example using R^2. If the fit is satisfactory, you are done. Otherwise, choose a different functional form for the forecasting function, i.e. go back to item 2.

den Haan, Judd, and Juillard (2010) and other paper in that issue consider different algorithms to solve a heterogeneous agent model with aggregate uncertainty. Generally, it could be said that more accurate and faster algorithms than this one exist, but the strength of Algorithm 5 is that it is easy to implement and it is robust.

6 The intertemporal choice of households under incomplete markets in general equilibrium: Concluding remarks

In the last 20 years macroeconomics has developed from the study of aggregate dynamics to the study of the dynamics of the entire distribution of allocations across economic actors . As a result the standard incomplete markets models – which were introduced in Section 4 – have become as a workhorse model for many questions in macroeconomics.

It is obvious that, for some questions, the incomplete markets model with heterogeneous agents is indispensable. However, there are some questions where it seems that heterogeneity does not, matter as shown by Caselli and Ventura (2000). Hence, the representative agent model seems to be an appropriate model for some questions. That is, the representative agent model is robust for some questions and this robustness can be tested by using the standard incomplete markets model with heterogeneous agents.

However, the standard incomplete markets model with heterogeneous agents has displaced the representative agent model in several questions studied in macroeconomics. The main problem with the representative agent model – in addition to the complete markets assumption – is that the agent is too rich. For example, a significant amount of aggregate consumption is done by people whose wealth holdings are at a quite low level. Moreover, issues related to taxation, social security or the cost of the business cycles cannot be studied with a representative agent since that agent has a significant buffer (assets) for "a rainy day". Furthermore, one of the most important questions is to explain the inequality that we observe in the data or why there is a significant amount of people who do not have a buffer against "a rainy day".

Heterogeneity and incomplete markets matter significantly for many macroeconomic phenomenons. Then there exists such issues as rich and poor people, involuntary unemployment and many more interesting aspects of the real world that are missed by the representative agent economy. DSGE models with incomplete markets provide a useful synthesis between a macro economically meaningful question and solid micro foundations – or heterogeneity among consumers. As a result, individuals' decisions are based on their innate characteristics and their luck. These decisions in turn define how the aggregate economy behaves. To summarize, I use the words of John M. Keynes: "Economics is a science of thinking in terms of models joined to the art of choosing models which are relevant to the contemporary world".

7 A summary for the essays

The study in Part II discusses the relationship between aggregate consumption and credit. In this paper, I extend the model by Krusell and Smith (1998) with stochastically time-varying liquidity constraint. Consumption and unsecured credit are correlated in the data. This fact has shaped a hypothesis which argues that the stochastically time-varying liquidity constraint – which may also be thought of as credit shocks – matters for the dynamics of aggregate consumption. If liquidity constraints matter for aggregate consumption, then procyclical credit shocks create a "financial accelerator" for the household sector. That is, credit shocks potentially cause fluctuations for other real variables by affecting aggregate consumption.

The simulations of the model imply that the time-varying credit constraint – or the credit shocks – do not matter for determination of aggregate consumption, even if the size of the shocks is set larger than the data support. The time-varying credit constraint matters only for a very few people, when the effects of other shocks, combined with the fixed liquidity constraint, determine the dynamics of consumption. I confirm that the time-varying liquidity constraint matters for poor people's consumption decisions, but their effect on the dynamics of aggregate consumption is insignificant. Hence, the positive correlation between credit and consumption does not originate from the behavior of liquidity constrained households, as argued in some studies. The positive correlation could be driven by the demand for credit – i.e. consumption – and the supply of credit merely adjusts to these changes. Moreover, there could be some other mechanism that generates the positive relationship.

The study in Part III considers the magnitude of saving which is done due to precautionary saving reasons. The question is an important one, since precautionary wealth possibly affects aggregate wealth accumulation, asset prices and optimal policy design. I use a general equilibrium setup, ála Aiyagari (1994), to derive a measure for the magnitude of precautionary wealth by using Finnish data. The measure is based on a similar Euler equation decomposition used by Gourinchas and Parker (2001) and Parker and Preston (2005) in a partial equilibrium framework. This approach keeps the analysis a fairly simple, since there is no need to estimate the actual magnitude of idiosyncratic shocks or solve the underlying model. The study provides a Solow-residual-type measure for the magnitude of precautionary wealth: the capital accumulation, which cannot be explained by the intertemporal reason of saving, must be a consequence of the precautionary saving motive. This unexplained part gives the magnitude of precautionary savings or the precautionary wealth in the economy.

The results from the paper are twofold. First, the magnitude of wealth, which is a consequence of precautionary saving motives, can directly be observed just by focusing on the aggregate Euler equations when the unknown parameters of the utility function are estimated. Second, the use of the logarithmic utility function is a consistent choice with an assumption about market completeness, which is the key assumption for aggregation in the standard growth model. Hence, the calibration of utility function is rationalized by an assumption that precautionary motives do not matter for capital accumulation in the long run. Then, the choice of IES is not based on micro studies, since these parameters may not be used in general equilibrium models directly, as argued Browning, Hansen, and Heckman (1999).

The last study, in Part IV, considers social security and disability insurance in particular. The disability insurance program is one of the largest social insurance programs in the United States and it is well known that the costs of any social insurance programs come from two sources: first, the social insurance program has substantial undesirable effects on incentives and, therefore, on economic performance. Second, these programs are often financed by a proportional tax rate, which further distorts the economy. The model introduced in this study provides a new extension for the standard textbook Ramsey model. I extend the Ramsey model by including a precautionary saving motive for households. That is, there is a risk for the permanent loss of job, which captures the uncertainty associated with disability. Moreover, in the model government provides the social insurance which affects households' economic behavior by removing the self-insurance (precautionary saving) motive. This program is financed by a proportional tax rate, which reduces the incentives of households to accumulate capital and supply labor. Hence, this model provides a tool to measure the cost of the disability insurance program in the framework of the standard Ramsey model.

The model implies that closing the current disability insurance program would increase per capita consumption by 2.5%. One-third of this burden is caused by higher tax rates and 2/3 comes from the change in economic behavior, i.e. from the removed precautionary saving motive. It is surprising that the labor supply decision does not depend on the level of disability insurance, even if taxes are increased when the level of insurance is increased. The supply of labor, however, is almost constant. Nonetheless, self-insurance works poorly against permanent shocks, and, therefore, the social insurance program increases welfare by providing a higher level of consumption for disabled households. But the real problem is the incompleteness of the private insurance markets, since the cost of this incompleteness can be almost two times larger than the cost arising from the social insurance. If the perfect insurance against permanent disability were provided by private insurance companies, rather than the social insurance program which is financed by the proportional tax rate, per capita consumption would increase by 3.5-7% depending on the elasticity of labor supply. The result implies that optimizing the tax-financed social insurance systems is not the best way to improve welfare. Rather, completing the markets by removing impediments to the private provision of insurance would generate a much more higher increase in welfare. Another way to interpret this result is that the cost generated by problems associated with imperfect information – which prevents market-based solutions – are indeed very large.

The solutions to the models have been done using MATLAB, Excel and Mathematica. All codes are available at the author's homepage:
`http://taloustiede.utu.fi/laitos/henkilokunta/kortela/index.html`
or upon a request: `tomi.kortela@utu.fi`.

Appendices for Part I

A The evolution of asset under the certainty equivalence

Assume an environment given in Section 3.1.1. The certainty equivalence implies a following consumption function

$$c_t = \frac{r}{1+r}a_t + \frac{r}{1+r}E_t\sum_{j=0}^{\infty}(1+r)^{-j}y_{t+j}. \tag{A1}$$

Further, define saving for period t as

$$s_t = \frac{r}{1+r}a_t + y_t - c_t. \tag{A2}$$

Now substituting equation (A1) into the definition of s_t (equation (A2)) and taking out y_t from equation I get

$$-s_t = -\frac{y_t}{1+r} + \frac{r}{1+r}\sum_{j=1}^{\infty}\frac{E_t y_{t+j}}{(1+r)^j} \tag{A3}$$

where I have multiplied equation by -1. Then take out from the summation term $\frac{E_t y_{t+1}}{(1+r)}$ and plug the term $\pm\frac{E_t y_{t+1}}{(1+r)^2}$ inside the equation:

$$-s_t = \frac{E_t\Delta y_{t+1}}{1+r} - \frac{E_t y_{t+1}}{(1+r)^2} + \frac{r}{1+r}\sum_{j=2}^{\infty}\frac{E_t y_{t+j}}{(1+r)^j} \tag{A4}$$

where Δ is difference operator ($\Delta x_{t+n} = x_{t+n} - x_{t+n-1}$). I can continue forward at the same method, and finally get

$$s_t = -\sum_{j=1}^{\infty}\frac{E_t\Delta y_{t+j}}{(1+r)^j} \tag{A5}$$

which is the famous "saving for a rainy day" equation given by Cambell (1987). So, saving should increase when expected income is decreasing and vise versa. Now rewrite the budget constraint (29) which yields

$$\Delta a_{t+1} = (1+r)(y_t - c_t) + ra_t, \tag{A6}$$

where the right hand side of equation equals to $(1 + r)s_t$, see equation (A2). Hence, using the definition of s_t, given by equation (A5), I get

$$\Delta a_{t+1} = -(1 + r) \sum_{j=1}^{\infty} \frac{E_t \Delta y_{t+j}}{(1 + r)^j}. \tag{A7}$$

If income follows i.i.d. with zero mean, I have for $j = 1$ $E_t \Delta y_{t+1} = -u_t$ and for $j > 1$ $E_t \Delta y_{t+j} = 0$. Hence,

$$\Delta a_{t+1} = u_t. \tag{A8}$$

That is, the evolution of asset follows a random walk.

B A sufficient condition for the upper bound of the state space

Assume that $\frac{1+r}{1+\rho} < 1$ and income shocks are i.i.d.[35] Let $c(x)$ be a policy function for consumption, where x is the cash on hand. Here I focus on the upper limit of the state space when I assume that the borrowing constraint do not bind. Then Euler equation (46) can be rewritten as

$$U_c\left(c(x_t)\right) = \frac{1+r}{1+\rho}E_t\left[U_c\left(c(x_{t+1})\right)\right] \tag{A9}$$

$$= \frac{1+r}{1+\rho}\frac{E_t\left[U_c\left(c(x_{t+1})\right)\right]}{U_c\left(c(x_{t+1}^{max})\right)}U_c\left(c(x_{t+1}^{max})\right), \tag{A10}$$

where x_{t+1}^{max} is the cash on hand on tomorrow when $y_{t+1} = y^{max}$ has realized.

Assume that following holds

$$\lim_{x\to\infty}\frac{E_t\left[U_c\left(c(x_{t+1})\right)\right]}{U_c\left(c(x_{t+1}^{max})\right)} = 1. \tag{A11}$$

Then, with large enough x_t, equation (A10) gives

$$U_c\left(c(x_t)\right) = \frac{1+r}{1+\rho}U_c\left(c(x_{t+1}^{max})\right) < U_c\left(c(x_{t+1}^{max})\right) \tag{A12}$$

which in turn implies (given that U is concave and c is monotone) that

$$c(x_{t+1}^{max}) < c(x_t) \Rightarrow x_{t+1}^{max} < x_t, \tag{A13}$$

since $c_x(x) > 0$. Hence, for x_t large enough $x_{t+1} < x_t$, i.e. there is an upper limit for assets for sure.

The lasting to do is to show under which circumstances the limit given by equation (A11) holds. To do that take a first-order Taylor approximation of $U_c\left(c(x_{t+1})\right)$ around $x_{t+1} = x_{t+1}^{max}$:

$$U_c\left(c(x_{t+1})\right) \approx U_c\left(c(x_{t+1}^{max})\right) + U_{cc}\left(c(x_{t+1}^{max})\right)c_x(x_{t+1}^{max})(x_{t+1} - x_{t+1}^{max}) \tag{A14}$$

and take a conditional expectation from both sides, which yields after some mod-

[35] This section is based on Kjetil Storesletten's lecture notes.

ifications

$$E_t\left[U_c\left(c(x_{t+1})\right)\right] \approx U_c\left(c(x^{max}_{t+1})\right) - U_{cc}\left(c(x^{max}_{t+1})\right) c_x(x^{max}_{t+1}) \left(y^{max}_{t+1} - E_t\left[y_{t+1}\right]\right). \quad \text{(A15)}$$

To derive equation (A15) from (A14) I used the facts that $x^{max}_{t+1} - x_{t+1} \equiv y^{max}_{t+1} - y_{t+1}$ and x^{max}_{t+1} is deterministic since it is defined by y^{max}_{t+1} and $a(x)$ which is a policy function for next period assets holdings. Now divide both sides with the term $U_c\left(c(x^{max}_{t+1})\right)$:

$$\frac{E_t\left[U_c\left(c(x_{t+1})\right)\right]}{U_c\left(c(x^{max}_{t+1})\right)} \approx 1 + \hat{\xi}\left(c(x^{max}_{t+1})\right)\left(y^{max}_{t+1} - E_t\left[y_t\right]\right) c_x(x^{max}_{t+1}), \quad \text{(A16)}$$

where

$$\hat{\xi}\left(c(x^{max}_{t+1})\right) = \frac{-U_{cc}\left(c(x^{max}_{t+1})\right)}{U_c\left(c(x^{max}_{t+1})\right)}.$$

That is, $\hat{\xi}$ is the coefficient of absolute risk aversion. Note that terms $\left(y^{max}_{t+1} - E_t\left[y_t\right]\right)$ and $c_x(x^{max}_{t+1})$ are both positive and finite when to satisfy the limit given by equation (A11) it must be the case that in equation (A16)

$$\lim_{x \to \infty} \hat{\xi}\left(c(x^{max}_{t+1})\right) = 0. \quad \text{(A17)}$$

Hence, if one want to have an upper bound for the state space, the utility function must be such that the absolute risk aversion is decreasing with asset holdings.

C Approximating an AR(1)-process with a Markov chain

I follow Tauchen (1986) to approximate an AR(1)-process with a finite state Markov chain. Consider an AR(1)-process with unconditional mean μ and an autocorrelation parameter η:

$$Z_{t+1} = (1-\eta)\mu + \eta Z_t + \epsilon_t \quad \text{with} \quad \epsilon_t \sim N(0, \sigma_\epsilon^2), \tag{A18}$$

when unconditional standard deviation for the process is $\sigma_Z = \frac{\sigma_\epsilon}{\sqrt{1-\eta^2}}$.

To begin, choose a grid which is evenly-spaced along the real line $z_1 < z_2 < \ldots < z_m$. The upper and lower bounds on the range are a multiple, λ, of the standard deviation of the autoregressive process. That is,

$$z_1 = \mu - \lambda\sigma_Z \tag{A19}$$

$$z_m = \mu + \lambda\sigma_Z. \tag{A20}$$

Now let the size of the step in the grid be $s = z_j - z_{j-1}$. Moreover, Tauchen's method is based on the fact that conditional on Z_t, Z_{t+1} is normally distributed with mean $(1-\eta)\mu + \eta Z_t$ and standard deviation σ_ϵ. Hence, the variable $v = \frac{z-[(1-\eta)\mu+\eta z_i]}{\sigma_\epsilon}$ has a standard normal distribution. Thus, the probability, p_{ij}, is a probability to move from point z_i to point z_j for $j = 2, 3, \ldots, m-1$, which equals to the area under the density function of the standard normal distribution in the interval

$$\left[\frac{z_j - [(1-\eta)\mu + \eta z_i] - s}{\sigma_\epsilon}, \frac{z_j - [(1-\eta)\mu + \eta z_i] + s}{\sigma_\epsilon}\right].$$

The probability to arrive at state z_1 is the area under the density in the interval $[-\infty, z_1 + s]$ and finally the probability to go from any state i to z_m is $p_{im} = 1 - \sum_{j=1}^{m-1} p_{ij}$.

The method can be summarized by a following algorithm:

Algorithm 6. *Approximating AR(1)-process with a finite state Markov ála Tauchen (1986).*

1. *Compute the unconditional mean μ and standard deviation σ_Z. Moreover, calculate autocorrelation parameter η and conditional standard deviation σ_ϵ.*

2. *Decide the number of grid points m and set a grid $z_1 < z_2 < \ldots < z_m$ over the state space as indicated by (A19) and (A20) when $s = z_j - z_{j-1}$.*

3. *Compute the transition matrix $P = (p_{ij})$ utilizing the cumulative distribution function of the standard normal distribution Φ. For $i = 1, \ldots, m$ set*

$$p_{ij} = \begin{cases} \Phi\left(\frac{z_1 - [(1-\eta)\mu + \eta z_i]}{\sigma_\epsilon} + \frac{s}{2\sigma_\epsilon}\right) & \text{for } \ j = 1 \\ \Phi\left(\frac{z_j - [(1-\eta)\mu + \eta z_i]}{\sigma_\epsilon} + \frac{s}{2\sigma_\epsilon}\right) - \Phi\left(\frac{z_j - [(1-\eta)\mu + \eta z_i]}{\sigma_\epsilon} - \frac{s}{2\sigma_\epsilon}\right) & \text{for } \ 1 < j < m \\ 1 - \sum_{j=1}^{m-1} p_{ij} & \text{for } \ j = m. \end{cases}$$

Closer the η gets to one the worser is the quality of approximation. Other methods that do the same approximation are given by Tauchen and Hussey (1991) and Rouwenhorst (1995). Kopecky and Suen (2010) compares the performance of these methods.

Part II

Essay I: Do credit shocks matter for aggregate consumption?

Do credit shocks matter for aggregate consumption?

Tomi Kortela*

Abstract

Consumption and unsecured credit are correlated in the data. This fact has created a hypothesis which argues that stochastically varying liquidity constraints – or credit shocks – matter for aggregate consumption. According to the hypothesis, there are a significant number of households who are liquidity constrained and, therefore, they change the level of consumption according to the availability of credit. However, I conclude in a general equilibrium framework that credit shocks are not quantitatively important for the aggregate economy. Hence, typical fluctuations in credit do not matter for the dynamics of aggregate consumption, but the existence of fixed liquidity constraint, combined with other shocks, dominate the dynamics of aggregate consumption.

JEL Codes: E21, E32, E44 ,E51

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1 Introduction

Issues concerning credit and consumption are highlighted by the current financial crises, since the level of credit and consumption have been sharply decreased. This fact could be interpreted so that the availability of credit has at least partly caused the drop in consumption.[1] Generally, it seems that credit market conditions matter for business cycle dynamics and a channel for credit shocks to propagate wider in the economy is through aggregate consumption.

One mechanism for how the availability of credit matters for the dynamics of aggregate consumption is based on the significance of liquidity constraints when households make their consumption decisions (Deaton, 1991; Huggett, 1993; Ludvigson and Michaelides, 2001; Gourinchas and Parker, 2002).[2] The availability of credit matters for the dynamics of aggregate consumption if a significant number of households are liquidity constrained or the presence of liquidity constraint affects their behavior powerfully, since it might bind in the future. When credit conditions vary jointly with current aggregate circumstance, this will cause the constrained households', i.e. households that like to consume more – but cannot – since the liquidity constraint is binding (or will bind), to change their level of consumption according to the availability of credit. Hence, credit market conditions may matter for business cycle dynamics through aggregate consumption. However, there is controversy over the importance of credit (or liquidity constraints) for consumption behavior. Empirical evidence based on micro and macro data suggest that the availability of credit matters significantly for consumption (Bacchetta and Gerlach, 1997; Ludvigson, 1999; Gross and Souleles, 2002; Alessie, Hochguertel, and Weber, 2005), but other studies have found the effect for aggregate consumption to be quite small (see, Ludvigson, 1998; Leth-Petersen, 2010).

[1] Moreover, the decreased level of credit has been one motivation to stimulate aggregate demand via fiscal policy, as Mankiw (2010) puts it: " [The Obama Administration] thought that, because of the credit crisis, people were not able to obtain loans; and, because people were not able to obtain loans, there was insufficient aggregate demand."

[2] Here, I just focus on the direct effects of credit shocks for the consumption of households, i.e. extra credit provides more resources for households to consume. However, there are other ways that credit may affect aggregate consumption. First, credit could matter for aggregate consumption through the financial accelerator mechanism (Kiyotaki and Moore, 1997; Bernanke, Getler, and Gilchirst, 1999) or via financial sector shocks (Jermann and Quadrini, 2009). Both of these mechanisms works through the production or investment sector and, in the general equilibrium, they matter also for aggregate consumption. Second, credit can be an alternative to money in exchange when shocks to the banking sector matter for business cycle dynamics and aggregate consumption (see, Benk, Gillman, and Kejak, 2005, 2008).

This paper takes a new approach to study the significance of liquidity constraints for the dynamics of aggregate consumption. The focus is on business cycle dynamics: this paper tries to test whether the fluctuating availability of credit matters for the dynamics of aggregate consumption. To study the quantitative importance of stochastically time-varying liquidity constraints – or credit shocks – for the dynamics of aggregate consumption, I utilize a general equilibrium model by Krusell and Smith (1998), to which I add stochastically time-varying liquidity constraints. To analyze the quantitative significance of credit shocks, productivity and employment shocks are needed to capture fluctuations in the incomes of households. Then, one can add the credit shocks to see their contribution to the dynamics of aggregate consumption.

There are several advantages in this approach compared to previous ones. First, even if it is well known that there is a dependency between consumption and credit the key question is which way the causality is running. It may be that the supply of credit only adjusts the changes in the demand for credit, which in turn results from the changes in aggregate consumption. This type of causality question is difficult to solve when aggregate data is used.[3] Moreover, there could be some other mechanism which generates the dependency between credit and consumption that has nothing to do with liquidity constraints. However, due to improper identification one could still accept the hypothesis that a stochastically varying liquidity constraint matters for aggregate consumption. Second, evidence from micro data, where causality and identification issues are more easy to deal with using a good instrument or natural experiment, leaves the relevance of the results at the aggregate level open, since a proper way to aggregate is missing.[4] These types of problems can be avoided by using a calibrated general equilibrium model. Within this approach, I can control the causality and identification issues since I can control the shocks. Moreover, I can also discuss the relevance of the stochastically time-varying liquidity constraints at the aggregate level, since the wealth distribution is endogenously determined and is roughly equal to the wealth distribution observed in the data. This solves the issues in the aggregation.

The simulations of the model imply that the credit shocks do not matter for the determination of aggregate consumption, even if the size of the shocks is set

[3] See Bacchetta and Gerlach (1997) and Ludvigson (1998).

[4] Studies using micro data commonly just speculate on the relevance of their results at the aggregated level (see, Ludvigson, 1999; Gross and Souleles, 2002; Alessie, Hochguertel, and Weber, 2005; Leth-Petersen, 2010).

larger than the data support. Further, I generate a large amount of consumers who demand credit, but still credit shocks do not matter for aggregate consumption. The stochastically varying liquidity constraint matters only for a very few people, when the effects of other shocks, combined with fixed liquidity constraint, determine the dynamics of consumption. I confirm that the time-varying liquidity constraints matter for poor people's consumption decisions, but their effect on the dynamics of aggregate consumption is insignificant. Hence, the interpretation that there are a significant number of households who will adjust their consumption according to the availability of credit is difficult to justify. The correlation between aggregate consumption and unsecured credit may be caused by a causality which runs from the demand of credit to the supply of credit, where the demand of credit is driven by consumption. Furthermore, there could be other mechanisms which create a positive correlation between credit and aggregate consumption. These alternative hypotheses are briefly discussed in this paper.

The paper is organized as follows: Section 2 shows the model and discusses the effects of stochastically time-varying liquidity constraint on agents' behavior. Section 3 delivers the results of simulations and interprets the results of the paper. Finally, Section 4 concludes the paper.[5]

2 The general equilibrium model with stochastically time-varying liquidity constraints

2.1 Environment

2.1.1 Production

At period t, the aggregate output Y_t is produced according to the Cobb-Douglas production function of capital input K_t, which depreciates at the rate $\delta \in [0, 1]$, and labor input L_t:

$$Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}, \tag{1}$$

[5] Appendix A provides some stylized facts from the data concerning credit and aggregate consumption.

with $\alpha \in [0,1]$ and $z_t \in Z = \{z_b, z_g\}$ which is (the aggregate) productivity shock which follows a first-order Markov structure. There are two aggregate states: either the state is good, $z_t = z_g$, when the economy is in a boom, or it is bad, $z_t = z_b$, when the economy is in a recession.

Factor and production markets are competitive which implies the factor prices:

$$\begin{aligned} w_t &= z_t(1-\alpha)K_t^{\alpha}L_t^{-\alpha} && (2)\\ r_t &= z_t\alpha K_t^{\alpha-1}L_t^{1-\alpha}-\delta. && (3) \end{aligned}$$

2.1.2 Stochasticity

Assume that there is a continuum of infinitely-lived agents of measure one. Each agent in this economy faces productivity shocks $\epsilon_t \in \Upsilon = \{0,1\}$ for their labor. When $\epsilon_t = 1$ the agent is employed, and in the case of $\epsilon_t = 0$ the agent is unemployed. ϵ is statistically independent across agents and follows a first-order Markov structure, but it is correlated with the aggregate state. Hence, the joint evolution of the exogenous states follows a Markov process with transition matrix Π, with $\Pi_{zz'\epsilon\epsilon'}$ stating

$$\Pi\left(z',\epsilon'\,|z,\epsilon\right) = \Pr\left(z_{t+1}=z',\epsilon_{t+1}=\epsilon'\mid z_t=z,\epsilon_t=\epsilon\right). \tag{4}$$

The transition probabilities for ϵ_{t+1} depend on z_t, i.e. agents have a higher job finding probability in good times than in bad times, but, controlling for Z, individual shocks are independently distributed.

2.1.3 The problem of agents

The agents' maximization problem is as follows:

$$\begin{aligned} \max_{\{c_t\}_{t=0}^{\infty}} E_tU\left(c_t\right) &= E_t\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma} && (5)\\ \text{s.t.}\quad a_{t+1}+c_t &= \left(1+r_t\right)a_t+\epsilon_t w_t\bar{l}+(1-\epsilon_t)\phi_0, && (6)\\ a_{t+1} &\geq D_t, && (7)\\ c_t &\geq 0 \quad \forall t. && (8) \end{aligned}$$

Hence, agents maximize their expected discounted utility conditional today's information by choosing the level of consumption c_t. Moreover, $\beta \in (0,1)$ is the discount factor and $1/\sigma$ gives the intertemporal elasticity of substitution. Agents receive income from working, $\epsilon_t w_t \bar{l}_t$, if they are employed, or they receive $(1-\epsilon_t)\phi_0$ when they are unemployed, which is the value of their nonmarket activity – or home-produced output.[6] I will calibrate the Markov processes, with the transition probabilities given by (4), in such a way that the number of agents who are unemployed is u_b in a recession and u_g in a boom ($u_g < u_b$) and the labor supply is fixed for the agents at the level $\bar{l}$.[7] These assumptions imply that the aggregate labor supply, L_t, is known for every period.

Agents collect income from the services of their capital holdings a_t, which is the only asset in the economy. Assets can be held as a store of value or agents may hold assets as a means of self-insurance against income shocks. The asset markets are incomplete in two different ways when compared with the Arrow-Debrau economy: first, there are no state contingent claims, and second, there are liquidity constraints. In order to rule out Ponzi schemes and to guarantee that loans are paid back, I restrict capital holdings to satisfy $a \in A \equiv [\ \underline{a}_g, \infty\ )$, where $\underline{a}_g$ is the lowest possible level of liquidity constraint in the economy.

2.1.4 The stochastically time-varying liquidity constraint

In this model, the liquidity constraint varies stochastically over time depending on the aggregate state. The liquidity constraint, D_t, gets the value $\underline{a}_g$, when $z_t = z_g$, and when $z_t = z_b$, the value of D_t is $\underline{a}_b$. Thus, $D_t \in A = \left\{\underline{a}_g, \underline{a}_b\right\}$. I do not want to add the number of states, so I assume that liquidity constraint follows the same first-order Markov than did the productivity shocks.[8] Thus, credit shocks and productivity shocks are perfectly correlated. It is assumed that $\underline{a}_g \leq \underline{a}_b \leq 0$, which implies that, in a boom, agents may carry a larger amount of debt than in a recession. That is, when an economy is moving from boom to recession, agents whose assets are $\underline{a}_g \leq a < \underline{a}_b$ must decrease their level of

[6] I do not want to add the government to this model, so the value of home-produced output can be thought of an unemployment benefit or it could be some type of partial insurance against idiosyncratic risk. A more detailed discussion of household production can be found, for example, in Greenwood, Rogerson, and Wright (1995). Adding a government, which transfers income from the employed to the unemployed and runs a balanced budged, is straightforward, but it does not add any substance to the model.

[7] Appendix C extends the model by endogenisizing the labor supply decision of agents.

[8] However, nothing prevents us from specifying a new Markov structure for liquidity shocks.

debt, i.e. save, so that their next period level of assets are at least at the level $\underline{a}_b$. I define *credit* as follows: an agent demands credit when her capital holdings – or net worth – is negative. Moreover, I refer to the changes in the liquidity constraint as *credit shocks*.

One way to interpret these credit shocks is to assume that there is a bank that decides what is the maximum level of debt that can be held in an economy. When times are good, the bank allows its customers to hold more debt than in bad times. For instance, assume that a bank's loanable funds increase during a boom and decrease in a recession, which makes the bank's supply of credit vary with the aggregate state. Hence, the time-varying liquidity constraint can be seen as changes in the supply of credit by banks. Moreover, Lown and Morgan (2006) provide evidence that the standards in loan supply vary strongly with GDP and they conclude that some sort of friction in lending markets leads lenders to ration loans via changes in standards rather than through changes in rates. Thus, this type of modeling seems to be appropriate.[9]

Defaults are not allowed. There are several papers focusing on defaults in credit markets: for example, see Athreya (2002); Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007) and Atherya, Tam, and Young (2009), to name but a few. In these studies, the authors combine endogenously determined default behavior with the standard incomplete markets model of Huggett (1993) or Aiyagari (1994). These types of studies are associated with this paper since they focus on the dynamics of unsecured credit markets, and the model framework is close to the one used here. A crucial exception is that I have aggregate shocks in the model, but the studies cited above only use idiosyncratic shocks. Hence, those studies focus on a steady state situation in the economy, but the focus in this paper is on business cycle dynamics.[10] I assume that defaults are not allowed, which allows me to focus on the business cycle dynamics, since this assumption enables me to include aggregate shocks within the model.

The assumption about not allowing defaults implies that agents must always be able to to reach the higher liquidity constraint $\underline{a}_b$. Hence, given $\underline{a}_b$ the budged

[9]Furthermore, there is a strong positive correlation between the aggregate measure of unsecured credit and GDP. This fact supports the modeling of credit shocks in the way described above.

[10]Solving only for the steady state in the model, where defaults are possible, is very time consuming. I do not know of paper which combines a general equilibrium model with incomplete markets, options to default and aggregate uncertainty. However, allowing defaults would be an important extension to the model. But, it should also be noted that many countries' legislation do not allow defaults.

constraint implies that, when $\{c_t\}_{t=0}^{\infty} \simeq 0$, then the lowest possible value for $\underline{a}_g$ is given by

$$\underline{a}_g = \frac{\underline{a}_b - \phi_0}{1 + r^{max}}, \tag{9}$$

where r^{max} is the highest possible interest rate in the economy. If $\underline{a}_g = \underline{a}_b$, then the liquidity constraint is equal to the natural borrowing limit àla Aiyagari (1994): $-\phi_0/r^{max}$. If we assume that $\underline{a}_g = \underline{a}_b = 0$ (and $\phi_0 = 0$), the model is the same as Krusell and Smith (1998).

Finally, it is good to notice that the credit defined here and the measured credit by statistics (i.e., the data on credit) are not equivalent. Here, I assume that credit is only demanded when agents' net worth is negative, but a significant part of outstanding credit is held by households having a positive net worth, as documented by Gross and Souleles (2002). Those households have debt (or credit) and assets such that the value of assets is greater than the value of debt, so the net worth is positive.[11] However, Ludvigson (1999) also followed the same type of modeling as here: she used data from only those households that had a low level of assets to estimate the effects of credit shocks to aggregate consumption, since it is not credible to assume that liquidity constraint would matter for households with many assets.

2.2 Computation

I use the same method as Krusell and Smith (1998) to solve the model, but I solve the agents' problem by using the endogenous gridpoint method by Carroll (2006), where the time varying liquidity constraint is easy to accommodate.[12] The definition of recursive competitive equilibrium and a more detailed discussion on the computation of this model are provided in Appendix B.

[11] This phenomenon is known as the *credit card debt puzzle*. A more detailed discussion of the credit card debt puzzle can be found in Section 3.3.

[12] There are different ways to compute this type of models. See den Haan, Judd, and Juillard (2010), and other papers in that issue, and Ríos-Rull (1999). Moreover, a detailed description of this type of model without credit shocks is given, for example, by Krusell and Smith (2006).

2.3 The time varying-liquidity constraint and decision rules

2.3.1 Parameter selection

To illustrate the effects of credit shocks, I set most of parameters as in Krusell and Smith (1998), which are standard in the literature. However, these choices do not generate realistic wealth distribution. Here, though, I demonstrate the effects of credit shocks for agents' decision rules. In Section 3 I change the model such that it generates a realistic wealth distribution and I focus on the aggregation, but here I set $\beta = 0.9894$, $\alpha = 0.36$, $\delta = 0.025$, $\sigma = 1$, $\bar{l} = 0.333$. Furthermore, I set $\underline{a}_b$ to -2.2: hence, the agents' borrowing limit in a recession is about half of their annual income. The final parameter is ϕ_0 and I set it to $\phi_0 = 0.35$ when the income in the unemployed state is about 45% that of an employed agent's labor income. This choice also sets $\underline{a}_g$, when $\underline{a}_b$ is fixed, as indicated by equation (9).

The Π is calibrated to roughly mimic fluctuations in the macroeconomic aggregates in the observed postwar U.S. time series. The unemployment rate in a recession, u_b, is 10% when the average duration of the unemployment is 2.5 quarters; in a boom, the unemployment rate, u_g, is 4% and the average unemployment spell is 1.5 quarters. Moreover, the average duration of boom and recession is eight quarters, with parameter values $z_g = 1.01$ and $z_b = 0.99$.[13]

2.3.2 The effects of the time-varying liquidity constraint

It is well known that a lack of insurance against idiosyncratic shocks, combined with liquidity constraint or prudence, causes a precautionary saving motive for agents (see, Deaton, 1991; Carroll, 1997). This, in turn, implies a concave consumption function, as shown in Figure 1.

The consumption function has more curvature at the low level of assets – or it may be said that the high marginal propensity to consume (MPC) applies only for poor people – and when an agent gets richer, the consumption function is almost linear. So, at high levels of wealth, the MPC approaches the MPC implied by the representative agent model. Hence, the consumption function

[13] However, to pin down all of the probabilities in the Π matrix, I need the following restriction: $\pi_{00z_gz_b} = 1.25\pi_{00z_bz_b}$ and $\pi_{00z_bz_g} = 0.75\pi_{00z_gz_g}$ for transition probabilities $\pi_{\epsilon_t\epsilon_{t+1}z_tz_{t+1}}$ in Π. The calibration of Π follows Krusell and Smith (1998).

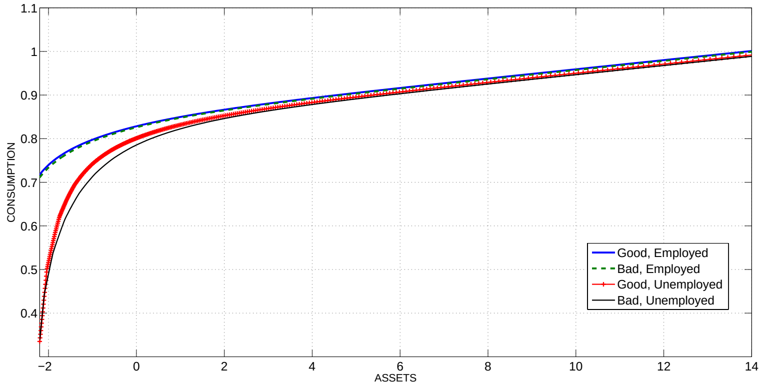


Figure 1: A sample of consumption function.

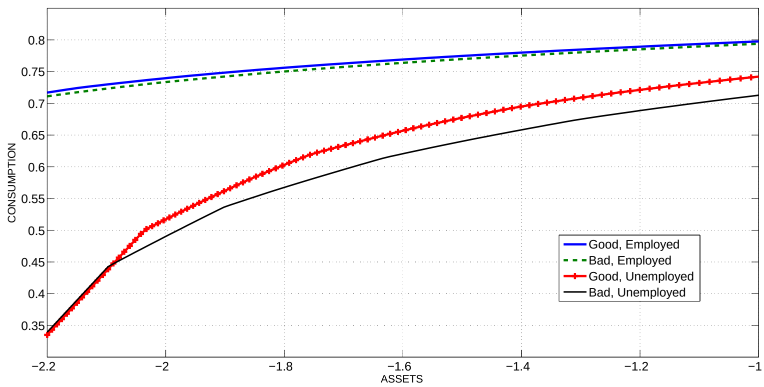
can be approximated in a linear fashion at high levels of wealth.[14] However, at low levels of wealth the consumption function is concave, and when a significant number of consumers hold practically no wealth, this fact questions the validity of linear consumption function as an approximation of the aggregate consumption function. Below, I focus only on low levels of wealth, since at the high levels of wealth the marginal propensities are the same for employed and unemployed agents, as it is shown by Huggett (1993); Aiyagari (1994); Krusell and Smith (1998) and Figure 1.

The time-varying liquidity constraints (or credit shocks) matter for agents' decisions to save and consume at low levels of wealth through two different sources. First, credit shocks directly matter in terms of the availability of current resources which households can consume. When the aggregate state is good, there are more resources – which are a consequence of the availability of extra credit – which can be consumed. Second, expectations about variations in the level of credit matter also for the households' consumption.

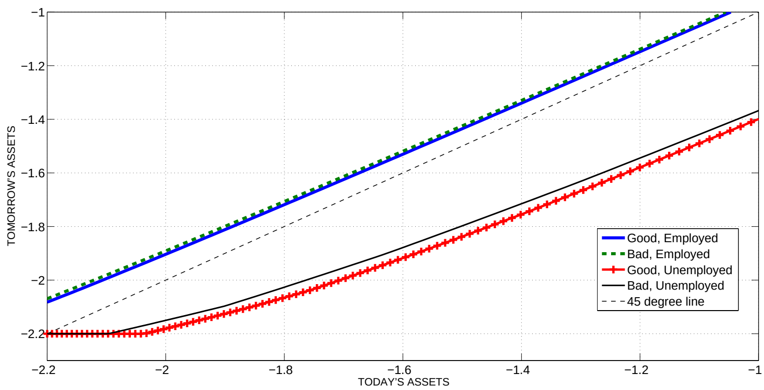
Figures 2 and 3 include samples of consumption functions and the decision rules. In Figure 2, the liquidity constraint is constant, whereas in Figure 3 the liquidity constraint is time-varying, i.e. it includes credit shocks. Decision rules tell the amount of capital which is carried into the next period, a_{t+1}, as a func-

[14] The steady state level of the aggregate capital stock is 11.49 in the representative agent model. Hence, the linear approximation around the steady state value gives a good approximation of the consumption function around that point.

tion of today's capital stock , a_t, and (z_t, ϵ_t). If a decision rule is above the 45 degree line, the agent is saving. A decision rule below that line implies that the agent's consumption is higher than her current income. Thus, decision rules describe the evolution of assets. Consumption functions, in turn, describe the level of consumption as a function of assets and (z_t, ϵ_t). There the differences in MPCs between the two cases can be observed. Both models imply practically the same aggregate capital stock and wealth distribution when the differences in the decision rules are not generated by differences in aggregate circumstances.



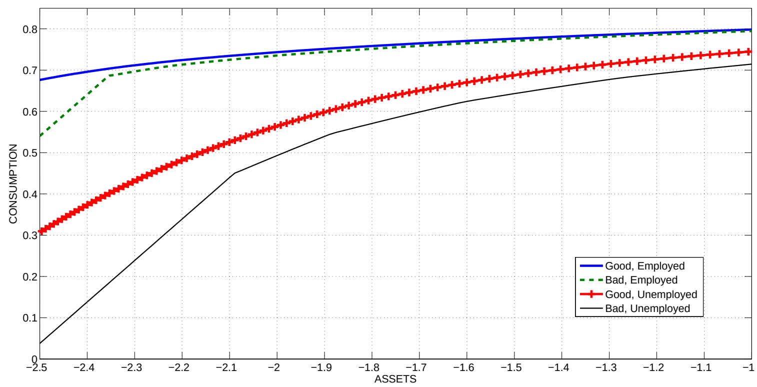
(a) Consumption functions



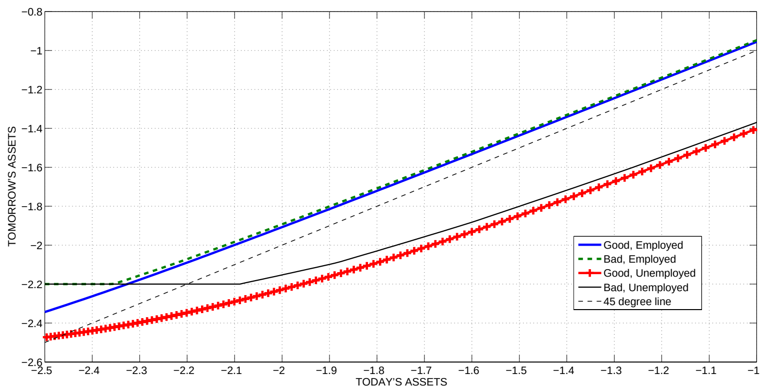
(b) Decision rules

Figure 2: A sample of consumption functions and decision rules in the case of constant liquidity constraint. The figure is the same as in Figure 1, but it only focuses on the low level of assets.

In Figure 2, the flat part of the decision rules implies that the liquidity constraint restrains consumption. This happens only for unemployed agents and then the MPC is 1, but the employed agents' MPCs are much lower. Further, poor agents change their consumption significantly when their employment status changes, but when agents have more assets, i.e. when agents have an insurance against income shocks, the changes in employment status generate a smaller change in consumption. Generally, the effects of liquidity constraint on consumption disappear relatively quickly when the agent accumulates more assets, which can be confirmed from Figure 1.



(a) Consumption functions



(b) Decision rules

Figure 3: A sample of consumption functions and decision rules with time-varying liquidity constraint

In Figure 3, it can be seen that the variation in liquidity constraint (i.e., credit shocks) only matters for the poorest agents in the economy. Then, it determines almost completely the consumption of unemployed agents whose assets are below the level $\underline{a}_b = -2.2$. Thus, changes in the aggregate state – and in the liquidity constraint – also change the consumption level of an unemployed agent. Thus, the MPC out of credit is high, as documented by Gross and Souleles (2002). Moreover, the level of an employed agent consumption is also restrained by the liquidity constraint in a bad state. For instance, assume that economy is in a good state and that an employed agent's assets are at the level -2.4, and then a recession comes and her credit is cut off when she has to drop her level of consumption (see Figure 3). The same applies also to an unemployed agent. Hence, the availability of current resources – or credit – matter for the agents' consumption levels, but in a very limited way, since it only matters for the poorest ones.

If the two cases are compared, several differences can be found. First, there is a higher MPC at the low levels of assets when the liquidity constraint is time-varying. The higher liquidity constraint ”forces” agents to keep extra balances in the bad aggregate state compared to the good state. Since agents have these extra balances, they can consume more from their increased income, which implies the higher MPC. Thus, these extra balances boost the growth of consumption when agents' incomes increase. However, when the assets reach the level of 1.2, the difference in marginal propensities between the two cases is practically zero.

Second, the most important difference is that when the liquidity constraint is constant, only the individual state defines agents' levels of consumption (and the evolution of assets). As in Figure 2, the level of consumption mainly depends on the individual state, i.e. the agents' employment status. But, when the liquidity constraint is time-varying, the aggregate state matters for the level of consumption, since the liquidity constraint varies with the aggregate state. This obviously only holds for agents who are influenced by the change in the liquidity constraint. However, the supply of credit is correlated perfectly with the movements of the aggregate state when the effects of the changes in the aggregate state are amplified by the changes in the supply of credit. So, there are larger ”jumps” in the consumption function between different aggregate states when the liquidity constraint is time-varying (see, Figures 2 (a) and 3 (a)). Basically, this is the mechanism that causes the aggregate consumption to fluctuate more tightly with the GDP. Hence, credit shocks now potentially matter for business cycle dynamics by affecting the dynamics of consumption.[15]

The last observation comes from Figure 3 (b): if an agent is liquidity constrained, she will use the extra credit, when it becomes available, only if she is unemployed at the same time. If the agent receives a positive shock, i.e. she is employed, she will start to save. In the case of employment the agent's MPC out of increased income is high, but she will accumulate assets since she would want to have the assets to smooth consumption in the future. Hence, the precautionary saving motive is pushing agents away from the situation where the liquidity constraint is binding. This obviously restrains the number of agent who are increasing their consumption due to the availability of credit.

I cannot conclude anything about the quantitative importance of credit shocks for the dynamics of aggregate consumption just by studying consumption functions and decision rules. There could be a large number of agents with a low level of assets when they matter for the determination of aggregate consumption, or most people could be in the linear part of the consumption function where the effects of credit shocks do not matter. In any case, it is evident that we need a model with a realistic wealth distribution, or we may as well say that aggregation matters for these types of questions.

3 Simulations

3.1 Matching the wealth distribution

Here, I generate a realistic wealth distribution within the model, but it requires some changes. The target for parameter ϕ_0 can be chosen in two different ways. This parameter is important since it defines the magnitude of credit shock, or the gap between $\underline{a}_g$ and $\underline{a}_b$, as indicated by equation (9). First, I set ϕ_0 in such a way that it generates a reasonable credit shock and I set $\phi_0 = 0.1$. Now, the time-variation in the liquidity constraint is about 10% of agents' labor income. This choice is in line with the estimates for time-varying liquidity constraint given by Ludvigson (1999). She estimates that the upper limit for the variation in the

[15] With smaller values of ϕ_0 the effects of the time-varying liquidity constraint becomes smaller, since the difference between $\underline{a}_g$ and $\underline{a}_b$ is smaller, as implied by equation (9). However, the conclusions are the same as here. Decision rules with a smaller value of ϕ_0 and with an endogenously determined labor supply are discussed in Appendix D.

amount of credit for ”poor” agents is 12.5%, and the lower limit is 6.7% of their labor income. However, the income of an unemployed agent is about 13% of the employed agent’s labor income, which is too low. Hence, I consider another target for ϕ_0, which is the level of unemployment insurance provided for agents, and I set $\phi_0 = 0.35$. In that case, the income in the unemployed state is about 45% of employed agents’ labor income, which is in line with the data. Moreover, it is important to note that with this choice, I also create a much larger credit shock than the data supports. The other parameters are as given in Section 2.3.

For calibrating the wealth distribution, I use the facts provided by Budria, Diaz-Gimménez, Quadrini, and Ríos-Rull (2002).[16] Generating the large group of poor agents is quite straightforward: just increase the magnitude of income of unemployed agent, ϕ_0, which then generates more poor people. Thus, this ”social security” removes the agent’s need for saving, as suggested by Hubbard, Skinner, and Zeldes (1995). However, the generation of a realistic right tale for the wealth distribution is problematic and I use the stochastic-β model (see, Krusell and Smith, 1998; Krusell, Mukoyama, Sahin, and Smith, 2009), where the discount factor is stochastic, which enables a thick right tale for the wealth distribution.[17]

So, I add another aspect of heterogeneity into the model: agents’ discount factors are ex ante identical, but they follow a Markov process. One interpretation could be that the discount factor may vary between the generations of the dynasty. More precisely, I assume that β can get three different values and I keep the average value of β at the same level as previously, i.e. 0.9894. The distribution is symmetric around its mean, when the high and low values of β are ± 0.0036 from the average. The transition probabilities are set as follows: 1) in the invariant distribution, 80% of the agents have the average value of β and 10% are at the other values of β; 2) there are no transition between the extreme values of β; and 3) the average duration of the highest and lowest β’s is 50 years, which is roughly the lenght of one generation in the dynasty.

The first set of models considers versions of the model, which were introduced

[16] They define wealth as the net worth of households, where the definition includes the value of financial and real assets of all kinds net of various kinds of debts.

[17] There are also other ways to generate a realistic wealth distribution. Huggett (1996) shows that a life-cycle model generates a quite realistic wealth distribution and De Nardi (2004), using the same framework, shows that voluntary bequest can explain the concentration of wealth. Further, one might let the rate of returns differ between agents, as shown by Quadrini (2000) and Cagetti and De Nardi (2006), or there could be a drastic dispersion in wages (see, Castañeda, Díaz-Giménez, and Ríos-Rull, 2003).

in Section 2. I consider five different versions of it:

- *Complete Markets.* This is a RBC-model where the supply of labor of agents is fixed, but the aggregate labor varies, as described in Section 2. However, there is a perfect insurance against idiosyncratic shocks.

- *Incomplete Markets.* This is the model used by Krusell and Smith (1998), where $\phi_0 = 0.1$ and the liquidity constraint is fixed.

- *Credit Shocks.* This is the model introduced in Section 2 with $\phi_0 = 0.1$. Note that the time-varying liquidity constraint – or credit shocks – are now added to the *Incomplete Markets* model.

- *Incomplete Markets II.* This is the same model as the *Incomplete Markets* model, but now $\phi_0 = 0.35$.

- *Credit Shocks II.* This is the same model as the *Incomplete Markets II*, but now I have added the credit shocks. Note that the larger value of ϕ_0 implies larger credit shocks. Hence, credit shocks in the *Credit Shocks II* model are larger than in the *Credit Shocks* model.

The second set of models are versions of the stochastic-β -model. I consider the last four cases from the first set of models. Table 1 summarizes the aspects of the wealth distribution of models in which the simulations were 5000 periods long.

Table 1 shows that wealth is very unequally distributed in the U.S.: the richest percentage hold 35% of all the wealth while the poorest 40% only hold 1% of the wealth, which implies a high Gini-coefficient.[18]

All the benchmark models generate wealth distributions in which the wealth is too equally distributed. As I noted above, it is difficult to generate an adequate number of rich households. As expected, increasing the value of income in the unemployed state (see *Credit Shocks II* and *Incomplete Markets II*) increases the number of poor people. The stochastic-β model (with $\phi_0 = 0.1$) generates a quite realistic wealth distribution in which we have more rich people, which results from the fact that they have a lower discount factor than the poor people. This is the

[18] The gini-coeffecient is calculated from the simulated data by using the following formula:

$$\text{Gini-coeffiecient} = \frac{1}{N}\left[N+1-2\left(\frac{\sum_{i=1}^{N} N+1-a_i}{\sum_{i}^{N} a_i}\right)\right],$$

where a_i is in ascending order and N is number of observations.

Table 1: The distribution aspects of wealth

Model	Mean K_t	Std. K_t	% of wealth hold by top 1%	20%	60%	Fraction with wealth ≤ 0	Gini coeff.
Benchmark:							
Complete Markets	11.49	0.29					
Incomplete Markets	11.57	0.26	13%	58%	87%	1%	0.53
Credit Shocks	11.57	0.26	13%	58%	87%	1%	0.53
Incomplete Markets II	11.56	0.20	13%	85%	97%	8%	0.80
Credit Shocks II	11.56	0.20	13%	85%	97%	8%	0.80
Stochastic-β:							
Incomplete Markets	12.02	0.26	35%	89%	98%	11%	0.84
Credit Shocks	12.02	0.25	35%	89%	98%	11%	0.84
Incomplete Markets II	12.00	0.22	48%	102%	103%	55%	0.98
Credit Shocks II	12.00	0.22	48%	102%	104%	55%	0.98
Data			35%	82%	99%	10%	0.80

first choice for studing the dynamics of consumption since it generates a realistic wealth distribution, which is essential. The agents' consumption decisions depend crucially on their level of wealth, which makes the aggregation – or the shape of wealth – an important part of the model.[19]

Based on the conclusions made regarding the shape of decision rules, it is expected that the credit shocks do not have any effect on the distribution of wealth. Credit shocks only matter for poor people who do not have assets. Hence, the wealth distribution is the same with or without the credit shocks. However, credit shocks may matter for the dynamics of aggregate consumption, since a significant number of households hold practically no wealth but they are responsible for a large part of aggregate consumption.

[19] In the extreme case (stochastic-β with $\phi_0 = 0.35$), the high income in the unemployed state combined with a variation in the discount factor generates a wealth distribution in which the wealth is too unequally distributed when compared against the values provided by the data. Half of the people have a negative net worth, which implies that the capital holdings of rich people are greater than the productive capital stock K_t. This explains why the richest 60% hold more than 100% of the wealth.

3.2 The time series properties of aggregate consumption with and without credit shocks

One way to find out about the effects of credit shocks on business cycle dynamics is to contrast a set of aggregate statistics generated by a model, where credit shocks do not exist, against aggregate statistics generated by the same model with credit shocks. If credit shocks do matter for business cycle dynamics, the consumption's relative standard deviation to the standard deviation of GDP should be higher than in the case without the credit shocks. Moreover, the cross-correlation between consumption and GDP should increase. Both of these effects come from the fact that poor households do not matter for capital accumulation or the formation of GDP, but they are responsible for a significant amount of consumption. Hence, consumption should be more volatile when credit shocks do exist. Further, when credit shocks and productivity shocks are perfectly correlated, the cross-correlation between consumption and GDP should increase if credit shocks matter for aggregate consumption. Furthermore, I have reported the autocorrelation function of consumption (3 lags) to see whether or not credit shocks matter for it. Table 2 considers the time-series properties of consumption and GDP from the same simulated data as used in Table 1. I have used the same shocks in all simulations when the results from the models can be compared to each other.

The result in Table 2 is quite unambiguous: credit shocks do not matter for aggregate consumption. The relative standard deviation between consumption and GDP is the same regardless of the existence of credit shocks. The only exception is the case between the *Incomplete Markets* and *Credit Shocks* models, but even then the difference is small. Moreover, a stochastic-β model with $\phi_0 = 0.10$, which generates a realistic wealth distribution, shows that credit shocks do not matter for aggregate consumption. Finally, it should be noticed that, even if I let the credit shocks be larger than the data implies, the previous conclusion holds. Moreover, if I generate a large number of households who demand credit (see the last two cases) credit shocks still do not matter for the dynamics of consumption.

The robustness of the simulations is discussed in Appendix E. I consider two extensions: First, I set $\sigma = 5$ and, second, I consider a model where the leisure is valued. The conclusions made in this section also apply in these extensions. Thus,

Table 2: Time series properties of aggregate consumption

Model	The relative std. of C_t to the std. of Y_t	Autocorrelation of C_t with C_{t-1}	C_{t-2}	C_{t-3}	Cross-correlation of C_t with Y_t	Y_{t-1}	Y_{t-2}
Benchmark:							
Complete Markets	37%	0.99	0.97	0.95	0.68	0.69	0.70
Incomplete Markets	38%	0.93	0.90	0.87	0.78	0.72	0.70
Credit Shocks	40%	0.93	0.90	0.87	0.78	0.72	0.70
Incomplete Markets II	47%	0.83	0.75	0.69	0.93	0.75	0.67
Credit Shocks II	47%	0.83	0.75	0.68	0.93	0.75	0.67
Stochastic-β:							
Incomplete Markets	42%	0.89	0.85	0.80	0.84	0.75	0.70
Credit Shocks	42%	0.89	0.85	0.80	0.84	0.74	0.70
Incomplete Markets II	51%	0.80	0.71	0.63	0.94	0.75	0.65
Credit Shocks II	51%	0.80	0.70	0.62	0.95	0.75	0.65

these results apply even if I allow a lower intertemporal elasticity of substitution and if I let the supply of labor be endogenously determined.

Appendix F compares the values of the simulations against the data. It can be said that the representative agent model generates a correlation between consumption and GDP that is much too low. This can be fixed by introducing incomplete markets, idiosyncratic shocks and $\phi_0 = 0.10$, when the correlation between consumption and GDP is 0.80. Moreover, if the stochastic-β model is used with the same parametrization, the correlation is 0.86. In the data the correlation is 0.9. Adding the credit shocks to these models does not generate a larger correlation between consumption and GDP. Further, it should be noticed that if a lot of poor households are generated (stochastic-β models with $\phi_0 = 0.35$), the correlation between consumption and GDP is almost one, but the wealth distribution is unrealistic. However, even then the addition of the credit shocks does not matter for the dynamics of aggregate consumption.

So, the simulations imply that the changes in the supply of credit do not matter for aggregate consumption. Thus, the correlation between consumption and credit in the data does not originate from households' responses to the changes in the level of liquidity constraint. In other words, we may say that the shocks in the supply of credit do not matter for aggregate consumption, at least in the case where extra credit provides more resources for households to consume. The effects of time-varying liquidity constraint disappear fast when agents accumulate assets. Hence, the dynamics of aggregate consumption is dominated by the agents who have assets such that the variations of liquidity constraint do not matter for them. However, even though the presence of liquidity constraint matters for them, the changes in the liquidity constraint are not significant. That is, there are very few people who are so poor that the liquidity constraint is actually binding for them, and hence, its variations are not significant for the dynamics of aggregate consumption. Rather, the fluctuations in consumption are generated by fluctuations in employment and by fluctuations in the risk of being unemployed. That is, circumstances in the labor market matter more than circumstances in the credit market. Note that a shock for employment directly affects agents' income, but a credit shock only changes the availability of insurance provided by the markets.

The result in this paper does not support the conclusions of Bacchetta and Gerlach (1997); Ludvigson (1999); Gross and Souleles (2002). These papers conclude that variations in credit matter for the dynamics of aggregate consumption since liquidity constrained agents change their consumption according to the availability of credit. The difference in results could be caused by improper aggregation methods and by problems in the causality question. However, the result of this paper supports the conclusions of Ludvigson (1998) and Leth-Petersen (2010), who argue authors conclude that variation in the supply of credit may be quantitatively quite small for the aggregate economy. Here I showed that fluctuations in credit do not basically matter at all for the dynamics of consumption.

3.3 Discussion

The simulations (Section 3.2) and analysis of the decision rules (Section 2.3.2) showed that there are not enough consumers whose consumption directly depends on the availability of credit. In this paper, however, I only focused on one particular mechanism, which has been suggested to create a dependency between credit and consumption. However, the result does not mean that, in general, credit is irrelevant in the determination of consumption since there are many possible ways in which credit might matter for consumption.

First, different types of credit or default behavior could alter this result. This paper only focused on unsecured credit, but collateralized credit, especially mortgages, could have a different type of effect on aggregate consumption. This type of analysis should model explicitly the dynamics of house prices, but this is a topic for another paper. Moreover, modeling default behavior – as it is done by Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007) – would represent an important extension of this model.

Second, credit could matter for firms. Generally, there could be another mechanism in general equilibrium which would allow credit to matter for aggregate consumption. For instance, variations in the supply of credit might matter for firms and this mechanism is known as the financial accelerator mechanism (Kiyotaki and Moore, 1997; Bernanke, Getler, and Gilchirst, 1999). Moreover, there also could be financial sector shocks which matter for the decisions of firms, as shown by Jermann and Quadrini (2009). Hence, conditions in the credit market might matter for the employment decisions of firms which in turn matters for the dynamics of aggregate consumption due to incomplete insurance against these employment shocks. Modeling this channel explicitly seems to be a fruitful approach for future research.

Third, credit can be an alternative to money in exchange. In particular, many households hold liquid assets yielding low returns (1 or 2%) and credit card debt, paying on average 14% interest on their debt at the same time (Gross and Souleles, 2002). This phenomenon is known as the *credit card debt puzzle*. In my opnion, the most plausible explanation is given by Telyukova and Wright (2008), who provide a monetary model where money and credit are needed in exhange.[20] In their model, households hold both liquidity and credit since sometimes households can do trade using credit, but sometimes this option is not available and households must use money. The model focuses only on the steady state, but, potentially, in this type of setup higher consumption will increase the demand for credit – since there is more trade – and, as a result, there will be a positive correlation between credit and consumption. It should be emphasized that issues with liquidity in their model are related to the actual exhange process, whereas in this model issues with liquidity are related to the resources that households can consume.[21]

[20] There are other explanations for this puzzle: for example, Bertaut, Haliassos, and Reiter (2009) argued that with credit card debt, one can control a spouse's consumption behavior.

[21] In addition, there are models where shocks to the banking sector may matter for business cycle dynamics and aggregate consumption (see, Benk, Gillman, and Kejak, 2005, 2008).

At the end of this discussion I must note that this paper delivers a disappointing result, since, within this model, I cannot create a positive correlation between credit and aggregate consumption. I can only say that the correlation which is observed from the data was not generated by a mechanism where liquidity constrained households change their level of consumption according to the availability of credit, but I do not have an explanation for why this correlation occurs. In fact, the correlation between credit and consumption in this model is a negative one – as it should be within this type of environment.[22] The positive correlation between credit and consumption would imply a type of behavior where households' would consume all of their disposable income, and in addition to that would even borrow in order to consume more. Moreover, this all should happen in a situation where the current income of households is above the households' permanent income. Hence, it seems that we need a monetary model to explain the positive correlation between credit and consumption when frictions in trade causes the correlation between consumption and credit.

4 Conclusion

The well-known positive correlation between unsecured credit and consumption has created a hypothesis according to which unsecured credit matters for the dynamics of consumption. One mechanism which has been suggested to deliver this correlation is based on the significance of liquidity constraints when households make their consumption decisions. That is, the stochastic time variation in the liquidity constraint, which can be seen as fluctuations in the supply of credit, matters for the dynamics of aggregate consumption and amplifies the effects of credit shocks more widely within the economy.

The analysis of decision rules showed that the stochastically time-varying liquidity constraints matters for the consumption decisions of households. Thus, the empirical and theoretical findings, that the time-varying liquidity constraint

[22] Here, consumption behavior is studied with standard preferences and the behavior is based on optimization. This implies that households will increase their savings when their income is above the permanent income, i.e. they are saving when they receive good shocks. It is the reverse when households receive bad shocks or their income is below the permanent income, since households will know that income will rise in the future. In this way, households can smooth consumption, and as a result, consumption and credit should be negatively correlated with each other. That is, when consumption is above its average level, households should also decrease their debt holdings, since income is also above its average level. Actually, between the years 1991 and 2007 credit and consumption were negatively correlated.

matters, are basically correct. However, they do not matter for the dynamics of *aggregate* consumption since the stochastic time variation in the liquidity constraint only affects the poorest households in the economy and the contribution of these poor households is insignificant for the aggregate economy. Moreover, the stochastically time-varying liquidity constraints do not matter for determination of aggregate consumption, even if I let the shocks be larger than the data implies or if there are a significant amount of consumers who demand credit.

The paper showed that typical fluctuations in credit cannot affect aggregate consumption by means of changing the availability of liquidity for households. It seems that the variations in productivity and labor market conditions, combined with a *fixed* liquidity constraint, generate the most fluctuations in aggregate consumption. Thus, the existence of liquidity constraint matters, but its typical variations in time do not generate additional variations in aggregate consumption. Hence, the positive correlation between credit and consumption does not originate from the behavior of liquidity constrained households, as argued by some studies. The positive correlation could be driven by the demand for credit, which in turn is affected by changes in consumption, and the supply of credit is merely adjusted to meet these changes. Moreover, there could be some other mechanism that generates the positive relationship.

There are many potential paths for future research which may generally cause credit to matter for the dynamics of aggregate consumption. First, different types of credit – for instance, collateralized credit – should be considered. Dealing with default behavior might also represent an important extension to this line of research. Second, credit could matter for the employment decisions of firms. These employment decisions in turn matter for the dynamics of aggregate consumption, since insurance against these shocks is incomplete and households face a liquidity constraint. Third, credit could mainly be used as a medium of exchange. Understanding the positive correlation between unsecured credit and consumption could be associated with understanding the credit card debt puzzle. A promising way to understand this type of behavior is use the monetary model suggested by Telyukova and Wright (2008).

Appendices for Part II

A Aggregate consumption and the availability of credit

A.1 Data description

Data sources: the time-series of consumption and GDP are from SourceOECD National Accounts Statistics: Quarterly National Accounts Vol 2009 release 11 and the time series of credit is from Federal Reserve Board's G19 statistical release. The consumption is measured from the national accounts as private consumption expenditure and the credit is the outstanding consumer credit which covers most short- and intermediate-term credit extended to individuals, excluding loans secured by real estate. When data is deflated, the GDP deflator is used. Data is quarterly data which is seasonally adjusted and the range is from 1955 to 2009q3.

A.2 Stylized facts from the data

A fraction of credit relative to GDP has doubled during the 20th century in the United States. In the end of the 20th century, the value of outstanding unsecured credit is about 18% of GDP (see Figure A1). Moreover, issues concerning credit and consumption are especially relevant due to the current financial crises, since the level of credit and consumption have been sharply decreased. This fact could be interpreted such that the availability of credit has at least partly caused the drop in consumption (see Figure A2).

Given this magnitude – at least potentially – changes in the aggregate volume of unsecured credit may matter for the performance of the whole economy. Especially, a well-known fact is that the aggregate measure of unsecured credit and the aggregate consumption are correlated as shown by Ludvigson (1999) with data from the U.S., and Bacchetta and Gerlach (1997) provides international evidence.[23] Figure A3 shows the deviations of aggregate private consumption expenditure and the outstanding consumer credit from their trend levels, where

[23] A more comprehensive analysis on credit and economic fluctuations can be found from Schularick and Taylor (2009).

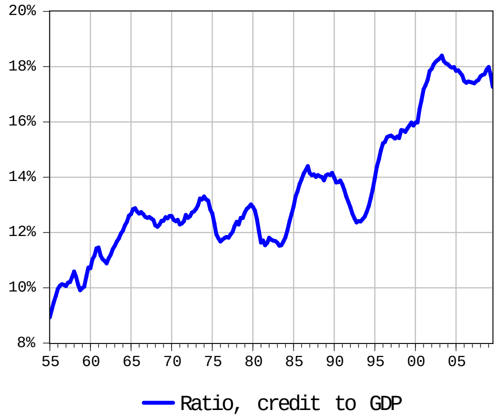


Figure A1: The ratio, credit to GDP.

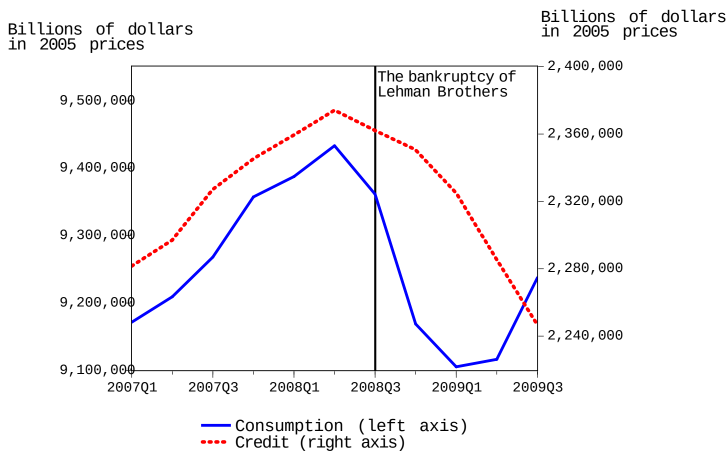
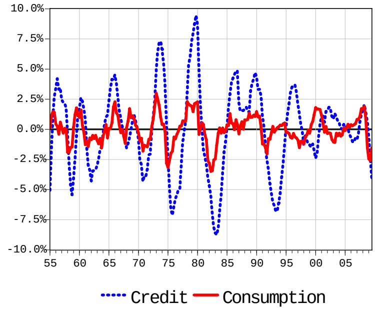
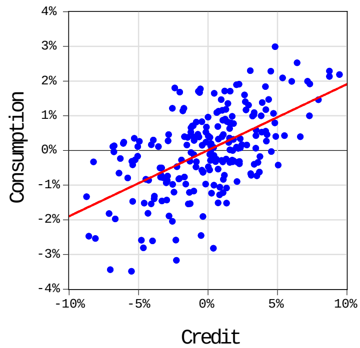


Figure A2: The volume of consumption and credit during the current financial crises.

the trends and cycle series are generated by using the Hodrick-Prescott filter. Credit covers most short- and intermediate-term credit extended to individuals which are not covered with real estate. Correlation between these two series is almost 0.6, but the main question is: which way the causality is running? Figure A3 shows that there is a dependency between consumption and credit and the hypothesis says that the variations in the supply of credit affects the aggregate consumption. However, it may be that the supply of credit only adjusts the



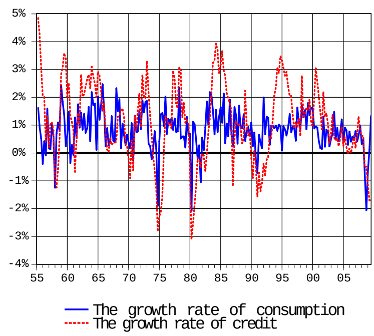
(a) The deviations from the trend.



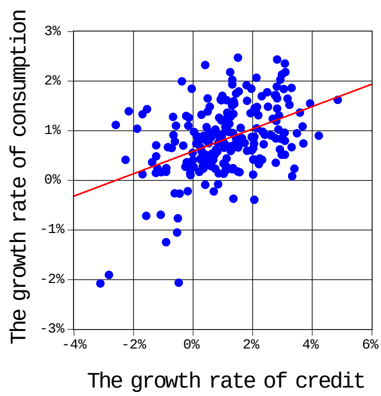
(b) Scatter between the cycle series and a regression line.

Figure A3: The deviations of consumption and credit from they trend levels in the U.S from 1955 to 2009 quarter 3.

changes in the demand of credit, which in turn results from the changes in the aggregate consumption. The dependency can be provided also by utilizing growth rates (see Figure A4) or between credit and GDP (see Figure A5)



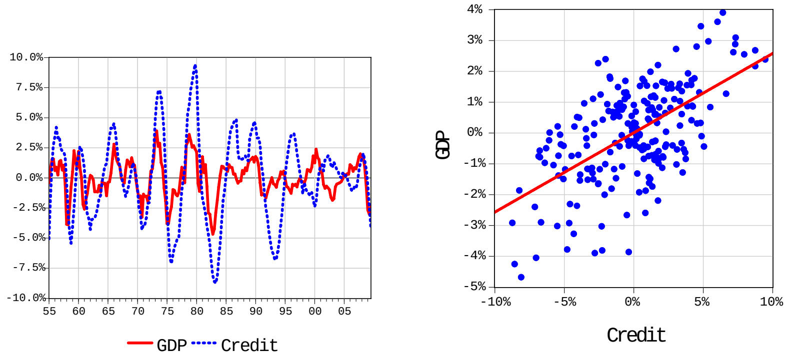
(a) Growth rates



(b) Scatter between growth rates and a regression line.

Figure A4: The growth rate of consumption and credit in the U.S from 1955 to 2009 quarter 3.

However, it should be noticed that consumption and credit are negatively correlated between years 1991 - 2007 (see Figure A6).



(a) The deviations from the trend.

(b) Scatter between the cycle series and a regression line.

Figure A5: The deviations of GDP and credit from they trend levels in the U.S from 1955 to 2009 quarter 3.

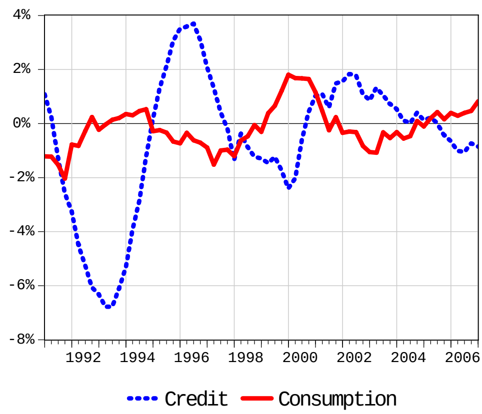


Figure A6: The deviations of consumption and credit from they trend levels in the U.S from 1991 to 2007

B The details of the solution method

B.1 A recursive competitive equilibrium

To solve the model, it must be defined in a recursive form. An agent's position at a point of time is described by individual state vector $s_t \in S$, where $s_t = (a_t, \epsilon_t, z_t)$ and $S = A \times \Upsilon$. Further, let $\mathcal{B}$ be the Borel σ-algebra in S and a probability measure μ over $\mathcal{B}$ describe how many types of agents there are in the economy at time t for any interval $B \in \mathcal{B}$, hence $\mu_t(s_t, B)$.

Different agents have different amount of wealth when they have different propensities to save. However, agents must be able to predict tomorrow's factor prices. That is, they must predict K_{t+1} in order to know the relevant prices (factor prices) for decision making. Hence, these prices depend on μ_t and z_t, and therefore, the relevant aggregate state is (μ_t, z_t). Moreover, K_{t+1} depend on stochastic evolution of μ_t. To formulate the problem recursively I need transition function Γ for μ which makes possible to predict K_{t+1}. That is,

$$\mu_{t+1} = \Gamma\left(\mu_t, z_t, z_{t+1}\right) \tag{A1}$$

where Γ also depend on z_{t+1} since the fraction of agents which are employed or unemployed tomorrow depend on z_{t+1}.

Now agents' problem given by equations (5)-(8) can be rewritten in a recursive form

$$\begin{aligned} V\left(a, \epsilon, \mu, z\right) &= \max_{c \in \mathbb{R}^+, a' \in A} \left\{ U(c) + \beta \sum_{\epsilon' \in \Upsilon, z' \in Z} \Pi_{zz'\epsilon\epsilon'} V\left(a', \epsilon', \mu', z'\right) \right\} && \text{(A2)} \\ \text{s.t.} \quad a' + c &= \left[1 + r(K, L, z)\right] a + w(K, L, z)\bar{l}\epsilon + (1-\epsilon)\phi_0 && \text{(A3)} \\ a' &\geq D(z) && \text{(A4)} \\ \mu' &= \Gamma(\mu, z, z') && \text{(A5)} \end{aligned}$$

where pricing functions $w(K, L, z)$ and $r(K, L, z)$ are given by equations (2) and (3). Problem of this type has a solution that I denote by the set of decision rules: $c = g^c(a, \epsilon, \mu, z)$ and $a' = g^{a'}(a, \epsilon, \mu, z)$. The recursive competitive equilibrium can be defined as foolows:

Definition A1. ***Recursive competitive equilibrium*** *consist of value function $V\left(a, \epsilon, \mu, z\right)$, optimal decision rules $c = g^c(a, \epsilon, \mu, z)$ and $a' = g^{a'}(a, \epsilon, \mu, z)$, pricing functions $w(K, L, z)$, $r(K, L, z)$ and the law of motion for μ: Γ, such that following conditions holds:*

1. ***Agents optimize****: given $w(K, L, z)$ and $r(K, L, z)$ value function $V\left(a, \epsilon, \mu, z\right)$ solves the problem given by equations (A2)-(A5) and $c = g^c(a, \epsilon, \mu, z)$ and $a' = g^{a'}(a, \epsilon, \mu, z)$ are the associated decision rules for all (a, ϵ, μ, z).*

2. ***The firm optimizes****: given prices w and r the representative firm chooses K and L optimally, as given by equations (2) and (3), for all (z, μ).*

3. ***Consistency condition between aggregate and individual behavior****:*

   - *The law of motion for μ, Γ, is generated by exogenous probabilities Π from the Markov chain and policy function $g^{a'}(a, \epsilon, \mu, z)$ as follows:*

   $$\mu'(s', B) = \Gamma_B(\mu, z, z') = \int_S Q(s, B)\mu(ds, B), \quad \textit{where}$$
   $$Q(s, B) = \sum_{\epsilon' \in B_\epsilon} \Pi_{zz'\epsilon\epsilon'} \mathbf{I}_{g^{a'}(a,\epsilon,\mu,z) \in B_a}$$

     for all $B \in \mathcal{B}$ and (μ, z, z'), with $\mathbf{I}$ being an indicator function that takes the value of one if the statement is true and otherwise zero.

   - *The aggregate asset holdings are given by $A = \int_S a\mu(ds, B)$ and $A' = \int_S g^{a'}(a, \epsilon, \mu, z)\mu(ds, B)$,*

   - *aggregate consumption is given by $C = \int_S g^c(a, \epsilon, \mu, z)d\mu(ds, B)$, for all (z, μ).*

4. ***Markets clear****:*

   - *the asset market clears: $K = A$,*

   - *the goods market clears: $C + K' = zF(K, L) + (1 - \delta)K$, where $zF(K, L)$ is given by equation (1) and*

   - *the labor market clears: $L = \int_S \epsilon\bar{l}d\mu(ds, B)$*

   for all (z, μ).

B.2 Computational strategy

I cannot solve the agents' optimization problem since it depends on μ, which is endogenous state variable, and it is, in principle, infinite dimensional object. To compute the equilibrium I need some way to present the distribution of assets holdings and I use approximation given by Krusell and Smith (1998).[24]

[24] For recent discussion about this approach see Maliar, Maliar, and Valli (2010).

It is assumed that agents only use partial information from μ when they predict future prices. To be exact I assume that agents just use the first moment, m_1, of μ (i.e. K) in addition to z. One way to interpret this method is to say that agents are boundedly rational since agents do not use all available information. However, it can be shown that the information, which is not used for forecasting prices, is simply not useful – agents are not boundedly rational at all, but unknown variables (prices) depend only on the first moment of μ.[25] Moreover, I need to specify the law of motion Γ. I assume that Γ have a simple log-linear form and I denote this functions as $K' = \Gamma_K(K, z, z')$. That is, I define $K' = \Gamma_K(K, z, z')$ as follows:

$$\log K_{t+1} = \begin{cases} \gamma_{gg0} + \gamma_{gg1} \log K_t & \text{when} \quad z_t = z_g, z_{t+1} = z_g \\ \gamma_{gb0} + \gamma_{gb1} \log K_t & \text{when} \quad z_t = z_g, z_{t+1} = z_b \\ \gamma_{bg0} + \gamma_{bg1} \log K_t & \text{when} \quad z_t = z_b, z_{t+1} = z_g \\ \gamma_{bb0} + \gamma_{bb1} \log K_t & \text{when} \quad z_t = z_b, z_{t+1} = z_b \end{cases} \tag{A6}$$

Then the agents' recursive maximization problem, i.e. equations (A2)-(A5), can be rewritten with partial information as follows:

$$\begin{aligned} V(a, \epsilon, K, z) &= \max_{c \in \mathbb{R}^+, a' \in A} \left\{ U(c) + \beta \sum_{\epsilon' \in \Upsilon, z' \in Z} \Pi_{zz'\epsilon\epsilon'} V(a', \epsilon', K', z') \right\} && \text{(A7)} \\ \text{s.t.} \quad a' + c &= [1 + r(K, L, z)]\, a + w(K, L, z)\bar{l}\epsilon + (1 - \epsilon)\phi_0 && \text{(A8)} \\ a' &\geq D(z) && \text{(A9)} \\ K' &= \Gamma_K(K, z, z') && \text{(A10)} \end{aligned}$$

I use endogenous gridpoint method by Carroll (2006) to solve the problem, where the time-varying liquidity constraint is easy to accommodate (see Section B.3). This maximization problem gives a decision rule $g^{a'}(a, \epsilon, z, K)$, which could be used in simulations. In simulations I obtain time series for K_t which in turn can be used to update the initial guess for Γ_K. When the parameters of Γ_K are converged, the model is solved.

I used a following setup in the simulations: the number of households was 5000 and the number of simulated periods was 5000. I used only observations from 1000-5000 in the OLS-estimations, since the first 1000 observations may be

[25] A short discussion about different interpretations can be found from Young (2010).

influenced by the initial conditions of simulation.

B.3 The details of the endogenous gridpoint method

Note that w_t, r_t, w_{t+1} and r_{t+1} are given by K, z and Γ_K, and these values are known by the agent when she solves the maximization problem. The Euler equations for agents' problem given by equations (A7)-(A10) are

$$c_t^{-\sigma} \;\geq\; \beta E_t\left[(1+r_{t+1})c_{t+1}^{-\sigma}\right] \quad \forall \quad t, K, z, \epsilon. \tag{A11}$$

If I now substitute the budget constraint to the Euler equations for consumption, I get

$$\begin{aligned} &[(1+r_t)a_t + w_t - a_{t+1}]^{-\sigma} \\ &\qquad \geq \beta E_t\Big\{(1+r_{t+1})\left[(1+r_{t+1})a_{t+1} + w_{t+1}\bar{l} - a_{t+2}\right]^{-\sigma}\Big\} \quad \text{if} \quad \epsilon_t = 1, \\ &[(1+r_t)a_t + \phi_0 - a_{t+1}]^{\sigma} \\ &\qquad \geq \beta E_t\left\{(1+r_{t+1})\left[(1+r_{t+1})a_{t+1} + \phi_0 - a_{t+2}\right]^{-\sigma}\right\} \qquad \text{if} \quad \epsilon_t = 0, \end{aligned} \tag{A12}$$

for all t, K, z.

Let me now fix the gridpoint for a_{t+1} and $a_{t+2} = g^{a_t+2}(a_{t+1}, \epsilon_{t+1}, z_{t+1}, K_{t+1})$ is the policy function. Now I can solve a_t as function of the fixed gridpoint and other exogenous variables from (A12). That is,

$$\begin{aligned} a_t &= \frac{\left\{\beta E_t\Big\{(1+r_{t+1})\left[(1+r_{t+1})a_{t+1}+w_{t+1}\bar{l}-a_{t+2}\right]^{-\sigma}\Big\}\right\}^{-\frac{1}{\sigma}} - w_t\bar{l} + a_{t+1}}{1+r_t} \quad \text{if} \quad \epsilon_t = 1 \\ a_t &= \frac{\left\{\beta E_t\left\{(1+r_{t+1})[(1+r_{t+1})a_{t+1}+\phi_0-a_{t+2}]^{-\sigma}\right\}\right\}^{-\frac{1}{\sigma}} - \phi_0 + a_{t+1}}{1+r_t} \quad \text{if} \quad \epsilon_t = 0, \end{aligned} \tag{A13}$$

for all t, K, z. With a help of Π I can evaluate the conditional expectation. Hence, I have defined the endogenous gridpoints a_t for a_{t+1}.

The last phase is updating the policy function $g^{a_{t+2}}$ by interpolation, when I can notice the effects of the liquidity constraint. Iteration may be stopped when $\max\left\{\left|g_n^{a_{t+2}} - g_{n+1}^{a_{t+2}}\right|\right\} <$ some predetermined error tolerance.

C Endogenous labor supply

The utility function of agents is following when leisure is valued

$$U\left(c_t, 1-l_t\right) = \sum_{t=0}^{\infty} \beta^t \frac{\left[c_t^{\phi}\left(1-l_t\right)^{1-\phi}\right]^{1-\sigma}}{1-\sigma}, \tag{A14}$$

where $1-\phi$ is the share parameter for leisure in the composite commodity. Each agent is endowed with one unit of time when the amount of leisure is $1-l_t$. Given that $\epsilon_t = 1$, i.e. agent has opportunity to work, she can decide the supply of labor. Agent receives income from working ,$\epsilon_t w_t l_t$, if she is employed ($\epsilon_t = 1$).

When the supply of labor is endogenous variable, the aggregate labor supply, L_t, is unknown for each individual since they can only observe their own labor supply decision. Hence, I need a forecasting function Θ for L_t:

$$L_t = \Theta(\mu_t, z_t), \tag{A15}$$

which depends on the distribution of agents μ_t as well as aggregate state. I can not use μ_t when I solve the model, and hence, I use the same partial information approach as previously. Agents' only use the mean of μ to predict the current aggregate labor supply. That is,

$$\begin{aligned} L_t &= \Theta_K(K_t, z_t) \\ \log L_t &= \begin{cases} \theta_{g0} + \theta_{g1} \log K_t & \text{when} \quad z_t = z_g \\ \theta_{b0} + \theta_{b1} \log K_t & \text{when} \quad z_t = z_b. \end{cases} \end{aligned} \tag{A16}$$

With this approximation I can solve agents' problem recursively.

Note that Θ_K does not forecast L_t perfectly – for perfect aggregation I should have μ as an argument for agents problem – which implies that market clearing would never hold exactly. Market clearing would require that L_t is known perfectly. This is only a problem when leisure is valued since agent know K_t, ϵ_t and z_t at the beginning of every period, when today's prices (r_t and w_t) are known, if L_t is known. However, in this case L_t is unknown and approximated by Θ_K, which implies that markets would not clear every period.

There are two options: if deviations from market clearing are not large one could accept those, but a more attractive option is clear market for every t. Hence, I must modify the decision rules such that agents can react when they observe today's prices. With these decision rules I can confirm that markets clear at every point of time in simulations. More precisely, I let value function (or decision rules) explicitly depend on L, $V^n(a,\epsilon,z,K,L)$, but when this value function (or decision rule) is updated, I set $V^{n+1}(a,\epsilon,z,K,L) = V^n(a,\epsilon,z,K;\Gamma_K,\Theta_K)$. Hence, agents view unknown prices as given by Γ_K and Θ_K, but when they observe the unknown prices they change optimally their behavior such that markets clear every period.

I use endogenous gridpoint method by Carroll (2006) here as well to solve the problem,[26] where the time varying liquidity constraint is easy to accommodate (see Section C.1). This maximization problem gives decision rules $g^{a'}(a,\epsilon,z,K,L)$ and $g^l(a,1,z,K,L)$, which could be used in simulations. In simulations I obtain time series for $\{K_t\}_{t=0}^{5000}$ and $\{L_t\}_{t=0}^{5000}$ which can be used to update the initial guess for Γ_K and Θ_K. To obtain a series for L I must clear labor market at every period by iterating, i.e. I must find L which solves

$$L = \int_S g^l(a,\epsilon,z,K,L)\mu(ds,B), \tag{A17}$$

which also matters for K' through $g^{a'}(a,\epsilon,z,K,L)$.

C.1 The details of the endogenous gridpoint method for the model with valued leisure

Note that w_t, r_t, w_{t+1} and r_{t+1} are given by K, z, Γ_K and Θ_L, which are given by the agent when she solves the maximization problem. This enables a significant simplification for Euler equations. The Euler equations for agents' problem with

[26] The standard RBC-model can be solved by using a generalization of this method by Barillas and Fernández-Villaverde (2007).

endogenous labor supply are

$$
\begin{aligned}
& c_t^{\phi(1-\sigma)-1}(1-l_t)^{(1-\phi)(1-\sigma)} \geq \\
& \beta E_t\left[(1+r_{t+1})c_{t+1}^{\phi(1-\sigma)-1}(1-l_{t+1})^{(1-\phi)(1-\sigma)}\right] && \text{(A18)} \\
& l_t = 1-\frac{1-\phi}{\phi}\frac{c_t}{w_t} && \text{(A19)} \\
& \text{if} \quad \epsilon_t = 1, \quad \text{and} \\
& c_t^{\phi(1-\sigma)-1} \geq \\
& \beta E_t\left[(1+r_{t+1})c_{t+1}^{\phi(1-\sigma)-1}(1-l_{t+1})^{(1-\phi)(1-\sigma)}\right] && \text{(A20)} \\
& \text{if} \quad \epsilon_t = 0, \\
& \quad \forall \quad t, K, z, L.
\end{aligned}
$$

Moreover, I can substitute (A19) into the budged constraint, which yields

$$
\begin{aligned}
c_t &= \phi\left[(1+r_t)a_t + w_t - a_{t+1}\right] \quad \text{if} \quad \epsilon_t = 1, \\
c_t &= (1+r_t)a_t + \phi_0 - a_{t+1} \quad \text{if} \quad \epsilon_t = 0,
\end{aligned} \tag{A21}
$$

for all t, K, z, L

If I now substitute (A19) and (A21) to the Euler equations for consumption, I get

$$
\begin{aligned}
& \{\phi\left[(1+r_t)a_t + w_t - a_{t+1}\right]\}^{-\sigma}\left(\tfrac{1-\phi}{\phi w_t}\right)^{(1-\phi)(1-\sigma)} \\
& \qquad \geq \beta E_t\left[(1+r_{t+1})c_{t+1}^{\phi(1-\sigma)-1}(1-l_{t+1})^{(1-\phi)(1-\sigma)}\right] \quad \text{if} \quad \epsilon_t = 1, \\
& \left[(1+r_t)a_t + \phi_0 - a_{t+1}\right]^{\phi(1-\sigma)-1} \\
& \qquad \geq \beta E_t\left[(1+r_{t+1})c_{t+1}^{\phi(1-\sigma)-1}(1-l_{t+1})^{(1-\phi)(1-\sigma)}\right] \quad \text{if} \quad \epsilon_t = 0,
\end{aligned} \tag{A22}
$$

for all t, K, z, L. Note that leading (A21) one period forward I can write

$$
\begin{aligned}
& c_{t+1}^{\phi(1-\sigma)-1}(1-l_{t+1})^{(1-\phi)(1-\sigma)} \\
& \qquad = \{\phi\left[(1+r_{t+1})a_{t+1} + w_{t+1} - a_{t+2}\right]\}^{-\sigma}\left(\tfrac{1-\phi}{\phi w_{t+1}}\right)^{(1-\phi)(1-\sigma)} \quad \text{if} \quad \epsilon_{t+1} = 1 \\
& c_{t+1}^{\phi(1-\sigma)-1}(1-l_{t+1})^{(1-\phi)(1-\sigma)} \\
& \qquad = \left[(1+r_{t+1})a_{t+1} + \phi_0 - a_{t+2}\right]^{\phi(1-\sigma)-1} \quad \text{if} \quad \epsilon_{t+1} = 0,
\end{aligned} \tag{A23}
$$

for all t, K, z, L.

Fix the gridpoint for a_{t+1} and $a_{t+2} = g^{a_t+2}(a_{t+1}, \epsilon_{t+1}, z_{t+1}, K_{t+1})$, which is the policy function. Then I can solve a_t as function of the fixed gridpoint and other exogenous variables from (A22). That is,

$$a_t = \frac{\phi^{-1}\left\{\frac{\beta E_t\left[(1+r_{t+1})c_{t+1}^{\phi(1-\sigma)-1}(1-l_{t+1})^{(1-\phi)(1-\sigma)}\right]}{\left(\frac{1-\phi}{\phi w_t}\right)^{(1-\phi)(1-\sigma)}}\right\}^{-\frac{1}{\sigma}} - w_t + a_{t+1}}{1+r_t} \quad \text{if} \quad \epsilon_t = 1 \tag{A24}$$

$$a_t = \frac{\left\{\beta E_t\left[(1+r_{t+1})c_{t+1}^{\phi(1-\sigma)-1}(1-l_{t+1})^{(1-\phi)(1-\sigma)}\right]\right\}^{\frac{1}{\phi(1-\sigma)-1}} - \phi_0 + a_{t+1}}{1+r_t} \quad \text{if} \quad \epsilon_t = 0,$$

for all t, K, z, L. With a help of (A23) and Π I can evaluate the conditional expectation. Hence, I have defined the endogenous gridpoints a_t for a_{t+1}. The last phase is updating the policy function by interpolation.

D Decision rules for extended models

D.1 Decision rules with the smaller value of income in the unemployed state

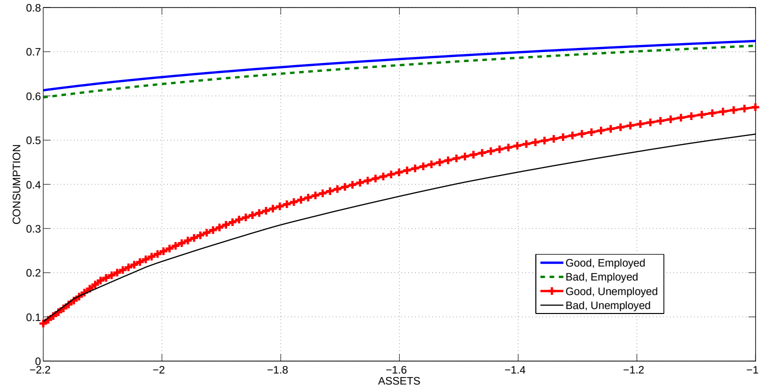
Here I have set $\phi_0 = 0.1$. Other parameters are the same as in Section 2.3. Figures A7 and A8 show the results, which are almost equivalent to the results in Section 2.3, but the effects of liquidity constraint is harder to see since the variation in the liquidity constraint is smaller.

The only difference is that the liquidity constraint is not binding for employed households when the aggregate state is bad. When $\phi_0 = 0.35$, the liquidity constraint was binding (at very low levels of assets) also for employed households in the bad aggregate state, this effect is now missing. This is caused by the smaller variations between the liquidity constraints. Expect for this departure conclusions given in Section 2.3 also applies here.

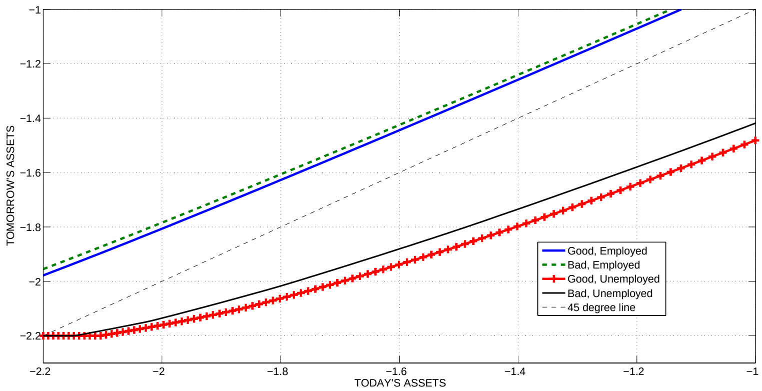
D.2 Decision rules and endogenous labor supply

The parameters are the same as in the Section 2.3 and but $\phi_0 = 0.1$. Figure A9 provides the decision rules when the liquidity constraint is constant and Figure A10 shows the same rules when there is the time-varying liquidity constraint.

The same conclusions apply here as in the baseline case.



(a) Consumption functions



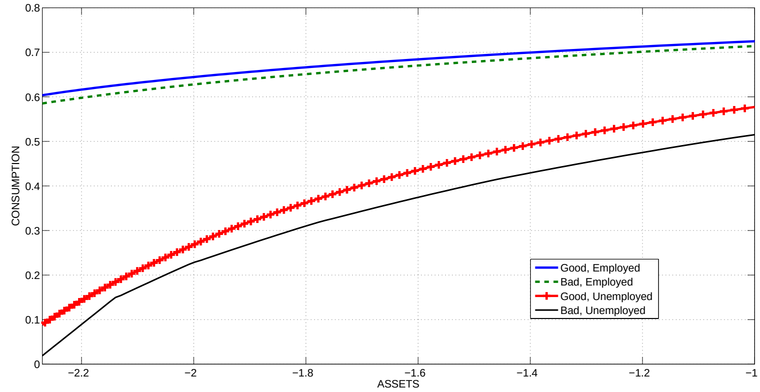
(b) Decision rules

Figure A7: A sample of consumption functions and decision rules in the case of constant liquidity constraint.

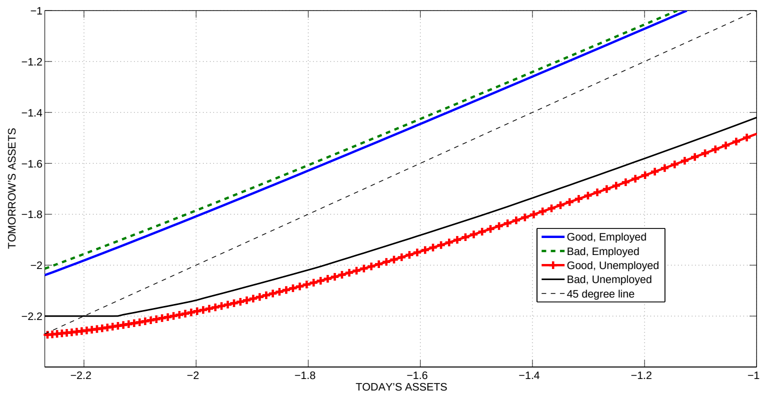
E The robustness of the simulations

E.1 The results of simulations with the lower intertemporal elasticity of substitution

In the baseline simulations were generated under log-utility, i.e. $\sigma = 1$. Here I increase the value of σ when $\sigma = 5$, which is also a quite commonly used value. Everything else is kept the same as in the simulations of Section 3. Tables A1 and A2 show the results.



(a) Consumption functions



(b) Decision rules

Figure A8: A sample of consumption functions and decision rules with time-varying liquidity constraint

Now agents' utility is lowered more by the fluctuation of consumption than in the case of log-utility. This increases the precautionary saving motive and the aggregate capital stock is higher than in the baseline case with log-utility. However, credit shocks do not matter in this case either since models with credit shocks deliver the same key statistics as model without credit shocks. Hence, the conclusion made in Section 3 holds here as well.

E.2 The results of simulations when leisure is valued

Now I let leisure be valued. Let $\sigma = 1$ and $\phi_0 = 0.1$ and the rest of parameters are the same as in Section 3. Tables A3 and A4 show the results.

The approximate aggregation does not hold as well here as it did in the baseline model where the leisure was not valued. R^2-statistics were 0.98 and 0.97 for the forecasting function of aggregate labor supply, i.e. for equations (A16). The behavior of aggregate labor supply is almost similar between the complete markets model (standard RBC-model) and the incomplete markets model. However, the capital holdings are reduced significantly in the incomplete markets model. The poor people supply more labor when their assets are at the low level, which implies that they have quite a good insurance against fluctuations in income. If we set $\phi_0 = 0$, the mean K_t is almost the same as in the case of complete markets. Thus, the importance of ϕ_0 is emphasized when the leisure is valued, but credit shocks do not matter in this case since models with credit shocks deliver the same key statistics as model without credit shocks. Hence, conclusion made in Section 3 holds here as well.

Table A1: The distribution aspects of wealth

Model	Mean K_t	Std. K_t	% of wealth hold by top 1%	20%	60%	Fraction with wealth ≤ 0	Gini coeff.
Benchmark:							
Complete Markets	11.53	0.52					
Incomplete Markets	11.95	0.45	10%	38%	77%	0%	0.29
Credit Shocks	11.95	0.45	10%	38%	77%	0%	0.29
Incomplete Markets II	11.80	0.33	13%	77%	93%	2%	0.71
Credit Shocks II	11.79	0.33	13%	77%	93%	2%	0.72
Stochastic-β:							
Incomplete Markets	12.12	0.44	22%	50%	81%	0%	0.41
Credit Shocks	12.13	0.44	22%	49%	81%	0%	0.42
Incomplete Markets II	12.04	0.33	70%	85%	97%	5%	0.84
Credit Shocks II	12.04	0.33	70%	85%	97%	5%	0.84
Data			35%	82%	99%	10%	0.80

Table A2: Time series properties of aggregate consumption

Model	The relative std. of C_t to the std. of Y_t	Autocorrelation of C_t with			Cross-correlation of C_t with		
		C_{t-1}	C_{t-2}	C_{t-3}	Y_t	Y_{t-1}	Y_{t-2}
Benchmark:							
Complete Markets	31%	0.99	0.52	0.32	0.74	0.72	0.72
Incomplete Markets	35%	0.86	0.83	0.81	0.81	0.72	0.69
Credit Shocks	35%	0.86	0.83	0.81	0.81	0.72	0.69
Incomplete Markets II	44%	0.78	0.69	0.62	0.96	0.75	0.65
Credit Shocks II	44%	0.78	0.69	0.62	0.96	0.75	0.65
Stochastic-β:							
Incomplete Markets	35%	0.84	0.82	0.79	0.81	0.72	0.68
Credit Shocks	35%	0.84	0.82	0.79	0.81	0.72	0.68
Incomplete Markets II	45%	0.74	0.66	0.59	0.94	0.73	0.63
Credit Shocks II	45%	0.74	0.66	0.59	0.94	0.73	0.63

F Comparison between the data and models

In these comparisons each model is simulated 1000 times with each simulation being 200 periods long to match the number of observations underlying the statistics reported from data. The data and simulated data were in logarithms and filtered by Hodrick-Prescott filter to give us the representation of the business cycles. Table A5 shows the results from models given in Section 3. Table A6 gives result generated by models described by Section E.1 and Table A7 considers the valued leisure case, i.e. it considers models presented in Section E.2.

Table A3: The distribution aspects of wealth

Model	Mean K_t	Std. K_t	% of wealth hold by top			Fraction with wealth ≤ 0	Gini coeff.
			1%	20%	60%		
Complete Markets	11.48	0.13					
Incomplete Markets	11.15	0.25	7%	44%	80%	0%	0.36
Credit Shocks	11.15	0.25	7%	44%	80%	0%	0.36
Data			35%	82%	99%	10%	0.80

Table A4: Time series properties of aggregate consumption

Model	The relative std. of C_t to the std. of Y_t	Autocorrelation of C_t with C_{t-1}	C_{t-2}	C_{t-3}	Cross-correlation of C_t with Y_t	Y_{t-1}	Y_{t-2}
Complete Markets	31%	0.98	0.96	0.93	0.63	0.64	0.65
Incomplete Markets	32%	0.94	0.92	0.90	0.63	0.63	0.64
Credit Shocks	32%	0.94	0.92	0.90	0.63	0.63	0.64

Table A5: Time series properties of aggregate consumption: Comparison against the data with the baseline calibration

Model	The relative std. of C_t to the std. of Y_t	Autocorrelation of C_t with C_{t-1}	C_{t-2}	C_{t-3}	Cross-correlation of C_t with Y_t	Y_{t-1}	Y_{t-2}
Benchmark:							
Complete Markets	20%	0.83	0.64	0.44	0.66	0.66	0.60
Incomplete Markets	26%	0.66	0.41	0.22	0.80	0.64	0.48
Credit Shocks	28%	0.65	0.40	0.21	0.80	0.63	0.48
Incomplete Markets II	48%	0.61	0.33	0.13	0.97	0.63	0.38
Credit Shocks II	48%	0.61	0.33	0.13	0.97	0.63	0.38
Stochastic-β:							
Incomplete Markets	34%	0.66	0.40	0.21	0.86	0.66	0.48
Credit Shocks	34%	0.66	0.40	0.21	0.86	0.66	0.48
Incomplete Markets II	54%	0.62	0.34	0.13	0.98	0.64	0.39
Credit Shocks II	54%	0.62	0.34	0.13	0.98	0.64	0.38
Data	77%	0.84	0.65	0.42	0.89	0.76	0.56

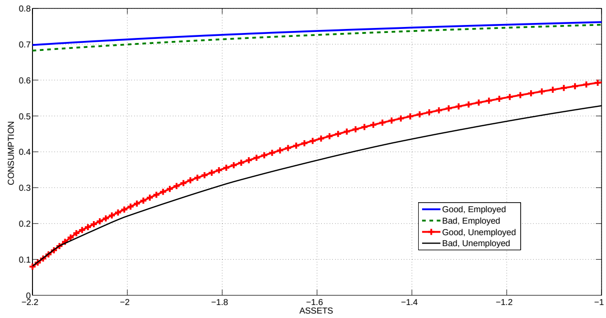
In all the models the standard RBC-model generates way too low cross-correlation between consumption and GDP. However, the RBC-model can generate quite realistic autocorrelation function for consumption. Incomplete markets model do not generate enough autocorrelation for consumption, but cross-correlation with GDP is quite close the one observed from the data. Generally, in all the models consumption is too smooth, i.e. the relative standard deviation of consumption to the standard deviation of GDP is too small, when it is compared against the value implied by data.

Table A6: Time series properties of aggregate consumption: Comparison against the data with the higher intertemporal elasticity of substitution

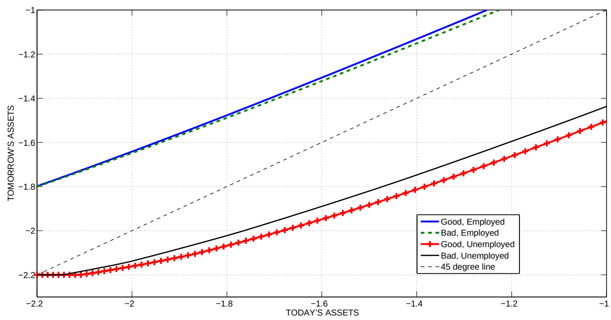
Model	The relative std. of C_t to the std. of Y_t	Autocorrelation of C_t with C_{t-1}	C_{t-2}	C_{t-3}	Cross-correlation of C_t with Y_t	Y_{t-1}	Y_{t-2}
Benchmark:							
Complete Markets	12%	0.75	0.52	0.32	0.80	0.67	0.54
Incomplete Markets	22%	0.57	0.29	0.11	0.85	0.58	0.38
Credit Shocks	22%	0.57	0.29	0.11	0.85	0.58	0.38
Incomplete Markets II	47%	0.59	0.29	0.09	0.99	0.60	0.32
Credit Shocks II	48%	0.58	0.29	0.09	0.99	0.60	0.32
Stochastic-β:							
Incomplete Markets	23%	0.57	0.29	0.10	0.86	0.58	0.37
Credit Shocks	23%	0.57	0.29	0.10	0.86	0.58	0.37
Incomplete Markets II	48%	0.58	0.29	0.09	0.99	0.60	0.32
Credit Shocks II	48%	0.58	0.29	0.09	0.99	0.60	0.32
Data	77%	0.84	0.65	0.42	0.89	0.76	0.56

Table A7: Time series properties of aggregate consumption: Comparison against the data when leisure is valued

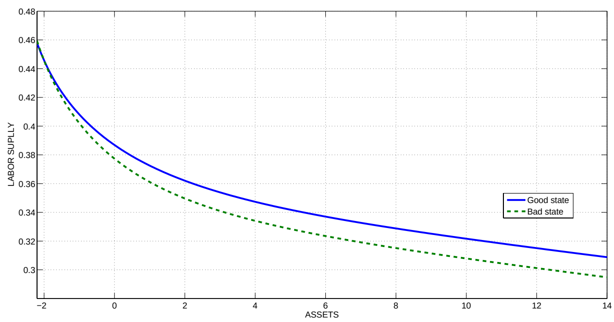
Model	The relative std. of C_t to the std. of Y_t	Autocorrelation of C_t with C_{t-1}	C_{t-2}	C_{t-3}	Cross-correlation of C_t with Y_t	Y_{t-1}	Y_{t-2}
Complete Markets	20%	0.84	0.65	0.45	0.63	0.65	0.61
Incomplete Markets	35%	0.97	0.93	0.89	0.62	0.66	0.67
Credit Shocks	35%	0.97	0.93	0.89	0.62	0.66	0.67
Data	77%	0.84	0.65	0.42	0.89	0.76	0.56



(a) Consumption functions

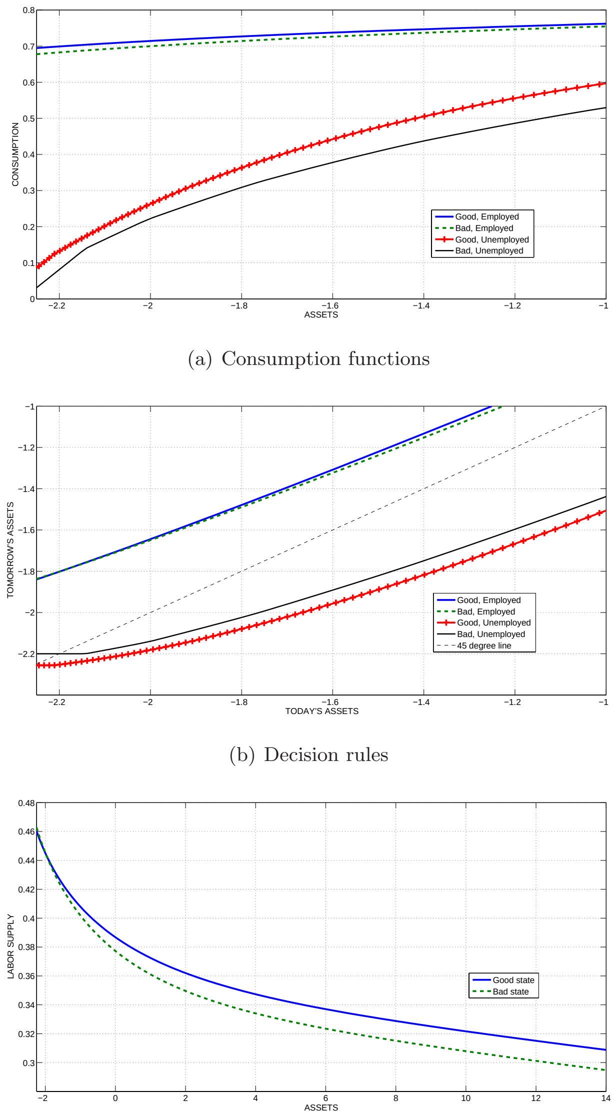


(b) Decision rules



(c) Labor supply functions

Figure A9: A sample of consumption functions and decision rules in the case of constant liquidity constraint.



(a) Consumption functions

(b) Decision rules

(c) Labor supply functions

Figure A10: A sample of consumption functions and decision rules with time-varying liquidity constraint

Part III

Essay II: Is the relative risk aversion low or precautionary savings high?

Is the relative risk aversion low or precautionary savings high?

Tomi Kortela*

Abstract

The steady state interest rate for the Finnish economy is calculated from the national accounts data and is compared to the interest derived from a dynamic general equilibrium in which the markets are complete. By exploiting a Euler equation decomposition, I show that the difference in interest rates gives the magnitude of precautionary savings. Hence, this paper provides a new method for measuring precautionary savings. Furthermore, a high relative risk aversion implies high precautionary savings, or, it can be concluded that the log-utility is a proper calibration for the utility function of the representative agent in a standard growth model.

JEL Codes: C81, E13, E17, E22, E25

Keywords: Precautionary saving, Euler equation

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1 Introduction

Precautionary savings is a stock of wealth, which is a result of past savings done for precautionary reasons. This "precautionary wealth" is generated by the past responses of current consumption to future risk since there is a lack of insurance against income uncertainty. Thus, the steady state capital stock is higher, i.e. the interest rate is lower, in an economy where idiosyncratic risks are not perfectly insured in contrast to the capital stock in a complete markets economy, as showed by Aiyagari (1994); Huggett (1997); Huggett and Ospina (2001).[1] The question presented in this paper is a quantitative one: To what extent do wealth holdings arise from precautionary reasons? An extensive amount of research has been done in order to reach a conclusion, but the estimates range from few percentages points to 60%.[2] However, the question is an important one, since precautionary wealth affects possibly aggregate wealth accumulation, asset prices and optimal policy design.

In this paper, I use a general equilibrium setup to derive a measure for the magnitude of precautionary wealth. The method is a general method and I use Finnish data to illustrate it, but the method described here can easily be applied to other countries' data as well. The measure is based on a similar Euler equation decomposition used by Gourinchas and Parker (2001) and Parker and Preston (2005) in a partial equilibrium framework. This approach keeps the analysis at a fairly simple level, since there is no need to estimate the actual magnitude of idiosyncratic shocks or solve the underlying model. In particular, it has been found difficult to identify the magnitude of uninsurable idiosyncratic shocks from the data (see the discussion and results of, for example, Storesletten, Telmer, and Yaron, 2004; Krueger and Perri, 2006; Blundell, Pistaferri, and Preston, 2008). This, however, is essential for the strength of the precautionary saving motive. Hence, this paper provides a new method for measuring the magnitude

[1] Flodén (2008) provides a detailed discussion on the measurement of precautionary wealth.

[2] Skinner (1988) and Caballero (1991) argued that about 50-60% of households' total wealth holdings can be explained by a precautionary saving motive and similar conclusions have recently been given by Cagetti (2003) and Gourinchas and Parker (2002). Moreover, Carroll and Samwick (1998) and Engen and Gruber (2001) argued that precautionary wealth is about 20-50% of households' total wealth holdings, but Guiso, Jappelli, and Terlizzese (1992), Dynan (1993) and Lusardi (1998) showed that precautionary saving or precautionary wealth is not a significant factor for wealth accumulation. Simulations of a general equilibrium model by Huggett (1993), Aiyagari (1994) and Krusell and Smith (1998) also showed that the contribution of the precautionary saving motive is small for capital accumulation within the aggregate economy. For a recent review of the literature, see Carroll and Kimball (2007) and Browning and Lusardi (1996).

of precautionary wealth just by using data provided by the national accounts, which avoids some problems encountered in previous estimates.

To derive the measure for the magnitude of precautionary savings, I begin by assuming that there are two reasons for holding capital: an intertemporal reason and a precautionary reason. The steady state interest rate, which is measured from the national accounts data, includes both intertemporal and precautionary motives for saving. This rate of return is compared to the interest rate, which is derived from a model in which I have assumed complete markets. In this case, capital accumulation results purely from an intertemporal reason and it depends on the parameters of the utility function and the growth rate of per capita income. The decomposition of the Euler equation in turn implies that the difference between the two interest rates gives us the magnitude of precautionary wealth. Hence, this paper provides a Solow-residual-type measure for the magnitude of precautionary wealth: the capital accumulation, which cannot be explained by the intertemporal reason of saving, must be a consequence of the precautionary saving motive. This unexplained part gives the magnitude of precautionary savings or the level of precautionary wealth in the economy.

The result depends on the two unknown parameters, which are the relative risk aversion or the intertemporal elasticity of substitution (IES) and the subjective time preference of households. It is shown that a high relative risk aversion (or a low IES) will imply that a significant amount of capital accumulation is a consequence of the precautionary saving motive, but with the log-utility – and with a feasible rate of subjective time preference – the precautionary saving motive is not important for capital accumulation. Hence, the results of the paper are twofold.

First, the magnitude of wealth, which is a consequence of precautionary saving motives, can directly be observed just by focusing on the aggregate Euler equations when the unknown parameters of the utility function are estimated. Hence, I provide a new and simple way to measure the magnitude of precautionary savings from aggregated data.

Second, the results provide a new rationalization for calibrated models which focus on long-run capital accumulation. This type of model is, for example, a standard growth model with a representative agent. The standard approach is to take parameters for these models from micro data, but Browning, Hansen, and Heckman (1999) argued that these parameters should not be used in general equilibrium models directly. Here, I show that the use of the logarithmic utility function is a consistent choice with an assumption about market completeness, which is the key assumption for aggregation in these models. Hence, the calibration of utility function is rationalized by an assumption that precautionary motives do not matter for long-run capital accumulation when the economy's long-run behavior could be approximated by assuming complete markets. This is actually suggested by many simulations within a general equilibrium framework.[3]

The paper is constructed as follows: Section 2 introduces the model; Section 3 derives the Euler equations and gives the measure for the magnitude of precautionary wealth; Section 4 constructs series from the national accounts which enables the estimations of interest rates; Section 5 shows the results; and, finally, Section 6 concludes the paper.

2 The model and the steady state

2.1 The model

I use a dynamic general equilibrium model similar to the one used by Aiyagari (1994) to describe the Finnish economy. The production side of the economy is described by the Cobb-Douglas production function and I assume a constant returns to scale and a perfect competition. Thus, the output of the economy is given as follows:

$$Y_t = A_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha}, \tag{1}$$

where α represents the capital share and A_t measures the growth of labor productivity, which is given by $A_t = (1+g)^t A_0$. Hence, g represents a constant growth rate of labor productivity. K_t represents capital and L_t represents the working age population, which is given by $L_t = (1+n)^t L_0$. I assume that the working-age population is growing at a constant rate n, which is equal to the growth rate of

[3]Notably Krusell and Smith (1998) showed that so-called approximate aggregation holds when only the rich households matter for aggregate capital stock, and they do not have a precautionary saving motive due to the large amount of wealth they possess.

the population. Moreover, factor prices are given by the equations

$$r_t = \alpha \frac{Y_t}{K_t} \tag{2}$$

$$w_t = (1-\alpha)\frac{Y_t}{L_t}. \tag{3}$$

The environment for households is as follows. There is a continuum of infinitely living households and the total mass of households is equalized to one. I assume that labor supply is inelastic when the households supply one unit of labor every period.[4] The problem of representative household j is

$$\max_{\{c_t^j\}_{t=0}^{\infty}} \quad E_t\left[U(c_t^j)\right] = E_t\left[\sum_{t=0}^{\infty} L_0^j\left[\beta(1+n)\right]^t \frac{(c_t^j)^{1-\theta}}{1-\theta}\right] \tag{4}$$

$$\text{s.t.} \quad c_t^j + (1+n)k_{t+1}^j - (1-\delta)k_t^j = w_t\epsilon_t^j + r_t k_t^j, \tag{5}$$

$$c_t^j \geq 0, \tag{6}$$

$$k_{t+1}^j \geq \underline{k}, \tag{7}$$

where E_t is a conditional expectation operator. The CRRA-utility function is chosen when θ^{-1} is the IES and θ gives the relative risk aversion, $\beta = (1+\rho)^{-1}$ gives the subjective discount rate and the household face a liquidity constraint $k_{t+1}^j \geq \underline{k}$. δ represents the depreciation rate, r_t represents the rental rate of capital or the interest rate and w_t represents the wage rate. There is one asset, productive capital k, which households can hold for smoothing consumption across time (an intertemporal reason to save), or they might hold capital as a means of self-insurance against idiosyncratic shocks (a precautionary reason to save). Moreover, asset holdings are restricted by an appropriate No-Ponzi scheme condition. These features are common for all households.[5]

[4]This assumption is questionable since labor supply decisions and precautionary saving are connected to each other. Pijoan-Mas (2006) shows that, under reasonable parametrization, a measure which does not consider labor supply decisions may overestimate the magnitude of precautionary saving. That is, working longer hours when receiving a low productivity shock gives households a self-insurance mechanism which lowers the demand for precautionary savings. Hence, the working hours are higher, but the productivity of labor is lower when markets are incomplete in contrast to the complete market case. However, Marcet, Obiols-Homs, and Weil (2007) model a situation where *working opportunity* is stochastic and markets are incomplete. The result is that, under reasonable calibration, working hours are lower in the case of incomplete markets in contrast to the complete markets case. Precautionary savings generates a wealth effect which reduces the working hours of households who are employed. Obviously, the role of labor supply and its effects on the magnitude of precautionary saving are not settled and the results depend on the modeling strategy. Thus, I leave labor supply as an exogenous variable.

[5]Some comments are in order. I choose the constant relative risk aversion (CRRA) utility function

Households begin with the working-age population L_0^j, and the working-age population of all households grows at the same rate n. Small letters are defined as follows: $\frac{X_t^j}{L_t^j} = x_t^j$. Hence, the variables of each household are divided by the working-age population of the household. Thus, c_t^j stands for consumption, k_t^j represents capital and $(1+n)k_{t+1}^j - (1-\delta)k_t^j$ represents investment i_t^j or saving. All households face an idiosyncratic shock ϵ_t^j to the wage rate and these shocks are not fully insurable.

2.2 The steady state

Here, I consider the steady state of the economy, which could also be referred to as stationary equilibrium.[6] In the steady state the distribution of wealth, which is measured over the state variables of households, has reached its ergodic distribution. Thus, $K_t = K_{t+1} = K$ which implies that $r_t = r_{t+1} = r$. But households' asset holdings are not constant, since households are affected by idiosyncratic shocks.

Now I can write the Bellman equation for the problem of household j given by equations (4)-(7):

$$\begin{aligned} V(k_t^j, \epsilon_t^j) \quad = \quad & \max_{k_{t+1}^j \geq \underline{k}, c_t \geq 0} \Big\{ U\left(w\epsilon_t^j + rk_t^j + (1-\delta)k_t^j - (1+n)k_{t+1}^j\right) \\ & + \beta(1+n)E\left[V\left(k_{t+1}^j, \epsilon_{t+1}^j\right) \mid \epsilon_t^j\right] \Big\}, \end{aligned} \tag{8}$$

where r and w are given by equations (2) and (3).

The problem's necessary condition can be described by using the Euler equation, which is

$$U'(c_t^j) \geq \beta R E_t \left[U'(c_{t+1}^j)\right], \tag{9}$$

where $R = 1 + r - \delta$, which is the rate of return for savings, and equation

because it is the most commonly used utility function and it is needed for keeping the state space bounded among other assumptions. Its special case is $\theta = 1$ when the utility function is equal to the log utility. Moreover, I focus on long-run capital accumulation and assume that an infinite horizon is a suitable approximation for the planning horizon of households. Finally, the constraint $\underline{k}$ can be set in various ways, but the key element is that households avoid a positive probability of consuming negative consumption; see Aiyagari (1994) for a detailed discussion.

[6] More details can be found in Aiyagari (1994). For a standard textbook treatment of the subject, see Ljungquist and Sargent (2004) and Heer and Maussner (2009).

(9) holds with equal sign if $k_{t+1}^j > \underline{k}$. Finally, note that when aggregated over the households' type distribution, the economy's feasibility constraint can be rewritten as

$$Y_t = C_t + I_t. \tag{10}$$

3 Interest rates and precautionary wealth

3.1 Incomplete markets

Here, I approximate the Euler equation in the steady state and I assume that there is incomplete insurance against idiosyncratic shocks, i.e. markets are incomplete. Equation (9) gives a necessary condition which must hold for every t. The equation can be rewritten as follows:

$$1 \geq \beta R E_t \left[\left(\frac{c_{t+1}^j}{c_t^j} \right)^{-\theta} \right]. \tag{11}$$

Further, assume that there is always a $\tilde{g}_{t+1}^j$, for which $c_{t+1}^j = (1 + \tilde{g}_{t+1}^j) c_t^j$. To approximate consumption growth, I take a second-order Taylor approximation of equation (11) around point $(1+g)$, where g represents the growth rate of permanent income for household j. g is equal to the growth rate of labor productivity, which I assume the same for all households.[7] Moreover, there is an equivalence between the consumption growth rate and the income growth rate, which comes from the assumption of the balanced growth path.[8] The Euler equation then becomes:

$$\begin{aligned} 1 \;\; \gtrsim \;\; & \beta R E_t \Bigg[(1+g)^{-\theta} \Bigg(1 - \theta (1+g)^{-1} (\tilde{g}_{t+1}^j - g) \\ & + \frac{1}{2} \theta (\theta + 1) \left(\frac{1 + \tilde{g}_{t+1}^j}{1+g} - 1 \right)^2 \Bigg) \Bigg]. \end{aligned} \tag{12}$$

[7] If g could differ in the steady state among households, the ergodic type distribution would not exist.

[8] The equivalence between the consumption growth rate and the income growth rate in the steady state could also occur in a partial equilibrium setup when it comes from the buffer-stock behavior of households (see, Carroll, 1997).

Using the conditional expectation operator, I get $E_t\left[\tilde{g}^j_{t+1} - g\right] = 0$ and define $\left(\frac{1+\tilde{g}^j_{t+1}}{1+g} - 1\right)^2 = (\sigma^j_{t+1})^2$, which is the variance of consumption growth for household j and $\frac{1}{2}\theta(\theta+1) = \phi$. Thus, the approximated Euler equation is

$$1 \gtrsim (1+g)^{-\theta}(1+\rho)^{-1}(1-\delta+r)\left(1+\phi E_t\left[(\sigma^j_{t+1})^2\right]\right). \tag{13}$$

Note that when the consumption growth rate is equal to the income growth rate, then the equilibrating variable in equation (13) is $(\sigma^j_{t+1})^2$, that is, the household's variance of consumption growth.

Now I assume that the liquidity constraint is set at the loosest possible, i.e. it is set to the natural borrowing limit àla Aiyagari (1994), when it is not binding for any household. Too strict liquidity constraint will produce less consumption insurance than observed from the data, but the natural borrowing limit produces a consistent amount of consumption insurance (see, Kaplan and Violante, 2010). Moreover, assume that the expectation errors are approximately zero and the approximation error from the use of the Taylor approximation is close to zero.[9] Hence, I can aggregate over the type distribution of households when the *aggregate* Euler equation holds with equal sign and is given by the equation

$$1 = (1+g)^{-\theta}(1+\rho)^{-1}(1-\delta+r)\left(1+\phi\int E_t\left[(\sigma^j_{t+1})^2\right]d\mu\right), \tag{14}$$

where μ represents the invariant type distribution of households. Moreover, $(\sigma^j_{t+1})^2$ depends on the type of household, i.e. it depends on the idiosyncratic shock ϵ and on the level of wealth of household j.

I define the term $1+\phi\int E_t\left[(\sigma^j_{t+1})^2\right]d\mu = 1+\Psi$, which captures the effects of precautionary saving and the curvature of the utility function. Thus, the interest rate is now given by equation (14)

$$1 + r^{im} = (1+g)^{\theta}(1+\rho)(1+\Psi)^{-1} + \delta, \tag{15}$$

where the precautionary saving motive exists if $\Psi > 0$ and r^{im} implies the interest

[9] The choice of the natural borrowing limit is not crucial here and simulations by Huggett (1993), Aiyagari (1994) and Krusell and Smith (1998), among others, suggested that liquidity constraint is not binding for most households at the steady state. The rest of the assumptions are more or less correct and harmless. However, we needed a well-defined lower bound for income. Note also that with the natural borrowing limit, precautionary savings occurs from prudence, as shown by Kimball (1990), and because households also save to avoid being hit by liquidity constraint, see Huggett (1993).

rate under incomplete markets. The term $(1+g)^{\theta}(1+\rho)$ gives the intertemporal motive for holding capital and the term $(1+\Psi)^{-1}$ captures the precautionary saving motive for capital holdings. The higher the variance of consumption growth rate for households, the higher Ψ becomes, which indicates a lower interest rate resulting from higher precautionary savings. This type of decomposition is also given by Gourinchas and Parker (2001) and Parker and Preston (2005).

3.2 Complete markets and aggregate uncertainty

Consider now that markets are complete, i.e. there is a perfect insurance against idiosyncratic shocks and liquidity constraints do not exist. However, assume that there is an aggregate risk which could result from aggregate productivity shock. This implies that the variance of consumption growth is the same for all households.

The aggregate economy could now be described by the standard stochastic growth model and households face only the risk observed in the aggregate consumption data, which is measured by Ψ. However, the unconditional variance in the growth rate of Finnish real consumption per working age population is about 0.00047, $\Psi \approx 0$, and the impact of precautionary saving on consumption is trivial. Thus, the interest rate at the steady state can be given under perfect insurance against idiosyncratic risk – i.e. under markets completeness – by the following Euler equation

$$1 + r^{cm} = (1+g)^{\theta}(1+\rho) + \delta, \tag{16}$$

where r^{cm} implies the interest rate under complete markets and this result is parallel to the certainty equivalence. The interest rate is now generated by the intertemporal reasons for holding capital.

3.3 The difference between interest rates and precautionary wealth

By comparing equations (15) and (16), it can be seen that under incomplete markets the interest rate is lower, $r^{im} < r^{cm}$ (i.e., aggregate capital stock is higher) than under complete markets when $\Psi > 0$. If precautionary saving reasons can be ignored, then $r^{im} = r^{cm}$, since $\Psi = 0$. Moreover, the difference between the interest rates gives a measure for the magnitude of precautionary wealth, since from r there is mapping to K.

Rewriting equation (2) in the steady state for the complete and the incomplete markets cases gives the following:

$$\bar{K}^{cm} = \left(\frac{r^{cm}}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{17}$$

$$\bar{K}^{im} = \left(\frac{r^{im}}{\alpha}\right)^{\frac{1}{\alpha-1}}, \tag{18}$$

where $\bar{K} = \frac{K}{AL}$, i.e. capital is in an intensive form or capital is scaled by the efficient working-age population, $(AL)^{-1}$. Moreover, it was previously shown that $\bar{K}^{im}$ is generated by the intertemporal and precautionary motives for holding capital, but $\bar{K}^{cm}$ only consists of the intertemporal reason for holding capital (see equations (15) and (16)). Therefore, $\bar{K}^{im} - \bar{K}^{cm}$ gives the magnitude of capital stock, which is held only for precautionary saving reasons. This is the contribution of Ψ in equation (15). That is, $\bar{K}^{im} - \bar{K}^{cm}$ is a Solow-residual-type quantity of capital holdings which explains the capital holdings that cannot be explained by the intertemporal reason for holding capital. The difference in capital holdings must be generated by the precautionary reason for holding capital since there is only the intertemporal and the precautionary motives for capital holdings and the former is captured by $\bar{K}^{cm}$. To get some metric for the result, the answer is given as a percentage of aggregate capital holdings.

That is, the magnitude of precautionary wealth at the steady state is now defined as

$$\frac{\bar{K}^{im} - \bar{K}^{cm}}{\bar{K}^{im}} = 1 - \left(\frac{r^{cm}}{r^{im}}\right)^{\frac{1}{\alpha-1}}. \tag{19}$$

Hence, the left-hand side of equation (19) defines the magnitude of precautionary wealth. The capital stock is larger when households have no access to insurance markets, as compared to the capital stock in a complete market economy and this "additional wealth" can be thought of as the economy's aggregate amount of precautionary wealth, as argued by Huggett and Ospina (2001). This is now defined by $\bar{K}^{im} - \bar{K}^{cm}$. Hence, equation (19) answers the question: How large is the precautionary wealth? The right-hand side of equation (19) uses equations (17) and (18) and shows that the left-hand side of the equation can be rewritten just by using the interest rates. That is, the results are driven by the differences in interest rates.

Starting from Aiyagari (1994), a number of studies have measured precautionary wealth in a general equilibrium setup by using a calibrated model, but here I use data provided by national accounts to estimate the magnitude of precautionary wealth. Hence, the next step is to measure r^{im} and r^{cm} from the data.

4 Measuring interest rates from the data

First, I need an estimate for r^{im}, but it cannot be measured using equation (15) since Ψ is unknown. However, since the model in Section 2 implied that all savings are turned into capital holdings, I can use the data to measure the actual savings by following capital accumulation. Moreover, there are only two motives for holding capital: hence, capital accumulation captures the intertemporal and precautionary motives for saving, i.e. it gives $\bar{K}^{im}$. It can be converted into an interest rate by exploiting the neoclassical distribution of national income. Thus, r^{im} is given by equation (2) and a measure is needed for Y_t, K_t and α. Second, r^{cm} can be derived from equation (16). I treat θ and ρ as unknown parameters which are left to vary. Estimates are then needed for g and δ. When I need a steady state measure, I use the average of the variable in question.

The model in Section 2 is a model for a closed economy without government. Therefore, the data provided by national accounts must be adapted to match the model. I need an empirical counter part for output (Y_t) and its per capita growth rate (g). Moreover, I need to generate the time-series of capital stock (K_t) and its depreciation rate (δ) and, finally, I must define the labor share (α). There are several strategies for doing this, but here I follow the strategy used by Conesa, Kehoe, and Ruhl (2007). The sample is from the years 1975 to 2007 and the data is provided by the Statistics Finland.

4.1 The definition of output

I measure output, Y_t, as gross national product (GNP). The GNP adds the foreign income of domestic residents to gross domestic product (GDP) and subtracts the income within the country of foreign residents. For a small open country such as Finland, GNP is a better measure for income which can be consumed or saved.

The saved income is invested in Finland or abroad, since the fundamental open economy identity must hold: $S_t = I_{t,\text{domestic}}$ + Net capital outflow. Thus, GNP is equal to investment and consumption, as given in equation (10), and I have only defined that $I_t = I_{t,\text{domestic}}$ + Net capital outflow, where government consumption is assumed to be a perfect substitute for private consumption. That is, public roads, schools and health care (to name but a few) are equal to their private counterparts.[10] Moreover, corporate saving is included within households' savings.

4.2 Estimating the growth rate of per capita income

In the steady state, the growth of aggregate income per working-age population must be equal to the growth of consumption per working age population. The growth rate of consumption and income are equal to the growth rate of A_t. So, the trend growth rate is the result of growth in the stock of world knowledge. This growth rate is defined in the model by g.

I measure the steady-state growth rate of consumption from the growth of y_t, which is GNP per working-age population. There is no simple way to find a steady state growth rate. Therefore, three different ways of finding a steady state growth rate are utilized. First, the nonlinear OLS regression is fitted to actual data, which gives the estimate of trend growth.[11] Second, the Hodrick-Prescott (H-P) filter is used and the trend given by the filter is plugged into following equation:

$$g = \frac{1}{32} \sum_{t=1975}^{2007} \Delta \log(y_t), \tag{20}$$

where Δ is the difference operator. Third, I use the actual data of y_t in equation (20). Figure 1 shows the actual y_t, the estimated steady state path and the trend given by the H-P filter. Both the estimated and the H-P filtered steady state paths seem to fit nicely.

[10] A similar strategy is used by Hayashi and Prescott (2002, 2007) for Japan.

[11] The estimated nonlinear OLS regression is $y_t = \varphi_1(1+\varphi_2)^{t-1975} + e_t$, where $\varphi_1 = y_{1975}$ and $\varphi_2 = g$, t =time and e_t is the residual .

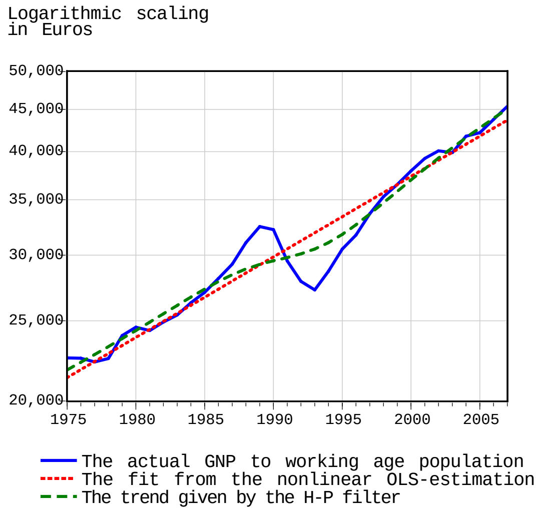

Figure 1: Actual GNP (measured in 2000 prices) to working-age population and the estimated steady state paths.

The actual steady state growth rate of y_t, given by equation (20), is 0.0219, the OLS-estimate for g is 0.0226 and the average growth rate of the H-P filtered data is 0.0227. The average of these estimates is 0.022, which is used as an estimate for g. The estimated growth rate is 2.2%, which is close to the estimated growth rate for the U.S. The growth rate of the U.S. is generally estimated at 2% (see, for example, Prescott, 2002), which could also be used as Finland's steady state growth rate.[12]

4.3 Generating the capital stock and its depreciation

The most difficult question pertains to the measure of wealth or capital stock. The observable capital stock is a result of past saving, which is, by definition, done according to the intertemporal and precautionary saving reasons. Note that the interest rate, given by equation (2), is a price which equilibriates the market of loanable funds when the supply and the demand of loanable funds are equal. Now the supply of loanable funds is $Y_t - C_t = S_t -$ Net capital outflow and the

[12]The U.S. was the industrial leader of the 20th century, and if (or when) the stock of knowledge is not country specific, the industrial leader set the trend growth rate. For more discussion on defining the trend growth rate, see Kehoe and Prescott (2002).

demand is $I_{t,\text{domestic}}$. It is assumed that in the steady state, the interest rate must be at a level which households are willing to accept. Otherwise, households would change their saving. The effects of saving in other sectors, affects households savings throughout the market of loanable funds. Thus, I assume that households are ricardian – at least in the long-run – and that corporate savings are a perfect substitute for household savings. Moreover, the capital stock measured by Statistics Finland does not match this theoretical counterpart, so I must construct this series.

The perpetual inventory method is used to estimate the capital holdings of Finnish residents. I use the law of motion for capital, as given by equation (5), which can be rewritten after aggregation as follows:

$$K_{t+1} = (1-\delta)K_t + I_t, \tag{21}$$

where the starting value of capital and δ are unknown and must be calibrated. I_t is calculated from the data and is given by $Y_t - C_t = I_t$, where Y_t is GNP and C_t is consumption (private and public): hence, I_t is equal to national saving and gross national investments. By cumulating national savings, I get a measure for national wealth, and this wealth is accumulated now to smooth consumption across time and states. Note that, $I_t = I_{t,\text{domestic}} + \text{Net capital outflow}$ when it also captures the investments to abroad by residents of Finland.

The value of δ is chosen to be consistent with the average ratio of depreciation to GNP, which is observed in the data over the period used for calibration purposes. The national accounts data gives:

$$\frac{1}{23}\sum_{t=1985}^{2007}\frac{\delta K_t}{Y_t} = 0.1654. \tag{22}$$

This condition must hold when δ and $\{K_t\}_{t=1985}^{2007}$ are chosen. A rule for choosing the value of capital stock at the beginning is still missing and I use a rule where the capital-output ratio of the initial period should match the average capital-output ratio over the calibration period. I choose the years 1975-1984 as the calibration period: hence, the following condition must hold

$$\frac{K_{1975}}{Y_{1975}} = \frac{1}{8}\sum_{t=1975}^{1984}\frac{K_t}{Y_t}. \tag{23}$$

The method for calculating the capital stock is more or less arbitrary, but it yields a reasonable result. In figure 2, the constructed capital stock or wealth measure used in this paper is compared to the Finland's domestic capital stock measured by Statistics Finland. In the late 1980s and early 1990s the measured

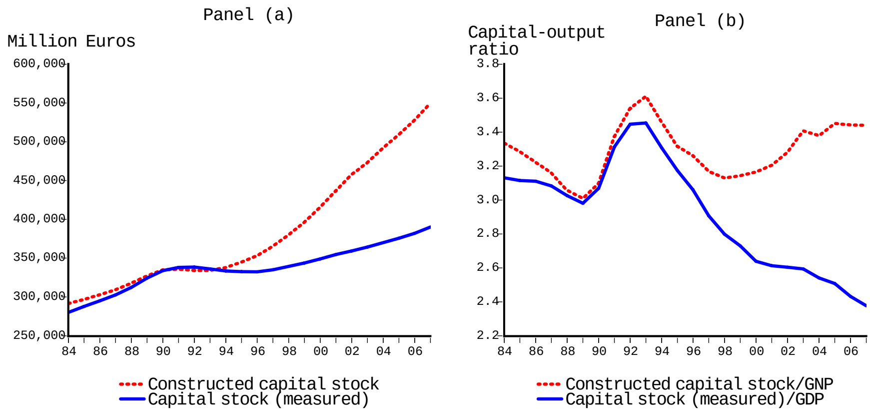

Figure 2: In Panel (a), the constructed and the measured capital stock are compared (in 2000 prices). In Panel (b), the constructed capital stock is divided by GNP and the measured capital stock is divided by GDP.

domestic capital stock was equal to the constructed capital stock. After 1994, the two series diverged from one another. This is consistent with the fact that Finland's net exports were highly positive at that time, which caused net capital outflow. Thus, Finnish investments and capital/wealth holdings were increasingly directed abroad.

When the constructed capital-GNP ratio is compared to the measured capital-GDP ratio, the difference between the two measures can once again be clearly observed. The drop in the capital-GDP ratio is significant and difficult to explain within a neoclassical growth framework.[13] The constructed capital stock to GNP seems to be stationary, fitting the facts given by the neoclassical growth framework, when the economy fluctuates around a steady state path.

Figure 2 shows that when a small open economy is considered, the *domestic* capital stock is not necessarily a good measure for wealth, which generates

[13]One explanation could be that the capital-GDP ratio has been very high in Finland and now it is converging to match values observed generally throughout the OECD countries, which are around 2.5. Thus, the behavior of capital stock can be explained if it is assumed that Finland's economy is on a transitory path towards a new steady state.

capital income. Of course, the domestic capital stock is useful when the capital used in domestic production is measured. However, the constructed capital stock used in this paper fits the facts given by the neoclassical growth framework (i.e., capital-output ratio should be constant at the steady state) and can be used when the national income is distributed according to the factors of production, which defines r^{im}.

4.4 The definition of labor share

The last parameter to estimate is α, the capital share of production. The labor income share, $1-\alpha$, can be directly measured from the data, but some adjustments must be done. As shown by Gollin (2002), it is a bad idea to measure the labor income share as the compensation of employees to GDP because some payments to self-employed workers and unremunerated family workers are not included in the labor income share. To correct this bias, I define labor share as

$$\text{Labor share} = \frac{\text{Compensation of employees}}{\text{GNP-Household net mixed income}}.$$

Now the ambiguous category is subtracted from GNP, which is an equivalent procedure for sharing the ambiguous category in the same proportions as in the rest of the economy.[14] The calculations give an average value of the labor income share of 0.637 for the period 1985-2007, which produces $\alpha = 0.363$.

4.5 Parameter selection for the relative risk aversion and subjective time preference

Generally, the calibration of standard real business cycle model gives the values of β, which are close to one (measured from quarterly data). Along these same lines, I define

$$\rho \in \{0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07\}$$

all of which could be reasonable choices.

[14]There is no rule for how the ambiguous category should be divided between labor and capital income, since the shares vary widely from country to country. However, this estimate is in line with Gollin (2002) and Conesa, Kehoe, and Ruhl (2007).

Moreover, researchers have found a weak connection between the real interest rate and growth in consumption, which implies a low value for the IES. However, structural estimates using micro data have defined the value of θ as slightly over one.[15] Further, Kilponen (2009) has estimated the IES based on Finnish data and the results imply that θ is in the range of 1.5-3.5. Thus, I define

$$\theta \in \{1, 2, 3, 4, 5\}$$

to match different estimates for θ.

5 The results

5.1 Is the estimate for the rate of return for the Finnish economy feasible?

The most important variable, which is needed for the results is the estimate for r^{im}, which basically generates the result. Hence, before jumping into the results, the question must be asked: Is the estimate for the rate of return for the Finnish economy, r^{im}, feasible? The data generated in Section 4 gives the estimate for r^{im}, and the average over the period 1985-2007 is used to estimate the r^{im}, which is based on equation (2). That is,

$$1 + r^{im} = 1 + \frac{1}{22} \sum_{t=1985}^{2007} r_t = 1 + \frac{1}{22} \alpha \sum_{t=1985}^{2007} \frac{Y_t}{K_t} = 1.110, \tag{24}$$

where Y_t is GNP and K_t is constructed as described in Section 4.3. When $\delta = 0.05$, the interest rate is 6%, which is a reasonable value. The answer reflects a situation at the steady state when cyclical variations are not considered.

To evaluate this result, compare it to the similar estimate provided by McGrattan and Prescott (2003). McGrattan and Prescott (2003) also argue that the return of Treasury Bills is not the rate at which households intertemporally substitute consumption. They used national accounts and capital stock to create the correct measure for the rate of return. For the U.S., the after tax average

[15] For papers that report a low value for the IES, see Hall (1988); Campbell and Mankiw (1989); Attanasio and Weber (1993); Ludvigson (1999); Cambell (1999). Papers that report the IES as being close to one include, for example, Gourinchas and Parker (2002); Cagetti (2003); Parker and Preston (2005).

return is about 4%. Here the expected after tax return is 4.1%, which seems very reasonable.[16] Hence, the measure for the interest rate, r^{im}, fit the fact that Finland is a small open economy and that the factor prices are given from the outside at least in the steady state.

5.2 The magnitude of precautionary wealth

Now I have an estimate for r^{im}. Then, the different values of r^{cm}, which are functions of θ and ρ, give the magnitude of precautionary wealth, as indicated by equation (19). If precautionary savings are unimportant for capital accumulation, then the term $1 + \Psi$ from Euler equation (15) can be ignored. Thus, the value given by equation (24), i.e. r^{im}, should be equal to $(1+g)^{\theta}(1+\rho)+\delta$. Then, all of the savings in Finland are caused – at least in the long-run – by the intertemporal reason for saving. However, if $r^{cm} > r^{im}$, then $\Psi > 0$, which implies that the precautionary saving motive also matters for long-run capital accumulation, as shown by the Euler equation composition. However, the value of r^{cm} depends on θ and ρ, which are unknown, and they can be led to vary. Using equation (19), one can derive the magnitude of precautionary wealth as a function of θ and ρ: the amount of precautionary wealth is now given as a percentage of aggregate wealth holdings. Table 1 shows the results.

Table 1: The steady state magnitudes of precautionary wealth in Finland for a period 1984-2007, given as a percentage of aggregate wealth holdings for different values of θ and ρ.

	ρ						
	0.01	0.02	0.03	0.04	0.05	0.06	0.07
θ							
1	-58%	-32%	-12%	3%	16%	25%	34%
2	-7%	7%	19%	28%	36%	42%	48%
3	22%	31%	38%	44%	50%	54%	58%
4	40%	46%	51%	56%	59%	63%	66%
5	52%	57%	60%	65%	69%	71%	71%

Table 1 shows that most of the combinations of θ and ρ give a positive and significant value for the magnitude of precautionary wealth, which implies that

[16] The marginal tax rate for capital is taken as an average over 1985-2005, which was calculated by Conesa, Kehoe, and Ruhl (2007). The expected after tax return is given by $(r^{im} - \delta)(1 - \tau_k) = (0.11 - 0.05)(1 - 0.3185) = 0.041$, where τ_k is the marginal tax rate for capital (the data for the marginal taxes rate is available from `http://www.greatdepressionsbook.com/`).

the term $1 + \Psi$ should be included in the aggregate Euler equation. Only the combinations of small values from both parameters are not feasible (i.e., the negative numbers), but the rest of the values are basically feasible. However, it can be seen that it is possible to choose θ and ρ in such a way that the precautionary saving motive can be ignored. Thus, it is feasible to conclude that precautionary savings do not affect the level of capital. Generally, the magnitude of precautionary wealth is very sensitive, even with quite small variations in the values of θ and ρ, which could explain the large differences between the estimates given the related literature (see Section 1).

Further, it is evident from Table 1 that high values of relative risk aversion, or low IES, imply high precautionary savings. If the IES is low, as suggested by many studies, households are not willing to accept large swings in their levels of consumption. Consequently, households will borrow against growing income – households want to use their future high income to consume more today since they like smooth consumption – which in turn causes a lower capital stock and a higher interest rate for the economy. This can be seen from Euler equation (16): the term $(1 + g)^{\theta}$ has higher values when θ increases, which causes a higher r^{cm}. The difference between the interest rate observed in the data, which households use when they intertemporally substitute consumption, and an interest generated by the high value θ combined with growing income must be explained somehow. Here I showed, by using the composition of Euler equation, that this difference can be explained by wealth holdings, which are motived by precautionary saving reasons. Thus, the uncertainty associated with the growth rates of household incomes prevents households from borrowing against their higher expected income in the future. Hence, the high relative risk aversion, or low IES, will imply a high level of precautionary savings.

The second interpretation of the results in Table 1 is that, by using the logarithmic utility function (when $\theta = 1$) and a reasonable value for the subjective time preference ($\rho = 3.75\%$), we get a result where $r^{im} = r^{cm}$. Hence, it can be concluded that precautionary saving does not matter for long-run capital accumulation. This result is suggested by many simulations: for instance, see Aiyagari (1994) and Krusell and Smith (1998). Perhaps rich people are responsible for capital accumulation and they do not have a precautionary saving motive since they are wealthy enough that the uncertainty associated with labor income does not matter for them. If one is willing to accept this, then a proper calibration for the utility function of the representative agent in a standard growth model, which considers issues associated with capital accumulation, is the logarithmic utility function. The calibration of θ has been one of the most difficult parameters to pin down in a typical growth model (see, discussion by Lucas, 1980; Prescott, 1986). The result given in this paper provides one rationalization for using the log-utility: then it can be argued that the vital assumption about the completeness of markets is a good approximation since $r^{im} = r^{cm}$, or that the precautionary saving motive does not matter for capital accumulation. Hence, the value of θ would not have been based on micro estimates, which cannot be used in general equilibrium models directly, as argued Browning, Hansen, and Heckman (1999), but, rather, it is rationalized by other assumption used in the model, namely the assumption of complete markets.

Appendix A shows the robustness of the result. The growth rate of 2% is used, which is the steady state growth rate of the U.S. economy, and hence, it could also be Finland's steady state growth rate. Moreover, different measures for labor share $(1-\alpha)$ are also considered. However, the results given in this section are robust and not altered by these changes. However, the form of production function represents a critical choice since it gives the mapping between the capital stocks and the interest rates. Moreover, a different type of model or changes in the utility function could give different results.

In particular, using the Kreps-Porteus nonexpected utility preferences, which were introduced for macroeconomics by Epstein and Zin (1989) and Weil (1990), would be an interesting alternative to the preferences used in this paper, since then one could allow for separation between the IES and risk aversion. However, as far as I know, there is no work which uses Kreps-Porteus preferences with a decreasing absolute risk aversion in general equilibrium setup.[17] Kimball and Weil (2009) recently provided the first systematic treatment of precautionary saving using Kreps-Porteus preferences. They gave the measure for local relative prudence $\mathcal{P}$ as follows:

$$\mathcal{P} = \theta \left(1 + \text{IES}\right). \tag{25}$$

Kreps-Porteus preferences are very popular, since with those preferences, one

[17] Works by Weil (1993) and van der Ploeg (1993) used constant absolute risk aversion preferences, but a decreasing absolute risk aversion is needed in order for the precautionary saving motive to decrease when the value of assets increases.

can explain numerous asset price puzzles, as shown by Bansal and Yaron (2004). In particular, with the parametrization $\theta = 10$ and IES $= 1.5$, one can explain the equity premium puzzle (Bansal and Yaron, 2004, Table 4). This parametrization gives $\mathcal{P} = 25$, which implies a strong precautionary saving motive for the representative agent. This in turn *could* imply that a large share of wealth holdings arises out of precautionary saving reasons. However, it is too early to draw any definite conclusions. Nonetheless, the use of Kreps-Porteus preferences in this type of setup seems to generate an extremely interesting research question for future papers.

6 Conclusions

The paper showed that if the level of interest, which calculate form data, is not the same as the interest rate implied by the model, which is based on a complete markets assumption, when only the intertemporal reason for capital holdings is relevant, then the difference is caused by precautionary reasons for holding capital. This measure for precautionary savings is based on very simple and fundamental definitions of economic theory. That is, the measure was given by using Euler equations where the decomposition of Euler equations showed that the possible difference in interest rates is caused by the precautionary saving reason for holding capital. The mapping between the interest rates and capital stock is based on the distribution of national income. The measure is a Solow-residual-type measure where the unexplained part of capital accumulation must be explained by the unknown parameter. It was shown that this parameter captures the precautionary saving motive for capital accumulation.

However, the magnitude of precautionary savings depends on the unknown parameters of the utility function. Generally, the magnitude of precautionary wealth is very sensitive to variations in these parameters, which could explain the large differences in the estimates given in the literature. Hence, the interpretations of the result were twofold. On the one hand, it can be shown that assuming a high relative risk aversion, or low IES, which is suggested in many studies, will directly imply that capital accumulation results to a great extent from incomplete insurance against idiosyncratic income uncertainty. On the another hand, the result can be interpreted by favoring the log-utility specification for the standard growth model with a representative agent since log-utility with a proper choice of the subjective time-preference implies that the precautionary saving motive does not matter for long-run capital accumulation. Hence, the problematic – but important – choice of the IES can be rationalized by arguing that with this choice the important assumption of complete markets is a good first order approximation for a model which concentrates on issues associated with capital accumulation in the long-run. Then, the choice of IES is not based on micro studies, since these parameters may not be directly used in general equilibrium models, as argued Browning, Hansen, and Heckman (1999).

This paper utilized data from the Finnish economy, but the same method applies to other countries as well. It would be interesting to see if the result differs in other countries. Moreover, the use of Kreps-Porteus preferences in this type of setup would be a valuable extension to the model presented in this paper.

Appendix for Part III

A The robustness of results

First, I use the steady state growth rate of the U.S. as Finland's steady state growth rate. The results are in Table A1.

Table A1: The steady state magnitude of precautionary wealth in Finland for a period 1984-2007, given as a percentage of aggregate wealth holdings for different values of θ and ρ, when steady state growth rate is 2%.

	ρ						
	0.01	0.02	0.03	0.04	0.05	0.06	0.07
θ							
1	-64%	-37%	-16%	0%	13%	24%	32%
2	-15%	1%	14%	24%	33%	40%	46%
3	14%	25%	33%	40%	46%	51%	55%
4	33%	40%	46%	52%	56%	60%	63%
5	47%	52%	56%	60%	63%	66%	68%

A lower growth rate causes a lower precautionary wealth, but the results are more or less the same as observed with baseline parameter values.

Second, I change the definition of labor share. I add to the households' net mixed income the net mixed income from farming and forestry. Both of these activities are often run by one family and it is unclear how these tranches should be dived between labor and capital income. I calculate the labor share using the formula given in Section 4.4.

The average over a period 1985-2007 gives a labor share 0.656, which implies that $\alpha = 0.343$. The value is a bit lower than the baseline value. I keep the steady state growth rate at 2.2%, but I use the value of α given above. Table A2 provides the results.

The magnitude of precautionary wealth is somewhat higher than the results from the baseline estimation.

Table A2: The steady state magnitude of precautionary wealth in Finland for a period 1984-2007, given as a percentage of aggregate wealth holdings for different values of θ and ρ, when $\alpha = 0.343$.

	ρ						
	0.01	0.02	0.03	0.04	0.05	0.06	0.07
θ							
1	-43%	-20%	-3%	11%	22%	31%	38%
2	1%	14%	25%	33%	40%	46%	51%
3	27%	36%	42%	48%	53%	57%	60%
4	44%	49%	54%	58%	62%	65%	67%
5	55%	59%	63%	66%	68%	70%	72%

Part IV

Essay III: On the costs of disability insurance

On the costs of disability insurance

Tomi Kortela*

Abstract

The costs of social insurance come from two sources: first, the social insurance changes the behavior of individuals, and second, taxes that are levied to finance these programs create further losses. We extend the standard Ramsey model by a precautionary saving motive and examine the disability insurance program in the United States. A baseline calibration implies that the program lowers per capita consumption by 2.5%: 1/3 of this burden is caused by higher taxes and 2/3 comes from the change in economic behavior. However, precautionary savings are inefficient at insuring people against permanent disability: therefore, social insurance increases welfare. But, a perfect *private* insurance program would provide a 3.5-7% higher per capita consumption than the current disability insurance program.

JEL Codes: E13, E61, H55

Keywords: Incomplete markets, Social insurance, The Ramsey model

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1 Introduction

The disability insurance program is one of the largest social insurance programs in the United States: in 2007, the program covered almost 9 million individuals, its costs were $99 billion and it constituted 17% of social security benefits. The size of the program is about three times larger than that of the unemployment insurance program and its size far surpasses that of any other similar program (SSA, 2009). Given these facts, it is important to study the cost of the disability insurance program for the U.S. economy.

It is well known that the costs of any social insurance program come from two sources: first, the social insurance program has substantial undesirable effects on incentives and, therefore, on economic performance. Second, these programs are often financed by a proportional tax rate, which causes a further distortion for the economy.[1] However, there no estimate – as far as we know – on the importance of these sources in generating the costs. Moreover, one could ask: What would happen if this program was shut down? Would aggregate output and consumption increase or decrease and would these effects result from the change in economic behavior or the change in the tax rate? Further, could households insure themselves against the risk of disability? Or, is the program extremely important for disabled households? These are the questions that this paper aims to discuss.

The model introduced in this paper gives a new extension for the standard textbook Ramsey model:[2] we extend the Ramsey model by including a precautionary saving motive for households. That is, there is a risk for the permanent loss of job, which captures the uncertainty associated with disability. Moreover, in the model the government provides the social insurance which affects households' economic behavior by removing the self-insurance (precautionary saving) motive. This program is financed by a proportional tax rate, which reduces the incentives for households to accumulate capital and supply labor. Hence, this model provides a tool to measure the cost of the disability insurance program within the framework of the standard Ramsey model.

The types of models which enable incomplete markets – i.e. where there is a lack of insurance against idiosyncratic uncertainty – are standard in a gen-

[1] For a more detailed discussion, see Feldstein and Liebman (2002) and Feldstein (2005).

[2] We refer here, for example, to Barro and Sala-i-Martin (2004, chap. 2).

eral equilibrium framework. Aiyagari (1994) and Huggett (1997) were the first to introduce idiosyncratic labor endowment risk with incomplete markets into dynamic stochastic general equilibrium (DSGE) models. The result from these models is that households save more than they would if there was no uncertainty and this additional saving can be seen as a measure for aggregate precautionary savings (Huggett and Ospina, 2001). However, an important point is made by Marcet, Obiols-Homs, and Weil (2007), who claim: a higher capital stock does not lead to higher output when labor supply is elastic, since a higher capital stock implies a wealth effect which reduces the supply of labor.[3] The model presented in this paper captures these aspects within the standard Ramsey model. The origin of the model is taken from Toche (2005) where he gives a tractable model of precautionary saving.[4] We expand this partial equilibrium model to a general equilibrium. A distinction between this model and the DSGE models cited above is that, whereas we consider the effects of uncertainty associated with rare events, generally the DSGE models are focused more on moderate fluctuations of uncertain labor income. Therefore, this model is a new one and fills a gap in the existing literature.

The second branch of literature associated with the paper is the economics of disability and the analysis of disability insurance targeted for these people.[5] Furthermore, works by, for example, Diamond and Sheshinski (1995), Autor and Duggan (2003) and Chandra and Samwick (2009), have modeled the economic effects of the risk of disability within a partial equilibrium setting. Moreover, many studies have taken an information approach to this subject and have analyzed optimal planning problems in which some key information is only privately observed. There, the focus is on implementing an optimal mechanism that minimizes the distortions caused by these insurance programs. See, for example, Golosov and Tsyvinski (2006) and references therein. This paper contributes to this literature by using a growth model and studying the effects of disability within a general equilibrium framework. Moreover, here we try to capture the magnitude of the welfare question related to the disability insurance program rather than provide

[3]It is assumed that a household can only adjust its labor supply if it is given the (stochastic) opportunity to work. Pijoan-Mas (2006) uses a model where idiosyncratic shocks are associated with individuals productivity. Then, precautionary saving can be replaced by longer working hours. However, we assume here, as it is traditionally assumed, that stochasticity is associated with the opportunity to work.

[4]See also the recent papers by Carroll and Toche (2009) and Carroll and Jeanne (2009).

[5]A review of this literature is given by Bound and Burkhauser (1999).

an optimal mechanism for this program. Note that we do not consider any type of strategic behavior when the paper gives a lower limit for the costs of the disability program.[6]

Our model implies that closing the current disability insurance program would increase per capita consumption by 2.5%. One-third of this burden is caused by higher tax rates and 2/3 comes from the change in economic behavior, i.e. from the removed precautionary saving motive. It is surprising that the labor supply decision does not depend on the level of disability insurance, even if taxes are increased when the level of insurance is increased. Rather, the supply of labor is almost constant. However, self-insurance works poorly against permanent shocks, and therefore, the social insurance program increases welfare by providing a higher level of consumption for disabled households. But the real problem is the incompleteness of the private insurance markets: if the perfect insurance against permanent disability is provided by private insurance companies rather than the current social insurance program which is financed by proportional tax rate, per capita consumption would increase by 3.5-7% depending on the Frisch elasticity of labor supply. The result implies that optimizing the tax financed social insurance systems is not the best way to improve welfare. Instead, completing the markets by removing impediments to the private provision of insurance would generate a much higher increase in welfare. Another interpretation of the result is that the cost generated by problems associated with imperfect information – which prevents market-based solutions – is indeed very large.

The rest of the paper is organized as follows: Section 2 introduces the model and Section 3 gives the aggregate variables, derives the steady state and analyzes its stability; Section 4 discusses the calibration of the model and reports the numerical results of simulations; finally, Section 5 concludes the paper.

2 The model

The model introduced in this paper is a standard neoclassical growth model or the Ramsey model in continuous time with an endogenous labor supply,[7] but

[6]Strategic behavior created by the disability insurance program causes additional costs to the ones considered here. See, for example, Rust and Phelan (1997) for a more detailed discussion.

[7]The classical references for the Ramsey model are Ramsey (1928), Cass (1965) and Koopmans (1965).

there are two significant exception compared to the baseline model. First, members of households face the constant probability of losing their jobs permanently throughout their lives. Then, the model captures the uncertainty associated with rare and permanent income losses of households, i.e. the model captures the uncertainty associated with a disability. These types of events could, for instance, be a severe injury or compulsory retirement. Second, the government only provides partial insurance against this uncertainty, and this disability insurance is financed by a proportional income tax. Since insurance against uncertainty is only partial, households have a precautionary saving motive and households can self-insure by holding a single asset – physical capital.

2.1 Demographics

There are two types of households: workers and disabled workers, and the latter do not work. At every moment working households get nL_0e^{nt} new members to their households, where L_0 is normalized to 1. The size of the population at time t is $P_t = \int_{-\infty}^{t} ne^{nv}dv = e^{nt}$, since no one dies. However, every instant there is the constant instantaneous probability μ of permanent disability for each working member of household. The transition from the employed state to the disability state follows a Poisson process with arrival rate μ, which implies that the expected time in the employed state is $\frac{1}{\mu}$. Since μ is also a fraction of households which face permanent disability at each instant, the size of working households is given by $L_t = \int_{-\infty}^{t} ne^{nv}e^{-\mu(t-v)}dv = \theta e^{nt}$, where $\theta = \frac{n}{n+\mu}$. A fraction θP_t of the entire population are workers, which implies that the size of disabled households is $Z_t = P_t - L_t = (1-\theta)e^{nt} = \lambda e^{nt}$, where $\lambda = \frac{\mu}{n+\mu}$.

2.2 Production and factor prices

The Cobb-Douglas production function is assumed and there exists a perfect competition among firms. The economy's production per efficient capita Y_t is given by

$$Y_t = \Theta K_t^{\alpha} l_t^{1-\alpha}, \tag{1}$$

where $\Theta = \theta^{1-\alpha}$. It is a scaling factor, since production is in the terms of per efficient capita, but only a fraction θ of the entire population is working.[8] Moreover, K_t is the capital stock, α is the capital share and l_t is labor.

Profit maximization of the representative firm gives following factor prices:

$$r_t = \alpha\Theta K_t^{\alpha-1} l_t^{1-\alpha} - \delta \tag{2}$$
$$w_t = (1-\alpha)\theta^{-\alpha} K_t^{\alpha} l_t^{-\alpha}, \tag{3}$$

where δ is the depreciation rate of capital stock. If $\mu = 0$, then the production side of the economy falls to the baseline neoclassical growth model. However, if $\mu > 0$, it decreases the number of workers, which lowers the interest rate and increases the wage rate when everything else is kept constant.

2.3 The behavior of the government

The government offers social insurance for disabled households. Every disabled household receives a social insurance benefit b_t, which is a lump sum transfer, for every period. To finance this program, the government must lay an income tax for capital income ($r_t A_t$) and labor income ($\theta w_t l_t$), which are taxed at the rate τ_t. Moreover, the government is running a balanced budget for every period, or a pay-as-you-go social insurance program, which implies the following budget constraint:

$$\lambda b_t = \tau_t r_t A_t + \tau_t \theta w_t l_t. \tag{4}$$

The left-hand side of the equation shows the aggregate transfer for disabled households (in per efficient capita form). It is assumed that the government sets a rate η, which is the replacement ratio of after-tax labor income when $0 \leq \eta \leq 1$. Therefore, the level of social insurance is defined as $b_t = \eta(1-\tau_t)w_t l_t$. The right-hand side of the equation (4) shows the net income of the government, which is composed of the taxed labor and capital incomes of households. Note that the aggregate assets are given by $A_t = A_t^e + A_t^d$, where $A_t^e \equiv \theta a_t^e$ and $A_t^d \equiv \lambda a_t^d$, where superscript indexes the individual's state: e stands for the employed state and d

[8]Production per efficient capita means that we have divided the actual production by the term $P_t T_t$, where T_t describes technological progress, for which the growth rate is g. Moreover, our Cobb-Douglas production function is in a Harrold-neutral form where effective labor is defined by $T_t L_t l_t$.

stands for the disabled state. Thus, the level of asset holdings of the representative (or average) agent in both states is weighted according to the population share associated with the state in question.

Rewrite the government budget constraint (4), which now defines the tax rate τ_t by

$$\tau_t = \frac{\lambda\eta w_t l_t}{r_t A_t + (\eta\lambda + \theta) w_t l_t}. \tag{5}$$

Hence, the income tax rate is an endogenous variable, and only the replacement ratio η is decided by the government. That is, the government decides how generous insurance it wants to provide for the disabled households.

2.4 The problem of households

Assume that households start their living within this economy at time $t = 0$. The representative households maximize their own and their prospective descendants' expected utility by making consumption and labor supply decisions. The problem of the representative household is given by

$$\max_{c_t, l_t} \quad E_0 U = E_0 \int_0^\infty e^{(n-\rho)t} U(c_t, l_t)\, dt \tag{6}$$

$$\text{s.t.} \quad \dot{a}_t = \begin{cases} [(1-\tau_t) r_t - n - g]\, a_t - c_t + (1-\tau_t) w_t l_t, & \text{if} \quad \epsilon = e \\ [(1-\tau_t) r_t - n - g]\, a_t - c_t + b_t, & \text{if} \quad \epsilon = d \end{cases} \tag{7}$$

$$a_t \geq 0 \quad \forall t \tag{8}$$

$$\text{and} \quad \lim_{t \to \infty} \left[a_t e^{-\int_0^t (1-\tau_s) r_s - g - n ds} \right] = 0. \tag{9}$$

E_0 is the conditional expectation operator and household get utility from consumption per adult person c_t and the supply of labor l_t produce disutility for household. However, in the disabled state $l_t = 0 \ \forall \ t$ by the definition of disability. n is the growth rate of population in the economy, ρ is the rate of time preference, g is the growth rate of productivity and variables are scaled per efficient capita. Household faces a budget constraint where r_t is the interest rate and w_t is the wage rate. When household is employed ($\epsilon = e$), it receives capital income from assets holdings a_t and labor income from working. Both incomes are taxed with a rate τ_t by the government. When household is disabled ($\epsilon = d$), it receives a social insurance benefit b_t which are tax free but its capital income is still taxed. Moreover, household face a liquidity constraint $a_t \geq 0$ and the last constraint ensures that budget constraint must finally hold with equality.

We can solve the problem by using the familiar tools of optimal control theory, for which Toche (2005) provided the following key insight: The only source of uncertainty in this problem is the timing of transition from the employment state to the disability state. Thus, uncertainty is restricted to a single transition, and after the transition, the problem is deterministic, i.e. disabled households do not face any kind of uncertainty. The key assumption is the persistence of this transition. Therefore, it is possible to solve the full problem by using "backward induction": first, solve the deterministic problem of disabled households; second, use that solution when solving the problem of employed household.[9]

We assume that new members within the economy were born into working households and that they are loved. Thus, the newborn members enter into the economy with an amount of assets which equal to the asset holdings of those currently working in the economy. The result of these assumptions is that the liquidity constraint is not binding for employed households, but it will be binding at some point of time for disabled households since, in the disabled state, households start to dis-save. Without the liquidity constraint, the assets of disabled households would approach minus infinity. To prevent that, we assume that disabled households cannot go into debt.

According to backward induction, we must first solve the problem of the disabled household, which is deterministic. But the household faces a liquidity constraint. We can provide a closed-form solution for this problem: first, we ignore the state variable inequality constraint, and next, take into account the effects of liquidity constraint.[10] Hence, we start by ignoring the liquidity constraint and then solve the problem of the representative household given by equations (6), (7) and (9).

[9] For a more detailed discussion, see Toche (2005), Carroll and Kimball (2007) and Carroll and Toche (2009).

[10] This method is given by Kamien and Schwartz (1981, chap. 17) and also utilized by Park (2006).

2.4.1 Solving the problem without the liquidity constraint

Assume that households have the following utility function:

$$U(c_t, l_t) = \log c_t - \gamma \frac{l_t^{1+\phi}}{1+\phi}, \tag{10}$$

where the Frisch elasticity is equal to $\frac{1}{\phi}$.[11] Appendix A provides a detailed derivation for the Euler equations. The Euler equations for consumption are given by:

$$\frac{\dot{c}_t^e}{c_t^e} = (1-\tau_t)r_t - \rho - g + \mu\left(\frac{c_t^e}{c_t^d} - 1\right) \tag{11}$$

$$\frac{\dot{c}_t^d}{c_t^d} = (1-\tau_t)r_t - \rho - g. \tag{12}$$

where $\dot{x}_t = \frac{dx_t}{dt}$. In the disabled state, households' Euler equations are in the standard form, but in the employed state there is a precautionary saving motive, which is given by the last term in equation (11). Note that the precautionary saving motive disappears when $\mu = 0$ or $c_t^e = c_t^d$. That is, if the probability of a permanent transition to the disability state is zero, or if the levels of consumption are the same in both states when there is a perfect insurance against uncertainty, the precautionary saving motive disappears.[12] The higher the μ or the greater the difference in the levels of consumption between the states (i.e. the lower the level of insurance), the higher the precautionary saving motive.

An employed household must also decide upon its labor supply. The first order condition for labor supply is given by

$$\gamma l_t^{\phi} = \frac{(1-\tau_t)w_t}{c_t^e}. \tag{13}$$

Thus, the marginal disutility from working must be equal to the proportion of after-tax wage to consumption, which is the standard result. Now we have derived Euler equations for the problem of households, but we must still define the value of c_t^d. We can do this by solving the problem of disabled households.

[11] As shown by King, Plosser, and Rebelo (1988), the form of utility function given by equation (10) is required for the balanced growth path. Moreover, log-utility for consumption implies that utility is separable with respect to consumption and labor. We assume log-utility for consumption for two reasons: first, we want to reduce the amount of calibrated parameters; second, we try to avoid overestimating the precautionary saving motive, which could be done by assuming a lower intertemporal elasticity of substitution for consumption.

[12] It is obvious that $c_t^e \geq c_t^d$.

2.4.2 The problem of disabled households

Consider a representative household which was disabled at time v. The budget constraint can be written for this household as follows:

$$\begin{aligned} \dot{a}^d_{t,v} &= [(1-\tau_t)r_t - n - g]\, a^d_{t,v} - c^d_{t,v} + b_t \\ a^d_{t=v,v} &= a^e_v \\ a^d_{t,v} &\geq 0 \quad \forall t. \end{aligned} \tag{14}$$

Now the modification is that the assets and consumption are function of time v. At moment v the household was disabled and its assets were at the level a^e_v in the employed state at that moment. Moreover, now we must also consider the effects of the liquidity constraint. First, we solve the households' problem without the effects of liquidity constraint. Second, we show that the liquidity constraint will be binding in the future. Third, we show the optimal consumption plan for disabled household.

The first part has already been done previously. The Euler equation for a household that is disabled, but not liquidity constrained, is given by equation (12) and straightforward integration yields

$$c^d_{t,v} = c^d_v e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v} - \rho - g](t-v)}, \tag{15}$$

where $(1-\bar{\tau}_{t,v})\bar{r}_{t,v} = \frac{1}{t-v}\int_v^t (1-\tau_s)r_s ds$. Euler equation (12) gives the optimal time path for consumption when the liquidity constraint is not binding and equation (15) gives the level of consumption along that path. However, we still must solve c^d_v in order to define the optimal level of consumption when the liquidity constraint is not binding. Plug equation (15) into household intertemporal budget constraint (14) which then gives

$$c^d_v = (\rho - n)\left[a^e_v + \tilde{b}_v\right], \tag{16}$$

where $\tilde{b}_v = \int_v^\infty b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v} - n - g](t-v)} dt$, i.e. $\tilde{b}_v$ is the present value for social insurance income.

Equation (16) now defines the level of consumption at time v, i.e. at the time when the household was disabled. Moreover, from the viewpoint of *still employed* household, we can replace v with t since the point v is not realized for it, i.e. we get c_t^d. Since the risk for disability is continuous and does not depend on time, equation (16) shows the level of consumption for employed household if it would fall into disability at the present moment. Thus, equation (16) defines – with c_t^e – the magnitude of the precautionary saving motive in equation (11).[13]

The second task was to show that the liquidity constraint will be binding in the future, which can be done by following the evaluation for the assets. The evaluation of assets over time can be derived from equation (14) and details of this derivation are given in Appendix B. This yields

$$a_{t,v}^d = \left[a_v^e + \tilde{b}_v\right] e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)} - \tilde{b}_v. \tag{17}$$

It is assumed that most people work and that only a small number of people are disabled. Further, working households are richer than disabled households, and therefore, more influential when the aggregate economy is considered. Hence, the interest rate and the capital stock are mainly set by the decisions of the workers. The Euler equation of workers (11) includes the term $\mu\left(\frac{c_t^e}{c_t^d} - 1\right)$, which gives the precautionary saving motive and lowers the level of r_t. However, disabled households face the same interest rate r_t in their Euler equation, which implies that $(1-\tau_t)r_t < \rho + g$ and this causes the dis-saving of disabled households (or a descending path of consumption). Moreover, the cause of continuous dis-saving is that the liquidity constraint will be binding. This can be shown by using equation (17): when $t = v$, equation (17) implies that $a_v^d = a_v^e$. When $t > v$ then $a_{t,v}^d < a_v^e$, since the term $e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)}$ has values ranging from one to zero as t increases, which results from the fact that $(1-\bar{\tau}_t)\bar{r}_t < \rho + g$. Thus, $a_{t,v}^d$ approaches $-\tilde{b}_v$ as times goes on, and hence, the liquidity constraint is binding when $a_{t,v}^d = 0$.

Note that the dis-saving of disabled households and the saving of employed households are results of the general equilibrium setup of the model. We could say that disabled households are "impatient": the after-tax interest rate is smaller than their efficient discount rate, which causes a descending time path for con-

[13]From the viewpoint of the employed household, we can write equation (16) as

$$c_t^d = (\rho - n)\left[a_t^e + \tilde{b}_t\right], \quad \text{where}$$
$$\tilde{b}_t = \int_t^\infty b_t e^{-[(1-\bar{\tau}_t)\bar{r}_t-n-g]t} dt.$$

sumption. The key assumption here is that the per capita social insurance benefit grows at the rate of g. Thus, disabled households also enjoy the benefits of growing productivity. However, "impatience" does not originate from the higher subjective discount factor compared to employed households, but, rather, from the fact that disabled households do not have a precautionary saving motive. Therefore, the prevailing interest rate in the economy – which is set by households that have a precautionary saving motive – causes the dis-saving of disabled households or their "impatience".

Now we can give the optimal path for consumption. Assume now that the liquidity constraint is binding at time T for a household which is disabled at time v. Then the budget constraint (equation (14)) implies that consumption is $c^d_{T,v} = b_t$, since, by definition, $\dot{a}^d_{T,v} = a^d_{T,v} = 0$. The moment when the liquidity constraint is binding, T, is still unknown, but it can be solved from equation (17). Furthermore, we can write the solution as $T = \Psi_t + v$, where Ψ_t is a function of the variables given in equation (17).[14]

It is now possible to give the optimal consumption plan over t for a household which became a disabled household at time v. It is as follows:

$$c^d_{t,v} = \begin{cases} c^d_v e^{[(1-\bar{\tau}_t)\bar{r}_t - \rho - g](t-v)} & \text{if} \quad t \in [\, v, T\,) \\ b_t & \text{if} \quad t \in [\, T, \infty\,)\,. \end{cases} \tag{18}$$

That is, before the liquidity constraint becomes binding, consumption follows the optimal time path implied by Euler equation (12). From the moment when the liquidity constraint becomes binding to infinity, the household consumes its social insurance benefits. This consumption plan maximizes the utility of the representative disabled household. Note that, when $b_t = 0$ (i.e. $\eta = 0$), the liquidity constraint is not bind – or is binding only asymptotically – and assets asymptotically converge to the lower limit: hence, $T = \infty$.

[14]Now we have assumed that Ψ_t does not depend on v. We will come back to this assumption in section 3.4, and actually, this assumption holds only approximately. Anyhow, this approximation is feasible.

3 Aggregation and the steady state

3.1 Aggregate consumption for the disability state

We start the aggregation from the disability state. According to equation (18), we can separate the consumption of households in the disability state into two parts depending on the asset holdings of the household. If the liquidity constraint is binding, the household's consumption is given by $c^d_{t,v} = b_t$ for all t, and this does not depend on v. However, the level of consumption for the disabled household which has assets depends on v, i.e. consumption depends on the moment when the household became disabled.

Assume now that the economy is living at moment t. Hence, households that have arrived at the disability state between times t and $t - \Psi_t$ are not liquidity constrained.[15] In order to capture consumption at every point on that optimal path, we must integrate from t to $t - \Psi_t$. Note that we add the levels of consumption between times t and $t - \Psi_t$, thus we must divide the sum by Ψ_t in order to get the average level of consumption at time t. Hence, by integrating equation (18) from t to $t - \Psi_t$ we can give the average consumption of disabled, *but unconstrained*, households as

$$c^{d,u}_t = \frac{1}{\Psi_t} \int_{t-\Psi_t}^{t} c^d_{t,v} dv = \frac{1}{\Psi_t} \int_{t-\Psi_t}^{t} c^d_v e^{[(1-\bar{\tau}_t)\bar{r}_t - \rho - g](t-v)} dv \tag{19}$$

where the second u in the superscript represents unconstrained households.

Now we must find the fraction of liquidity constrained households and the fraction of households that have assets. The number of disabled households at time t is Z_t and it is increasing at rate n. Thus, we can divide Z_t as follows:

$$\begin{aligned} Z_t = \int_{-\infty}^{t} Z e^{-n(t-v)} dv &= \int_{t-\Psi_t}^{t} Z e^{-n(t-v)} dv + \int_{-\infty}^{t-\Psi_t} Z e^{-n(t-v)} dv \\ &= (1 - e^{-n\Psi_t}) + e^{-n\Psi_t} = (1-\kappa_t) + \kappa_t, \end{aligned} \tag{20}$$

where we have now normalized $\frac{Z}{n} = 1$. Since L_0 was normalized to unity, we may

[15] The liquidity constraint is binding at the point of time $t = \Psi_t + v$ for a household that reached the disability state at time v. So, we can write $v = t - \Psi_t$ which tells the point of time when the household, for which the liquidity constraint for the first time became binding at t, became disabled. If $v < t - \Psi_t$, the liquidity constraint was already binding for the household and the household is consuming just b_t. Thus, the liquidity constraint is not binding for households which reached the disability state between $t \geq v \geq t - \Psi_t$.

normalize Z again without a loss of generality. Moreover, we have defined $\kappa_t \equiv e^{-n\Psi_t}$, when the fraction κ_t from the *disabled households* are liquidity constrained and the fraction $(1 - \kappa_t)$ is not.

Now we can give the aggregate consumption of disabled households as a weighted sum:

$$c_t^{d,a} = (1 - \kappa_t)c_t^{d,u} + \kappa_t b_t, \tag{21}$$

where the weights are just shares of disabled households, as defined by equation (20). The fraction $1 - \kappa_t$ does have assets and they consume, on average, $c_t^{d,u}$ (see equation (19)). The fraction κ_t does not have assets when the level of consumption is b_t.

3.2 Aggregate consumption and the Euler equation

The aggregate behavior of employed households can be described by the behavior of the representative household, which was defined in Section 2.4. When variables are defined in per efficient capita form, we only have to multiply these variables by the share of the population which is employed. That is, we can write $\theta c_t^e \equiv C_t^e$. Moreover, equation (21) defines the aggregate consumption of disabled households in terms of efficient capita and we can define $\lambda c_t^{d,a} \equiv C_t^d$. Hence, the aggregate consumption in per efficient capita form C_t can be defined by the following equation:

$$C_t = C_t^e + C_t^d = \theta c_t^e + \lambda c_t^{d,a}. \tag{22}$$

In order to derive the aggregate Euler equation for consumption, which tells the optimal time path for aggregate consumption, we must differentiate equation (22) with respect to t. This yields

$$\dot{C}_t = \dot{C}_t^e + \dot{C}_t^d = \theta \dot{c}_t^e + \lambda \dot{c}_t^{d,a}. \tag{23}$$

When equation (11) is multiplied by θc_t^e, it defines $\theta \dot{c}_t^e$. However, the definition of $\lambda \dot{c}_t^{d,a}$ is more complex.

Differentiating (21) with respect to time gives

$$\dot{c}_t^{d,a} = (1-\kappa_t)\dot{c}_t^{d,u} + \kappa_t \dot{b}_t - \dot{\kappa}_t(c_t^{d,u} - b_t). \tag{24}$$

Disabled households who do not have assets consume b_t for every t and, thus, for the fraction κ_t the Euler equation for consumption is $\dot{b}_t$. Further, the optimal time path of consumption for disabled households which have assets (a fraction $1-\kappa_t$) is given by Euler equation (12) and the average level of consumption associated with that path is $c_t^{d,u}$ (see equation (19)). Thus, Euler equation (12) defines $\dot{c}_t^{d,u}$. The last term in equation (24) captures the transitions of κ_t in time.

Hence, the aggregate Euler equation (23) for consumption can be rewritten as

$$\begin{aligned}\dot{C}_t &= \left[(1-\tau_t)r_t - \rho - g + \mu\left(\frac{c_t^e}{c_t^d} - 1\right)\right]\theta c_t^e \\ &\quad + \left[(1-\tau_t)r_t - \rho - g\right]\lambda(1-\kappa_t)c_t^{d,u} + \lambda\kappa_t\dot{b}_t - \lambda\dot{\kappa}_t(c_t^{d,u} - b_t). \end{aligned} \tag{25}$$

3.3 The capital stock and its evolution

Since the economy is closed, aggregate assets must equal aggregate capital. Thus, we can write $A_t = K_t = K_t^e + K_t^d$, where $K_t^e \equiv \theta k_t^e$ and $K_t^d \equiv \lambda k_t^d$. The asset holdings of households in the employed state are easy to aggregate: we only have to multiply the asset holdings of the representative agent by population share θ. However, the aggregate asset holdings of disabled households are more complicated to define.

Liquidity constrained disabled households do not have assets, which means that the fraction $1-\kappa_t$ represents households in the disability state which own capital. Therefore, $k_t^d = (1-\kappa_t)k_t^{d,u}$, where $k_t^{d,u}$ represents the average asset holdings of unconstrained households in the disability state at time t. We can define $k_t^{d,u}$ by using equation (17) and we apply the same aggregation method that we used for consumption in Section 3.1. Doing this, we get

$$k_t^{d,u} = \frac{1}{\Psi_t}\int_{t-\Psi_t}^{t} a_{t,v}^d dv = \frac{1}{\Psi_t}\int_{t-\Psi_t}^{t}\left[k_v^e + \tilde{b}_v\right] e^{[(1-\bar{\tau}_t)\bar{r}_t - \rho - g](t-v)} - \tilde{b}_v dv. \tag{26}$$

Now $k_t^{d,u}$ defines the average capital holdings of an unconstrained household in the disabled state at time t. Thus, the aggregate capital holdings of disabled households is given by $K_t^d \equiv \lambda k_t^d = \lambda(1-\kappa_t)k_t^{d,u}$, where the average capital holding of disabled households ($k_t^{d,u}$) is just multiplied by the share of disabled households which have assets ($1-\kappa_t$). This gives the amount of capital for the representative (or average) disabled household k_t^d. Finally, we must multiply it by the population weight of disabled households (λ).

Now we have defined the level of capital. The next task is to derive an equation which gives its evolution in time. We can easily aggregate equation (7), which gives

$$\dot{K}_t \equiv \theta \dot{a}_t^e + \lambda \dot{a}_t^d = \left[(1-\tau_t)r_t - g - n\right] K_t - C_t + \theta w_t l_t + \lambda b_t,$$

where we have used definition (22). The government was running a balanced budget, or a pay-as-you-go social insurance system, and all of output of was exhausted to households. That is, by using equations (1), (2), (3) and (4) we can write the capital evolution equation in the following way:

$$\dot{K}_t = Y_t - C_t - (\delta + g + n)K_t. \tag{27}$$

Thus, the evolution equation of aggregate capital stock is the standard one.

3.4 The steady state

The dynamics of the economy can now be described by the system of three differential equations, which are the aggregate Euler equation for consumption (25), the evolution equation for the aggregate capital stock (27) and the evolution equation for the assets of disabled households (7). Since the aggregate Euler equation depends on K_t and k_t^d, we also need the evolution equation for k_t^d in addition to the two standard equations. Moreover, we must take into account the effects of the endogenous labor supply so that these equations can be rewritten as

$$\begin{aligned}
\dot{\hat{C}}_t &= \left[(1-\tau_t)r_t - \rho - g + \mu\left(\frac{\hat{c}_t^e}{\hat{c}_t^d} - 1\right)\right]\theta \hat{c}_t^e \\
&\quad + \left[(1-\tau_t)r_t - \rho - g\right]\lambda(1-\kappa_t)\hat{c}_t^{d,u} \\
&\quad + \lambda\kappa_t \dot{\hat{b}}_t - \lambda\dot{\kappa}_t(\hat{c}_t^{d,u} - \hat{b}_t) - \frac{\dot{l}_t}{l_t}\hat{C}_t
\end{aligned} \tag{28}$$

$$\dot{\hat{K}}_t = \hat{Y}_t - \hat{C}_t - (\delta + g + n)\hat{K}_t - \frac{\dot{l}_t}{l_t}\hat{K}_t \tag{29}$$

$$\dot{\hat{k}}_t^d = [(1-\tau_t)r_t - n - g]\,\hat{k}_t^d - (1-\kappa_t)\left(\hat{c}_t^{d,u} - \hat{b}_t\right) - \frac{\dot{l}_t}{l_t}\hat{k}_t^d \tag{30}$$

where the hat indicates that variables are divided by the term $T_t P_t l_t$.[16] The steady state implies that $\dot{l}_t = \dot{\hat{C}}_t = \dot{\hat{K}}_t = \dot{\hat{b}}_t = \dot{\kappa}_t = 0$. Because of this, the steady state capital stock is given by equation (28) and steady state consumption is given by equation (29). Moreover, the steady state value of $\hat{k}_t^d$ is given by equation (26).

The steady state values are labeled with a star. The steady state capital stock is now defined by three equations. First, we need the definition of aggregate capital stock: $\hat{K}_\star = \hat{K}_\star^e + \hat{K}_\star^d$. Second, we use the aggregate Euler equation (28). Third, we have defined $\hat{k}_\star^d = (1-\kappa_\star)\hat{k}_\star^{d,u}$, and $\hat{k}_\star^{d,u}$ is given by equation (26). Moreover, we have used the approximation that $\hat{k}_{v,\star}^e$ is a constant, i.e. $\hat{k}_{v,\star}^e \approx \hat{k}_\star^e$.[17] Now we have three equations which are as follows:

$$\hat{K}_\star = \theta\hat{k}_\star^e + \lambda\hat{k}_\star^d \tag{31}$$

$$\begin{aligned} 0 = &\left[(1-\tau_\star)r_\star - \rho - g + \mu\left(\frac{\hat{c}_\star^e}{\hat{c}_\star^d} - 1\right)\right]\theta\hat{c}_\star^e \\ &+ [(1-\tau_\star)r_\star - \rho - g]\,\lambda(1-\kappa_\star)\hat{c}_\star^{d,u} \end{aligned} \tag{32}$$

$$\begin{aligned} \hat{k}_\star^d = (1-\kappa_\star)&\left\{\frac{\hat{k}_\star^e + \frac{\hat{b}_\star}{(1-\tau_\star)r_\star - n - g}}{\Psi_\star\left[\rho + g - (1-\tau_\star)r_\star\right]}\left[1 - e^{-[\rho+g-(1-\tau_\star)r_\star]\Psi_\star}\right]\right. \\ &\left. - \frac{\hat{b}_\star}{(1-\tau_\star)r_\star - n - g}\right\}. \end{aligned} \tag{33}$$

Now we substitute $\tau_\star$, $r_\star$, $\hat{c}_\star^d$, $\kappa_\star$, $\hat{c}_\star^{d,u}$, $\Psi_\star$ and $\hat{b}_\star$ into equations (32) and (33). Then use equation (22), which gives $\hat{c}_\star^e = \frac{1}{\theta}\left(\hat{C}_\star - \lambda\hat{c}_\star^{d,a}\right)$, where $\hat{C}_\star$ is defined by equation (29) and $\hat{c}_\star^{d,a}$ by equation (21).[18] Then there are three unknowns, $\hat{K}_\star$,

[16] We also need the equation for the growth of labor supply, which can be derived by using equations (5), (11), (13) and (27). More details are given in Appendix C.

[17] Actually, $\hat{k}_{v,\star}^e$ is not exactly constant. Since disabled households are decreasing their asset levels (dis-saving), then employed households must increase their asset levels in order to keep aggregate capital stock constant. Thus, the level of $\hat{k}_{v,\star}^e$ depends on v, i.e. a point of time when the household arrived at the disability state. However, the growth rate of the level of assets (per efficient capita) is very close to zero. Our baseline parameters imply that the growth rate is 0.00004. Thus, we may well approximate that $\hat{k}_{v,\star}^e$ is a constant. This approximation also implied that $\Psi_\star$ was only a function of t, but not a function of v as well.

[18] The rest of the steady state variables are given by the following equations: $r_\star$ is given by (2), $\tau_\star$ is

$\hat{k}^e_\star$ and $\hat{k}^u_\star$, and three equations. Finally, we use equation (31), which implies that $\hat{k}^e_\star = \frac{1}{\theta}\left(\hat{K}_\star - \lambda \hat{k}^u_\star\right)$, and substitute $\hat{k}^e_\star$ into equations (32) and (33). Hence, these two equations jointly determine $\hat{K}_\star$ and $\hat{k}^u_\star$ and we can numerically solve these equations. Figure 1 shows the solution.[19]

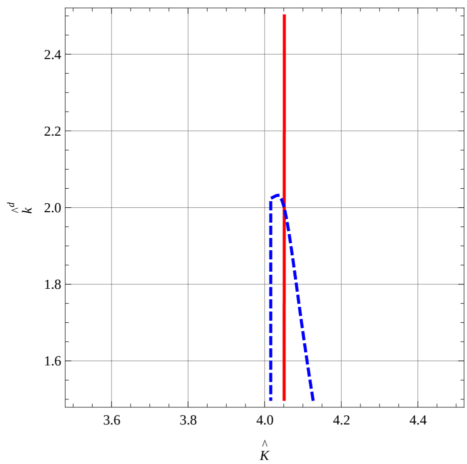

Figure 1: The solution for the steady state in $\hat{K}$-$\hat{k}^d$-space. The solid line shows the values of $\hat{K}$ and $\hat{k}^d$ when the aggregate Euler equation (32) holds. The dashed line shows the values of $\hat{K}$ and $\hat{k}^d$ when equation (33) holds. The intersection of the solid and the dashed line defines the steady state values of $\hat{K}_\star$ and $\hat{k}^d_\star$.

The solid line in Figure 1 gives the solution for aggregate Euler equation (32) with respect to the different values of $\hat{K}$ and $\hat{k}^d$. The line is (almost) vertical, which means that the steady state value of $\hat{K}$ does not depend significantly on $\hat{k}^d$. That is, the value of aggregate capital stock $\hat{K}$ can be solved very precisely even if the value of $\hat{k}^d$ is unknown. Since disabled households are in a minority and they are, on average, poorer than working households, they do not matter significantly in the determination of aggregate variables. However, the intersection of the solid and the dashed line determines the steady state values of aggregate capital stock and the capital holdings of disabled households.

given by (5), $\kappa_\star$ is given by (20), $\hat{c}^{d,u}_\star$ is given by (19), $\hat{b}_\star = \eta(1-\tau_\star)\hat{w}_\star$ and $\Psi_\star$ can be solved using equation (17). Finally, we use equations (1) and (3) to substitute $\hat{w}_\star$ and $\hat{Y}_\star$ into previous definitions. Appendix D gives a detailed derivation of these steady state measures.

[19] The parameter values used in Figure 1 are given in Table 1.

3.5 The stability of the steady state and the reduced system

Let us now define the system of equations (28), (29) and (30) as

$$\dot{\mathbf{x}}_t = F\left(\mathbf{x}_t\right), \tag{34}$$

where $\mathbf{x} = (\hat{C}, \hat{K}, \hat{k}^d)$ denotes coordinates in the phase space and F represent a smooth map that describes the evolution of the dynamic system. The stability of the system can be locally analyzed by taking a first-order Taylor approximation around the steady state when the real part of eigenvalues ν of the Jacobian matrix determine the nature of the steady state. By applying this procedure to equation (34), we can write our differential equations as

$$\dot{\mathbf{x}}_t = \mathbf{A}\left(\mathbf{x}_t - \mathbf{x}_\star\right), \tag{35}$$

where $\mathbf{A} = DF(\mathbf{x}_\star)$ is the Jacobian matrix and the eigenvalues of the Jacobian are $\nu_1 > 0$, $\nu_2 < 0$ and $\nu_3 > 0$. Thus, the Grobman-Hartman Theorem implies that we have a locally saddle path stable steady state. Moreover, the eigenvector associated with the negative eigenvalue corresponds to the stable arm (or manifold) of the linearized system and eigenvectors associated with positive eigenvalues correspond to the unstable arms (or manifolds). The uniques of the steady state can be shown in the usual way by using the tranversality condition.

We can use this system, but we can also reduce its dimension, and still get an illustration for the dynamics of the system. Gomis-Porqueras and Haro (2009) show that we may find *preserved quantities* that help us simplify the study of the evolution of the system, but the computation of preserved quantities is very model dependent. In this model, we can find a phase space which determines the evolution of aggregate variables. The evolution of aggregate variables $\hat{C}$ and $\hat{K}$ does not depend on equation (30), as suggested by Figure 1. Thus, the dynamic of the model is captured by equations (28) and (29), i.e. the dynamics of the model boil down to the dynamics of the Ramsey model. Moreover, this system has one stable and one unstable manifold (or arm) which implies a saddle path stability. Appendix E provides a more formal discussion of the topics covered here.

4 Simulations

We set most of the parameters on the values that are traditionally used in the existing literature, and those values are as follows: $n = 0.01$, $\alpha = \frac{1}{3}$, $\rho = 0.05$, $g = 0.02$, $\delta = 0.05$. Parameters that control the supply of labor are ϕ and γ, where the Frisch elasticity is given by $\frac{1}{\phi}$, and we assume that labor supply is elastic when $\phi = 0.5$. Finally, γ controls the level of labor supply. It is set so that in the baseline Ramsey model where markets are perfect (i.e, there is a perfect insurance against uncertainty), $l_\star = 0.33$ when $\gamma = 5$.

Now we are left with parameters that are essential for the paper: the arrival rate μ and the replacement ratio η. Note that it is almost impossible to calibrate μ from the data. However, we can observe the value of $\lambda = \frac{\mu}{n+\mu}$, and when $n = 0.01$, we get $\mu = \frac{\lambda n}{1-\lambda}$. λ is the fraction of disabled households (or individuals) out of the working-age population. Since our model does not take into account any aspects of strategic behavior, we should find out the number of "genuine" disabled individuals. But, when people behave strategically, this is obviously difficult. Moreover, it must be noticed that the definition of disability is not very exact. Anyhow, we use self-reported disability status as an estimate for λ.[20] Bound and Burkhauser (1999, Table 2) summarize evidence from five different data sets. For working-age males, the percentage of the population with disabilities varies from 8.1% to 11.7%. For women, the corresponding numbers are 7.8% and 11.6%. Given these numbers, we set $\lambda = 0.1$, which seems to be an appropriate number. This then implies that $\mu = 0.0011\ldots$ or that the expected working life is $\frac{1}{\mu} = 900$ years. Hence, permanent disability is a rare event!

The second important parameter was η, which gives the replacement ratio, i.e. how many per cents is the disability insurance benefits of the current after tax labor income. Autor and Duggan (2003, Table 1) report the replacement ratio (including in-kind Medicare benefits) for non-elderly males at various percentiles of the wage distribution and ages for the year 1999. The replacement ratio varies significantly depending on age and earnings: the highest value is 104% and the lowest is 22%. We use a very harsh method to deal with this heterogeneity and just use a mean of that sample which is 50%. That is, we set $\eta = 0.5$, which

[20] These numbers need not reflect the true status of the individuals, but we assume that there is a much weaker reason to misreport the status in an anonymous survey. The same strategy is used by Golosov and Tsyvinski (2006) and Benitez-Silva, Buchinsky, Chan, Cheidvasser, and Rust (2004).

could be thought of as a replacement ratio for a representative agent. Table 1 summarize our baseline parameter values.

Table 1: The baseline parameter values

Parameter	Value	Explanation
n	0.01	Growth rate of population
α	$\frac{1}{3}$	Capital share
ρ	0.05	Subjective time preference
g	0.02	Growth rate of productivity
δ	0.05	Depreciation rate of capital
ϕ	0.5	Frisch elasticity is $\frac{1}{\phi} = 2$
γ	5	Defines the level of labor supply
λ	0.1	Percent of population with disabilities
θ	0.9	Percent of population without disabilities
η	0.5	Replacement ratio

4.1 The results

Now we consider the numerical values at the steady state implied by the calibrated model. Table 2 reports the results.[21] To measure the effects of market imperfection, we also report values from the Ramsey model, where we have only scaled output by Θ. This is the complete market case (CM) and the values are reported in column (1). There, we assume that households distribute consumption equally to everyone, i.e. there is a perfect insurance against permanent disability.[22]

The case of incomplete markets (IM) is a case in which only the government provides social insurance by deciding the replacement ratio. We give the values $\eta = 0$, $\eta = 0.5$ and $\eta = 1$ to capture the cost and benefits of social insurance in this model, and columns (2), (3) and (4) report these values. However, the incentive-compatible constraint is such that $U^e \geq U^d$. Hence, the utility of working households must be higher than the utility of disabled households, and this

[21] Appendix F shows the results when $\lambda = 0.3$. The results are parallel to these results only the magnitudes are much higher due to a degree of higher uncertainty and the higher number of disabled households. Second, we also considered changes in the Frisch elasticity. We set the Frisch elasticity lower and higher than in the baseline case, i.e. we set $\phi = 1$ and $\phi = \frac{1}{3}$. The lower Frisch elasticity implies that the disability insurance program generates a one percentage point smaller costs. The higher Frisch elasticity, in turn, gives 1 percentage point higher costs. Finally, we set $\phi = 2$, which gives a lower bound for welfare costs. Then, replacing the current system with a prefect private one would increase aggregate consumption by 3.5%.

[22] Obviously, this kind of arrangement could be based on altruism or some other mechanism which could provide a perfect insurance against disability, or there are private competitive insurance companies which provide insurance at an actuarially fair price.

implies that the maximum value of $\eta = 0.565$. Hence, the higher values of η are only hypothetical. When $\eta = 0$ there is no social insurance and people must live on their assets in the disability state. In the case of $\eta = 1$, the government provides a perfect insurance when we are back in the Ramsey model, but there is a positive tax rate and $\eta = 0.5$ gives the result from our baseline calibration.

In column (5) the reported values are generated by a model where the tax rate, τ, is not endogenously determined, but it is set instead at the same level as in the case of $\eta = 0.5$. However, here we set $\eta = 0$ when we can separate the sources of distortion caused by the disability insurance program. We can now measure how much the higher tax rate, which is needed to finance the disability insurance program, affects the performance of the economy under the baseline calibration.

By comparing columns (2) and (3), we can measure the burden generated by the social insurance program for the economy. By closing the social insurance program, the government could increase per capita consumption by 2.5%. Moreover, by comparing columns (2) and (5) we see the distortion originating from the increase in the tax rate. Obviously, higher tax rates reduce incentives to save, which lowers the level of capital stock. By comparing columns (3) and (5), we can measure a distortion which results from the reduced precautionary saving motive of households, since in column (5), we have controlled the effects of taxes. The higher level of social insurance reduces the self-insurance (or precautionary saving) motive of working households. Since c_t^u rises, it reduces the value of term $\frac{c_t^e}{c_t^u}$.[23] Thus, we can summarize these comparisons by concluding that 1/3 of the distortion (2.5% lower per capita consumption) is caused by a higher tax rate and 2/3 comes from the change in economic behavior, i.e. from the reduced precautionary saving motive.

Second, the magnitude of precautionary wealth can be calculated by comparing columns (1) and (2). When the markets are perfect, representative working household capital holdings are $k_\star^e = 1.379$. In the case of imperfect markets, when disabled households capital holdings are approximately zero, we get $k_\star^e = \frac{1}{\theta} K_\star = 1.562$. This additional wealth can be seen as a result of the precautionary saving motive. Hence, the precautionary saving motive increases the asset holdings of working households' by 13%. Chandra and Samwick (2009) gives sim-

[23]This result is generally well known. See, for example, Hubbard and Judd (1987), Hubbard, Skinner, and Zeldes (1995) and Engen and Gruber (2001).

Table 2: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1)	(2)	(3)	(4)	(5)
Market setting	CM	IM	IM	IM	IM
Replacement ratio		$\eta = 0$	$\eta = 0.5$	$\eta = 1$	$\eta = 0$
Tax rate		$\tau =$endo.	$\tau =$endo.	$\tau =$endo.	$\tau = 0.041$
Output, $Y_\star$	0.497	0.480	0.461	0.455	0.475
Agg. consumption, $C_\star$	0.386	0.369	0.360	0.359	0.366†
Capital, $K_\star$	1.379	1.406	1.257	1.203	1.356
Tax rate, $\tau_\star$	0	0	0.041	0.079	0.041
After tax int., $(1-\tau_\star)r_\star$	0.070	0.064	0.069	0.070	0.064
After tax wage, $(1-\tau_\star)w_\star$	1.111	1.141	1.056	0.999	1.080
Saving rate, $1-\frac{C_\star}{Y_\star}$	0.222	0.234	0.218	0.212	0.228
Labor supply, $l_\star$	0.3311	0.3118	0.3101	0.3107	0.3121
Cons. for $\epsilon = e$, $c_\star^e$	0.386	0.409	0.379	0.359	0.387
Cons. for $\epsilon = d$, $c_\star^{d,a}$	0.386	0.001	0.191	0.359	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.523	-1.973	-1.556	-1.545	-2.132

† Here aggregate consumption is composed of private and public consumption.

ilar estimates for the magnitude of precautionary savings which are attributable to disability risk. However, we can conclude that social insurance income is important for disabled households, and hence, the precautionary savings do not insure households against the risk of disability. By increasing the level of social insurance, η, we can raise the consumption of disabled households and welfare. This result is consistent with Deaton (1991), who shows that the effectiveness of savings as an insurance mechanism against shocks to labor earnings declines as the persistence of these shocks rises.

Third, by comparing columns (1) and (4) we can see the difference between two perfect insurance systems. The values in column (1) give the steady state values under perfect markets when there is perfect private insurance. Column (4) shows a case where the government provides the perfect insurance against permanent disability and the program is financed by proportional taxes. It is evident that the public system works much worse than the private one. Actually, per capita consumption is about 7% higher when insurance is provided by private competitive insurance companies. Moreover, the same conclusion can be made when the current system (see column (3)) and complete markets are compared against one another. However, this result depend on the Frisch elasticity of labor supply and when it is set at 0.5 the increase in the amount of aggregate consumption would be 3.5% (see Appendix F). In any case, the results imply that optimizing the tax-financed social insurance systems is not the best way to improve welfare, but that completing the markets by removing impediments to the private provision of insurance would generate a much higher increase in welfare.[24] For example, Golosov and Tsyvinski (2006) showed that moving from the current system in the U.S. to an asset-tested disability insurance system increases consumption by about 0.5%. Another interpretation for the result is that the costs generated by problems associated with imperfect information – which prevents market-based solutions – are indeed very large. Further, one can see that this as an estimate (probably an upper limit) for a consumption loss which is generated by government institutions since these institutions crowd out privately provided insurance against permanent disability.

Moreover, imperfect markets cause a significant loss of welfare. If we consider columns (1) and (2), we see that completing the markets would increase output by 3.5% and that aggregate consumption would rise by about 4.5%. But why does market incompleteness decrease welfare? When there is no social insurance, workers do not have to share their consumption with disabled households, which increases their level of consumption by 6%. This "wealth effect" reduces their labor supply by almost 6%, which then reduces aggregate output, even if the level of capital increases in the economy. Thus, completing markets would reduce workers' consumption, but it would increase aggregate consumption and output. As emphasized by Marcet, Obiols-Homs, and Weil (2007), incomplete markets increase the level of capital stock and then lead to higher output (the so-called Aiygari-Huggett effect) *if labor supply is inelastic.* But, when labor supply is

[24]The similar policy recommendation was also expressed by Atkeson, Chari, and Kehoe (1999) when they examined capital taxation.

elastic, the ex-post wealth effect dominates the Aiygari-Huggett effect. Thus, economies with less developed financial markets also have a lower level of output and welfare.

Finally, we can conclude by noting a surprising result: higher tax rates, which are levied to finance social insurance programs, do not cause any distortion to the supply of labor. Labor supply is constant when tax rates get higher, even if labor supply is very elastic (see, Table 1). A higher level of social insurance reduces the precautionary saving motive of working households, which lowers their consumption, since capital income is now at a lower level. Hence, the supply of labor is constant even though the after-tax wage has decreased. The decreased level of the after tax-wage rate and consumption balance each other out, which implies that the supply of labor does not depend on the level of social insurance. Hence, the social insurance program causes the distortion to the economy only by reducing the level of capital stock.

5 Conclusions

The model in this paper extends the Ramsey model by using a precautionary saving motive. Households face the constant probability of permanently losing their jobs throughout their lives and this uncertainty is only partially insured by a social insurance policy that is provided by the government. Thus, the model captures the uncertainty associated with the rare and permanent income losses of households. In this paper, we focused on the disability insurance in the United States.

With the calibrated model, we can measure the costs of social insurance that come from two sources: the extent to which the social insurance changes individuals' behavior and the extent to which the taxes that are levied to finance this program. The disability insurance program lowers per capita consumption by 2.5%. One-third of this burden is caused by higher taxes and 2/3 comes from the change in economic behavior. That is, the social insurance removes households' precautionary saving motive, which lowers the level of capital stock, and the higher tax rate further lowers the level of capital stock. However, the supply of labor does not significantly depend on the level of disability insurance, even if we take into account the effects of higher taxes that are needed to finance the disability insurance program. That is, the extent to which the disability insurance program distorts the economy mainly comes from the lower level of capital stock.

Finally, it is easy to conclude that the leading way to improve welfare is to figure out how impediments that prevent the private provision of insurance against disability can be removed. The private provision of perfect insurance against permanent disability would lead to (depending on the Frisch elasticity of labor supply) 3.5-7% higher per capita consumption than the current disability insurance program. Another interpretation for the result is that the costs generated by problems associated with imperfect information – which prevents market-based solutions – are indeed very large. Yet another alternative interpretation for the result is that the government institutions (i.e. social security) crowd out private solutions at insurance markets. In the past, when financial markets were underdeveloped, government social security institutions provided insurance. However, nowdays financial markets are developed and there could be privately provided insurance, but the government institutions crowd out these solutions. Given the estimates in this paper, the crowding out may result in a large welfare loss.

Appendices for Part IV

A Solving the problem of representative household without liquidity constraint

In this section we ignore the effects of liquidity constraint, hence the problem of representative household is given by equations (6), (7) and (9). The problem can be solved by using the standard tools of optimal control theory. Good sources of solution methods for these types of problems are given by Dixit and Pindyck (1994, chap. 3 and 4), Bertsekas (2005, chap. 3) and Acemoglu (2009, chap. 6).

First, we suppose that every period of time is length of Δt, when we can write Bellman equation for the problem by

$$V(a_t, t, \epsilon) = \max_{c_t, l_t} \left\{ U(c_t, l_t)\Delta t + (1 + (\rho - n)\Delta t)^{-1} E_t V(a_{t+\Delta t}, t + \Delta t, \epsilon_{t+\Delta t}) \right\} \tag{A1}$$

Assume that V is continuous and differentiable when we can write

$$\begin{aligned} E_t V(a_{t+\Delta t}, t + \Delta t, \epsilon_{t+\Delta t}) &= \frac{\partial V(a_t, t, \epsilon)}{\partial a_t} \dot{a}_t \Delta t + \frac{\partial V(a_t, t, \epsilon)}{\partial t} \Delta t \\ &\quad + E_t \left[V(a_t, t, \epsilon_t) \Delta t \right]. \end{aligned} \tag{A2}$$

Remeber that uncertainty associated to this problem is given by a Poisson process which implies that

$$E_t \left[V(a_t, t, \epsilon_t) \Delta t \right] = \begin{cases} (1 - \mu \Delta t) V^e(a_t, t) + \mu \Delta t V^d(a_t, t), & \text{if } \epsilon = e \\ V^d(a_t, t), & \text{if } \epsilon = d \end{cases} \tag{A3}$$

Now we can write the Bellman equation for both states by substituting equations (A2) and (A3) into (A1). This gives

$$\begin{aligned} (1 + (\rho - n)\Delta t) V^e(a_t, t) &= \max_{c_t^e, l_t} \left\{ U(c_t^e, l_t)(1 + (\rho - n)\Delta t)\Delta t + \frac{\partial V^e(a_t, t)}{\partial a_t} \dot{a}_t^e \Delta t \right\} \\ &\quad + \frac{\partial V^e(a_t, t)}{\partial t} \Delta t \\ &\quad + (1 - \mu \Delta t) V^e(a_t, t) + \mu \Delta t V^d(a_t, t) \end{aligned} \tag{A4}$$

$$(1+(\rho-n)\Delta t)V^d(a_t,t) = \max_{c_t^d}\left\{U(c_t^d,0)(1+(\rho-n)\Delta t)\Delta t + \frac{\partial V^d(a_t,t)}{\partial a_t}\dot{a}_t^d\Delta t\right\} + \frac{\partial V^d(a_t,t)}{\partial t}\Delta t + V^d(a_t,t), \quad \text{(A5)}$$

where $\dot{a}_t^e = [(1-\tau_t)r_t - n - g]\, a_t - c_t^e + (1-\tau_t)w_t l_t$ and $\dot{a}_t^d = [(1-\tau)r_t - n - g]\, a_t - c_t^d + b_t$.

Simplify and divide both equations by Δt. After this let $\Delta t \to 0$. Then we get the standard Hamilton-Jacobi-Bellman equations for both states:

$$0 = \max_{c_t^e, l_t}\left\{U(c_t^e, l_t) + \frac{\partial V^e(a_t,t)}{\partial a_t}\dot{a}_t^e\right\} + \frac{\partial V^e(a_t,t)}{\partial t} + \mu\left[V^d(a_t,t) - V^e(a_t,t)\right] - (\rho-n)V^e(a_t,t) \quad \text{(A6)}$$

$$0 = \max_{c_t^d}\left\{U(c_t^d) + \frac{\partial V^d(a_t,t)}{\partial a_t}\dot{a}_t^d\right\} + \frac{\partial V^d(a_t,t)}{\partial t} - (\rho-n)V^d(a_t,t). \quad \text{(A7)}$$

Note that equation (A7) does not depend on equation (A6) which is the key to the tractable solution of this problem. This feature is generated by the assumption that the transition between the states occur only once and the only source of uncertainty is the timing of that transition. Moreover, the term $\mu\left[V^d(a_t,t) - V^e(a_t,t)\right]$ in equation (A7) gives the expected reduction in the value function due to income loss from permanent disability.

The maximization of equations (A6) and (A7) yields the envelopment relations. For the employed state we get

$$U'_{c_t^e} = -\frac{\partial V^e(a_t,t)}{\partial a_t} \quad \text{(A8)}$$

$$-U'_{l_t} = (1-\tau_t)w_t\frac{\partial V^e(a_t,t)}{\partial a_t}, \quad \text{(A9)}$$

and for the disability state we get

$$U'_{c_t^d} = -\frac{\partial V^d(a_t,t)}{\partial a_t} \quad \text{(A10)}$$

where $U_x^n = \frac{\partial^n U}{\partial x^n}$. Differentiating envelopment conditions, which are associated with consumption, respect to time and assuming that $U''_{c_t^e,l_t} = 0$ (see equation (10)) we get:

$$U''_{c_t^e}\dot{c}_t^e = -\frac{\partial^2 V^e(a_t,t)}{\partial a_t^2}\dot{a}_t^e \tag{A11}$$

$$U''_{c_t^d}\dot{c}_t^d = -\frac{\partial^2 V^d(a_t,t)}{\partial a_t^2}\dot{a}_t^d. \tag{A12}$$

Given sufficient time the value function converges its stationary form implying $\frac{\partial V(a_t,t)}{\partial t} = 0$. Differentiating equations (A6) and (A7) respect to a_t yields

$$\begin{aligned} 0 &= \frac{\partial^2 V^e(a_t,t)}{\partial a_t^2}\dot{a}_t^e + \frac{\partial V^e(a_t,t)}{\partial a_t}((1-\tau_t)r_t - n - g) - (\rho - n)\frac{\partial V^e(a_t,t)}{\partial a_t} \\ &\quad +\mu\left(\frac{\partial V^d(a_t,t)}{\partial a_t} - \frac{\partial V^e(a_t,t)}{\partial a_t}\right) \end{aligned} \tag{A13}$$

$$\begin{aligned} 0 &= \frac{\partial^2 V^d(a_t,t)}{\partial a_t^2}\dot{a}_t^d + \frac{\partial V^d(a_t,t)}{\partial a_t}((1-\tau_t)r_t - n - g) \\ &\quad -(\rho - n)\frac{\partial V^d(a_t,t)}{\partial a_t}. \end{aligned} \tag{A14}$$

Substitute equations (A8), (A10), (A11) and (A12) into equations (A13) and (A14). This yields Euler equation for both states:

$$\frac{\dot{c}_t^e}{c_t^e} = -\frac{U'_{c_t^e}}{c_t^e U''_{c_t^e}}\left[(1-\tau_t)r_t - \rho - g + \mu\left(\frac{U'_{c_t^d}}{U'_{c_t^e}} - 1\right)\right] \tag{A15}$$

$$\frac{\dot{c}_t^d}{c_t^d} = -\frac{U'_{c_t^d}}{c_t^d U''_{c_t^d}}\left[(1-\tau_t)r_t - \rho - g\right] \tag{A16}$$

Finally, we substitute equation (A8) into equation (A9) which gives

$$\frac{U'_{l_t}}{U'_{c_t^e}} = (1-\tau_t)w_t. \tag{A17}$$

Using the utility function $U(c_t, l_t) = \log c_t - \gamma\frac{l_t^{1+\phi}}{1+\phi}$ equations (A15), (A16) and (A17) imply equations (11), (12) and (13) in the text.

B The derivation of asset evaluation in the disabled state

Start by substituting equation (15) and (16) into equation (14). This gives

$$\dot{a}^d_{t,v} = [(1-\tau_t)r_t - n - g]\, a^d_t - (\rho - n)\left[a^e_v + \tilde{b}_v\right] e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)} + b_t$$

Bring the term $[(1-\tau_t)r_t - n - g]\, a^d_t$ on the left hand side of the equation and multiply both sides by term $e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)}$. Use Leibniz's rule and take an integral over t which then gives

$$\begin{aligned}\int \frac{d}{dt} a^d_{t,v} e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} dt &= -\int (\rho - n)\left[a^e_v + \tilde{b}_v\right] e^{-(\rho-n)(t-v)} dt \\ &\quad + \int b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} dt. \qquad \text{(A18)}\end{aligned}$$

Do the integration and multiply both sides with $e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)}$ which gives then equation

$$\begin{aligned}a^d_{t,v} &= \left[a^e_v + \tilde{b}_v\right] e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)} \\ &\quad + e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} \int b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} dt. \qquad \text{(A19)}\end{aligned}$$

Now the term $e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} \int b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_t-n-g](t-v)} dt$ is unknown but it can be shown that it equals to $-\tilde{b}_v$. Hence, it gives the present value of social insurance transfers. This can be verified by assuming $t = v$ for terms $a^d_{t,v}$ and $\left[a^e_v + \tilde{b}_v\right] e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-\rho-g](t-v)}$ when these terms can be rewritten as a^e_v and $\left[a^e_v + \tilde{b}_v\right]$. Since we are dealing with a budget constraint, our unknown term $e^{[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} \int b_t e^{-[(1-\bar{\tau}_{t,v})\bar{r}_{t,v}-n-g](t-v)} dt = -\tilde{b}_v$, otherwise the budget constraint would not hold. Since budget constraint must hold for every t, it implies that previous statement must hold for every t as well. When this substitution is done, we get equation (17) in the text.

C The growth rate of labor supply

Equation (13) defined the optimal labor supply, which now can be rewritten as

$$l_t = \left(\frac{(1-\tau_t)\hat{w}_t}{\gamma \hat{c}_t^e} \right)^{\frac{1}{1+\phi}}, \tag{A20}$$

where $\hat{w}_t = (1-\alpha)\theta^{-\alpha}\hat{K}_t^{\alpha}$. This equation implies that the growth rate of l is given by

$$\frac{\dot{l}_t}{l_t} = \frac{1}{1+\phi}\left[\alpha \frac{\dot{\hat{K}}_t}{\hat{K}_t} - \frac{\dot{\hat{\tau}}_t}{1-\hat{\tau}_t} - \frac{\dot{\hat{c}}_t^e}{\hat{c}_t^e}\right]. \tag{A21}$$

We can derive the term $\frac{\dot{\hat{\tau}}_t}{1-\hat{\tau}_t}$ from equation (5), which gives

$$\frac{\dot{\hat{\tau}}_t}{1-\hat{\tau}_t} = \hat{\Omega}_t \frac{\dot{\hat{K}}_t}{\hat{K}_t} \quad \text{, where} \tag{A22}$$

$$\hat{\Omega}_t = \frac{\lambda\eta\delta\theta^{\alpha}\hat{K}_t^{1-\alpha}}{\left(\lambda\eta + \theta + \frac{\alpha\theta}{1-\alpha} - \frac{\delta\theta^{\alpha}}{1-\alpha}\hat{K}_t^{1-\alpha}\right)\left(\theta + \frac{\alpha\theta}{1-\alpha} - \frac{\delta\theta^{\alpha}}{1-\alpha}\hat{K}_t^{1-\alpha}\right)}. \tag{A23}$$

Now rewrite equation (A21)

$$\frac{\dot{l}_t}{l_t} = \frac{1}{1+\phi}\left[\frac{\dot{\hat{K}}_t}{\hat{K}_t}\left(\alpha - \hat{\Omega}_t\right) - \frac{\dot{\hat{c}}_t^e}{\hat{c}_t^e}\right]. \tag{A24}$$

The term $\frac{\dot{\hat{K}}_t}{\hat{K}_t}$ can be defined by using equation (27) and $\frac{\dot{\hat{c}}_t^e}{\hat{c}_t^e}$ can be derived by using equation (11). Then the equation (A24) can be rewritten as

$$\begin{aligned}\frac{\dot{l}_t}{l_t} &= \frac{\alpha - \hat{\Omega}_t}{\phi + \alpha - \hat{\Omega}_t}\left[\frac{\hat{Y}_t}{\hat{K}_t} - \frac{\hat{C}_t}{\hat{K}_t} - (\delta + g + n)\right] \\ &\quad - \frac{1}{\phi + \alpha - \hat{\Omega}_t}\left[(1-\tau_t)r_t - \rho - g + \mu\left(\frac{\hat{c}_t^e}{\hat{c}_t^d} - 1\right)\right],\end{aligned} \tag{A25}$$

which now gives the growth rate of labor supply. When equation (A25) is plugged in equations (28), (29) and (30) we have defined our system.

D A detailed derivation of the steady state measures

In the steady state $\dot{K}_t = \dot{l}_t = 0$, hence $K_t = K_\star$ and $l_t = l_\star$ are constants, which implies that $r_\star$, $w_\star$, $\tau_\star$ are constants. This can easily be verified from equations (2), (3) and (5). Let us continue by defining the steady state values for the disability state.

The value of $\tilde{b}_t$ in the steady state is given by

$$\tilde{b}_\star = \int_v^\infty b_\star e^{-\int_v^t (1-\tau_\star)r_\star - g - n ds} dt = \frac{b_\star}{(1-\tau_\star)r_\star - g - n},$$

where $b_\star = \eta(1-\tau_t)w_t l_t = \eta(1-\tau_\star)w_\star l_\star$, which is constant at the steady state. Next we focus on equation (17) or equation (A19) and substitute $\tilde{b}_t = \tilde{b}_\star$ into equation which yields

$$a^d_{v,\star} = \left[a^e_{v,\star} + \frac{b_\star}{(1-\tau_\star)r_\star - g - n}\right] e^{\int_v^t (1-\tau_\star)r_\star - \rho - g ds} - \frac{b_\star}{(1-\tau_\star)r_\star - g - n}.$$

Moreover, the liquidity constraint was binding at time T for a household who was disabled at time v, when we get in the steady state

$$a^d_{T,v} = \left[a^e_{v,\star} + \frac{b_\star}{(1-\tau_\star)r_\star - g - n}\right] e^{\int_v^T (1-\tau_\star)r_\star - \rho - g ds} - \frac{b_\star}{(1-\tau_\star)r_\star - g - n}.$$

Since the liquidity constraint ($a^d_{t,v} \geq 0$) is binding when $t = T$, it implies that $a^d_{T,v} = 0$. Thus, we can easily solve T:

$$\begin{aligned} T &= \Psi_\star + v \quad \text{where} \\ \Psi_\star &\equiv \frac{\log\left[\frac{\frac{b_\star}{(1-\tau_\star)r_\star - n - g}}{a^e_\star + \frac{b_\star}{(1-\tau_\star)r_\star - n - g}}\right]}{(1-\tau_\star)r_\star - \rho - g}. \end{aligned} \tag{A26}$$

Note that we have now assumed that in the steady state $a^e_{v,\star} = a^e_\star$, i.e. $\dot{c}^e_t = \dot{a}^e_t = 0$ in the steady state. This holds only approximately and actually our baseline parameter values imply that in the steady state $\dot{a}^e_t = 0.00004 \approx 0$. Hence, this approximate is quite correct. When λ get larger values this approximation is not so good anymore. However, without this approximation the model is significantly more complicated to solve. Finally,

$$c^d_\star = (\rho - n)\left[a^e_\star + \frac{b_\star}{(1-\tau_\star)r_\star - g - n}\right] \tag{A27}$$

is given by equation (16).

Now we can give the steady state values of our aggregate measures. Let us start from equation (20) which gives the steady state value for $\kappa_\star = e^{-n\Psi_\star}$ where $\Psi_\star$ is defined by (A26). The consumption of disabled but unconstrained households for every t was given by equation (19), which in the steady state gives

$$\begin{aligned} c^{d,u}_\star &= \frac{1}{\Psi_\star}\int_{t-\Psi_\star}^{t} c^d_\star e^{\int_v^t (1-\tau_\star)r_\star - g - \rho ds} dv \\ &= \frac{c^d_\star}{\Psi_\star\left[\rho + g - (1-\tau_\star)r_\star\right]}\left(1 - e^{-[\rho+g-(1-\tau_\star)r_\star]\Psi_\star}\right), \end{aligned} \tag{A28}$$

where equations (A26) and (A27) defines $\Psi_\star$ and $c^d_\star$. The capital holdings of disabled households are given by equation (26) which yields

$$\begin{aligned} k^{d,u}_\star &= \frac{1}{\Psi_\star}\int_{t-\Psi_\star}^{t}\left[k^e_\star + \frac{b_\star}{(1-\tau_\star)r_\star - n - g}\right] e^{\int_v^t (1-\tau_\star)r_\star - g - \rho ds} - \frac{b_\star}{(1-\tau_\star)r_\star - n - g} dv \\ &= \frac{k^e_\star + \frac{b_\star}{(1-\tau_\star)r_\star - n - g}}{\Psi_\star\left[\rho + g - (1-\tau_\star)r_\star\right]}\left(1 - e^{-[\rho+g-(1-\tau_\star)r_\star]\Psi_\star}\right) - \frac{b_\star}{(1-\tau_\star)r_\star - n - g}. \end{aligned}$$

Finally, we can define $l_\star$ from equation (13) or (A20) which gives the steady state value of the supply of labor as follows

$$l_\star = \left(\frac{(1-\tau_\star)w_\star}{\gamma c^e_\star}\right)^{\frac{1}{1+\phi}},$$

where equation (22) defines $c^e_\star = \frac{1}{\theta}\left(C_\star - \lambda c^{d,a}_\star\right)$. We can get the value of $C_\star = Y_\star - (n+g+\delta)K_\star$, from equation (29), where $Y_\star$ is defined by equation (1). Moreover, $c^{d,a}_\star$ is given by (21) which gives

$$c^{d,a}_\star = (1-\kappa_\star)c^{d,u}_\star + \kappa_\star b_\star,$$

where $c^{d,u}_\star$ is defined by equation (A28).

E The preserved quantities and the reduced system

In this model, we can show that given the baseline parameter values the stable and unstable manifolds (or arms), which are associated with aggregate variables, are restricted to the level set spanned by a vector $(\hat{C}, \hat{K})$. Thus, our system has 2-dimensional invariant and unique manifolds which are the level sets of the preserved quantity and this subset of phase space defines the flow of our aggregate variables. Hence, we can ignore the effects of $\hat{k}^d$ when we are interested in aggregate behavior of the economy.[25]

Let us consider a general case where a dynamic system is given $\dot{\mathbf{x}}_{\mathbf{t}} = F(\mathbf{x}_t)$ where the phase space is $\mathcal{B} \subset \mathbb{R}^n$ and $F : \mathcal{B} \subset \mathbb{R}^n \to \mathbb{R}^n$. To find preserved quantities, we can consider manifolds that are defined implicitly by a set of equations. To define these manifolds, given a smooth map $H : \mathcal{B} \subset \mathbb{R}^n \to \mathbb{R}^{n-m}$, and given by $H_0 \in \mathbb{R}^{n-m}$, the level set

$$\Omega_{H_0} = \{\mathbf{x} \in \mathcal{B} \,|\, H(\mathbf{x}) = H_0\}$$

is an $n-m$-dimensional manifold if for all each point $\mathbf{x} \in \Omega_{H_0}$ the rank of the differential $DH(\mathbf{x})$ is $n-m$. Then, note that the level set Ω_{H_0} is invariant under the dynamic system (34), i.e. $\dot{\mathbf{x}}_{\mathbf{t}} = F(\mathbf{x}_t)$, if and only if

$$H(\mathbf{x}) = H_0 \Rightarrow H(F(\mathbf{x})) = H_0.$$

Hence, the map H is a preserved quantity for the dynamic system $\dot{\mathbf{x}}_{\mathbf{t}} = F(\mathbf{x}_t)$ if and only if for all $\mathbf{x} \in \mathcal{B}, H(F(\mathbf{x})) = H(\mathbf{x})$. Then, given an initial state $\mathbf{x_0}$ of the dynamic system, its motion $\mathbf{x}_t = F^t(\mathbf{x}_0)$ can be restricted to the level set of $H_0 = H(\mathbf{x}_0)$. Moreover, the existence of preserved quantity restricts the level set where invariant manifolds can exist, which is a result of following proposition.

Proposition A1. *(Gomis-Porqueras and Haro (2009, Proposition 2)). Let the preserved quantity $H : \mathcal{B} \to \mathbb{R}^{n-m}$ be a smooth map, preserved by a dynamic system $\dot{\mathbf{x}}_{\mathbf{t}} = F(\mathbf{x}_t)$, where $F : \mathcal{B} \subset \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{x}_\star \in \mathcal{B}$ be a steady state of $\dot{\mathbf{x}}_{\mathbf{t}} = F(\mathbf{x}_t)$. Then, for all $\mathbf{x} \in \mathcal{B}$ s.t. $\lim_{t\to+\infty} F^t(\mathbf{x}) = \mathbf{x}_\star$ or $\lim_{t\to-\infty} F^t(\mathbf{x}) = \mathbf{x}_\star$, then $\mathbf{x} \in \Omega_{H_\star}$ where $H_\star = H(\mathbf{x}_\star)$.*

[25] The following discussion is heavily based on Gomis-Porqueras and Haro (2009). Generally, a good source for the analysis of dynamic systems is Guckenheimer and Holmes (1983).

Proof. (Following Gomis-Porqueras and Haro (2009)). Proof is based on a simply continuity argument. Assume that the preserved quantity $H : \mathcal{B} \to \mathbb{R}^{n-m}$ is a continuous function. Moreover, assume that $\mathbf{x} \in \mathcal{B}$ is such its orbits $\mathbf{x}_t = F^t(\mathbf{x})$ converges to the steady state $\mathbf{x}_\star \in \mathcal{B}$ in the future. That is, assume that $\lim_{t\to+\infty} F(\mathbf{x}_t) = \mathbf{x}_\star$. Since $H_0 = H(\mathbf{x}) = H(\mathbf{x}_t) \quad \forall t$, then $H_0 = \lim_{t\to+\infty} H(\mathbf{x}_t) = H(\mathbf{x}_\star)$. The study of $\lim_{t\to-\infty} F^t(\mathbf{x}) = \mathbf{x}_\star$ is analogous. $\square$

The result is that all the points in set H_0, that convergence in the steady state, belong the same level surface of such a steady state.

Let us now turn back to our model. Since linearized system (35) has distinct real eigenvalues ν_1, ν_2 and ν_3, we have the unique solution to (35). Moreover, we can find a 2-dimensonal preserved quantity given by H for the flow of linearized $\dot{\mathbf{x}}_\mathbf{t} = F(\mathbf{x}_t)$ and with H we can describe the flow of $\hat{K}$,$\hat{C}$. The eigenvectors associated with eigenvalues $\nu_{\hat{C}}$ and $\nu_{\hat{K}}$ span the stable E^s and unstable E^u subspaces and in this space there is a subspace $\Omega_{H_0} \in \mathcal{B}$. In this subspace the stable and unstable manifolds (W^s and W^u) exist since the eigenvector associated with $\nu_{\hat{k}^d}$ span only one dimensional space. That is, it can be shown that the eigenvalue associated with $\hat{k}^d$ do not matter for the dynamics of $\hat{K}$ and $\hat{C}$. Hence, as stated by proposition A1, we have a preserved quantity H and this describes the flow of $\hat{K}$,$\hat{C}$.

To demonstrate previous discussion we consider a numerical example generated by our baseline parameter values which are $n = 0.01$, $\alpha = \frac{1}{3}$, $\rho = 0.05$, $g = 0.02$, $\delta = 0.05$, $\lambda = 0.1$, $\theta = 0.9$, $\eta = 0.5$, $\gamma = 5$, $\phi = 0.5$. Now we may write the solution of the linearized system as follows

$$\begin{aligned}
\mathbf{x}_t &= \sum_{j=1}^{3} c_j e^{\nu_j t} \mathbf{v}_{\nu_j} + \mathbf{x}_0 \\
\mathbf{x}_t &= c_1 e^{-0.13t} \mathbf{v}_{\nu_1} + c_2 e^{0.34t} \mathbf{v}_{\nu_2} + c_3 \mathbf{v}_{\nu_3} e^{0.04t} + \mathbf{x}_0, \quad \text{where}
\end{aligned}$$

$$\mathbf{v}_{\nu_1} = \begin{bmatrix} -0.80 \\ -0.08 \\ -0.60 \end{bmatrix}, \quad \mathbf{v}_{\nu_2} = \begin{bmatrix} 0.64 \\ -0.45 \\ -0.62 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_{\nu_3} = \begin{bmatrix} -0.0001 \\ 0.0001 \\ 1 \end{bmatrix}$$

Thus, when we use approximation $-0.0001 \approx 0.0001 \approx 0$, we can write

$$\begin{aligned}\hat{K}_t &= -0.80e^{-0.13t}c_1 + 0.64e^{0.34t}c_2 + \hat{K}_0 \\ \hat{C}_t &= -0.08e^{-0.13t}c_1 - 0.45e^{0.34t}c_2 + \hat{C}_0 \\ \hat{k}_t^d &= -0.60e^{-0.13t}c_1 - 0.62e^{0.34t}c_2 + e^{0.04t}c_3 + \hat{k}_0^d,\end{aligned}$$

where c_1, c_2 and c_3 must be solved by using the transversality conditions and $\mathbf{x}_0$. Hence, when we are only interested in the evolution of aggregate variables, i.e. $\mathbf{x}^r = (\hat{K},\hat{C})$, we can reduce the dimension of our initial system. Thus, we can give our reduced system as

$$\dot{\mathbf{x}}_t^r = H\left(F\left(\mathbf{x}_t\right)\right) = H(\mathbf{x}_t).$$

Thus, when we have found our steady state, we may analyze the dynamics of the aggregate variables only by focusing on 2-dimensional system which determines the dynamics of aggregate consumption and capital stock. This is the reduced system. In other words, our reduced system corresponds equations (28) and (29). Moreover, when we linearize our reduced system $H(\mathbf{x}_t)$, given $\hat{k}_\star^d$, the eigenvalues (ν_1 and ν_2) are the same as in the case of original system and eigenvectors are linear combinations of $\mathbf{v}_{\nu_1}$ and $\mathbf{v}_{\nu_2}$.

Finally note that, when λ get unfeasible large values like $\lambda = 0.3$ we cannot reduce the dimension of our system. That is obvious, since then the behavior of disabled households also matters for the behavior of the aggregate variables, due to the fact that a large share of population in the economy are disabled when they matter also for the behavior of the aggregate variables.

F The robustness of the results

We only change the value of $\lambda = 0.3$ when $\mu = 0.0043$ or $\frac{1}{\mu} = 233$ and calibrate $\gamma = 6.45$, when under complete markets $l_\star = 0.33$. Table A1 report the results.

Second, we consider *lower* Frisch elasticity and we set $\phi = 2$. But, we still keep $l_\star = 0.33$, which then implies that $\gamma = 26$. Results are given by Table A2.

Third, we consider a Frisch elasticity equal to 1 when $\phi = 1$. But, we still keep $l_\star = 0.33$, which then implies that $\gamma = 8.5$. Results are given by Table A3.

Table A1: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1) CM	(2) IM $\eta = 0$ τ =endo.	(3) IM $\eta = 0.5$ τ =endo.	(4) IM $\eta = 1$ τ =endo.	(5) IM $\eta = 0$ $\tau = 0.141$
Output, $Y_\star$	0.385	0.345	0.298	0.285	0.339
Agg. consumption, $C_\star$	0.300	0.253	0.236	0.231	0.258†
Capital, $K_\star$	1.070	1.144	0.772	0.665	1.014
Tax rate, $\tau_\star$	0	0	0.141	0.244	0.141
After tax int., $(1-\tau_\star)r_\star$	0.070	0.050	0.068	0.070	0.053
After tax wage, $(1-\tau_\star)w_\star$	1.11	1.214	0.922	0.770	0.990
Saving rate, $1 - \frac{C_\star}{Y_\star}$	0.222	0.265	0.207	0.187	0.239
Labor supply, $l_\star$	0.3303	0.2706	0.2645	0.2660	0.280
Cons. for $\epsilon = e$, $c_\star^e$	0.300	0.362	0.278	0.231	0.290
Cons. for $\epsilon = d$,$c_\star^{d,a}$	0.300	0.001	0.139	0.231	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.776	-3.205	-1.897	-1.876	-3.585

† Here aggregate consumption is defined as private+public consumption.

Fourth, we consider a more *higher* Frisch elasticity and we set $\phi = \frac{1}{3}$. But, we still keep $l_\star = 0.33$, which then implies that $\gamma = 4.15$. Results are given by Table A4.

Table A2: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1) CM	(2) IM $\eta = 0$ τ =endo.	(3) IM $\eta = 0.5$ τ =endo.	(4) IM $\eta = 1$ τ =endo.	(5) IM $\eta = 0$ $\tau = 0.141$
Output, $Y_\star$	0.498	0.496	0.478	0.471	0.490
Agg. consumption, $C_\star$	0.387	0.380	0.374	0.371	0.378†
Capital, $K_\star$	1.384	1.454	1.303	1.246	1.401
Tax rate, $\tau_\star$	0	0	0.041	0.079	0.041
After tax int., $(1-\tau_\star)r_\star$	0.070	0.064	0.069	0.070	0.064
After tax wage, $(1-\tau_\star)w_\star$	1.111	1.141	1.056	0.999	1.080
Saving rate, $1-\frac{C_\star}{Y_\star}$	0.222	0.234	0.218	0.212	0.228
Labor supply, $l_\star$	0.3321	0.3223	0.3214	0.3217	0.3225
Cons. for $\epsilon = e$, $c_\star^e$	0.387	0.422	0.393	0.371	0.400
Cons. for $\epsilon = d$,$c_\star^{d,a}$	0.387	0.001	0.198	0.371	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.234	-1.721	-1.261	-1.251	-1.838

† Here aggregate consumption is defined as private+public consumption.

Table A3: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1) CM	(2) IM $\eta = 0$ τ =endo.	(3) IM $\eta = 0.5$ τ =endo.	(4) IM $\eta = 1$ τ =endo.	(5) IM $\eta = 0$ $\tau = 0.141$
Output, $Y_\star$	0.502	0.493	0.473	0.467	0.487
Agg. consumption, $C_\star$	0.391	0.377	0.370	0.368	0.376†
Capital, $K_\star$	1.395	1.444	1.292	1.236	1.391
Tax rate, $\tau_\star$	0	0	0.041	0.079	0.041
After tax int., $(1-\tau_\star)r_\star$	0.070	0.064	0.069	0.070	0.064
After tax wage, $(1-\tau_\star)w_\star$	1.111	1.141	1.056	0.999	1.080
Saving rate, $1-\frac{C_\star}{Y_\star}$	0.222	0.234	0.218	0.212	0.228
Labor supply, $l_\star$	0.335	0.320	0.319	0.319	0.320
Cons. for $\epsilon = e$, $c^e_\star$	0.391	0.419	0.390	0.368	0.397
Cons. for $\epsilon = d$,$c^{d,a}_\star$	0.391	0.001	0.226	0.368	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.369	-1.850	-1.399	-1.388	-1.975

† Here aggregate consumption is defined as private+public consumption.

Table A4: The steady state allocations in the complete markets setting (CM) and under incomplete markets (IM).

	(1) CM	(2) IM $\eta = 0$ τ =endo.	(3) IM $\eta = 0.5$ τ =endo.	(4) IM $\eta = 1$ τ =endo.	(5) IM $\eta = 0$ $\tau = 0.141$
Output, $Y_\star$	0.497	0.477	0.458	0.451	0.472
Agg. consumption, $C_\star$	0.387	0.366	0.358	0.356	0.364†
Capital, $K_\star$	1.382	1.398	1.249	1.196	1.348
Tax rate, $\tau_\star$	0	0	0.041	0.079	0.041
After tax int., $(1-\tau_\star)r_\star$	0.070	0.064	0.069	0.07	0.064
After tax wage, $(1-\tau_\star)w_\star$	1.111	1.141	1.056	0.999	1.080
Saving rate, $1-\frac{C_\star}{Y_\star}$	0.222	0.234	0.218	0.212	0.228
Labor supply, $l_\star$	0.3316	0.3100	0.3081	0.3087	0.3103
Cons. for $\epsilon = e$, $c_\star^e$	0.387	0.406	0.377	0.356	0.385
Cons. for $\epsilon = d$,$c_\star^{d,a}$	0.387	0.001	0.190	0.356	0.001
Utility $U = \theta U^e + \lambda U^d$	-1.593	-2.097	-1.628	-1.617	-2.203

† Here aggregate consumption is defined as private+public consumption.

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