Stochastic Processes and their Applications 172 (2024) 104351

Contents lists available at ScienceDirect

Stochastic Processes and their Applications

journal homepage: www.elsevier.com/locate/spa

Solutions for Poissonian stopping problems of linear diffusions via extremal processes

Luis H.R. Alvarez E. [a,*], Jukka Lempa [b], Harto Saarinen [b], Wiljami Sillanpää [b]

[a] Department of Accounting and Finance, Turku School of Economics, FIN-20014 University of Turku, Finland
[b] Department of Mathematics and Statistics, University of Turku, FIN-20014 University of Turku, Finland

ARTICLE INFO

MSC:
60G40
60J60
49K45

Keywords:
Poissonian timing
Optimal stopping
Linear diffusions
Extremal processes

ABSTRACT

We develop a general yet simple technique for solving Poissonian timing problems of linear diffusions by relying on the close connection of the extremal processes and the first passage times of the underlying diffusion. We provide a closed-form representation of the expected value gained by employing an ordinary first passage time-based stopping strategy. This approach simplifies the determination of the optimal policy, transforming it into an analysis of ordinary first-order optimality conditions. We relate our findings to various existing approaches for solving stopping problems of linear diffusions and express the optimality conditions in a single boundary setting in a form familiar from optimal stopping of Lévy-processes.

1. Introduction

As is intuitively clear, higher investment timing flexibility increases value by enlarging the set of possible exercise policies. Consequently, the highest value is attained in the continuous limit where timing opportunities are abundant and investment options can be exercised at any date. Unfortunately, in practice there are typically various factors resulting into constraints ranging from lack of sufficient liquidity to imperfect observability impeding the continuous timing of investment opportunities. Motivated by this observation, we consider in this study a general class of solvable optimal stopping problems in the presence of imperfect timing ability. Instead of being able to stop the underlying diffusion at any stopping date, we assume that the decision maker can stop only at random dates modeled as a sequence of IID exponential times. In this way our approach can be interpreted as optimal stopping based on random sampling of the underlying diffusion.

Poissonian timing problems were originally introduced in Rogers and Zane [33], where the authors considered the classical Merton problem of optimal investment and the Poisson times model liquidity constraints for balancing the portfolio (see also Pham and Tankov [32], Gassiat et al. [12] and references therein). Related optimal stopping problems subject to Poissonian timing constraints have been studied during the recent years. This line of research was initiated by Dupuis and Wang [10], where the underlying is a geometric Brownian motion and the exercise payoff is of American call option-type. The results of Dupuis and Wang [10] were extended in Lempa [21] to cover a broader class of payoff functions and underlying linear diffusion dynamics. The approach developed in Lempa [21] was reconsidered in an real options setting in Alvarez E. et al. [3] focusing also on endogenous information acquisition. The convexity and monotonicity properties of the value function for a class of stopping problems with state dependent Poisson constraint and diffusion dynamics is analyzed in Hobson [15]. The studies Menaldi and Robin [27] and Lange and K. [20] provide further generalizations. In Menaldi and Robin [27] the state variable can be a time-inhomogeneous Markov process with a locally compact state space and the interarrival times are non-exponential, while in Lange and K. [20] the problem

* Corresponding author.
E-mail addresses: lhralv@utu.fi (L.H.R. Alvarez E.), jumile@utu.fi (J. Lempa), hoasaa@utu.fi (H. Saarinen), wtasil@utu.fi (W. Sillanpää).

https://doi.org/10.1016/j.spa.2024.104351
Received 30 August 2023; Received in revised form 4 February 2024; Accepted 22 March 2024
Available online 27 March 2024
0304-4149/

is considered when the underlying is multidimensional. Further, the studies Albrecher et al. [2] and Albrecher and Ivanovs [1] are of interest since they initiated a more comprehensive study of Poisson constraint in the class of Lévy processes by studying the fluctuation theory of Lévy processes observed at Poisson arrival times. Their results have been subsequently applied to Poisson constrained stopping problems for spectrally negative Lévy processes in Pérez et al. [30],Pérez and Yamazaki [31], and Palmowski et al. [29]. Lastly, optimal stopping games have also been considered in this framework. These studies include Liang and Sun [24,25], Hobson et al. [16], Palmowski et al. [29], and Lempa and Saarinen [22], where the problem is considered in various degrees of generality.

Other related works are Guo and Liu [13], Liang and Wei [26], Hobson and Zeng [17], Arai and Takenaka [5], and Hobson and Zeng [18]. Guo and Liu [13] study a problem in which the goal is to maximize a payoff upon the maximum of a geometric Brownian motion. Liang and Wei [26] consider an optimal switching problem where the switching opportunities appear only at the event times of a Poisson process. In Arai and Takenaka [5] Poisson constrained optimal stopping with regime switching geometric Brownian motion is studied. Hobson and Zeng [18] considers a setting where it is possible to increase the liquidity by increasing the rate of the Poisson process to generate more frequent stopping opportunities when needed, and in Hobson and Zeng [17], the stopping opportunities given by the Poisson process are taken with a state dependent probability. We remark that there is also an increasing amount of related literature focusing on other type of bounded variation stochastic control problems in the presence of Poissonian timing, see Saarinen [34] for a comprehensive list of references.

Instead of solving the considered class of stopping problems directly, we first concentrate on deriving explicit representations for the expected present values accrued from following elementary boundary policies, namely, first exit times from open intervals. By relying on techniques from the classical theory of linear diffusions, we are able to express these values in terms of known functionals of the extremal processes for the underlying diffusion. In this way, we develop a simple and relatively straightforward technique for computing explicitly expected present values accrued at first exit times despite the presence of potentially significant overshoot. Interestingly, the derived representations can be interpreted in terms of path decomposition results, even though the derivation of the values does not rely on path decompositions. A second advantage of the developed approach is that, since the derived representations depend only on the current state and the boundaries of the intervals, candidate optimal stopping policies can be derived by relying on ordinary first order conditions. We find that the class of excessive mappings for the underlying stochastic processes differ significantly from the ones arising in the continuous limit. First of all, a mapping which is excessive for the underlying diffusion in the continuous limit is excessive in the discrete Poissonian setting as well. However, the opposite is not true and, therefore, the solution of an optimal stopping problem under imperfect timing ability may result into an optimal policy which differs significantly form the one in the continuous limit despite the known convergence of the optimal policy and its value. The principal reason for this is that excessivity in the Poissonian timing setting is global and cannot be localized. Second, the objective functional under imperfect timing ability is smoother than in the continuous limit. This implies that problems resulting into corner solutions in the continuous limit may possess a smooth interior solution in the discrete Poissonian setting for all finite intensities. This observation shows that even though the optimal policies and their values converge to their continuous limits as the sampling intensity tends to infinity, convergence may be pronouncedly slow.

Our paper is related to the original study Lempa [21]. However, we extend the analysis of that study in several significant ways. First, as stated above, we develop a relatively simple approach for computing the expected present values of rewards with respect to first passage times from arbitrary open intervals (independently of whether they are optimal or not). These expressions are then utilized for expressing necessary conditions for the optimality of a considered boundary policy. Second, instead of focusing on single boundary stopping problems, we consider multiple boundary problems as well and present explicit solutions in some particular cases arising in the literature on optimal stopping. Third, we verify the optimality of the proposed stopping policies by characterizing general circumstances under which they indeed satisfy the associated *Bellman equation* guaranteeing the optimality of the proposed strategies. In Lempa [21] this condition *was taken as given* and was not verified as part of the proofs.

The contents of this paper are as follows. In Section 2 we characterize the underlying random dynamics, present the considered class of optimal stopping problems, and state auxiliary results needed later in the analysis of the considered stopping policies. Our main findings on the optimal stopping rules and their values are then stated in Section 3. We illustrate our key findings explicitly in various different settings in Section 4 and Section 5 finally concludes our study.

2. Optimal stopping problem

2.1. Underlying dynamics and problem setting

Our main objective is to study and present closed form solutions for a relatively large class of optimal stopping problems in the presence of timing constraints. To this end, we assume that the underlying state variable follows a linear, time homogeneous, and regular diffusion process defined on a complete filtered probability space $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ satisfying the usual conditions. The stochastic dynamics of the underlying state process are assumed to evolve on the state space $\mathcal{I} = (a, b) \subseteq \mathbb{R}$ and governed by the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{1}$$

where the infinitesimal coefficients $\mu : \mathcal{I} \mapsto \mathbb{R}$ and $\sigma : \mathcal{I} \mapsto \mathbb{R}_+$ are assumed to be continuous and to satisfy the conditions

(A) $\sigma^2(x) > 0$ for all $x \in \mathcal{I}$,
(B) $(|\mu(x)| \vee 1)/\sigma^2(x)$ is locally integrable on $\mathcal{I}$.

These assumptions guarantee that the stochastic differential Eq. (1) has a weak solution defined up to an explosion time. Moreover, the solution is unique in law (cf. Borodin and Salminen [6], p. 49). In order to avoid interior singularities, we also assume that the diffusion does not die inside $\mathcal{I}$, thus implying that the boundaries a and b are either natural, entrance, exit or regular. If the boundary is regular, we assume that the process is killed at that boundary (for a comprehensive characterization of the boundary behavior of diffusions, see pp. 14–21 in Borodin and Salminen [6]). As usual, we denote by L_r^1 the class of measurable mappings $f : \mathcal{I} \mapsto \mathbb{R}$ satisfying the uniform integrability condition

$$\mathbb{E}_x \int_0^{\tau_{\mathcal{I}}} e^{-rs} |f(X_s)| ds < \infty,$$

where $\tau_{\mathcal{I}} = \inf\{t \geq 0 : X_t \notin \mathcal{I}\}$ denotes the potentially finite life time of the underlying diffusion.

In order to model the imperfect timing ability of the decision maker, we assume that $(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ supports a Poisson process $N = (N_t, \mathcal{F}_t)$ with known intensity λ. We refer to N as an underlying signal process and assume that it is independent of X. Given the underlying signal process N, we denote its jump times by $T_i, i \geq 0$ and assume that $T_0 = 0$ and $T_\infty = \infty$. We now define the class of admissible stopping times as

$$\mathcal{T}_0 = \{\tau : \text{ for all } \omega \in \Omega, \tau(\omega) = T_n(\omega) \text{ for some } n \in \mathbb{N}_0\}.$$

It is worth noticing that this assumption has a profound impact on the value of standard barrier policies, since the decision maker is able to stop the underlying diffusion immediately.

Our objective is now to consider the optimal stopping problem

$$V_\lambda(x) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right], \tag{2}$$

where $r > 0$ is a known exogenously determined discount rate and the exercise payoff $g : \mathcal{I} \mapsto \mathbb{R}$ is assumed to be a lower semicontinuous mapping satisfying for all $x \in \mathcal{I}$ the uniform integrability condition $g \in L_r^1$.

2.2. Auxiliary results

In what follows, we let $\theta = r + \lambda$ and denote the fundamental solutions of the ODE

$$(\mathcal{G}_\theta u)(x) = \frac{1}{2}\sigma^2(x)u''(x) + \mu(x)u'(x) - \theta u(x) = 0$$

by $\psi_\theta(x)$ and $\varphi_\theta(x)$. We also denote the constant Wronskian (w.r.t. scale S) of the fundamental solutions by B_θ (for a thorough characterization of the fundamental solutions and their boundary behavior, see pp. 18–20 in Borodin and Salminen [6]). Before proceeding in our analysis we introduce the operator

$$(\mathcal{L}_{h_\theta} h_r)(x) = \frac{h_r'(x)}{S'(x)} h_\theta(x) - \frac{h_\theta'(x)}{S'(x)} h_r(x).$$

It is now worth noticing that if $h_\theta(x)$ satisfies $(\mathcal{G}_\theta h_\theta)(x) = 0$ and $h_r(x)$ satisfies $(\mathcal{G}_r h_r)(x) = 0$, then $(\mathcal{L}_{h_\theta} h_r)(x)$ satisfies the condition $(\mathcal{L}_{h_\theta} h_r)(x) = -(\mathcal{L}_{h_r} h_\theta)(x)$. Moreover, standard differentiation yields

$$(\mathcal{L}_{h_\theta} h_r)'(x) = -\lambda h_\theta(x) h_r(x) m'(x)$$

implying that

$$(\mathcal{L}_{h_\theta} h_r)(z) - (\mathcal{L}_{h_\theta} h_r)(y) = -\lambda \int_y^z h_\theta(t) h_r(t) m'(t) dt \tag{3}$$

for $y < z$. This identity plays a crucial role in the simplifications of the optimality conditions expressed later.

We denote the expected cumulative present value of the flow $f \in L_\theta^1$ as

$$(R_\theta f)(x) = \mathbb{E}_x \int_0^\infty e^{-\theta s} f(X_s) ds$$

and recall that it can be re-expressed in terms of the fundamental solutions as

$$(R_\theta f)(x) = B_\theta^{-1} \varphi_\theta(x) (\Psi_\theta f)(x) + B_\theta^{-1} \psi_\theta(x) (\Phi_\theta f)(x), \tag{4}$$

where the functionals Ψ_θ and Φ_θ are defined as

$$(\Psi_\theta f)(x) = \int_a^x \psi_\theta(t) f(t) m'(t) dt$$

and

$$(\Phi_\theta f)(x) = \int_x^b \varphi_\theta(t) f(t) m'(t) dt.$$

Moreover, $(\mathcal{L}_{\psi_\theta} R_\theta f)(x) = -(\Psi_\theta f)(x)$ and $(\mathcal{L}_{\varphi_\theta} R_\theta f)(x) = (\Phi_\theta f)(x)$ on the states x where f is continuous. A useful result needed later for the verification of optimality is now summarized in the following.

Lemma 1. *Assume that $f \in L^1_\theta$, that $v_r(x)$ is a positive r-harmonic function for X, and that the function $F_r(x) := f(x)/v_r(x)$ is right-continuous and of bounded variation. Assume also that $\lim_{x\to a+}(\mathcal{L}_{v_r}\psi_\theta)(x) = 0$ and $\lim_{x\to b-}(\mathcal{L}_{\varphi_\theta}v_r)(x) = 0$. Then,*

$$\lambda(R_\theta f)(x) = f(x) + B_\theta^{-1}\varphi_\theta(x)\int_a^x (\mathcal{L}_{\psi_\theta}v_r)(t)dF_r(t) + B_\theta^{-1}\psi_\theta(x)\int_x^b (\mathcal{L}_{\varphi_\theta}v_r)(t)dF_r(t). \tag{5}$$

Proof. Since $f(x) = v_r(x)F_r(x)$ we notice by utilizing the representation (4) that

$$\lambda(R_\theta f)(x) = \lambda B_\theta^{-1}\varphi_\theta(x)\int_a^x \psi_\theta(t)v_r(t)F_r(t)m'(t)dt + \lambda B_\theta^{-1}\psi_\theta(x)\int_x^b \varphi_\theta(t)v_r(t)F_r(t)m'(t)dt.$$

Under our assumptions

$$\lambda\int_a^x \psi_\theta(t)v_r(t)(F_r(t) - F_r(x) + F_r(x))m'(t)dt = (\mathcal{L}_{v_r}\psi_\theta)(x)F_r(x) - \int_a^x (\mathcal{L}_{v_r}\psi_\theta)(t)dF_r(t)$$

and

$$\lambda\int_x^b \varphi_\theta(t)v_r(t)(F_r(t) - F_r(x) + F_r(x))m'(t)dt = (\mathcal{L}_{\varphi_\theta}v_r)(x)F_r(x) + \int_x^b (\mathcal{L}_{\varphi_\theta}v_r)(t)dF_r(t).$$

Noticing now that $B_\theta^{-1}\left(\varphi_\theta(x)(\mathcal{L}_{v_r}\psi_\theta)(x) + \psi_\theta(x)(\mathcal{L}_{\varphi_\theta}v_r)(x)\right)F_r(x) = f(x)$ then proves the alleged result. □

As we will later establish in Section 3 a candidate value $f(x)$ has to satisfy the condition $\lambda(R_\theta f)(x) \le f(x)$. Lemma 1 reduces the analysis of the validity of this condition into the analysis of the behavior of the ratio $f(x)/v_r(x)$ for an appropriately chosen positive r-harmonic function $v_r(x)$.

A set of useful identities characterizing the special cases associated with the fundamental solutions is now stated in the following.

Lemma 2. *It holds that $\psi_r, \varphi_r \in L^1_\theta$. Moreover,*

$$(\mathcal{L}_{\psi_r}\psi_\theta)(x) = \lambda(\Psi_\theta\psi_r)(x), \tag{6}$$

$$(\mathcal{L}_{\varphi_\theta}\varphi_r)(x) = \lambda(\Phi_\theta\varphi_r)(x), \tag{7}$$

$$(\mathcal{L}_{\varphi_r}\psi_\theta)(x) = (\mathcal{L}_{\varphi_r}\psi_\theta)(a+) + \lambda(\Psi_\theta\varphi_r)(x), \tag{8}$$

$$(\mathcal{L}_{\varphi_\theta}\psi_r)(x) = (\mathcal{L}_{\varphi_\theta}\psi_r)(b-) + \lambda(\Phi_\theta\psi_r)(x) \tag{9}$$

for all $x \in \mathcal{I}$. Especially, $(\mathcal{L}_{\varphi_\theta}\psi_r)(b-) = 0$ when b is natural for X and $(\mathcal{L}_{\varphi_r}\psi_\theta)(a+) = 0$ when a is natural for X.

Proof. We first notice that the process $\{e^{-rt}\psi_r(X_t); t < \tau_\mathcal{I}\}$ is a positive local martingale and, therefore, a supermartingale. Consequently,

$$\mathbb{E}_x\int_0^{T\wedge\tau_\mathcal{I}} \lambda e^{-(r+\lambda)s}\psi_r(X_s)ds = \int_0^T \lambda e^{-\lambda s}\mathbb{E}_x[e^{-rs}\psi_r(X_s); s < \tau_\mathcal{I}]ds \le \psi_r(x)\left(1 - e^{-\lambda T}\right)$$

for all $x \in \mathcal{I}$ and $T > 0$. Letting $T \uparrow \infty$ now proves that $\psi_r \in L^1_\theta$. The case of φ_r can be treated analogously. Consider now the function $(\mathcal{L}_{\psi_r}\psi_\theta)(x)$. It is clear that under our assumptions on the boundary behavior of the underlying diffusion X that $\lim_{x\to a+}(\mathcal{L}_{\psi_r}\psi_\theta)(x) = 0$. Since $(\mathcal{L}_{\psi_r}\psi_\theta)'(x) = \lambda\psi_\theta(x)\psi_r(x)m'(x)$ the alleged claim (6) follows from the fundamental theorem of calculus. Establishing (7) is completely analogous. Consider now the function $(\mathcal{L}_{\varphi_r}\psi_\theta)(x)$. Since $(\mathcal{L}_{\varphi_r}\psi_\theta)(x) > 0$ for all $x \in \mathcal{I}$ and $(\mathcal{L}_{\varphi_r}\psi_\theta)'(x) = \lambda\psi_\theta(x)\varphi_r(x)m'(x) > 0$ we notice that the limit $\lim_{x\to a+}(\mathcal{L}_{\varphi_r}\psi_\theta)(x)$ exists. Identity (8) now follows from the fundamental theorem of calculus. Establishing identity (9) is completely analogous. Finally, (6) implies that $\psi_\theta'(x)/\psi_\theta(x) \ge \psi_r'(x)/\psi_r(x)$ for all $x \in \mathcal{I}$ and, therefore, that

$$(\mathcal{L}_{\varphi_\theta}\psi_r)(x) = \psi_r(x)\left[\frac{\psi_r'(x)}{S'(x)}\frac{\varphi_\theta(x)}{\psi_r(x)} - \frac{\varphi_\theta'(x)}{S'(x)}\right] < B_\theta\frac{\psi_r(x)}{\psi_\theta(x)}.$$

Consequently, if b is a natural boundary for X, then (cf. Lemma 2.1 in Lempa [21])

$$0 \le \lim_{x\to b-}(\mathcal{L}_{\varphi_\theta}\psi_r)(x) < B_\theta\lim_{x\to b-}\frac{\psi_r(x)}{\psi_\theta(x)} = 0$$

proving the alleged claim. Establishing that $(\mathcal{L}_{\varphi_r}\psi_\theta)(a+) = 0$ when a is natural for X is completely analogous. □

In what follows, the extremal processes of the underlying diffusion play an important role. We denote by $M_t = \sup\{X_s; s \le t\}$ the running supremum and by $I_t = \inf\{X_s; s \le t\}$ the running infimum of the underlying diffusion X. As is well-known from the literature on stochastic processes, these processes are closely associated with the first passage times of the underlying diffusion process. If $\tilde{T} \sim \exp(\theta)$ is independent of the underlying diffusion, then (see Section II.19 in Borodin and Salminen [6])

$$\mathbb{P}_x\left[X_{\tilde{T}} \in dy | M_{\tilde{T}} = z\right] = \frac{\psi_\theta(y)m'(y)dy}{\int_a^z \psi_\theta(t)m'(t)dt}$$

and

$$\mathbb{P}_x\left[X_{\tilde{T}} \in dy | I_{\tilde{T}} = z\right] = \frac{\varphi_\theta(y)m'(y)dy}{\int_z^b \varphi_\theta(t)m'(t)dt}.$$

It is worth pointing out that both probabilities are independent of the current state x. Moreover, if $g : \mathcal{I} \mapsto \mathbb{R}$ satisfies the uniform integrability condition $g \in L^1_\theta$, then

$$\mathbb{E}_x\left[g(X_{\tilde{T}}) | M_{\tilde{T}} = z\right] = \frac{\int_a^z g(y)\psi_\theta(y)m'(y)dy}{\int_a^z \psi_\theta(t)m'(t)dt} = \frac{(\Psi_\theta g)(z)}{(\Psi_\theta 1)(z)} \tag{10}$$

and

$$\mathbb{E}_x\left[g(X_{\tilde{T}}) | I_{\tilde{T}} = z\right] = \frac{\int_z^b g(y)\varphi_\theta(y)m'(y)dy}{\int_z^b \varphi_\theta(t)m'(t)dt} = \frac{(\Phi_\theta g)(z)}{(\Phi_\theta 1)(z)}. \tag{11}$$

Especially, if X is a twice continuously differentiable monotone function of Brownian motion (or a Lévy-process which is not a subordinator or a compound Poisson process), then

$$\mathbb{P}_x\left[X_{\tilde{T}} \in dy | M_{\tilde{T}} = z\right] = \mathbb{P}_z\left[I_{\tilde{T}} \in dy\right]$$

and

$$\mathbb{P}_x\left[X_{\tilde{T}} \in dy | I_{\tilde{T}} = z\right] = \mathbb{P}_z\left[M_{\tilde{T}} \in dy\right].$$

In that case, we observe that

$$\mathbb{E}_x\left[g(X_{\tilde{T}}) | M_{\tilde{T}} = z\right] = \mathbb{E}_z\left[g(I_{\tilde{T}})\right] \tag{12}$$

and

$$\mathbb{E}_x\left[g(X_{\tilde{T}}) | I_{\tilde{T}} = z\right] = \mathbb{E}_z\left[g(M_{\tilde{T}})\right]. \tag{13}$$

We denote as $T^y = \inf\{T_n, n \in \mathbb{Z}_+ : X_{T_n} \geq y\}$ the first Poisson passage time at which the underlying process exceeds the threshold y and is observed and as $T_y = \inf\{T_n, n \in \mathbb{Z}_+ : X_{T_n} \leq y\}$ the first Poisson passage time at which the underlying process falls below the threshold y and is observed. As usual, we denote as $\tau_y = \inf\{t \geq 0 : X_t = y\}$ the first hitting time of the underlying diffusion to the state y under continuous sampling. As is clear, if $x \geq y$ then $T_y \geq \tau_y$ and if $x \leq y$ then $T^y \geq \tau_y$ almost surely.

Let $(y, z) \subset \mathcal{I}$ and consider the function $u : \mathcal{I} \mapsto \mathbb{R}$ defined as

$$u(x) = g(x)\mathbb{1}_{\mathcal{I}\setminus(y,z)}(x) + v_r(x)\mathbb{1}_{(y,z)}(x) \tag{14}$$

where $v_r(x) = c_1\psi_r(x) + c_2\varphi_r(x)$ and $c_1, c_2 \in \mathbb{R}$ are unknown constants. As was established in Lemma 2.1 of Lempa [21], a function $f : \mathcal{I} \mapsto \mathbb{R}$ satisfying the identity $f(x) = \lambda(R_\theta f)(x)$ on an open interval (y, z) is necessarily r-harmonic on (y, z). In light of this observation, let us now consider the following question:

Can the constants c_1 and c_2 be chosen so that $u(x) = \lambda(R_\theta u)(x)$ for all $x \in (y, z)$?

The answer to this question is positive as proven in our next Lemma.

Lemma 3. *Let $v_r(x) = H(x, y, z)$, where*

$$\begin{aligned} H(x, y, z) := &\frac{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)\varphi_r(x) - (\mathcal{L}_{\varphi_\theta}\varphi_r)(z)\psi_r(x)}{(\mathcal{L}_{\varphi_\theta}\varphi_r)(z)(\mathcal{L}_{\psi_\theta}\psi_r)(y) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\lambda(\Psi_\theta g)(y) \\ &+ \frac{(\mathcal{L}_{\psi_\theta}\psi_r)(y)\varphi_r(x) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)\psi_r(x)}{(\mathcal{L}_{\psi_\theta}\psi_r)(y)(\mathcal{L}_{\varphi_\theta}\varphi_r)(z) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\lambda(\Phi_\theta g)(z). \end{aligned} \tag{15}$$

Then the function $u(x)$ defined by (14) satisfies the identity $u(x) = \lambda(R_\theta u)(x)$ for all $x \in (y, z)$.

Proof. To prove that the answer to this question is positive, we first notice that identity $v_r(x) = \lambda(R_\theta u)(x)$ can be re-expressed as

$$\begin{aligned} v_r(x) = &\lambda B_\theta^{-1}\varphi_\theta(x)(\Psi_\theta g)(y) + \lambda B_\theta^{-1}\psi_\theta(x)(\Phi_\theta g)(z) \\ &+ \lambda B_\theta^{-1}\varphi_\theta(x)\int_y^x \psi_\theta(t)v_r(t)m'(t)dt + \lambda B_\theta^{-1}\psi_\theta(x)\int_x^z \varphi_\theta(t)v_r(t)m'(t)dt. \end{aligned}$$

Utilizing now (8) and (9) of Lemma 2 results into equation

$$\begin{aligned} v_r(x) &= \lambda B_\theta^{-1}\varphi_\theta(x)(\Psi_\theta g)(y) + \lambda B_\theta^{-1}\psi_\theta(x)(\Phi_\theta g)(z) \\ &\quad + B_\theta^{-1}\varphi_\theta(x)((\mathcal{L}_{\psi_\theta}v_r)(y) - (\mathcal{L}_{\psi_\theta}v_r)(x)) + B_\theta^{-1}\psi_\theta(x)((\mathcal{L}_{\varphi_\theta}v_r)(x) - (\mathcal{L}_{\varphi_\theta}v_r)(z)) \\ &= v_r(x) + B_\theta^{-1}\varphi_\theta(x)\left[\lambda(\Psi_\theta g)(y) + (\mathcal{L}_{\psi_\theta}v_r)(y)\right] + B_\theta^{-1}\psi_\theta(x)\left[\lambda(\Phi_\theta g)(z) - (\mathcal{L}_{\varphi_\theta}v_r)(z)\right] \end{aligned}$$

implying that we necessarily have

$$c_1(\mathcal{L}_{\psi_\theta}\psi_r)(y) + c_2(\mathcal{L}_{\psi_\theta}\varphi_r)(y) = -\lambda(\Psi_\theta g)(y), \tag{16}$$

$$c_1(\mathcal{L}_{\varphi_\theta}\psi_r)(z) + c_2(\mathcal{L}_{\varphi_\theta}\varphi_r)(z) = \lambda(\Phi_\theta g)(z). \tag{17}$$

Utilizing the identities in Lemma 2 demonstrate that

$$(\mathcal{L}_{\varphi_r}\psi_\theta)(y)(\mathcal{L}_{\varphi_\theta}\psi_r)(z) - (\mathcal{L}_{\psi_r}\psi_\theta)(y)(\mathcal{L}_{\varphi_\theta}\varphi_r)(z) > \lambda^2(\Phi_\theta 1)(z)(\Psi_\theta 1)(y)(\psi_r(z)\varphi_r(y) - \varphi_r(z)\psi_r(y)) > 0,$$

which, in turn, proves that Eqs. (16) and (17) have a unique root which reads as

$$c_1 = \lambda \frac{-(\mathcal{L}_{\varphi_\theta}\varphi_r)(z)(\Psi_\theta g)(y) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)(\Phi_\theta g)(z)}{(\mathcal{L}_{\psi_\theta}\psi_r)(y)(\mathcal{L}_{\varphi_\theta}\varphi_r)(z) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)(\mathcal{L}_{\varphi_\theta}\psi_r)(z)},$$

$$c_2 = \lambda \frac{(\mathcal{L}_{\psi_\theta}\psi_r)(y)(\Phi_\theta g)(z) + (\mathcal{L}_{\varphi_\theta}\psi_r)(z)(\Psi_\theta g)(y)}{(\mathcal{L}_{\psi_\theta}\psi_r)(y)(\mathcal{L}_{\varphi_\theta}\varphi_r)(z) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}.$$

Substituting these constants to the function $v_r(x) = c_1\psi_r(x) + c_2\varphi_r(x)$ proves identity (15). □

Lemma 3 shows how the locally r-harmonic function $v_r(x)$ should be chosen in order to guarantee that (14) is harmonic on the interval (y, z) with respect to the exponential clock and the underlying diffusion in a two-boundary setting. The two special cases focusing on a single boundary are proved in the following corollary.

Corollary 1 (*A*). *Assume that $x \in (a, z)$, where $z \in \mathcal{I}$, and let*

$$u(x) = g(x)\mathbb{1}_{[z,b)}(x) + \frac{\lambda(\Phi_\theta g)(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\psi_r(x)\mathbb{1}_{(a,z)}(x). \tag{18}$$

Then the function $u(x)$ satisfies the identity $u(x) = \lambda(R_\theta u)(x)$ for all $x \in (a, z)$.

(B) Assume that $x \in (y, b)$, where $y \in \mathcal{I}$, and let

$$u(x) = g(x)\mathbb{1}_{(a,y]}(x) + \frac{\lambda(\Psi_\theta g)(y)}{(\mathcal{L}_{\varphi_r}\psi_\theta)(y)}\varphi_r(x)\mathbb{1}_{(y,b)}(x). \tag{19}$$

Then the function $u(x)$ satisfies the identity $u(x) = \lambda(R_\theta u)(x)$ for all $x \in (y, b)$.

Proof. The alleged claim follows from Eqs. (16) and (17) by letting $y \downarrow a$ and $z \uparrow b$, respectively. □

Having shown how the function (14) has to be chosen in order to guarantee the validity of the global r-harmonicity condition $\lambda(R_\theta u)(x) = u(x)$ we are now in position to state our first result on the expected present value of the exercise payoff accrued at the first exit dates from arbitrary open intervals on $\mathcal{I}$.

Theorem 1. *Assume that $x \in (y, z)$, where $a < y < z < b$. Then,*

$$\mathbb{E}_x\left[e^{-rT_y\wedge T^z} g\left(X_{T_y\wedge T^z}\right)\right] = v_r(x), \tag{20}$$

where the function $v_r(x)$ is defined as in Lemma 3. If the boundaries a and b are natural for X and $x \in (y, z)$, then $v_r(x)$ can be re-expressed as

$$\begin{aligned} v_r(x) = &\left(\frac{\frac{\psi_r(x)}{\varphi_r(x)} - \frac{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}}{\frac{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]} - \frac{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}}\right)\frac{\mathbb{E}_x\left[g(X_{\tilde{T}})|M_{\tilde{T}}=y\right]}{\mathbb{E}_x\left[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}}=y\right]}\varphi_r(x) \\ &+ \left(\frac{\frac{\varphi_r(x)}{\psi_r(x)} - \frac{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}}{\frac{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]} - \frac{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}}\right)\frac{\mathbb{E}_x\left[g(X_{\tilde{T}})|I_{\tilde{T}}=z\right]}{\mathbb{E}_x\left[\psi_r(X_{\tilde{T}})|I_{\tilde{T}}=z\right]}\psi_r(x). \end{aligned} \tag{21}$$

Especially, if X is a twice continuously differentiable monotone function of Brownian motion, then $v_r(x)$ can be further simplified to the form

$$v_r(x) = \left(\frac{\frac{\psi_r(x)}{\varphi_r(x)} - \frac{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}}{\frac{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]} - \frac{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}}\right)\frac{\mathbb{E}_y\left[g(I_{\tilde{T}})\right]}{\mathbb{E}_y\left[\varphi_r(I_{\tilde{T}})\right]}\varphi_r(x) + \left(\frac{\frac{\varphi_r(x)}{\psi_r(x)} - \frac{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}}{\frac{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]} - \frac{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}}\right)\frac{\mathbb{E}_z\left[g(M_{\tilde{T}})\right]}{\mathbb{E}_z\left[\psi_r(M_{\tilde{T}})\right]}\psi_r(x).$$

Proof. Assume that $x \in (y, z)$, where $a < y < z < b$, and let $\hat{T} = T^z \wedge T_y$. Consider the function

$$U(x) = \mathbb{E}_x\left[e^{-r\hat{T}}g(X_{\hat{T}})\right].$$

Invoking the tower property of conditional expectations yields

$$U(x) = \mathbb{E}_x\left[e^{-rT_1}\mathbb{E}_{(T_1,X_{T_1})}\left[e^{-r(\hat{T}-T_1)}g(X_{\hat{T}})\right]\right].$$

The memoryless property of the exponential distribution and the strong Markov property of X now shows that

$$U(x) = \mathbb{E}_x\left[e^{-rT_1}U(X_{T_1})\right] = \lambda(R_\theta U)(x)$$

for all $x \in (y,z)$. Since $U(x) = g(x)$ for all $x \notin (y,z)$, the alleged identity follows from Lemma 3. The rest of the claim follows from the representations (10) and (11). □

Theorem 1 states an explicit representation of the expected present value of the exercise payoff accrued from an exercise strategy characterized as a first exit time from an open interval on $\mathcal{I}$. As is clear from Theorem 1, the expected present value admits a relatively simple representation in terms of well-known functionals of the extremal processes whenever the boundaries of the state space of the underlying diffusion are natural. According to our lemma, this representation can be simplified further whenever the underlying is a twice continuously differentiable monotone function of Brownian motion. In this way our findings present an interesting connection between the first passage times and functionals of extremal processes in the presence of imperfect timing ability.

In sharp contrast with the continuous diffusion limit, we observe that this expected value is not necessarily continuous across the boundaries. The reason for this observation is the *overshoot* associated with the underlying dynamics in the presence of imperfect timing ability. More precisely, since $T_y \geq \tau_y$ and $T^z \geq \tau_z$ almost surely, the decision maker knows that $X_{T_y+} \in (a,y)$ and $X_{T^z+} \in (z,b)$ a.s. It is precisely the random quantities $y - X_{T_y+}$ and $X_{T^z+} - z$ which generate the discontinuity in the expected present value. It is also worth mentioning that the expected value (21) is also closely connected to the representations arising in approaches relying on functional concavity originally introduced in Dynkin and Yushkevich [11] within a Brownian motion setting and later extended into a regular diffusion setting in Dayanik and Karatzas [9]. An interesting special case of Theorem 1 is stated in our next corollary.

Corollary 2. *Assume that $x \in (y,z)$, where $a < y < z < b$. Then,*

$$\begin{aligned}\mathbb{E}_x\left[e^{-rT_y\wedge T^z}\right] &= \frac{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)\varphi_r(x) - (\mathcal{L}_{\varphi_\theta}\varphi_r)(z)\psi_r(x)}{(\mathcal{L}_{\varphi_\theta}\varphi_r)(z)(\mathcal{L}_{\psi_\theta}\psi_r)(y) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\lambda(\Psi_\theta 1)(y) \\ &+ \frac{(\mathcal{L}_{\psi_\theta}\psi_r)(y)\varphi_r(x) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)\psi_r(x)}{(\mathcal{L}_{\psi_\theta}\psi_r)(y)(\mathcal{L}_{\varphi_\theta}\varphi_r)(z) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y)(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\lambda(\Phi_\theta 1)(z).\end{aligned} \tag{22}$$

Especially, if X is a twice continuously differentiable monotone function of Brownian motion, then (22) can be re-expressed as

$$\mathbb{E}_x\left[e^{-rT_y\wedge T^z}\right] = \left(\frac{\frac{\psi_r(x)}{\varphi_r(x)} - \frac{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}}{\frac{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]} - \frac{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}}\right)\frac{\varphi_r(x)}{\mathbb{E}_y\left[\varphi_r(I_{\tilde{T}})\right]} + \left(\frac{\frac{\varphi_r(x)}{\psi_r(x)} - \frac{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}}{\frac{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]} - \frac{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}}\right)\frac{\psi_r(x)}{\mathbb{E}_z\left[\psi_r(M_{\tilde{T}})\right]}.$$

Proof. The alleged claim follows by choosing $g(x) \equiv 1$ in Theorem 1. □

Theorem 1 and Corollary 2 focus on two-sided exit policies. An interesting subclass of these policies arise when the stopping strategy is one-sided. These cases are treated in our next theorem and its corollary.

Theorem 2 (*A*). *Assume that $x \in (a,z)$, where $z \in \mathcal{I}$. Then,*

$$\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right] = \frac{\lambda(\Phi_\theta g)(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\psi_r(x).$$

If b is a natural boundary for X, then

$$\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right] = \frac{(\Phi_\theta g)(z)}{(\Phi_\theta\psi_r)(z)}\psi_r(x) = \frac{\mathbb{E}_x[g(X_{\tilde{T}})|I_{\tilde{T}} = z]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}} = z]}\psi_r(x). \tag{23}$$

Especially, if X is a twice continuously differentiable monotone function of Brownian motion, then

$$\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right] = \frac{\mathbb{E}_z[g(M_{\tilde{T}})]}{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}\psi_r(x) \tag{24}$$

for all $x \in (a,z)$.

(*B*) *Assume that $x \in (y,b)$, where $y \in \mathcal{I}$. Then,*

$$\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right] = \frac{\lambda(\Psi_\theta g)(y)}{(\mathcal{L}_{\varphi_r}\psi_\theta)(y)}\varphi_r(x).$$

If a is a natural boundary for X, then

$$\mathbb{E}_x\left[e^{-rT_y}g(X_{T_y})\right] = \frac{(\Psi_\theta g)(y)}{(\Psi_\theta\varphi_r)(y)}\varphi_r(x) = \frac{\mathbb{E}_x[g(X_{\tilde{T}})|M_{\tilde{T}} = y]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}} = y]}\varphi_r(x). \tag{25}$$

Especially, if X is a twice continuously differentiable monotone function of Brownian motion, then

$$\mathbb{E}_x\left[e^{-rT_y}g(X_{T_y})\right] = \frac{\mathbb{E}_y[g(I_{\tilde{T}})]}{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}\varphi_r(x) \tag{26}$$

for all $x \in (y,b)$.

Proof. The claims follow from Corollary 1 and Theorem 1. □

Corollary 3 (*A*). *Assume that* $x \in (a, z)$, *where* $z \in \mathcal{I}$. *Then,*

$$\mathbb{E}_x\left[e^{-rT^z}\right] = \frac{\lambda(\Phi_\theta 1)(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\psi_r(x).$$

If b *is a natural boundary for* X, *then*

$$\mathbb{E}_x\left[e^{-rT^z}\right] = \frac{(\Phi_\theta 1)(z)}{(\Phi_\theta\psi_r)(z)}\psi_r(x) = \frac{\psi_r(x)}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}} = z]}. \tag{27}$$

Especially, if X *is a twice continuously differentiable monotone function of Brownian motion, then*

$$\mathbb{E}_x\left[e^{-rT^z}\right] = \frac{\psi_r(x)}{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}. \tag{28}$$

(B) Assume that $x \in (y, b)$, *where* $b \in \mathcal{I}$. *Then,*

$$\mathbb{E}_x\left[e^{-rT_y}\right] = \frac{\lambda(\Psi_\theta 1)(y)}{(\mathcal{L}_{\varphi_r}\psi_\theta)(y)}\varphi_r(x).$$

If a *is a natural boundary for* X, *then*

$$\mathbb{E}_x\left[e^{-rT_y}\right] = \frac{(\Psi_\theta 1)(y)}{(\Psi_\theta\varphi_r)(y)}\varphi_r(x) = \frac{\varphi_r(x)}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}} = y]}. \tag{29}$$

Especially, if X *is a twice continuously differentiable monotone function of Brownian motion, then*

$$\mathbb{E}_x\left[e^{-rT_y}\right] = \frac{\varphi_r(x)}{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}. \tag{30}$$

Proof. The alleged claims follow Corollary 1 and Theorem 2. □

Remark 2.1. The expected present values (27) and (29) can be utilized in the analysis of the almost sure finiteness of the first passage times T^z and T_y. In the case where the boundaries are natural we have

$$\mathbb{P}_x[T^z < \infty] = \lim_{r\to 0+}\frac{(\Phi_\theta 1)(z)}{(\Phi_\theta\psi_r)(z)}\psi_r(x)$$

and

$$\mathbb{P}_x[T_y < \infty] = \lim_{r\to 0+}\frac{(\Psi_\theta 1)(y)}{(\Psi_\theta\varphi_r)(y)}\varphi_r(x).$$

If the boundaries a and b are killing boundaries, then

$$\mathbb{P}_x[T^z < \tau_a] = \lim_{r\to 0+}\frac{\lambda(\Phi_\theta 1)(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\psi_r(x)$$

and

$$\mathbb{P}_x[T_y < \tau_b] = \lim_{r\to 0+}\frac{\lambda(\Psi_\theta 1)(y)}{(\mathcal{L}_{\varphi_r}\psi_\theta)(y)}\varphi_r(x).$$

The two latter identities are useful in the analysis of the probability of ruin in risk theoretic models arising in the literature on insurance.

It is worth emphasizing that our results demonstrate that expected present values accrued from following a standard boundary policy admit an interesting decomposition in terms of the expected present value of an unit of account accrued at the exercise date and the expected value of the exercise payoff associated with the extremal processes of the underlying diffusion (a path decomposition result). In the single boundary setting we notice that

$$\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right] = \mathbb{E}_x\left[e^{-r\tau_z}\right]\mathbb{E}_z\left[e^{-rT^z}\right]\mathbb{E}_x[g(X_{\tilde{T}})|I_{\tilde{T}} = z] \tag{31}$$

for $x \in (a, z)$ and

$$\mathbb{E}_x\left[e^{-rT_y}g(X_{T_y})\right] = \mathbb{E}_x\left[e^{-r\tau_y}\right]\mathbb{E}_y\left[e^{-rT_y}\right]\mathbb{E}_x[g(X_{\tilde{T}})|M_{\tilde{T}} = y] \tag{32}$$

for $x \in (y, b)$. Again, if X is a twice continuously differentiable monotone function of Brownian motion, these representation can be simplified further into the forms

$$\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right] = \mathbb{E}_x\left[e^{-r\tau_z}\right]\mathbb{E}_z\left[e^{-rT^z}\right]\mathbb{E}_z[g(M_{\tilde{T}})] \tag{33}$$

for $x \in (a, z)$ and

$$\mathbb{E}_x\left[e^{-rT_y}g(X_{T_y})\right] = \mathbb{E}_x\left[e^{-r\tau_y}\right]\mathbb{E}_y\left[e^{-rT_y}\right]\mathbb{E}_y[g(I_{\tilde{T}})] \tag{34}$$

for $x \in (y, b)$. Consequently, we notice that the evaluation of the expected values can be expressed in terms of the product of three functions. The first term captures the expected present value of a unit of account accrued from waiting continuously up to the first hitting time to the boundary. The second term, in turn, captures the expected present value of a unit of account accrued at the actual discrete first entrance date when the process is started from the boundary. The third term then captures the expected value of the exercise payoff once the boundary has been crossed.

It is at this point worth emphasizing that the derived decompositions (31) and (32) can be alternatively obtained by relying on the law of total probability. To see that this is indeed the case, we first observe that identity (11) can be re-expressed as

$$\mathbb{E}_x\left[g(X_{\tilde{T}})|I_{\tilde{T}}=z\right]=\frac{\mathbb{E}_x\left[e^{-rT_1}g(X_{T_1});X_{T_1}\geq z\right]}{\mathbb{E}_x\left[e^{-rT_1};X_{T_1}\geq z\right]}$$

Hence,

$$\begin{aligned}\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right]&=\sum_{k=1}^{\infty}\mathbb{E}_x\left[e^{-rT_k}g(X_{T_k})|T^z=T_k\right]\mathbb{P}_x(T^z=T_k)\\&=\sum_{k=1}^{\infty}\mathbb{E}_x\left[e^{-rT_{k-1}}\mathbb{E}_{X_{T_{k-1}}}\left[e^{-rT_1}g(X_{T_1})|X_{T_1}\geq z\right]|T^z>T_{k-1}\right]\mathbb{P}_x(T^z=T_k)\\&=\sum_{k=1}^{\infty}\mathbb{E}_x\left[e^{-rT_{k-1}}\mathbb{E}_{X_{T_{k-1}}}\left[e^{-rT_1}|X_{T_1}\geq z\right]|T^z>T_{k-1}\right]\mathbb{P}_x(T^z=T_k)\mathbb{E}\left[g(X_{\tilde{T}})|I_{\tilde{T}}=z\right]\\&=\sum_{k=1}^{\infty}\mathbb{E}_x\left[e^{-rT_k}|T^z=T_k\right]\mathbb{P}_x(T^z=T_k)\mathbb{E}\left[g(X_{\tilde{T}})|I_{\tilde{T}}=z\right]\\&=\mathbb{E}_x\left[e^{-rT^z}\right]\mathbb{E}\left[g(X_{\tilde{T}})|I_{\tilde{T}}=z\right].\end{aligned}$$

Establishing (32) is completely analogous.

In the two boundary setting a similar decomposition can be obtained when the boundaries a and b are natural for the underlying diffusion. In that case we find that

$$\begin{aligned}\mathbb{E}_x\left[e^{-rT_y\wedge T^z}g\left(X_{T_y\wedge T^z}\right)\right]&=\left(\frac{\frac{\psi_r(x)}{\varphi_r(x)}-\frac{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}}{\frac{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}-\frac{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}}\right)\mathbb{E}_x\left[e^{-rT_y}g(X_{T_y})\right]\\&+\left(\frac{\frac{\varphi_r(x)}{\psi_r(x)}-\frac{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}}{\frac{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}}=z]}-\frac{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|M_{\tilde{T}}=y]}}\right)\mathbb{E}_x\left[e^{-rT^z}g(X_{T^z})\right]\end{aligned}$$

for all $x \in (y, z)$. Again, if X is a twice continuously differentiable monotone function of Brownian motion, this representation can be simplified further into the form

$$\begin{aligned}\mathbb{E}_x\left[e^{-rT_y\wedge T^z}g\left(X_{T_y\wedge T^z}\right)\right]&=\left(\frac{\frac{\psi_r(x)}{\varphi_r(x)}-\frac{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}}{\frac{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}-\frac{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}}\right)\mathbb{E}_y[g(I_{\tilde{T}})]\mathbb{E}_x\left[e^{-r\tau_y}\right]\mathbb{E}_y\left[e^{-rT_y}\right]\\&+\left(\frac{\frac{\varphi_r(x)}{\psi_r(x)}-\frac{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}}{\frac{\mathbb{E}_z[\varphi_r(M_{\tilde{T}})]}{\mathbb{E}_z[\psi_r(M_{\tilde{T}})]}-\frac{\mathbb{E}_y[\varphi_r(I_{\tilde{T}})]}{\mathbb{E}_y[\psi_r(I_{\tilde{T}})]}}\right)\mathbb{E}_z[g(M_{\tilde{T}})]\mathbb{E}_x\left[e^{-r\tau_z}\right]\mathbb{E}_z\left[e^{-rT^z}\right].\end{aligned}$$

3. Optimal stopping rules and their values

Before stating our main findings on the considered class of optimal stopping problems, consider the dynamic programming equation (i.e. *Bellman equation*)

$$V_\lambda(x)=\max\left\{g(x),\mathbb{E}_x\left[e^{-rT_1}V_\lambda(X_{T_1})\right]\right\}=\max\{g(x),\lambda(R_\theta V_\lambda)(x)\} \tag{35}$$

associated with the optimal stopping problem (2). Denote by V_∞ the value of the optimal stopping policy in the continuous limit and by C_∞ and Γ_∞ the associated continuation and stopping regions, respectively. We can now establish the following useful result.

Lemma 4. *$V_\infty(x) \geq V_\lambda(x)$ for all $x \in \mathcal{I}$ and $\lambda < \infty$. Moreover, $\Gamma_\infty \subset \Gamma_\lambda$.*

Proof. $V_\infty(x)$ constitutes the smallest r-excessive majorant of the payoff $g(x)$ and, therefore, satisfies the inequality $V_\infty(x) \geq \max\{g(x), \lambda(R_\theta V_\infty)(x)\}$ for all $x \in \mathcal{I}$. Lemma 2.3 in Lempa [21] now implies that $V_\infty(x) \geq V_\lambda(x)$ for all $x \in \mathcal{I}$. The second claim follows after noticing that if $x \in \Gamma_\infty$, then $V_\infty(x) = g(x) \leq V_\lambda(x) \leq V_\infty(x)$ implying that $x \in \Gamma_\lambda$ as well. □

In order to understand the general structure of the solutions of (35), we first divide the state space in the continuation (inaction) region $C_\lambda = \{x \in \mathcal{I} : V_\lambda(x) > g(x)\} \subset \mathcal{I} = (a,b) \subseteq \mathbb{R}$ where stopping is suboptimal and in the stopping (action) region $\Gamma_\lambda = \{x \in \mathcal{I} : V_\lambda(x) = g(x)\} \subset \mathcal{I}$ where it pays to exercise. In the generic case the value can be expressed as

$$V_\lambda(x) = g(x)\mathbb{1}_{\Gamma_\lambda}(x) + \lambda(R_\theta V_\lambda)(x)\mathbb{1}_{C_\lambda}(x).$$

As was established in Lemma 2.1 of Lempa [21], identity $V_\lambda(x) = \lambda(R_\theta V_\lambda)(x)$ implies that the value is r-harmonic on the continuation set and, therefore, can be alternatively expressed in the form

$$V_\lambda(x) = g(x)\mathbb{1}_{\Gamma_\lambda}(x) + v_r(x)\mathbb{1}_{C_\lambda}(x),$$

where $v_r(x)$ is an r-harmonic function for the underlying diffusion X. It is now clear that if $x \in C_\lambda$, then

$$\begin{aligned} v_r(x) &= \mathbb{E}_x\left[e^{-rT}g(X_T)\mathbb{1}_{\Gamma_\lambda}(X_T)\right] + \mathbb{E}_x\left[e^{-rT}v_r(X_T)\mathbb{1}_{C_\lambda}(X_T)\right] \\ &= \mathbb{E}_x\left[e^{-rT}(g(X_T) - v_r(X_T))\mathbb{1}_{\Gamma_\lambda}(X_T)\right] + v_r(x). \end{aligned}$$

Hence, we observe that

$$\mathbb{E}_x\left[e^{-rT}(g(X_T) - v_r(X_T))\mathbb{1}_{\Gamma_\lambda}(X_T)\right] = 0 \tag{36}$$

for all $x \in C_\lambda$. It is at this point worth pointing out two major differences of condition (36) with the standard optimality conditions in the continuous limit. First of all, (36) does not involve the derivative of the exercise payoff. Instead, it is based on its expected value which, as an integral, is smoother than the payoff. In this way we notice that Poissonian timing results into a smoother objective functional than in the continuous limit. Second, condition (36) is global since it is expressed as an integral over the entire state–space $\mathcal{I}$. Consequently, the verification of optimality of a proposed policy cannot be carried out by relying on local arguments. Fortunately, Lemma 1 is helpful in many cases since it reduces the problem into the analysis of known ratios associated with the limiting continuous timing setting. More precisely, if the conditions of Lemma 1 are met, then

$$\Delta(x) = B_\theta^{-1}\varphi_\theta(x)\int_a^x (\mathcal{L}_{\psi_\theta}v_r)(t)\mathbb{1}_\Gamma(t)d\left(\frac{g(t)}{v_r(t)}\right) + B_\theta^{-1}\psi_\theta(x)\int_x^b (\mathcal{L}_{\varphi_\theta}v_r)(t)\mathbb{1}_\Gamma(t)d\left(\frac{g(t)}{v_r(t)}\right), \tag{37}$$

where $\Delta(x) := \lambda(R_\theta V_\lambda)(x) - V_\lambda(x)$. An interesting auxiliary result characterizing states which are included in the stopping set is summarized in the following.

Lemma 5. *Assume that $v_r : \mathcal{I} \mapsto \mathbb{R}_+$ is a r-harmonic function on $\mathcal{I}$. Assume also that*

$$y \in \operatorname*{argmax}_{x\in\mathcal{I}}\left\{\frac{g(x)}{v_r(x)}\right\}.$$

Then $y \in \Gamma_\lambda = \{x \in \mathcal{I} : V_\lambda(x) = g(x)\}$.

Proof. The proof of the alleged result is essentially analogous with the proof in the continuous limit originally treated in Christensen and Irle [8] (see also Christensen [7]). The r-harmonicity of v_r implies that $e^{-rt}v_r(X_t)$ constitutes a local martingale. The nonnegativity of v_r then guarantees that it is a supermartingale. Noticing now that

$$g(x) \le V_\lambda(x) = \sup_{\tau\in\mathcal{T}_0}\mathbb{E}_x\left[e^{-r\tau}\frac{g(X_\tau)}{v_r(X_\tau)}v_r(X_\tau)\right] \le \frac{g(y)}{v_r(y)}\sup_{\tau\in\mathcal{T}_0}\mathbb{E}_x\left[e^{-r\tau}v_r(X_\tau)\right] \le \frac{g(y)}{v_r(y)}v_r(x)$$

and letting $x \to y$ proves the alleged result. □

Along the lines of the original findings in Christensen and Irle [8] and Christensen [7], Lemma 5 shows how interior points in the stopping region can be found by investigating familiar ratios arising in the literature on the optimal stopping of linear diffusions. It is, however, worth emphasizing that as proven in Lemma 4, the stopping region of the limiting continuous timing setting is included in the stopping region of the Poissonian problem. Hence, the stopping region Γ_λ cannot be determined by simply focusing on the ratios $g(x)/v_r(x)$.

3.1. Single boundary case

We can now establish the following general result characterizing the optimal exercise strategies in a single boundary setting.

Theorem 3 (A)**.** *Assume that there exists a unique threshold*

$$\{z_\lambda^*\} = \operatorname*{argmax}_{z\in\mathcal{I}}\left\{\frac{\lambda(\Phi_\theta g)(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\right\} \in (a,b).$$

Assume also that the function

$$V_\lambda(x) = g(x)\mathbb{1}_{[z_\lambda^*,b)}(x) + \frac{\lambda(\Phi_\theta g)(z_\lambda^*)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z_\lambda^*)}\psi_r(x)\mathbb{1}_{(a,z_\lambda^*)}(x) \tag{38}$$

satisfies the conditions $V_\lambda(x) \geq g(x)$ *for all* $x \in (a, z^*_\lambda)$ *and* $g(x) \geq \lambda(R_\theta V_\lambda)(x)$ *for* $x \geq z^*_\lambda$. *Then,* $V_\lambda(x)$ *constitutes the value of the optimal stopping time* $T^{z^*_\lambda}$. *In the special case, where the optimal stopping boundary* z^*_λ *is attained at a point where* $g(x)$ *is continuous, the value is continuous and can be re-expressed as*

$$V_\lambda(x) = g(x)\mathbb{1}_{[z^*_\lambda, b)}(x) + \frac{g(z^*_\lambda)}{\psi_r(z^*_\lambda)}\psi_r(x)\mathbb{1}_{(a, z^*_\lambda)}(x). \tag{39}$$

(B) Assume that there exists a unique threshold

$$\{y^*_\lambda\} = \underset{y\in\mathcal{I}}{\operatorname{argmax}} \left\{ \frac{\lambda(\Psi_\theta g)(y)}{(\mathcal{L}_{\varphi_r}\psi_\theta)(y)} \right\} \in (a, b).$$

Assume also that the function

$$V_\lambda(x) = g(x)\mathbb{1}_{(a, y^*_\lambda)}(x) + \frac{\lambda(\Psi_\theta g)(y^*_\lambda)}{(\mathcal{L}_{\varphi_r}\psi_\theta)(y^*_\lambda)}\varphi_r(x)\mathbb{1}_{(y^*_\lambda, b)}(x). \tag{40}$$

satisfies the conditions $V_\lambda(x) \geq g(x)$ *for all* $x \in (y^*_\lambda, b)$ *and* $g(x) \geq \lambda(R_\theta V_\lambda)(x)$ *for* $x \leq y^*_\lambda$. *Then,* $V_\lambda(x)$ *constitutes the value of the optimal stopping time* $T_{y^*_\lambda}$. *In the special case, where the optimal stopping boundary* y^*_λ *is attained at a point where* $g(x)$ *is continuous, the value is continuous and can be re-expressed as*

$$V_\lambda(x) = g(x)\mathbb{1}_{(a, y^*_\lambda)}(x) + \frac{g(y^*_\lambda)}{\psi_r(y^*_\lambda)}\varphi_r(x)\mathbb{1}_{(y^*_\lambda, b)}(x). \tag{41}$$

Proof. Denote the proposed value function (38) as $\tilde{V}(x)$. Part (A) of Corollary 1 implies that

$$\tilde{V}(x) = \mathbb{E}_x\left[e^{-rT^{z^*_\lambda}} g\left(X_{T^{z^*_\lambda}}\right)\right]$$

Since $T^{z^*_\lambda}$ is an admissible stopping strategy, we notice that $V_\lambda(x) \geq \tilde{V}(x)$ for all $x \in \mathcal{I}$. In order to reverse this inequality we first observe that our assumptions on the proposed value function imply that it satisfies the dynamic programming equation $\tilde{V}(x) = \max\{g(x), \lambda(R_\theta\tilde{V})(x)\}$. The alleged inequality now follows from Lemma 2.4 in Lempa [21]. Finally, since

$$\frac{d}{dz}\frac{\lambda(\Phi_\theta g)(z)}{(\mathcal{L}_{\varphi_\theta\psi_r})(z)} = \frac{\lambda\varphi_\theta(z)\psi_r(z)m'(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)}\left[\frac{\lambda(\Phi_\theta g)(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)} - \frac{g(z)}{\psi_r(z)}\right]$$

at the points where $g(x)$ is continuous, we notice that in that case

$$\frac{\lambda(\Phi_\theta g)(z^*_\lambda)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z^*_\lambda)} = \frac{g(z^*_\lambda)}{\psi_r(z^*_\lambda)} \tag{42}$$

at an optimal boundary proving (39). Establishing part (B) is completely analogous. □

Theorem 3 states a set of general conditions under which the optimal stopping strategy is a single boundary policy with an explicitly known value. By imposing appropriate boundary conditions, the representations in Theorem 3 can be further refined as is done in our next corollary.

Corollary 4 (*A*). *Assume that the conditions of part (A) of Theorem 3 are met. Assume also that* b *is a natural boundary for* X. *In that case, the value of the optimal stopping strategy* $T^{z^*_\lambda}$ *can be expressed for* $x \in (a, z^*_\lambda)$ *as*

$$V_\lambda(x) = \frac{\mathbb{E}_x[g(X_{\tilde{T}})|I_{\tilde{T}} = z^*_\lambda]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}} = z^*_\lambda]}\psi_r(x), \tag{43}$$

where the optimal boundary z^*_λ *satisfies the optimality condition*

$$\frac{\mathbb{E}_x[g(X_{\tilde{T}})|I_{\tilde{T}} = z^*_\lambda]}{\mathbb{E}_x[\psi_r(X_{\tilde{T}})|I_{\tilde{T}} = z^*_\lambda]} = \frac{g(z^*_\lambda)}{\psi_r(z^*_\lambda)}.$$

If X *is a twice continuously differentiable monotone function of Brownian motion, then*

$$V_\lambda(x) = \frac{\mathbb{E}_{z^*_\lambda}[g(M_{\tilde{T}})]}{\mathbb{E}_{z^*_\lambda}[\psi_r(M_{\tilde{T}})]}\psi_r(x) \tag{44}$$

for all $x \in (a, z^*_\lambda)$. *In this case, the optimal boundary* z^*_λ *satisfies the optimality condition*

$$\frac{\mathbb{E}_{z^*_\lambda}[g(M_{\tilde{T}})]}{\mathbb{E}_{z^*_\lambda}[\psi_r(M_{\tilde{T}})]} = \frac{g(z^*_\lambda)}{\psi_r(z^*_\lambda)}. \tag{45}$$

(B) Assume that the conditions of part (B) of Theorem 3 are met. Assume also that a *is a natural boundary for* X. *In that case, the value of the optimal stopping strategy* $T^{y^*_\lambda}$ *can be expressed for* $x \in (y^*_\lambda, b)$ *as*

$$V_\lambda(x) = \frac{\mathbb{E}_x[g(X_{\tilde{T}})|M_{\tilde{T}} = y^*_\lambda]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}} = y^*_\lambda]}\varphi_r(x), \tag{46}$$

where the optimal boundary y^*_λ satisfies the optimality condition

$$\frac{\mathbb{E}_x[g(X_{\tilde{T}})|M_{\tilde{T}} = y^*_\lambda]}{\mathbb{E}_x[\varphi_r(X_{\tilde{T}})|M_{\tilde{T}} = y^*_\lambda]} = \frac{g(y^*_\lambda)}{\varphi_r(y^*_\lambda)}.$$

If X is a twice continuously differentiable monotone function of Brownian motion, then

$$V_\lambda(x) = \frac{\mathbb{E}_{y^*_\lambda}[g(I_{\tilde{T}})]}{\mathbb{E}_{y^*_\lambda}[\varphi_r(I_{\tilde{T}})]}\varphi_r(x) \tag{47}$$

for all $x \in (y^*_\lambda, b)$. In this case, the optimal boundary y^*_λ satisfies the optimality condition

$$\frac{\mathbb{E}_{y^*_\lambda}[g(I_{\tilde{T}})]}{\mathbb{E}_{y^*_\lambda}[\varphi_r(I_{\tilde{T}})]} = \frac{g(y^*_\lambda)}{\varphi_r(y^*_\lambda)}. \tag{48}$$

Proof. The alleged results follow from Theorem 3 and Theorem 2. □

Corollary 4 shows that in the case where the boundaries of the state space of X are natural, the value of the optimal policy admits a compact representation in terms of known functionals of the extremal processes of the underlying diffusion. It is worth emphasizing that the expressions in Corollary 4 are related to the representation of the values of perpetual options for Lévy processes (cf. Mordecki [28]).

A set of relatively weak sufficient conditions under which our general findings stated in Theorem 3 are satisfied are now summarized in the following.

Theorem 4 (*A*)**.** *Assume that b is a natural boundary for X, that $g \in C(\mathcal{I})$ and that there is a unique $x_0 \in \mathcal{I}$ s.t. $g(x) \gtreqless 0$ when $x \gtreqless x_0$. Assume also that there is a unique $\{\hat{x}_r\} = \text{argmax}_{x\in\mathcal{I}}\left\{g(x)/\psi_r(x)\right\}$ so that $g(x)/\psi_r(x)$ is nondecreasing on $(a, \hat{x}_r)$ and nonincreasing on $[\hat{x}_r, b)$. Then, there exists a unique exercise threshold $z^*_\lambda \in (x_0, \hat{x}_r)$ satisfying the ordinary first order condition (42). If also*

$$\int_{z^*_\lambda}^{x} \Big(\varphi_\theta(x)(\mathcal{L}_{\psi_r}\psi_\theta)(t) + \psi_\theta(x)(\mathcal{L}_{\varphi_\theta}\psi_r)(t)\Big)\, d\,\frac{g(t)}{\psi_r(t)} > 0 \tag{49}$$

for all $x > \hat{x}_r$, then the conditions of part (A) of Theorem 3 are met and the value of the optimal stopping strategy $T^{z^*_\lambda}$ reads as in (39). In the special case where $V_\infty(x) = g(x)$ for all $x \geq \hat{x}_r$ inequality (49) is always satisfied.

(B) Assume that a is a natural boundary for X, that $g \in C(\mathcal{I})$ and that there is a unique $x_0 \in \mathcal{I}$ s.t. $g(x) \gtreqless 0$ when $x \lesseqgtr x_0$. Assume also that there is a unique $\{\tilde{x}_r\} = \text{argmax}_{x\in\mathcal{I}}\left\{g(x)/\varphi_r(x)\right\}$ so that $g(x)/\varphi_r(x)$ is nondecreasing on $[\tilde{x}_r, b)$ and nonincreasing on $(a, \tilde{x}_r)$. Then, there exists a unique exercise threshold $y^*_\lambda \in (\tilde{x}_r, x_0)$ satisfying the ordinary first order condition

$$\frac{\lambda(\Psi_\theta g)(y^*_\lambda)}{(\mathcal{L}_{\varphi_r}\psi_\theta)(y^*_\lambda)} = \frac{g(y^*_\lambda)}{\varphi_r(y^*_\lambda)}.$$

If also

$$\int_{x}^{y^*_\lambda} \Big(\varphi_\theta(x)(\mathcal{L}_{\varphi_r}\psi_\theta)(t) + \psi_\theta(x)(\mathcal{L}_{\varphi_\theta}\varphi_r)(t)\Big)\, d\,\frac{g(t)}{\varphi_r(t)} < 0 \tag{50}$$

for all $x < \tilde{x}_r$, then the conditions of part (B) of Theorem 3 are met and the value of the optimal stopping strategy $T_{y^*_\lambda}$ reads as in (40). In the special case where $V_\infty(x) = g(x)$ for all $x \leq \tilde{x}_r$ inequality (50) is always satisfied.

Proof. Denote the proposed value function by $\tilde{V}(x)$. Since $T^{z^*_\lambda}$ is an admissible stopping strategy we find that

$$V_\lambda(x) \geq \mathbb{E}_x\left[e^{-rT^{z^*_\lambda}} g(X_{T^{z^*_\lambda}})\right] = \tilde{V}(x)$$

for all $x \in \mathcal{I}$. In order to prove the opposite inequality, we first observe that the assumed continuity of g implies that

$$\frac{d}{dz}\frac{\lambda(\Phi_\theta g)(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(z)} = \frac{\lambda\varphi_\theta(z)\psi_r(z)m'(z)}{(\mathcal{L}_{\varphi_\theta}\psi_r)^2(z)} I(z),$$

where

$$I(z) = \lambda\int_z^b \varphi_\theta(t)g(t)m'(t)dt - \frac{g(z)}{\psi_r(z)}(\mathcal{L}_{\varphi_\theta}\psi_r)(z).$$

Since

$$I(x_0) = \lambda\int_{x_0}^b \varphi_\theta(t)g(t)m'(t)dt > 0$$

and

$$I(z) < -\frac{g(z)}{\psi_r(z)}(\mathcal{L}_{\varphi_\theta}\psi_r)(b-) < 0$$

for all $z \geq \hat{x}_r$, we notice that equation $I(z) = 0$ has a root on $(x_0, \hat{x}_r)$. Noticing now that

$$dI(z) = -(\mathcal{L}_{\varphi_\theta}\psi_r)(z)d\left(\frac{g(z)}{\psi_r(z)}\right) < 0$$

for all $z \in (x_0, \hat{x}_r)$ then proves the uniqueness of z^*_λ. Moreover, since z^*_λ is attained on the set where the ratio $g(x)/\psi_r(x)$ is increasing, we find that $\tilde{V}(x) \geq g(x)$ for all $x \in \mathcal{I}$. In order to prove that $\lambda(R_\theta\tilde{V})(x) \leq \tilde{V}(x)$ for all $x \in \mathcal{I}$, we first observe that under our assumptions $\lim_{x\to b-}(\mathcal{L}_{\varphi_\theta}\psi_r)(x) = 0$. Moreover, noticing that

$$\frac{g(x)}{\psi_r(x)} = \frac{g(x)}{\psi_r(x)}\mathbb{1}_{(a,\hat{x}_r)}(x) - \left(-\frac{g(x)}{\psi_r(x)}\mathbb{1}_{[\hat{x}_r,b)}(x)\right)$$

proves that $g(x)/\psi_r(x)$ is of bounded variation and Lemma 1 applies. Consequently, we notice that

$$\begin{aligned}\lambda(R_\theta\tilde{V})(x) - \tilde{V}(x) &= B_\theta^{-1}\varphi_\theta(x)\int_{z^*_\lambda}^x(\mathcal{L}_{\psi_\theta}\psi_r)(t)d\frac{g(t)}{\psi_r(t)} + B_\theta^{-1}\psi_\theta(x)\int_x^b(\mathcal{L}_{\varphi_\theta}\psi_r)(t)d\frac{g(t)}{\psi_r(t)}\\ &= -B_\theta^{-1}\int_{z^*_\lambda}^x\left(\varphi_\theta(x)(\mathcal{L}_{\psi_r}\psi_\theta)(t) + \psi_\theta(x)(\mathcal{L}_{\varphi_\theta}\psi_r)(t)\right)d\frac{g(t)}{\psi_r(t)} \leq 0\end{aligned}$$

for all $x \in (z^*_\lambda, \hat{x}_r)$ since $g(x)/\psi_r(x)$ is nondecreasing on $(z^*_\lambda, \hat{x}_r)$ and

$$\int_{z^*_\lambda}^b(\mathcal{L}_{\varphi_\theta}\psi_r)(t)d\frac{g(t)}{\psi_r(t)} = 0.$$

Imposing inequality (49) then guarantees that $\lambda(R_\theta\tilde{V})(x) \leq \tilde{V}(x)$ for all $x > \hat{x}_r$ as well. Consequently, the proposed value function satisfies the dynamic programming equation $\tilde{V}(x) = \max\{g(x), \mathbb{E}_x[e^{-rT}\tilde{V}(X_T)]\}$ and, therefore, dominates the value of the optimal policy by Lemma 2.4 in Lempa [21]. This proves that $V_\lambda(x) = \tilde{V}(x)$ and that $T^{z^*_\lambda}$ is an optimal stopping strategy. In the special case where $V_\infty(x) = g(x)$ for all $x \geq \hat{x}_r$ we notice by utilizing Lemma 4 that $V_\infty(x) = V_\lambda(x) = g(x)$ for all $x \geq \hat{x}_r$. Since $V_\infty(x)$ is r-excessive for X and expectation preserves ordering, we notice directly that $\lambda(R_\theta V_\lambda)(x) \leq \lambda(R_\theta V_\infty)(x) \leq V_\infty(x) = V_\lambda(x)$ for all $x \geq \hat{x}_r$. Establishing part (B) is completely analogous. □

3.2. Two-boundary setting

Having considered the single boundary setting, we now proceed in our analysis and investigate the two-boundary case. As in the single boundary setting, we first observe that if a pair $(y^*_\lambda, z^*_\lambda)$ maximizing the representation (15) exists and the exercise payoff is continuous at the boundaries y^*_λ, z^*_λ, then ordinary differentiation shows that the boundaries satisfy the ordinary first order conditions

$$\begin{aligned}&u_2(y^*_\lambda)\lambda(\Phi_\theta g)(z^*_\lambda) - u_1(y^*_\lambda)\lambda(\Psi_\theta g)(y^*_\lambda) = g(y^*_\lambda),\\ &u_2(z^*_\lambda)\lambda(\Phi_\theta g)(z^*_\lambda) - u_1(z^*_\lambda)\lambda(\Psi_\theta g)(y^*_\lambda) = g(z^*_\lambda),\end{aligned} \tag{51}$$

where

$$\begin{aligned}u_1(x) &= \frac{\psi_r(x)(\mathcal{L}_{\varphi_\theta}\varphi_r)(z^*_\lambda) - \varphi_r(x)(\mathcal{L}_{\varphi_\theta}\psi_r)(z^*_\lambda)}{(\mathcal{L}_{\psi_\theta}\psi_r)(y^*_\lambda)(\mathcal{L}_{\varphi_\theta}\varphi_r)(z^*_\lambda) - (\mathcal{L}_{\psi_\theta}\varphi_r)(y^*_\lambda)(\mathcal{L}_{\varphi_\theta}\psi_r)(z^*_\lambda)},\\ u_2(x) &= \frac{\varphi_r(x)(\mathcal{L}_{\psi_\theta}\psi_r)(y^*_\lambda) - \psi_r(x)(\mathcal{L}_{\psi_\theta}\varphi_r)(y^*_\lambda)}{(\mathcal{L}_{\varphi_\theta}\varphi_r)(z^*_\lambda)(\mathcal{L}_{\psi_\theta}\psi_r)(y^*_\lambda) - (\mathcal{L}_{\varphi_\theta}\psi_r)(z^*_\lambda)(\mathcal{L}_{\psi_\theta}\varphi_r)(y^*_\lambda)}.\end{aligned}$$

We again notice that if a pair satisfying the optimality conditions (51) exists, then the value of the policy is continuous across the stopping boundaries, that is, then $\lim_{x\to z^*_\lambda-} v_r(x) = g(z^*_\lambda)$ and $\lim_{x\to y^*_\lambda+} v_r(x) = g(y^*_\lambda)$. The optimality conditions (51) can be simplified further to the form

$$\begin{aligned}\lambda(\Psi_\theta g)(y^*_\lambda) &= \frac{g(z^*_\lambda)u_2(y^*_\lambda) - g(y^*_\lambda)u_2(z^*_\lambda)}{u_1(y^*_\lambda)u_2(z^*_\lambda) - u_1(z^*_\lambda)u_2(y^*_\lambda)},\\ \lambda(\Phi_\theta g)(z^*_\lambda) &= \frac{g(z^*_\lambda)u_1(y^*_\lambda) - g(y^*_\lambda)u_1(z^*_\lambda)}{u_2(z^*_\lambda)u_1(y^*_\lambda) - u_2(y^*_\lambda)u_1(z^*_\lambda)}.\end{aligned}$$

We can now establish the following result.

Theorem 5. *Assume that there exists a pair of boundaries y^*_λ, z^*_λ maximizing the value (15). Assume also that the function*

$$V_\lambda(x) = g(x)\mathbb{1}_{\mathcal{I}\setminus(y^*_\lambda,z^*_\lambda)}(x) + H(x, y^*_\lambda, z^*_\lambda)\mathbb{1}_{(y^*_\lambda,z^*_\lambda)}(x)$$

satisfies the inequality $V_\lambda(x) \geq g(x)$ for all $x \in (y^*_\lambda, z^*_\lambda)$ and the inequality $g(x) \geq \lambda(R_\theta V_\lambda)(x)$ for all $x \notin (y^*_\lambda, z^*_\lambda)$. Then, $V_\lambda(x)$ is the value of the optimal stopping strategy $T^{z^*_\lambda} \wedge T_{y^*_\lambda}$. If the exercise payoff is continuous at the optimal boundaries y^*_λ, z^*_λ, then $H(x, y^*_\lambda, z^*_\lambda)$ can be re-expressed in the familiar form

$$H(x, y^*_\lambda, z^*_\lambda) = \mathbb{E}_x\left[e^{-r\tau_{y^*_\lambda}\wedge\tau_{z^*_\lambda}}g(X_{\tau_{y^*_\lambda}\wedge\tau_{z^*_\lambda}})\right] = g(y^*_\lambda)\frac{\hat{\varphi}_r(x)}{\hat{\varphi}_r(y^*_\lambda)} + g(z^*_\lambda)\frac{\hat{\psi}_r(x)}{\hat{\psi}_r(z^*_\lambda)} \tag{52}$$

where $\hat{\psi}_r(x) = \varphi_r(y^*_\lambda)\psi_r(x) - \psi_r(y^*_\lambda)\varphi_r(x)$ *and* $\hat{\varphi}_r(x) = \psi_r(z^*_\lambda)\varphi_r(x) - \varphi_r(z^*_\lambda)\psi_r(x)$ *denote the fundamental solutions associated with the diffusion killed at* $\tau_{y^*_\lambda} \wedge \tau_{z^*_\lambda}$.

Proof. Denote the proposed value function as $\tilde{V}(x)$. Since it can be expressed as

$$\tilde{V}(x) = \mathbb{E}_x\left[e^{-rT^{z^*_\lambda}\wedge T_{y^*_\lambda}} g(X_{T^{z^*_\lambda}\wedge T_{y^*_\lambda}})\right]$$

and $T^{z^*_\lambda} \wedge T_{y^*_\lambda}$ is an admissible stopping strategy, we find that $V_\lambda(x) \geq \tilde{V}(x)$ for all $x \in \mathcal{I}$. On the other hand, the assumptions of our Theorem guarantee that the proposed value satisfies the dynamic programming equation $\tilde{V}(x) = \max\{g(x), \lambda(R_\theta\tilde{V})(x)\}$ on $\mathcal{I}$. Inequality $V_\lambda(x) \leq \tilde{V}(x)$ now follows from Lemma 2.4 in Lempa [21]. □

Theorem 5 states a set of sufficient conditions under which a candidate pair of boundaries maximizing the representation (15) constitutes the optimal stopping boundaries and, consequently, under which $T^{z^*_\lambda} \wedge T_{y^*_\lambda}$ constitute the optimal stopping time. It is of interest to notice that (52) demonstrates that under the optimal policy the identity

$$\mathbb{E}_x\left[e^{-rT^{z^*_\lambda}\wedge T_{y^*_\lambda}} g(X_{T^{z^*_\lambda}\wedge T_{y^*_\lambda}})\right] = \mathbb{E}_x\left[e^{-r\tau_{y^*_\lambda}\wedge\tau_{z^*_\lambda}} g(X_{\tau_{y^*_\lambda}\wedge\tau_{z^*_\lambda}})\right]$$

holds for all $x \in (y^*_\lambda, z^*_\lambda)$. Consequently, the optimal Poissonian timing policy generates an expected value which coincides with the one attained by following a continuous (suboptimal) timing policy.

4. Explicit illustrations

In this section we illustrate our main results explicitly both in a general as well as in parameterized settings. These illustrations simultaneously also characterize the principal differences between the problems in the presence of imperfect timing ability and the ones in the continuous limit.

4.1. General linear diffusion

4.1.1. Discontinuous payoff

Consider the case where the upper boundary b is unattainable for X and the exercise payoff is discontinuous and reads as $g(x) = \mathbb{1}_{[z,b)}(x)$, where $z \in \mathcal{I}$ is an exogenously given constant state. Consider now the function

$$J(x) := \frac{\lambda(\Phi_\theta g)(x)}{(\mathcal{L}_{\varphi_\theta}\psi_r)(x)}. \tag{53}$$

in the present setting. It is clear that

$$J'(x) = \frac{\lambda\varphi_\theta(x)m'(x)}{(\mathcal{L}_{\varphi_\theta}\psi_r)^2(x)}\left[\frac{r}{\theta}\frac{\varphi'_\theta(x)}{S'(x)}\psi_r(x) - \frac{\psi'_r(x)}{S'(x)}\varphi_\theta(x)\right] < 0$$

for $x \in (z, b)$ and

$$J'(x) = \frac{\lambda^2\varphi_\theta(x)\psi_r(x)m'(x)}{(\mathcal{L}_{\varphi_\theta}\psi_r)^2(x)}(\Phi_\theta 1)(z) > 0$$

for $x \in (a, z)$. Consequently, we notice that $\{z\} = \operatorname{argmax}_{x\in\mathcal{I}}\{J(x)\}$. Let

$$\tilde{V}(x) = \mathbb{1}_{[z,b)}(x) + J(z)\psi_r(x)\mathbb{1}_{(a,z)}(x)$$

denote the proposed value function. Since $J(z) > 0$, we notice that $\tilde{V}(x) \geq \mathbb{1}_{[z,b)}(x)$ for all $x \in \mathcal{I}$. On the other hand, since

$$\lambda(R_\theta\tilde{V})(x) = \frac{\lambda}{\theta}\left(1 - \frac{\psi'_r(z)}{\psi'_r(z)\varphi_\theta(z) - \varphi'_\theta(z)\psi_r(z)}\varphi_\theta(x)\right),$$

and

$$\lambda(R_\theta\tilde{V})'(x) = -\frac{\lambda}{\theta}\frac{\psi'_r(z)}{\psi'_r(z)\varphi_\theta(z) - \varphi'_\theta(z)\psi_r(z)}\varphi'_\theta(x) > 0$$

for all $x \in (z, b)$ and

$$\lim_{x\to z+}\tilde{V}(x) - \lambda(R_\theta\tilde{V})(x) = \frac{r}{\theta} - \frac{\lambda}{\theta}\frac{\psi'_r(z)\varphi_\theta(z)}{\varphi'_\theta(z)\psi_r(z) - \psi'_r(z)\varphi_\theta(z)} > 0$$

we notice that $\tilde{V}(x) \leq \lambda(R_\theta\tilde{V})(x)$ for all $x \in \mathcal{I}$ as well. Consequently, we notice that the conditions of part (A) of Theorem 3 are met and, therefore, that $\tilde{V}(x)$ is the value of the (trivial) stopping policy T^z. Note that we cannot utilize part (A) of Theorem 4, since the exercise payoff is discontinuous.

It is worth noticing that while the optimal stopping boundary coincides in this case with the one in the continuous limit the values are significantly different. In the Poissonian setting the value is discontinuous across the boundary z since

$$\tilde{V}(z-) = \frac{\lambda}{\theta}\frac{\varphi_\theta'(z)\psi_r(z)}{\varphi_\theta'(z)\psi_r(z) - \psi_r'(z)\varphi_\theta(z)} < 1.$$

4.1.2. Optimal timing of a call spread

Consider now the optimal exercise timing of a call spread when the underlying diffusion evolves on $\mathbb{R}_+$ and the upper boundary ∞ is unattainable for X. In this case the exercise payoff reads as

$$g(x) = (x-K)^+ - (x-M)^+,$$

where K, M are known strike prices satisfying the assumption $0 < K < M$. For the sake of simplicity, we assume that the function $\mu(x) - r(x-K)$ is either nonnegative on $[K, M]$ or satisfies the inequality $\mu(x) - r(x-K) \gtreqless 0$ for $x \lesseqgtr x_0$, where $x_0 \in [K, M]$.

Consider first the limiting continuous time setting. In that case the ratio

$$\frac{g(x)}{\psi_r(x)} = \begin{cases} \frac{M-K}{\psi_r(x)}, & x > M, \\ \frac{x-K}{\psi_r(x)}, & K < x < M, \\ 0, & x \in (0, K). \end{cases}$$

is decreasing on (M, ∞) and satisfies the identity

$$\frac{d}{dx}\frac{g(x)}{\psi_r(x)} = \frac{S'(x)}{\psi_r^2(x)}\left[\frac{\psi_r(K)}{S'(K)} + \int_K^x (\mu(t) - r(t-K))\psi_r(t)m'(t)dt\right]$$

for all $x \in (K, M)$. In light of our assumptions on the function $\mu(x) - r(x-K)$ we notice that two cases may arise. If

$$\frac{\psi_r(K)}{S'(K)} + \int_K^M (\mu(t) - r(t-K))\psi_r(t)m'(t)dt \geq 0,$$

then

$$\{M\} = \underset{\in\mathbb{R}_+}{\operatorname{argmax}}\left\{\frac{g(x)}{\psi_r(x)}\right\}$$

and the optimal policy is to exercise the option at the strike price M (a corner solution). If there is, however, a threshold $\hat{x}_r \in (x_0, M)$ satisfying the identity

$$\frac{\psi_r(K)}{S'(K)} + \int_K^{\hat{x}_r} (\mu(t) - r(t-K))\psi_r(t)m'(t)dt = 0,$$

then

$$\{\hat{x}_r\} = \underset{x\in\mathbb{R}_+}{\operatorname{argmax}}\left\{\frac{g(x)}{\psi_r(x)}\right\}$$

and the optimal policy is to exercise the option at the threshold $\hat{x}_r$.

Consider now the Poissonian case instead. The continuity of the exercise payoff implies that the functional $J(x)$ introduced in (53) is continuously differentiable. Standard differentiation yields

$$J'(x) = \frac{\lambda\varphi_\theta(x)m'(x)P(x)}{(\mathcal{L}_{\varphi_\theta}\psi_r)^2(x)},$$

where

$$P(x) := \lambda\psi_r(x)\int_x^\infty \varphi_\theta(t)g(t)m'(t)dt - g(x)(\mathcal{L}_{\varphi_\theta}\psi_r)(x).$$

It is now clear that

$$P(x) = (M-K)\left[\frac{r}{\theta}\psi_r(x)\frac{\varphi_\theta'(x)}{S'(x)} - \varphi_\theta(x)\frac{\psi_\theta'(x)}{S'(x)}\right] < 0$$

for all $x \geq M$. On the other hand, since

$$P(x) = \lambda\psi_r(x)\int_K^M \varphi_\theta(t)(t-K)m'(t)dt - \frac{\lambda}{\theta}\psi_r(x)(M-K)\frac{\varphi_\theta'(M)}{S'(M)} > 0$$

for all $x \in (0, K]$, we notice that equation $P(x) = 0$ has at least one root $z_\lambda^* \in (K, M)$. In order to establish the uniqueness of the root, we first observe that

$$\frac{d}{dx}\left[\frac{P(x)}{\psi_r(x)}\right] = -\frac{d}{dx}\left[\frac{g(x)}{\psi_r(x)}\right](\mathcal{L}_{\varphi_\theta}\psi_r)(x)$$

for all $x \in (K, M)$. If $g(x)/\psi_r(x)$ is increasing on (K, M), then uniqueness follows from the monotonicity and continuity of $P(x)/\psi_r(x)$ on (K, M). Assume instead that $\hat{x}_r \in (K, M)$. In that case $\hat{x}_r = \operatorname{argmin}\{P(x)/\psi_r(x)\}$ and uniqueness follows from the monotonicity and continuity of $P(x)/\psi_r(x)$ and inequality $P(x) < 0$ for $x \geq M$. Hence, we find that there is a unique maximizing threshold

$$\{z_\lambda^*\} = \underset{x \in \mathbb{R}_+}{\operatorname{argmax}}\{P(x)\}.$$

It is at this point worth noticing that in this case, *the stopping boundary is always attained at a state below the strike price M and no corner solution arises*. Moreover, it is clear that $z_\lambda^* < \hat{x}_r$. Invoking part (A) of Theorem 4 shows that in this case the value reads as

$$V_\lambda(x) = g(x)\mathbb{1}_{[z_\lambda^*,\infty)}(x) + \frac{(z_\lambda^* - K)}{\psi_r(z_\lambda^*)}\psi_r(x)\mathbb{1}_{(0,z_\lambda^*)}(x).$$

4.2. Brownian motion

When X is an ordinary Brownian motion, we have $\psi_\theta(x) = x^{k_\theta}$, $\varphi_\theta(x) = x^{-k_\theta}$, $S'(x) = 1$, and $m'(x) = 2/\sigma^2$, where $k_\theta = \sqrt{2\theta/\sigma^2}$. In this case

$$\mathbb{E}_x\left[e^{-rT_y}g(X_{T_y})\right] = (k_\theta - k_r)e^{-k_r(x-y)}\int_{-\infty}^{y} e^{k_\theta(t-y)}g(t)dt$$

for $x > y$ and

$$\mathbb{E}_x\left[e^{-rT^y}g(X_{T^y})\right] = (k_\theta - k_r)e^{k_r(x-y)}\int_{y}^{\infty} e^{-k_\theta(t-y)}g(t)dt$$

for $x < y$. Choosing $g = 1$ yields

$$\mathbb{E}_x\left[e^{-rT_y}\right] = \left(1 - \frac{k_r}{k_\theta}\right)e^{-k_r(x-y)}$$

for $x > y$ and

$$\mathbb{E}_x\left[e^{-rT^y}\right] = \left(1 - \frac{k_r}{k_\theta}\right)e^{k_r(x-y)}$$

for $x < y$.

Remark 4.1. Consider now the special case where the underlying is a BM killed at the origin. In that case the fundamental solution reads as $e^{k_r x} - e^{-k_r x}$ and, consequently,

$$\mathbb{E}_x\left[e^{-rT^y}; T^y < \tau_0\right] = \frac{\lambda}{r+\lambda}\frac{k_\theta\frac{\sinh(k_r x)}{k_r}}{\cosh(k_r y) + k_\theta\frac{\sinh(k_r y)}{k_r}}$$

for all $x \in (0, y)$. Since $\sinh(k_r x)/k_r \to x$ as $r \to 0$ we find in line with the statements of Remark 2.1 that

$$\mathbb{P}_x\left[\tau_0 < T^y\right] = 1 - \frac{x}{\frac{1}{k_\lambda} + y}$$

coinciding with the expression derived in Li et al. [23] by utilizing a different approach.

In the two-boundary setting, we have

$$\begin{aligned} v_r(x) = {} & \frac{2\lambda}{\sigma^2}\int_{-\infty}^{y} e^{k_\theta(t-y)}g(t)dt\,\frac{(k_\theta - k_r)e^{k_r(x-z)} - (k_\theta + k_r)e^{k_r(z-x)}}{(k_\theta - k_r)^2e^{k_r(y-z)} - (k_\theta + k_r)^2e^{k_r(z-y)}} \\ & + \frac{2\lambda}{\sigma^2}\int_{z}^{\infty} e^{-k_\theta(t-z)}g(t)dt\,\frac{(k_\theta - k_r)e^{k_r(y-x)} - (k_\theta + k_r)e^{k_r(x-y)}}{(k_\theta - k_r)^2e^{k_r(y-z)} - (k_\theta + k_r)^2e^{k_r(z-y)}}, \end{aligned} \tag{54}$$

where

$$v_r(x) = \mathbb{E}_x\left[e^{-rT_y\wedge T^z}g\left(X_{T_y\wedge T^z}\right)\right]$$

and $x \in (y, z)$. Along the lines of our results on the general representation of the expected present values attained at first passage times, we notice that the single boundary representations follow from (54) by letting $y \to -\infty$ and $z \to \infty$. In order to illustrate the expected value (54) explicitly, assume that the exercise payoff is an even function $g : \mathbb{R} \mapsto \mathbb{R}$ and that $y = -z$, where $z > 0$. In that case the value is of the separable form

$$v_r(x) = \frac{(k_\theta - k_r)(k_\theta + k_r)}{k_\theta + k_r\tanh(k_r z)}\frac{\cosh(k_r x)}{\cosh(k_r z)}\int_{z}^{\infty} e^{-k_\theta(t-z)}g(t)dt$$

for $x \in (-z, z)$.

The first order conditions (51) now read as

$$u_2(y^*_\lambda)\int_{z^*_\lambda}^{\infty} e^{-k_\theta t}g(t)dt - u_1(y^*_\lambda)\int_{-\infty}^{y^*_\lambda} e^{k_\theta t}g(t)dt = \frac{g(y^*_\lambda)}{(k_\theta^2-k_r^2)},$$
$$u_2(z^*_\lambda)\int_{z^*_\lambda}^{\infty} e^{-k_\theta t}g(t)dt - u_1(z^*_\lambda)\int_{-\infty}^{y^*_\lambda} e^{k_\theta t}g(t)dt = \frac{g(z^*_\lambda)}{(k_\theta^2-k_r^2)},$$

where

$$u_1(x) = \frac{(k_\theta-k_r)e^{-(k_\theta+k_r)z^*_\lambda+k_r x} - (k_\theta+k_r)e^{(k_r-k_\theta)z^*_\lambda - k_r x}}{(k_\theta-k_r)^2 e^{-(k_\theta+k_r)(z^*_\lambda-y^*_\lambda)} + (k_\theta+k_r)^2 e^{-(k_\theta-k_r)(z^*_\lambda-y^*_\lambda)}},$$
$$u_2(x) = \frac{(k_\theta+k_r)e^{(k_\theta-k_r)y^*_\lambda+k_r x} - (k_\theta-k_r)e^{(k_\theta+k_r)y^*_\lambda - k_r x}}{(k_\theta-k_r)^2 e^{-(k_\theta+k_r)(z^*_\lambda-y^*_\lambda)} + (k_\theta+k_r)^2 e^{-(k_\theta-k_r)(z^*_\lambda-y^*_\lambda)}}$$

4.2.1. *A class of symmetric stopping problem*

In order to illustrate our findings in a two-boundary setting, consider now the case where $g(x) = |x|$ and

$$dX_t = \sigma dW_t, \quad X_0 = x, \tag{55}$$

where $\sigma > 0$. It is well-known that in this case $\psi_\theta(x) = e^{k_\theta x}$, $\varphi_\theta(x) = e^{-k_\theta x}$, $S'(x) = 1$, $m'(x) = 2/\sigma^2$, and $B_\theta = 2k_\theta$.

Given the symmetry of the process as well as the exercise payoff, we guess that the solution of the problem is symmetric as well and, therefore, that $y^*_\lambda = -z^*_\lambda < 0$. The optimality condition

$$\frac{2\lambda}{\sigma^2}\int_{z^*_\lambda}^{\infty} e^{-k_\theta t} t dt = (\mathcal{L}_{\varphi_\theta} v_r)(z^*_\lambda)$$

can now be expressed as

$$\left(k_r \tanh(k_r z^*_\lambda) + \frac{2r}{\sigma^2 k_\theta}\right) z^*_\lambda = \frac{\lambda}{\theta}. \tag{56}$$

Since the continuously differentiable function

$$f(x) = \left(k_r \tanh(k_r x) + \frac{2r}{\sigma^2 k_\theta}\right) x$$

is monotonically increasing and satisfies the limiting conditions $\lim_{x\downarrow 0} f(x) = 0$ and $\lim_{x\uparrow\infty} f(x) = \infty$ we find that Eq. (56) has a unique root on $\mathbb{R}_+$. Moreover, since

$$k_r z^*_\lambda \tanh(k_r z^*_\lambda) = \frac{\lambda}{\theta} - \frac{2r}{\sigma^2 k_\theta} z^*_\lambda < 1,$$

we notice that $z^*_\lambda < z^*_\infty$, where $z^*_\infty > 0$ denotes the unique positive root of equation

$$k_r z^*_\infty \tanh(k_r z^*_\infty) = 1$$

characterizing the upper optimal stopping boundary in the case continuous stopping is possible. To see that this is indeed the case, we notice that

$$\lim_{\lambda\to\infty}\left(\left(k_r \tanh(k_r z^*_\lambda) + \frac{2r}{\sigma^2 k_\theta}\right) z^*_\lambda - \frac{\lambda}{\theta}\right) = k_r z^*_\lambda \tanh(k_r z^*_\lambda) - 1.$$

Consequently, $\lim_{\lambda\to\infty} z^*_\lambda = z^*_\infty$. The candidate value reads now

$$V_\lambda(x) = \begin{cases} |x|, & x \in (-\infty, -z^*_\lambda] \cup [z^*_\lambda, \infty) \\ z^*_\lambda \frac{\cosh(k_r x)}{\cosh(k_r z^*_\lambda)}, & x \in (-z^*_\lambda, z^*_\lambda). \end{cases}$$

To see that this indeed constitutes the value of the considered stopping problem we first notice that since

$$\{-z^*_\infty, z^*_\infty\} = \underset{x\in\mathbb{R}}{\operatorname{argmax}}\left\{\frac{|x|}{\cosh(k_r x)}\right\}$$

and $z^*_\lambda \in (0, z^*_\infty)$ where $x/\cosh(k_r x)$ is monotonically increasing, we find that $V_\lambda(x) \geq |x|$ for all $x \in \mathbb{R}$. Invoking now Lemma 1 with $v_r(x) = \cosh(k_r x)$ yields

$$\lambda(R_\theta V_\lambda)(x) - V_\lambda(x) = \frac{1}{2k_\theta}\int_{z^*_\lambda}^{x} U_x(t) d\left(\frac{t}{v_r(t)}\right)$$

where

$$U_x(t) = e^{-k_\theta x}(\mathcal{L}_{\psi_\theta} v_r)(t) - e^{k_\theta x}(\mathcal{L}_{\varphi_\theta} v_r)(t)$$

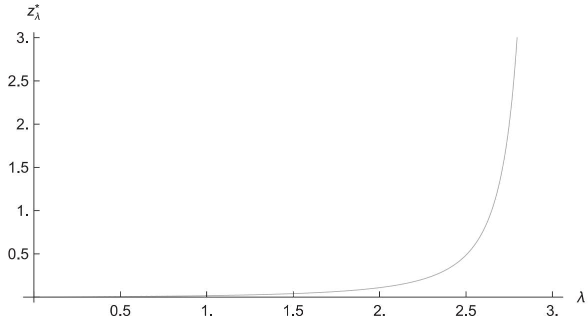

Fig. 1. The optimal stopping boundary z^*_λ.

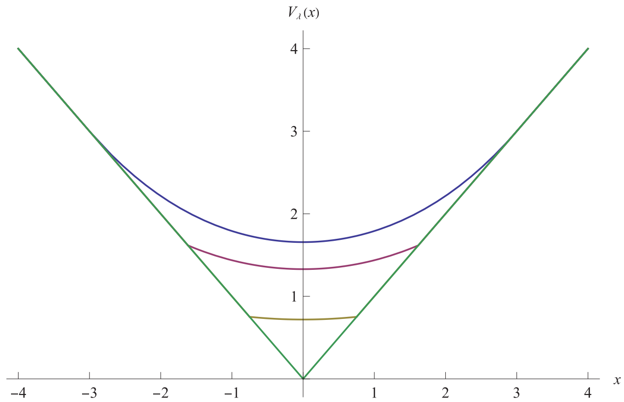

Fig. 2. The values $V_\lambda(x)$ of the optimal stopping strategies.

for all $x \geq z^*_\lambda$. Noticing that $U_x(x) = -\cosh(k_r x)/(2k_\theta) <$ and $U'_x(t) = 2\lambda v_r(t)(e^{k_\theta(x-t)} - e^{k_\theta(t-x)})/\sigma^2 > 0$ for all $t \leq x$ demonstrates that $U_x(t) \leq 0$ for all $t \leq x$. Since $x/\cosh(k_r x)$ is increasing on $[z^*_\lambda, z^*_\infty]$ we notice that $\lambda(R_\theta V_\lambda)(x) \leq V_\lambda(x)$ for all $x \in [z^*_\lambda, z^*_\infty]$. Completely analogous arguments show that $\lambda(R_\theta V_\lambda)(x) \leq V_\lambda(x)$ for all $x \in [-z^*_\infty, -z^*_\lambda]$ as well. Finally, since $V_\lambda(x) = V_\infty(x) = |x|$ for all $x \notin [-z^*_\infty, z^*_\infty]$ and expectation preserves ordering, we notice that $\lambda(R_\theta V_\lambda)(x) \leq \lambda(R_\theta V_\infty)(x) \leq V_\infty(x) = |x| = V_\lambda(x)$ for all $x \notin [-z^*_\infty, z^*_\infty]$ as well, thus demonstrating that $V_\lambda(x)$ satisfies the conditions of Theorem 5 and, therefore, constitutes the value of the optimal policy.

The optimal stopping boundary is illustrated in Fig. 1 as a function of the arrival intensity under the parameter assumptions $r = 0.02$ and $\sigma = 0.5$.

The values of the optimal stopping strategies are, in turn, illustrated in Fig. 2 for various arrival intensities ($\lambda = 0, 0.01, 0.05, 0.5, \infty$) under the parameter assumptions $r = 0.02$ and $\sigma = 0.5$.

It is worth pointing out that the findings based on the payoff $g(x) = |x|$ can be extended to a large class of symmetric payoffs satisfying a set of relatively easily verifiable sufficient conditions. More precisely, if there is a unique threshold $z^*_\infty > 0$ so that the function $g(x)/\cosh(k_r x)$ is increasing on $(0, z^*_\infty)$ and decreasing on (z^*_∞, ∞), and $g(z^*_\infty) < \infty$, then the value of the optimal policy reads as

$$V_\lambda(x) = \begin{cases} g(x), & x \in (-\infty, -z^*_\lambda] \cup [z^*_\lambda, \infty) \\ g(z^*_\lambda)\frac{\cosh(k_r x)}{\cosh(k_r z^*_\lambda)}, & x \in (-z^*_\lambda, z^*_\lambda), \end{cases}$$

where the optimal exercise threshold $z^*_\lambda \in (0, z^*_\infty)$ constitutes the unique positive root of equation

$$g(z^*_\lambda)\left(k_\theta + k_r \tanh(k_r z^*_\lambda)\right) = (k_\theta^2 - k_r^2)\int_{z^*_\lambda}^{\infty} e^{-k_\theta(t - z^*_\lambda)} g(t)dt$$

It is again worth emphasizing that in contrast to the continuous limit the optimal boundary z^*_λ does not constitute the state at which the ratio $g(x)/\cosh(k_r x)$ is maximized, that is, $\{z^*_\lambda\} \neq \mathrm{argmax}_{x\in\mathbb{R}}\{g(x)/\cosh(k_r x)\}$ when $\lambda < \infty$.

4.2.2. A periodic example

Consider the optimal stopping problem in the case where the exercise payoff reads as $g(x) = -\cos(x)$ and the underlying follows a BM characterized by (55). It is clear that $g(x)$ is a periodic and even function. By utilizing the approach developed in (cf. Christensen and Irle [8] and Christensen [7]) we observe that in the continuous limit ($\lambda \to \infty$) the value of the optimal policy is even and reads as

$$V_\lambda(x) = \begin{cases} -\frac{\cos(x^*)}{\cosh(k_r x^*)}\cosh(k_r(x-2i\pi)), & x \in (2i\pi - x^*, x^* + 2i\pi), i \in \mathbb{Z} \\ -\cos(x), & x \notin \cup_{i=-\infty}^{\infty}(2i\pi - x^*, x^* + 2i\pi), \end{cases}$$

where the optimal boundary

$$\{x^*\} = \underset{x\in(0,\pi)}{\operatorname{argmax}}\left\{\frac{-\cos(x)}{\cosh(k_r x)}\right\}$$

constitutes the unique root on $(0,\pi)$ of the first order optimality condition

$$\sin(x^*) + k_r \tanh(k_r x^*)\cos(x^*) = 0. \tag{57}$$

These observations indicate that in the discrete setting where $\lambda < \infty$ the continuation region should have the form

$$C_\lambda = \cup_{i=-\infty}^{\infty}(2i\pi - z_\lambda^*, z_\lambda^* + 2i\pi),$$

where the optimal boundary z_λ^* constitutes the unique root on $(0, x^*)$ of the condition

$$\frac{k_\theta}{1+k_\theta^2}(1+k_r^2)\cos(z_\lambda^*) + \frac{k_\theta^2 - k_r^2}{1+k_\theta^2}\sin(z_\lambda^*) + k_r\tanh(k_r z_\lambda^*)\cos(z_\lambda^*) = 0. \tag{58}$$

Moreover, the value of the proposed candidate optimal policy reads

$$V_\lambda(x) = \begin{cases} -\frac{\cos(z_\lambda^*)}{\cosh(k_r z_\lambda^*)}\cosh(k_r(x-2i\pi)), & x \in (2i\pi - z_\lambda^*, z_\lambda^* + 2i\pi), i \in \mathbb{Z} \\ -\cos(x), & x \notin \cup_{i=-\infty}^{\infty}(2i\pi - z_\lambda^*, z_\lambda^* + 2i\pi). \end{cases}$$

To see that the candidate value actually constitutes the value of the optimal stopping policy we first observe based on the behavior of the function $-\cos(x)/\cosh(k_r x)$ that $V_\lambda(x) \geq -\cos(x)$ for all $x \in \mathbb{R}$. To see that the proposed value function $V_\lambda(x)$ satisfies the inequality $\lambda(R_\theta V_\lambda)(x) \leq V_\lambda(x)$ for all $x \in \Gamma_\lambda$ as well, consider the difference $\Delta(x) = \lambda(R_\theta V_\lambda)(x) - V_\lambda(x)$ on Γ_λ. Since $(\mathcal{G}_\theta R_\theta V_\lambda)(x) = -V_\lambda(x)$ for all $x \in \mathbb{R}$ and $V_\lambda(x) = -\cos(x)$ for all $x \in \Gamma_\lambda$, we notice that

$$(\mathcal{G}_\theta \Delta)(x) = \lambda(\mathcal{G}_\theta R_\theta V_\lambda)(x) - (\mathcal{G}_\theta V_\lambda)(x) = -(\mathcal{G}_r V_\lambda)(x) = -\left(\frac{1}{2}\sigma^2 + r\right)\cos(x) > 0$$

for all $x \in \Gamma_\lambda$. Let

$$\hat{\tau}_i = \tau_{2i\pi - z_\lambda^*} \wedge \tau_{z_\lambda^* + 2i\pi} = \inf\{t \geq 0 : X_t \notin (2i\pi - z_\lambda^*, z_\lambda^* + 2i\pi)\}$$

denote the first exit time from the set $(2i\pi - z_\lambda^*, z_\lambda^* + 2i\pi)$. Invoking Dynkin's theorem yields

$$\mathbb{E}_x\left[e^{-\theta\hat{\tau}_i}\Delta(X_{\hat{\tau}_i})\right] > \Delta(x)$$

for all $x \in (2i\pi - z_\lambda^*, z_\lambda^* + 2i\pi)$. Since $\Delta(2i\pi - z_\lambda^*) = \Delta(z_\lambda^* + 2i\pi) = 0$ we notice that

$$\Delta(x) < \mathbb{E}_x\left[e^{-\theta\tau_{2i\pi - z_\lambda^*}}\Delta(2i\pi - z_\lambda^*); \tau_{2i\pi - z_\lambda^*} < \tau_{z_\lambda^* + 2i\pi}\right] + \mathbb{E}_x\left[e^{-\theta\tau_{z_\lambda^* + 2i\pi}}\Delta(z_\lambda^* + 2i\pi); \tau_{2i\pi - z_\lambda^*} > \tau_{z_\lambda^* + 2i\pi}\right] = 0$$

for all $x \in (2i\pi - z_\lambda^*, z_\lambda^* + 2i\pi)$. Since $\Gamma_\lambda = \cup_{i=-\infty}^{\infty}[2i\pi - z_\lambda^*, z_\lambda^* + 2i\pi]$ we notice that $\Delta(x) < 0$ for all $x \in \Gamma_\lambda$ demonstrating that the proposed value function is indeed the value of the optimal policy.

Letting the arrival intensity tend to infinity in (58) yields the optimality condition (57) demonstrating again how the discrete stopping problem approaches its continuous limit as the arrival intensity increases. The value and exercise payoff of the optimal timing strategy are illustrated in Fig. 3 for various arrival intensities ($\lambda = 0.0075, 0.05, 0.1, \infty$) under the parameter assumptions $r = 0.04$ and $\sigma = 0.5$.

4.3. Geometric Brownian motion

We know that if X is a geometric Brownian motion, then $\psi_\theta(x) = x^{\alpha_\theta}$, $\varphi_\theta(x) = x^{\beta_\theta}$, $S'(x) = x^{\alpha_\theta+\beta_\theta-1}$, and $m'(x) = 2x^{-\alpha_\theta-\beta_\theta-1}/\sigma^2$, where

$$\alpha_\theta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\theta}{\sigma^2}}$$

and

$$\beta_\theta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\theta}{\sigma^2}}$$

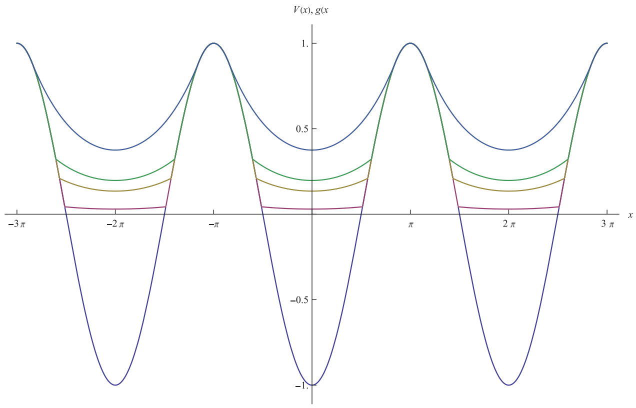

Fig. 3. The values $V_\lambda(x)$ of the optimal stopping strategies.

denote the positive and negative roots of the characteristic equation $\sigma^2 p(p-1) + 2\mu p - 2\theta = 0$, respectively. In this case

$$\mathbb{E}_x\left[e^{-rT_y} g(X_{T_y})\right] = (\beta_r - \beta_\theta) y^{\beta_\theta - \beta_r} x^{\beta_r} \int_0^y t^{-\beta_\theta - 1} g(t) dt$$

for $x > y$ and

$$\mathbb{E}_x\left[e^{-rT^y} g(X_{T^y})\right] = (\alpha_\theta - \alpha_r) x^{\alpha_r} y^{\alpha_\theta - \alpha_r} \int_y^\infty t^{-\alpha_\theta - 1} g(t) dt$$

for $x < y$. Choosing $g = 1$ yields

$$\mathbb{E}_x\left[e^{-rT_y}\right] = \left(1 - \frac{\beta_r}{\beta_\theta}\right)\left(\frac{x}{y}\right)^{\beta_r}$$

for $x > y$ and

$$\mathbb{E}_x\left[e^{-rT^y}\right] = \left(1 - \frac{\alpha_r}{\alpha_\theta}\right)\left(\frac{x}{y}\right)^{\alpha_r}$$

for $x < y$.

In the two-boundary setting, we have

$$\begin{aligned} v_r(x) = &\frac{(\alpha_\theta - \alpha_r)(x/z)^{\alpha_r} - (\alpha_r - \beta_\theta)(x/z)^{\beta_r}}{(\alpha_\theta - \alpha_r)^2 (y/z)^{\alpha_r} - (\alpha_r - \beta_\theta)^2 (y/z)^{\beta_r}} \frac{2\lambda}{\sigma^2} \int_0^y g(t) y^{\beta_\theta} t^{-\beta_\theta - 1} dt \\ &+ \frac{(\alpha_\theta - \alpha_r)(x/y)^{\beta_r} - (\alpha_r - \beta_\theta)(x/y)^{\alpha_r}}{(\alpha_\theta - \alpha_r)^2 (z/y)^{\beta_r} - (\alpha_r - \beta_\theta)^2 (z/y)^{\alpha_r}} \frac{2\lambda}{\sigma^2} \int_z^\infty g(t) z^{\alpha_\theta} t^{-\alpha_\theta - 1} dt, \end{aligned}$$

where

$$v_r(x) = \mathbb{E}_x\left[e^{-rT_y \wedge T^z} g\left(X_{T_y \wedge T^z}\right)\right]$$

and $x \in (y, z)$. The first order conditions (51) now read as

$$u_2(y_\lambda^*) \int_{z_\lambda^*}^\infty t^{\alpha_\theta - 1} g(t) dt - u_1(y_\lambda^*) \int_0^{y_\lambda^*} t^{-\beta_\theta - 1} g(t) dt = \frac{g(y_\lambda^*)}{(\alpha_r \beta_r - \alpha_\theta \beta_\theta)},$$

$$u_2(z_\lambda^*) \int_{z_\lambda^*}^\infty t^{\alpha_\theta - 1} g(t) dt - u_1(z_\lambda^*) \int_0^{y_\lambda^*} t^{-\beta_\theta - 1} g(t) dt = \frac{g(z_\lambda^*)}{(\alpha_r \beta_r - \alpha_\theta \beta_\theta)},$$

where

$$u_1(x) = \frac{(\beta_r - \beta_\theta) {z_\lambda^*}^{\beta_\theta - \alpha_r} x^{\alpha_r} - (\alpha_r - \beta_\theta) {z_\lambda^*}^{\beta_\theta - \beta_r} x^{\beta_r}}{(\alpha_\theta - \alpha_r)(\alpha_\theta - \beta_r)({y_\lambda^*}^{\alpha_\theta - \beta_r} {z_\lambda^*}^{\beta_\theta - \alpha_r} - {y_\lambda^*}^{\alpha_\theta - \alpha_r} {z_\lambda^*}^{\beta_\theta - \beta_r})},$$

$$u_2(x) = \frac{(\alpha_\theta - \beta_r) {y_\lambda^*}^{\alpha_\theta - \alpha_r} x^{\alpha_r} - (\alpha_\theta - \alpha_r) {y_\lambda^*}^{\alpha_\theta - \beta_r} x^{\beta_r}}{(\alpha_\theta - \alpha_r)(\alpha_\theta - \beta_r)({y_\lambda^*}^{\alpha_\theta - \beta_r} {z_\lambda^*}^{\beta_\theta - \alpha_r} - {y_\lambda^*}^{\alpha_\theta - \alpha_r} {z_\lambda^*}^{\beta_\theta - \beta_r})}.$$

4.3.1. *Optimal timing of a call spread*

To illustrate the valuation and timing of a call spread considered in Section 4.1.2 explicitly, assume that X is a geometric Brownian motion with parameters (μ, σ) and assume that $\mu < r$. In that case we have

$$\hat{x}_r = \frac{\alpha_r}{\alpha_r - 1} K \wedge M.$$

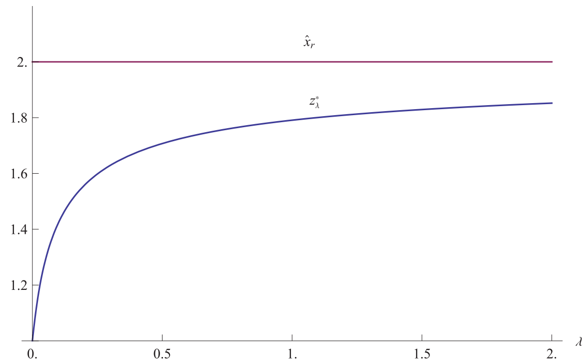

Fig. 4. The optimal exercise boundaries $\hat{x}_r$ and z_λ^*.

On the other hand, the optimal exercise threshold $z_\lambda^* \in (K, M)$ constitutes the unique root of equation

$$z_\lambda^* - \left(1 - \frac{1}{\alpha_\theta}\right)\frac{\alpha_r K}{\alpha_r - 1} + \frac{(\alpha_\theta - \alpha_r)M}{\alpha_\theta(\alpha_r - 1)}\left(\frac{z_\lambda^*}{M}\right)^{\alpha_\theta} = 0.$$

It is again worth emphasizing that while $\lim_{\lambda\to\infty} z_\lambda^* = \hat{x}_r$, $z_\lambda^* < \hat{x}_r$ for all $\lambda < \infty$. The optimal exercise boundaries are illustrated in Fig. 4 under the assumptions that $\sigma = 0.1, r = 0.04, \mu = 0.03, K = 1$, and $M = 2$.

4.3.2. *Investment timing of a risk averse investor*

In order to illustrate our findings in a asymmetric setting, we consider the investment timing problem of a risk averse investor first considered in Guo and Shepp [14] and later analyzed in Alvarez E. et al. [4] in a 2-dimensional setting. To this end, we assume that the exercise payoff reads as $g(x) = \max(x, K)$, where $K > 0$ is a known reservation level, and that the underlying follows an ordinary geometric Brownian motion characterized by the SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x.$$

In order to guarantee the finiteness of the considered expectations, we also assume that $r > \mu$.

The discontinuity of the timing opportunities has now a profound impact on the optimal policy. In contrast with the continuous setting immediate stopping may be always optimal depending on the precise parametrization of the problem. To see that this is indeed the case, let us first consider the expected value $\mathbb{E}_x[e^{-rT} g(X_T)] = \lambda(R_\theta g)(x)$. Standard integration yields

$$\lambda(R_\theta g)(x) = \frac{2\lambda}{\sigma^2(\alpha_\theta - \beta_\theta)}\begin{cases} \frac{(\alpha_\theta - \beta_\theta)x}{(\alpha_\theta - 1)(1-\beta_\theta)} - \frac{1}{\beta_\theta(1-\beta_\theta)} x^{\beta_\theta} K^{1-\beta_\theta}, & x \geq K \\ \frac{\beta_\theta - \alpha_\theta}{\alpha_\theta \beta_\theta} K + \frac{1}{\alpha_\theta(\alpha_\theta - 1)} x^{\alpha_\theta} K^{1-\alpha_\theta}, & x < K. \end{cases}$$

The monotonicity and convexity of the exercise payoff $g(x) = \max(x, K)$ implies that $\lambda(R_\theta g)(x)$ is monotonically increasing and convex as well. Since

$$\lim_{x\to 0+} \lambda(R_\theta g)(x) = \frac{\lambda}{r + \lambda} K < K$$

and

$$\lim_{x\to\infty} \lambda(R_\theta g)'(x) = \frac{\lambda}{\lambda + r - \mu} < 1$$

we notice that

$$\{K\} = \underset{x\in\mathbb{R}_+}{\operatorname{argmax}} \left\{\lambda(R_\theta g)(x) - \max(x, K)\right\}.$$

Direct computations yield

$$\lambda(R_\theta g)(K) - K = \frac{u(\lambda)}{(\alpha_\theta - \beta_\theta)\beta_\theta(1 - \alpha_\theta)} K,$$

where

$$u(\lambda) = \beta_\theta^2 - 2\alpha_r\beta_r\beta_\theta - \alpha_r\beta_r + \alpha_r\beta_r(\alpha_r + \beta_r)$$

satisfies the conditions $u(0) = (\alpha_r - \beta_r)(\alpha_r - 1)\beta_r < 0$ and $\lim_{\lambda\to\infty} u(\lambda) = +\infty$. Consequently, equation $u(\lambda) = 0$ has at least one root on $\mathbb{R}_+$. To show that this root is indeed unique, we notice that $u(\hat{\lambda}) = 0$ provided that

$$\beta_{\hat{\lambda}} = -\frac{2}{\sigma^2}(r + \sqrt{r(r - \mu)}).$$

Solving this equation results into the critical intensity

$$\hat{\lambda} = \frac{2}{\sigma^2}(r + \sqrt{r(r-\mu)})^2 + (\alpha_r + \beta_r)(r + \sqrt{r(r-\mu)}) - r.$$

We have thus found that immediate stopping is optimal and the value reads as $V_\lambda(x) = \max(x, K)$ when $\lambda \leq \hat{\lambda}$.

Assume now that $\lambda > \hat{\lambda}$. In this case the candidate optimal boundaries and constants c_1, c_2 satisfy the conditions

$$\begin{aligned} c_1 y_\lambda^{*\alpha_r} + c_2 y_\lambda^{*\beta_r} &= K, \\ c_1 z_\lambda^{*\alpha_r} + c_2 z_\lambda^{*\beta_r} &= z_\lambda^*, \\ c_1(\alpha_\theta - \alpha_r) y_\lambda^{*\alpha_r} + c_2(\alpha_\theta - \beta_r) y_\lambda^{*\beta_r} &= -\frac{2\lambda K}{\sigma^2 \beta_\theta}, \\ c_1(\alpha_r - \beta_\theta) z_\lambda^{*\alpha_r} + c_2(\beta_r - \beta_\theta) z_\lambda^{*\beta_r} &= \frac{2\lambda z_\lambda^*}{\sigma^2(\alpha_\theta - 1)}. \end{aligned}$$

Solving these equations result into the boundaries

$$\begin{aligned} y_\lambda^* &= \frac{(\alpha_\theta - 1)(\alpha_\theta - \beta_r)}{(\alpha_\theta - \alpha_r)(1 - \beta_r)} \frac{\beta_r}{\beta_\theta} P_\lambda^{\alpha_r - 1} K, \\ z_\lambda^* &= \frac{(\alpha_\theta - 1)(\alpha_\theta - \beta_r)}{(\alpha_\theta - \alpha_r)(1 - \beta_r)} \frac{\beta_r}{\beta_\theta} P_\lambda^{\alpha_r} K, \end{aligned}$$

where

$$P_\lambda = \left(\frac{\alpha_\theta - \alpha_r}{\alpha_\theta - \beta_r}\right)^{\frac{2}{\alpha_r - \beta_r}} \left(\left(1 - \frac{1}{\beta_r}\right)\frac{\alpha_r}{\alpha_r - 1}\right)^{\frac{1}{\alpha_r - \beta_r}}$$

is an increasing function satisfying the limiting conditions $\lim_{\lambda \to 0+} P_\lambda = 0$ and

$$\lim_{\lambda \to \infty} P_\lambda = \left(\left(1 - \frac{1}{\beta_r}\right)\frac{\alpha_r}{\alpha_r - 1}\right)^{\frac{1}{\alpha_r - \beta_r}} > 1.$$

The monotonicity of P_λ implies that $P_\lambda > 1$ for all $\lambda > \hat{\lambda}$. Consequently, $z_\lambda^* > y_\lambda^*$ when $\lambda > \hat{\lambda}$. Especially, we notice that in the continuous limit

$$\begin{aligned} \lim_{\lambda \to \infty} y_\lambda^* &= \frac{\beta_r}{\beta_r - 1}\left(\left(1 - \frac{1}{\beta_r}\right)\frac{\alpha_r}{\alpha_r - 1}\right)^{\frac{\alpha_r - 1}{\alpha_r - \beta_r}} K, \\ \lim_{\lambda \to \infty} z_\lambda^* &= \frac{\beta_r}{\beta_r - 1}\left(\left(1 - \frac{1}{\beta_r}\right)\frac{\alpha_r}{\alpha_r - 1}\right)^{\frac{\alpha_r}{\alpha_r - \beta_r}} K. \end{aligned}$$

Note that these expressions coincide with the ones originally derived in Guo and Shepp [14].

It remains to establish that the proposed value function

$$V_\lambda(x) = \begin{cases} x, & x \in [z_\lambda^*, \infty), \\ \frac{x_r^\alpha z_\lambda^{*1-\alpha_r}}{(\alpha_r - \beta_r)} \frac{(1-\beta_r)(\alpha_\theta - \alpha_r)}{(\alpha_\theta - 1)} + \frac{x_r^\beta z_\lambda^{*1-\beta_r}}{(\alpha_r - \beta_r)} \frac{(\alpha_r - 1)(\alpha_\theta - \beta_r)}{(\alpha_\theta - 1)}, & x \in (y_\lambda^*, z_\lambda^*), \\ K, & x \in (0, y_\lambda^*], \end{cases}$$

satisfies inequalities $\lambda(R_\theta V_\lambda)(x) \leq V_\lambda(x)$ and $V_\lambda(x) \geq \max(x, K)$ for all $x \in \mathbb{R}_+$. To see that this is indeed the case, we first observe that the proposed value function is increasing, convex, and satisfies the identities $\lambda(R_\theta V_\lambda)(y_\lambda^*) = V_\lambda(y_\lambda^*)$ and $\lambda(R_\theta V_\lambda)(z_\lambda^*) = V_\lambda(y_\lambda^*)$. Consequently, since $\lambda(R_\theta V_\lambda)'(x) - V_\lambda'(x) = \lambda(R_\theta V_\lambda)'(x) > 0$ for all $x \in (0, y_\lambda^*)$ we observe that $\lambda(R_\theta V_\lambda)(x) \leq V_\lambda(x)$ for all $x \in (0, y_\lambda^*]$. On the other hand, since

$$\lim_{x \to \infty} \lambda(R_\theta V_\lambda)'(x) = \frac{\lambda}{\theta - \mu} < 1$$

we notice that

$$\lambda(R_\theta V_\lambda)'(x) - V_\lambda'(x) = \lambda(R_\theta V_\lambda)'(x) - 1 < 0$$

for all $x > z_\lambda^*$ demonstrating that $\lambda(R_\theta V_\lambda)(x) \leq V_\lambda(x)$ for all $x \in [z_\lambda^*, \infty)$ as well. Since $\lambda(R_\theta V_\lambda)(x) = V_\lambda(x)$ for all $x \in (y_\lambda^*, z_\lambda^*)$ we notice that $\lambda(R_\theta V_\lambda)(x) \leq V_\lambda(x)$ for all $x \in \mathbb{R}_+$. Finally, inequality $V_\lambda(x) \geq \max(x, K)$ for all $x \in \mathbb{R}_+$ follows from the convexity of the proposed value function after noticing that $V_\lambda'(y_\lambda^*+) > 0$ and

$$V_\lambda'(z_\lambda^*-) = 1 - \frac{(\alpha_r - 1)(1 - \beta_r)}{\alpha_\theta - 1} < 1.$$

5. Conclusions

We developed an approach based on functionals of the running supremum and infimum of the underlying linear diffusion for solving optimal stopping problems within a constrained Poissonian timing setting. We presented a relatively simple and straightforward method for computing expected present values accrued from following standard first exit policies from open intervals within a Poissonian timing setting by exploiting the close connection of the extremal processes with first passage times. In that way we transformed the original problem into a static optimization problem where the optimized variables are the boundaries. This was shown simplify the analysis of the considered stopping problems considerably, since it directly provides a closed form expression for the candidate value in terms of admissible stopping policies. Since the value of the optimal policy dominates all the values of admissible stopping policies, the verification of optimality is reduced into proving that the candidate value satisfies the dynamic programming equation (i.e. the Bellman equation).

There are various ways towards which our analysis could be extended. A natural direction would be to consider bounded variation control problems in the presence of Poisson timing constraints. Since that class of control problems is closely connected to the considered class of stopping problems, it would be natural to expect that our results should be possible to extend into that setting as well. A second direction towards which our analysis could be extended is to consider optimal stopping of Lévy processes in the presence of Poisson timing constraints. Even though a general treatment appears to be unfeasible, single boundary problems subject to spectrally one-sided jumps may be doable due to the close-form expressions of the Laplace transforms of first passage times in terms of the scale functions of the underlying Lévy process (see, for example, Chapter 8 in Kyprianou [19]). Both of these extensions are still unanswered problems outside the scope of the present study and left for future research.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors are grateful to an *anonymous referee* for constructive comments and suggested improvements. The financial support by the Foundation for Economic Education (*Liikesivistysrahasto*), Finland is gratefully acknowledged. H. Saarinen also acknowledges *Emmy.network* for continued support.

References

[1] H. Albrecher, J. Ivanovs, Strikingly simple identities relating exit problems for Lévy processes under continuous and Poisson observations, Stochastic Process. Appl. 127 (2) (2017) 643–656.
[2] H. Albrecher, J. Ivanovs, X. Zhou, Exit identities for Lévy processes observed at Poisson arrival times, Bernoulli 22 (3) (2016) 1364–1382.
[3] L.H.R. Alvarez E., J. Lempa, R. Stenbacka, Investment timing with imperfect information, 2010, Preprint.
[4] L.H.R. Alvarez E., J. Lempa, R. Stenbacka, Selection between competing projects: The effect of timing constraints, 2012, Preprint.
[5] T. Arai, M. Takenaka, Constrained optimal stopping under a regime-switching model, 2022, arXiv:2204.07914.
[6] A.N. Borodin, P. Salminen, Handbook of Brownian Motion-Facts and Formulae, Birkhäuser, 2015.
[7] S. Christensen, Optimal decision under ambiguity for diffusion processes, Math. Methods Oper. Res. 77 (2) (2013) 207–226.
[8] S. Christensen, A. Irle, A harmonic function technique for the optimal stopping of diffusions, Stochastics 83 (4–6) (2011) 347–363.
[9] S. Dayanik, I. Karatzas, On the optimal stopping problem for one-dimensional diffusions, Stochastic Process. Appl. 107 (2) (2003) 173–212.
[10] P. Dupuis, H. Wang, Optimal stopping with random intervention times, Adv. Appl. Prob. 34 (1) (2002) 141–157.
[11] E.B. Dynkin, A.A. Yushkevich, Markov Processes: Theorems and Problems, Plenum Press, New York, 1969, p. x+237, Translated from the Russian by James S. Wood.
[12] P. Gassiat, F. Gozzi, H. Pham, Investment/consumption problem in illiquid markets with regime-switching, SIAM J. Control Optim. 52 (2014) 1761–1786.
[13] X. Guo, J. Liu, Stopping at the maximum of geometric Brownian motion when signals are received, J. Appl. Probab. 42 (2005) 826–838.
[14] X. Guo, L. Shepp, Some optimal stopping problems with nontrivial boundaries for pricing exotic options, J. Appl. Probab. 38 (2001) 647–658.
[15] D. Hobson, The shape of the value function under Poisson optimal stopping, Stochastic Process. Appl. 133 (2021) 229–246.
[16] D. Hobson, G. Liang, H. Sun, Callable convertible bonds under liquidity constraints, 2021, arXiv:2111.02554.
[17] D. Hobson, M. Zeng, Randomised rules for stopping problems, J. Appl. Probab. 57 (2020) 981–1004.
[18] D. Hobson, M. Zeng, Constrained optimal stopping, liquidity and effort, Stochastic Process. Appl. 150 (2022) 819–843.
[19] A.E. Kyprianou, Fluctuations of Lévy Processes with Applications: Introductory Lectures, Springer Science & Business Media, 2014.
[20] R.D. Lange, S. K., Real-option valuation in multiple dimensions using Poisson optional stopping times, J. Financ. Quant. Anal. 55 (2020) 653–677.
[21] J. Lempa, Optimal stopping with information constraint, Appl. Math. Optim. 66 (2012) 147–173.
[22] J. Lempa, H. Saarinen, A zero-sum Poisson stopping game with asymmetric signal rates, Appl. Math. Optim. 87 (3) (2023) 35.
[23] Y. Li, Y. Chen, S. Wang, Z. Peng, Exit identities for diffusion processes observed at Poisson arrival times, Front. Math. China 15 (2020) 507–528.
[24] G. Liang, H. Sun, Dynkin games with Poisson random intervention times, SIAM J. Control Optim. 57 (2019) 2962–2991.
[25] G. Liang, H. Sun, Risk-sensitive dynkin games with heterogeneous Poisson random intervention times, 2020, arXiv:2008.01787.
[26] G. Liang, W. Wei, Optimal switching at Poisson random intervention times, Discrete Contin. Dyn. Syst. Ser. B 21 (2016) 1483–1505.
[27] J.L. Menaldi, M. Robin, On some optimal stopping problems with constraint, SIAM J. Control Optim. 54 (2016) 2650–2671.
[28] E. Mordecki, Optimal stopping and perpetual options for Lévy processes, Finance Stochast. 6 (4) (2002) 473–493, http://dx.doi.org/10.1007/s007800200070.
[29] Z. Palmowski, J. Pérez, K. Yamazaki, Double continuation regions for American options under Poisson exercise opportunities, Math. Finance 31 (2021) 722–771.
[30] J.L. Pérez, N. Rodosthenous, K. Yamazaki, Non-zero-sum optimal stopping game with continuous versus periodic observations, 2021, arXiv:2107.08243.
[31] J.L. Pérez, K. Yamazaki, American options under periodic exercise opportunities, Statist. Probab. Lett. 135 (2018) 92–101.
[32] H. Pham, P. Tankov, A model of optimal consumption under liquidity risk with random trading times, Math. Finance 18 (2008) 613–627.
[33] L.C.G. Rogers, O. Zane, A simple model of liquidity effects, in: Advances in Finance and Stochastics, Springer-Verlag, Berlin, Heidelberg, 2002, pp. 161–176.
[34] H. Saarinen, On Poisson Constrained Control of Linear Diffusions (Ph.D. thesis), Annales Universitatis Turkuensis, Ser. AI, 689, Astronomica-Chemica-Physica-Mathematica, 2023.