PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 142, Number 2, February 2014, Pages 559–573
S 0002-9939(2013)11781-8
Article electronically published on October 25, 2013

BERNOULLI INEQUALITY AND HYPERGEOMETRIC FUNCTIONS

RIKU KLÉN, VESNA MANOJLOVIĆ, SLAVKO SIMIĆ, AND MATTI VUORINEN

(Communicated by Walter Van Assche)

Abstract. Bernoulli type inequalities for functions of logarithmic type are given. These functions include, in particular, Gaussian hypergeometric functions in the zero-balanced case $F(a, b; a + b; x)$.

1. Introduction

The Bernoulli inequality [Mit, p. 34] is often used in the following form: For $c > 1, t > 0$,

$$\log(1 + ct) \le c \log(1 + t)\,. \tag{1.1}$$

Recently, in the study of geometric function theory, the following variant of this classical result was proved in [KMV], where it was applied to estimate distortion under quasiconformal mappings.

Theorem 1.2 ([KMV, Lemma 3.1 (7)]). *For $0 < a \le 1 \le b$ let $\varphi(t) = \max\{t^a, t^b\}$. Then for $c \ge 1$ and all $t > 0$,*

$$\log(1 + c\varphi(t)) \le c \max\{\log^a(1 + t), b \log(1 + t)\}\,.$$

Note that for $a = b = 1$, Theorem 1.2 yields the classical Bernoulli inequality (1.1) as a particular case.

The goal of this paper is to study various generalizations of Theorem 1.2. The key problem is to find classes of functions which are of logarithmic type so that a counterpart of Theorem 1.2 holds. We formulate the following question. We write $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

Question 1.3. For $\phi(x) := \max\{x^a, x\},\ 0 < a < 1, x \in \mathbb{R}_+$, do there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1 \le \frac{\log^p(1 + \phi(x))}{\phi(\log^p(1 + x))} \le c_2,\ p > 0, \tag{1.4}$$

$$c_3 \le \frac{\log(1 + \phi(x)) \log(1 + \log(1 + \phi(x)))}{\phi(\log(1 + x) \log(1 + \log(1 + x)))} \le c_4\,? \tag{1.5}$$

Received by the editors July 21, 2011 and, in revised form, March 20, 2012.
2010 *Mathematics Subject Classification.* Primary 26D07, 33C05.
Key words and phrases. Log-convexity, hypergeometric functions, inequalities.
The fourth author was supported by the Academy of Finland, project 2600066611.

Our first main result is Theorem 1.6, which settles this question in the affirmative.

Theorem 1.6. *The inequalities* (1.4) *and* (1.5) *hold with the constants* $c_1 = (\log 2)^{p(1-a)}, c_2 = 1,\ c_3 = (\log 2 \log(1+\log 2))^{1-a}, c_4 = 1$.

The following proposition gives precise monotonicity intervals, and the proof of Theorem 1.6 is based on it.

Theorem 1.7. *Let* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be a differentiable function and for* $c \neq 0$ *define*

$$g(x) := \frac{f(x^c)}{(f(x))^c}.$$

We have the following:

1. *if* $h(x) := \log(f(e^x))$ *is a convex function, then* $g(x)$ *is monotone increasing for* $c, x \in (0,1)$ *or* $c, x \in (1,\infty)$ *or* $c < 0,\ x > 1$ *and monotone decreasing for* $c \in (0,1),\ x > 1$ *or* $c > 1, x \in (0,1)$ *or* $c < 0,\ x \in (0,1)$;
2. *if* $h(x)$ *is a concave function, then* $g(x)$ *is monotone increasing for* $c \in (0,1),\ x > 1$ *or* $c > 1,\ x \in (0,1)$ *or* $c < 0,\ x \in (0,1)$ *and monotone decreasing for* $c, x \in (0,1)$ *or* $c > 1, x > 1$ *or* $c < 0,\ x > 1$;
3. *if* $h(x)$ *is neither convex nor concave on* $\mathbb{R}_+$, *then* $g(x)$ *is not monotone on* $\mathbb{R}_+$.

Next we turn our attention to the Gaussian hypergeometric functions ${}_2F_1(a, b; c; x)$. Below we also use the simpler notation $F(a, b; c; x)$, omitting the subscripts. As is well-known, these functions have a logarithmic singularity at $x = 1$ for real positive triples (a, b, c) with $a+b = c$; see (2.7). Because of this logarithmic behavior in the zero-balanced case $c = a + b$, it is natural to expect that we might have a counterpart of Theorem 1.2 in this case, under suitable restrictions on (a, b, c). Our second main result reads as follows.

Theorem 1.8. *For* $c, d > 0$ *with* $1/c + 1/d \geq 1$, *the function defined for* $r \in (0,1)$ *and* $p > 0$ *by*

$$\omega(c, d, p, r) = \left(\frac{r^p}{1+r^p} F\left(c, d; c+d; \frac{r^p}{1+r^p}\right)\right)^{1/p}$$

is increasing in p. *In particular,*

$$\frac{\sqrt{r}}{1+\sqrt{r}} F\left(c, d; c+d; \frac{\sqrt{r}}{1+\sqrt{r}}\right) \leq \left(\frac{r}{1+r} F\left(c, d; c+d; \frac{r}{1+r}\right)\right)^{1/2}.$$

As we will explain in Section 3, (3.4), this result may be regarded as a Bernoulli type inequality for the zero-balanced hypergeometric function.

2. Properties of $F(a, b; c; x)$

In this section, we study some monotonicity properties of the function $F(a, b; c; x)$ and certain of its combinations with other functions. We first recall some well-known properties of this function which will be used in the sequel.

It is well-known that hypergeometric functions are closely related to the classical *gamma function* $\Gamma(x)$, the *psi function* $\psi(x)$, and the *beta function* $B(x, y)$. For $\operatorname{Re} x > 0,\ \operatorname{Re} y > 0$, these functions are defined by

$$\Gamma(x) \equiv \int_0^\infty e^{-t} t^{x-1} dt,\ \psi(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)},\ B(x, y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{2.1}$$

respectively (cf. [AS]). We recall the *difference equation* [AS, Chap. 6]

$$\Gamma(x+1) = x\Gamma(x), \tag{2.2}$$

and the *reflection property* [AS, 6.1.15]

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} = B(x, 1-x). \tag{2.3}$$

We shall also need the function

$$R(a,b) \equiv -2\gamma - \psi(a) - \psi(b), \;\; R(a) \equiv R(a, 1-a), \;\; R\left(\frac{1}{2}\right) = \log 16, \tag{2.4}$$

where γ is the *Euler-Mascheroni constant* given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = 0.577215\ldots. \tag{2.5}$$

For $|x| < 1$ the hypergeometric function is defined by the series expansion

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!},$$

where $(a,0) = 1, (a,n) = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1), n = 1, 2, \ldots,$ is the *Appell symbol* and $a, b, c \in \mathbb{R} \setminus \{0\}$. The differentiation formula ([AS, 15.2.1]) reads

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1, b+1; c+1; x). \tag{2.6}$$

An important tool for our work is the following classification of the behavior of the hypergeometric function near $x = 1$ in the three cases $a + b < c, a + b = c$, and $a + b > c$:

(2.7)
$$\begin{cases} F(a,b;c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \;\; a+b<c, \\[2ex] B(a,b)F(a,b;a+b;x) + \log(1-x) = R(a,b) + \mathrm{O}((1-x)\log(1-x)), \;\; a+b=c, \\[2ex] F(a,b;c;x) = (1-x)^{c-a-b}F(c-a, c-b; c; x), \;\; c<a+b. \end{cases}$$

Some basic properties of this series may be found in standard handbooks; see for example [AS]. For some rational triples (a, b, c), the function $F(a, b; c; x)$ can be expressed in terms of well-known elementary functions. For what follows, an important particular case is [AS, 15.1.3]

$$g(x) \equiv xF(1,1;2;x) = \log \frac{1}{1-x}. \tag{2.8}$$

It is clear that for $a, b, c > 0$ the function $F(a, b; c; x)$ is a strictly increasing map from $[0, 1)$ into $[1, \infty)$. For $a, b > 0$ we see by (2.7) that $F(a, b; a + b; x)$ defines an increasing homeomorphism from $[0, 1)$ onto $[1, \infty)$.

Theorem 2.9 ([ABRVV], [AVV1, Theorem 1.52])**.** *For $a, b > 0$, let $B = B(a, b)$ be as in* (2.1)*, and let $R = R(a, b)$ be as in* (2.4)*. Then the following are true:*

(1) *The function $f_1(x) \equiv \frac{F(a,b;a+b;x)-1}{\log(1/(1-x))}$ is strictly increasing from $(0, 1)$ onto $(ab/(a + b), 1/B)$.*

(2) *The function* $f_2(x) \equiv BF(a,b;a+b;x) + \log(1-x)$ *is strictly decreasing from* $(0,1)$ *onto* (R,B).
(3) *The function* $f_3(x) \equiv BF(a,b;a+b;x) + (1/x)\log(1-x)$ *is increasing from* $(0,1)$ *onto* $(B-1,R)$ *if* $a,b \in (0,1)$.
(4) *The function* f_3 *is decreasing from* $(0,1)$ *onto* $(R,B-1)$ *if* $a,b \in (1,\infty)$.
(5) *The function*

$$f_4(x) \equiv \frac{xF(a,b;a+b;x)}{\log(1/(1-x))}$$

is decreasing from $(0,1)$ *onto* $(1/B,1)$ *if* $a,b \in (0,1)$.
(6) *If* $a,b>1$, *then* f_4 *is increasing from* $(0,1)$ *onto* $(1,1/B)$.
(7) *If* $a=b=1$, *then* $f_4(x)=1$ *for all* $x \in (0,1)$.

We also need the following refinement for some parts of Theorem 2.9.

Lemma 2.10 ([PV, Cor. 2.14]). *For* $c,d>0$, *denote*

$$f(x) \equiv \frac{xF(c,d;c+d;x)}{\log(1/(1-x))}.$$

(1) If $c \in (0,\infty)$ *and* $d \in (0,1/c]$, *then the function* f *is decreasing with range* $(1/B(c,d),1)$.

(2) If $c \in (1/2,\infty)$ *and* $d \geq c/(2c-1)$, *then* f *is increasing from* $(0,1)$ *to the range* $(1,1/B(c,d))$.

Lemma 2.11 ([K, Thm. 1.5]). *If* $\max\{a,b\} \leq c$, *then the coefficients of the Maclaurin power series expansion of the ratio* $F(a+1,b+1,c+1;x)/F(a,b,c;x)$ *form a monotone decreasing and convex sequence.*

3. Heuristic considerations

We now apply Theorem 2.9 to demonstrate that the behavior of the hypergeometric function $F(a,b;c;x)$ in the zero-balanced case $c=a+b$ is nearly logarithmic in the sense that some basic identities of the logarithm yield functional inequalities for the zero-balanced function.

Fix $x \in (0,1)$ and, for a given $p>0$, a number $z \in (0,1)$ such that

$$\log\frac{1}{1-x} = xF(1,1;2;x) = \log\left(\frac{1}{1-z}\right)^p = p\log\frac{1}{1-z}\,.$$

Therefore $z = 1-\sqrt[p]{1-x}\,$.

Lemma 3.1. *For* $c,d \in (0,1]$ *define* $h(x) = xF(c,d;c+d;x)/\log(1/(1-x)), x \in [0,1)$ *and let* $p \geq 1$, $B=B(c,d)$. *Then for all* $x \in (0,1), z = 1-\sqrt[p]{1-x}$,

$$B \geq B\,h(z) \geq B\,h(x) \geq 1 \quad \text{and} \quad F(c,d;c+d;z) \geq (1/p)F(c,d;c+d;x)\,,$$

with equality for $c=d=1$.

Proof. Observe that for $p \geq 1$,

$$0 < z = 1-\sqrt[p]{1-x} \leq x,$$

and hence the result follows from Theorem 2.9 (5). The equality statement follows from Theorem 2.9 (7). ∎

Next, writing the basic addition formula for the logarithm

$$\log z + \log w = \log(zw), \quad z, w > 0\,,$$

in terms of the function g in (2.8), we have

$$g(x) + g(y) = g(x + y - xy), \quad x, y \in (0, 1).$$

Based on this observation and some computer experiments we pose the following question:

Question 3.2. (1) Fix $c, d > 0$ and let $g(x) = xF(c, d; c + d; x)$ for $x \in (0, 1)$ and set

$$h(x, y) = \frac{g(x) + g(y)}{g(x + y - xy)}$$

for $x, y \in (0, 1)$. For which values of c and d is this function bounded from below and above?

(2) Is it true that

a) $h(x, y) \geq 1$, if $cd \leq 1$?
b) $h(x, y) \leq 1$, if $c, d > 1$?

(3) Can the difference

$$d(x, y) = g(x) + g(y) - g(x + y - xy)$$

be bounded by some constants depending on c, d only?

Recall that by the Bernoulli type inequality of Theorem 1.2 we have

$$\log(1 + \sqrt{r}) \leq \log^{1/2}(1 + r) \tag{3.3}$$

for all $r \in (0, 1)$. In terms of (2.8) this reads as

$$\frac{\sqrt{r}}{1 + \sqrt{r}}\, F\left(1, 1; 2; \frac{\sqrt{r}}{1 + \sqrt{r}}\right) \leq \left(\frac{r}{1 + r}\, F\left(1, 1; 2; \frac{r}{1 + r}\right)\right)^{1/2} \tag{3.4}$$

for all $r \in (0, 1)$.

Question 3.5. Fix $c, d \in (0, 1]$ and let

$$\omega(c, d, p, r) = \left(\frac{r^p}{1 + r^p}\, F\left(c, d; c + d; \frac{r^p}{1 + r^p}\right)\right)^{1/p} \tag{3.6}$$

for $r \in (0, 1), p > 0$. Is it true that for each r, $\omega(c, d, p, x)$ is increasing in p? If this holds, then (3.4) would be a special case of it.

The answer to this question is given in Theorem 4.4.

According to formula (2.7) the function $F(c, d; c+d; x)$ has logarithmic behavior when x is close to 1. This suggests that we may expect a Bernoulli type inequality to hold for this function.

For what follows we fix $a, b \in (0, \infty)$ with $0 < a \leq 1 \leq b$ and write for $t > 0$

$$\varphi(t) = \max\{t^a, t^b\}. \tag{3.7}$$

Next we will rewrite the inequality in Theorem 1.2 for the function g in (2.8) when $c = 1$ and denote

$$g(x) = \log(1 + \varphi(r)) \equiv A,$$

implying $r = \varphi^{-1}(x/(1-x))$. We now also require, in concert with Theorem 1.2, that

$$A \le b \max\{\log^a(1+r), \log(1+r)\}$$

or, equivalently,

$$\begin{aligned} g(x) &\le b \max\left\{\log^a\left(1+\varphi^{-1}\left(\frac{x}{1-x}\right)\right), \log\left(1+\varphi^{-1}\left(\frac{x}{1-x}\right)\right)\right\} \\ &= b \max\left\{g^a\left(\frac{u}{1+u}\right), g\left(\frac{u}{1+u}\right)\right\}, \end{aligned}$$

where $u = \varphi^{-1}(x/(1-x))$ and g is given in (2.8). Now set $\varphi(s) = x/(1-x)$, i.e. $x = \varphi(s)/(1+\varphi(s))$, and we have for the function g in (2.8)

$$g\left(\frac{\varphi(s)}{1+\varphi(s)}\right) \le b \max\left\{g^a\left(\frac{s}{1+s}\right), g\left(\frac{s}{1+s}\right)\right\} \tag{3.8}$$

for $s > 0$. On the basis of this discussion we ask the following question:

Question 3.9. Let $c, d > 0$ and $g(x) = xF(c, d; c+d; x)$. Under which conditions on c and d do we have that for all $s > 0$,

$$g\left(\frac{\varphi(s)}{1+\varphi(s)}\right) \le \frac{b^2}{a} \max\left\{g^a\left(\frac{s}{1+s}\right), g\left(\frac{s}{1+s}\right)\right\}, \tag{3.10}$$

where $\varphi(s)$ is as in (3.7)?

On the basis of (3.8) we expect that there are numbers $c_1, c_2 \in (0, \infty)$ such that $0 < c_1 \le 1 \le c_2$ and (3.10) holds for all $c, d \in (c_1, c_2)$.

Question 3.11. Let g be as in Question 3.9. Is the generalized version of the Bernoulli inequality,

$$g(x) \le b(1+b-a)\varphi\left(g\left(\frac{\varphi^{-1}(x/(1-x))}{1+\varphi^{-1}(x/(1-x))}\right)\right),$$

where $\varphi(x) = \max\{x^a, x^b\}$, $\varphi^{-1}(x) = \min\{x^{1/a}, x^{1/b}\}$, $c, d \in (0,1)$ and $0 < a < 1 < b$, true?

Mathematica tests show that the function

$$t(x) = \frac{g(x)}{b\varphi\left(g\left(\frac{\varphi^{-1}(x/(1-x))}{1+\varphi^{-1}(x/(1-x))}\right)\right)}$$

consists of three parts: $(0, \min\{\alpha, \beta\})$, $(\min\{\alpha, \beta\}, \max\{\alpha, \beta\})$ and $(\max\{\alpha, \beta\}, 1)$. We easily obtain that $\alpha = 1/2$, because then $\varphi^{-1}(x/(1-x)) = 1$. Note that β is the solution of

$$g\left(\frac{\varphi^{-1}(x/(1-x))}{1+\varphi^{-1}(x/(1-x))}\right) = 1.$$

(1) When is $\beta > 1/2$?

Is it true that

(2) $t(x)$ is monotone on each interval $(0, \min\{\alpha, \beta\})$, $(\min\{\alpha, \beta\}, \max\{\alpha, \beta\})$ and $(\max\{\alpha, \beta\}, 1)$?
(3) $t(x) \ge \min\{t(1/2), t(1-)\}$?
(4) $t(x) \le t(\beta)$?

4. Answers to the questions of Section 3

Putting $\frac{x}{1-x} = s$, we have to show that $t(s) \le b$ with

$$t(s) := \frac{g(\frac{s}{1+s})}{\varphi\left(g\left(\frac{\varphi^{-1}(s)}{1+\varphi^{-1}(s)}\right)\right)}, s \in (0, \infty).$$

The main tool for determining the best possible bounds of $t(s)$ is given by Theorem 1.7. Therefore we have to investigate convexity/concavity property of the function $G(u) := \log g\left(\frac{e^u}{1+e^u}\right)$.

We are in a position to formulate the following result.

Theorem 4.1. *Let $c, d > 0$ and $g(x) = xF(c, d; c + d; x)\,, x \in (0, 1)\,$. The function $G(u) := \log g\left(\frac{e^u}{1+e^u}\right)$ is concave on $(-\infty, +\infty)$ if and only if $1/c + 1/d \ge 1$.*

Proof. Let us consider the function G' with $\frac{e^u}{1+e^u} = y,\ y \in (0, 1)$, and write it as

$$G'(y) = 1 - y + y(1 - y)\frac{F'(y)}{F(y)}.$$

Since $F'(x) = \frac{cd}{c+d}F(c + 1, d + 1; c + d + 1; x)$, applying Lemma 2.11 we get

$$\frac{F'(x)}{F(x)} = \sum_0^\infty a_n x^n, \tag{4.2}$$

where the sequence $\{a_n\}$ is monotone decreasing and convex, with $a_0 = \frac{cd}{c+d}$.

Hence

$$G'(y) = 1 + (a_0 - 1)y + \sum_1^\infty (a_n - a_{n-1})y^{n+1}$$

and

$$G''(y) = \frac{cd}{c+d} - 1 + \sum_1^\infty (n + 1)(a_n - a_{n-1})y^n < 0,$$

since $\frac{cd}{c+d} \le 1$.

Therefore $\log g(y)$ is concave on $(0, 1)$ and, consequently, $G(u)$ is concave on $(-\infty, +\infty)$. The proof is complete. □

Remark 4.3. Note that if in the proof of Theorem 4.1, $\frac{cd}{c+d} = 1+\epsilon, \epsilon > 0$, then $G''(y)$ is positive for sufficiently small y and G has an inflection point since $\lim_{y\to 1^-} G''(y) < 0$. Therefore the condition $c + d \ge cd$ is necessary and sufficient for G to be concave over the whole interval.

The necessary tool for answering Questions 3.5-3.11 is established.

We shall give in the sequel a positive answer to Question 3.5 under the condition that $1/c + 1/d \ge 1$, which includes the proposed case $c, d \in (0, 1]$.

Theorem 4.4. *Under the condition $1/c + 1/d \ge 1$, the function ω, defined above in Question 3.5, is monotone increasing in p.*

Proof. Denote equivalently

$$\omega(p) = \left(g\left(\frac{e^{pt}}{1+e^{pt}}\right)\right)^{1/p}, \quad t<0, p>0,$$

where $g(x) := xF(c,d,c+d;x) = xF(x)$.

We get

$$\frac{\omega'}{\omega} = (\log\omega)' = \left(\frac{\log(g(\frac{e^{pt}}{1+e^{pt}}))}{p}\right)' = \frac{\Omega(p)}{p^2},$$

with

$$\Omega(p) := pt\frac{e^{pt}}{(1+e^{pt})^2}\frac{g'\left(\frac{e^{pt}}{1+e^{pt}}\right)}{g\left(\frac{e^{pt}}{1+e^{pt}}\right)} - \log g\left(\frac{e^{pt}}{1+e^{pt}}\right).$$

Changing the variable $\frac{e^{pt}}{1+e^{pt}} := x, x \in (0,1/2)$ and recalling the definition of g, we obtain

$$\begin{aligned}\Omega(x) &= x(1-x)\left(\frac{F(x)+xF'(x)}{xF(x)}\right)\log\frac{x}{1-x} - \log(xF(x)) \\ &= \left(1-x+x(1-x)\frac{F'(x)}{F(x)}\right)\log\frac{x}{1-x} - \log x - \log F(x).\end{aligned}$$

From the proof of Lemma 2.11, we derive the following inequalities:

$$(1-x)\frac{F'(x)}{F(x)} < a_0; \quad \log F(x) < -a_0\log(1-x).$$

Noting that $\log\frac{x}{1-x} < 0$ for $x \in (0,1/2)$, we get

$$\begin{aligned}\Omega(x) &> (1-x+a_0x)\log\frac{x}{1-x} - \log x + a_0\log(1-x) \\ &= (a_0-1)\left(x\log\frac{x}{1-x} + \log(1-x)\right) \\ &= (1-a_0)\left(x\log\frac{1}{x} + (1-x)\log\frac{1}{1-x}\right) \geq 0,\end{aligned}$$

since

$$1-a_0 = 1-\frac{cd}{c+d} = \frac{cd}{c+d}\left(\frac{1}{c}+\frac{1}{d}-1\right) \geq 0.$$

Therefore $\omega' > 0$ and $\omega(p)$ is monotone increasing, as required. □

Remark 4.5. It is evident from the proof of Theorem 4.4 that the function $\omega(p) := (g(x^p/(1+x^p)))^{1/p}, 0<x<1, p>0$, is monotone increasing in p for any $g(x) = x{}_2F_1(a,b,c;x), a,b,c>0$ and $ab \leq c, \max\{a,b\} \leq c$.

The same is valid for the conclusion of Theorem 4.1.

Note also that the function $\omega(p)$ is not monotone in p for $x>1$. For example, in the case $(c,d,x) = (1,1,4)$, when g is as in (2.8), we obtain $\omega(1) = \log 5 \approx 1.61$, $\omega(2) = (\log 17)^{1/2} \approx 1.68$ and $\omega(4) = (\log 257)^{1/4} \approx 1.53$.

Remark 4.6. From Theorem 4.1 it follows that the function $g(e^x/(1+e^x))$ is log-concave on $\mathbb{R}$. In particular, the function $g(x^p/(1+x^p))$ is log-concave in p, that is,

$$g\left(\frac{x^p}{1+x^p}\right)g\left(\frac{x^q}{1+x^q}\right) \leq g^2\left(\frac{x^{(p+q)/2}}{1+x^{(p+q)/2}}\right), x>0, p,q \in \mathbb{R}.$$

Also, from Theorem 4.4 we get that the function $\frac{\log g\left(\frac{x^p}{1+x^p}\right)}{p}$ is monotone increasing in p; that is, $\log g(\frac{x^p}{1+x^p})$ is sub-additive on $\mathbb{R}_+$. Hence,

$$g\left(\frac{x^p}{1+x^p}\right) g\left(\frac{x^q}{1+x^q}\right) \le g\left(\frac{x^{p+q}}{1+x^{p+q}}\right), p, q > 0, 0 < x < 1.$$

Both corollaries are valid for the class of functions g defined in Remark 4.5.

An answer to part (1) of Question 3.11 is given by the following assertion.

Theorem 4.7. *Let β be as in Question 3.11. We have that*
(1) $\beta > 1/2$ *if* $(a_0 - 1)/h \le c_0$*;*
(2) $\beta < 1/2$ *if* $(a_0 - 1)/h \ge c_1$*,*

where

$$c_0 = 1 - \frac{1}{2\log 2} \approx 0.27865;\ c_1 = \frac{1}{\log 2} - 1 \approx 0.4427 \text{ and } a_0 = \frac{cd}{c+d},\ h = \frac{a_0^2}{c+d+1}.$$

Proof. (1) First note that the functions g, φ and φ^{-1} are strictly increasing. Therefore $\beta > 1/2$ if $g(1/2) < 1$ and vice versa.

From the relation (4.2) we obtain

$$(1-t)\frac{F'(t)}{F(t)} = a_0 + \sum_1^\infty (a_n - a_{n-1})t^n \le a_0 - (a_0 - a_1)t,$$

since $\{a_n\}$ is a monotone decreasing sequence.

Therefore,

$$\frac{F'(t)}{F(t)} \le a_0 \frac{1}{1-t} - (a_0 - a_1)\frac{t}{1-t},$$

and, integrating over $[0, x]$, we get

$$\log F(x) - \log F(0) \le -a_0 \log(1-x) + (a_0 - a_1)(x + \log(1-x)).$$

Putting $x = 1/2$, one can see that the condition $g(1/2) \le 1$ is satisfied if

$$a_0 \log 2 + (a_0 - a_1)(1/2 - \log 2) \le \log 2,$$

i.e.,

$$\frac{a_0 - 1}{a_0 - a_1} \le 1 - \frac{1}{2\log 2} = c_0.$$

By the above remark we have that in this case $\beta > 1/2$.

(2) Since $\{a_n\}$ is a convex sequence we conclude that $\{a_{n-1} - a_n\}$ is a monotone decreasing sequence.

Hence

$$(4.8) \qquad (1-t)\frac{F'(t)}{F(t)} = a_0 - \sum_1^\infty (a_{n-1} - a_n)t^n$$
$$\ge a_0 - (a_0 - a_1)t(1 + t + t^2 + \cdots) = a_0 - (a_0 - a_1)\frac{t}{1-t}.$$

Therefore,

$$\frac{F'}{F} \ge (2a_0 - a_1)\frac{1}{1-t} - (a_0 - a_1)\frac{1}{(1-t)^2},$$

and, integrating over $t \in [0, x]$, we get

$$\log F(x) \ge -(2a_0 - a_1)\log(1-x) - (a_0 - a_1)\frac{x}{1-x}.$$

Putting that $x = 1/2$ we see that the condition $g(1/2) \geq 1$ is satisfied if

$$(2a_0 - a_1)\log 2 - (a_0 - a_1) \geq \log 2,$$

which is equivalent to

$$\frac{a_0 - 1}{a_0 - a_1} \geq \frac{1}{\log 2} - 1 = c_1.$$

From (4.8) we get

$$(1-t)F'(t) = F(t)\left(a_0 - \sum_1^\infty (a_{n-1} - a_n)t^n\right),$$

i.e.,

$$\frac{cd}{c+d}F(c+1, d+1, c+d+1; t) = F(c, d, c+d; t)\left(a_0 - \sum_1^\infty (a_{n-1} - a_n)t^n\right),$$

and comparing power series coefficients, we easily obtain

$$a_0 = \frac{cd}{c+d}; \quad a_0 - a_1 = h = \frac{c^2d^2}{(c+d)^2(c+d+1)}.$$

□

Corollary 4.9. *We have that $g(1/2) < 1$ if $a_0 \leq 1$.*

Remark 4.10. Note that the condition $cd \leq 1$ implies $1/c + 1/d \geq 1$, that is, $a_0 \leq 1$. Therefore Theorems 4.4 and 1.7 could be applied to the expression $T(s)$. Moreover, by Corollary 4.9 we have

$$g(1/2) < 1 = g\left(\frac{\gamma}{1+\gamma}\right),$$

where γ is the unique solution of the equation $g(\frac{s}{1+s}) = 1$, and, since g is a monotone increasing function, we conclude that $\gamma > 1$.

The following assertion is a counterpart to Theorem 4.4.

Lemma 4.11. *For fixed $s > 0$ and $c, d > 0$ with $cd \leq 1$, the function $\frac{g(\frac{s^p}{1+s^p})}{p}$ is monotone decreasing in p, $p \in (0, \infty)$.*

Proof. Denote equivalently

$$w(p) = \frac{g(\frac{e^{pt}}{1+e^{pt}})}{p}, \ p > 0, \ t \in \mathbb{R}.$$

We have

$$p^2 w'(p) = pt\frac{e^{pt}}{(1+e^{pt})^2}g'(\frac{e^{pt}}{1+e^{pt}}) - g(\frac{e^{pt}}{1+e^{pt}}) := A(p).$$

Changing the variable $\frac{e^{pt}}{1+e^{pt}} = x$, we get

$$A(x) = x(1-x)\log\frac{x}{1-x}g'(x) - g(x), \ 0 < x < 1.$$

Now, Lemma 2.10 tells us that the function $\frac{g(x)}{-\log(1-x)}$ is monotone decreasing if $cd \le 1$, and this is equivalent to $g(x) \ge -(1-x)\log(1-x)g'(x)$. Therefore,

$$\begin{aligned} A(x) &\le (1-x)g'(x)(x\log\frac{x}{1-x} + \log(1-x)) \\ &= -(1-x)g'(x)(x\log\frac{1}{x} + (1-x)\log\frac{1}{1-x}) \\ &\le 0, \end{aligned}$$

since g is an increasing function. □

An immediate consequence is the next corollary.

Corollary 4.12. *For $b \ge a > 0$ and $cd \le 1$, the inequality*

$$1 \le \frac{g(\frac{s^b}{1+s^b})}{g(\frac{s^a}{1+s^a})} \le \frac{b}{a}$$

holds for arbitrary $s \ge 1$.

Another interesting result follows from Lemma 4.11.

Corollary 4.13. *For $c, d > 0$ with $cd \le 1$, the inequality*

$$g(\frac{s^p}{1+s^p}) + g(\frac{s^q}{1+s^q}) \ge g(\frac{s^{p+q}}{1+s^{p+q}})$$

holds true for arbitrary $s, p, q > 0$.

An answer to Question 3.9 with improved constant is given in the next theorem.

Theorem 4.14. *Let $0 < a \le 1 \le b < \infty$ and let $\varphi(s)$ be defined as in* (3.7) *and $c, d > 0$ with $cd \le 1$. Then the inequality*

$$g\left(\frac{\varphi(s)}{1+\varphi(s)}\right) \le \frac{b}{a}\max\left\{g^a\left(\frac{s}{1+s}\right), g\left(\frac{s}{1+s}\right)\right\} \tag{4.15}$$

holds for each $s > 0$.

Proof. Analyzing the structure of the above inequality, we decide that our task is to find an upper bound for the expression $T(s)$ given by

$$T(s) = \begin{cases} g\left(\frac{s^a}{1+s^a}\right) \Big/ g^a\left(\frac{s}{1+s}\right), & 0 < s \le 1; \\ g\left(\frac{s^b}{1+s^b}\right) \Big/ g^a\left(\frac{s}{1+s}\right), & 1 < s \le \gamma; \\ g\left(\frac{s^b}{1+s^b}\right) \Big/ g\left(\frac{s}{1+s}\right), & \gamma < s, \end{cases}$$

where γ is the unique solution of the equation $g(\frac{s}{1+s}) = 1$. By Remark 4.10, $\gamma > 1$.

Applying the second part of Theorem 1.7 we see that $T(s)$ is monotone decreasing for $s \in (0,1)$. Therefore, in this case we have

$$T(s) < \lim_{s\to 0^+} T(s) = \lim_{s\to 0^+} \frac{s^a/(1+s^a)}{(s/(1+s))^a} \frac{F(c,d,c+d;s^a/(1+s^a))}{(F(c,d,c+d;s/(1+s)))^a} = 1.$$

Now, for $1 < s \le \gamma$, write

$$T(s) = \frac{g(\frac{s^a}{1+s^a})}{g^a(\frac{s}{1+s})} \frac{g(\frac{s^b}{1+s^b})}{g(\frac{s^a}{1+s^a})} = T_1(s)T_2(s).$$

By Theorem 1.7, $T_1(s) = \frac{g(\frac{s^a}{1+s^a})}{g^a(\frac{s}{1+s})}$ is monotone increasing in s. Therefore, for $1 < s \leq \gamma$, we get

$$T_1(s) \leq T_1(\gamma) = g(\frac{\gamma^a}{1+\gamma^a}) \leq g(\frac{\gamma}{1+\gamma}) = 1$$

and

$$T_2(s) = \frac{g(\frac{s^b}{1+s^b})}{g(\frac{s^a}{1+s^a})} \leq \frac{b}{a} \tag{4.16}$$

by Corollary 4.12.

Analogously, in the case $s > \gamma$ we have

$$T(s) = \frac{g(\frac{s^b}{1+s^b})}{g(\frac{s}{1+s})} \leq b. \tag{4.17}$$

The assertion follows from (4.16) and (4.17). □

Finally, an answer to Question 3.11 is given in the next theorem.

Theorem 4.18. *The inequality*

$$g(x) \leq b\ \varphi\left(g\left(\frac{\varphi^{-1}(x/(1-x))}{1+\varphi^{-1}(x/(1-x))}\right)\right)$$

holds for $x \in (0,1)$*, where* $\varphi(y) = \max\{y^a, y^b\}$*,* $\varphi^{-1}(y) = \min\{y^{1/a}, y^{1/b}\}$*,* $0 < a < 1 < b$ *and* $c, d > 0,\ cd \leq 1$.

Proof. Changing variable $\frac{x}{1-x} = \varphi(s),\ s \in \mathbb{R}_+$, we get

$$t(s) = \begin{cases} g\left(\frac{s^a}{1+s^a}\right) \Big/ g^a\left(\frac{s}{1+s}\right), & 0 < s \leq 1; \\ g\left(\frac{s^b}{1+s^b}\right) \Big/ g^a\left(\frac{s}{1+s}\right), & 1 < s \leq \gamma; \\ g\left(\frac{s^b}{1+s^b}\right) \Big/ g^b\left(\frac{s}{1+s}\right), & \gamma < s, \end{cases}$$

where γ is the unique solution of the equation $g(\frac{s}{1+s}) = 1$.

Proceeding similarly as above, we obtain

$$t(s) < \lim_{s \to 0^+} t(s) = 1,$$

in the case $0 < s \leq 1$. For $1 < s \leq \gamma$, we have

$$t(s) = \frac{g\left(\frac{s^b}{1+s^b}\right)}{g\left(\frac{s}{1+s}\right)} g^{1-a}\left(\frac{s}{1+s}\right) \leq b\ g^{1-a}\left(\frac{\gamma}{1+\gamma}\right) = b,$$

by Corollary 4.12. For $s > \gamma$, by Theorem 1.7 we get

$$t(s) < t(\gamma) = \frac{g\left(\frac{\gamma^b}{1+\gamma^b}\right)}{g\left(\frac{\gamma}{1+\gamma}\right)} \leq b,$$

by Corollary 4.12 again.

Therefore $t(s) \leq b$, which proves Theorem 4.18. □

5. Proofs of the main theorems

In this section we give the proofs of the main theorems.

Proof of Theorem 1.7. We shall prove part (1) only. The proof of part (2) is similar, and the assertion of (3) follows from the former considerations.

Since h is convex, $h' = \frac{e^x f'(e^x)}{f(e^x)}$ is an increasing function, that is, if $u > v$, then

$$\frac{e^u f'(e^u)}{f(e^u)} > \frac{e^v f'(e^v)}{f(e^v)}. \tag{5.1}$$

Now,

$$\begin{aligned}\frac{g'}{g} &= c\left(\frac{x^{c-1}f'(x^c)}{f(x^c)} - \frac{f'(x)}{f(x)}\right)\\ &= ce^{-\log x}\left(\frac{e^{c\log x}f'(e^{c\log x})}{f(e^{c\log x})} - \frac{e^{\log x}f'(e^{\log x})}{f(e^{\log x})}\right),\end{aligned}$$

and by (5.1), the conclusion of part (1) follows by comparing $c\log x$ with $\log x$. □

Applying Theorem 1.7 we are able to give an answer to Question 1.3 and prove Theorem 1.6. Before that, we introduce the following lemma.

Lemma 5.2.

(1) *The expression* $w(x) := e^x + r(r(x))(e^x - 1 - r(x)), r(x) = \log(1+e^x)$, *is positive for* $x \in \mathbb{R}$.
(2) *The function* $v(x) := r(r(x)) = \log(1 + \log(1+e^x))$ *is log-concave.*

Proof. (1) Putting $1 + e^x = e^t, t > 0$, we obtain

$$\begin{aligned}w(t) &= e^t - 1 + \log(1+t)(e^t - 2 - t) = (e^t-1)(1+\log(1+t)) - (1+t)]\log(1+t)\\ &> t(1+\log(1+t)) - (1+t)\log(1+t) = t - \log(1+t) > 0.\end{aligned}$$

(2) By differentiation, we get

$$\frac{d^2\log(v(x))}{dx^2} = \frac{vv'' - (v')^2}{v^2} = -\frac{e^x w(x)}{((1+e^x)(1+\log(1+e^x)))^2},$$

which is negative by (1). □

Proof of Theorem 1.6. Since $\phi(u) = u^a$ for $0 < u \le 1$ and $\phi(u) = u$ if $u \ge 1$, we easily get

$$f_1(x) = \frac{\log^p(1+\phi(x))}{\phi(\log^p(1+x))} = \begin{cases}\frac{\log^p(1+x^a)}{(\log^p(1+x))^a}, & 0 < x \le 1;\\ (\log(1+x))^{p(1-a)}, & 1 < x \le e-1;\\ 1, & x > e-1.\end{cases} \tag{5.3}$$

For the proof of (1.4), we will apply Theorem 1.7 and show first that the function $r(x) := \log(1+e^x)$ is log-concave on $\mathbb{R}$.

Indeed, since

$$r'(x) = \frac{e^x}{1+e^x}, \quad r''(x) = \frac{e^x}{(1+e^x)^2},$$

we get

$$rr'' - (r')^2 = \frac{e^x}{(1+e^x)^2}(\log(1+e^x) - e^x) < 0,$$

because $\log(1+t) < t,\ t > 0$.

Since $r(x)$ is log-concave, Theorem 1.7 gives

$$f_1(1) = (\log 2)^{p(1-a)} \leq f_1(x) < 1 = \lim_{x\to 0} f_1(x),\ x \in (0,1].$$

Also,

$$f_1(1) = (\log 2)^{p(1-a)} < f_1(x) \leq 1 = f_1(e-1),\ x \in (1, e-1].$$

Hence, $c_1 = (\log 2)^{p(1-a)}, c_2 = 1$ and those bounds are best possible.

Answering (1.5), we proceed analogously. Denote

$$s(x) := \log(1+x)\log(1+\log(1+x))$$

and let $x_0,\ x_0 \approx 2.4555$, be the unique positive solution of the equation $s(x) = 1$. By the definition of ϕ, we get

$$f_2(x) = \begin{cases} \frac{s(x^a)}{(s(x))^a}, & 0 < x \leq 1; \\ (s(x))^{1-a}, & 1 < x \leq x_0; \\ 1, & x > x_0. \end{cases} \tag{5.4}$$

By Lemma 5.2 (2) it is evident that $s(e^x) = r(x)r(r(x)) = r(x)v(x)$ is a log-concave function since it is represented by the product of two log-concave functions.

Applying the second part of Theorem 1.7, we get

$$\frac{s(x^a)}{(s(x))^a} < \lim_{x\to 0} \frac{s(x^a)}{(s(x))^a} = 1;$$

$$\frac{s(x^a)}{(s(x))^a} \geq \frac{s(1)}{(s(1))^a} = (\log 2 \log(1+\log 2))^{1-a},$$

for $0 < x \leq 1$. Since $s(x)$ is an increasing function, it follows that

$$(s(1))^{1-a} < f_2(x) \leq (s(x_0))^{1-a} = 1,$$

for $1 < x \leq x_0$.

Hence for $x > 0$,

$$(\log 2 \log(1+\log 2))^{1-a} \leq f_2(x) \leq 1,$$

and those bounds are best possible. □

Remark 5.5. Although Question 1.3 can be solved by the method of [KMV, Lemma 3.1], an application of Theorem 1.7 gives the result more efficiently.

Remark 5.6. An affirmative answer to Question 3.2 is given in [SV].

References

[AS] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964. MR0167642 (29 #4914)

[ABRVV] G. D. Anderson, R. W. Barnard, K. C. Richards, M. K. Vamanamurthy, and M. Vuorinen, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1713–1723, DOI 10.2307/2154966. MR1264800 (95m:33002)

[AVV1] Glen D. Anderson, Mavina K. Vamanamurthy, and Matti K. Vuorinen, *Conformal invariants, inequalities, and quasiconformal maps*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons Inc., New York, 1997. With 1 IBM-PC floppy disk (3.5 inch; HD); A Wiley-Interscience Publication. MR1462077 (98h:30033)

[AVV2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl. **335** (2007), no. 2, 1294–1308, DOI 10.1016/j.jmaa.2007.02.016. MR2346906 (2008i:33012)

[KMV] R. Klén, V. Manojlović, and M. Vuorinen, *Distortion of normalized quasiconformal mappings.* Manuscript, 20 pp., arXiv:0808.1219 [math.CV]

[K] Reinhold Küstner, *Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order α*, Comput. Methods Funct. Theory **2** (2002), no. 2, 597–610. MR2038140 (2004k:33008)

[Mit] D. S. Mitrinović, *Analytic inequalities.* In cooperation with P. M. Vasić, Die Grundlehren der mathematischen Wissenschaften, Band 165, Springer-Verlag, New York, 1970. MR0274686 (43 #448)

[PV] S. Ponnusamy and M. Vuorinen, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika **44** (1997), no. 2, 278–301, DOI 10.1112/S0025579300012602. MR1600537 (99b:33004)

[SV] Slavko Simić and Matti Vuorinen, *On quotients and differences of hypergeometric functions*, J. Inequal. Appl. **2011**, 2011:141, 10 pp., DOI 10.1186/1029-242X-2011-141. MR2890390

Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland

E-mail address: `ripekl@utu.fi`

Faculty of Organizational Sciences, University of Belgrade, Jove Ilica 154, 11000 Belgrade, Serbia

E-mail address: `vesnam@fon.rs`

Mathematical Institute SANU, Kneza Mihaila 36, 11000 Belgrade, Serbia

E-mail address: `ssimic@turing.mi.sanu.ac.rs`

Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland

E-mail address: `vuorinen@utu.fi`