TURUN YLIOPISTO
UNIVERSITY OF TURKU

METRICS AND QUASICONFORMAL MAPS

Rahim Kargar

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University of Turku

Faculty of Science
Department of Mathematics and Statistics
Mathematics
Doctoral Programme in Exact Sciences (EXACTUS)

Supervised by

Professor Matti Vuorinen
Department of Mathematics and Statistics, University of Turku, Finland

Professor Peter Hästö
Department of Mathematics and Statistics, University of Turku, Finland

Reviewed by

Professor Vasudevarao Allu
School of Basic Sciences, Indian Institute of Technology Bhubaneswar, India

Dr. Jiaolong Chen
Key Laboratory of Computing and Stochastic Mathematics, Hunan Normal University, China

Opponent

Professor Swadesh Kumar Sahoo
Department of Mathematics, Indian Institute of Technology Indore, India

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ABSTRACT

Geometric Function Theory (GFT) is a mathematical discipline dedicated to examining the geometric characteristics of functions. Its applications extend across diverse domains, such as physics, engineering, and computer science. Some of the most important subjects in GFT are Möbius transformation, univalent functions, Schwarz lemma, and conformal invariants. This research focuses on subdomains within the n-dimensional Euclidean space, exploring their geometrical characteristics and the classes of functions defined on these subdomains. The function classes examined include conformal maps, analytic functions, as well as quasiconformal and quasiregular maps. Several metrics are also used in this research, including but not limited to the hyperbolic metric, the quasihyperbolic metric, the visual angle metric, the Harnack metric, and the triangular ratio metric.

The dissertation comprises six papers that have been published. Preliminary versions of these papers can be accessed on the web pages of the arXiv.org preprint server. In the following, we summarize what we have done in each paper.

In paper [I], our research compares the hyperbolic metric with intrinsic metrics within convex polygonal domains on the complex plane. We examine the triangular ratio metric, which is an intrinsic metric, within rectangles. We also study the conformal radius of a planar domain, evaluated at a point of the domain, and its dependence on the distance of the point to the domain boundary.

In [II], we study the approximation of special functions frequently appearing in GFT and quasiconformal mapping theory. The functions we study can be represented as ratios of complete elliptic integrals or as inverses of such ratios. Specifically, we focus on the distortion function $\varphi_K(r)$, $K \geq 1$, which gives a majorant for $|f(x)|$ when $f : \mathbb{B}^2 \to \mathbb{B}^2$ with $f(0) = 0$, is a K-quasiconformal mapping of the unit disk $\mathbb{B}^2$ onto itself. Surprisingly, our approximation method is straightforward: employing just five steps of Landen iteration prove sufficient to attain machine precision.

In paper [III], we establish several new formulas expressing the visual angle metric of the unit disk in terms of the hyperbolic metric. These formulas then yield a sharp Schwarz lemma for the visual angle metric under quasiregular mappings.

Paper [IV] demonstrates that the modulus metric lacks Hölder continuity with respect to the hyperbolic metric.

Paper [V] explores the $(s, C(s))$-Harnack inequality within a domain $G \subset \mathbb{R}^n$, where $s \in (0, 1)$ and $C(s) \geq 1$. We also study a set of inequalities concerning $(s, C(s))$-Harnack functions. Additionally, we examine the behavior of the Harnack metric under K-quasiconformal and K-quasiregular mappings, where $K \geq 1$.

Furthermore, we present a form of the harmonic Schwarz lemma.

Paper [VI] introduces a measure of the non-univalency of a harmonic mapping and uses this measure to find the best approximation of a harmonic mapping by a univalent harmonic mapping.

Finally, we remark that the findings of this thesis suggest several conjectures and open questions for further investigation.

KEYWORDS: Metric Space, Quasiconformal Maps, Quasiregular Maps, Harmonic Mappings

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TIIVISTELMÄ

Geometrisen Funktioteorian (GFT) ala on matemaattinen tutkimusala, joka keskittyy tutkimaan funktioiden geometrisia ominaisuuksia. Sen sovellukset ulottuvat monille eri aloille, kuten fysiikkaan, tekniikkaan ja tietojenkäsittelytieteisiin. GFT:ssä keskeisiä aiheita ovat Möbius-kuvaukset, univalentit funktiot, Schwarzin lemma, ja konformi-invariantit. Tutkimus keskittyy n-ulotteisen euklidisen avaruuden osa-alueisiin, niiden geometrisiin ominaisuuksiin ja näillä osa-alueilla määriteltyjen funktioiden luokkiin. Tutkittuja funktioluokkia ovat konformikuvaukset, analyyttiset funktiot sekä kvasikonformiset ja kvasisäännölliset kuvaukset.

Tässä tutkimuksessa käytetään myös useita metriikoita, mukaan lukien hyperbolinen metriikka, kvasihyperbolinen metriikka, visuaalinen metriikka, Harnackin metriikka ja kolmiosuhde metriikka. Väitöskirja koostuu kuudesta julkaistusta artikkelista. Näiden artikkelien alustavat versiot ovat saatavilla arXiv.org-palvelun verkkosivuilla. Seuraavaksi esitämme yhteenvedon siitä, mitä olemme tehneet kussakin artikkelissa.

Artikkelissa [I] tutkimuksemme keskittyy vertaamaan hyperbolista metriikkaa intrinsisiin metriikoihin konveksilla monikulmioalueilla kompleksitasossa. Tutkimme erityisesti kolmisuhdemetriikkaa suorakulmioilla. Lisäksi analysoimme tasoalueen pisteeseen liittyvän konformisen säteen riippuvuutta pisteen etäisyydestä alueen reunaan.

[II]:ssa tutkimme laskennallisia tekniikoita erikoisfunktioiden approksimointiin. Tutkittavat funktiot esiintyvät usein GFT:ssä ja kvasikonformisten kuvauksien teoriassa. Tutkimamme funktiot voidaan esittää elliptisten integraalien suhteina tai tällaisten suhteiden käänteisfunktioina. Keskitymme erityisesti funktioon $\varphi_K(r)$, $K \geq 1$, joka antaa ylärajan $|f(x)|$:lle, kun $f : \mathbb{B}^2 \to \mathbb{B}^2$ ja $f(0) = 0$, ja on K-kvasikonforminen kuvaus yksikkökiekolta yksikkökiekolle. Yllättäen menetelmämme on tehokas: viiden Landenin iterointikiekkoksen käyttö osoittautuu riittäväksi konetarkkuuteen.

[III]:ssa esitämme useita uusia kaavoja, jotka ilmaisevat yksikkökiekon visuaalisen kulmametriikan hyperbolisen metriikan avulla. Nämä kaavat antavat siten tarkan Schwarzin lemman visuaalisen kulmametriikan tapauksessa kvasisäännöllisissä kuvauksissa.

[IV]:ssa osoitamme, että eräällä metriikalla ei ole Hölder-jatkuvuutta hyperbolisen metriikan suhteen.

[V]:ssa tutkimme $(s, C(s))$-Harnackin epäyhtälöä alueella $G \subset \mathbb{R}^n$, missä $s \in (0, 1)$ ja $C(s) \geq 1$. Esitämme joukon epäyhtälöitä $(s, C(s))$-Harnack-funktioista. Lisäksi tutkimme Harnackin metriikan käyttäytymistä K-kvasikonformisten ja K-

kvasisäännöllisten kuvauksien yhteydessä, missä $K \geq 1$. Lisäksi esitämme Schwarzin lemman.

Artikkelissa [VI] esittelemme harmonisen kuvauksen harmonisille funktioille mittarin ja käytämme sitä parhaan approksimaation löytämiseen analyyttiselle funktiolle yksikkökiekkoon univalenttina harmonisena kuvauksena.

Huomautamme vielä, etta tämän väitöskirjan tulokset antavat useita avoimia kysymyksiä jatkotutkimukseen.

ASIASANAT: Metrinen avaruus, Kvasikonformiset kuvaukset, Kvasisäännölliset kuvaukset, Harmoniset kuvaukset

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June 2024
Rahim Kargar

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Notations

Symbol	**Description**	**Page**
$\mathbb{D}$ or $\mathbb{B}^2$	The unit disk	4, 15, 48
$\mathbb{B}^n$	The unit ball	4
$\mathbb{R}^n$	The n-dimensional Euclidean space	4
$\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$	The Möbius spaces	4
$d_G(x)$	The Euclidean distance between x and ∂G	4
$\mathbb{H}^n$	The upper half space	4
$L[x, y]$	The line through x in the direction of y	4
$T_a(z)$	The Möbius transformation	5
$B_\rho(x, M)$	The hyperbolic ball with center x and radius M	5
$A[x, y]$	The Ahlfors bracket	5
x_*, y_*	The endpoints of the hyperbolic geodesic through x and y	5
$q(x, y)$	The spherical metric	6
$\|a, b, c, d\|$	The absolute ratio metric	6
$\rho(x, y)$	The hyperbolic metric	6
$k_G(x, y)$	The quasihyperbolic metric	7
$j_G(x, y)$	The distance ratio metric	7
$h_G(x, y)$	The Harnack metric	8
$s_G(x, y)$	The triangular ratio metric	8
$v_G(x, y)$	The visual angle metric	9
$\mathsf{M}_p(\Gamma)$, $\mathsf{M}(\Gamma)$	The (p-)modulus of a curve family	9
$\lambda_G(x, y)$	The Ferrand's metric	10
$\mu_G(x, y)$	The Modulus metric	11
$\delta_G(x, y)$	The Seittenranta metric	11
γ_n	The Grötzsch capacity	10
τ_n	The Teichmüller capacity	10
$\mu(r)$	The modulus of the Grötzsch ring	12
$\mathcal{K}(r)$	The complete elliptic integral of the first kind	12
$\varphi_K(r)$	The distortion function	13
c_n	The constant in the spherical cap inequality	13
$J_f(x)$	The Jacobian determinant	14
$r_D(z_0)$	The conformal radius at z_0	15

List of Original Publications

This dissertation is based on the following original publications, which are referred to in the text by their Roman numerals:

I Dautova, D., Kargar, R., Nasyrov, S., and Vuorinen, M.: *Intrinsic metrics in polygonal domains*. Math. Nachr. **296** (2023), no. 11, 4961–4977.arXiv: 2206.03744v1[math.CV]

II Kargar, R., Rainio, O., and Vuorinen, M.: *Landen transformations applied to approximation*. Pure Appl. Funct. Anal. **9** (2024), no. 2, 503–516.arXiv: 2212.09336v3[math.CV]

III Fujimura, M., Kargar, R., and Vuorinen, M.: *Formulas for the visual angle metric*. to appear in J. Geom. Anal. arXiv:2304.04485v4[math.CV]

IV Kargar, R., and Rainio, O.: *Conformally invariant metrics and lack of Hölder continuity*. Bull. Malays. Math. Sci. Soc. (2024) 47:48.arXiv: 2304.11588v2[math.CV]

V Kargar, R.: *On Harnack inequality and harmonic Schwarz lemma*. Canad. Math. Bull. (2024), 1–15. DOI: 10.4153/S0008439524000298 arXiv: 2312.15232v3[math.CV]

VI Kargar, R.: *Harmonic approximations of analytic functions*. Complex Var. Elliptic Equ. **68** (2023), no. 10, 1751–1774.

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1 Introduction

The topic of this Ph.D. thesis is Geometric Function Theory (GFT), a branch of classical analysis. The study of the functions of a complex variable and its geometric characteristics is the focus of GFT [59; 4]. Central questions are to investigate how certain geometric transformations, such as rotations, dilations, and conformal mappings, affect the behavior of various functions. Several branches of mathematics use these functions, such as potential theory, number theory, and differential equations. GFT has applications in many areas of mathematics and science, including physics, engineering, and computer science. It is used in studying fluid dynamics, electromagnetism, and the behavior of materials, among other things.

Conformal mappings, also called univalent functions (analytic and injective functions) [60, p. 382, Theorem 1.2], are important concepts in GFT. Conformal mappings are transformations mapping a planar domain onto another such domain, preserving local shapes and angles. They map one complex domain to another and can be applied to investigate the characteristics of complex analytical functions. These mappings are crucial in understanding and generalizing geometric properties in higher-dimensional spaces. Quasiconformal and quasiregular mappings in $\mathbb{R}^n$ generalize conformal and analytic functions in the complex plane $\mathbb{C}$, respectively. Quasiregular mappings are defined in the same way as quasiconformal mappings, but without the requirement of being a homeomorphism; see [71; 73; 82; 51; 38; 30]. The reader may wish to take a look at the various available collections of surveys on quasiconformal mappings, namely [17; 65], and [49].

Another important concept in GFT is the notion of a metric space. In GFT, a metric space plays a fundamental role as a mathematical framework for studying the geometric or topological properties of functions. Notions from metric spaces, such as distances between points, spheres, and balls, are natural tools for GFT. In this Ph.D. thesis, there is a significant concept called intrinsic metric. The intrinsic metric considers not only the distance between two points in a domain G but also the distances to the boundary of G. The hyperbolic metric and several other metrics are commonly used to study these distances. Examples of these other metrics include the quasihyperbolic metric, the triangular ratio metric, the visual angle metric, and the Harnack metric studied here.

Now, let us briefly examine the history and literature surrounding quasiconformal, quasiregular mappings, and hyperbolic geometry. For almost a century, there has been an ongoing study of plane quasiconformal mappings. In 1928, Grötzsch [28] first introduced the concept, which involves determining the most nearly conformal homeomorphism between pairs of topologically equivalent plane configurations with one conformal invariant. In particular, he studied mapping a square onto a rectangle with the corners matching. Later, in a fundamental paper on covering surfaces, L.V. Ahlfors [1] also explored quasiconformal mappings. From 1938 to 1943, Teichmüller [78] extended Grötzsch's study to include mappings between closed Riemann surfaces. He obtained a natural parameter space for surfaces of fixed genus $p > 1$, which is homeomorphic to $\mathbb{R}^{6(p-1)}$ [80; 79]. At the same time, Lavrent'ev [50] and Morrey [58] generalized a classical result due to Gauss on the existence of isothermal coordinates. They established versions of what is now known as the measurable Riemann mapping theorem for quasiconformal mappings.

The progress in the research of quasiconformal maps was interrupted by World War II. After the war, the efforts to advance this field were revitalized through the contributions of Ahlfors [2; 3], L. Bers, I.N. Vekua, and their students. Moreover, in 1961, F.W. Gehring [24] and J. Väisälä [81] initiated the systematic study of quasiconformal mappings in $\mathbb{R}^n$, while Yu. G. Reshetnyak explored quasiregular mappings in 1966 [70; 72]. In a series of highly significant papers published between 1966 and 1969, Reshetnyak established the fundamental properties of quasiregular mappings, utilizing modern tools from differential geometry, nonlinear PDE theory, and the theory of Sobolev spaces. From 1969 to 1972, O. Martio, S. Rickman, and J. Väisälä [54; 55; 56; 83] developed a new approach to the theory of quasiregular mappings. This approach relied on certain findings of Reshetnyak, in particular on his result that a non-constant quasiregular mapping possesses the characteristics of being both discrete and open.

One of the topics of GFT is to study how distances between points are changed or distorted under mappings. For this purpose, many ways to measure distances have been introduced. In particular, invariant or quasi-invariant metrics are such tools. An ideal metric for plane domains is the conformally invariant hyperbolic metric because conformal mappings form a large class of mappings in the plane. Due to the rigidity of conformal mappings in $\mathbb{R}^n$, $n \geq 3$, conformal mappings are Möbius transformations; their family is much more narrow for $n \geq 3$ than in the plane. Therefore, it is natural to look for a class of mappings, wider than Möbius transformations, and to study metrics that are nearly invariant or quasi-invariant under the class. For instance, quasi-invariance under bi-Lipschitz mappings is a very useful property. F.W. Gehring and B.P. Palka [27] introduced two metrics that have become standard tools in modern GFT: the quasihyperbolic and distance ratio metrics of a domain $G \subset \mathbb{R}^n$, $n \geq 2$, see [25], [29], [30], and [84]. Later on, several other metrics were introduced, and we list some of them here:

- Ferrand's Möbius invariant version of the quasihyperbolic metric [19], and

Seittenranta's Möbius metric, a Möbius invariant version of the distance ratio metric [75];

- the modulus metric and Ferrand's modulus metric, see [52; 18], [87, p. 193], and [7, p. 182];
- the Apollonian metric Beardon [11], Hästö [34], Hästö-Ibragimov [35], Hästö-Ibragimov-Lindén [36], Hästö-Ibragimov-Minda-Ponnusamy-Sahoo [37], and Gehring-Hag [25];
- the triangular ratio metric of Hästö [33; 44; 67; 68], Dovgoshey-Hariri-Vuorinen metric [16; 22], and the Barrlund metric [21];
- the visual angle metric of Klén-Lindén-Vuorinen-Wang [48; 20; 23];
- the Harnack metric, see [69; 85; 30].

The structure of this Ph.D. thesis is as follows: in Chapter 2, we present some definitions and background information essential to understanding the theoretical framework of our study. In Chapter 3, we study inequalities between the hyperbolic metric and intrinsic metrics within convex polygonal domains. In Chapter 4, we explore computational techniques for approximating specialized functions that frequently arise in GFT and the theory of quasiconformal mapping. In Chapter 5, we prove several new formulas for the visual angle metric of the unit disk in terms of the hyperbolic metric. These formulas are then employed to demonstrate a sharp Schwarz lemma for the visual angle metric under quasiconformal mapping. In Chapter 6, we demonstrate that the modulus metric lacks Hölder continuity with respect to the hyperbolic metric. In Chapter 7, we study the Harnack metric and $(s, C(s))$-Harnack inequality in a proper subdomain G of $\mathbb{R}^n$, where $s \in (0,1)$ and $C(s) \geq 1$. Also, a new version of the Schwarz lemma, known as the harmonic Schwarz lemma, has been presented. Finally, in Chapter 8, we find the best approximation of a harmonic mapping by univalent harmonic mapping.

2 Background and Definitions

This chapter overviews the foundational concepts and terminology relevant to geometric function theory.

2.1 Complex Numbers

Let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$ denote the complex, the real, and the integer numbers, respectively, and let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$, where $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. The usual notation for the standard unit vectors $\mathbb{R}^n$, with $n \geq 2$, is $e_1, \ldots, e_n$. Each of these vectors is in the direction of a specific coordinate axis. For $x, y \in \mathbb{R}^n$ the inner product is defined by $x \cdot y = \sum_{n=1}^{n} x_i y_i$. The metric space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, equipped with the chordal metric is called the Möbius space. The open ball and sphere with center $x \in \mathbb{R}^n$ and radius $r > 0$ are denoted by $B^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, and $S^{n-1}(x, r) = \{y \in \mathbb{R}^n : |x - y| = r\}$, respectively. We employ the abbreviations $\mathbb{B}^n = B^n(0, 1)$ and $S^{n-1} = S^{n-1}(0, 1)$.

Let G be a proper subdomain in $\mathbb{R}^n$. We denote by $d_G(x)$ the Euclidean distance, $\mathrm{dist}(x) = \inf\{|x - z| : z \in \partial G\}$ between the point x and the boundary of G. Also, let $\mathbb{H}^n = \{(x_1, \ldots, x_n) : x_n > 0\}$ denote the upper half space in $\mathbb{R}^n$. Let Ω_n and ω_{n-1} stand for the n-dimensional volume of the unit ball $m_n(\mathbb{B}^n)$, and the $(n-1)$-dimensional surface area of S^{n-1}, respectively. Let $L[a, b]$ denote the line through a and b $(\neq a)$. If $L[a, b]$ and $L[c, d]$ are two lines having a unique point of intersection e, then $e := LIS[a, b, c, d] = u/v$, where (see, e.g. [30, Ex. 4.3(1), p. 57 and p. 373])

$$\begin{cases} u = (\overline{a}b - a\overline{b})(c - d) - (a - b)(\overline{c}d - c\overline{d}); \\ \\ v = (\overline{a} - \overline{b})(c - d) - (a - b)(\overline{c} - \overline{d}). \end{cases}$$

Consider the circle passing through the distinct noncollinear points a, b, and c as $C[a, b, c]$. Then the center $m(a, b, c)$ of $C[a, b, c]$ is given by

$$m(a, b, c) = \frac{|a|^2(b - c) + |b|^2(c - a) + |c|^2(a - b)}{a(\overline{c} - \overline{b}) + b(\overline{a} - \overline{c}) + c(\overline{b} - \overline{a})}.$$

Sometimes, we write $a^* = a/|a|^2 = 1/\overline{a}$ for $a \in \mathbb{C} \setminus \{0\}$ instead. We put $0^* = \infty$

and $\infty^* = 0$. The following formula,

$$w(z) = \frac{a-b}{\overline{a}-\overline{b}}\,\overline{z} - \frac{a\overline{b}-\overline{a}b}{\overline{a}-\overline{b}},$$

gives the reflection of a point z in a line through two distinct points a and b. We use the Ahlfors bracket notation $A[a,b]$ for $a, b \in \mathbb{B}^2$ (see [30, p. 38])

$$A[a,b] = \sqrt{|a-b|^2 + (1-|a|^2)(1-|b|^2)} = |1 - a\overline{b}|. \tag{1}$$

A hyperbolic ball $B_\rho(x, M)$ with $x \in \mathbb{B}^n$ and $M > 0$ defined in the unit ball $\mathbb{B}^n$ is equal to the Euclidean ball $B^n(y, r)$, where (see, [30, (4.20), p. 56])

$$y = \frac{x(1-t^2)}{1-|x|^2t^2}, \quad r = \frac{(1-|x|^2)t}{1-|x|^2t^2}, \quad \text{and} \quad t = \operatorname{th}(M/2).$$

Möbius Transformations. Let $a, b, c, d, z \in \mathbb{C}$, and $ad - bc \neq 0$. A mapping of the type,

$$z \mapsto \frac{az+b}{cz+d}, \quad z \neq -d/c,$$

is a Möbius transformation. The special Möbius transformation

$$T_a(z) = \frac{z-a}{1-\overline{a}z}, \quad a \in \mathbb{B}^2 \setminus \{0\}, \tag{2}$$

maps the unit disk $\mathbb{B}^2$ onto itself with $T_a(a) = 0$ and $T_a(\pm a/|a|) = \pm a/|a|$. For $a, b \in \mathbb{B}^2 \setminus \{0\}$, the endpoint $\operatorname{ep}(a, b)$ on the unit circle of the hyperbolic line passing through points a and b is denoted by

$$\operatorname{ep}(a,b) := T_{-b}(T_b(a)/|T_b(a)|).$$

Let $C[a, b]$ be the circle orthogonal to the unit circle, containing two distinct points $a, b \in \mathbb{C}$. If $a, b \in \mathbb{B}^2$, then $C[a, b] \cap \partial\mathbb{B}^2 = \{a_*, b_*\}$, where the points are labeled in such a way that a_*, a, b, and b_* appear in this order on $C[a, b]$, and $\operatorname{ep}(a, b) = a_*$ and $\operatorname{ep}(b, a) = b_*$. If the hyperbolic geodesic segment connecting the two distinct points $a, b \in \mathbb{B}^2$ is denoted by $J[a, b]$, then $J[a, b]$ is a subarc of $C[a, b] \cap \mathbb{B}^2$. Additionally, the hyperbolic line is $J^*[a_*, b_*] = C[a, b] \cap \mathbb{B}^2$.

Uniformly Perfect Set. For $0 < c < 1$, let $UP_n(c)$ denote the collection of compact sets E in $\mathbb{R}^n$ with $\operatorname{card}(E) \geq 2$ satisfying the condition

$$\{x \in E : cr < |x - a| < r\} \neq \emptyset$$

for all $a \in E$ and $0 < r < d(E)/2$, where $d(E) = \sup\{d(x, y) : x, y \in E\}$. We say that a set is *uniformly perfect* if it is in the class $UP_n(c)$ for some $c \in (0, 1)$; see, e.g., [66].

2.2 Metrics

This section lists all the metrics employed in this thesis.

Spherical Metric. The spherical (chordal) metric q in $\overline{\mathbb{R}}^n$ is defined by

$$q(x,y)=\begin{cases}\dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x\neq\infty\neq y;\\[2ex] \dfrac{1}{\sqrt{1+|x|^2}}, & y=\infty.\end{cases}$$

Absolute Ratio Metric. The absolute (cross) ratio of an ordered quadruple a, b, c, and d of distinct points in $\overline{\mathbb{R}}^n$ is defined by

$$|a,b,c,d|=\frac{q(a,c)q(b,d)}{q(a,b)q(c,d)}.$$

By the definition of spherical metric for distinct point a, b, c, and d in $\mathbb{R}^n$ we have

$$|a,b,c,d|=\frac{|a-c||b-d|}{|a-b||c-d|}.$$

The absolute ratio metric is invariant under the Möbius transformation [10]. On the other hand, if f is a Möbius transformation, then

$$|f(a),f(b),f(c),f(d)|=|a,b,c,d|$$

for all distinct points a, b, c, and d in $\overline{\mathbb{R}}^n$.

Let sh, ch and th denote the hyperbolic functions $\sinh$, $\cosh$, and $\tanh$, respectively.

Hyperbolic Metric. Let $\mathbb{R}^n_+ := \{x=(x_1,\ldots,x_n) : x_n>0\}$. If $x,y\in\mathbb{R}^n_+$, then the hyperbolic distance between x and y is defined by

$$\rho(x,y)=\inf_{\gamma\in\Gamma_{xy}}\int_\gamma\frac{|\mathrm{d}x|}{x_n},$$

where Γ_{xy} stands for the collection of all rectifiable curves in $\mathbb{R}^n_+$ joining x and y. The hyperbolic metric $\rho_{\mathbb{H}^n}$ and $\rho_{\mathbb{B}^n}$ of the upper half space $\mathbb{H}^n$ and of the unit ball $\mathbb{B}^n$ can also be expressed as follows ([10, p. 35]):

$$\mathrm{ch}\rho_{\mathbb{H}^n}(x,y)=1+\frac{|x-y|^2}{2x_ny_n},\quad x,y\in\mathbb{H}^n=\mathbb{R}^n_+ \tag{3}$$

and ([10, p. 40])

$$\mathrm{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}=\frac{|x-y|}{\sqrt{1-|x|^2}\sqrt{1-|y|^2}},\quad x,y\in\mathbb{B}^n, \tag{4}$$

respectively. It should be noted by (3) that the hyperbolic metric $\rho(x,y)$ is completely determined once the Euclidean distances $x_n = d(x, \partial\mathbb{H}^n)$, $y_n = d(y, \partial\mathbb{H}^n)$, and $|x-y|$ are known. It follows also from (4) that

$$\mathrm{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|}{A[x,y]}, \quad x,y \in \mathbb{B}^n,$$

where $A[x,y]$ is the Ahlfors bracket defined in (1), see [30, Exercise 4.20]. We note that the hyperbolic metric is invariant under Möbius transformations. Indeed, if f is a Möbius transformation such that $f\mathbb{B}^n = \mathbb{B}^n$, then $\rho_{\mathbb{B}^n}(x,y) = \rho_{\mathbb{B}^n}(f(x), f(y))$ for all $x, y \in \mathbb{B}^n$. Also, if g is an Möbius transformation such that $g\mathbb{B}^n = \mathbb{H}^n$, then $\rho_{\mathbb{B}^n}(x,y) = \rho_{\mathbb{H}^n}(g(x), g(y))$ for all $x, y \in \mathbb{B}^n$, see [30, p. 56].

It is possible to simplify the hyperbolic metric formulas for the two cases $\mathbb{H}^2$ and $\mathbb{B}^2$ in terms of complex numbers as follows:

$$\mathrm{th}\frac{\rho_{\mathbb{H}^2}(x,y)}{2} = \mathrm{th}\left(\frac{1}{2}\log\left(\frac{|x-\overline{y}|+|x-y|}{|x-\overline{y}|-|x-y|}\right)\right) = \frac{|x-y|}{|x-\overline{y}|}, \quad x,y \in \mathbb{H}^2$$

and

$$\mathrm{th}\frac{\rho_{\mathbb{B}^2}(x,y)}{2} = \mathrm{th}\left(\frac{1}{2}\log\left(\frac{|1-x\overline{y}|+|x-y|}{|1-x\overline{y}|-|x-y|}\right)\right) = \frac{|x-y|}{|1-x\overline{y}|}, \quad x,y \in \mathbb{B}^2,$$

respectively; see [30, Exercise 4.17 & 4.20].

Quasihyperbolic Metric. The quasihyperbolic distance between two points x and y is defined by

$$k_G(x,y) = \inf_{\gamma}\int_{\gamma}\frac{|\mathrm{d}x|}{d_G(x)},$$

where the infimum is taken over all rectifiable curves $\gamma \subset G$ containing both x and y in a proper subdomain G of $\mathbb{R}^n$. Gehring and Palka first introduced the metric mentioned here in their work [27]. We know also from [27, p. 174] that

$$k_{\mathbb{H}^n}(x,y) = \rho_{\mathbb{H}^n}(x,y) \quad \text{and} \quad k_{\mathbb{B}^n}(x,y) \le \rho_{\mathbb{B}^n}(x,y) \le 2k_{\mathbb{B}^n}(x,y). \tag{5}$$

Unlike the hyperbolic metric, the quasihyperbolic metric is not preserved under Möbius transformations. This means that the distance between two points might change after applying a Möbius transformation. However, the change is bounded by a constant factor, see [87, Lemma 3.10].

The Distance Ratio Metric. Let $\Omega \subsetneq \mathbb{R}^n$ be any open set in $\mathbb{R}^n$. The distance ratio metric is defined by

$$j_\Omega(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d_\Omega(x), d_\Omega(y)\}}\right), \quad x,y \in \Omega.$$

Gehring and Palka introduced the distance ratio metric [27], and M. Vuorinen provided the above simple formula, see [86]. Likewise, the quasihyperbolic metric, the distance ratio metric, is not invariant under a Möbius transformation. Gehring, Osgood, and Palka proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [26, proof of Theorem 4] and [27, Corollary 2.5].

Moreover, when $\Omega \in \{\mathbb{B}^n, \mathbb{H}^n\}$ then we have for all $x, y \in \Omega$ (see, [30, Lemma 4.9])

$$j_\Omega(x,y) \le \rho_\Omega(x,y) \le 2 j_\Omega(x,y). \tag{6}$$

Uniformity. Let $G \subsetneq \mathbb{R}^n$ be a domain. We say that G is *uniform* if there exists a constant $A = A(G) \ge 1$ such that $k_G(x,y) \le A j_G(x,y)$ for all $x, y \in G$. It is clear from relations (5) and (6) that the unit ball $\mathbb{B}^n$ and the upper half-space $\mathbb{H}^n$ are examples of uniform domains with the constant 2 (see [30, Definition 6.1]).

Let $\mathbb{R}_+$ denote the set $\{x \in \mathbb{R} : x > 0\}$.

$(s, C(s))$**-Harnack Inequality.** (see [85]) Consider a proper subdomain G of $\mathbb{R}^n$, and let $u : G \to \mathbb{R}_+ \cup \{0\}$ be a continuous function. For $s \in (0,1)$ and $C(s) \ge 1$ we say that u satisfies the $(s, C(s))$-Harnack inequality in a domain $G \subset \mathbb{R}^n$, if

$$\max_{B_x} \; u(z) \le C(s) \min_{B_x} \; u(z) \tag{7}$$

holds, whenever $B^n(x,r) \subset G$, and $B_x := \overline{B}^n(x, sr)$. A function satisfying (7) is called a $(s, C(s))$-Harnack function. It is well known that non-negative harmonic functions satisfy (7); see [85].

The classical Harnack inequality for harmonic functions is expressed as follows: Let $B^n(x,r)$ be the Euclidean ball such that the concentric ball $B^n(x, 2r)$ is contained in a domain $G \subset \mathbb{R}^n$, $n \ge 2$. Then there exists a positive constant $C := C(n)$ such that

$$\sup_{B^n(x,r)} \; u \le C \inf_{B^n(x,r)} \; u$$

for all nonnegative harmonic functions $u : G \to \mathbb{R}_+ \cup \{0\}$. The Harnack inequality provides a basis for defining the Harnack metric.

Harnack Metric. For arbitrary $x, y \in G$, the Harnack metric is defined by

$$h_G(x,y) = \sup \left| \log \frac{u(x)}{u(y)} \right|,$$

where the supremum is taken over all positive harmonic functions u in G; see [69, p. 14, Definition 1.3.4], [32], and [40].

Triangular Ratio Metric. The triangular ratio metric $s_G : G \times G \to [0,1]$ is defined by

$$s_G(x,y) = \frac{|x-y|}{\inf_{z \in \partial G}(|x-z| + |z-y|)}, \quad x, y \in G.$$

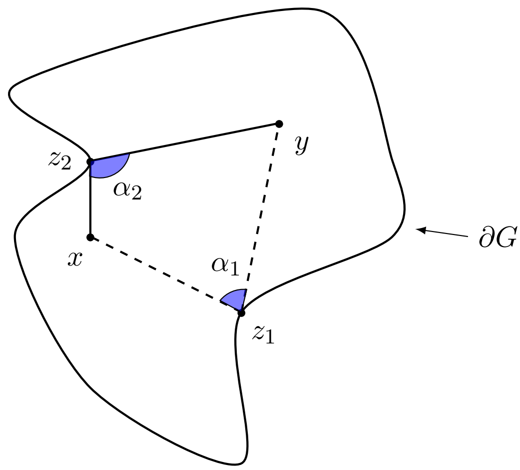


Figure 1. The definition of visual angle metric; $v_G(x, y) = \max\{\alpha_1, \alpha_2\} = \alpha_2$.

This metric was introduced by P. Hästö [33], and studied in [31; 30; 67]. Chen, Hariri, Klén and Vuorinen [14, Corollary 3.30] proved that if f is a Möbius transformation from $\mathbb{H}^n$ onto $\mathbb{H}^n$, then

$$s_{\mathbb{H}^n}(f(x), f(y)) = s_{\mathbb{H}^n}(x, y)$$

for all $x, y \in \mathbb{H}^n$. Also, if f is a Möbius transformation from $\mathbb{B}^n$ onto $\mathbb{B}^n$, then

$$s_{\mathbb{B}^n}(f(x), f(y)) \leq 2s_{\mathbb{B}^n}(x, y)$$

for all $x, y \in \mathbb{B}^n$.

Visual Angle Metric. Let G be a proper subdomain of $\mathbb{R}^n$ such that ∂G is not a proper subset of a line. The visual angle metric for $x, y \in G$ is defined by

$$v_G(x, y) = \sup\{\measuredangle(x, z, y), z \in \partial G\},$$

where the notation $\measuredangle(x, z, y)$ denotes the angle in range $[0, \pi]$ between the segments $[x, z]$ and $[z, y]$. The above supremum is attained for $z \in \partial G$ such that the angle $\measuredangle(x, z, y)$ is largest, see Figure 1. This metric was introduced and studied very recently by Klén et al. in [48]. They also proved that if $f : \mathbb{B}^n \to \mathbb{B}^n$ is a Möbius transformation onto $\mathbb{B}^n$, then

$$v_{\mathbb{B}^n}(f(x), f(y)) \leq 2v_{\mathbb{B}^n}(x, y)$$

for all $x, y \in \mathbb{B}^n$.

Modulus of a Curve Family. Let Γ be a family of curves in $\mathbb{R}^n$. Also, let $\mathcal{F}(\Gamma)$ denote the family of all non-negative Borel-measurable functions $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$

such that $\int_\gamma \rho ds \geq 1$ for each locally rectifiable curve $\gamma \in \Gamma$. The modulus of a curve family $\Gamma \subset \mathbb{R}^n$ is defined by (see [30, p. 104])

$$\mathsf{M}(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^n \mathrm{d}m,$$

where m stands for the n-dimensional Lebesgue measure. If $\mathcal{F}(\Gamma) = \emptyset$, we set $\mathsf{M}(\Gamma) = \infty$. This case occurs only if there is a constant path in Γ because otherwise, the constant function ∞ is in $\mathcal{F}(\Gamma)$.

We denote by $\Delta(E, F; G)$ the family of all closed non-constant curves joining two non-empty sets E and F in a domain G, where E, F, and G are subsets of $\overline{\mathbb{R}}^n$.

Capacity. Let $A \subset \mathbb{R}^n$ be an open set and $C \subset A$ be compact. The pair $E = (A, C)$ is called a *condenser*. If E is a bounded condenser in $\mathbb{R}^n$, then its *capacity* is given by (see [30, Definition 9.2 & Theorem 9.6])

$$\text{cap}(A, C) = \mathsf{M}(\Delta(C, \partial A; A)).$$

Our next step is to define two decreasing homeomorphisms. The Grötzsch capacity $\gamma_n : (1, \infty) \to (0, \infty)$ and the Teichmüller capacity $\tau_n : (0, \infty) \to (0, \infty)$ are defined by the following formulas (see [30, p. 121]):

$$\gamma_n(s) = \mathsf{M}(\Delta(\overline{\mathbb{B}}^n, [se_1, \infty]; \mathbb{R}^n)), \quad s > 1 \tag{8}$$

and

$$\tau_n(s) = \mathsf{M}(\Delta([-e_1, 0], [se_1, \infty]; \mathbb{R}^n)), \quad s > 0,$$

respectively. For $s > 1$, we have $\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1)$, $\lim_{s \to 1+} \gamma_n(s) = \infty$, and $\lim_{s \to \infty} \gamma_n(s) = 0$, see [30, Theorem 7.31].

Ferrand's Metric or Ferrand's Modulus Metric. Suppose that G is a proper subdomain of $\overline{\mathbb{R}}^n$, and $x, y \in G$ such that $x \neq y$. We define λ_G as follows:

$$\lambda_G(x, y) = \inf_{C_x, C_y} \mathsf{M}(\Delta(C_x, C_y; G)),$$

where the infimum is taken over all pairs of continua C_x and C_y such that $x \in C_x$, $y \in C_y$, $\overline{C_x} \cap \partial G \neq \varnothing$, and $\overline{C_y} \cap \partial G \neq \varnothing$; see Figure 2 for more details. It follows from [30, Lemma 7.10] that λ_G is invariant under conformal mappings of G. Ferrand proved that $\lambda_G^{-1/n}$ is a metric; see [52]. To avoid confusion with the other metric named after her, we will refer to it as Ferrand's metric or Ferrand's modulus metric. The proof that the expression $\lambda_G^{1/(1-n)}$ is a metric for the case $G = \mathbb{B}^n$ was presented in [6], and the question of whether this holds for all domains was discussed in [7, p. 193]. A. Yu. Solynin [76] and J. Jenkins [43] independently solved this question for $n = 2$. The general case $n \geq 3$ was solved by J. Ferrand, who proved $\lambda_G^{1/(1-n)}$

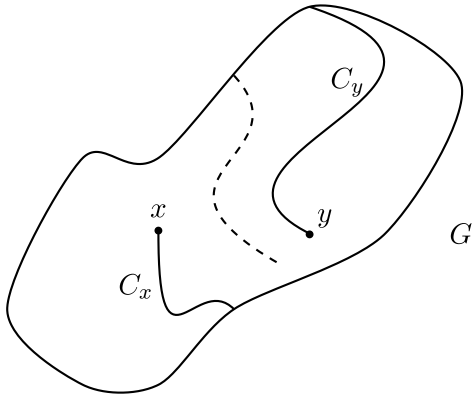


Figure 2. Conformal invariant $\lambda_G(x, y)$.

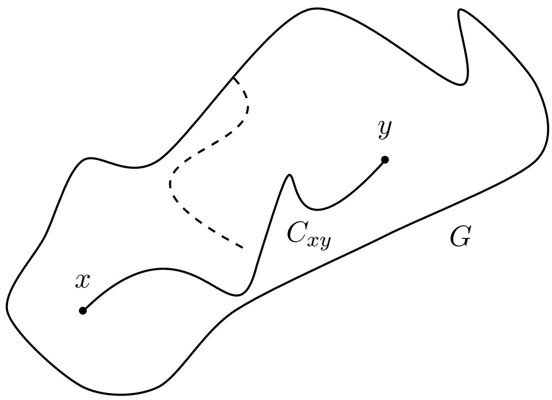


Figure 3. Conformal invariant $\mu_G(x, y)$.

defines a metric on G for all proper subdomains G of $\mathbb{R}^n$; see [18] or [30, Theorem 10.3].

Modulus Metric. Let G be a proper subdomain of $\overline{\mathbb{R}}^n$. Define

$$\mu_G(x, y) = \inf_{C_{xy}} \mathsf{M}(\Delta(C_{xy}, \partial G; G)),$$

where the infimum is taken over all continuous paths C_{xy} in G joining x and y, represented by a continuous function $\gamma : [0, 1] \to G$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$; see Figure 3. The metric μ_G is also invariant under conformal mappings of G. We note that μ_G is a metric if $\operatorname{cap} \partial G > 0$, see [30, p. 174]. If $\operatorname{cap} \partial G > 0$, we call μ_G the modulus metric or conformal metric of G.

The Möbius Metric or Seittenranta Metric. For distinct points a, b, c and d in $\overline{\mathbb{R}}^n$ let

$$m(a, b, c, d) = \max\{|a, b, d, c|, |a, c, d, b|\}.$$

If $G \subset \overline{\mathbb{R}}^n$ is a domain with card $(\overline{\mathbb{R}}^n \setminus G) \geq 2$, then let

$$m_G(b,c) = \sup\{m(a,b,c,d) : a,d \in \partial G\}.$$

Then the Möbius metric or Seittenranta metric $\delta_G(x,y)$ is defined by

$$\delta_G(x,y) = \log\left(1 + m_G(x,y)\right).$$

This metric is defined in a domain with at least two boundary points and is invariant under Möbius transformations (see [75]). It is also monotone with respect to the domain, i.e., if $G \subsetneq \overline{\mathbb{R}}^n$, and $G' \subset G$ are domains, then $\delta_G(x,y) \leq \delta_{G'}(x,y)$ for all $x,y \in G'$.

2.3 Special Functions

Elliptic Integrals and $\gamma_2(s)$. If we let $n = 2$ in (8), the capacities can be computed with the following formula (see [30, p. 122]]):

$$\gamma_2(1/r) = \frac{2\pi}{\mu(r)}, \quad 0 < r < 1,$$

where

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)} \quad \text{and} \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}.$$

The function $\mathcal{K}(r)$ is called a complete elliptic integral of the first kind, which satisfies the following Landen identities:

$$\begin{cases} \mathcal{K}\left(\dfrac{2\sqrt{r}}{1+r}\right) = (1+r)\,\mathcal{K}(r) = \mathcal{K}'\left(\dfrac{1-r}{1+r}\right); \\[2ex] \mathcal{K}\left(\dfrac{1-r}{1+r}\right) = \dfrac{1}{2}(1+r)\,\mathcal{K}'(r) = \mathcal{K}'\left(\dfrac{2\sqrt{r}}{1+r}\right), \end{cases}$$

where $\mathcal{K}'(r) = \mathcal{K}(\sqrt{1-r^2})$, see [30, p. 122]. From now on, we will assume that $r' = \sqrt{1-r^2}$ whenever we are not at risk of confusion. Also, the following three functional identities are satisfied by the modulus $\mu(r)$ (see [30, p. 122]):

$$\begin{cases} \mu(r')\mu(r) = \dfrac{\pi^2}{4}; \\[2ex] \mu(r)\mu\left(\dfrac{1-r}{1+r}\right) = \dfrac{\pi^2}{2}; \\[2ex] \mu(r) = 2\mu\left(\dfrac{2\sqrt{r}}{1+r}\right) = \dfrac{1}{2}\mu\left(\left(\dfrac{r}{1+r'}\right)^2\right). \end{cases} \tag{9}$$

The following inequalities hold for $r \in (0,1)$ (see [30, p. 122]):

$$\log\frac{1}{r} < \log\frac{1+3r'}{r} < \log\frac{(1+\sqrt{r'})^2}{r} < \mu(r) < \log\frac{2(1+r')}{r} < \log\frac{4}{r}. \tag{10}$$

As a result of Jacobi's work, the following inequalities hold for $0 < r < 1$ [7, p. 91, (5.30)]

$$\log\frac{(1+\sqrt{r'})^2}{r} < \operatorname{arth}\sqrt[4]{r'} < \mu(r) < \frac{\pi^2}{4\operatorname{arth}\sqrt[4]{r}}. \tag{11}$$

The Grötzsch capacity has an upper bound in terms of the function $\mu(r)$ as follows (see [30, Theorem 9.17(2)]):

$$2^{n-1}c_n\log\left(\frac{s+1}{s-1}\right) \le \gamma_n(s) \le 2^{n-1}c_n\mu\left(\frac{s-1}{s+1}\right) < 2^{n-1}c_n\log\left(4\frac{s+1}{s-1}\right), \tag{12}$$

where the constant c_n is given by (see Ref. [7, p. 41]):

$$c_n := \left(2\int_0^{\pi/2}\sin^{\frac{2-n}{n-1}}t\,dt\right)^{1-n}\omega_{n-2} \tag{13}$$

for $n \ge 2$. In particular, $c_2 = 2/\pi$, and $c_3 = 4\pi^2/\Gamma(1/4)^4 = 0.22847329\ldots$.

The Distortion Function. For $r \in (0,1)$ and $K > 0$, we define a homeomorphism $\varphi_K : [0,1] \to [0,1]$ by

$$\varphi_K(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))} = \varphi_{K,n}(r), \tag{14}$$

with $\varphi_K(0) = 0$, and $\varphi_K(1) = 1$. This function will occur in the quasiregular version of the Schwarz lemma as well as in its many applications. The paper [8] presents a survey of these special functions. The basic estimate for $\varphi_K(r)$, $K \ge 1$, $r \in (0,1)$ is [7, Theorem 10.9(1)]

$$r^{1/K} < \varphi_K(r) < 4^{1-1/K}r^{1/K}. \tag{15}$$

If we let $n = 2$, $K > 0$, and $r \in (0,1)$, then we have (see [7, Eq. (5.35)])

$$\varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K).$$

Jacobian Elliptic Functions. The function $\operatorname{sn}(z,\lambda)$ is the inverse of the elliptic integral of the first kind and is defined implicitly by

$$z = \int_0^{\operatorname{sn}(z,\lambda)}\frac{du}{\sqrt{(1-u^2)(1-\lambda^2u^2)}}, \quad \lambda^2 \le 1.$$

The functions $\operatorname{cn}(z,\lambda)$ and $\operatorname{dn}(z,\lambda)$ are defined by

$$\operatorname{sn}^2(z,\lambda) + \operatorname{cn}^2(z,\lambda) = 1 \quad \text{and} \quad \lambda^2\operatorname{sn}^2(z,\lambda) + \operatorname{dn}(z,\lambda)^2 = 1. \tag{16}$$

In particular, $\operatorname{cn}(0,\lambda) = \operatorname{dn}(0,\lambda) = 1$.

2.4 Mappings

Absolutely Continuous on Lines (ACL). Let $\mathbb{R}^{n-1}_j := \{x \in \mathbb{R}^n : x_j = 0\}$, where $j = 1, 2, \ldots, n$. Furthermore, let $T_j : \mathbb{R}^n \to \mathbb{R}^{n-1}_j$ be an onto orthogonal projection $T_j x = x - x_j e_j$. Let $Q = \{x \in \mathbb{R}^n : a_j \leq x_j \leq b_j\}$ be a closed n-interval. A mapping $u : Q \to \mathbb{R}$ is called *absolutely continuous on lines*, abbreviated as ACL if it is absolutely continuous on almost every line segment in Q, parallel to the coordinate axes $e_1, \ldots, e_n$. More precisely, if E_j is the set of all $x \in T_j Q$ such that the mapping $t \mapsto u(x + te_j)$ is not absolutely continuous on $[a_j, b_j]$, then $m_{n-1}(E_j) = 0$ for all $j = 1, \ldots, n$.

Let Ω be an open set in $\mathbb{R}^n$. An ACL mapping $u : \Omega \to \mathbb{R}$ is said to be ACL^p, $p \geq 1$, if u is locally L^p-integrable in Ω and if the partial derivatives $\partial_j u$ (which exist a.e. and are measurable) of u are locally L^p-integrable as well; see Ref. [57, p. 22].

K-Quasiregular Mappings. Let $G \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. A mapping $f : G \to \mathbb{R}^n$ is K*-quasiregular* if f belongs to ACL^n and there exists a constant $K \geq 1$ satisfying the inequality

$$|f'(x)|^n \leq K J_f(x), \quad \text{where} \quad |f'(x)| = \max_{|\phi|=1} |f'(x)\phi|,$$

almost everywhere in G. Here, $f'(x)$ and $J_f(x)$ represent the formal derivative and the Jacobian determinant of f at the point x, respectively; see [30, p. 288].

K-Quasiconformal Mappings. A homeomorphism $f : G \to G'$ between two domains $G, G' \subset \overline{\mathbb{R}}^n$, $n \geq 2$, is called K*-quasiconformal* with some constant $K \geq 1$, if the two-sided inequality

$$\mathsf{M}(\Gamma)/K \leq \mathsf{M}(f(\Gamma)) \leq K\mathsf{M}(\Gamma)$$

holds for every curve family Γ in G; see [30, p. 289].

Hölder Continuity. Let (X, d_1) and (Y, d_2) be metric spaces, and $\beta \in (0, 1]$. A mapping $f : X \to Y$ is said to be β-Hölder continuous if there exists a constant C such that $d_2(f(x), f(y)) \leq C d_1(x, y)^\beta$ for all $x, y \in X$. Further, f is said to be Hölder continuous if and only if it is β-Hölder continuous for some $\beta \in (0, 1]$; see [74, p. 143].

Bi-Lipschitz Mappings. A mapping $f : X \to Y$ between two metric spaces (X, d_1) and (Y, d_2) is called *bi-Lipschitz*, or C-bi-Lipschitz to emphasize the constant, if there is a constant $C \geq 1$ such that

$$\frac{1}{C} d_1(x, y) \leq d_2(f(x), f(y)) \leq C d_1(x, y)$$

for every pair of points $x, y \in X$; see [74, p. 147].

Conformal Radius. Let D be a simply connected domain in the complex plane with a non-degenerate boundary. The *conformal radius* $r_D(z_0)$ at a point $z_0 \neq \infty$ is the radius of a disk centered at the origin, which can be mapped onto D conformally by a function F with the normalization $F(0) = z_0$ and $F'(0) > 0$; see Ref. [63, p. 162]. Let $g : D \to \mathbb{B}^2$ be a conformal mapping of D onto $\mathbb{B}^2$ and f be its inverse. If $g(z_0) = \zeta$, then

$$r_D(z_0) = |f'(\zeta)|(1 - |\zeta|^2) = \frac{1 - |g(z_0)|^2}{|g'(z_0)|}$$

and if $G : D \to \mathbb{H}^2$ is a conformal mapping of D onto the upper half plane, F is its inverse, and $G(z_0) = w$, then

$$r_D(z_0) = 2 \operatorname{Im} w \, |F'(w)| = \frac{2 \operatorname{Im} G(z_0)}{|G'(z_0)|}.$$

Harmonic Mappings. Let Ω be a domain and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. We say that an analytic and normalized ($f(0) = 0 = f'(0) - 1$) function f is *starlike* in $\mathbb{D}$ if $f(\mathbb{D})$ is a set that is starlike with respect to the origin. In other words, the straight line joining any point in $f(\mathbb{D})$ to the origin lies in $f(\mathbb{D})$. In addition, Ω is *convex* if any line segment joining two points in Ω also lies in Ω. A univalent function f is called convex if $f(\mathbb{D})$ is a convex set. A planar *harmonic* mapping $f : \Omega \to \mathbb{C}$ is a complex-valued function of the form $f = u(z) + iv(z)$, where $z = x + iy$, and u, v are real harmonic functions. It is known that each complex-valued harmonic mapping f on $\mathbb{D}$ admits the decomposition $f = h + \overline{g}$, where both h and g are analytic in $\mathbb{D}$, see [15]. We denote by $\mathcal{H}$ the class of all normalized ($f(0) = 0$ and $f_z(0) = 1$) harmonic mappings f having the form $f = h + \overline{g}$, where

$$h(z) := z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) := \sum_{n=1}^{\infty} b_n z^n. \tag{17}$$

The Jacobian of the mapping f is defined as $J_f(z) := |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$. We notice that (see [53]) a harmonic mapping $f = h + \overline{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if, and only if, $J_f(z) > 0$. We denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f \in \mathcal{H}$ which are *sense-preserving* and *univalent* in $\mathbb{D}$.

Let $z = re^{i\theta} \in \mathbb{D}$, $0 \leq r < 1$, and $0 \leq \theta \leq 2\pi$. A function $f \in \mathcal{H}$ is called a harmonic *starlike* function of order α, $\alpha \in [0, 1)$, if

$$\frac{\partial}{\partial \theta}\left(\arg f(re^{i\theta})\right) > \alpha.$$

The class of harmonic starlike functions f of order α in $\mathbb{D}$ will be denoted by $\mathcal{S}^*_{\mathcal{H}}(\alpha)$. Likewise, a function $f \in \mathcal{H}$ is called harmonic *convex* mapping of order α, $\alpha \in [0, 1)$, if

$$\frac{\partial}{\partial\theta}\left\{\arg\left(\frac{\partial}{\partial\theta}\left(\arg f(re^{i\theta})\right)\right)\right\} > \alpha.$$

We denote by $\mathcal{K}_{\mathcal{H}}(\alpha)$ the class of all harmonic convex mappings f of order α in $\mathbb{D}$.

3 Intrinsic Metrics in Polygonal Domains

Intrinsic metrics are the most useful metrics in geometric function theory. These types of metrics do not only consider the distance between points, but also their distance from the domain boundary. The hyperbolic metric $\rho_G(x, y)$, $x, y \in G$, of planar domains G is ideal for this purpose, especially due to their conformal invariance. Considering the planar case, estimating the hyperbolic distances of a given domain is challenging [12]. This chapter aims to study the case when the domain is polygonal in the plane and to compare the value of the hyperbolic metric to the triangular ratio metric s_G. As a special case, we consider the rectangle-shaped polygonal domain. Thus, it is natural to study, for two given points in a rectangle, finding the extremal point for s-metric.

3.1 Nearest Point For a Rectangle

Let R be a rectangle with vertices $\pm k \pm i$, where $k \geq 1$. By the definition of the s-metric, if a point $x \in R$ is fixed, then the infimum $|x - z| + |z - y|$ is attained at some points z lying on a side of ∂R. This side depends on the location of y. For a fixed $x \in R$ we have four subdomains, two trapezoids, and two triangles. Indeed, the lower and the upper subdomains are trapezoids if and only if $\mathrm{Re}\, x^2 - \mathrm{Im}\, x^2 < k^2 - 1$; in the case of $\mathrm{Re}\, x^2 - \mathrm{Im}\, x^2 > k^2 - 1$ trapezoids are the left and right subdomains. The case $\mathrm{Re}\, x^2 - \mathrm{Im}\, x^2 = k^2 - 1$ corresponds to a subdivision of R into four triangles. Figure 4 shows a subdivision for $k = 1.5$ and $x = 0.8 - 0.4i$. It also shows that if y belongs to the right triangle, then z belongs to the blue segment (the right width).

The main goal of this chapter is to find the best positive constants C_1 and C_2 such that for the rectangle R the following inequalities hold:

$$C_1 s_R(u, v) \leq \mathrm{th}\, \frac{\rho_R(u, v)}{2} \leq C_2 s_R(u, v), \quad u, v \in R.$$

Based on our methodology, we investigate the relationship between conformal radius and boundary distance for a domain. We will find the limit

$$\lim_{u \to v} \frac{\mathrm{th}\, (\rho_R(u, v)/2)}{s_R(u, v)}, \quad u, v \in R.$$

In the following, we give the geometric interpretation of the above limit.

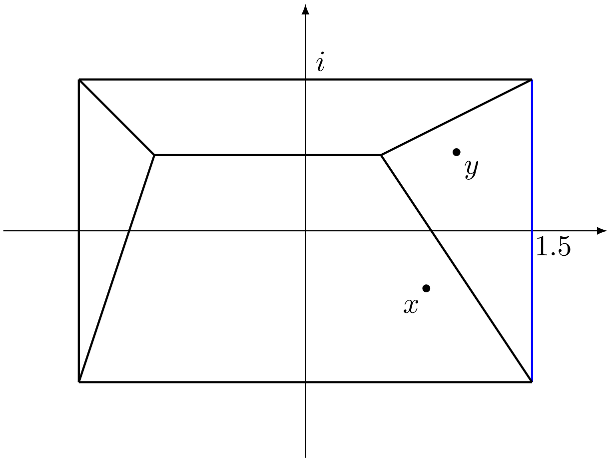


Figure 4. The rectangle $[-1.5, 1.5] \times [-i, i]$ with a fixed point $x = 0.8 - 0.4i$.

3.2 Geometric Interpretation

Let the segments ℓ_1 and ℓ_2 be parts of the bisectors of the angles at the vertices $-k-i$ and $-k+i$ of the rectangle R, respectively, connecting these vertices with the point $-(k-1)$ on the real axis. Similarly, let ℓ_3 and ℓ_4 join the vertices $k \pm i$ with the point $k-1$ on the real axis. A segment with the endpoints $\pm(k-1)$ is denoted by ℓ_0. It is clear that these five segments subdivide R into four subdomains, two triangles, and two trapezoids; see Figure 5 in the case of $k = 2$. We see that if a point v is in one of these subdomains, then the point ζ_0, where the minimal value of $|\zeta - v|$, $\zeta \in \partial R$, is attained, lies on the side of R which is common with this subdomain. If v is on one of the segments described above and $v \neq \pm(k-1)$, then the shortest distance is attained at two sides of R. At last, if $v = \pm(k-1)$, then there are three sides where the minimum value of $|\zeta - v|$, $\zeta \in \partial R$, is attained; in the case $k = 1$, we have the square, the points $\pm(k-1)$ coincide with its center, and the minimal distance is attained at all four of the sides.

We can describe this situation as follows: If v is an interior point of one subregion, then the largest disk contained in R and centered at v touches ∂R at one point lying on the surface of R which is common with this subdomain (the dotted circle). If $v \in \cup_{j=0}^{4} \ell_j$ and $v \neq \pm(k-1)$, then the largest disk touches two of the sides (the dashed circle), and if $v = \pm(k-1)$, then it touches ∂R at three points (the right circle). These three cases are shown in Figure 5. (If $k = 1$, then the largest disk touches all four sides of R).

3.3 Main Results

We begin this section with the following lemma:

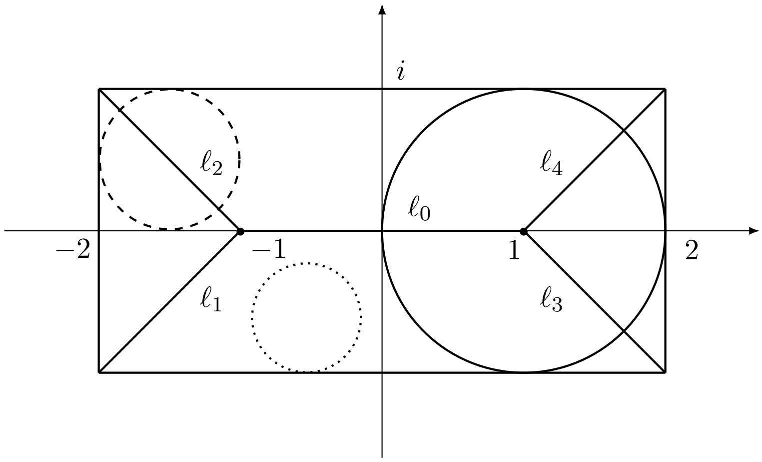


Figure 5. Decomposition of a rectangle R, and five segments ℓ_j ($0 \le j \le 4$) in the case $k = 2$.

***Lemma* 18.** *[I, Lemma 4.1] We have*

$$\lim_{u\to v} \frac{\operatorname{th}(\rho_R(u,v)/2)}{s_R(u,v)} = \frac{2d_R(v)}{r_R(v)},$$

where $d_R(v) = dist(v, \partial R)$ and $r_R(v)$ is the conformal radius of R at the point v.

***Lemma* 19.** *[I, Lemma 4.2] Let D be a convex planar domain with nonempty boundary ∂D and suppose that one of the following conditions is valid:*

(i) The circle centered at a point $z_0 \in D$ of radius $d_D(z_0)$ touches ∂D at some point ζ_0;

(ii) The boundary ∂D contains two linear segments $[a, \zeta_0]$ and $[\zeta_0, b]$ and the circle centered at a point $z_0 \in D$ of radius $d_D(z_0)$ touches ∂D at two points $\zeta_1 \in [a, \zeta_0]$ and $\zeta_2 \in [\zeta_0, b]$.

Then for every z_1 which is an interior point of the segment with endpoints z_0 and ζ_0 we have

$$\frac{d_D(z_1)}{r_D(z_1)} \le \frac{d_D(z_0)}{r_D(z_0)}. \tag{20}$$

Moreover, if either (i) holds and D is distinct from a half-plane, or (ii) is valid and ∂D is distinct from the union of two rays coming from the point ζ_0 and passing through the points a and b, then the inequality in (20) *is strict.*

The main result of this chapter is as follows:

***Theorem* 21.** *[I, Theorem 4.3] If R is the rectangle $[-k, k] \times [-1, 1]$, $k \ge 1$, then for all $v \in R$*

$$1 < \frac{2d_R(v)}{r_R(v)} \le C(\lambda), \tag{22}$$

where
$$C(\lambda) = \mathcal{K}(\lambda) \left| \frac{\operatorname{cn}(i\,\mathcal{K}(\lambda), \lambda) \operatorname{dn}(i\,\mathcal{K}(\lambda), \lambda)}{\operatorname{sn}(i\,\mathcal{K}(\lambda), \lambda)} \right|. \tag{23}$$
Both inequalities of (22) *are sharp. Moreover, equality on the right-hand side of* (22) *holds if and only if* $v = \pm(k-1)$.

Since $\operatorname{sn}(i\,\mathcal{K}(\lambda), \lambda)$ is a pure imaginary number and by (16), we can write (23) in the form
$$C(\lambda) = \mathcal{K}(\lambda) \frac{\sqrt{(1+a^2(\lambda))(1+\lambda^2 a^2(\lambda))}}{a(\lambda)},$$
where $a(\lambda) := |\operatorname{sn}(i\,\mathcal{K}(\lambda), \lambda)|$.

***Corollary* 24.** *[I, Corollary 4.4] We have*
$$\frac{2}{C(\lambda)} d_R(v) \leq r_R(v) < 2 d_R(v), \quad v \in R$$
and the inequalities are sharp.

Based on the above results, we formulate the following conjecture:

***Conjecture* 25.** *[I, Conjecture 4.5] Let* $C(\lambda)$ *be defined as in* (23). *Then for all points* u *and* v *lying in the rectangle* $R = [-k, k] \times [-1, 1]$, $k \geq 1$, *we have*
$$s_R(u, v) \leq \operatorname{th} \frac{\rho_R(u, v)}{2} \leq C(\lambda) s_R(u, v).$$
These inequalities are sharp.

***Remark* 26.** *[I, Remark 4.6] The inequality* $s_R(u, v) \leq \operatorname{th}(\rho_R(u, v)/2)$ *and its sharpness are evident.*

It should be noted that we can also consider a rectangle with $k < 1$. For this purpose, it is enough to consider the mapping $z \mapsto iz/k$ which maps R onto the rectangle $[-1/k, 1/k] \times [-1, 1]$.

It should also be noted that the method described above, based on Lemma 19, can be used to obtain sharp two-sided estimations of $2d_D(\cdot)/r_D(\cdot)$ for other convex planar domains D. First, consider a bounded convex n-gon P and its decomposition into parts $P = \cup_{j=1}^n P_j$ such that $z \in P_j$ if and only if z is closer to the j-th side of P than to other sides. Then we investigate the set $\cup_{j=1}^n \partial P_j$. It is a graph and is open after removing the boundary points of P from it, i.e., without endpoints, edges that have a nonempty intersection with ∂P. Then we obtain its subgraph and denote it by $Gr(P)$. For example, if we take P as a rectangle $R = [-k, k] \times [-1, 1]$, then $Gr(P)$ consists of the segment ℓ_0. If P admits an inscribed circle (a circle that is tangent to all sides of P), then $Gr(P)$ consists of the point that is the center of the circle.

Lemma 19 immediately implies the following result:

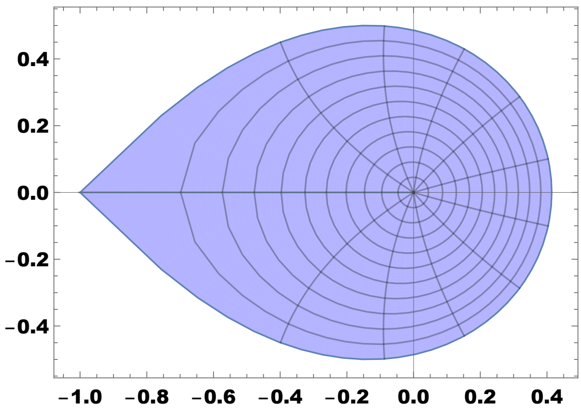


Figure 6. The image of the unit disk under the function $\phi_1(z)$, i.e. $\phi_1(\mathbb{B}^2) =: D_1$.

Theorem 27. *[I, Theorem 6.1] The maximum of* $2d_P(u)/r_P(u)$, $u \in P$, *is attained at some point of the graph* $Gr(P)$.

Analyzing some concrete domains and classes of domains (see Example 30 below), we can suggest the following conjecture:

Conjecture 28. *[I, Conjecture 6.2] The maximum of* $2d_P(u)/r_P(u)$, $u \in P$, *is attained at a vertex of the graph* $Gr(P)$.

Before computing $2d_D(\cdot)/r_D(\cdot)$, we give an example that determines the maximum value of the conformal radius.

Example 29. [I, Example 5.1] Let $w = \phi_1(z) = \sqrt{1+z} - 1$, $z \in \mathbb{B}^2$, where the branch of the square root is chosen such that $\sqrt{1} = 1$. The function ϕ_1 maps the open unit disk $\mathbb{B}^2$ onto the domain D_1 which is the right-half of the lemniscate of Bernoulli; see Figure 6. Hence, the inverse mapping $z = \varphi_1(w) = w(w+2)$ maps D_1 onto $\mathbb{B}^2$. Since $\varphi_1(0) = 0$ and $\varphi_1'(0) = 2 > 0$, we have $r_{\phi_1(\mathbb{B}^2)}(0) = 0.5$. Now we will find the maximum of the conformal radius of D_1. If $w = \phi_1(z)$, where $z = te^{i\theta}$, $t \in [0, 1)$ and $\theta \in [0, 2\pi)$, then by the definition of conformal radius we obtain

$$r_{D_1}(w) = \frac{1}{2}|1+z|^{-1/2}(1-|z|^2) \le \frac{1}{2}(1-t)^{-1/2}(1-t^2) = \frac{1}{2}(1-t)^{1/2}(1+t)$$

and the equality holds if and only if $\theta = \pi$. The maximal value of the function $(1-t)^{1/2}(1+t)/2$ is attained at $t_0 = 1/3$ and is equal to $2\sqrt{6}/9$. Thus, we see that the maximum of $r_{D_1}(w)$ in $\mathbb{B}^2$ is equal to $2\sqrt{6}/9 = 0.544331\ldots$ which is attained at the point $w = \phi_1(-1/3) = \sqrt{6}/3 - 1 = -0.183503\ldots$.

We are now ready to compute $2d_D(\cdot)/r_D(\cdot)$ for the non-polygonal domain described in Example 29.

Example 30. [I, Example 6.14] The domain D_1 which is the image of the unit disk under the mapping $w = \phi_1(z) = \sqrt{1+z} - 1$ is convex. It follows from Lemma 19 that the maximum of the ratio $2d_{D_1}(w)/r_{D_1}(w)$, $w \in D_1$, is attained at some point w that is real, i.e., $-1 < w < \sqrt{2} - 1$. Now we will find $d_{D_1}(w)$ for such w.

Denote $t = w + 1$, $0 < t < \sqrt{2}$. Then

$$\sqrt{1+e^{i\theta}} = e^{i\theta/4}\sqrt{2\cos(\theta/2)}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

and

$$\begin{aligned} A(\theta) :&= |\phi_1(e^{i\theta}) - w|^2 = |\sqrt{1+e^{i\theta}} - t|^2 \\ &= t^2 - 2\sqrt{2}t\cos(\theta/4)\sqrt{\cos(\theta/2)} + 2\cos(\theta/2). \end{aligned}$$

If we put $\tau = \cos(\theta/4)$, then we obtain $A(\theta) = f(\tau)$, where

$$f(\tau) = t^2 - 2\sqrt{2}t\tau\sqrt{2\tau^2 - 1} + 2(2\tau^2 - 1), \quad \sqrt{2}/2 \le |\tau| \le 1.$$

Analyzing $f(\tau)$, we find that for $2\sqrt{2}/3 \le t < \sqrt{2}$ we have $f(\tau) \ge f(1) = (\sqrt{2}-t)^2$, therefore, the distance from w to the boundary of D_1, i.e., $d_{D_1}(w)$, equals to $\sqrt{2} - t$. If $0 < t < 2\sqrt{2}/3$, then $f(\tau)$ attains its minimum at the point τ_0 such that

$$\tau_0^2 = \frac{1}{4}\left(1 + \frac{1}{\sqrt{1-t^2}}\right) \tag{31}$$

and then $d_{D_1}(w) = \sqrt{\sqrt{1-t^2} - (1-t^2)}$. Therefore,

$$d_{D_1}(w) = \begin{cases} \sqrt{\sqrt{1-t^2} - (1-t^2)}, & 0 < t < 2\sqrt{2}/3, \\ \\ \sqrt{2} - t, & 2\sqrt{2}/3 \le t < \sqrt{2}, \end{cases} \tag{32}$$

where $t = 1 + w$.

Let $\tau_0 = \cos(\theta_0/4)$. Analyzing (31), we see that when t increases from 0 to $2\sqrt{2}/3$ the point θ_0 decreases from π to 0. Therefore, for every boundary point, $\phi_1(\theta)$, $0 < \theta < \pi$, there is a $t \in [0, 2\sqrt{2}/3]$ such that the maximal circle in D_1 centered at the point $w = t - 1$ touch $\phi_1(\theta)$. Because of the symmetry of D_1 with respect to the real axis, a similar fact is valid for $-\pi < \theta < 0$. Then, from Lemma 19 we conclude that the maximal value of $2d_{D_1}(w)/r_{D_1}(w)$, $w \in D_1$, is attained at some point on the real axis.

For the conformal radius at the points $w = \phi_1(z)$, $-1 < z < 1$, we have $r_{D_1}(w) = (1/2)(1-z)^{1/2}(1+z)$. Taking into account that $z = (w+1)^2 - 1 = t^2 - 1$ we have

$$r_{D_1}(w) = \frac{1}{2}t\left(2 - t^2\right). \tag{33}$$

Simple analysis of the function $2d_{D_1}(w)/r_{D_1}(w)$ with the help of (32) and (33) shows that it has a unique maximum at the point $w_0 = \sqrt{1-u_0^2}-1 = -0.109718\ldots$, where $u_0 = 0.45541\ldots$ is a unique root of the cubic equation $4u^3+3u^2-1=0$. The maximum equals $1.85318\ldots$. Therefore, in D_1 we have the sharp inequality

$$\frac{2d_{D_1}(w)}{r_{D_1}(w)} \leq 1.85318\ldots, \quad w \in D_1.$$

4 Landen Transformations Applied to Approximation

Landen's transformation is a mapping of the parameters of an elliptic integral that is useful for numerically evaluating elliptic functions. John Landen first defined this transformation, which played an important role in the work of A.M. Legende, C.F. Gauss, and many other researchers. The *Landen transformation* is defined by

$$r \mapsto \frac{2\sqrt{r}}{1+r},$$

where $0 < r < 1$. Also, a *Landen sequence* of functions, recursively defined in terms of this transformation, is closely related to the elliptic functions. This chapter aims to show that some of the well-known special functions of geometric function theory can be efficiently computed using a few steps of the Landen iteration. These functions include the functions $\mu(r)$ and $\varphi_K(r)$.

Additionally, this chapter provides a recursive scheme for numerical approximations of several functions. The approximations were obtained after five iterations with errors close to the machine epsilon. There are only a few lines of code, and it can be implemented in any programming language; no programming libraries are needed. For instance, the first iteration for the function φ_K improves the classical majorant function $4^{1-1/K}r^{1/K}$ for $\varphi_K(r)$, where $K > 1$.

Ascending and Descending Landen Sequences. For $r \in (0,1)$, let $L(r,0) = r$,

$$L(r,p+1) = \frac{2\sqrt{L(r,p)}}{1+L(r,p)} \quad \text{and} \quad L(r,-p-1) = \left(\frac{L(r,-p)}{1+\sqrt{1-L(r,-p)^2}}\right)^2,$$

for $p = 0,1,2,3,\ldots$. The recursively defined sequences $\{L(r,p)\}$ and $\{L(r,-p)\}$ are called *ascending* and *descending Landen sequences*, respectively. It is clear that each of the Landen functions $L(\cdot,p) : (0,1) \to (0,1)$ is an increasing homeomorphism with

$$L(r,p) < L(r,p+1) \quad \text{and} \quad L(r,p+q) = L(L(r,p),q)$$

for all $r \in (0,1)$ and $p,q \in \mathbb{Z}$. In particular,

$$y = L(r,p) \Leftrightarrow r = L(y,-p).$$

Therefore, $L(\cdot,-p)$ is the inverse of $L(\cdot,p)$. There is a close connection between Landen transformations, elliptic integrals, and elliptic functions. In particular, the complete integrals $\mathcal{K}(r)$ satisfy functional identities associated with the Landen transformation, such that they can be expressed as infinite products whose factor functions correspond to Landen sequences. Moreover, it follows from (9) that

$$\mu(r) = 2^{-p}\mu(L(r,-p)) \tag{34}$$

for $r \in (0,1)$ and $p \in \mathbb{Z}$. As shown in [7, Section 5] one could use (34) to refine inequalities for $\mu(r)$ such as (11).

Let us start the results of this chapter with the following proposition:

***Proposition* 35.** *[II, Proposition 2.1] Let $r \in (0,1)$ and $p = 0,1,2,\ldots$. Then*

$$L(r,-p-1) < r^{2^p} \leq r \tag{36}$$

and

$$L(r,p+1) > r^{2^{-p}} \geq r.$$

Next, we find two identities for Landen's sequences.

***Proposition* 37.** *[II, Proposition 2.2] The following identities hold for all $r \in (0,1)$ and $p = 0,1,2,3,\ldots$.*

$$L(r,p)^2 + L(r',-p)^2 = 1; \tag{38}$$

$$L(r,p-1) = \frac{1-L(r',-p)}{1+L(r',-p)}.$$

***Remark* 39.** *[II, Remark 2.3] It follows from* (38) *and inequality of* (36) *that*

$$L(r,p+1) = \sqrt{1-L(r',-p-1)^2} > \sqrt{1-(r')^{2^{p+1}}} := \ell(r,p).$$

Computer experiment shows that this lower bound, i.e., $\ell(r,p)$, is better than the lower bound in (35)*, i.e. $r^{2^{-p}}$, when r is close to* 1.

Arithmetic-Geometric Mean. For $0 < b < a$ define $a_0 = a$, $b_0 = b$, $a_{n+1} = (a_n+b_n)/2$ and $b_{n+1} = \sqrt{a_n b_n}$. The common limit of these sequences

$$AG(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

is the *arithmetic geometric mean* of a and b.

***Lemma* 40.** *[II, Lemma 2.6] Consider the arithmetic geometric mean iteration with $a = 1$, $b \in (0,1)$, and $\alpha \in [0,1]$, and let b_n be the n^{th} iterate of the b-sequence. Then $b_n < L(b^\alpha,n)$ for $n = 1,2,3\ldots$.*

Table 1. Lower bounds of μ and their inverses

j	$u_j(r)$	$u_j^{-1}(y)$
1	$\operatorname{arth}\sqrt[4]{r'}$	$\sqrt{1-\operatorname{th}^8(y)}$
2	$\log\dfrac{(1+\sqrt{r'})^2}{r}$	$-$
3	$\log\dfrac{1+3r'}{r}$	$\dfrac{\exp(y)+3\sqrt{8+\exp(2y)}}{9+\exp(2y)}$
4	$\log(1/r)$	$\exp(-y)$
5	$\log\dfrac{1+r'}{r}$	$\dfrac{2\exp(y)}{1+\exp(2y)}$

Table 2. Upper bounds of μ and their inverses

j	$v_j(r)$	$v_j^{-1}(y)$
1	$\dfrac{\pi^2}{4\operatorname{arth}\sqrt[4]{r}}$	$\operatorname{th}^4(\pi^2/4y)$
2	$\log\dfrac{2(1+r')}{r}$	$\dfrac{4\exp(\max\{y,\log 2\})}{4+\exp(2\max\{y,\log 2\})}$
3	$\log(4/r)$	$4\exp(-\max\{y,\log 4\})$

By (10) and (11), we summarize the lower and upper bounds of $\mu(r)$ with their inverses in Table 1 and Table 2, respectively. We omitted the inverse of $u_2(r)$ in Table 1 due to its complicated formula.

In the next result, applying upper and lower bounds for μ, we find a new minorant and majorant for φ_K.

***Lemma* 41.** *[II, Lemma 2.8] Assume that $u, v : (0,1) \to (0,\infty)$ are decreasing homeomorphisms with*

$$u(r) < \mu(r) < v(r), \quad 0 < r < 1. \tag{42}$$

Then

$$u^{-1}(v(r)/K) < \varphi_K(r) < v^{-1}(u(r)/K)$$

for all $K > 1$ and $r \in (0,1)$.

***Corollary* 43.** *[II, Corollary 2.9] Let $u : (0,1) \to (0,\infty)$ and $v : (0,1] \to [c,\infty)$, $c > 0$, be decreasing homeomorphisms which satisfy* (42). *Then*

$$u^{-1}(v(r)/K) < \varphi_K(r) < v^{-1}(\max\{u(r)/K, c\})$$

for all $K > 1$ and $r \in (0,1)$.

Example 44. [II, Example 2.10] Consider u_1 and v_3 as above. Since v_3 is a decreasing homeomorphism from $(0, 1]$ onto $[\log 4, \infty)$, therefore,

$$u_1^{-1}(v_3(r)/K) < \varphi_K(r) < v_3^{-1}(\max\{u_1(r)/K, \log 4\})$$

for all $K > 1$ and $r \in (0, 1)$.

We recall the following lemma from [7, p. 17]:

***Lemma* 45.** *Let f be a decreasing homeomorphism from $(0, 1)$ onto $(0, \infty)$, and let g, h be strictly decreasing continuous functions from $(0, 1)$ into $(0, \infty)$ with $h(0+) = \infty$ such that $g(r) < f(r) < h(r)$. Let $C > 1$ and $s = f^{-1}(f(r)/C)$. Then*

$$g(r) > Cg(s) \quad \text{and} \quad h(r) < Ch(s), \quad 0 < r < 1,$$

if and only if $f(r)/g(r)$ and $h(r)/f(r)$ are strictly increasing on $(0, 1)$. In particular, if both $h^{-1}(h(r)/C)$ and $g^{-1}(g(r)/C)$ are defined, then

$$g^{-1}(g(r)/C) < s < h^{-1}(h(r)/C), \quad 0 < r < 1.$$

As an application of Lemma 45 we obtain:

***Lemma* 46.** *[II, Lemma 2.12] Let $u_1(r)$ and $v_2(r)$ be defined as in Tables 1 and 2, respectively, where $r \in (0, 1)$. Then*

$$u_1^{-1}(u_1(r)/K) < \varphi_K(r) < v_2^{-1}(v_2(r)/K)$$

for all $K > 1$ and $r \in (0, r_0)$, where $r_0 \in (0, 1)$ is such that both $u_1^{-1}(u_1(r)/K)$ and $v_2^{-1}(v_2(r)/K)$ are defined. Moreover, the left-hand side inequality is sharp in the sense that $u_1^{-1}(u_1(r)/K) \to r$ as $K \to 1$.

***Remark* 47.** *[II, Remark 2.13] It is worth mentioning that the new upper and lower bounds for φ_K are more accurate than those bounds in* (15). *Indeed,*

$$r^{1/K} < u_1^{-1}(u_1(r)/K) < \varphi_K(r) < v_2^{-1}(v_2(r)/K) < 4^{1-1/K} r^{1/K}$$

for all $K > 1$ and $r \in (0, 1)$.

4.1 Approximations

The three functions μ, μ^{-1} and φ_K were extensively investigated in [7] with computer implementations in languages such as Mathematica, MATLAB, and C on the accompanying diskette. Here, our goal is to show that for a large range of the arguments, we obtain results with accuracy to those in [7], now only using Mathematica. The methods applied in [7] for the numerical evaluation of $\mathcal{K}$ and μ were based on the arithmetic-geometric, while for μ^{-1} and φ_K a Newton iteration was used. Here, we show that the Landen sequences yield approximations with errors close to machine epsilon agreement with the results of [7] when the recursion level is moderate, 4, or 5.

Our starting point is the following lemma:

***Lemma* 48.** *[II, Lemma 3.1] (1) The function $\mu(r) + \log r$ is a monotone decreasing function from $(0,1)$ onto $(0, \log 4)$.*
(2) For $0 < r < 1$, and $p \in \mathbb{Z}$

$$2^{-p} \log \frac{1}{L(r,-p)} < \mu(r) < 2^{-p} \log \frac{4}{L(r,-p)},$$

and, in particular, for $p = 1$ we have

$$u_5(r) < \mu(r) < v_2(r),$$

where u_5 and v_2 are defined as in Tables 1 and 2, respectively.

***Proposition* 49.** *[II, Proposition 3.2] Let $u, v : (0,1) \to (0,\infty)$ be continuous functions with $|u(r) - v(r)| < M$ for some constant M and for all $r \in (0,1)$. Also let $u(r) < \mu(r) < v(r)$ for all $r \in (0,1)$. Then*

$$\lim_{p\to-\infty} 2^p u(L(r,p)) = \lim_{p\to-\infty} 2^p v(L(r,p)) = \mu(r).$$

Next, based on the Landen sequences, we will find the best and simplest approximation of $\mu^{-1}(y)$.

***Proposition* 50.** *[II, Proposition 3.4] Let $u, v : (0,1) \to (0,\infty)$ be decreasing homeomorphism with $u(r) < \mu(r) < v(r)$ for all $r \in (0,1)$. Then for $y > 0$ and $r = \mu^{-1}(y)$ we have*

$$L(u^{-1}(2^{-p}y), -p) < r = \mu^{-1}(y) < L(v^{-1}(2^{-p}y), -p).$$

***Remark* 51.** *[II, Remark 3.5] Applying Proposition 50 with $u_i(r)$ $(i = 1,3,4)$, $v_j(r)$ $(j = 1,2,3)$, and their inverses, we obtain the following approximations:*

$$\mu^{-1}(y) \approx L\left(u_1^{-1}(2^{-p}y), -p\right) =: f_1(y,p);$$

$$\mu^{-1}(y) \approx L\left(v_1^{-1}(2^{-p}y), -p\right) =: g_1(y,p);$$

$$\mu^{-1}(y) \approx L\left(u_3^{-1}(2^{-p}y), -p\right) =: f_2(y,p);$$

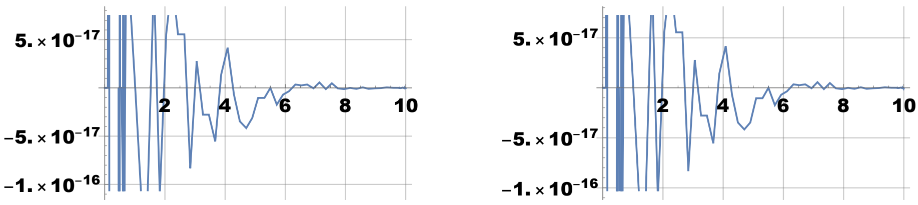
$$\mu^{-1}(y) \approx L\left(v_2^{-1}(2^{-p}y), -p\right) =: g_2(y,p);$$

$$\mu^{-1}(y) \approx L\left(u_4^{-1}(2^{-p}y), -p\right) =: f_3(y,p);$$

and

$$\mu^{-1}(y) \approx L\left(v_3^{-1}(2^{-p}y), -p\right) =: g_3(y,p).$$

Computer experiments show that $g_2(y,-4)$ and $g_3(y,-5)$ are the best approximations for $\mu^{-1}(y)$. We note that $\mu^{-1}(y) - g_2(y,-4)$ and $\mu^{-1}(y) - g_3(y,-5)$ have an error value of order $10^{-14}, \ldots, 10^{-24}$ in the interval $(0.2, 20)$; see Figure 7. For the computation of $\mu^{-1}(y)$ we use the Mathematica "cip.m" file from [7, Appendix B].



(a) (b)

Figure 7. (a): The graph of $\mu^{-1}(y) - g_2(y, -4)$, where $0 < y < 10$ (b): The graph of $\mu^{-1}(y) - g_3(y, -5)$, where $0 < y < 10$.

4.1.1 Approximations of $\varphi_K(r)$

In the study of Hölder continuity of quasiconformal mappings of the plane, the special function $\varphi_K(r)$ defined as (14) has an important role. Based on Proposition 49 and Remark 51 we study the approximation [7, Theorem 5.43]

$$\varphi_K(r) = L(\varphi_K(L(r, -p)), p) \approx L\left(4^{1-1/K} L(r, -p)^{1/K}, p\right) =: L_\varphi(K, r, p) \quad (52)$$

for various values of p with $4^{1-1/K} L(r, -p)^{1/K} < 1$. Table 3 shows a structural formula for $L_\varphi(K, r, p)$, where $p = 0, 1$.

Table 3. The function $L_\varphi(K, r, p)$ for $p = 0, 1$

p	$L_\varphi(K, r, p)$	c
0	$4^{1-1/K} r^{1/K}$	$-$
1	$\dfrac{2\sqrt{4^{1-1/K}\ c^{1/K}}}{1 + 4^{1-1/K}\ c^{1/K}}$	$\left(\dfrac{r}{1+r'}\right)^2$

Here we note that $L_\varphi(K, r, p)$ is a majorant for the function $\varphi_K(r)$. We also study the following approximation by applying Remark 51 and Lemma 48(2)

$$\varphi_K(r) \approx g_3\left(2^{-p} \log(4/L(r, -p))/K, -5\right) =: LM(K, r, p),$$

where $K > 1$, and $r \in (0, 1)$. Computer experiments show that $LM(K, r, 5)$ is the best approximation for $\varphi_K(r)$, see Figure 8.

Remark 53. *[II, Remark 3.4] The next few lines of Mathematica code*

```
L[s_, p_] :=  Module[{j=0, y=s},
 While[((j<Abs[p])),
  If[p<0, y=(y/(1+Sqrt[1-y^2]))^2,
    y = 2*Sqrt[y]/(1+y)]; j++]; y];
LPhi[K_, r_, p_] := L[4*Exp[(1/K)*Log[L[r, -p]/4]], p];
```

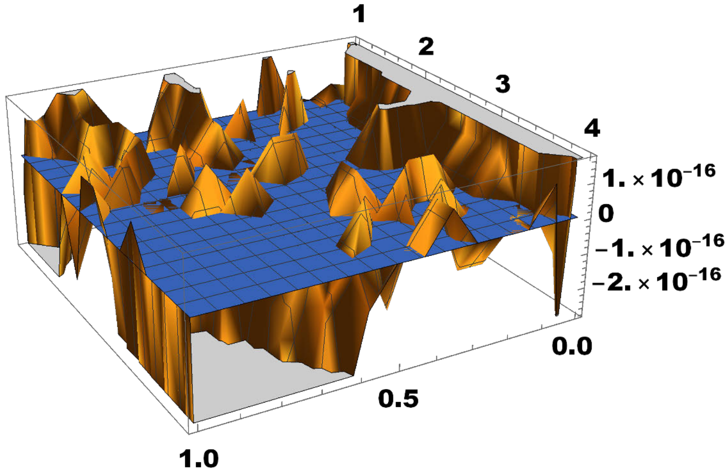


Figure 8. The 3D plot of $\varphi_K(r) - LM(K, r, 5)$ for $1 < K < 4$ and $0 < r < 1$.

define an approximation for φ_K. This function satisfies

```
LPhi[K,r,1] < 4^{1-1/K}r^{1/K}
```

as we see using the command

```
Plot3D[{0,4^(1-1/K)r^(1/K)-LPhi[K,r,1]},{r,0,1},{K,1,3}].
```

It follows from (15) *that* $4^{1-1/K}r^{1/K}$ *is a majorant for* φ_K*, where* $K > 1$*, and* $r \in (0, 1)$*. On the other hand,*

```
LPhi[K,r,1],    K>1,  0<r<1,
```

is not well-defined, e.g. for $K = 2$ *and* $r = 0.9$*. Because*

$$4^{1-1/2} L(0.9, -1)^{1/2} = 1.25358 > 1.$$

4.2 Open Problems

Computational experiments have led us to formulate the following questions:
(1) Let $L_\varphi(K, r, p)$ be defined as in (52). Then

$$L_\varphi(K, r, p) \leq 4^{1-\frac{1}{K}} r^{\frac{1}{K}}$$

for $1 < K < 4.6$, $r \in (0, 0.7]$, and $p = 0, 1, 2, \ldots$.
Motivation. Considering that $p = 0$ is obvious, we may assume that $p = 1, 2, 3, \ldots$. Since $L(\cdot, p) : (0, 1) \to (0, 1)$ is an increasing homeomorphism, we are looking for $r \in (0, 1)$ and $K > 1$ such that $4^{1-1/K} L(r, -p)^{1/K} < 1$. Computer experiments

show that $4^{1-1/K}L(r,-p)^{1/K} < 1$ holds true for all $r \in (0, 0.7]$, $1 < K < 4.6$, and $p = 1, 2, \ldots$.
(2) Remark 53 only deals with the case $p = 1$. What about $p = 2$? Can we find some pair of functions u, v, where u is a minorant of μ and v a majorant of μ such that the corresponding $L_\varphi(K, r, 1)$ would be a majorant of φ_K?

5 Formulas For the Visual Angle Metric

Geometric function theory has significantly benefited from developing intrinsic metrics for planar domains in recent decades. One of the properties of intrinsic metrics is that they share some properties with the hyperbolic metric of the unit disk, but they are also simpler. Intrinsic metrics are often quasi-invariant under subclasses of conformal maps, for example, under Möbius transformations. In this chapter, we study such an intrinsic metric, the *visual angle metric*, introduced by Klén et al. in [48]. We also prove several new formulas for the visual angle metric of the unit disk in terms of the hyperbolic metric. Then we apply these to establish a sharp Schwarz lemma for the visual angle metric under quasiconformal mapping.

Let $S(x, r) = \{y \in \mathbb{R}^2 : |x - y| = r\}$ be the circle centered at $x \in \mathbb{R}^2$ with the radius $r > 0$.

We begin the results of this chapter with the following proposition:

***Proposition* 54.** *[III, Proposition 2.6] Let* $0 < r < s < 1$*,* $r_1 = 1/r$*, and* $s_1 = 1/s$*. Also, let* $c \in \mathbb{B}^2$ *with* $\operatorname{Re} c = (s + r)/2$*,* $c_1 \in \mathbb{C}$ *with* $\operatorname{Re} c_1 = (s_1 + r_1)/2$*, and* $\arg c = \arg c_1$*. If* $a \in S(c, |c - r|)$ *and* $b \in S(c_1, |c_1 - r_1|)$*, then* $2\measuredangle(s, a, r) = 2\measuredangle(r_1, b, s_1) = \measuredangle(r, c, s)$*.*

Figure 9 gives more details on Proposition 54.

***Lemma* 55.** *[III, Lemma 2.12] Suppose that* $a, b \in \mathbb{B}^2 \setminus \{0\}$ *are two points noncollinear with* 0 *and* $|a| \neq |b|$*. Then the inversion* $h : \mathbb{B}^2 \to \mathbb{B}^2$ *with* $h(a) = b$ *is given by*

$$h(z) = \frac{c\overline{z} - 1}{\overline{z} - \overline{c}}, \quad \textit{where} \quad c = LIS[a, b, a^*, b^*] = \frac{a - b + ab(\overline{a} - \overline{b})}{|a|^2 - |b|^2},$$

and h *maps the chord* $L(a, b) \cap \mathbb{B}^2$ *onto itself. Moreover, the circle* $S(c, \sqrt{|c|^2 - 1})$ *orthogonal to the unit circle* $S(0, 1)$ *intersects the unit circle at the points* $(1 \pm i\sqrt{|c|^2 - 1})/\overline{c}$*.*

Figure 10 illustrates this lemma in more detail.

To begin our discussion of the visual angle metric, we consider points on the same radius of the unit disk.

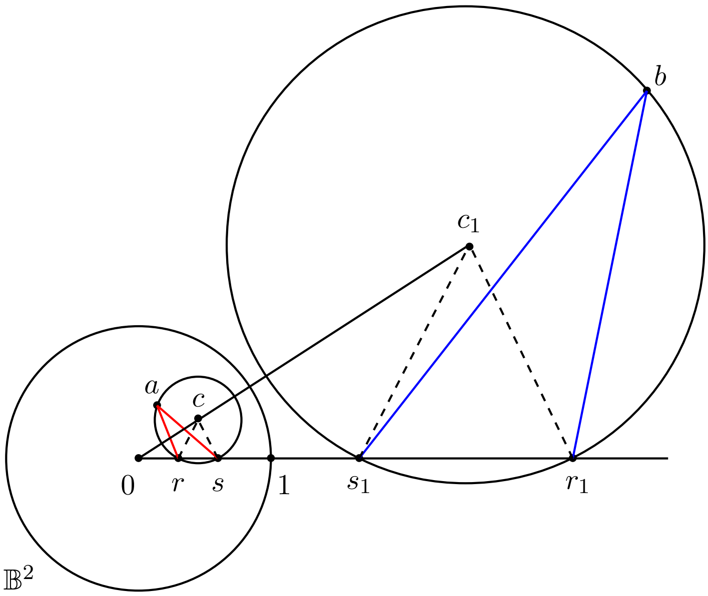


Figure 9. Here $s_1 = 1/s$ and $r_1 = 1/r$. The key points are that the triangles $\triangle(r, s, c)$ and $\triangle(s_1, r_1, c_1)$ are similar, and hence the angles $\measuredangle(r, a, s)$ and $\measuredangle(s_1, b, r_1)$ are equal.

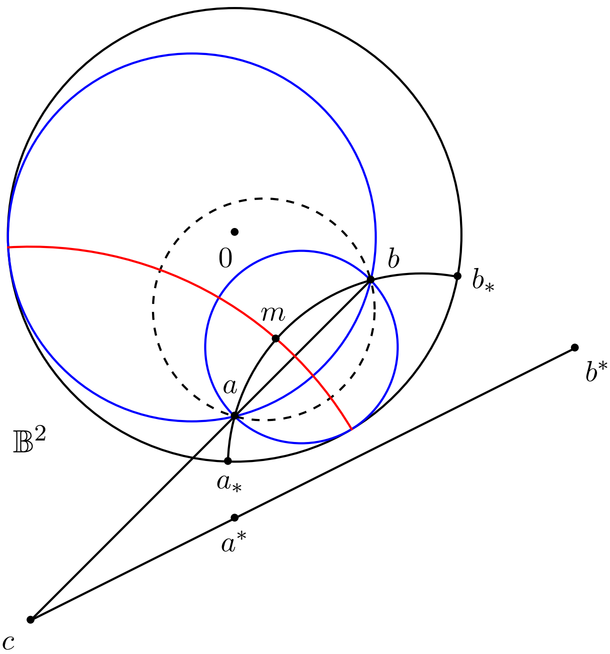


Figure 10. The inversion $h(z) = (c\overline{z} - 1)/(\overline{z} - \overline{c})$ with $c = LIS[a, b, a^*, b^*]$ maps the unit disk onto itself and the chord containing a and b onto itself. The point m is the hyperbolic midpoint of a and b, and the hyperbolic circle through a and b centered at m is drawn with a dashed line.

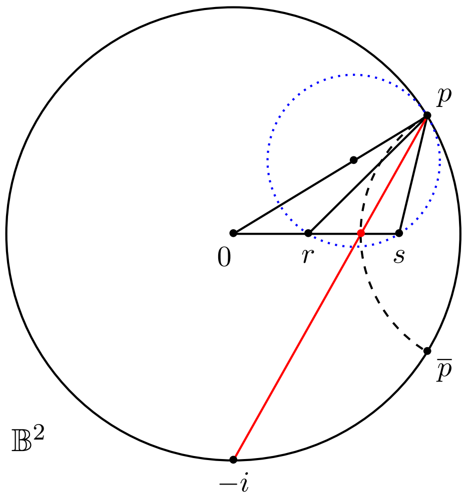


Figure 11. The segment $[-i, p]$ bisects the angle $\measuredangle(r, p, s)$. The dashed circle orthogonal to the unit circle at the points p and $\overline{p}$ passes through the hyperbolic midpoint of the segment $[r, s]$. Note that $\mathrm{Re}\{p\} = (r+s)/(1+rs) = \mathrm{th}((\rho_{\mathbb{B}^2}(0,r) + \rho_{\mathbb{B}^2}(0,s))/2)$.

5.1 The Case of Radial Points $v_{\mathbb{B}^2}(r, s)$, $0 < r < s < 1$

Writing

$$c = (r+s)/2, \quad d = \sqrt{(1-r^2)(1-s^2)}/2, \quad c^2 + d^2 = \left(\frac{1+rs}{2}\right)^2,$$

we easily see that the circle $S(c+id, (1-rs)/2)$ passes through the points $(r, 0)$ and $(s, 0)$ and is internally tangent to $S(0, 1)$ at the point

$$p = \left(\frac{r+s}{1+rs}, \frac{\sqrt{(1-r^2)(1-s^2)}}{1+rs}\right).$$

Therefore

$$v_{\mathbb{B}^2}(r, s) = \arcsin\frac{s-r}{1-rs}.$$

Moreover, the segment $[-i, p]$ bisects the angle $\measuredangle(r, p, s)$ and the circle orthogonal to the unit circle at the points p and $\overline{p}$ passes through the hyperbolic midpoint

$$\frac{r+s}{1+rs+\sqrt{(1-r^2)(1-s^2)}}$$

of the segment $[r, s]$, see Figure 11, for more details.

Lemma 56. *([48, Lemma 3.10]) For $a, b \in \mathbb{B}^2$ collinear with 0 we have*

$$\tan v_{\mathbb{B}^2}(a, b) = \mathrm{sh}\frac{\rho_{\mathbb{B}^2}(a, b)}{2}.$$

The next theorem generalizes the above observations connected with the visual angle metric of two points on the same radius.

Theorem 57. *[III, Theorem 3.2] For $a, b \in \mathbb{B}^2$ non-collinear with 0 and with $|a| \neq |b|$, the circle $S(c, \sqrt{|c|^2-1})$ centered at $c = LIS[a, b, a^*, b^*]$ is orthogonal to $\partial\mathbb{B}^2$. Let $u = S(c, \sqrt{|c|^2-1}) \cap L[a, b]$ and let $v \in S(c, \sqrt{|c|^2-1}) \cap \partial\mathbb{B}^2$ be the point maximizing the visual angle metric. Then*

$$v_{\mathbb{B}^2}(a, b) = \measuredangle(a, v, b) \quad \textit{and} \quad \measuredangle(a, v, u) = \measuredangle(u, v, b).$$

Figure 12 illustrates this theorem in more detail.

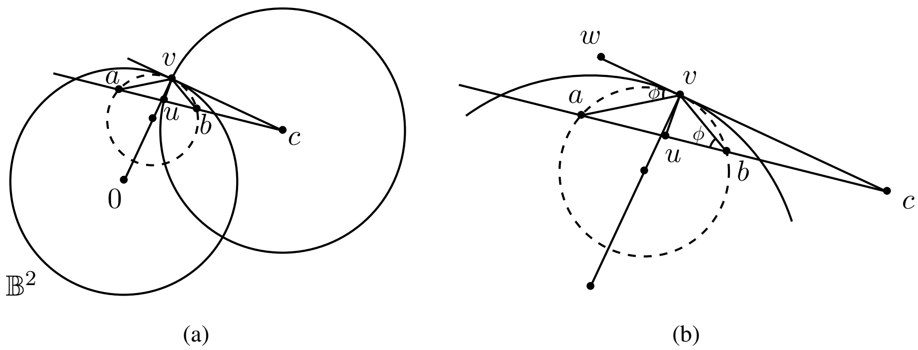


Figure 12. (a): Angle bisection visualized. (b): Angle bisection visualized, detail. Here $\measuredangle(w, v, a) = \measuredangle(v, b, a) = \phi$.

5.2 A Functional Identity Between $\rho_{\mathbb{B}^2}(a, b)$ and $v_{\mathbb{B}^2}(a, b)$

First, we shall prove a new formula for $v_{\mathbb{B}^2}(a, b)$, but we first give an auxiliary lemma.

Lemma 58. *[III, Lemma 3.4] Let $m_1, m_2 \in \mathbb{B}^2$ be non-collinear with 0, and let $|m_1| \neq |m_2|$. Then, there exists an inversion $h : \mathbb{B}^2 \to \mathbb{B}^2 = h(\mathbb{B}^2)$ with the following properties:*

$$i) \quad h(m_1) = m_2; \qquad ii) \quad h(L[m_1, m_2] \cap \mathbb{B}^2) = L[m_1, m_2] \cap \mathbb{B}^2.$$

The following theorem provides a geometric construction for the extremal point z; see Figure 13, for more details.

Theorem 59. *[III, Theorem 1.2] Let $a, b \in \mathbb{B}^2$ with $|a| \neq |b|$ and $0 \notin L[a, b]$. Then*

$$v_{\mathbb{B}^2}(a, b) = \max\{\measuredangle(a, z_1, b), \measuredangle(a, z_2, b)\},$$

where z_1 and z_2 are the points of intersection of the unit circle and an orthogonal circle

$$S(0, 1) \cap S(c, \sqrt{|c|^2-1}), \quad \textit{where} \quad c = \frac{a(1-|b|^2) - b(1-|a|^2)}{|a|^2 - |b|^2}.$$

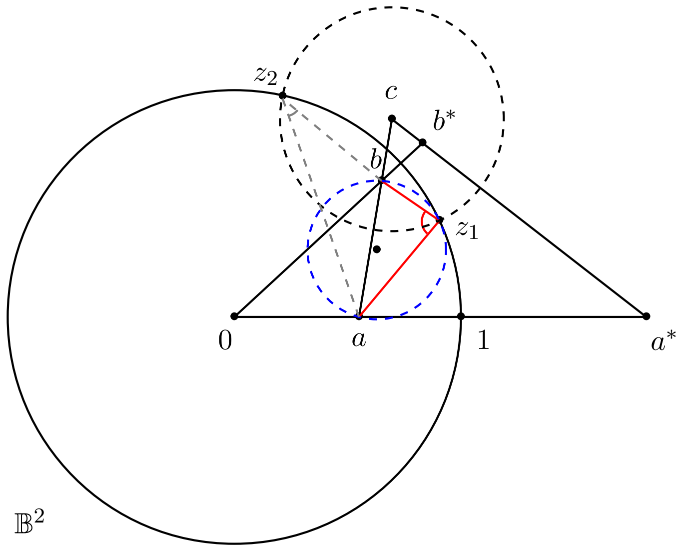


Figure 13. If $c = LIS[a, b, a^*, b^*]$ and $z_1 = S(c, \sqrt{|c|^2 - 1}) \cap S(0, 1)$ is a point in the sector with vertex c and sides $L[c, a]$ and $L[c, a^*]$, then $v_{\mathbb{B}^2}(a, b) = \measuredangle(a, z_1, b)$.

Moreover, $\{z_1, z_2\} = (1 \pm i\sqrt{|c|^2 - 1})/\overline{c}$*. In the case* $|a| = |b|$

$$v_{\mathbb{B}^2}(a, b) = 2\arctan\left(\frac{|a-b|}{2 - |a+b|}\right).$$

***Theorem* 60.** *[III, Theorem 1.3] For* $a, b \in \mathbb{B}^2$ *let* $L[a, b]$ *be the line through* a *and* b*. We have*

$$\tan\frac{v_{\mathbb{B}^2}(a, b)}{2} = \frac{(1 + |m|)u}{1 + \sqrt{1 + (1 - |m|^2)u^2}}, \quad u = \operatorname{sh}\frac{\rho_{\mathbb{B}^2}(a, b)}{2},$$

where $m = (\overline{a}b - a\overline{b})/(2(\overline{a} - \overline{b}))$ *is the midpoint of the chord of the unit disk containing the two points* a *and* b*. Hence* $|m| = d(L[a, b], \{0\})$*.*

The idea of the proof of Theorem 60 is visualized in Figure 14. The key points are that the triangles $\triangle(b, a, d)$ and $\triangle(h(a), h(b), h(d))$ are similar, and hence the angles $\measuredangle(h(b), h(v), h(a))$ and $\measuredangle(a, v, b)$ are equal.

Observe that in the case when $a, b, 0$ are collinear, Theorem 60 reduces to Lemma 56 and that Corollary 61 (2) below gives an equivalent form of Lemma 56.

***Corollary* 61.** *[III, Corollary 3.11] (1) For* $a, b \in \mathbb{B}^2$ *and* m *as in Theorem 60, we have*

$$\sin v_{\mathbb{B}^2}(a, b) = \frac{(1 + |m|)(1 + \sqrt{1 + (1 - |m|^2)u^2})u}{1 + \sqrt{1 + (1 - |m|^2)u^2} + (1 + |m|)u^2}, \quad u = \operatorname{sh}\frac{\rho_{\mathbb{B}^2}(a, b)}{2}.$$

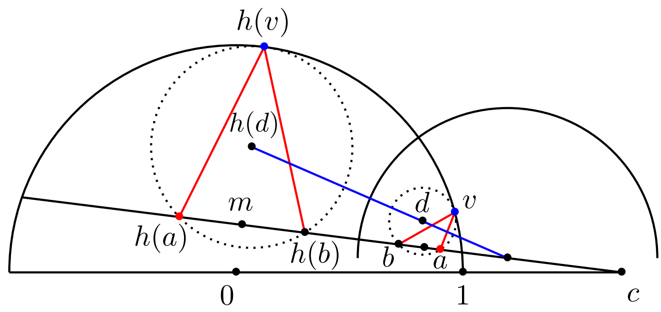


Figure 14. Illustration of Theorem 60.

(2) For $m=0$ we have

$$\sin v_{\mathbb{B}^2}(a,b) = \operatorname{th}\frac{\rho_{\mathbb{B}^2}(a,b)}{2} = \frac{|a-b|}{\sqrt{|a-b|^2+(1-|a|^2)(1-|b|^2)}}.$$

Proposition 62. *[III, Proposition 3.13] For $m\in(0,1)$, $r>0$, and $u=\operatorname{sh}(r/2)$*

$$(1+m)\operatorname{th}\frac{r}{4} \le \frac{(1+m)\,u}{1+\sqrt{1+(1-m^2)\,u^2}} \le \min\left\{\frac{(1+m)u}{2}, \sqrt{\frac{1+m}{1-m}}\operatorname{th}\frac{r}{4}\right\}.$$

The next corollary yields, as a special case, Theorem 3.11 of [48].

Corollary 63. *[III, Corollary 3.14] For $a,b\in\mathbb{B}^2$ and m as in Theorem 60, we have*

$$(1+|m|)\operatorname{th}\frac{\rho_{\mathbb{B}^2}(a,b)}{4} \le \tan\frac{v_{\mathbb{B}^2}(a,b)}{2}$$

$$\le \min\left\{\frac{1+|m|}{2}\operatorname{sh}\frac{\rho_{\mathbb{B}^2}(a,b)}{2}, \sqrt{\frac{1+|m|}{1-|m|}}\operatorname{th}\frac{\rho_{\mathbb{B}^2}(a,b)}{4}\right\}.$$

To prove the next result, we need some basic facts about quasiregular mappings, see [30; 7]. In the following, we use the quasiregular Schwarz lemma in a specific form with detailed information about the distortion function, φ_K.

Lemma 64. *[III, Lemma 3.15] (1) Let $f:\mathbb{B}^2\to\mathbb{B}^2$ be a K-quasiregular mapping, where $K\ge 1$, and $a,b\in\mathbb{B}^2$. Then*

$$\operatorname{th}\frac{\rho_{\mathbb{B}^2}(f(a),f(b))}{2} \le \varphi_K\left(\operatorname{th}\frac{\rho_{\mathbb{B}^2}(a,b)}{2}\right) \le 4^{1-1/K}\left(\operatorname{th}\frac{\rho_{\mathbb{B}^2}(a,b)}{2}\right)^{1/K}.$$

(2) The function φ_K, $K\ge 1$, satisfies for $0<r<1$

$$\frac{\varphi_K(r)}{1+\sqrt{1-\varphi_K(r)^2}} = \sqrt{\varphi_K\left(\left(\frac{r}{1+\sqrt{1-r^2}}\right)^2\right)}.$$

Our next result yields a sharp Schwarz lemma for the visual angle metric.

***Theorem* 65.** *[III, Theorem 1.5] Let $f : \mathbb{B}^2 \to \mathbb{B}^2 = f(\mathbb{B}^2)$ be a non-constant K-quasiregular mapping, where $K \geq 1$. For $a, b \in \mathbb{B}^2$ let m_1 and m_2 be the midpoints of the chords of the unit disk containing $f(a), f(b)$ and a, b, respectively. Then we have*

$$\tan\frac{v_{\mathbb{B}^2}(f(a), f(b))}{2} \leq 2^{1-1/K}\, c \left(\tan\frac{v_{\mathbb{B}^2}(a,b)}{2}\right)^{1/K}$$

with equality for $K = 1$ and $m_1 = m_2 = 0$, where

$$c = \sqrt{\frac{1+|m_1|}{1-|m_1|}}\frac{1}{(1+|m_2|)^{1/K}}.$$

***Corollary* 66.** *[III, Corollary 3.17] Let $T_w : \mathbb{B}^2 \to \mathbb{B}^2$ be a Möbius transformation as in* (2)*. Let $a, b, w \in \mathbb{B}^2$, $|m_2| = d(L[a,b], 0)$, and $|m_1| = d(L[T_w(a), T_w(b)], 0)$. Then*

$$\tan\frac{v_{\mathbb{B}^2}(T_w(a), T_w(b))}{2} \leq c(m_1, m_2)\tan\frac{v_{\mathbb{B}^2}(a,b)}{2}$$

with equality for $m_1 = m_2 = 0$, where

$$c(m_1, m_2) = \frac{1}{1+|m_2|}\sqrt{\frac{1+|m_1|}{1-|m_1|}}.$$

Next, we consider an evenly separated sequence of collinear points.

5.3 Evenly Separated Collinear Points

Consider a collinear sequence of points a_j, $j = 1, 2, 3, \ldots$ in the unit disk, with a constant hyperbolic distance $\rho_{\mathbb{B}^2}(a_j, a_{j+1}) = \mathrm{const.}$ It turns out that also $v_{\mathbb{B}^2}(a_j, a_{j+1})$ is a constant; see Figure 15 for more details.

5.4 Additional Identities For $v_{\mathbb{B}^2}(a, b)$

In the following, we give two analytic formulas for $v_{\mathbb{B}^2}(a, b)$. In the first formula, we use the recent implicit formula for the hyperbolic midpoint of two points, while in the second formula, we use symbolic computations. Due to their lengthy expressions, these formulas are best used for computer work.

***Theorem* 67.** *[III, Theorem 4.4] For given $a, b \in \mathbb{B}^2$, let $m \in \mathbb{B}^2$ be their hyperbolic midpoint. Then*

$$v_{\mathbb{B}^2}(a,b) = \max\{\measuredangle(a, q_1, b), \measuredangle(a, q_2, b)\},$$

where

$$q_1 = T_m^{-1}(Q), \quad q_2 = T_m^{-1}(-Q),$$

and $Q = i(T_m(a) - T_m(b))/|T_m(a) - T_m(b)|$.

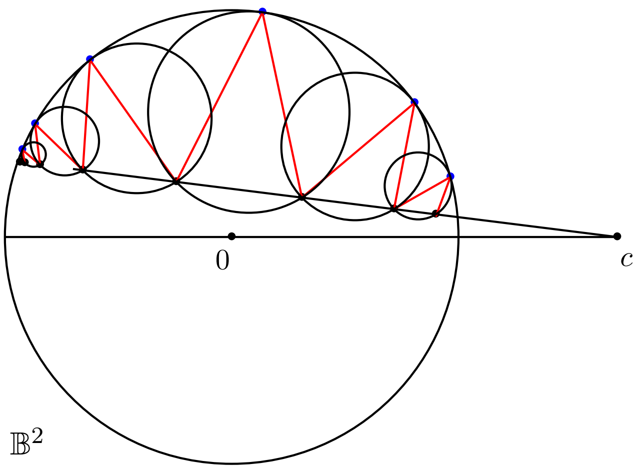


Figure 15. Here $\rho_{\mathbb{B}^2}(a_j, a_{j+1}) = \mathrm{const}$ and also $v_{\mathbb{B}^2}(a_j, a_{j+1})$ is a constant. It follows from Proposition 54 that the sectors with vertices on the unit circle have equal angles.

An illustration of Theorem 67 is shown in Figure 16.

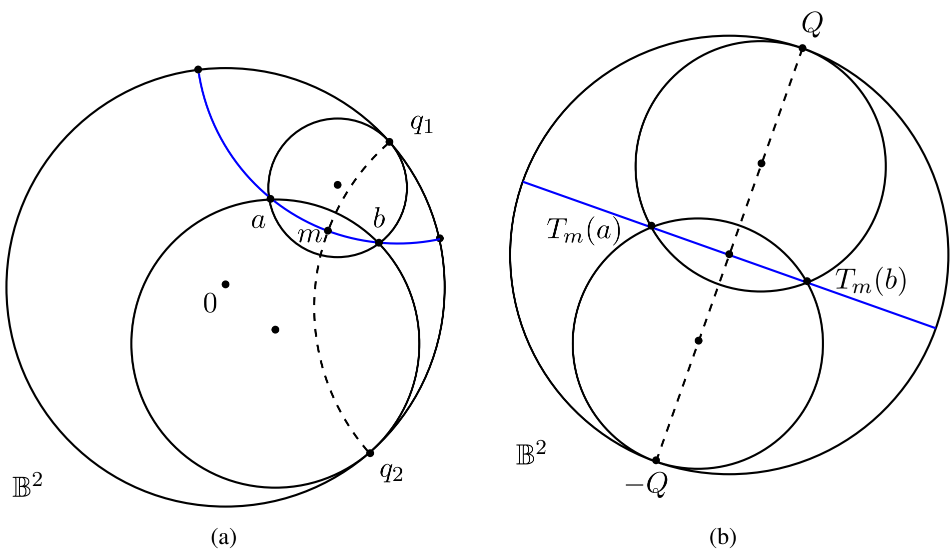


Figure 16. (a): Before Möbius transformation T_m. (b): After Möbius transformation T_m.

Finally, we have:

Theorem 68. *[III, Theorem 4.6] For given $a, b \in \mathbb{B}^2$, let $q \in S(0,1)$ be the point that gives $v_{\mathbb{B}^2}(a,b) = \measuredangle(a,q,b)$ and set $p = a + t(b-a)i$. If a right triangle $\triangle(a,b,p)$ is inscribed in the circle passing through three points a, b, and p, then t is given as*

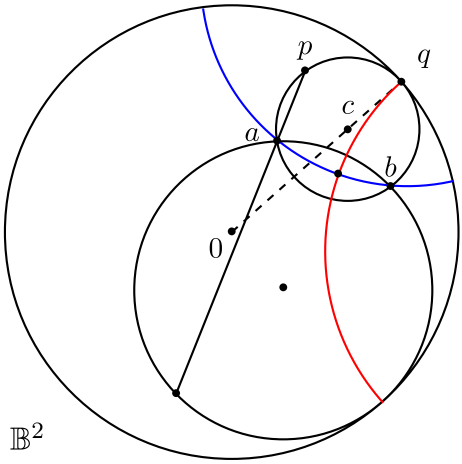


Figure 17. $|c| + |p - c| = 1$ yields an equation to find p using RISA. Then $v_{\mathbb{B}^2}(a, b) = \measuredangle(a, q, b) = \measuredangle(a, p, b)$.

the solution with the smaller absolute value of the following equation

$$\big((a\overline{b} - \overline{a}b)^2 + 4(\overline{a} - \overline{b})(a - b)\big)t^2 + 2(a\overline{b} - \overline{a}b)(a\overline{b} + \overline{a}b - 2)it$$
$$- \big((a\overline{b} + \overline{a}b)^2 - 4(a\overline{a} + b\overline{b} - 1)\big) = 0.$$

Figure 17 gives more details on Theorem 68.

6 Conformally Invariant Metrics and Lack of Hölder Continuity

In their landmark paper [5] L. Ahlfors and A. Beurling introduced the modulus of a curve family. During the past 70 years, this conformal invariant has become a key tool in geometric function theory. The conformal modulus of a curve family is frequently used to analyze the distortion of distances between points under quasiconformal mappings because of its conformal invariance. For basic curve families, it is occasionally sufficient to combine majorization or minorization with rough estimates. However, there are two disadvantages to using rough estimates: Firstly, it necessitates knowledge of the moduli of curve families. What is more, imprecise estimations result in information loss. It is frequently beneficial to simplify the estimating problem to classical extremal problems and apply these systematically rather than utilizing rough estimates. We now go over two traditional extremal situations that have seen this application. In particular, H. Grötzsch and O. Teichmüller, who contributed to the development of conformal invariant in the 1920s and 1930s, have already examined these issues.

6.1 Introduction

In this chapter, we prove some new inequalities for the modulus metric μ_D, where $D \in \{\mathbb{H}^n, \mathbb{B}^n\}$, and compare this metric to other intrinsic metrics. Also, we prove that μ_D metric and λ_D^{-1} metric are not Hölder continuous with respect to the hyperbolic metric, where $D \in \{\mathbb{H}^2, \mathbb{B}^2\}$.

Let us start by recalling the following theorems:

Theorem 69. *(1) If* $\mathrm{cap}(\partial G) > 0$*, then* μ_G *is a metric.*
(2) λ_G^p *is a metric if and only if* $p \in [-1/(n-1), 0)$*.*

Theorem 69(1)'s proof is straightforward, but part 2 has an interesting background, see [30, Section 10.2]. The next two theorems provide a foundation for multiple applications of the theorem 69.

Theorem 70. *([30, Theorem 10.4]) For* $D \in \{\mathbb{H}^n, \mathbb{B}^n\}$ *and all distinct points* $x, y \in D$*,*
(1) $\mu_D(x,y) = \gamma_n(1/\mathrm{th}(\rho_D(x,y)/2))$*;*

(2) $\lambda_D(x,y) = 2^{-n}\gamma_n(\mathrm{ch}(\rho_D(x,y)/2))$*,*
where ρ *stands for the hyperbolic metric and* γ_n *is the Grötzsch capacity defined in* (8).

***Theorem* 71.** *([7, Collorary 10.19]) Let* f *be a* K*-quasiconformal homeomorphism, where* $K \geq 1$*. Then for all distinct* $x, y \in G$*,*

$$\mu_G(x,y)/K \leq \mu_{G'}(f(x), f(y)) \leq K\mu_G(x,y)$$

and

$$\lambda_G(x,y)/K \leq \lambda_{G'}(f(x), f(y)) \leq K\lambda_G(x,y).$$

In other words, written as mappings between metric spaces,

$$f : (G, \mu_G) \to (G', \mu_{G'}), \quad \textit{or} \quad f : (G, \lambda_G^{1/(1-n)}) \to (G', \lambda_{G'}^{1/(1-n)}),$$

the mapping f *is bi-Lipschitz. For the case of* μ_G*, we assume that* $\mathrm{cap}(\partial G) > 0$ *and, for the case of* λ_G*, let* $\mathrm{card}(\overline{\mathbb{R}}^n \backslash G) \geq 2$.

6.2 Main Results

It is natural to ask whether the metrics ρ_D, μ_D, and $\lambda_D^{1/(1-n)}$ in $D \in \{\mathbb{H}^n, \mathbb{B}^n\}$ are comparable in any other way given that they are all conformally invariant. The Schwarz lemma for quasiconformal maps $f : \mathbb{B}^n \to f\mathbb{B}^n \subset \mathbb{B}^n$ follows from Theorem 71, and it also shows that these mappings are Hölder continuous with respect to the hyperbolic and Euclidean metrics. In this section, we will compare metrics to Euclidean and hyperbolic metrics because the local structure of both spaces depends on the special function γ_n.

Let $G \subset \mathbb{R}^n$ be a domain with $\mathrm{card}(\partial G) \geq 2$. Then the boundary ∂G is uniformly perfect, if and only if there exists a constant $b > 0$ such that for all $x, y \in G$ the inequality (see [77, Theorem 1.2])

$$\mu_G(x,y) \geq c\delta_G(x,y), \tag{72}$$

holds. The estimates in (12) also yield

$$2^{n-1}c_n\rho_D(x,y) \leq \mu_D(x,y) \leq 2^{n-1}c_n\mu(1/e^{\rho_D(x,y)}) < 2^{n-1}c_n(\rho_D(x,y) + \log 4), \tag{73}$$

where $D \in \{\mathbb{H}^n, \mathbb{B}^n\}$. The same inequality can also be written for λ_D:

$$c_n \log(t) \leq \lambda_D(x,y) \leq \frac{c_n}{2}\mu\left(t^{-2}\right) < c_n \log(2t), \quad \text{with} \quad t = \frac{e^{\rho_D(x,y)/2}+1}{e^{\rho_D(x,y)/2}-1}.$$

Because $\rho_{\mathbb{B}^n} = \delta_{\mathbb{B}^n}$, the lower bound in (73) is compatible with (72) up to a constant factor, but it can still be improved for small values of the hyperbolic distance $\rho_{\mathbb{B}^2}(x,y)$

in the two-dimensional case. Note that, for $n = 2$, it follows from the inequality (73) that

$$\frac{4}{\pi}\rho_D(x,y) \leq \mu_D(x,y) \leq \frac{4}{\pi}(\rho_D(x,y) + \log 4). \tag{74}$$

We consider the following preliminary result:

***Lemma* 75.** *[IV, Lemma 3.4] For all $t > 0$ and $p > 0$, the expression $[\mathrm{arth}\left((\mathrm{th}\, t)^{1/p}\right)]^p$ is strictly increasing with respect to p.*

As an application of Lemma 75 we prove the following:

***Corollary* 76.** *[IV, Corollary 3.7] For all $x, y \in D \in \{\mathbb{H}^2, \mathbb{B}^2\}$, the inequality*

$$\mu_D(x,y) \geq \frac{8}{\pi\sqrt[4]{2}}\rho_D(x,y)^{1/4}$$

holds.

***Remark* 77.** *[IV, Remark 3.8] Corollary 76 gives a better lower bound for the modulus metric than the inequality (74) if and only if the hyperbolic distance $\rho_{\mathbb{B}^2}(x,y)$ is less than 2. See Figure 18 for more details. We note that in Figure 18(a), $\rho_{\mathbb{B}^2}(x,0) < 2$ if and only if $0 < x \leq 0.75$. Also, in Figure 18(b), $\rho_{\mathbb{B}^2}(x,0) \geq 2$ if and only if $0.75 < x < 1$.*

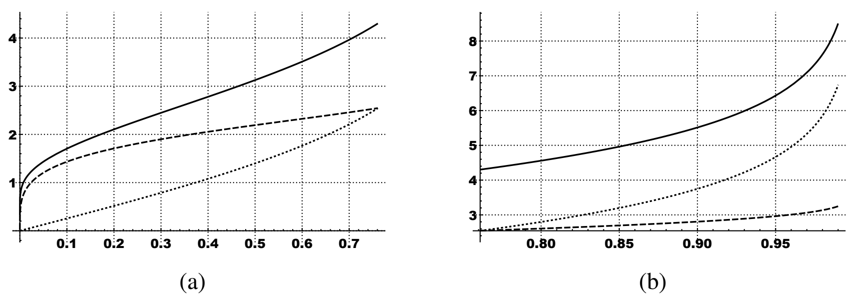


Figure 18. (a): The graph of $\mu_{\mathbb{B}^2}(x,0)$, its lower bound in Corollary 76 (dashed), and its lower bound in (74) (dotted) when $\rho_{\mathbb{B}^2}(x,0) < 2$, where $0 < x \leq 0.75$. (b): The graph of $\mu_{\mathbb{B}^2}(x,0)$, its lower bound in (74) (dotted), and its lower bound in Corollary 76 (dashed) when $\rho_{\mathbb{B}^2}(x,0) \geq 2$, where $0.75 < x < 1$.

The Euclidean midpoint rotation was introduced in article [67] as a new technique for determining upper and lower bounds for the triangular ratio metric. The midpoint rotation involves rotating two distinct points $x, y \in \mathbb{B}^2$ around their Euclidean midpoint $(x+y)/2$ so that the smaller angle ν between lines $L(x,y)$ and $L(0,(x+y)/2)$ varies on the interval $[0, \pi/2]$, see Figure 19. Here, we assume that $x \neq -y$, because otherwise the midpoint $(x+y)/2$ is the origin, and the hyperbolic distance $\rho_{\mathbb{B}^2}(x,y)$ is invariant under rotations around the origin.

In the next result, we give upper and lower bounds for $\rho_{\mathbb{B}^2}(x,y)$.

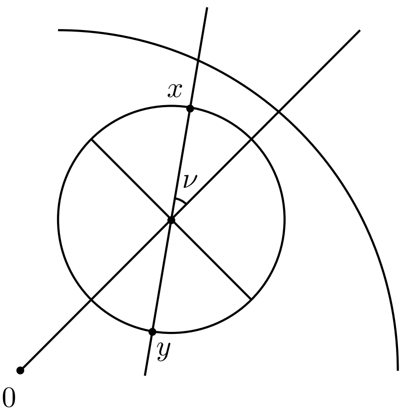


Figure 19. In the Euclidean midpoint rotation, two distinct points x, y in the unit disk $\mathbb{B}^2$ are rotated around their midpoint $(x+y)/2$ so that the smaller angle ν between the lines $L(x,y)$ and $L(0,(x+y)/2)$ varies.

***Theorem* 78.** *[IV, Theorem 3.9] For all distinct points $x, y \in \mathbb{B}^2$ such that $x \neq -y$, the hyperbolic distance $\rho_{\mathbb{B}^2}(x,y)$ decreases as x and y are rotated around their Euclidean midpoint so that the smaller angle between lines $L(x,y)$ and $L(0,(x+y)/2)$ increases. Furthermore,*

$$\frac{2|x-y|}{\sqrt{4-8x\cdot y+(|x|^2+|y|^2)^2}} \leq \operatorname{th}\frac{\rho_{\mathbb{B}^2}(x,y)}{2} \leq \frac{|x-y|}{1-x\cdot y},$$

where equality holds in the first inequality if and only if $|x| = |y|$ and in the second inequality if and only if x, y are collinear with the origin. Note also that $|x-y|/(1-x\cdot y) < 1$ if and only if $|x+y|+|x-y| < 2$.

By [7, 7.64 (26-27), p. 156],

$$\begin{aligned} &\frac{|x-y|}{\min\{|x-y|+\sqrt{1-|x|^2}\sqrt{1-|y|^2}, 1+|x||y|\}} \leq \operatorname{th}\frac{\rho_{\mathbb{B}^2}(x,y)}{2} \\ &\leq \frac{|x-y|}{\max\{|x-y|+(1-|x|)(1-|y|), 1-|x||y|\}}. \end{aligned} \tag{79}$$

The bounds of Theorem 78 are sometimes better, but not always, when compared to these bounds. When $x = 0.6+0.3i$ and $y = 0.1+0.1i$, both the upper and lower bounds of Theorem 78 are better than those of inequality (79) (Table 4), but the bounds of Theorem 78 are worse for $x = -0.7+0.7i$ and $y = 0.65-0.6i$ (Table 5).

***Corollary* 80.** *[IV, Corollary 3.11] For all distinct points $x, y \in \mathbb{B}^2$ such that $x \neq -y$, the distance $\mu_{\mathbb{B}^2}(x,y)$ decreases as x and y are rotated around their Euclidean midpoint so that the smaller angle between lines $L(x,y)$ and $L(0,(x+y)/2)$ increases.*

Table 4. Upper and lower bounds of $\mathrm{th}\,(\rho_{\mathbb{B}^2}(x,y)/2)$ by Theorem 78 and Inequality (79) when $x = 0.6 + 0.3i$ and $y = 0.1 + 0.1i$

L.H.S Thm 78	L.H.S (79)	R.H.S Thm 78	R.H.S (79)
0.575624	0.491855	**0.591776**	0.594959

Table 5. Upper and lower bounds of $\mathrm{th}\,(\rho_{\mathbb{B}^2}(x,y)/2)$ by Theorem 78 and Inequality (79) when $x = -0.7 + 0.7i$ and $y = 0.65 - 0.6i$

L.H.S Thm 78	L.H.S (79)	R.H.S Thm 78	R.H.S (79)
0.997999	**0.999183**	0.999555	**0.999381**

Furthermore, if $|x+y|+|x-y|<2$,

$$\gamma_2\left(\frac{\sqrt{4-8x\cdot y+(|x|^2+|y|^2)^2}}{2|x-y|}\right)\leq \mu_{\mathbb{B}^2}(x,y)\leq \gamma_2\left(\frac{1-x\cdot y}{|x-y|}\right),$$

where equality holds in the first inequality if and only if $|x|=|y|$ and in the second inequality if and only if x, y are collinear with the origin. If $|x+y|+|x-y|\geq 2$, only the first inequality above holds.

Finally, we prove that $\mu_{\mathbb{B}^2}(x,y)$ and $\lambda_{\mathbb{B}^2}^{-1}$ are not Hölder continuous.

***Lemma* 81.** *[IV, Lemma 3.12] The modulus metric defined in the unit disk is not Hölder continuous with respect to the Euclidean metric.*

***Theorem* 82.** *[IV, Theorem 1.6] The μ_D metric, $D \in \{\mathbb{H}^2, \mathbb{B}^2\}$, is not Hölder continuous with respect to ρ_D.*

***Remark* 83.** *It follows from Lemma 81 that the modulus metric is not Hölder continuous with respect to the hyperbolic metric because the hyperbolic metric and the Euclidean metric are locally similar.*

***Corollary* 84.** *[IV, Corollary 3.13] The metric $\lambda_{\mathbb{B}^2}^{-1}(x,y)$ defined in the unit disk is not Hölder continuous with respect to the hyperbolic metric.*

7 On Harnack Inequality and Harmonic Schwarz Lemma

Harnack's inequality stands as a fundamental result in the study of partial differential equations (PDEs), with several applications across various branches of mathematics, particularly in the theory of elliptic and parabolic equations. The Harnack inequality usually applies to positive solutions of elliptic or parabolic equations in divergence form. In the case of elliptic equations describing steady-state problems like heat conduction or electrostatics, the inequality establishes bounds on the solutions by comparing maximum and minimum values within a domain. Additionally, the German mathematician Axel Harnack is credited with developing the original formulation of this inequality for harmonic functions in the plane; see [47] for more details. It should be noted that this inequality was first published in 1887 in the book [32].

This chapter aims to investigate the $(s, C(s))$-Harnack inequality for all values of s between 0 and 1, where $C(s)$ is greater than or equal to 1. It also derives the constant $C(s)$ in the case when u is a harmonic function on the ball $B^n(x, r)$ and on the unit ball $\mathbb{B}^n$. Additionally, we prove that if $f : G \to \mathbb{R}^n$ is a quasiregular mapping, then the function $u(x) = d_{fG}(f(x))$, for $x \in G$, satisfies the $(s, C(s))$-Harnack inequality for all $s \in (0, 1)$, where G is a proper subdomain of $\mathbb{R}^n$. Next, we examine how quasiconformal and quasiregular mappings affect the behavior of the Harnack metric. Lastly, we present a type of Schwarz lemma.

7.1 Harnack Inequality

The first result of this chapter is as follows:

***Lemma* 85.** *[V, Lemma 3.1] All positive harmonic functions on $B^n(x, r) \subset \mathbb{R}^n$ are $(s, C(s))$-Harnack with*

$$C(s) := C(s, n) = \frac{1}{1 - s^2}\left(\frac{1 + s}{1 - s}\right)^n$$

for all $s \in (0, 1)$.

Next, we investigate the $(s, C(s))$-Harnack inequality on the unit ball $\mathbb{B}^n$.

Theorem 86. *[V, Theorem 3.2] (i) Let $s \in (0,1)$ and $u : \mathbb{B}^n \to (0,\infty)$ be a Harnack function. Then for all $x, y \in \mathbb{B}^n$*

$$u(x) \leq C(s)^{1+t} u(y), \qquad t = \frac{\log((1+r)/(1-r))}{\log((1+s)/(1-s))},$$

where $r = \operatorname{th}(\rho_{\mathbb{B}^n}(x,y)/2)$ and $C(s) \geq 1$.
(ii) If u is a positive harmonic function, $x \in \mathbb{B}^n$, $s \in (0,1)$ and $y \in S^{n-1}(x, s(1-|x|))$, then

$$u(x) \leq \frac{1}{1-s^2}\left(\frac{1+s}{1-s}\right)^n u(y), \qquad s < \exp(\rho_{\mathbb{B}^n}(x,y)) - 1.$$

We present the next outcome regarding quasiregular mappings, specifically demonstrating that if a quasiregular mapping $f : G \to \mathbb{R}^n$ is considered, then the function $u(x) = d_{fG}(f(x))$, for $x \in G$, fulfills the $(s, C(s))$-Harnack inequality for all $s \in (0,1)$.

Theorem 87. *[V, Theorem 3.4] Let G be a proper subdomain of $\mathbb{R}^n$, and $f : G \to \mathbb{R}^n$ be a quasiregular mapping such that $fG \subset \mathbb{R}^n$ is a A-uniform domain. Also, let ∂fG be connected such that it consists of at least two points. Then, the function $u(x) = d_{fG}(f(x))$, $(x \in G)$, satisfies the $(s, C(s))$-Harnack inequality with the constant*

$$C(s) := \exp\left(\frac{AK_I(f)}{c_n}\omega_{n-1}\left(\log\frac{sd_G(x)}{|x-y|}\right)^{1-n}\right), \quad s \in (0,1),$$

for $y \in B_s := B^n(x, sd_G(x))$, where ω_{n-1} is the $(n-1)$-dimensional surface area of S^{n-1}, $K_I(f)$ is the inner dilatation of f, and c_n is a constant number depending only on n defined in (13).

The connectedness of ∂fG is a crucial component of Theorem 87. However, it is important to note that the theorem's statement can be challenged by the presence of an analytic function $f : \mathbb{B}^2 \to \mathbb{B}^2 \setminus \{0\} = f\mathbb{B}^2$ given by:

$$f(z) = \exp\left(\frac{z+1}{z-1}\right), \quad z \in \mathbb{B}^2.$$

7.2 Harnack Metric

This section investigates the Harnack metric $h_G(x,y)$, where G is a proper subdomain of $\mathbb{R}^n$.

***Theorem* 88.** *[V, Theorem 3.6] Let $s \in (0,1)$ and $C(s) \geq 1$. (i) If G is a proper subdomain of $\mathbb{R}^n$, then*

$$h_G(x,y) \leq \left(1 + \frac{k_G(x,y)}{2\log(1+s)}\right) \log C(s).$$

(ii) If $G = \mathbb{B}^n$ or $G = \mathbb{H}^n$, then we have

$$h_G(x,y) \leq \left(1 + \frac{\rho_G(x,y)}{\log[(1+s)/(1-s)]}\right) \log C(s).$$

***Theorem* 89.** *[V, Theorem 3.9] i) If $f : \mathbb{B}^n \to f\mathbb{B}^n$ is a nonconstant K-quasiregular mapping with $f\mathbb{B}^n \subset \mathbb{B}^n$, where $K \geq 1$, then the inequality*

$$h_{f\mathbb{B}^n}(f(x), f(y)) \leq 2K(h_{\mathbb{B}^n}(x,y)/2 + \log 4)$$

holds for all $x, y \in \mathbb{B}^n$.
ii) If $f : \mathbb{B}^n \to f\mathbb{B}^n = \mathbb{B}^n$ is a K-quasiconformal mapping, where $K \geq 1$, then the inequality

$$h_{f\mathbb{B}^n}(f(x), f(y)) \leq b \max\{h_{\mathbb{B}^n}(x,y), 2^{1-\alpha} h_{\mathbb{B}^n}(x,y)^{\alpha}\}$$

holds, where $\alpha = K^{1/(1-n)}$ and b is a constant depending on K and n. Here, b tends to 1 as K tends to 1.

***Theorem* 90.** *[V, Theorem 3.10] Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be a nonconstant K-quasiregular mapping such that $f\mathbb{H}^n \subset \mathbb{H}^n$, where $K \geq 1$. Then*

$$h_{f\mathbb{H}^n}(f(x), f(y)) \leq K(h_{\mathbb{H}^n}(x,y) + \log 4).$$

***Theorem* 91.** *[V, Theorem 3.11] If $f : \mathbb{B}^2 \to \mathbb{B}^2$ is a nonconstant K-quasiregular mapping, where $K \geq 1$, then*

$$h_{\mathbb{B}^2}(f(x), f(y)) \leq c(K) \max\{h_{\mathbb{B}^2}(x,y), 2^{1-1/K} h_{\mathbb{B}^2}(x,y)^{1/K}\}$$

for all $x, y \in \mathbb{B}^2$, where $c(K) = 2\mathrm{arth}(\varphi_K(\mathrm{th}\,(1/2)))$. In particular, $c(1) = 1$.

7.3 Schwarz Lemma

Researchers have extensively explored harmonic functions in the past few years, uncovering numerous findings extending the analytic functions' principles. An example of this adaptation includes the identification of a counterpart to the Schwarz lemma in this particular domain, as outlined in [39].

***Lemma* 92.** *Let $u : \mathbb{B}^2 \to \mathbb{B}^2$ be a harmonic function with $u(0) = 0$. Then*

$$|u(z)| \leq \frac{4}{\pi} \arctan |z|.$$

The inequality is sharp for each point $z \in \mathbb{B}^2$.

We formulate the following theorem, extending the Schwarz lemma mentioned earlier.

***Theorem* 93.** *[V, Theorem 4.3] Let $0 < r < R$ and $M > 0$. If u is a complex-valued harmonic mapping in the disk $B^2(a, R)$ such that $|u(w)| \leq M$ for all $w \in B^2(a, R)$, then*

$$\left| u(a+z) - \frac{R^2 - |z|^2}{R^2 + |z|^2} u(a) \right| \leq \frac{2M}{\pi} \arctan \left(\frac{2R|z|}{R^2 - |z|^2} \right), \quad z = re^{it}.$$

The result is sharp.

Let ∇u be the gradient of u at x defined by

$$\nabla u(x) = (\partial u / \partial x_1, \ldots, \partial u / \partial x_n).$$

Kalaj and Vuorinen formulated the theorem below; consult [45, Theorem 1.8] for details.

***Theorem* 94.** *Let u be a real harmonic function of the unit disk into $(-1, 1)$. Then the following sharp inequality holds:*

$$|\nabla u(z)| \leq \frac{4}{\pi} \frac{1 - |u(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{B}^2.$$

Next, we extend the theorem mentioned above as follows:

***Theorem* 95.** *[V, Theorem 4.7] Let α and β be two real numbers such that $\alpha < \beta$. If $u : \mathbb{B}^2 \to (\alpha, \beta)$ is a real harmonic function, then we have*

$$|\nabla u(z)| \leq \frac{2(\beta - \alpha)}{\pi} \frac{1 - \frac{4}{(\beta-\alpha)^2} \left| u(z) - \frac{\alpha+\beta}{2} \right|^2}{1 - |z|^2}, \quad z \in \mathbb{B}^2.$$

The result is sharp.

The following result was obtained by Chen [13, Theorem 1.2]:

***Theorem* 96.** *Let u be a real harmonic mapping of $\mathbb{B}^2$ into the open interval $(-1, 1)$. Then*

$$|\nabla u(z)| \leq \frac{4}{\pi} \frac{\cos\left(\frac{\pi}{2} u(z)\right)}{1 - |z|^2}$$

holds for $z \in \mathbb{B}^2$. The inequality is sharp for any $z \in \mathbb{B}^2$ and any value of $u(z)$, and the equality occurs for some point in $\mathbb{B}^2$ if and only if $u(z) = (4\mathrm{Re}\{\arctan\, f(z)\})/\pi$, $z \in \mathbb{B}^2$ with a Möbius transformation f of $\mathbb{B}^2$ onto itself.

Our Chen's extending result is as follows:

***Theorem* 97.** *[V, Theorem 4.8] If $u : \mathbb{B}^2 \to (\alpha, \beta)$ is an into harmonic mapping, then*

$$|\nabla u(z)| \leq \frac{2(\beta - \alpha)}{\pi} \frac{\cos\left(\frac{\pi}{\beta-\alpha}\left(u(z) - \frac{\alpha+\beta}{2}\right)\right)}{1 - |z|^2},$$

where α and β are real numbers such that $\alpha < \beta$. The result is sharp.

8 Approximation of Certain Harmonic Mappings

The problem of approximating a non-univalent analytic function's distance from a subclass of univalent functions is relatively new. In 2012, Pascu and Pascu conducted a study on the best approximation of an analytic function within a subclass of starlike functions. They extended Karush-Kuhn-Tucker (KKT) conditions to semi-finite quadratic programming to solve this problem. In 2014, they also solved the problem for a specific subclass of convex functions, as described in Ref. [62]. Kargar et al. [46] investigated the best approximation of an analytic function with a locally univalent normalized analytic function in a 2017 study. Arora et al. [9] recently studied non-vanishing analytic functions of the form z/f, where f is a univalent function, and their approximations by functions z/g with g belonging to a certain subclass of starlike functions. Ponnusamy and Qiao [64] utilized a different method to approximate certain biharmonic mappings compared to the methods used in the aforementioned works. This chapter aims to consider this problem for certain subclasses of harmonic mappings.

8.1 The Best Approximation

To find the best approximation, we need the following lemmas:

***Lemma* 98.** *(see Ref. [42]) Let* $f = h + \overline{g}$ *be given by* (17) *and* $\alpha \in [0, 1)$. *If*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| + \sum_{n=1}^{\infty}(n+\alpha)|b_n| \le 1-\alpha, \tag{99}$$

then $f = h + \overline{g}$ *is harmonic univalent in* $\mathbb{D}$ *and* $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$. *The starlike harmonic mapping*

$$f_1(z) := z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n-\alpha} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{n+\alpha} \overline{y}_n \overline{z}^n,$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$ *shows that the inequality is sharp.*

***Lemma* 100.** *(see Ref. [41]) Let* $f = h + \overline{g}$ *be given by* (17) *and* $\alpha \in [0, 1)$. *If*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| + \sum_{n=1}^{\infty} n(n+\alpha)|b_n| \le 1-\alpha, \tag{101}$$

then $f = h + \overline{g}$ is harmonic univalent in $\mathbb{D}$ and $f \in \mathcal{K}_{\mathcal{H}}(\alpha)$. The result is sharp for

$$f_2(z) := z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n(n-\alpha)} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{n(n+\alpha)} \overline{y}_n \overline{z}^n,$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$.

We will denote by $\mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ and $\mathcal{K}^{*}_{\mathcal{H}}(\alpha)$ the subclasses of $\mathcal{S}^{*}_{\mathcal{H}}(\alpha)$ and $\mathcal{K}_{\mathcal{H}}(\alpha)$ consisting of functions f which satisfy (99) and (101), respectively.

The first preliminary result of this chapter is as follows:

***Lemma* 102.** *[VI, Lemma 2.1] Let $f : \mathbb{D} \to \mathbb{C}$ be analytic in $\mathbb{D}$ and have a series representation of the form $f = h + \overline{g}$, where*

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \textit{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then

$$\int_{\mathbb{D}} |f(x+iy)|^2 \mathrm{d}x\mathrm{d}y = \sum_{n=0}^{\infty} \frac{\pi}{n+1} \left(|a_n|^2 + |b_n|^2\right).$$

Next, we give a definition denoted by $\mathrm{dist}(f, \mathcal{S}^{*}_{\mathcal{H}}(\alpha))$. This definition shows that how "far" is a function f from the class $\mathcal{S}^{*}_{\mathcal{H}}(\alpha)$. Also, the same definition is true for $\mathrm{dist}(f, \mathcal{K}_{\mathcal{H}}(\alpha))$, $\mathrm{dist}(f, \mathcal{S}^{**}_{\mathcal{H}}(\alpha))$, and $\mathrm{dist}(f, \mathcal{K}^{*}_{\mathcal{H}}(\alpha))$.

***Definition* 103.** *[VI, Definition 2.1] For $f \in \mathcal{H}$, we define*

$$\mathrm{dist}(f, \mathcal{S}^{*}_{\mathcal{H}}(\alpha)) := \inf_{F \in \mathcal{S}^{*}_{\mathcal{H}}(\alpha)} \left(\iint_{\mathbb{D}} |f(x+iy) - F(x+iy)|^2 \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

We note that $\mathrm{dist}(f, \mathcal{S}^{*}_{\mathcal{H}}(\alpha))$ is a measure, and the following Theorem 104 shows that $\mathrm{dist}(\cdot, \mathcal{S}^{*}_{\mathcal{H}}(\alpha))$ is not a norm in $\mathcal{H}$.

***Theorem* 104.** *[VI, Theorem 2.1] Let $f \in \mathcal{H}$ be of the form* (17). *Then,* $\mathrm{dist}(f, \mathcal{S}^{*}_{\mathcal{H}}(\alpha)) = 0$ *if and only if $f \in \mathcal{S}^{*}_{\mathcal{H}}(\alpha)$.*

From now on, for convenience, in the second series (99)-(101) above, we replace "n" by "m" for $n \geq 1$. To find the best approximation of a harmonic mapping $f \in \mathcal{H}$ in the subclass $\mathcal{S}^{**}_{\mathcal{H}}(\alpha) \subset \mathcal{S}^{*}_{\mathcal{H}}(\alpha)$, by Lemma 102 and Definition 103, we consider the problem of finding

$$\inf \left\{ \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} + \sum_{m=1}^{\infty} \frac{(y_m - b_m)^2}{m+1} \right\}, \tag{105}$$

where $(a_n)_{n\geq 2}$ and $(b_m)_{m\geq 1}$ are given sequences of non-negative real numbers and the infimum is taken over all non-negative sequences $(x_n)_{n\geq 2}$ and $(y_m)_{m\geq 1}$ of real numbers satisfying

$$\sum_{n=2}^{\infty}(n-\alpha)x_n + \sum_{m=1}^{\infty}(m+\alpha)y_m \leq 1-\alpha. \tag{106}$$

We note that the above infimum (105) is 0 when $x_n = a_n$ $(n \geq 2)$ and $y_m = b_m$ $(m \geq 1)$. It is clear that the solution of the above problem is trivial if

$$\sum_{n=2}^{\infty}(n-\alpha)a_n + \sum_{m=1}^{\infty}(m+\alpha)b_m \leq 1-\alpha.$$

We will therefore consider the following additional hypothesis on the sequences $(a_n)_{n\geq 2}$ and $(b_m)_{m\geq 1}$

$$\sum_{n=2}^{\infty}(n-\alpha)a_n + \sum_{m=1}^{\infty}(m+\alpha)b_m > 1-\alpha.$$

Define the function

$$f(x,y) := \sum_{n=2}^{\infty}\frac{(x_n-a_n)^2}{n+1} + \sum_{m=1}^{\infty}\frac{(y_m-b_m)^2}{m+1},$$

where x denotes the non-negative sequence $(x_n)_{n\geq 2}$ and y denotes the non-negative sequence $(y_m)_{m\geq 1}$.

***Remark* 107.** *[VI, Remark 3.1] Let $(\gamma_n)_{n\geq 1}$ be a sequence of positive numbers with $\lim_{n\to\infty}\gamma_n = 0$. Then each of the intervals $[1,+\infty)$ and $[1/(1+s), 1/s)$, $s \geq 1$, contains only a finite number of the terms of the sequence. This led us to find a permutation $(i_n)_{n\geq 1}$ of the indices $\{1,2,3,\ldots\}$ such that $(\gamma_{i_n})_{n\geq 1}$ is a non-increasing sequence and $\lim_{n\to\infty}\gamma_{i_n} = 0$.*

Let α_n and β_m be defined by

$$\alpha_n := \frac{2a_n}{(n+1)(n-\alpha)}, \quad n \geq 2, 0 \leq \alpha < 1 \tag{108}$$

and

$$\beta_m := \frac{2b_m}{(m+1)(m+\alpha)}, \quad m \geq 1, 0 \leq \alpha < 1, \tag{109}$$

respectively.

One of the main results of this chapter is the following:

***Theorem* 110.** *[VI, Theorem 3.1] Let $(a_n)_{n\geq 2}$ and $(b_m)_{m\geq 1}$ be two sequences of positive real numbers with*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n+\sum_{m=1}^{\infty}(m+\alpha)b_m>1-\alpha$$

and

$$\lim_{n\to\infty}\frac{a_n}{n^2}=0=\lim_{m\to\infty}\frac{b_m}{m^2}.$$

Then there exist integers $N\geq 2$ and $M\geq 1$ such that the minimum of the quadratic problem (105)-(106) *is given by*

$$\sum_{n\in I^c}\frac{a_n^2}{n+1}+\sum_{m\in J^c}\frac{b_m^2}{m+1}+\frac{\left[\sum_{n\in I}(n-\alpha)a_n+\sum_{m\in J}(m+\alpha)b_m-(1-\alpha)\right]^2}{\sum_{n\in I}(n+1)(n-\alpha)^2+\sum_{m\in J}(m+1)(m+\alpha)^2},$$

and is attained for the sequences $(x_n)_{n\geq 2}$ and $(y_m)_{m\geq 1}$ respectively, given by

$$x_n=\begin{cases} a_n-\frac{1}{2}\mu_{N,M}(n+1)(n-\alpha), & n\in I;\\ 0, & n\in I^c\end{cases}$$

and

$$y_m=\begin{cases} b_m-\frac{1}{2}\mu_{N,M}(m+1)(m+\alpha), & m\in J;\\ 0, & m\in J^c,\end{cases}$$

where

$$\mu_{N,M}:=\frac{2\left[\sum_{n\in I}(n-\alpha)a_n+\sum_{m\in J}(m+\alpha)b_m-(1-\alpha)\right]}{\sum_{n\in I}(n+1)(n-\alpha)^2+\sum_{m\in J}(m+1)(m+\alpha)^2}, \qquad (111)$$

$n=2,3,\ldots,|\mathcal{P}|+1$, $m=1,2,\ldots,|\mathcal{Q}|$, $I=\{i_1,i_2,\ldots,i_N\}$ *and* $J=\{j_1,j_2,\ldots,j_M\}$. *Here $(i_n)_{n=2,\ldots,|\mathcal{P}|+1}$ and $(j_m)_{m=1,\ldots,|\mathcal{Q}|}$ are permutations of the indices in $\mathcal{P}=\{n\geq 2: a_n>0\}$ and respectively in $\mathcal{Q}=\{m\geq 1: b_m>0\}$ such that $(\alpha_n)_{n\geq 2}$ and $(\beta_m)_{m\geq 1}$ defined by* (108) *and* (109) *are non-increasing sequences, respectively.*

Moreover, N and M can be taken to be equal to

$$N=\min\{n\geq 2:\max\{\alpha_{n+1},\beta_{m+1}\}\leq\mu_{n,m}\leq\max\{\alpha_n,\beta_m\}\ \ \forall m\geq 1\},$$

and

$$M=\min\{m\geq 1:\max\{\alpha_{n+1},\beta_{m+1}\}\leq\mu_{n,m}\leq\max\{\alpha_n,\beta_m\}\ \ \forall n\geq 2\},$$

where

$$\mu_{n,m}=\frac{2\left[\sum_{k=2}^{n}(i_k-\alpha)a_{i_k}+\sum_{k=1}^{m}(j_k+\alpha)b_{j_k}-(1-\alpha)\right]}{\sum_{k=2}^{n}(i_k+1)(i_k-\alpha)^2+\sum_{k=1}^{m}(j_k+1)(j_k+\alpha)^2}.$$

8.2 Applications

As an application of Theorem 110, we will determine the best approximation of a harmonic mapping f in the subclass $\mathcal{S}_{\mathcal{H}}^{**}(\alpha)$, that is, we will find

$$\operatorname{dist}(f, \mathcal{S}_{\mathcal{H}}^{**}(\alpha)) = \inf_{F \in \mathcal{S}_{\mathcal{H}}^{**}(\alpha)} \left(\iint_{\mathbb{D}} |f(x+iy) - F(x+iy)|^2 \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}$$

for a given function $f \in \mathcal{H}$, and we will determine the extremal function $F \in \mathcal{S}_{\mathcal{H}}^{**}(\alpha)$ for which the minimum is attained. The result is the following:

***Theorem* 112.** *[VI, Theorem 4.1] Let the function $f \in \mathcal{H}$ have the series expansion* (17) *and*

$$\lim_{n\to\infty} \frac{|a_n|}{n^2} = 0 = \lim_{m\to\infty} \frac{|b_m|}{m^2}.$$

i) If

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| + \sum_{m=1}^{\infty} (m+\alpha)|b_m| \le 1-\alpha,$$

then

$$\operatorname{dist}(f, \mathcal{S}_{\mathcal{H}}^{**}(\alpha)) = 0,$$

which is attained for $F = f \in \mathcal{S}_{\mathcal{H}}^{**}(\alpha) \subset \mathcal{S}_{\mathcal{H}}^{*}(\alpha)$.
ii) If

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| + \sum_{m=1}^{\infty} (m+\alpha)|b_m| > 1-\alpha,$$

then

$$\operatorname{dist}(f, \mathcal{S}_{\mathcal{H}}^{**}(\alpha)) = \sqrt{\pi} \left(\sum_{n\in I^c} \frac{a_n^2}{n+1} \right.$$

$$\left. + \sum_{m\in J^c} \frac{b_m^2}{m+1} + \frac{\left[\sum_{n\in I}(n-\alpha)|a_n| + \sum_{m\in J}(m+\alpha)|b_m| - (1-\alpha)\right]^2}{\sum_{n\in I}(n+1)(n-\alpha)^2 + \sum_{m\in J}(m+1)(m+\alpha)^2} \right)^{\frac{1}{2}},$$

where I and J are given by Theorem 110. Moreover, the minimum value of $\operatorname{dist}(f, \mathcal{S}_{\mathcal{H}}^{**}(\alpha))$ *above is attained for the function F given by*

$$F(z) = H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n, \qquad G(z) = \sum_{m=1}^{\infty} B_m z^m,$$

$$A_n = \begin{cases} \left(|a_n| - \frac{1}{2}\mu_{N,M}(n+1)(n-\alpha)\right) e^{i\arg(a_n)}, & n \in I; \\ 0, & n \in I^c, \end{cases}$$

and

$$B_m = \begin{cases} \left(|b_m| - \frac{1}{2}\mu_{N,M}(m+1)(m+\alpha)\right) e^{i\arg(b_m)}, & m \in J; \\ \\ 0, & m \in J^c, \end{cases}$$

where $\mu_{N,M}$ *is given by* (111) *with* a_n *and* b_m *replaced by* $|a_n|$ *and* $|b_m|$, *respectively.*

***Remark* 113.** *[V, Remark 4.1] If we take* $\alpha = 0$ *and* $g \equiv 0$ *in the above Theorem 112, then we get Theorem 5 in [61].*

By the same argument, we have the following theorem:

***Theorem* 114.** *[VI, Theorem 4.2] Let the function* $f \in \mathcal{H}$ *have the series expansion* (17) *and*

$$\lim_{n\to\infty} \frac{|a_n|}{n^3} = 0 = \lim_{m\to\infty} \frac{|b_m|}{m^3}.$$

i) If

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| + \sum_{m=1}^{\infty} m(m+\alpha)|b_m| \le 1-\alpha,$$

then

$$\operatorname{dist}(f, \mathcal{K}^*_{\mathcal{H}}(\alpha)) = 0,$$

which is attained for $F = f \in \mathcal{K}^*_{\mathcal{H}}(\alpha) \subset \mathcal{K}_{\mathcal{H}}(\alpha)$.
ii) If

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| + \sum_{m=1}^{\infty} m(m+\alpha)|b_m| > 1-\alpha,$$

then

$$\operatorname{dist}(f, \mathcal{K}^*_{\mathcal{H}}(\alpha)) = \sqrt{\pi}\Bigg(\sum_{n\in I^c} \frac{a_n^2}{n+1} +$$

$$\sum_{m\in J^c} \frac{b_m^2}{m+1} + \frac{\left[\sum_{n\in I} n(n-\alpha)|a_n| + \sum_{m\in J} m(m+\alpha)|b_m| - (1-\alpha)\right]^2}{\sum_{n\in I} n^2(n+1)(n-\alpha)^2 + \sum_{m\in J} m^2(m+1)(m+\alpha)^2} \Bigg)^{\frac{1}{2}},$$

where I and J are given by Theorem 110. Moreover, the infimum value of $\operatorname{dist}(f, \mathcal{K}^*_{\mathcal{H}}(\alpha))$ *above is attained for the function* F *given by*

$$F(z) = H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n, \qquad G(z) = \sum_{m=1}^{\infty} B_m z^m,$$

$$A_n = \begin{cases} \left(|a_n| - \frac{1}{2}\mu_{N,M}(n)(n+1)(n-\alpha)\right) e^{i\arg(a_n)}, & n \in I; \\ \\ 0, & n \in I^c, \end{cases}$$

and

$$B_m = \begin{cases} \left(|b_m| - \frac{1}{2}\mu_{N,M}(m)(m+1)(m+\alpha)\right) e^{i\arg(b_m)}, & m \in J; \\ 0, & m \in J^c, \end{cases}$$

where $\mu_{N,M}$ is given by (111) *with a_n and b_m replaced by $|a_n|$ and $|b_m|$, respectively.*

***Remark* 115.** *[VI, Remark 4.2] Letting $\alpha = 0$ and $g \equiv 0$ in Theorem 114, gives Theorem 5 in [62].*

8.3 Examples

In this section, as an application of the previous theorems, we consider the following examples:

Example 116. [VI, Example 5.1] Consider the function $f_\gamma : \mathbb{D} \to \mathbb{C}$ defined by $f_\gamma(z) = z + \overline{\gamma}\,\overline{z}$ $(|\gamma| \neq 1)$, where $\gamma \in \mathbb{C}$ is a constant. We see that $(|\overline{\gamma}| = |\gamma|)$

- if $|\gamma| \leq (1-\alpha)/(1+\alpha)$, $0 \leq \alpha < 1$, then $f_\gamma \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$, and thus

$$\mathrm{dist}(f_\gamma, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = 0.$$

- if $|\gamma| > (1-\alpha)/(1+\alpha)$, then applying Theorem 112 it follows that $\mathcal{Q} = \{1\}$, $M = 1$ and $J = \{j_1\} = \{1\}$. Therefore,

$$\mathrm{dist}(f_\gamma, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = \sqrt{\frac{\pi}{2}}\left(|\gamma| - \frac{1-\alpha}{1+\alpha}\right), \quad 0 \leq \alpha < 1$$

  is attained for the function $F_{\alpha,\gamma} \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ defined by

$$F_{\alpha,\gamma}(z) = z + \left(\frac{1-\alpha}{1+\alpha} e^{i\arg\overline{\gamma}}\right)\overline{z}, \quad 0 \leq \alpha < 1, z \in \mathbb{D}.$$

Example 117. [VI, Example 5.2] Define the function $f_{n,\gamma}$ as follows:

$$f_{n,\gamma}(z) = z + \overline{\gamma}\,\overline{z}^n, \quad n \geq 1, \gamma \in \mathbb{C}.$$

Then for $n \geq 2$ (the case $n = 1$ is investigated in Example 116 above), we have:

- if

$$|\gamma| \leq \frac{1-\alpha}{n+\alpha}, \quad \alpha \in [0,1), n \geq 2,$$

  then $f_{n,\gamma} \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$, therefore $\mathrm{dist}(f_{n,\gamma}, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = 0$.

- if $|\gamma| > (1-\alpha)/(n+\alpha)$, from Theorem 112 we obtain that $M = 1$ and $J = \{j_1\} = \{n\}$. Then

$$\operatorname{dist}(f_{n,\gamma}, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = \sqrt{\frac{\pi}{n+1}}\left(|\gamma| - \frac{1-\alpha}{n+\alpha}\right), \quad 0 \le \alpha < 1, n \ge 2, \gamma \in \mathbb{C},$$

is attained for the function $F_{n,\alpha,\gamma} \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ defined by

$$F^0_{n,\alpha,\gamma}(z) = z + \left(\frac{1-\alpha}{n+\alpha} e^{i \arg \overline{\gamma}}\right) \overline{z}^n, \quad 0 \le \alpha < 1, n \ge 2, \gamma \in \mathbb{C}, z \in \mathbb{D}.$$

Figure 20 and Figure 21 show a comparison between the images of $\mathbb{D}$ under the function $f_{n,\gamma}$ and its best starlike approximation function $F^0_{n,\alpha,\gamma}$ for some values γ, $n = 2$ and $\alpha = 0$. Notice that in all cases the minimum of $\operatorname{dist}(f_{2,\gamma}, \mathcal{S}^{**}_{\mathcal{H}}(\alpha))$ is attained for $f_{2,0.5}$ and its rotations, and that $f_{2,0.5}$ is starlike univalent while $f_{2,1+i}$ and $f_{2,0.5+0.3i}$ are not starlike univalent mappings.

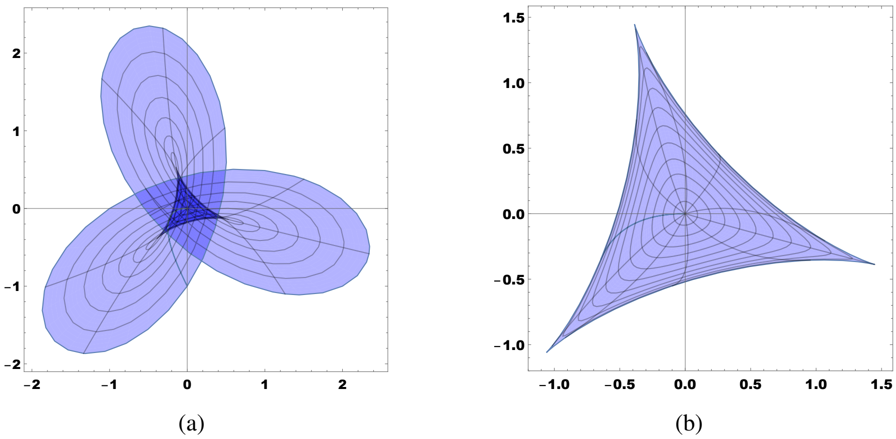


Figure 20. (a): The image of $\mathbb{D}$ under $f_{2,1+i}$ (non-starlike) (b): The image of $\mathbb{D}$ under $F^0_{2,0,1+i}$ (the best starlike approximation of $f_{2,1+i}$)

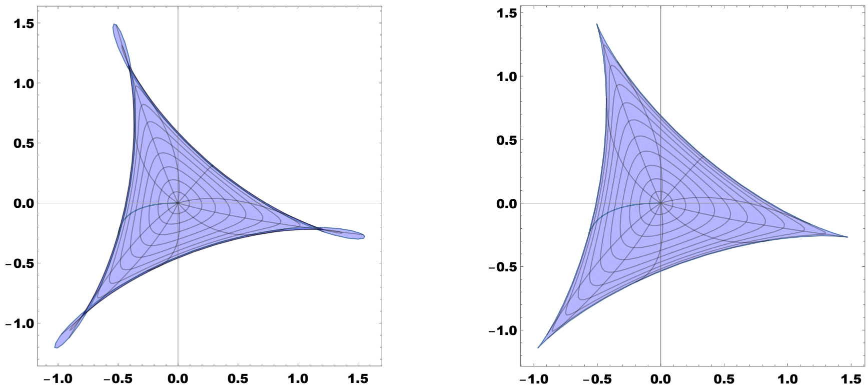
- if

$$|\gamma| \le \frac{1-\alpha}{n(n+\alpha)}, \quad \alpha \in [0,1), n \ge 2,$$

then $f_{n,\gamma} \in \mathcal{K}^*_{\mathcal{H}}(\alpha)$, therefore $\operatorname{dist}(f_{n,\gamma}, \mathcal{K}^*_{\mathcal{H}}(\alpha)) = 0$.

- if $|\gamma| > (1-\alpha)/n(n+\alpha)$, then applying Theorem 114 we get $M = 1$ and $J = \{j_1\} = \{n\}$. Therefore,

$$\operatorname{dist}(f_{n,\gamma}, \mathcal{S}^{**}_{\mathcal{H}}(\alpha)) = \sqrt{\frac{\pi}{n+1}}\left(|\gamma| - \frac{1-\alpha}{n(n+\alpha)}\right), \quad 0 \le \alpha < 1, n \ge 2, \gamma \in \mathbb{C},$$



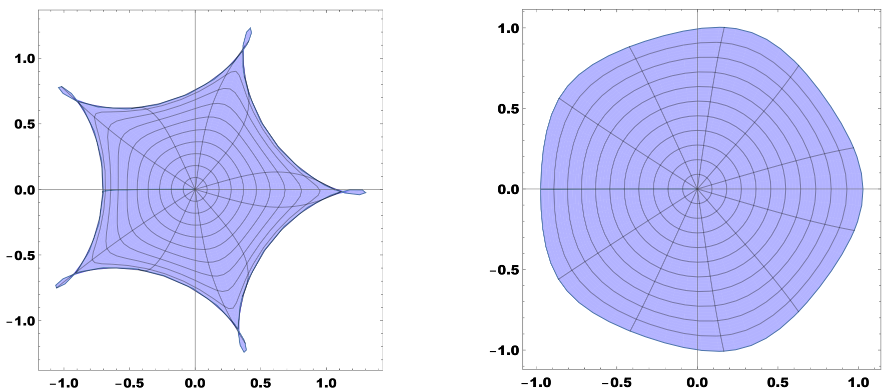
(a) (b)

Figure 21. (a): The image of $\mathbb{D}$ under $f_{2,0.5+0.3i}$ (non-starlike) (b): The image of $\mathbb{D}$ under $F^0_{2,0,0.5+0.3i}$ (the best starlike approximation of $f_{2,0.5+0.3i}$)

which is attained for the function $F_{n,\alpha,\gamma} \in \mathcal{S}^{**}_{\mathcal{H}}(\alpha)$ defined by

$$F^1_{n,\alpha,\gamma}(z) = z + \left(\frac{1-\alpha}{n(n+\alpha)} e^{i \arg \overline{\gamma}}\right) \overline{z}^n, \quad 0 \le \alpha < 1, n \ge 2, \gamma \in \mathbb{C}, z \in \mathbb{D}.$$

Figure 22 shows a comparison between the images of $\mathbb{D}$ under the function $f_{4,0.3+0.025i}$ and under its best convex approximation function $F^1_{4,1/2,0.3+0.025i}$ (here we note that for $\gamma = 0.3 + 0.025i$ we have $|\gamma| = 0.30104 > (1-1/2)/(4(4+1/2)) = 1/36 = 0.028$).



(a) (b)

Figure 22. (a): The image of $\mathbb{D}$ under $f_{4,0.3+0.025i}$ (non-convex) (b): The image of $\mathbb{D}$ under $F^1_{4,1/2,0.3+0.025i}$ (the best convex approximation of $f_{4,0.3+0.025i}$)

Some Biographical Data

J. Landen (1719–1790)
C.F. Gauss (1777–1855)
R.H. Nevanlinna (1895–1980)
M.A. Lavrent'ev (1900–1980)
H. Grötzsch (1902–1993)
I.N. Vekua (1907–1977)
C.B. Morrey (1907–1984)
L.V. Ahlfors (1907–1996)
O. Teichmüller (1913–1943)
L. Bers (1914–1993)
J. Ferrand (1918–2014)
K.I. Virtanen (1921-2006)
F.W. Gehring (1925–2012)
O. Lehto (1925–2020)
B. Fuglede (1925–2023)
Yu. G. Reshetnyak (1929–2021)
M. K. Vamanamurthy (1934–2009)
S. Rickman (1935–2017)
P.L. Duren (1935–2020)
R. Kühnau (1936–2022)
J. Borwein (1951–2016)
J. Heinonen (1960–2007)

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