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To my family

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ABSTRACT

Functions are a fundamental topic in school mathematics. Understanding the concept of functions requires students to understand various representations of functions and their interconnections. Finding relationships among different representations can be challenging because it often necessitates viewing multiple forms simultaneously. In traditional learning, where graphs are drawn by hand, plotting various graphs is time-consuming and may involve additional difficulties in calculation and drawing. In this context, using a computer program offers distinct advantages: it enables the rapid, simultaneous display of a function's graph and equation, allowing students to immediately observe how changes in one representation affect another. Nevertheless, it is also important for students to learn how to draw graphs by hand to understand how a graph emerges from another representation, such as an equation or a table.

The focus of the present dissertation was on lower secondary school students' learning of functions using the dynamic geometry program GeoGebra, in addition to more traditional teaching methods. The thesis aimed to investigate (1) the impact of the use of the GeoGebra dynamic program on the learning outcomes of the students who learn functions in basic school, (2) how do students participating in GeoGebra-assisted intervention and students in traditional instruction explain their answers when matching the equation of a linear function to the graph, and (3) students' attitudes and affects of using GeoGebra dynamic geometry program in two different contexts – inside school and outside school.

In order to test whether supplementing traditional instruction with GeoGebra can lead to a better conceptual understanding of functions and to explore how the use of GeoGebra influences students' attitudes toward mathematics, two experiments were conducted. The results of these experiments are presented in three empirical studies (Study I, II, and III). Additionally, three competitions were organized in which students created GeoGebra-based projects linking the topic of functions to everyday life. For example, in one competition, students had to create a fireworks display, in another competition, they had to create a movement in some real-life context. Students' opinions about using GeoGebra outside school are presented in Study IV.

Study I examined the effect of GeoGebra on learning linear functions and inverse variation among seventh-grade students. The quasi-experimental design involved

212 students, with 128 in the experimental group using GeoGebra in addition to traditional learning and 84 in the control group receiving only traditional instruction without computer programs. There was no statistically significant difference in the learning results between the experimental and control groups neither in the pre-test nor in the post-test. Students' ratings of the necessity, interest, and enjoyment of learning functions were lower while their difficulty ratings were higher in both the experimental and control classes than the same ratings of learning mathematics in general. However, students in the experimental group reported in the post-questionnaire that the use of computers made learning functions easier and more enjoyable. Notably, 42% of the experimental group students indicated that using GeoGebra improved their attitude towards mathematics.

Study II explored how GeoGebra impacts students' understanding of the relationships between different representations of linear functions and developing their graphing skills. In this study, two tasks from the final test of the first experiment were analysed. In the first task, the equation and graph of a linear function were matched and the basis for the match was explained. In the second task, the graph of a linear function was drawn based on the given equation. The results, involving the same student groups as Study I, indicated that the preferred reasoning of students using GeoGebra was using the values of the coefficients *a* and *b*, while students in the control class preferred to create a table. The experimental group performed significantly better at matching linear function graphs with their corresponding equation. However, both groups required further development in explaining. There was no difference between the experimental and control group students in the graph drawing task. This suggests that dynamic geometry software GeoGebra can be a valuable tool for teaching linear functions, provided that tasks are designed to help students discover links between different representations.

Study III explored the impact of computer use on learning quadratic functions in ninth grade. The quasi-experiment included five classes of 9th-graders using GeoGebra alongside traditional methods and five classes with only traditional methods when learning functions. A total of 199 students participated in the study, 105 of them belonged to the control classes and 94 to the experimental classes. The results indicated no significant differences in learning outcomes between the two groups. However, the students who used computers had better attitudes toward learning functions, and 33% of the experimental group students who responded to the question found that their attitudes towards mathematics improved. While using GeoGebra may not directly improve test scores, it can enhance student engagement and interest in mathematics.

Study IV highlighted the role of GeoGebra in increasing students' interest in mathematics through creative competitions. These competitions, held over three years, involved students creating GeoGebra applets on various topics such as patterns from graphs of functions, fireworks, and moving objects. With participation numbers of 232, 160, and 167 respectively, the competitions demonstrated that according to the students themselves, they learned more about GeoGebra and improved their mathematical skills. Participants reported that the competitions made mathematics more enjoyable and relevant, and they appreciated the opportunity to see the practical applications of mathematical concepts.

The results of the four empirical studies suggest that when teaching students about functions in lower secondary school the students can gain a better understanding when they have an opportunity to use the dynamic geometry program GeoGebra. In order to develop students' explanation skills, in addition to solving tasks, students could also be directed to explain their solution process or way of thinking. This gives the teacher valuable information about how the students have understood the topic and what needs to be paid attention to. Using the GeoGebra program makes learning more enjoyable for students. Solving creative tasks related to real life and art provides a lot of challenges and joy for doing mathematics.

KEYWORDS: Linear function, quadratic function, GeoGebra, mathematics

TURUN YLIOPISTO
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TIIVISTELMÄ

Funktiot ovat keskeinen osa koulumatematiikkaa. Funktioiden ymmärtäminen edellyttää, että oppilaat hallitsevat funktioiden eri esitysmuodot ja niiden väliset yhteydet. Näiden suhteiden oivaltaminen voi olla haastavaa, koska se edellyttää usein usean esitysmuodon tarkastelua samanaikaisesti. Perinteisessä opetuksessa, jossa kuvaajat piirretään käsin, useiden kuvaajien laatiminen vie aikaa ja voi aiheuttaa lisähaasteita laskennassa ja piirtämisessä. Tässä kontekstissa tietokoneohjelmien käyttö tarjoaa selviä etuja: niiden avulla voidaan nopeasti ja samanaikaisesti esittää funktion kuvaaja ja lauseke, jolloin oppilaat voivat heti havaita, miten muutos toisessa esityksessä vaikuttaa toiseen. Kuitenkin on myös tärkeää, että oppilaat oppivat piirtämään kuvaajia käsin, jotta he ymmärtäisivät, miten kuvaaja muodostetaan toisista representaatioista, kuten esimerkiksi lausekkeesta tai taulukosta.

Tämän väitöskirjan tavoitteena oli tutkia peruskoulun oppilaiden funktioiden oppimista hyödyntämällä dynaamista GeoGebra-geometriaohjelmaa perinteisen opetuksen rinnalla. Tutkimuksen tavoitteet olivat: (1) tutkia GeoGebran käytön vaikutusta peruskoululaisten funktioiden oppimistuloksiin; (2) selvittää, miten GeoGebra-avusteiseen opetukseen osallistuvat oppilaat ja perinteisessä opetuksessa olevat oppilaat selittävät vastauksiaan yhdistäessään lineaarisen funktion lausekkeen ja kuvaajan; (3) tutkia oppilaiden asenteita ja tunteita GeoGebra-ohjelman käyttöä kohtaan koulussa ja sen ulkopuolella.

Tutkimusta varten toteutettiin kaksi kokeellista tutkimusta. Tulokset esitetään kolmessa empiirisessä osatutkimuksessa (Tutkimus I, II ja III). Lisäksi järjestettiin kolme kilpailua, joissa oppilaat loivat GeoGebrassa projekteja, jotka liittyivät funktioihin ja arkielämään, kuten ilotulitukseen tai liikkeeseen tosielämässä. Oppilaiden mielipiteet GeoGebran käytöstä koulun ulkopuolella esitetään tutkimuksessa IV.

Tutkimus I tarkasteli GeoGebran vaikutusta lineaaristen funktioiden ja käänteislukusuhteen oppimiseen 7. luokan oppilailla. Kvasikokeelliseen tutkimukseen osallistui 212 oppilasta, joista 128 kuului kokeelliseen ryhmään ja 84 kontrolliryhmään. Oppimistuloksia koskevassa ennakkotestissä ja jälkitestissä ei ollut tilastollisesti merkitsevää eroa ryhmien välillä. Molemmissa ryhmissä funktioiden oppiminen koettiin vähemmän kiinnostavaksi, mielenkiintoisemmaksi ja miellyttäväksi mutta vaikeammaksi kuin matematiikan oppiminen yleensä. Kuitenkin kokeellinen ryhmä raportoi, että tietokoneiden käyttö helpotti ja teki oppimisesta miellyttävämpää. 42 % kokeellisista oppilaista ilmoitti, että GeoGebran käyttö paransi heidän suhtautumistaan matematiikkaan.

Tutkimus II tarkasteli, miten GeoGebra vaikuttaa oppilaiden kykyyn ymmärtää lineaaristen funktioiden eri esitysmuotojen suhteita ja kehittää kuvaajan piirtämistaitoja. Tässä tutkimuksessa analysoitiin ensimmäisen osatutkimuksen osallistujien kahden viimeisen loppumittauksen tehtävän vastauksia. Tulokset osoittivat, että GeoGebraa käyttävät oppilaat perustuivat vastauksissaan useammin kertoimien a ja b arvoihin, kun taas kontrolliryhmä preferoi taulukon laatimista. Ryhmien välillä ei ollut eroja kuvaajan piirtämistehtävässä. Kokeellinen ryhmä suoriutui merkittävästi paremmin lineaaristen funktioiden kuvaajien ja lausekkeiden yhdistämisessä, mutta molemmat ryhmät kaipasivat edelleen tukea selittämisessä. Tutkimus antaa viitteitä siitä, että GeoGebralla on arvoa lineaaristen funktioiden eri esitysmuotojen opettamisessa.

Tutkimus III käsitteli tietokoneiden käytön vaikutusta 9. luokan oppilaiden kvadraattisten funktioiden oppimiseen. Viidessä kokeellisessa luokassa käytettiin GeoGebraa ja viidessä kontrolliluokassa vain perinteisiä menetelmiä. 199 oppilasta osallistui tutkimukseen, 105 koeryhmään ja 94 kontrolliryhmään. Tulokset eivät osoittaneet tilastollisesti merkittäviä eroja oppimistuloksissa, mutta tietokonetta käyttäneet oppilaat suhtautuivat positiivisemmin funktioiden oppimiseen, ja 33 % ilmoitti suhtautumisensa matematiikkaan parantuneen. Vaikkei GeoGebran käyttö suoraan parantaisi oppimistuloksia se voi lisätä oppilaiden sitoutumista ja mielenkiintoa matematiikkaa kohtaan.

Tutkimus IV korosti GeoGebran roolia matematiikkakiinnostuksen lisääjänä luovien kilpailujen kautta. Oppilaat loivat GeoGebralla funktioiden kuvaajia käyttämällä kuvia eri aiheista, kuten ilotulituksista ja kohteiden liikkeestä kolmena vuonna järjestetyissä kilpailuissa. Osallistujia oli 232 ensimmäisenä, 160 toisena ja 167 kolmantena vuonna. Osallistujat kokivat oppineensa lisää GeoGebrasta ja parantaneensa matemaattisia taitojaan. Osallistuneiden mielestä kilpailut tekivät matematiikasta mielekkäämpää ja toivat esiin sen käytännön sovelluksia.

Kaikkien neljän tutkimuksen tulokset viittaavat siihen, että funktioiden opetuksessa yläkouluiässä GeoGebra voi parantaa oppimiskokemusta, ymmärrystä ja kiinnostusta. Selittämistaitojen kehittämiseksi oppilaita tulisi rohkaista kuvailemaan ajatteluprosessiaan. GeoGebra tekee oppimisesta hauskempaa, ja luovat, reaalielämään liittyvät tehtävät tarjoavat innostusta ja iloa matematiikan opiskeluun.

AVAINSANAT: Lineaarinen funktio, kvadraattinen funktio, GeoGebra, matematiikka.

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02.11.2025

Sirje Pihlap

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List of Original Publications

This doctoral thesis is based on the following studies reported in four original articles.

I Pihlap, Sirje (2021). “Mathematics is not as difficult as I thought”: The effects of computer use on learning linear functions and inverse variation. *ICERI2021 Proceedings: 14th annual International Conference of Education, Research and Innovation Online Conference. 8-9 November, 2021.* Ed. L. Gómez Chova, A. López Martínez, I. Candel Torres. IATED Academy, 7663−7671. (ICERI2021 Proceedings; 1). DOI: 10.21125/iceri.2021.1714

II Pihlap, S., Veermans, K. and Hannula-Sormunen, M. (2023). The impact of the use of a dynamic geometry program on students' understanding of the concept of linear function. *The International Journal for Technology in Mathematics Education, 30*(4), 247−254. DOI: 10.1564/tme_v30.4.7

III Pihlap, S. (2017). The Impact of Computer Use on Learning of Quadratic Functions. The *International Journal for Technology in Mathematics Education, 24*(2), 59−66. DOI: 10.1564/tme_v24.2.02

IV Pihlap, S., Kuresson, A., Kraav, T, Jukk, H, Orav-Puurand, K. (2024). Exploring Mathematics with GeoGebra. *International Journal for Technology in Mathematics Education, 31*(2), 89−95. DOI: 10.1564/tme_v31.2.07.

In Study II and Study IV Pihlap designed the study and was responsible for data collection, data analysis and the writing of the first draft, in addition to revising of the manuscript in collaboration with co-authors.

Declaration of generative AI and AI-assisted technologies: In the preparation of this dissertation, Pihlap utilised AI tools ChatGPT and Grammarly to assist with paraphrasing and enhancing readability. All content was subsequently reviewed and edited by the author, who assumes full responsibility for the final version of the publication.

1 Introduction

Functions are an important but also complex topic in school mathematics. The topic of functions is considered complex because it is related to other mathematical concepts, and because functions have different representations (Baki & Güveli, 2008; Clement, 2001; Llinares, 2000; Pierce, Stacey & Bardini, 2010; Sajka, 2003). For instance, algebraic and graphical representations are two fundamentally different symbolic systems that work together to define and shape the mathematical concept of a function. Neither the functions equations nor their corresponding graphs should be regarded as independent of one another (Leinhardt et al., 1990).

Therefore, effective instruction goes beyond teaching students to perform various procedures, such as finding the value of y given x or creating a graph from an equation. For a deep understanding of functions, it is crucial to present functions in multiple ways, helping students grasp algebraic, graphical, and other representations of functions, and to understand the connections between these representations, achieving fluency in transitioning among them (Kalchman & Koedinger, 2005). However, many students struggle with these connections between different representations of functions (Baki & Güveli, 2008; Wurning, 2009).

According to Godwin and Beswetherick (2003), students need to draw graphs on paper with a pen as it helps them understand how function values y are derived from argument values x. However, a major limitation of pen-and-paper drawing is the time it consumes and the static format of a graph, making it challenging to study the properties of functions and explore *what if* questions. Additionally, the drawing field is limited, and students cannot easily modify their graphs, leading to a restricted understanding of the concept of functions (Godwin & Beswetherick, 2003). It is crucial to observe different representations of functions simultaneously so that students can perceive them as equivalent representations of the same mathematical relation (Kalchman & Koedinger, 2005). The use of dynamic geometry software (such as GeoGebra, Gabri, Desmos) can enhance students' understanding of functions. These software tools offer interactive connections among various representations of functions (graphs, tables, equations), thereby helping students more clearly understand the relationships between them (Nocar & Zdrahal, 2016). Dynamic geometry software also facilitates plotting graphs of functions based on equations, allowing students to experiment and make discoveries about functions by fixing one coefficient and modifying another (Bray & Tangney, 2017; Godwin & Beswetherick, 2003; Ruthven et al., 2009).

An OECD report (2020) highlights the significance of innovative use of technology to foster deeper conceptual comprehension. Nevertheless, various studies indicate that technology has often been utilised primarily as a communication tool rather than effectively facilitating students' conceptual grasp of mathematics (Jukk et al., 2021; OECD, 2020, p. 290). PISA 2012 showed that computers are infrequently used during mathematics instruction. On average across OECD countries, for 14% of students, only teachers demonstrated the use of computers; 32% of students reported that they had used a computer to solve a math problem in the last month (OECD, 2015, p. 56-57). Estonian PISA 2022 results analysis reveals that teachers have the skills and interest to use ICT tools in teaching mathematics. However, often, this may be limited to the teacher's activities at the board rather than offering engaging digital tasks to students. It is possible that 57% of learners do not encounter the use of IT tools in mathematics classes (Lorenz, 2023).

Previous studies have found that the use of computers can be useful for a better understanding of the concept of function (Baki & Güveli, 2008; Cheung & Slavin, 2013; Godwin & Beswetherick, 2003; Reed et al., 2010; Wurning, 2009). As previous research has highlighted, the impact of computer programs in the classroom depends on how they are utilised (Göbel, 2021; McCoy, 1996), it is necessary to examine the various ways computers are used (Papanastasiou et al., 2003; Leinbach et al., 2002; Övez, 2018; Hillmayr et al., 2020). Also, it is important that teachers know students' errors or misconceptions and according to this knowledge identify appropriate teaching strategies (Tanışlı & Kalkan, 2018). Hillmayr et al. (2020) emphasise the importance of investigating how the use of digital tools can positively influence students' attitudes toward mathematics.

According to the Estonian National Curriculum (Põhikooli…, 2011), teachers should use computers when teaching the topic of functions. However, there are no previous studies in Estonia that investigate how to use computers for learning functions to make them effective or influence students' attitudes towards mathematics.

1.1 Learning functions

In Estonian schools, the topic of functions is taught in the 7th and 9th grades (Põhikooli ..., 2011). Topics covered before the concept of functions include constant and variable values, proportional values and their properties, inverse values, and the reading of graphs (such as motion and temperature graphs). Veelmaa (2010) recommends presenting the definition of functions in a language understandable to seventh-grade students, meaning that the definition should avoid using terms that are difficult for them to understand. For instance, in the 7th-grade textbook (Kaljas et al., 2023), a function is defined as follows: "If every value of the independent variable *x* corresponds to a unique value of the dependent variable *y*, then the dependent variable is called a function of the independent variable." A linear function is defined as a relationship represented by the equation $y = ax + b$ (Kaljas et al., 2023). The concepts of proportional and inverse variation and linear functions are explained using real-life examples. In the 7th grade, representations of functions in the form of equations, tables, and graphs are discussed, and the connections between different representations are highlighted. Students are required to manually plot function graphs, as well as using computer software, and interpret the coefficients *a* and *c* in the function equations (Põhikooli …, 2011).

In the 9th grade, quadratic functions are studied. Students are required to plot graphs both with pen and paper and with computer program, explain the dependence of the position and shape of the graph of a quadratic function $y = ax^2 + bx + c$ on the constant term *c* and the quadratic term *a*, find the zero points on both the graph and the equation, explain the meaning of the zero points, find the vertex of the parabola on the graph and calculate the coordinates of the parabola's vertex from the equation (Põhikooli …, 2011).

From the above, it can be seen that in both the 7th and 9th grade, the interpretation of the coefficients in the function equation, which means understanding the relationships between the function equation and its graphical representation, is an important part of learning the topic of functions. Establishing relationships between various pieces of information develops conceptual knowledge (Hiebert & Lefevre, 2013). To acquire a conceptual understanding of a function, it is important to know the different ways of representing a function, such as an equation, a graph, and a table, in each of which one can see how the value of one variable depends on the value of another. Conceptual understanding also involves understanding how different representations of a function are related (Chiu et al., 2001; Hiebert & Lefevre, 2013; Kalchman & Koedinger, 2005; Leinhardt et al., 1990). It is essential to recognise that transitioning between two representations of a function – such as an equation and a graph – can occur in both directions: from the equation to the graph and from the graph to the equation. Both directions are crucial for grasping the concept of a function, and instruction should also address both. Converting an equation into its graph is generally simpler: you need only compute point coordinates from the equation, plot them on a Cartesian grid, and connect the points. However, determining an equation from a graph is more complex, as it involves identifying patterns (Birgin, 2012; Leinhardt et al., 1990).

Students often struggle to comprehend the relationships between different representations of functions (Baki & Güveli, 2008; Koklu & Topcu, 2012).

Therefore, it is important to establish educational settings that motivate students to delve into the *why* and *how* aspects of their discussions and to explore various ways to represent functions (Tanışlı, & Kalkan, 2018). Adopting effective teaching methodologies can aid students in mastering different forms of function representation, such as equations, tables and graphs, and understanding the connections between these representations. (Goulding & Kyriacou, 2007; Kalchman & Koedinger, 2005; Leinhardt et al., 1990; Tanışlı, & Kalkan, 2018).

1.1.1 Learning Linear Functions

A linear function is a student's first experience working with an independent and dependent variable (Kalchman and Koedinger, 2005; Pierce, Stacey and Bardini, 2010). Besides comprehending the roles of variables x and y, students must also understand the meaning of the coefficients a and b in the equation of the linear function $y = ax + b$ (Pierce, Stacey & Bardini, 2010). It is the first topic where students are introduced to two different symbolic systems: the algebraic and graphical representations of a function. On one hand, these dual symbolic systems are essential for grasping the concept of a function; on the other hand, they present complexities for students to understand (Leinhardt et al., 1990). Kalchman and Koedinger (2005) emphasise the importance of presenting tables, graphs, and equations side by side at the same time so that students see them as equivalent forms of the same mathematical relationship. The focus is on smooth transitions between these representations and recognising the nature of their relationship to each other.

When students lack a clear understanding of linear functions, they struggle to connect different forms of representation (Koklu & Topcu, 2012). Studies have shown that these difficulties often manifest in several ways. One recurring issue is the inability to interpret information from graphs and equations, which becomes evident when students overlook inconsistencies between visual and symbolic representations. For example, students may not recognise the discrepancy if the line is rising but the equation has a negative x-coefficient. Recognising and correcting such discrepancies demonstrates the student's ability to accurately interpret the slope of an equation and effectively compare multiple representations – a sign of deeper conceptual understanding (Kalchman & Koedinger, 2005). Another common problem is the misunderstanding of fundamental geometric properties, such as the requirement that parallel lines must have identical slopes (Yerushalmy, 1991).

Students also struggle to recognise whether tabular representations correspond to linear relationships. In an exercise where the student was asked to create a table based on a provided graph, they failed to recognise that the table they produced did not accurately depict a linear function. This was because the change in y was inconsistent for each unit change in x. Such mistakes highlight a fundamental gap in the student's understanding of functions. It shows that the student either lacked or did not effectively use the knowledge necessary to discern key characteristics (like increases or decreases) across different representations of functions. (Kalchman & Koedinger, 2005)

Furthermore, learners often experience difficulties in interpreting the coefficients of the equation $y = ax + b$. For example, students have difficulty understanding and explaining how the coefficients a and b of the equation $y = ax + b$ relate to the x- and y-intercepts. This indicates a broader difficulty in connecting the basic elements of these linear equations (Birgin, 2012; Mosckhovich, 1996).

1.1.2 Learning quadratic functions

Quadratic functions can be expressed symbolically in different forms. Two common forms are the standard form $y = ax^2 + bx + c$ and the vertex form $y = a(x – b)^2 + c$, where (b, c) is the vertex of the parabola (Göbel, 2021). In Estonian school mathematics, the standard form is used (Põhikooli…, 2011). To acquire a conceptual understanding of a function, it is important to know the different ways of representing a function and understand how different representations of a function are related (Chiu et al., 2001; Kalchman & Koedinger, 2005; Leinhardt et al., 1990). According to the Estonian curriculum, the student must explain the dependence of the position and shape of the graph of a quadratic function $y = ax^2 + bx +c$ on the free term c and the coefficient of quadratic term a (Põhikooli…, 2011).

Ellis and Grinstead (2008) investigated how students relate the coefficients of the quadratic function $y = ax^2 + bx + c$ to the graph of this function. A TI-83 graphing calculator was used for learning quadratic functions. The two most common interpretations of c were that it denotes the y-intercept and that it shifts the graph. Some students thought that c is the x-intercept or c affects the vertex or is part of the vertex. These two answers indicate that the student has not understood the concept of a quadratic function. The students demonstrated four interpretations of the parameter a: a affects the shape of the parabola; a is the "stretch factor"; a determines whether the parabola faces up or down; a is the slope of the parabola. The last one is the unexpected result – parabola does not have a constant slope.

The errors and misunderstandings about linear and quadratic functions emphasise the need for more effective teaching strategies to bridge the gap between different representations of linear functions and enhance students' conceptual understanding.

1.2 Use of technology

Since their introduction, computers have been repeatedly highlighted as valuable tools in education, including mathematics (Hillmayr et al., 2020). A range of meta-analyses have investigated the effects of technology on student outcomes, generally indicating a positive impact, albeit with variations in effect size and methodological rigour.

For instance, Kulik and Kulik's (1987) meta-analysis of 199 studies showed that most reported a positive effect of computer-based instruction. The authors attributed these gains not only to the computers themselves but also to the well-constructed instructional materials that often accompany technology-based learning. Similarly, Cheung and Slavin (2013), in their meta-analysis of 74 studies, found that integrating technology yielded a modest positive effect on mathematics achievement (ES = 0.15). This effect was strongest (ES = 0.18) when technology supplemented traditional teaching, compared to comprehensive digital programs (ES = 0.07) or computer-managed learning (ES = 0.08). However, Cheung and Slavin (2013) warned that many studies lacked methodological rigour – such as control groups or comparable pre-tests – leading to inflated effect sizes.

Hillmayr et al.'s (2020) review of 92 studies focusing on mathematics and science confirmed these positive trends: 87% of effect sizes were favourable for learning outcomes, and the overall effect was medium-sized and statistically significant. Digital tools also had a mostly positive influence on students' attitudes toward their subjects (16 of 25 effect sizes were positive). Notably, teacher training in digital tools enhanced learning outcomes, and dynamic mathematics programs showed the largest positive effects. Although using technology as a supplement rather than a substitute was slightly more beneficial, this difference was not statistically significant. Furthermore, shorter-term interventions generally yielded stronger effects compared to those lasting over six months. While randomised experiments tended to show larger effects than quasi-experimental studies, the difference was again not statistically significant.

Based on a synthesis of 81 meta-analyses, Hattie (2013) found that the effect of computers on learning is no different from any other well-planned intervention.

In sum, research consistently suggests that technology can bolster learning outcomes and improve attitudes (Cheung & Slavin, 2013; Hillmayr et al., 2020; Kulik & Kulik, 1987), especially when used alongside traditional instructional methods (Cheung & Slavin, 2013; Hillmayr et al., 2020). Nevertheless, caution is warranted in interpreting these effects, as methodological limitations in some studies may overstate the benefits (Cheung & Slavin, 2013).

1.2.1 Use of technology for learning functions

The potential of integrating computers into learning functions has long been acknowledged, especially because of the unique benefits digital tools can offer (Baki & Güveli, 2008; Cheung & Slavin, 2013; Godwin & Beswetherick, 2003; Reed et al., 2010; Wurning, 2009). A central aspect of understanding functions is recognising how different function representations interrelate (Chiu et al., 2001; Kalchman & Koedinger, 2005; Leinhardt et al., 1990). In traditional teacher-centred settings, students often find it difficult to connect the equation of a function to its graphical representation (Birgin, 2012). Technology – such as graphing software, dynamic geometry tools, and graphics calculators – helps address this challenge by displaying tables, graphs, and equations concurrently, allowing students to see these as multiple views of the same relationship (Birgin, 2012; Kalchman & Koedinger, 2005). With digital tools, students can quickly generate graphs and alter function coefficients, giving them opportunities for experimentation and discovery (Birgin, 2012; Godwin & Beswetherick, 2003). Nevertheless, simply using computers does not guarantee better learning outcomes compared to traditional pen-and-paper methods. The effectiveness of technology depends on how it is applied (Goulding & Kyriacou, 2007; Göbel, 2021; McCoy, 1996). Teachers play a key role in effectively orchestrating both digital and non-digital resources to support the learning process (Goulding & Kyriacou, 2007).

Freeing students from lower-level cognitive tasks or those that exceed their mental capacity has been associated with the potential for enhanced intellectual performance (Salomon & Perkins, 2005). Technological tools – such as dynamic geometry software – can assume a significant portion of the cognitive processes involved in construction tasks, thereby reducing the mental burden compared to traditional paper-based approaches. This technological support enables students to allocate greater cognitive resources to higher-order processes, such as interpretation and detailed examination of function behaviour, which are often difficult to manage manually (Salomon & Perkins, 2005; Sivasubramaniam, 2000). In a comparative study, Sivasubramaniam (2000) examined students' abilities to interpret context-free graphs when using either a paper-and-pencil medium or a computer-based environment. The paper-based condition required students to first construct the graphs, followed by a shift in focus toward interpretation. Conversely, the computer-based environment automated the graph creation process, allowing students to concentrate more directly on interpretive tasks. Findings indicated that students using the computer-based medium exhibited a stronger focus on interpretation and demonstrated a greater ability to articulate their reasoning, in comparison to students using paper and pencil.

However, Godwin and Beswetherick (2003) warn that if the freed-up time is not devoted to deeper analysis, students may fail to grasp how the output values (y)

depend on the input values (x). Weigand and Weller (2001) likewise found that computer-generated images are often viewed only as static pictures unless teachers consistently prompt reflection on the underlying mathematical ideas. They recommend posing questions that steer students toward interpreting and explaining what they see on the screen.

In a study involving 23 students aged 14–15, Arnal-Palacián et al. (2022) explored the challenges associated with learning quadratic functions using traditional pencil-and-paper methods and investigated the potential of GeoGebra to address these difficulties. The authors identified several common errors, including computational mistakes, confusion regarding symmetry, misconceptions about roots, and inadequately plotted points. Through a comparative analysis of tasks completed via traditional pencil-and-paper methods and those carried out using GeoGebra by the same students, the researchers observed that the dynamic features of the software significantly reduced or eliminated many of these issues. GeoGebra facilitated clearer conceptual visualisation and minimised routine calculation errors, thereby allowing students to engage with more advanced tasks – such as parameter manipulation – within the same instructional timeframe.

Other research confirms the positive impact of technology on learning of functions. Koklu and Topcu (2012) observed that students who spent eight hours over two weeks using the dynamic geometry program Cabri achieved better results in understanding quadratic functions compared to a control group. Ogbonnaya and Mushipe (2020) also found that students using GeoGebra for learning linear functions performed better at drawing and interpreting graphs than those employing only pencil and paper. Meanwhile, Övez (2018) showed that students taught with GeoGebra and discovery-based worksheets were more successful in linking algebraic expressions to their graphical representations than their peers in a traditional setup. In a study conducted by Övez (2018), 62 students were randomly assigned to either an experimental group, which learned quadratic functions through GeoGebra and discovery worksheets, or a control group that followed traditional pencil-and-paper instruction. The findings indicated that the GeoGebra-based approach more effectively improved student achievement and facilitated the attainment of learning objectives compared to the standard tenth-grade curriculum. Observations of the control group suggested a primary focus on algebraic manipulations and formula memorisation, with a limited understanding of how equations connect to their graphical representations. By contrast, the experimental group demonstrated a stronger ability to relate algebraic expressions to graphs and interpret those relationships. Övez (2018) attributes this improvement to the guided discovery approach and the creation of examples in the GeoGebra environment, which allowed students to uncover underlying concepts for themselves.

Supplementary web-based resources can further enhance classroom instruction. Baki and Güveli (2008) divided 80 ninth-graders into two groups: one received only traditional instruction, while the other combined traditional methods with web-based mathematics teaching (WBMT). WBMT materials covered general concepts of linear and quadratic functions and offered multiple simultaneous function representations. With around 10 hours of weekly WBMT sessions across five weeks, the experimental group had greater improvement in post-tests. Students using online resources not only made more significant progress but also gained confidence and motivation by encountering multiple representations of linear and quadratic functions.

Göbel (2021) explored the impact of technology-assisted guided discovery on students' understanding of the parameters in a quadratic function. She concluded that dynamic visualisations, such as drag modes and sliders, appear more effective than static methods like function plotters or no technological support. However, students ultimately failed to explain how parameter changes affected the function, which Göbel suggests may be due to their tendency to solve tasks rather than explain their reasoning. Consequently, she recommends providing additional guidance to prompt students to explain and justify their insights. Also, PISA 2022 data from Estonia suggests that students who regularly explained their reasoning in class performed the PISA 2022 test above the national average (Jukk, 2023).

Ultimately, the teacher's role is pivotal in integrating technology effectively to direct students' attention to meaningful mathematical concepts (Godwin & Sutherland, 2004). If guided well, digital tools can foster deeper conceptual understanding, enable students to explore multiple representations of functions and promote higher-level thinking by alleviating the burden of routine tasks.

In sum, integrating digital tools into teaching functions offers significant potential for improving students' conceptual understanding by performing simultaneously different representations, such as equations, tables, and graphs (Birgin, 2012; Kalchman & Koedinger, 2005; Leinhardt et al., 1990). Digital resources, including dynamic geometry software, reduce students' cognitive load by automating routine tasks, thus allowing them to focus on interpretation and deeper analysis (Salomon & Perkins, 2005; Sivasubramaniam, 2000). Studies consistently show that students using technology exhibit fewer misconceptions, engage more readily in exploration, and demonstrate stronger abilities to relate algebraic expressions to graphical representations compared to traditional pencil-and-paper methods (Arnal-Palacián et al., 2022; Koklu & Topcu, 2012; Ogbonnaya & Mushipe, 2020; Övez, 2018). However, technology alone does not guarantee improved learning; students can treat dynamic visuals as static unless teachers actively encourage reflection and conceptual explanation (Godwin & Beswetherick, 2003; Weigand & Weller, 2001). Effective instructional guidance, including strategic questioning and explicit encouragement of reasoning, is essential to maximise the educational benefits of digital tools (Goulding & Kyriacou, 2007; Göbel, 2021). Thus, the teacher's role remains critical in orchestrating technology-enhanced learning environments that support meaningful mathematical exploration and understanding.

1.3 Students' attitudes towards mathematics

Attitudes and motivation play a key role in determining how willingly individuals continue with mathematics once it is no longer compulsory (Hannula et al., 2014). For a significant number of students, attitudes towards mathematics decrease notably during secondary education (Wen & Dubé, 2022). According to Brown et al. (2008), the reasons why students anticipated participating or not participating in mathematics classes were perceived difficulty and low self-confidence, which were the primary deterrents, alongside a lack of interest, boredom, and perceived irrelevance. Enjoyment was a key factor distinguishing schools with high mathematics enrolment from those with low enrolment (Brown et al., 2008).

Research on the link between learning outcomes and students' attitudes consistently shows that there is a correlation between students' attitudes towards mathematics and their learning outcomes in mathematics (Davadas & Lay, 2017; Grootenboer et al., 2015; Hannula et al., 2014; Manzana et al., 2019; Nicolaidou & Philippou, 2003; Reed et al., 2010; Singh & Imam, 2013; Wiberg et al., 2024). Students who enjoy studying mathematics and engage in related activities believe that learning mathematics leads to positive outcomes (such as school success and future job opportunities), and combined with trust in their own mathematical abilities, they are more likely to excel in the subject. Consequently, it is crucial for educators to examine students' attitudes toward mathematics and provide targeted support to foster a positive disposition, ultimately improving their chances of higher achievement (Hwang & Son, 2021).

Based on a systematic review of 95 studies about secondary students' attitudes towards mathematics and its relations with mathematics achievement, Wen and Dube (2022) argue that the improvement of mathematics-related attitudes is hindered by the lack of a clear and unified understanding of the concept of attitude. For example, only 20 studies out of 95 clearly stated their definition of mathematics attitudes, while the remaining articles did not define the term at all. Wen and Dube (2022) suggest adopting a multidimensional definition, according to which mathematics attitudes consist of specific cognitive aspects (value, gender roles/beliefs, confidence, self-concept), affects (enjoyment, anxiety), and behavioural intentions (i.e., willingness and tendency to spend more time learning mathematics). The review also examined the relationships between each subdimension and mathematics achievement. In general, anxiety and gender role beliefs were negatively associated with mathematics performance, while enjoyment, self-concept, confidence, perceived value, and behavioural intentions were positively related to achievement.

One way to change students' attitudes towards mathematics more positively is to use computer programs in teaching. Hillmayr et al. (2020) conducted an exploratory analysis to examine whether using digital tools in science or mathematics classes influenced students’ attitudes toward the subject, compared to classes without digital tools. Of the 92 studies analysed, 16 offered effect sizes related to student attitudes; these studies investigated attitudes in addition to student learning outcomes. The study did not clarify which aspects of attitudes were analysed. Overall findings indicated that employing digital tools had a small yet statistically significant positive effect on students’ attitudes toward the subject. Consequently, secondary school students who were taught with digital tools in science or mathematics reported significantly more positive attitudes than those who learned without digital tools.

Similarly, Pierce and Stacey (2011) found that using real-world situations within the dynamic geometry program GeoGebra led to positive changes in middle school students’ attitudes. They examined how students imported real-world photos into GeoGebra’s graphics view and then plotted lines or parabolas on those images. While students often began by guessing and checking, they quickly became more systematic once they realised how altering a function’s symbolic rule changes its graph. According to classroom observations reported by Pierce and Stacey (2011), students enjoyed the challenge of fitting a line precisely to an image. Presenting tasks in this way not only helps students see how real-world curves can be described by function rules, but also enables them to explore the influence of different parameters, practice finding equations of various lines and parabolas, and engage in friendly competition to find the “best fit”. Evidence of success stemmed primarily from lesson observations and teacher reports rather than detailed analyses of how students learned from these dynamic geometry activities.

In sum, cultivating positive student attitudes through supportive instructional strategies is crucial for enhancing mathematics achievement and participation.

2 Aims

The present thesis provides an examination of how using the dynamic geometry software GeoGebra impacts the learning of the concept of functions among 14- to 16-year-old Estonian students. According to the Estonian National Curriculum (Põhikooli …, 2011), teachers should use computers when teaching the topic of functions. However, there are no previous studies in Estonia that investigate how to use computers for learning functions to make them effective or influence students' attitudes towards mathematics.

Functions are a fundamental yet challenging topic in school mathematics due to their multiple representations. Effective learning requires students not only to perform procedural tasks but also to understand how different representations relate to each other, transitioning fluently among equations, graphs, and tables (Kalchman & Koedinger, 2005; Leinhardt et al., 1990). Many students struggle to connect algebraic and graphical forms, which limits their conceptual understanding of functions (Baki & Güveli, 2008; Wurning, 2009).

Dynamic geometry software, such as GeoGebra, has demonstrated considerable potential to address these learning difficulties. These digital tools provide simultaneous visualisations of functions' multiple representations, facilitating easier transitions between equations, graphs, and tables, and enabling students to explore mathematical relationships dynamically (Birgin, 2012; Nocar & Zdrahal, 2016). Moreover, digital environments allow students to experiment efficiently with function parameters and observe immediate changes, thereby deepening their conceptual understanding while freeing cognitive resources for higher-order thinking (Salomon & Perkins, 2005; Sivasubramaniam, 2000). However, the mere presence of technology does not guarantee improved outcomes. Teachers play a crucial role in ensuring technology supports rather than distracts from conceptual understanding. Effective instructional practices, such as guiding students' interpretations and prompting reflection, are essential to prevent superficial engagement with digital tools (Godwin & Beswetherick, 2003; Göbel, 2021).

Additionally, students' attitudes significantly impact their mathematics learning outcomes. Positive attitudes – such as enjoyment, perceived relevance, and self-efficacy – strongly correlate with higher mathematics achievement and greater engagement (Hannula et al., 2014; Singh & Imam, 2013). Conversely, negative attitudes characterised by anxiety, boredom, or low self-confidence often result in disengagement, leading to reduced participation in mathematics education and contributing to shortages of mathematically skilled professionals (Brown et al., 2008). Understanding how students' attitudes toward mathematics change when using dynamic geometry software is crucial to improving instruction and fostering sustained mathematical interest and achievement.

The aim of the thesis is to investigate

1) the impact of the use of the GeoGebra dynamic program on the learning outcomes of students who learn functions in lower secondary school *(Study I, II, III)*.

2) how do students participating in GeoGebra-assisted intervention and students in traditional instruction explain their answers when they match the equation of a linear function to the graph? Are there differences in these explanations? *(Study II)*.

3) students' attitude and affect towards mathematics when using the GeoGebra dynamic geometry program in two different contexts – inside the school and outside school (*Study I, III, IV)*.

3 Methods

For Studies I-III, a quasi-experiment design was selected. This enabled to carry out the study without changing the usual teaching arrangements. Three studies were conducted, of which the first two studies concerned the learning and teaching of functions in the 7th grade and the third study concerned the learning and teaching of functions in the 9th grade. It is important to note that Study I and Study II had the same sample and data, but had a different focus. In Study I the focus is on the effect of using GeoGebra dynamic geometry program on 7th-grade students' learning outcomes and attitudes towards learning mathematics generally and functions. In Study II the focus is on the understanding of different representations of a linear function in the experimental and control classes. Study III is about the effect of using GeoGebra on 9th-grade students' learning outcomes and attitudes towards learning mathematics generally and quadratic functions.

Study IV explores learning outside the classroom. The data come from three student competitions where participants were required to use the GeoGebra program to produce creative works based on a specified theme, using functions or other curved lines. The study aimed to clarify students' attitudes toward creating these competition works.

3.1 Participants

To find participants for Studies I-III the researcher approached experienced mathematics teachers from different regions of Estonia, inviting them to participate in the study together with their students.

Study I and II included twelve 7th-grade classes from eight schools. Seven were experimental and five were control classes. There were nine teachers in these classes. One teacher taught both the experimental and control classes and two teachers taught two experimental classes. Any students with missing pre- or post-test and experimental classes students who did not participate in all computer-based lessons were excluded. The total number of such students was 28. The total size of the sample was 212 students, with 84 students in the control group and 128 students in the experimental group.

Study III included ten 9th-grade classes from five schools. Five classes were experimental and five were control classes. There were nine teachers in total, one of them teaching two experimental classes. A total of 199 students participated in the study, 105 of them belonged to the control classes and 94 to the experimental classes. 173 (90 control and 83 experimental) students took both the pre and post-test, and 185 (95 control and 90 experimental) answered the questionnaires.

For Study IV, students from all over Estonia were invited to participate in the voluntary competition. The number of participants in the three competitions (creating patterns using graphs of functions, creating fireworks for Estonia's birthday, and creating moving objects) was 232, 160 and 167, respectively.

3.2 Research design

Two quasi-experiments were conducted for Studies I-III. The quasi-experiment was preferred to a randomized control trial because, in the case of a true experiment conducted in a school, it would be difficult to randomly divide students into experimental and control groups. In addition, it would make the learning situation of the study different from the usual one (an additional teacher would be needed, and there would be a change in the size and participants of the study group). Such changes could affect the outcome of the experiment and that would lower ecological validity. A quasi-experiment, on the other hand, makes it possible to carry out the study under usual school and class conditions in a naturalistic setting.

Before the studies, the researcher communicated with all participating teachers by e-mail. A generally suitable number of lessons for teaching the topic was agreed upon, and the number of lessons conducted in the computer classroom as well as the learning resources used there were discussed with the teachers. Some days before the study, the researcher met with all participating teachers, explaining the procedure of the study. The teachers received written guidelines on the whole procedure of the study. All teachers filled in a diary with information about absent students and a brief description of each lesson.

In line with previous research (Cheung & Slavin, 2013; Hillmayr et al., 2020; Khalili & Shashaani, 1994; Sinclair et al., 2004), which has demonstrated that supplementing traditional instruction with computer-based tools is more effective than relying solely on digital methods, it was decided that teachers participating in the study would integrate GeoGebra dynamic geometry software as a complementary tool rather than as a replacement for traditional teaching approaches. Consequently, in the experimental classes, students learned functions through traditional methods in combination with GeoGebra, whereas in the control classes, students engaged exclusively with traditional instruction and did not use computers. Teachers had the option to choose whether they wanted to have classes where students used computers when learning functions. Depending on their preference, their classes were allocated to either the experimental or control groups. One 7th-grade teacher taught both the experimental and control classes.

Study I and Study II were concerned with the teaching of the proportional and inverse variation and linear functions in the 7th grade. A total of 23 lessons were spent on the topic of functions in the 7th grade, of which the first two lessons were repeated topics necessary for learning functions. The third lesson was used for administering the pre-test. At the beginning of the lesson, after the pre-test, a pre-questionnaire was filled in. After the pre-test, the next 19 lessons were spent on learning the topic of functions. In the experimental group, four lessons out of 19 took place in the computer lab. In two experimental classes, three lessons were held in the computer lab, and the fourth lesson on using GeoGebra was assigned as homework. The last lesson was reserved for the post-test, and in the lesson following the post-test, all students filled out the post-questionnaire (Table 1).

The second quasi-experiment (Study III) concerned the teaching of the quadratic function in the 9th grade. The design of the experiment was similar to the first quasi-experiment. A total of 22 lessons were planned for the topic of the quadratic function (Table 1). Since quadratic functions were learned at the beginning of the school year, the first five hours were repeated topics that are necessary for learning quadratic functions. A pre-test followed on the topics repeated in the previous five lessons. This was followed by 15 lessons of studying quadratic functions. The students in the experimental group were scheduled to use the GeoGebra program for 3 of these 15 lessons. Due to various circumstances, two experimental classes conducted all three lessons in the computer lab, while two other classes completed one lesson in the lab and carried out the remaining two as independent home assignments. Additionally, one class had two lessons in the lab, with the third lesson completed as a home assignment.

Table 1. Research design for experimental and control groups.

Activity	**Study I, II**	**Study III**
Rehearsal topics necessary for learning functions	2 lessons	5 lessons
Pre-test	1 lesson	1 lesson
Pre-questionnaire (15 min) and learning functions	19 lessons	15 lessons
Post-test	1 lesson	1 lesson
Post-questionnaires	At the beginning of the first lesson after the post-test	At the beginning of the first lesson after the post-test

Study IV concerned dealing with the theme of functions in outschool activities. This study provided an overview of the student competitions held over the three years, the topics of which were “Patterns from Function Graphs,” “Movement,” and “Fireworks for the Anniversary of the Estonian Republic”. When creating these competition entries, students had to work with line equations and graphs in the GeoGebra environment. The participants received a guide from the competition organizers, which included technical instructions on how to use GeoGebra. Based on these guidelines, the students used their creativity to create competition entries corresponding to the theme. In addition to their competition submissions, students completed a short questionnaire where they were asked for their opinions on the process of creating their competition work: what they found useful and what brought them joy during the process.

3.3 Ethical aspects

Before starting the experiment, the researcher contacted mathematics teachers and school management. She introduced herself and the planned research explaining what data was planned to be collected, that all data would be kept confidential, and that the names of the schools and participants in the study were not going to be mentioned when publishing the research results.

Teachers participated in the study voluntarily. Teachers’ preferences concerning participation with their students in experimental or control groups were taken into account as well. Teachers make their decisions considering their own experiences and their institution’s possibilities to use computers for teaching mathematics.

When asking parents for permission for their children’s participation in the study, it was explained what the research was, what data was planned to be collected, and that the names of the schools and participants in the study would not be mentioned when publishing the research results, all data would be kept confidential and that participation in the study was voluntary for the students. The parents were informed and gave their informed consent via the e-school system.

The researcher did not meet the students. The teacher introduced them the study. The students received from the researcher a cover letter that explained the research, and data collection and asked for consent. The students were told that their tests and questionnaires were visible only to the teacher and the researcher and that their names would not be published anywhere.

All students were required to participate in the lessons, but they could forego answering the questions in the questionnaire if they did not want to participate in the research. The students were asked for a verbal consent in class.

In the study, the students were asked for their opinions on learning mathematics. They signed their names on the questionnaires in order to combine the data from the pre- and post-tests and the pre- and post-questionnaires.

The only direct personal data that was collected was the name of the students. After entering the data in Excel, the names column was removed and replaced by a student code. The entire database was kept only on the researcher's password-protected computer and was not used by anyone else. The researcher kept the data with names and codes in a separate file in a case it would be necessary to go back to the basic data, i.e., students' tests or questionnaires.

According to Gall et al. (1996, p. 98), "Participants in experimental research are placed in different treatment conditions. /.../ The treatment group is likely to receive /.../ the opportunity to participate in an innovative program, while the control group receives /.../ a conventional program". In this study, all control groups could use computer-based study materials after the end of this study. For this reason, all learning materials used in the experimental classes were made available to the teachers who taught control classes, and they were encouraged to use these materials.

The students participated in the competition voluntarily. A short questionnaire about the preparation of the competition work was also part of the competition work. The students were asked for permission to publish the competition work. The best entries were published with the names of the authors. The students were not asked for personal data other than their names. The names of the students were not added to the database that was used to analyse the answers to the questionnaires.

3.4 Measures

This section first describes the measures for Studies I-III and then for Study IV (see subsection 3.4.4).

In Studies I-III, a pre- and post-test was used to evaluate learning results before and after learning functions, and a pre- and post-questionnaire was used to clarify students' attitudes about learning mathematics. The measures of Studies I-III are presented in Table 2.

Table 2. Measures in Studies I-III.

Instrument	**7th grade**	**9th grade**
Pre-test for experimental and control groups.	Before learning functions. Maximum 41 points.	Before learning functions. Maximum 50 points.
Post-test for experimental and control groups.	After learning functions. Maximum 41 points.	After learning functions. Maximum 50 points.
Pre-questionnaire for experimental and control groups	At the beginning of the lesson following the pre-test.	At the beginning of the lesson following the pre-test.
Post-questionnaire for experimental and control groups	At the beginning of the lesson following the post-test.	At the beginning of the lesson following the post-test.
Post-questionnaire only for the experimental group	In the lesson following the post-test.	In the lesson following the post-test.

A more detailed overview of the measures is given below.

3.4.1 Pre- and post-tests

To measure learning outcomes in Studies I-III, data were collected with pre- and post-tests, which were compiled by the researcher according to the national curriculum. Both the pre-test and the post-test were paper-and-pencil tests. The pre- and post-tests were different tests. The results of the pre-tests were used to check that experimental and control groups would not differ significantly in terms of their previous knowledge.

Both the pre-tests and the post-tests were evaluated by two experienced mathematics teachers to ensure the content validity of these tests. Cronbach's alpha characterizing the reliability (an indicator of internal consistency) of the 7th- and 9th-grade pre-tests was 0.76 and 0.87 and in the post-test 0.79 and 0.88, respectively.

3.4.1.1 Pre-tests

Pre-tests for the 7th and the 9th grades consisted of previously studied topics necessary for learning functions. These topics were repeated for two lessons in the seventh grade and for five lessons in the ninth grade before the pre-test.

The pre-test for the 7th grade (Studies I and II) had 7 tasks for which a maximum of 41 points could be obtained. Descriptions of the pre-test tasks are presented in Table 3.

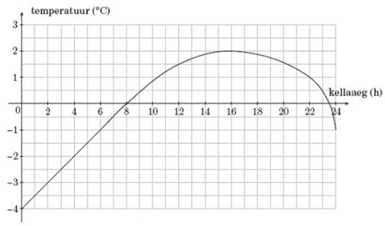
Table 3. Pretest for 7th grade.

- (7 points). Fill in the empty cells in the table.

x	-2	0.5		
$2 : x$			-2	
$-x + 2$			3	-2

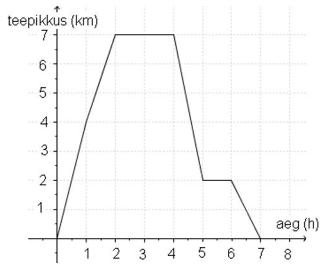
- (9 points). Look at the temperature graph and answer questions.
  - a) What was the air temperature at 12? What was it at 3?
  - b) When was the air temperature $-2°$? When was it $-1°$?
  - c) When was the temperature the lowest?
- (2.5 points). Give the previous result as a pair where the first number is the x value and the second is the corresponding y value: A(-2; ....), B(.....; .....), C(....; ....), D(....; ....), E(....; ....).
- (2.5 points). Draw points A, B, C, D and E, and connect the points by line.
- (7 points). Draw points $A(-2;-4)$ and $B(3;6)$ and a straight line through points A and B. Draw a point to this straight line and mark it with C. See the figure and write the coordinates of point C. Study the coordinates of points A, B and C. What similarities do you notice? Draw the points D and E so that the coordinates of D and E have the same characteristic as A, B and C.

- d) When was the air temperature positive?
- e) How many degrees did the air temperature change between 16 and 22 o'clock?



- (5 points). Calculate $y = 3x - 2$ for the values of
  $x = -2, -1, 0, 1, 2$. Complete the table.

- (8 points). The figure shows the movement of hikers. Check out this graph and write a little story based on this figure.



The pre-test for the 9th grade (Study III) had 9 tasks for which a maximum of 50 points could be obtained (Table 4).

Table 4. Pre-test for the 9th grade.

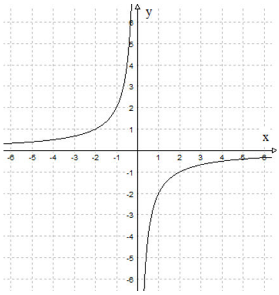
- (5 points). Simplify $(3x - 2)^2 + 2(6x - 2) - 18x^2$ and calculate its value when $x = -2$.

- (5 points) Fill in the table.

x	–3		0		0.5
$y = x^2 + 1$		1		5	

- (6 points) In the equation of the function y = $-2x+3$, the linear term is .... and the constant term is .... The graph of this function is ...., which intersects the y-axis at the point (......; ......). The value of the variable y of the function $y = -2x + 3$ is equal to zero if $x = \dots$
- (6 points) There is a graph of ...... dependence in the figure. This graph is called a ....

- (4 points) Solve the equation $2x^2 + 5x - 3 = 0$.
- (4 points) Draw the graph of the function $y = \frac{3}{x}$.
- (2 points) Draw in the same coordinate system the graph of the function $y = -4x$.
- (2 points) Mark the solution of the system of equations $y = -4x$ and y = $3/x$ on the plot and label it as A. The coordinates of point A are (....; .....).
- (8 points) Look at the figure and decide which line corresponds to which equation. Write the number of this line in the blank. Explain your decision.



Look at the figure and fill in the blanks:

if $x = -2$, then $y =$ ….;

if $x = 4$, then $y =$ ….;

if $y = -1$, then $x =$ …;

if $y = 2$, then $x =$ …;

- (4 points) In the previous exercise, the graph of the function $y = a/x$ is given. Using this figure find the value of a.
  a = …., so the equation of the function is $y =$…$/x$.

  Look at the figure and decide whether points

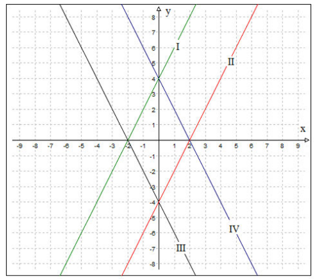
  $A(-5; -3)$ and $B(1; 2)$ lie on the graph of the function.

$y = -2x + 4$ ….;

$y = -2x - 4$ ….;

$y = 2x + 4$ ….;

$y = 2x - 4$ ….;



- (4 points) The graph of the function $y = ax$ passes through the point $C(2; 3)$. Calculate the value of a and determine whether this graph also passes through point $D(1; 6)$.

3.4.1.2 Post-tests

Post-tests in Studies I-III measured the learning results achieved. The post-tests were pencil and paper tests for both the control and experimental classes, no computer was used.

The post-test for the 7th grade (Studies I and II) had 7 tasks for which a maximum of 41 points could be obtained. Descriptions of the post-test tasks for 7th grade are presented in Table 5.

Table 5. Post-test for the 7th grade.

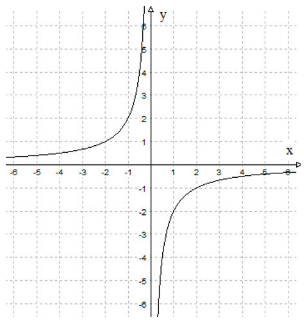
- (6 points) Fill in the table for the function $y = \frac{1}{3}x$

x	−3			0		2
y			1		−2	

- a) (4 points) Draw the graph of function $y = \frac{3}{x}$.
  b) (2 points) Draw the graph of function $y = -4x$.

- (8 points) In the figure you can see the graph of …. dependence, what is called …. .

  Fill in the blanks using the figure:



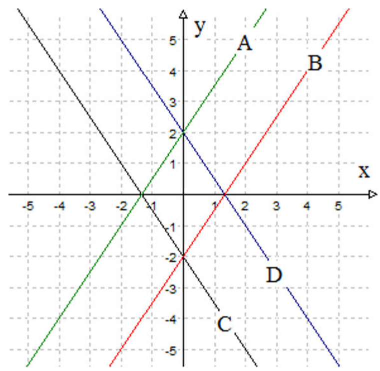
  if $x = -2, y = \cdots$; if $x = 4, y = \cdots$;
  if y= $-1, x = \cdots$; if y= $2, x = \cdots$.
  For which x values the y values are positive? ..... .

- (6 points) In the previous task is given the graph of function $y = \frac{a}{x}$; find the value of a using the figure.

  Draw the points $A(-5;\ -3)$ and $B(1;-2)$ on the figure of the previous task. See the figure and decide whether points $A(-5;\ -3)$; and $B(1;\ -2)$. are located on the graph of the function.

- (4 points) The graph of function $y=ax$ goes through point $C(2;\ -4)$. Calculate the value of a and decide if the graph of this function passes through point D (1; 4).
- (3 points) Without making the plot find the coordinates of three points so that they lie on the line $y = 7x - 2$.
- (8 points) See the figure and fit the equation and the line. Explain your choices.

  a) $y = -1{,}5x + 2$ …..;
  b) $y = 1{,}5x + 2$ …..;
  c) $y = -1{,}5x - 2$ …..;
  d) $y = 1{,}5x - 2$ …..;



The post-test for the 9th grade (Study III) had 9 tasks for which a maximum of 50 points could be obtained (Table 6).

Table 6. Post-test for the 9th grade.

- (5 points) Simplify $(3x-2)^2 + 2(6x-2) - 18x^2$ and compute its value when $x = -2$.
- (5 points) Fill in the table.

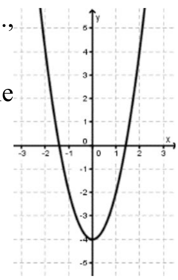
x	–3		–1		0,5
$y = x^2 + 1$		1		5	

- (6 points) Fill in the gaps.
- (4 points) In the previous task, the graph of the function $y = ax^2 + c$. Find the value of a and c using the figure:
  a = ..., c = ...; equation of the function is $y = \ldots$.

See the figure and decide whether these points are located on the graph of the function:

In the equation $y = x^2 - x - 6$, the square member is ...., the linear member is .... and the free member is ... . The graph of this function is ...., which crosses y-axes at the point where $y =$ .... The value of the function $y = x^2 - x - 6$ is equal to zero if x = ...

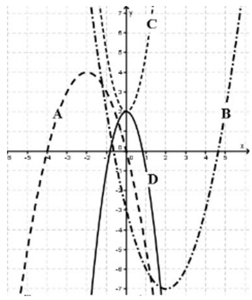
- (4 points) Draw the graph of the function $y = -x^2 + 4x$.
  (2 points) Draw at the same figure the graph of function $y = x$.
  (2 points) Mark and label the root(s) of the equation system $\begin{cases} y = x \\ y = -x^2 + 4x \end{cases}$ at the figure.
- (6 points) In the figure, you can see the graph of function y = ..., which is called .... .
  Fill in the gaps using the figure:
  if $x = 2$, then $y =$....;
  if $y = -2$, then $x =$....;
  zero crossings are ....;
  the vertex of the graph of the function is .... .



$A(-2;\ -3)$ .....; $B(-2; 4)$ ..... .

- (4 points) The graph of the function $y=ax^2$ passes through point $C(2, 8)$. Calculate the value of a and determine whether the graph passes point D (–1, 2).
- (8 points) See the figure and connect the equation and corresponding parabola. Justify the choice.

$y = 3x^2 + 2$ $\quad y = -3x^2 + 2$
$y = x^2 - 4x - 3$ $\quad y = -x^2 - 4x$



- (4 points) Graph of the function $y = ax^2$ passes through point C (2, 8). Calculate the value of a and determine whether the graph passes point D (–1, 2).

3.4.2 Questionnaires

In order to investigate students' attitudes towards learning mathematics and functions, pre- and post-questionnaires were used in Studies I and III. To explain how the use of the GeoGebra program influences students' attitudes in the experimental and control classes towards learning mathematics, both 7th- and 9th-grade students were asked to rate the cognitive components (necessity, difficulty) and affective components (interestingness and liking) of learning mathematics before studying the topic of functions. The students evaluated four statements on a five-point scale. In the post-questionnaire, they were asked to assess the same statements again, this time concerning their learning of the topic of functions (Table 7), to clarify what changes in students' attitudes are brought about by learning a more complex topic. Cronbach's alpha characterising the reliability of the questionnaires, in 7th-grade pre- and post-questionnaires was 0.74 and 0.80, respectively and in 9th-grade pre- and post-questionnaires, 0.80 and 0.77, respectively.

Table 7. Pre- and post-questionnaires for the 7th and 9th grade.

Question before studying the topic of functions	**Question after studying the topic of functions**
Studying maths is unnecessary (1)...necessary (5)	Studying functions is unnecessary (1)...necessary (5)
Studying mathematics is difficult (1)...easy(5)	Studying functions is difficult (1)...easy(5)
Studying mathematics is dull (1)...interesting(5)	Studying functions is dull (1)...interesting(5)
Do you like studying maths? I do not like (1)… I like (5)	Do you like studying functions? I do not like (1)… I like (5)

In addition, the students of the 7th- and 9th-grade experimental groups filled in a questionnaire with two multiple-choice questions about the lessons carried out in the computer lab and the learning resources used there. Also, there were three open questions where the students of the 7th- and 9th-grade experimental groups were asked what they liked or did not like while studying the topic of functions: "Have the lessons that took place in the computer lab changed your attitude towards mathematics? Please explain your answer," "Please describe what you liked about the lessons that took place in the computer lab," "Please describe what bothered you or what could have been different".

3.4.3 Study diary for teachers

All participating teachers in Studies I-III completed a study diary, received from the researcher. In their study diaries, teachers recorded for each lesson the names of students absent, student participation levels, the content of lessons, teaching methods used, and home assignments. Study diary data were used to exclude students who did not participate in the experimental class from the selection of the experimental class.

3.4.4 Questionnaire for Study IV

In Study IV, a questionnaire was used for each competition to obtain information on what the students thought they learned and what made them happy while doing the competition work. Students were asked two open-ended questions at each competition: "What useful things did you learn when creating the competition work?" and "What made you happy when creating the competition work?".

3.5 Learning materials for the experimental group

Worksheets for the experimental group were designed based on the national curriculum (Põhikooli …, 2011) and the results of previous studies (Godwin & Beswetherick, 2003; Weigand & Weller, 2001). For the conceptual understanding of functions, it is important to understand how different representations of functions are related (Kalchman & Koedinger, 2005). The teaching materials for the experimental lessons were designed to utilize GeoGebra dynamic geometry software, enabling students to actively explore and examine how modifying coefficients in a function's equation affects its graphical representation and how graphical changes relate back to the equation (Bray & Tangney, 2017; Godwin & Beswetherick, 2003; Kalchman & Koedinger, 2005; Nocar & Zdrahal, 2016; Ruthven et al., 2009). The worksheets included tasks and questions that, step by step, guided the students to explore functions, discover connections between different representations and draw conclusions using GeoGebra in order to avoid the students just observing the dynamic graphs in the GeoGebra environment without understanding the content. The worksheets themselves were paper-and-pencil-based, but the answers to the tasks on the worksheets needed to be found by using the GeoGebra program. For some tasks, the students had to create the graphs themselves using the GeoGebra program, for other tasks, a dynamic GeoGebra applet was prepared by the researcher. In these tasks, the coefficients in the equation of the graph could be changed using sliders and the change of the graph of the function as a result of moving the sliders could be observed at the same time. The learning resources prepared for the experimental classes are described below.

3.5.1 Learning materials for 7th grade (Study I and Study II)

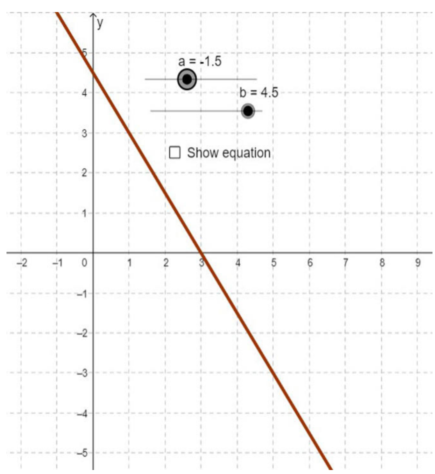
In the seventh grade, the first functions to be learned are proportional and inverse dependence and linear function. Worksheets were prepared for the experimental classes for four lessons in a computer lab. The rest of the classes were traditional.

The first experimental lesson was dedicated to proportional dependence $y = ax$. The students were asked to use GeoGebra program to investigate a proportional dependence graph, while the instructions and questions on the worksheet guided them towards the conclusion that the graph of proportional dependence $y = ax$ passes through points (0; 0) and (1; a). Next, a dynamic GeoGebra applet (www.geogebra.org/m/DJUg7CK2) was used to observe how changing the sign of the coefficient a in the equation $y = ax$ affects the location of the graph (straight line) in the Cartesian coordinates. The third task was more playful. The students had to guess the equation of the line shown on Geogebra, and then check the correctness of their answer.

In the second experimental lesson, the students investigated inverse dependency $y = a/x$. In the first task, the students had to use the GeoGebra dynamic applet (http://www.geogebra.org/m/WEk7375Q) to investigate how the shape and position of the graph (hyperbola) in the Cartesian coordinate system changes when the coefficient a changes. In the second task, the students investigated what happens if the coefficient a is zero. In the third task, the conclusion $xy = a$ is reached as a result of the observations. In the fourth task, the students had to guess the equation of the hyperbola shown on Geogebra, and then check the correctness of their answer.

In the third experimental lesson, the students investigated the linear function $y = ax + b$. They had to draw graphs of the functions $y = 2x – 5$, $y = 2x + 1$, $y = 2x – 4$, and $y = 2x$ in GeoGebra, the aim was that the students would notice that lines with equal linear terms ax are parallel. Next, the students drew the functions $y = x – 2$, $y = 2x – 2$, $y = –0.5x – 2$, $y = –4x – 2$ on GeoGebra and looking at graphs and equations they were led to notice that with an equal constant term b, all lines pass through point b on the y-axis. Next, the GeoGebra dynamic applet (Table 8) was studied. The values of the coefficients a and b were changed using the sliders and conclusions were made about the position of the line depending on the values of the coefficients a and b. Finally, the students had to guess the equation of the line shown on Geogebra, and then check the correctness of their answer.

Table 8. Example of the task from the third 7th-grade computer lesson.

Open www.geogebra.org/m/m4YVyuuF Move the point on slider b. This gives a particular value to the constant term b in the equation $y = ax + b$. Start changing the position of the point on slider a. This changes the value of the linear term coefficient a in the equation $y = ax + b$. Observe how changes the position of the line. Fill out the gaps. If $a < 0$, then the line is (*sloping up, down*) ... . If $a > 0$, then the line is ... . If $a = 0$, then the line is ... . In the case of the observed lines, the intersection of the line with ...-axis did not change, because ... did not change in the line equation. Now, use slider a to give a specific value to the linear term coefficient a and start changing the value of b. Observe how changes the position of the line. Fill out the gaps. All observed lines are ... to each other because ... did not change in the line equation. How are	

the values of the constant term and the intersection of the line with y-axis related to each other? …   Please read through everything above and fill out the gaps: Conclusion: The graph of a linear function is a ... and its equation is ... . The coefficient ... of linear member ... indicates ….. The constant term ... indicates ... .

In the fourth lesson, the experimental group students created patterns in GeoGebra using graphs of functions. To do this, the coefficients in the equations of the learned functions were changed using sliders, and the sliders were animated. A trace was left on the moving lines and the colour of the lines was made to depend on the values of the sliders. The animation was stopped at the moment when there was a beautiful pattern on the screen. The pattern was saved as an image (Figure 1). When doing such a task, the student needs to observe how changing the coefficients affects the shape of the lines and location in the coordinate system.

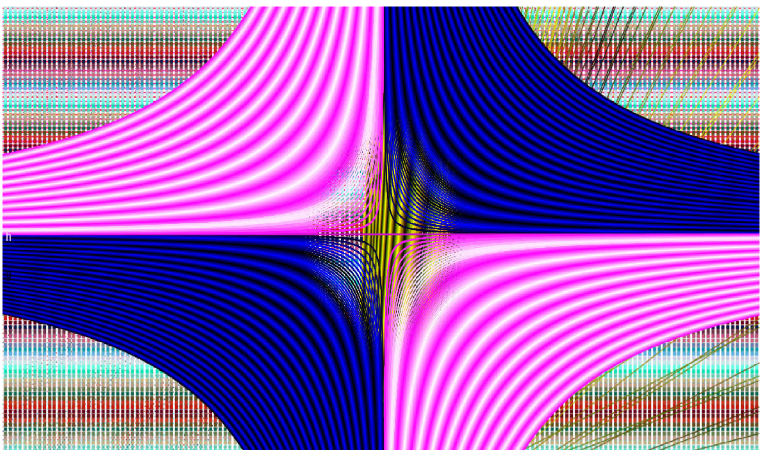


Figure 1. Example of a pattern (Pihlap, 2020).

3.5.2 Learning materials for 9th grade (Study III)

In the ninth grade, the quadratic function $y = ax^2 + bx + c$ was studied. Worksheets were prepared for three computer lessons. In the first lesson, the quadratic function was studied using the GeoGebra applet (Table 9). By changing the coefficients *a* and *b* using sliders, the position, shape, zeros, and peak of the parabola were examined.

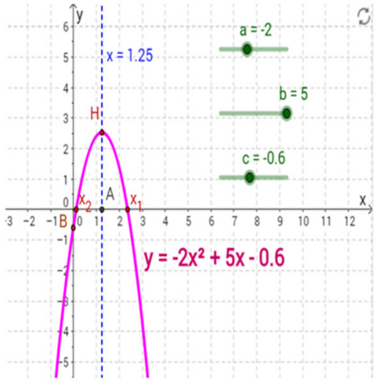
Table 9. Example of the task from the first 9th-grade computer lesson.

Open http://ggbtu.be/m319239

Leave sliders b and c at a particular value and start changing the value of a. Observe the changes in the equation and parabola and fill out the gaps:

If a has a positive value, then the parabola ….. .
If a has a negative value, then the parabola ….. .
The parabola narrows if …. .
The parabola widens if …. .
What happens on the graph if $a = 0$? Why does it happen? … .

Leave sliders a and b at a particular value and start changing the value of c. How does the parabola change (position, shape)? … .



In the second lesson, the students had to draw the graphs of the functions $y = 2x^2 + 1$ and $y = -2x^2 + 1$ in GeoGebra and change the position of these graphs in the Cartesian coordinates by dragging the graphs with the mouse. The students were directed to notice that the coefficient of the quadratic term does not change when the position of the parabola is changed in the coordinate system. Then the students had to draw the graphs of the functions $y = 3x^2 + 1$, $y = 3x^2 + 2$, $y = 3x^2 - 1$, $y = 3x^2 - 2$ and find what is the same in these equations and how this "same in the equations" could be seen in the figure. Also, what was different in these equations and how this "different in these equations" could be seen in the figure. Next, the same questions for the functions $y = 2x^2 - 1$, $y = 5x^2 - 1$, $y = 0.5x^2 - 1$, $y = -5x^2 - 1$, $y = -0.5x^2 - 1$, $y = -2x^2 - 1$ and then $y = -2x^2 + 3x + 2$, $y = 0.5x^2 + 2$, $y = -0.1x^2 + 2$, $y = 3x^2 - 7x + 2$ were asked.

In the third lesson, patterns from the graphs of learned functions were drawn in the GeoGebra program analogously to what was described in Study I.

3.6 Learning materials for the control group

In the control classes, traditional learning took place using textbooks, workbooks, and pen and paper. The students did not use computer programs. The textbooks and workbooks used were in line with the national curriculum. The same textbooks and workbooks were also used in the experimental classes. The main difference compared to the experimental group's GeoGebra lessons was that the control group did not use dynamic drawings.

3.7 Learning materials for students competitions

In Study IV, three competitions, "Patterns from Function Graphs," "Movement," and "Fireworks for the Birthday of the Estonian Republic," organized for students were introduced. When creating these competition works, the students needed to deal with function equations and graphs in the GeoGebra environment. For each competition, the students received instruction about the basic GeoGebra skills they used creatively in the competition work. Table 10 is a shortened example of one instruction. All three instructions are presented in the fourth article.

Table 10. Example of the instruction for creating patterns using graphs of functions.

Open *GeoGebra*. 1. Insert some *Sliders* in the *Graphics view*. Fill in the cells (*Min, Max, Increment*). Open the *Animation* tab and choose *Speed* and *Repeat*. 3. To make a pattern, type in an equation of a function in the *Input bar*. When doing this, use the slider names in the equation 4. Make a right-click in *Graphics view* and untick *Axes* and *Grid*. 5. Make a right-click on one of the graphs, choose *Object Properties* and follow the next steps: • Open the *Basic* tab, untick *Show Label* and tick *Show Trace.* • Open the *Style* tab and choose a *Line Thickness* and *Style.* • To get a nice pattern, you can make changes in the dynamic colours. For this open the *Advanced* tab and add *Dynamic Colors*. You can use the slider names you already have or insert new sliders. Make changes until you are happy with your picture. 6. Repeat the steps (see point 5) with other graphs you have created. 7. To animate the slider, right-click the slider and click *Animation On*. 9. Pause the animation when you have a picture that you like. 10. To export your picture, open the menu *File -> Export -> Graphics View as Picture*.

3.8 Data analysis methods

3.8.1 Statistical analysis

Three different kinds of statistical analyses were used in the Studies I-III (Table 11).

Table 11. Statistical analysis conducted in studies.

Statistical analysis	Study	Purpose of the analysis
Independent samples t-test	I, III	Comparison of control and experimental groups learning results. Comparison of control and experimental groups' attitudes.
Paired samples t-test	I, III	Comparison of the answers given to the questions before and after studying functions within the same group.
Chi-square test	II	Comparison of the distributions of the explanations in the experimental and control groups

To compare the average results of the experimental and control classes in the pre-test and the post-test, an independent samples t-test was performed (Study I and Study III). In addition, in Study III, based on the pre-tests, all students were classified into three groups to compare the impact of GeoGebra use on students with different performance levels. Instead of one standard deviation, it was decided to add and subtract half a standard deviation to avoid the weaker and the strongest groups being too small. To determine the middle group, half of the standard deviation (5.37) was deducted from and added to the average preliminary test score (32.26) so that the stronger and weaker groups would not be too small. Consequently, the weaker group (W) included students who scored less than 22 points in the preliminary test, the middle group scored 22–37.5 points, while the strongest group scored 38 or more points.

Before learning functions, both 7th- and 9th-grade students were asked to rate statements about the necessity, difficulty, interest and pleasantness of learning mathematics on a 5-point scale. In the final questionnaire, which was administered after learning the topic of functions, they were asked to rate the same statements, but about the necessity, difficulty, interest and pleasantness of learning functions. To compare the ratings given by the experimental and control groups for each statement, an independent sample t-test was performed for each statement. To compare scores given to statements about learning mathematics and functions within the same group, a one-sample t-test was performed for each statement (for example, the ratings given by the control group on the necessity of learning mathematics and learning functions were compared).

In Study II, to compare the percentages of correct and incorrect answers given by the students of the experimental and control groups in the task where the graph of a function and an equation had to be related, a chi-square test was performed. Also, the chi-square test was used to compare the distribution of explanations given by experimental and control students for how they matched the equation and graph of a linear function.

3.8.2 Content analysis

To analyse the open questions, inductive qualitative content analysis (Study I and Study III) or inductive quantitative content analysis (Study II) was performed. To increase the reliability of the coding, the students' solutions were coded by another researcher in addition to the author. To ensure confidentiality of the data, the co-coder did not see the names of the respondents; the names were replaced with codes. Differences in coding results were resolved through discussion until a consensus was reached. In Study II the students' solutions were transcribed, coded, and categorised. Responses within categories and subcategories were counted and descriptive statistics and diagrams were presented.

In Studies I and III, students were asked whether the use of computers had influenced their attitude toward mathematics. In this question, the students could also write an explanation of their answer. In Study I, the responses of seventh-grade students were divided into two main categories:

(1) Attitude improved (subcategories: better understanding; learning became easier, more convenient, and more interesting; less handwriting) and

(2) Attitude did not change (subcategories: attitude was already positive; it makes no difference whether learning takes place with or without computers; dislike of mathematics regardless of computer use; dislike of learning mathematics with computers).

Additionally, there was one student whose attitude toward mathematics became more negative.

In Study III, the responses of ninth-grade students were similarly divided into two categories:

(1) Attitude improved (subcategories: better understanding; learning became easier, more convenient, more enjoyable, and more engaging) and

(2) Attitude did not change (subcategories: attitude was already positive; it makes no difference whether learning takes place with or without computers; dislike of mathematics regardless of computer use; using computers was helpful).

There was also one student whose attitude toward mathematics became more negative.

In Study II, students were asked to explain their choices in a task that required matching the equation of a function ($y = ax + b$) with its corresponding graph. The students' explanations were classified into two main categories:

(1) Correct explanations (subcategories: based on coefficients a and b; using a table of x and y values) and

(2) Incorrect explanations (subcategories: misunderstanding of coefficient a; decision based only on coefficient a; decision based only on coefficient b; explanation unclear; no explanation provided).

4 An overview of empirical studies

In the following overview, four studies reported in the four articles are briefly described.

4.1 Study I

Pihlap, Sirje (2021). "Mathematics is not as difficult as I thought": The effects of computer use on learning linear functions and inverse variation. ***ICERI2021 Proceedings: 14th annual International Conference of Education, Research and Innovation Online Conference. 8-9 November, 2021.*** **Ed. L. Gómez Chova, A. López Martínez, I. Candel Torres. IATED Academy, 7663−7671.**

The article provides an overview of a study that explained the impact of using computers on learning outcomes and student attitudes when learning the topic of linear functions and inverse variation. Twelve grade 7 classes from different regions of Estonia participated in the study. The sample included a total of 212 students, of whom 84 were in the control group and 128 in the experimental group.

Before learning the topic of functions, a paper-and-pencil pre-test was taken to assess the students' prior knowledge necessary for learning the topic of functions. A comparison of the pre-test results revealed that the results of the experimental and control group students were not statistically significantly different. A total of 19 lessons were allocated to teach the topic of functions. In the experimental classes, students used the dynamic geometry program GeoGebra for 3-4 lessons of the allocated 19 lessons. The students using GeoGebra were directed to explore the relationships between different representations of a function using worksheets created by the researcher. The students in the control classes did not use computers to learn functions. The post-test was a paper-and-pencil test for everyone, and no computers were used. The results of the post-test in the experimental and control classes were not statistically significantly different.

The students were also asked to rate the necessity, difficulty, interest in and appeal of mathematics before and after learning the functions on a scale from one to five. In the pre-questionnaire in comparison between the experimental and control classes, there was a statistically significant difference in the assessment of the appeal of learning mathematics (means 3.64 and 3.14 respectively, $p = .002$). In the post-questionnaire, there was no statistically significant difference between the experimental and control groups in the evaluations of the enjoyability of learning the functions (means 2.89 and 2.69, respectively, t-test $p = .29$).

When comparing the answers given to the four aforementioned questions within the same group, the scores for learning mathematics in general were statistically significantly higher in both the experimental and control groups compared to the answers to the same questions about learning functions (paired samples t-test, $p<.001$ for all eight comparisons). According to the students, the topic of functions was less necessary, more boring, more difficult and less enjoyable than learning mathematics in general.

43% of the responders in the experimental group found that using computers changed their attitude towards mathematics. The attitude of one student had changed for worse, while the attitudes of the rest had improved. Four reasons emerged from those students whose attitudes had improved toward mathematics: (1) computers helped them understand the content better, noting that computers made learning easier; (2) they found the experience more convenient and interesting, which resulted in an increased interest in learning mathematics; (3) the students enjoyed that there was less manual writing in the computer lab classes; (4) for some students it was simply that they liked working with computers.

For the remaining 57% of respondents, the use of computers did not change their attitude towards mathematics. The following three reasons emerged: (1) they already had a positive attitude towards mathematics; (2) it made no difference whether computers were used for learning or not; (3) irrespective of the use of computers, they did not enjoy mathematics. The opinions of the students in the experimental group revealed that they felt that the computer lab classes helped them understand the topic and felt better about learning mathematics.

This study argues that these results support identifying suitable educational methods and resources that are crucial for maintaining students' motivation for learning. According to the opinions of the students in the experimental group, the lessons in a computer lab helped them understand the topic and enjoy learning mathematics. The fact that the attitude towards mathematics improved for 42% of the students in the experimental group after the computer lab lessons can be reported as significant since an improved attitude towards mathematics creates preconditions for better learning outcomes.

4.2 Study II

Pihlap, S., Veermans, K. and Hannula-Sormunen M. (2023). The impact of the use of a dynamic geometry program Geogebra on students' understanding of the concept of linear function. ***The International Journal for Technology in Mathematics Education,*** **30(4), 247−254.**

To develop the concept of a function, students must understand how the different representations of a function are related. Therefore, Study II aimed to clarify how the use of a dynamic geometry program, GeoGebra, in learning linear functions helps students understand the relationships between different representations of a linear function and the ability to plot a linear function.

In this study we analysed two tasks of the post-test of the study described in section 4.1. In the first analysed task, the students had to match the graph of a linear function and an equation. In addition to finding the correct pairs, the students had to explain how they arrived at their choice for what they believed to be the correct answer. 59.2% of the choices made by the students of the experimental group and 49.4% of the choices made by the students of the control group were correct. The result of the students in the experimental group was significantly better than that of the students in the control class (chi-square test, $p < .05$). The students' explanations were divided into correct and incorrect explanations. The students could make correct choices for both correct and incorrect explanations and wrong choices for both correct and incorrect explanations. Explanations were analysed independently of whether a student had made a correct or incorrect choice in matching the equation to the graph.

The correct explanations given by the students formed two subcategories. First, the students made their decisions according to the coefficient a of the linear term ax and the constant term b. Second, the students created a table of the values of the argument x and the function y. The most preferred method of explanation by the students in the experimental group was using the values of a and b (49.2%), while the students in the control group used the table the most (23.5%) (Figure 2).

The students who explained their choice based on the values of the coefficients a and b, used the following explanations: (a) the constant term b shows the point where the line intersects the y-axis and the sign of a shows whether the line is rising or falling; (b) the y-intercept of the line is the constant term b, to find the second point one move from the y-intersection point one unit to the right and a unit vertically; (c) b is the y-intercept and a indicates which the Cartesian coordinates quadrants that the line passes through; (d) the line is obtained from $y = ax$ by shifting it b units vertically.

The students who used a table to explain their reasoning did so in two ways: (a) they created a table based on the equation and looked at which line the found points belonged to; (b) they created a table based on the line and checked which equation the coordinates of the found points fit into.

Incorrect explanations (incorrect, unclear and unanswered) were divided into five categories: (a) it was assumed that in the equation $y = ax + b$, a is the intercept with the x-axis, and b is the intercept with the y-axis; (b) the decision was made only based on a constant term b; (c) only a was explained; (d) the wording of the explanation was unclear; or (e) no explanation was given.

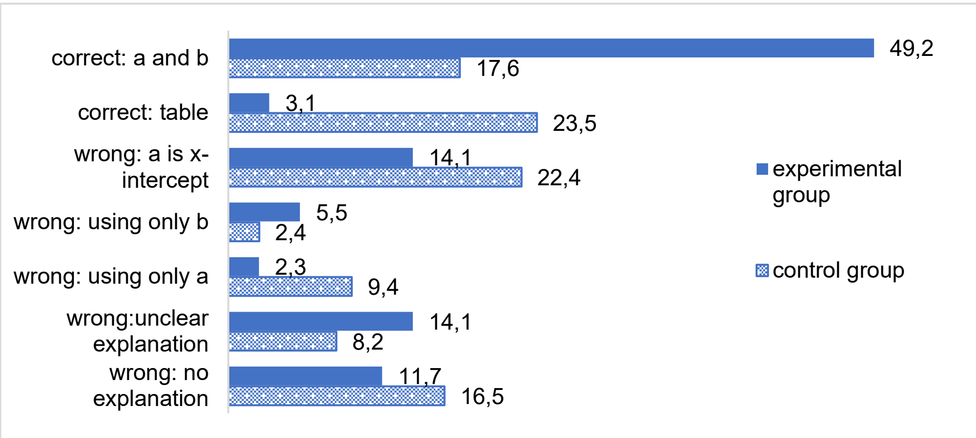


Figure 2. Percentages of explanations for the first task in the experimental and control groups.

In the second analysed task, the students had to draw a graph of a linear function given by an equation. The correct solutions were 65% in the experimental group and 63% in the control group. There was no statistically significant difference in the comparison of solution methods distributions (chi-square test, $p < .05$). Drawing pencil-and-paper graphs is an important skill for students to understand how the values of the independent variable x are related to the values of the dependent variable y (Godwin Beswetherick, 2003). In this study, in the experimental group, the GeoGebra program was used in addition to traditional teaching, which meant that the students also drew graphs with pen and paper and were still able to draw graphs without the use of a computer program.

The results of the study suggest that both the experimental and control groups of students require further development in their understanding and ability to explain linear functions. However, the experimental group performed better in matching the graph and equation of a linear function, indicating that teachers may benefit from integrating dynamic geometry programs in their instruction of linear functions. To ensure that computer use is effective, it is essential to design tasks that encourage students to seek connections and provide answers that make sense of what appears on the screen.

4.3 Study III

Pihlap, S. (2017). The Impact of Computer use on Learning of Quadratic Functions. The International Journal for Technology in Mathematics Education, 24 (2), 59–66. DOI: 10.1564/tme_v24.2.02.

Study III aimed to investigate the influence of computer use on the learning outcomes and the attitudes of 9^{th}-grade students who were learning quadratic functions. The quasi-experiment with pre-and post-tests was conducted across five Estonian schools, involving a total of 199 participants. Among them, 105 students were assigned to the control group while 94 students were assigned to the experimental group. In the experimental group, three out of fifteen lessons were dedicated to computer-assisted instruction, whereas the control group did not receive any computer-based instruction.

The pre-test consisted of tasks on topics that were necessary for learning the quadratic functions. The purpose of the pre-test was to clarify whether the prior knowledge of the experimental and control classes differed or not. The post-test had tasks on the topic of quadratic functions. A t-test was made to compare the mean scores of pre-and post-tests in experimental and control classes. The control group had somewhat better results in both the pre-and the post-test, but the difference was not statistically significant ($p > 0.05$).

Based on the pre-tests, the students were classified into three groups (weaker, medium and strongest) to compare the impact of computer use on students with different levels of performance. No statistically significant differences were found between the control and experimental groups in either the pre-test or the post-test in any level group.

The students were also asked to rate the necessity, difficulty, interest in and appeal of mathematics before and after learning the functions on a scale from one to five. There was no significant difference in the ratings of the necessity and difficulty of learning mathematics (functions) between the experimental and control classes. Also, there was no significant difference in the ratings of the interest in and appeal of learning mathematics, but the interest in and appeal of learning functions were rated higher by the students of the experimental classes, and this difference was statistically significant (t-test $p < .02$ and $p < .001$, respectively). The students who used computers found the learning of functions more interesting and they liked it more than the students who did not use computers.

Students in the experimental classes also completed a questionnaire on using computers as learning tools (n = 71). Among the 69 students who answered the main question, 35 (50.7 %) felt that computers greatly clarified lesson content and said they would like to keep using them; 30 (43.5 %) reported no noticeable difference between studying with or without computers; and 4 (5.8 %) considered computers a distraction and preferred not to use them for mathematics in the future.

When asked whether computer-based lessons had altered their overall attitude toward mathematics, 69 students responded. Twenty-four (34.8 %) said their attitude had changed – 23 for the better and 1 for the worse – while 45 indicated no change. Explanations from those whose attitudes improved included clearer understanding, easier learning, and lessons that were more interesting, enjoyable, convenient, or exciting. Students who reported no change typically said their attitude was already positive, that computer use made little difference, or that they simply did not like mathematics; a few added that working on a computer was helpful but that the number of computer lessons was too small to shift their views.

In sum, half of the respondents believed computers substantially deepened their understanding of functions, and roughly one-third said computer use improved their attitude toward mathematics. Given the widespread concern about students' limited appeal for mathematics noted in international studies (Lepmann 2006, 2011), this is a noteworthy finding.

4.4 Study IV

Pihlap, S., Kuresson, A., Kraav, T., Jukk, H. & Orav-Puurand, K. (2024). Exploring Mathematics with GeoGebra. *International Journal for Technology in Mathematics Education*, *31*(2), 89-95. DOI: 10.1564/tme_v31.2.07.

The article provides an overview of three student competitions organised in Estonia, the purpose of which was to popularise the use of the dynamic geometry program GeoGebra, to support the acquisition of the concept of function and the connection of mathematics with other subjects and everyday life. When creating these competition works, the participating students needed to deal with line equations and graphs in the GeoGebra environment. The study aimed to clarify the students' opinions about the preparation of the competition work: what useful pieces of knowledge did they learn and what made them happy in the process of creating the competition work.

Students from all over Estonia were invited to participate in the voluntary competition. The number of participants in the three competitions ("Patterns from Function Graphs," "Movement," and "Fireworks for the Anniversary of the Estonian Republic") was 232, 160 and 167, respectively.

In the competition “Patterns from Function Graphs”, the task was to create a pattern using graphs of functions learned in mathematics class, or some more interesting lines. To create such patterns, it was necessary to change the coefficients in the line's equation using sliders and then animate these sliders. In the GeoGebra program, it is possible to leave a trace on the line, which means that the positions of the line created by changing the coefficients in the equation remain visible. The animation was stopped at the moment when an interesting pattern emerged. When completing this task, a student needed to observe how changing the coefficients in the equation affects the shape and position of the graph in the coordinate system.

To participate in the competition “Fireworks for the Anniversary of the Republic of Estonia”, it was necessary to create a dynamic slide using the GeoGebra program, depicting fireworks. One way to create a fireworks effect is to use the sliders to move the point along a suitable line, such as a parabola. To make a point move along a line, a student had to understand how the coordinates of the point on the line were related to the equation of the line and be able to enter such coordinates in the GeoGebra program. Another option was to make the lines move and the points on the lines move along with them. If such lines were hidden, the points would still move. In creating this competition work, students had to understand how graphs are related to these equations and how point coordinates were related to line equations.

The task for the competition “Movement” was to animate an object or an image using the GeoGebra program, utilizing sliders, random numbers, or the animation of a point on a line.

To find out the students’ opinions, the competition work included a questionnaire, where the students were asked for their opinions about the creation of the competition work. The majority of the students found that they learned to use the GeoGebra program when creating the competition work. For some students the competition work was their first introduction to the program. Others had seen the program from afar, e.g., when the teacher had used it in class to visualise a topic, but the students had not used it themselves. Some students had previously used the program but learned about some new features.

By doing the competition work, the students learned to notice mathematics around them or the connection between mathematics and art. Their attitude towards mathematics improved, as it became clear that mathematics can be useful outside of the classroom. Some students realised that their mathematics skills improved while doing the competition work or they understood something that they had not grasped in class with paper and pencil.

In their responses, the students mentioned that they learned to be patient while working on the competition. Success came from staying calm and trying again if the initial result was not as expected.

In each competition, some students were delighted with the final result of their work. They enjoyed the feeling of accomplishment, even though the work might have been challenging initially. The students were pleased with the beauty of their completed work and the connection between mathematics and art; thus, participating in the competition fostered an integration between mathematics and art education.

The students were thrilled that, thanks to GeoGebra, they gained new knowledge about functions and hoped that they would have a better understanding of mathematics in the future.

The study concludes that participation in these competitions was beneficial and enjoyable for the students. According to Brown et al. (2008), "developing a positive attitude towards mathematics is a key aspect of students' learning, alongside attainment." This underscores the importance of fostering a positive perception of mathematics among students, as it influences their learning process.

The paper offers ideas for teachers on integrating such approaches into mathematics lessons, longer mathematics projects or projects integrating different subjects. The paper highlights possibilities to engage students with mathematics and make the subject more appealing.

5 Main findings and discussion

The present PhD thesis aimed to investigate (1) the impact of the use of the GeoGebra dynamic program on learning outcomes of the students who learn functions in lower secondary school *(Study I, II, III)*; (2) how do students participating in GeoGebra-assisted intervention and the students in traditional instruction explain their answers *(Study II)*; (3) students' attitudes and affects towards mathematics when using mathematical computer program Geogebra in two different contexts – inside school and outside school (*Study I, III, IV)*. In the following, the main results are presented based on the aims of the study.

First, in the seventh grade, the subject of linear function and inverse dependence was studied in experimental classes using the GeoGebra program. The control group did not use computers. Their learning took place traditionally using a textbook and paper and pencil. Overall, there was no statistically significant difference in the pre-tests or post-tests in the comparison between the experimental and control groups (*Study I*). However, in the task requiring students to match graphs corresponding to linear function equations, the experimental group's students performed significantly better in identifying correct matches compared to the control group's students (Study II). There was no significant intervention effect on the ability to draw a graph according to the equation using paper and pencil (*Study II*).

In the ninth grade, the quadratic function was studied and the control group did not use computers whereas in the experimental group, the students used the GeoGebra program in three of the lessons. There was no statistically significant difference in the pre-and post-test between the experimental and control groups (*Study III*).

Thus, the use of the GeoGebra program, in addition to traditional learning, did not affect the learning results significantly, except for the task where it was necessary to demonstrate a deeper understanding of the relationship between a linear function equation and a graph. In this task, the results of the students who used the GeoGebra program were significantly better compared to the students of the control group.

Second, to investigate how do students participating in computer-assisted intervention and the students in traditional instruction explain their answers, the students' explanations of the task where the equation of a linear function and a graph had to be matched were analysed (*Study II).* The students could make correct choices for both correct and incorrect explanations and wrong choices for both correct and incorrect explanations. The explanations were analysed independently of whether a student had made a correct or incorrect choice in matching the equation to the graph.

The explanations of the students who used the GeoGebra program were significantly different from the explanations of the students in the control group. While the preferred explanation method of the students of the experimental group was based on the meaning of the coefficients in the equation of a linear function, the students of the control group preferred to make a table of the values of the argument x and the function y and make a decision based on this table. There were also some similarities among the mistakes students made. One such mistake that was more common in the control group but also present in the experimental class, deserves attention – students thought that the coefficient a of the linear term in the equation of the linear function $y = ax + b$ indicates the intersection with the x-axis. There were also students in both the experimental and control groups who referred to only one coefficient in their explanation, either a or b instead of both. The analyses also revealed some other interesting results, for instance, that a quarter of the students in both the experimental and control groups either did not explain their choice, or the wording of the explanation was incomprehensible.

Third, to investigate students' attitudes and affects towards mathematics when using GeoGebra before starting with functions, the students of both 7th and 9th grade (*Study I, III*) were asked about necessity, difficulty, interest in and appeal of learning mathematics. After learning functions, the students were asked the same questions in connection to learning functions. In the 7th grade, a statistically significant difference between experimental and control groups was found in the assessment of the appeal of learning mathematics in the pre-questionnaire, where the assessment of the experimental group was significantly higher. However, there was no statistically significant difference between the experimental and control groups in the assessment of the appeal of learning functions in the post-questionnaire (*Study I*). In the 9th grade, there was no significant difference in evaluations for learning mathematics between the experimental and control groups in pre-questionnaire. However, in the post-questionnaire, the evaluations for learning quadratic functions were significantly higher in terms of interest and appeal in the experimental group compared to the control group. (*Study III*).

When comparing the answers given to the four aforementioned questions within the same group, the scores about learning mathematics were statistically significantly higher in both the experimental and control groups compared to the answers to the same questions about learning functions. Consequently, students in 7th and 9th grade rate learning functions as less necessary, more difficult, and more boring, and students like it less compared to how they rated mathematics in general before learning functions (*Study I, III*).

The students of the experimental group were asked whether the lessons with computers had changed their attitude towards mathematics. It turned out that 42% of the 7th-graders and 33% of the 9th-graders who responded to the question found that their attitudes improved (*Study I, III*). One seventh-grader and one ninth-grader had a worse attitude; the rest had a better attitude. The students whose attitude had improved explained that computers helped them understand the content better, they noted that GeoGebra made learning easier, more convenient and interesting, which resulted in an increased interest in learning mathematics. Some students just liked working with computers. The students whose attitude about mathematics did not change explained that they had already been positive about it. For some students, it was all the same regardless of whether computers were used for learning or not. Some students did not enjoy mathematics irrespective of the use of computers and some students did not like using computers in mathematics education.

Learning can also occur outside the classroom. In the student competitions, the task was to prepare a creative work related to real life or artwork using the GeoGebra program. This was done by using equations of functions or other lines and their graphs when preparing the artistic work (*Study IV*). The students who participated in the competitions reported that they gained useful knowledge about GeoGebra and mathematics, especially functions, and that their understanding of mathematics improved. Some, for instance, reported that they saw that mathematics is related to art and everyday life. The students also liked that they could relate mathematics to their field of interest. In addition, it turned out that they learned to be patient and to work hard for the goal. The students were delighted with the process of creating the competition work and the final result. Also, the knowledge that it was done, the feeling of success and pride in one's work.

5.1 Theoretical and methodological implications

The present set of studies has implications for theories on using ICT in learning functions.

First, many studies have found that learning with a dynamic geometry program helps students better understand the concept of functions because they can simultaneously observe changes in the coefficients of the function equation and the graph (Baki & Güveli, 2008; Göbel, 2021; Övez, 2018). The results of this study revealed that in the experimental group, where the GeoGebra program was used in addition to traditional learning, the post-test results were not statistically significantly different from those of the control group. However, students who learned using GeoGebra were better able to match the equation and graph of a linear function. Also, students themselves believe that they understood the topic of functions better when learning using GeoGebra, but this improved understanding was not significantly reflected in the post-test results. This discrepancy aligns with Soderstrom and Bjork's (2015) assertion that learning and performance should be distinguished: substantial learning may not always manifest in improved immediate performance, and conversely, immediate performance enhancements may not equate to genuine learning gains. Thus, it is possible that the final assessment tasks in this study did not effectively measure the specific knowledge and skills the students gained through the use of GeoGebra.

Second, according to Weigand and Weller (2001), the students who learned quadratic functions with computers (using the computer algebra program Derive) did not show better understanding, but these students developed a different understanding compared to those who worked with paper and pencil. In the present study, it was revealed that the students of the experimental and control groups related the equation of a linear function and the graph differently. The students in the experimental group preferred to look at the coefficients a and b in the equation of the line $y = ax + b$, while the students in the control group mostly made a table of x and y values and made decisions based on that. Since the GeoGebra lessons in the experimental group focused on studying the relationship between coefficients and graphs, this was the preferred method for the students in the post-test. Also, the possibility that the teachers' preference in explaining the topic could have influenced the students' explanations cannot be excluded.

Third, the Estonian PISA 2022 results revealed that students who explained their solutions in more than half of the lessons got above the national average result in mathematics (Jukk, 2023). Göbel (2021) investigated how technology-assisted guided discovery influences understanding the quadratic function's parameters. She found that the students did not succeed in explaining their solutions and more attention should be paid to this. Based on the results of the present study, it can be said that the explanations of the students of both the experimental and control groups about the relationship between the linear function equation coefficients and the graph need to be developed.

Fourth, several researchers (Hwang & Son, 2021; Imam, 2013; Manzana et al., 2019; Reed et al., 2010; Wiberg et al., 2024) have found that students' attitudes toward mathematics are positively related to learning outcomes. One way to make mathematics more enjoyable for students is to use technology. Hillmayr et al. (2020) conducted a meta-analysis, which showed that secondary school students who received instruction with the support of digital tools in science or mathematics developed notably more positive attitudes toward the subject compared to those who were taught without such technology. In the present study, it was found that 42% of the 7^{th}-grade and 33% of the 9^{th}-grade experimental group students reported an improvement in their attitudes towards mathematics after learning functions with GeoGebra. This is an important result from the point of view of learning mathematics, especially considering that the limited appeal of mathematics is often a problem identified in studies (Lepmann 2006, 2011; Reed et al, 2010).

Fifth, according to Reed et al. (2010), students' attitudes towards mathematics are influenced by their perception of the subject's difficulty. Based on the present study, students in 7th and 9th grade rate learning functions as more difficult compared to how they rated mathematics in general before learning functions, and students find learning functions less appealing. This may be because the topic of functions is more difficult than the previous topics studied in mathematics. If something is difficult to understand, it becomes less interesting, and students tend not to like it.

The present set of studies has implications for methodology on using ICT in learning functions. According to a meta-analysis by Hillmayr et al. (2020), the positive effects of using computers appeared more often in studies with imperfect methodologies, such as the absence of a control group or significant differences between the baseline levels of the experimental and control groups. In the present study, the experimental and control groups did not differ in terms of pre-test results. The lessons in the experimental group, which incorporated the GeoGebra program, were designed to emphasise conceptual understanding – specifically, the relationship between function coefficients and graphs (Chiu et al., 2001; Kalchman & Koedinger, 2005; Leinhardt et al., 1990). In this study, the students in the experimental group had the opportunity to see the graphs dynamically change according to the values of the equation coefficients in the GeoGebra environment.

The perception of improved understanding among the students might have enhanced their self-efficacy. According to Hannula et al. (2014), mathematical self-efficacy and achievement are interconnected, implying that students' increased confidence could positively influence their future performance. Therefore, simultaneously measuring judgement of learning, actual learning outcomes, and self-efficacy could provide more nuanced insights into the process and effectiveness of student learning.

5.2 Practical implications

The results of the four empirical studies have implications for practice.

First, the present results suggest that when teaching students about functions in lower secondary school, students can gain a better understanding when they have an opportunity to use a dynamic geometry program such as GeoGebra. The dynamic geometry program allows students to simultaneously look at different representations of a function (e.g., an equation and a graph) and investigate how changes in the coefficients of the equation affect the position of the graph in the coordinate system and, in the case of a quadratic function, the shape of the graph. The learning materials created for computer classes within the framework of this work give teachers ideas on how to guide students to explore and discover connections using the GeoGebra program. This is the basis for conceptual understanding of functions (Kalchman & Koedinger, 2005). Although ICT (information and communication technology) tools and opportunities to use them in mathematics education are changing rapidly, the learning and assessment materials used in the empirical studies of this dissertation can be easily adapted to new environments (e.g., GeoGebra Classroom, Amplify Classroom, etc.).

Second, the results of this work suggest that students' explanation skills need more attention. In order to develop these skills, in addition to solving tasks, students could also be directed to explain their solution process or way of thinking (Göbel, 2021). This gives the teacher valuable information about how students have understood the topic and what needs to be paid attention to. In addition, PISA 2022 Estonian results showed that students who regularly explained their reasoning in class performed the PISA 2022 test above the national average (Jukk, 2023).

Third, the present results, like Hwang and Son (2021), suggest that teachers investigate students' attitudes toward mathematics and provide appropriate support for developing a positive attitude toward mathematics. According to the results of this study, using the GeoGebra program helped students to feel that they understood the topic of functions better and made learning more enjoyable for them. According to the students who participated in the student competition, solving creative tasks related to real life and art provides a lot of challenge and joy from doing mathematics and teaches them patience and striving for a goal. The latter is also important in light of the results of PISA 2022, which revealed that the persistence of students has changed compared to 10 years ago. Less than a third (28%) of the students agreed with the statement that they would continue with the work they had started until they finished it. Ten years earlier, more than half of the students were so determined (Jukk, 2023).

Fourth, the results of the present study extend the current understanding of how to use the dynamic geometry program GeoGebra outside the classroom to learn functions. In this study, three student competitions were considered, where the GeoGebra program had to be used to prepare a creative work using the equations of functions or other lines (e.g., circles). From the opinions of the students who participated in the competitions, it was revealed that while doing the competition work, they began to see the connections between mathematics and art, and also saw that mathematics was all around them. The students felt that they began to understand mathematics better and started to like mathematics more. Pierce and Stacey (2011) note that many mathematics teachers seek to build on students' enjoyable experiences – particularly those linked to real-life or out-of-school contexts – in order to create a "halo effect," where the positive emotions associated with the experience carry over to the specific mathematical activity and, ultimately, foster a more favourable attitude toward mathematics as a whole. This study provides practical suggestions for teachers on how to incorporate such strategies into regular lessons, extended mathematics projects, or interdisciplinary activities.

5.3. Limitations

One limitation of the study lies in the variation of implementation across classes. In *Study III*, due to differing circumstances, two experimental classes were able to complete all three lessons in a computer lab; two other classes had access to the lab for only one lesson and completed the remaining two as independent home assignments; and one class held two lessons in the lab, with the third conducted as a home task. While this variation reduces the level of control in the study, it also reflects the realities of everyday school life, where unforeseen events and scheduling constraints impact lesson delivery.

Another limitation could be the small number of lessons where GeoGebra was used by the students. However, this number of lessons corresponded to real school life. In the present study, the GeoGebra program was used for 3-4 lessons. The technical capabilities of schools change quickly, and GeoGebra mobile applications have also been developed. It would be important to investigate how increasing the proportion of lessons that make use of GeoGebra in learning functions affects students' attitudes and learning outcomes.

The final tests for *Study I* and *Study III* included tasks on the entire topic of functions, including function-related skills. However, GeoGebra was only used to explore different representations of functions and the relationships between them. It is possible that a different final test would have better highlighted the learning effect of GeoGebra.

In addition, the assignment of students and teachers to experimental and control groups was not random, which means that some of the observed differences between the groups may be partly influenced by teacher-specific or classroom-related factors that cannot be entirely ruled out.

In the *Study II* post-test, the students were required to draw a graph of the linear function. A table for x and y values was also provided, allowing the students to calculate the coordinates of two points and draw a graph using these points. Without such a table, we would have gathered more insight into students' preferred methods for drawing a straight line.

In *Study III*, when the groups with different achievement levels were compared, the number of students in the weaker group of the control group was too small to produce accurate statistical results. To study students' attitudes, pre- and post-questionnaires were created (Study I, Study II), in which students were asked four questions about different aspects of attitudes (necessity, interestingness, difficulty, and pleasantness of mathematics/functions). The limitation of the questionnaires is the small number of questions. For example, Wen and Dube (2022) recommend a multidimensional definition of attitudes, according to which mathematics attitudes consist of specific cognitive aspects (value, gender roles/beliefs, confidence, self-concept), affects (enjoyment, anxiety), and behavioural intentions (i.e., willingness and tendency to spend more time learning mathematics). A more comprehensive questionnaire addressing a more extensive range of attitudes would help to better understand different aspects of students' attitudes.

Both *Study I* and *Study III* revealed that more than one-third of students' attitudes towards mathematics improved as a result of the lessons in the computer classroom. Here, it cannot be ruled out that students may have given the answer they thought was expected of them. The students did write about what they liked and what they did not like about the computer lessons, but to gain more clarity, the students could have also been interviewed, and the concept of attitude could have been discussed more broadly.

5.4. Challenges for future studies

The present set of studies investigated the impact of using the dynamic geometry program GeoGebra on learning functions in lower secondary school when GeoGebra was used in addition to traditional teaching. In Studies I-III, GeoGebra was used in addition to traditional teaching for 3-4 lessons, where the entire lesson was held in a computer lab using the GeoGebra program. Future research could examine how different ways of combining traditional teaching and GeoGebra use affect learning outcomes and students' attitudes towards learning mathematics. A future study could investigate how would learning outcomes and students' attitudes towards learning mathematics (functions) change if GeoGebra was used in more lessons, in different combinations with traditional teaching (e.g., some part of each lesson with GeoGebra).

Combining GeoGebra-based activities differently with traditional teaching approaches could also draw inspiration from the findings of Study IV, in which students completed creative works using GeoGebra. These works allowed the students to choose topics based on their personal interests or real-life phenomena (e.g., cycling, fireworks). Such an approach not only enhanced the students' proficiency with GeoGebra but also deepened their understanding of functions. Additionally, the students reported experiencing enjoyment alongside a sense of effort when working toward a goal. Based on these findings, integrating creative GeoGebra projects into traditional instruction could provide teachers with clear evidence of the educational value, helping them justify allocating additional classroom time to technology-enhanced activities.

While the possibility that the teachers' preference in explaining the topic could also have influenced the students' explanations cannot be excluded, it may also indicate that variations in instruction affect the strategies that students refer to or rely on in future interactions with similar problems. Therefore, if we have an idea of what we think students should do, then it may be of value to try to figure out an approach that makes it more likely.

In this study (see 5.1), students reported that using GeoGebra helped them better understand what they were learning. However, this was only slightly reflected in the learning outcomes. According to Soderstrom & Bjork (2015), significant learning can occur without any performance improvement. A future study could investigate the exact nature of the improved understanding that students reported gaining from the use of Geogebra. Furthermore, the perception of improved understanding among students might have enhanced their self-efficacy. According to Hannula et al. (2014), mathematical self-efficacy and achievement are interconnected, implying that students' increased confidence could positively influence their future performance. Therefore, simultaneously measuring judgement of learning, actual learning outcomes, and self-efficacy could provide more nuanced insights into the process and effectiveness of student learning.

It is also important to know what students find difficult and what mistakes they make. This will allow for appropriate changes in teaching (Tanışlı & Kalkan, 2018). Students' explanations of their solutions provide valuable information about errors. In the present study, the reason why some students thought that the coefficient a in the linear function expression $y = ax + b$ indicates the point of intersection of the line with the x-axis (i.e., the zero point) remained unclear. This would require further explanation.

Finally, a future study could also investigate how the use of GeoGebra affects learning in other topics (for example, geometry) and at other school levels.

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