# CONFORMAL INVARIANTS

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This master thesis investigates the moduli of families of curves and the capacities of the Grötzsch and Teichmüller rings, which are applied in the main parts of this master thesis. The extremal properties of these rings are discussed in connection with the spherical symmetrization.

Applications are given to the study of distortion of quasiconformal maps in the euclidean $n$-dimensional space.

**Key words: conformal invariants, moduli of curve families, quasiconformal mappings.**

Contents

1. **Introduction** 2
2. **Modulus of Curve family** 2
3. **Conformal invariants** 14
4. **Quasiconformal mapping** 24
5. **Distortion theory** 31
6. **Quadruples and quasiconformal maps** 35
**Acknowledgments** 39
References 39

## 1. **Introduction**

This master thesis is about conformal invariants and their application to quasiconformal mapping and to distortion theory.

The theory of quasiconformal mappings in the euclidean $n$-space $\mathbf{R}^n$ originated from the pioneering papers of F. W. Gehring and J. Väisälä in the early 1960's. Their work generalized the classical two-dimensional theory of quasiconformal maps due to H. Grötzsch 1928, O. Teichmüller in the period 1938-44, and L. Bers, L. V. Ahlfors from the early 1950's on. Some of the main tools in the higher dimensional case $n \geq 3$ are conformal invariants and conformal capacity, see Gehring [G2].

This master thesis is an introduction to certain topics in quasiconformal mappings and conformal invariants. The presentation closely follows the books [Va], [Vu], and [AVV].

This thesis consists of six sections. In the second section we introduce the modulus of families of curves, capacities of Grötzsch and Teichmüller rings, special functions and Möbius transformation. The third section is about the conformal invariants and some theorems which describe their behaviour and Mori's ring domain. In the fourth section, we introduce the notion of quasiconformal maps and we prove some theorems with the applications of conformal invariants. In last two sections we are discussing the behaviour of quasiconformal mappings with the distortion functions.

It is assumed that the the reader has knowledge of basic real analysis and calculus in $\mathbf{R}^n$. We also assume familiarity with basic topology.

## 2. **Modulus of Curve family**

A *path* in $\mathbf{R}^n$ ($\overline{\mathbf{R}}^n$) is a continuous mapping $\gamma : \Delta \to \mathbf{R}^n$ (resp. $\overline{\mathbf{R}}^n$) where $\Delta \subset \mathbf{R}$ is an interval. If $\Delta' \subset \Delta$ is an interval, we call $\gamma|\Delta'$ a sub path of $\gamma$. The path $\gamma$ is called *closed* (*open*) if $\Delta$ is *closed* (resp. *open*). (Note that according to this definition, e.g. the path $\gamma : [0,1] \to \mathbf{R}^n$ is closed and that it is not required that $\gamma(0) = \gamma(1)$). The *locus* (or *trace*) of the path $\gamma$ is the set $\gamma\Delta$. The locus is also denoted by $|\gamma|$ or simply by $\gamma$ if there is no danger of confusion. We use the word *curve* as the synonym for the *path*. The *length* $\ell(\gamma)$ of the curve $\gamma : \Delta \to \mathbf{R}^n$ is defined in the usual way, with the help of polygonal approximations and a passage to the limit (see [Va, pp. 1-8]). The path $\gamma : \Delta \to \mathbf{R}^n$ is called *rectifiable* if $\ell(\gamma) < \infty$ and *locally rectifiable* if each closed sub path of $\gamma$ is rectifiable. If $\gamma : [0,1] \to \mathbf{R}^n$ is a rectifiable path, then $\gamma$ has a parametrization by means of arc length, also called the *normal representation* of $\gamma$. The normal representation of $\gamma$ is denoted by $\gamma^o : [0,1] \to \mathbf{R}^n$. Making use of the normal representation we define the line integral over a rectifiable curve $\gamma$.

We are now ready to define the modulus of curve family. Suppose that $\Gamma$ is a curve family in $\overline{\mathbf{R}}^n$. That is, the elements of $\Gamma$ are curves in $\overline{\mathbf{R}}^n$. We denote by $\mathrm{F}(\Gamma)$ the set of all non-negative Borel functions $\rho : \overline{\mathbf{R}}^n \to \mathbf{R} \cup \{\infty\}$ such that $\int_\gamma \rho ds \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$. For each $p \geq 1$ we set

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{R^n} \rho^p dm.$$

If $F(\Gamma) = \phi$, we define $M_p(\Gamma) = \infty$. This happens only if $\Gamma$ contains a constant path (which will never occur in this thesis), because otherwise the constant function $\rho(x) = \infty$ belongs to $F(\Gamma)$. Clearly $0 \leq M_p(\Gamma) \leq \infty$. The number $M_p(\Gamma)$ is called the $p$-modulus of $\Gamma$. The most important case for our purpose is the case p = n. We shall denote $M_n(\Gamma)$ simply by $M(\Gamma)$ and call it the modulus of $\Gamma$. In the literature, one often uses the extremal length of $\Gamma$. This is simply equal to $M_p(\Gamma)^{1/(1-p)}$. The modulus is perhaps a more natural concept, for it has the following measure theoretic property.

2.1. **Theorem.** *$M_p$ is an outer measure in the space of all curves in $\overline{\mathbf{R}}^n$. That is,*

(1) $M_p(\phi) = 0$.
(2) $\Gamma_1 \subset \Gamma_2$ *implies* $M_p(\Gamma_1) \leq M_p(\Gamma_2)$.
(3) $M_p(\bigcup_{i=1}^{\infty}) \leq \displaystyle\sum_{i=1}^{\infty} M_p(\Gamma_i)$.

2.2. **Definition.** Let $\Gamma_1$ and $\Gamma_2$ be curves families in $\overline{\mathbf{R}}^n$. We say that $\Gamma_2$ is minorized by $\Gamma_1$ and denote $\Gamma_2 > \Gamma_1$ if every $\gamma \in \Gamma_2$ has a subcurve which belongs to $\Gamma_1$.

2.3. **Theorem.** *If $\Gamma_1 < \Gamma_2$, then $M_p(\Gamma_1) \geq M_p(\Gamma_2)$.*

2.4. **Definition.** The curve families $\Gamma_1, \Gamma_2, \ldots$ are called separate if there exist disjoint Borel sets $E_i$ in $\mathbf{R}^n$ such that if $\gamma \in \Gamma_i$ is locally rectifiable, then $\int_\gamma g_i ds = 0$ where $g_i$ is the characteristic function of $\chi_{E_i}$.

2.5. **Theorem.** *If $\Gamma_1, \Gamma_2, \ldots$ are separate and if $\Gamma < \Gamma_i$ for all $i$, then*

$$M_p(\Gamma) \geq \sum M_p(\Gamma_i).$$

2.6. **Example.** Given a curve family $\Gamma$, it is usually a very difficult task to compute $M_p(\Gamma)$. However, it is often easy to find an upper bound for $M_p(\Gamma)$, for if we take any $\rho \in F(\Gamma)$, then $M_p(\Gamma) \leq \int \rho^p dm$.

2.7. **Remark.** If $G = \mathbf{R}^n$ or $\overline{\mathbf{R}}^n$ we often denote $\Delta(E, F; G)$ by $\Delta(E, F)$. Curve families of this form are the most important for what follows. The following subadditivity property is useful. If $E = \bigcup_{j=1}^{\infty}$ and $c_E(F) = M_p(\Delta(E, F)) = c_F(E)$, then $c_F(E) \leq \sum c_F(E_j)$, see Theorem 2.1(2). More precisely if $G \subset \overline{\mathbf{R}}^n$ is a domain and $F \subset G$ is fixed, then $c_F^E(E) = M_p(\Delta(E, F; G))$ is an outer measure define for $E \subset G$. In a sense which will be made precise later on, $c_E(F)$ describes the mutual size and location of $E$ and $F$. Assume now that $D$ is an open set in $\overline{\mathbf{R}}^n$ and that $F \subset D$. It follows from Theorem 2.1(2) that

$$M_p(\Delta(F, \partial D; D \setminus F)) \leq M_p(\Delta(F, \partial D; D)) \leq M_p(\Delta(F, \partial D)).$$

On the other hand, because $\Delta(F, \partial D; D) < \Delta(F, \partial D)$ and $\Delta(F, \partial D; D \setminus F) < \Delta(F, \partial D; D)$, see Theorem 2.3

$$M_p(\Delta(F, \partial D)) = M_p(\Delta(F, \partial D; D)) = M_p(\Delta(F, \partial D; D \setminus F)). \tag{2.8}$$

2.9. **Theorem.** *Suppose that the curves of a family $\Gamma$ lie in a Borel set $G \subset \overline{\mathbf{R}}^n$ and that $\ell(\gamma) \geq r > 0$ for every locally rectifiable $\gamma \in \Gamma$. Then*

$$M_p(\Gamma) \leq \frac{m(G)}{r^p}.$$

2.10**. The cylinder.** Let $E$ be a Borel set in $\mathbf{R}^{n-1}$ and let $h > 0$. Set

$$G = \{x \in R^n | (x_1, \ldots, x_{n-1}) \in E \text{ and } 0 < x_n < h\}.$$

Then $G$ is a cylinder with base $E$ and $F = E + he_n$ and with height $h$. Set $\Gamma = \Delta(E, F, G)$. We show that

$$M_p(\Gamma) = \frac{m_{n-1}(E)}{h^{p-1}} = \frac{m(G)}{h^p}.$$

Since $l(\gamma) \geq h$ for every $\gamma \in \Gamma$, Theorem 2.9 implies $M_p(\Gamma) \leq m(G)/h^p$. Let $\rho$ be an arbitrary function in $F(\Gamma)$. For each $y \in E$ let $\gamma_y : [0, h] \to R^n$ be the vertical segment $\gamma(t) = y + te_n$. Then $\gamma_y \in \Gamma$. Assuming that $p > 1$ we obtain by Hölder's inequality

$$1 \leq \left( \int_{\gamma_y} \rho ds \right)^p \leq h^{p-1} \int_0^h \rho(y + te_n)^p \, dt.$$

Integration over $y \in E$ yields by Fubini's theorem

$$m_{n-1}(E) \leq h^{p-1} \int_E dm_{n-1} \int_0^h \rho(y + te_n)^p \, dt = h^{p-1} \int_G \rho^p dm \leq h^{p-1} \int \rho^p dm.$$

Since this holds for every $\rho \in F(\Gamma)$, we obtain $M_p(\Gamma) \geq m_{n-1}(E)/h^{p-1}$.
The proof for $p = 1$ is somewhat simpler.

2.11. **Lemma.** *Let $D$ and $D'$ be domains in $\overline{\mathbf{R}}^n$ and let $f : D \to D'$ be a conformal mapping. Then $M(f\Gamma) = M(\Gamma)$ for each curve $\Gamma$ in $D$ where $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$.*

2.12**. The spherical ring.** If $0 < a < b < \infty$, the domain $A = \mathbf{B}^n(b) \setminus \overline{\mathbf{B}}^n(a)$ is called a spherical ring. Let $E = S(a)$, $F = S(b)$ and $\Gamma_A = \Delta(E, F, A)$. We shall prove that

$$M(\Gamma_A) = \omega_{n-1} \left( \log \frac{b}{a} \right)^{1-n}. \tag{2.13}$$

Let $\rho \in F(\Gamma_A)$ and $\omega_{n-1}$ be the $(n-1)$ dimensional area of $S^{n-1}$. For each unit vector $y \in S^{n-1}$ we let $\gamma : [a, b] \to \mathbf{R}^n$ be the radial segment, defined by $\gamma_y(t) = ty$. By Hölder's inequality we obtain

$$1 \leq \left( \int_{\gamma_y} \rho \, ds \right)^n \leq \int_a^b \rho(ty)^n t^{n-1} \, dt \left( \int_a^b t^{-1} \, dt \right)^{n-1} = \left( \log \frac{b}{a} \right)^{n-1} \int_a^b \rho(ty)^n t^{n-1} \, dt.$$

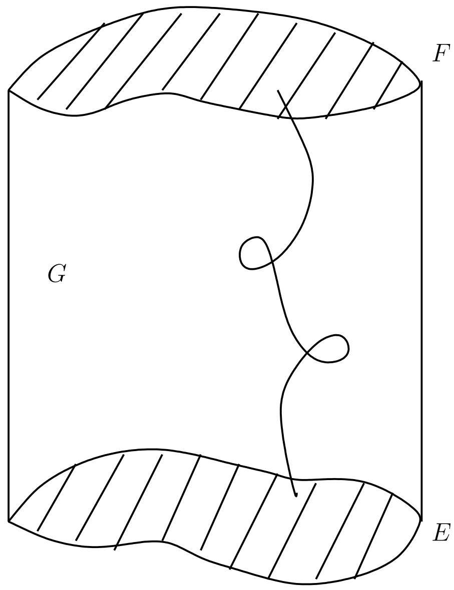


FIGURE 1. Cylinder $G$ with base $E$ and $F$.

Integrating over $y \in S^{n-1}$ yields

$$\omega_{n-1} \leq \left(\log \frac{b}{a}\right)^{n-1} \int \rho^n dm. \tag{2.14}$$

Taking the infimum over all $\rho \in F(\Gamma)$ we obtain

$$\omega_{n-1} \leq \left(\log \frac{b}{a}\right)^{n-1} M(\Gamma_A).$$

On the other hand, we have equality in (2.14) if we define $\rho(x) = 1/\left(|x| \log(b/a)\right)$ for $x \in A$ and $\rho(x) = 0$ otherwise. By (2.8) the formula (2.13) holds also if $A$ is replaced by $\mathbf{R}^n$. Letting $a \to 0$ we see by Theoerem 2.1 that

$$M(\Delta(F, \{0\})) = 0.$$

2.15. **Lemma.** *Let $\{\Gamma_j\}$ be separated curve families in $\overline{\mathbf{R}}^n$ with $\Gamma_j < \Gamma$ for all $j = 1, 2, \ldots$ . If $p > 1$ then*

$$M_p(\Gamma)^{1/(1-p)} \geq \sum_{j=1}^{\infty} M_p(\Gamma_j)^{1/(1-p)}.$$

2.16. **Lemma.** *Let $s \in (0, 1)$ and*

$$\Gamma_1 = \Delta([0, se_1], S^{n-1}, \mathbf{B}^n),\ \Gamma_2 = \Delta([0, se_1], [\frac{1}{s}e_1, \infty], \mathbf{R}^n).$$

*Then $M_p(\Gamma_1) = 2^{p-1} M_p(\Gamma_2)$ for $p > 1$.*

2.17. **Lemma.** *Let $\Gamma_1 = \Delta([0, e_1], [t^2 e_1, \infty))$ and $\Gamma_2 = \Delta([0, e], [t^2 e_1, \infty))$ where $e \in S^{n-1}$ and $t > 1$. Then $M(\Gamma_2) \leq M(\Gamma_1)$.*

2.18. **Lemma.** *Let $S = S^{n-1}(r), \varphi \in (0,\pi]$, Let $K$ be the spherical cap $S \cap C(\varphi)$, and let $E$ and $F$ be the non-empty subsets of $K$. Then*

(1) $M_n^s(\Delta(E,F;K)) \geq \dfrac{b_n}{r}$ *where $b_n$ is a positive number depending only on $n$* (*see* [Va, (10.4)]).

(2) *If $K = S$,i,e. $\varphi = \pi$, then $b_n$ may be replaced by $c_n = 2^n b_n$ in the above inequality.*

2.19. **Lemma.** *Let $0 < a < b$ and let $E$, $F$ be sets in $\mathbf{R}^n$ with*

$$E \cap S^{n-1}(t) \neq \emptyset \neq F \cap S^{n-1}(t)$$

*for $t \in (a,b)$. Then*

$$M(\Delta(E,F;\mathbf{B}^n(b) \setminus \mathbf{B}^n(a))) \geq c_n \log \frac{b}{a}.$$

*Equality holds if $E = (ae_1, be_1)$, $F = (-be_1, -ae_1)$.*

2.20**. The modulus of ring.** A domain $D$ in $\overline{\mathbf{R}}^n$ is termed a ring, if $\overline{\mathbf{R}}^n \setminus D$ has two components. If the components are $C_0$ and $C_1$ we write $D = R(C_0, C_1)$. The (*conformal*) *modulus* of a ring $R(C_0,C_1)$ is defined by

$$\mathrm{mod} R(C_0,C_1) = \left( \frac{M(\Delta(C_0,C_1))}{\omega_{n-1}} \right)^{1/(1-n)}.$$

The *capacity* of $R(C_0,C_1)$ is $M(\Delta(C_0,C_1))$. A ring is a special case of a condenser. A condenser is a pair $(A,C)$ of an open set $A \subset \mathbf{R}^n$ and a compact set $C \subset A$. If $R(C_0,C_1)$ is a ring and $C_0$ is bounded, then $(\mathbf{R}^n \setminus C_1, C_0)$ is a condenser. For the capacity of condenser see [Vu].

In the two-dimensional case the modulus of a ring $R$ has the following geometric interpretation: $\mathrm{mod} R = t$ if and only if $R$ can be mapped conformally onto the annulus $\{z \in R^2 : 1 < |z| < e^t\}$. Owing to this geometric interpretation the modulus of a ring often is convenient to use in the two-dimensional case. In the multidimensional case there is no such geometric interpretation for the modulus of a ring because of the rigidity of the class of conformal mappings in $R^n$, $n \geq 3$. On the other hand there is also a geometric way of looking at the capacity of a particular ring, the so-called Grötzsch ring, which is applicable to all dimensions $n \geq 2$. For this reason we shall prefer the capacity to the modulus of a ring.

2.21**. The Grötzsch and Teichmüller rings.** The complementary components of the *Grötzsch ring* $R_{G,n}(s)$ in $\mathbf{R}^n$ are $\overline{\mathbf{B}}^n$ and $[se_1, \infty]$, $s > 1$, while those of the *Teichmüller ring* $R_{T,n}(t)$ are $[-e_1, 0]$ and $[te_1, \infty]$, $t > 0$. We shall need two functions $\gamma_n(s)$, $s > 1$, and $\tau_n(t)$, $t > 0$, to designate the moduli of the families of all those curves which connect the complementary components of the Grötzsch and Teichmüller rings in $\mathbf{R}^n$, respectively.

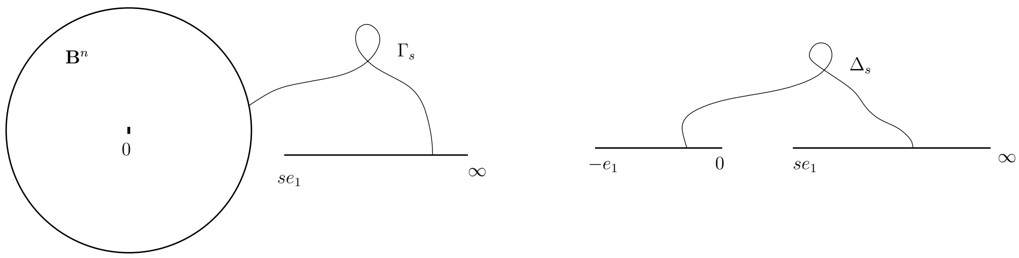


FIGURE 2. $\mathrm{cap}R_{G,n}(s) = M(\Gamma_s) = \gamma_n(s), \quad \mathrm{cap}R_{T,n}(t) = M(\Delta_t) = \tau_n(t).$

$$\left\{ \begin{array}{ll} \mathrm{cap}R_{G,n}(s) = \gamma_n(s) & = M(\Gamma_s), \\ \mathrm{cap}R_{T,n}(t) = \tau_n(t) & = M(\Delta_t). \end{array} \right. \tag{2.22}$$

Where $\Gamma_s = \Delta(\overline{\mathbf{B}}^n, [se_1, \infty])$ and $\Delta_t = ([-e_1, 0], [te_1, \infty])$. We shall refer to these functions as the Grötzsch capacity and the Teichmüller capacity.

2.23. **Lemma.** *The following functional inequalities hold:*

(1) $\tau(s) \le \gamma(1+2s) = 2^{n-1}\tau(4s^2+4s),\ s > 0,$

(2) $\tau(s) \le 2\tau(2s + 2s\sqrt{1+1/s}),\ s > 0,$

(3) $\tau(s) \le \tau(t) + \tau\left(\dfrac{s(1+t)}{t-s}\right),\ 0 < s < t < \infty,$

(4) $\tau(u) \le \tau\left(\dfrac{uv}{u+v+1}\right) \le \tau(u) + \tau(v),\ u, v > 0.$

The Grötzsch and Teichmüller rings $R_{G,n}(s)$ and $R_{T,n}(s)$ can also be understood as condensers in the following way:

$$\left\{ \begin{array}{lll} R_{G,n}(s) & = (\mathbf{R}^n \setminus te_1 : t \ge s, \overline{\mathbf{B}}^n), & \mathrm{s} \in (1, \infty), \\ R_{T,n}(s) & = (\mathbf{R}^n \setminus te_1 : t \ge s, [-e_1, 0]), & \mathrm{s} \in (0, \infty). \end{array} \right. \tag{2.24}$$

We define functions $\Phi = \Phi_n$ and $\Psi = \Psi_n$ by $\mathrm{mod}R_{G,n}(s) = \log\Phi(s)$ and $\mathrm{mod}R_{T,n}(s) = \log\Psi(s)$. Then

$$\left\{ \begin{array}{ll} \mathrm{cap}R_{G,n}(s) & = \omega_{n-1}(\log\Phi(s))^{1-n} = \gamma_n(s), \\ \mathrm{cap}R_{T,n}(s) & = \omega_{n-1}(\log\Psi(s))^{1-n} = \tau_n(s). \end{array} \right. \tag{2.25}$$

2.26. **Lemma.** *The function $\Phi(t)/t$ is increasing for $t > 1$ and $\Psi(t-1) = \Phi(\sqrt{t})^2$ for $t > 1$. Moreover, the functions $\gamma_n$ and $\tau_n$ are strictly decreasing.*

We define the Grötzsch constant $\lambda_n$ by

$$\log \lambda_n = \lim_{t\to\infty} (\log \Phi(t) - \log t) \tag{2.27}$$

by Lemma 2.26 there exists lim $\in \mathbf{R} \cup \{\infty\}$ for $n = 2 : \lambda_2 = 4$ and $\lambda_n \in [4, 2e^{n-1}], n \geq 3$.

2.28. **Lemma.** *For each $n \geq 2$ there exists a number $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$ such that*

(1) $t \leq \Phi(t) \leq \lambda_n t, \quad t > 1$ ,

(2) $t + 1 \leq \Psi(t) \leq \lambda_n^2 (t+1), \quad t > 1$ .

*Furthermore, $\lambda_n^{1/n} \to e$ as $n \to \infty$ and, in particular, $\lambda_n \to \infty$ as $n \to \infty$.*

For the statement of the next result we need the following notation

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}, \ \ \mathcal{K}(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-1/2} dx$$

for $0 < r < 1$.

The function $\mathcal{K}(r)$ is called a complete elliptic integral of the first kind and its values can be found in tables. The argument $r$ is sometimes called the modulus of the elliptic integral $\mathcal{K}(r)$. The complement of $r \in (0,1)$ is $r' = \sqrt{1-r^2}$. From the basic properties of $\mathcal{K}(r)$ see [AVV, pp.48-55], it follows easily that the normalized quotient

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}'(r)}{\mathcal{K}(r)} \tag{2.29}$$

is a strictly decreasing homeomorphism of the interval $(0,1)$ onto $(0,\infty)$, with limit values $\mu(0_+) = \infty, \mu(1_-) = 0$.

2.30. **Theorem.** *For $s \in (1,\infty)$ and $n \geq 2$*

(1) $\omega_{n-1}(\log \lambda_n s)^{1-n} < \gamma_n(s) \leq \omega_{n-1}\mu(1/s)^{1-n} < \omega_{n-1}(\log(s + 3\sqrt{s^2-1}))^{1-n}$

(2) $2^{n-1}c_n \log\left(\frac{s+1}{s-1}\right) \leq \gamma_n(s) \leq 2^{n-1}c_n\mu\left(\frac{s-1}{s+1}\right) < 2^{n-1}c_n \log\left(4\frac{s+1}{s-1}\right)$

*Moreover, if $s \in (0,\infty)$ and $a = 1 + 2(1+\sqrt{1+s})/s$, then*

(3) $c_n \log a \leq \tau_n(s) \leq c_n\mu(1/a) < c_n \log(4a)$,

*and $(1 + 1/\sqrt{s})^2 \leq a \leq (1 + 2/\sqrt{s})^2$ holds true. Furthermore, when $n = 2$, the second inequality in (1), the second inequality in (2), and the second inequality in (3) hold as identities.*

2.31. **Theorem.** *For $n \geq 2, s > 1$, and $x, c > 0$*

(1) $\gamma_n(s) = 2^{n-1}\tau_n(s^2-1)$,

(2) $\tau_n^{-1}(c\tau_n(x)) = (\gamma_n^{-1}(c\gamma_n(\sqrt{x+1})))^2 - 1$.

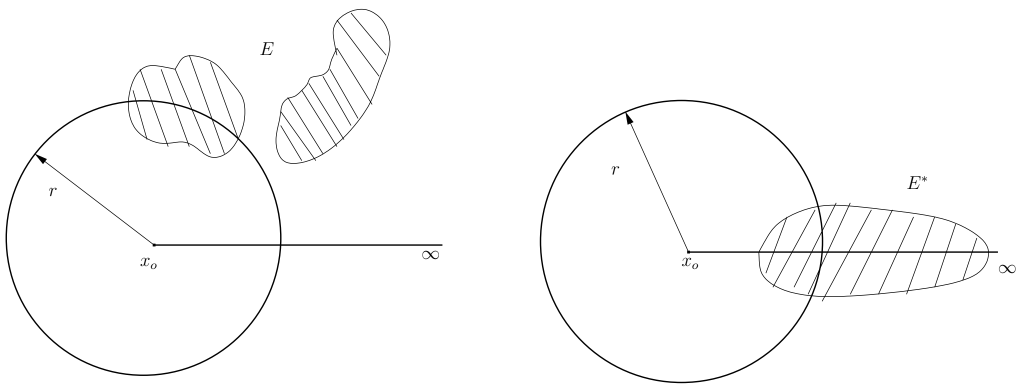


FIGURE 3. Spherical symmetrization $E^*$ of set $E$.

*The functions $\gamma_n$ and $\tau_n$ are continuous and strictly decreasing, with range* $(0,\infty)$.

2.32. **Hyperbolic metric and capacity.** Let $J[x,y]$ be a geodesic segment of the hyperbolic metric, $x,y \in \mathbf{B}^n$ and $T_x \in M(\mathbf{B}^n)$ [Vu, 2.25] implies that

$$\tag{2.33} \begin{cases} \operatorname{cap}(\mathbf{B}^n, J[x,y]) = \operatorname{cap}(\mathbf{B}^n, T_x J[x,y]) \\ = \operatorname{cap}(\mathbf{B}^n, [0, |T_x y| e_1]) = \gamma\left(1/\tanh\dfrac{\rho(x,y)}{2}\right) \end{cases}$$

substitution into 2.30(2) yields

$$\tag{2.34} \begin{cases} 2^{n-1} c_n \rho(x,y) \le \operatorname{cap}(\mathbf{B}^n, J[x,y]) \\ \le 2^{n-1} c_n \mu\left(\dfrac{1-\tanh\dfrac{\rho(x,y)}{2}}{1+\tanh\dfrac{\rho(x,y)}{2}}\right) = 2^{n-1} c_n \mu\left(\dfrac{\cosh\frac{\rho}{2} - \sinh\frac{\rho}{2}}{\cosh\frac{\rho}{2} + \sinh\frac{\rho}{2}}\right) \\ \le 2^{n-1} c_n \mu(e^{-\rho}) \le 2^{n-1} c_n \log(4e^{\rho(x,y)}) = 2^{n-1} c_n (\rho(x,y) + \log 4) \end{cases}$$

where the inequality $\mu(t) < \log\dfrac{4}{t}$ was also used (see [AVV, (5.3)]). For large values of $\rho(x,y)$ Lemma 2.34 is quite accurate. When $\rho(x,y)$ is small, better bounds follow from Theorem 2.30(1) and [Vu, 7.24].

2.35. **Spherical symmetrization.** If $x_0 \in \mathbf{R}^n, E \subset \overline{\mathbf{R}}^n$ and if $L$ is a ray from $x_0$ to $\infty$, then the *spherical symmetrization* $E^*$ of $E$ in $L$ is defined as follows:

(1) $x_0 \in E^*$ if and only if $x_0 \in E$.
(2) $\infty \in E^*$ if and only if $\infty \in E$.
(3) For each $r \in (0,\infty)$, $E^* \cap S^{n-1}(x_0,r) \ne \phi$ if and only if $E \cap S^{n-1}(x_0,r) \ne \phi$, in which case $E^* \cap S^{n-1}(x_0,r)$ is a closed spherical cap centered on $L$ with the same $m_{n-1}$ measure $E \cap S^{n-1}(x_0,r)$.

Let $(A, C)$ be a condenser and $x_0 \in \mathbf{R}^n$. Denoted by $C^*$ and $B$ the spherical symmetrizations of $C$ and $\overline{\mathbf{R}}^n \setminus E$ in two opposite rays $L$ and $L'$ emanating from $x_0$, and let $A^* = \mathbf{R}^n \setminus B$. Then it is easy to verify that $(A^*, C^*)$ is a condenser [S].

2.36. **Theorem.** *If $(A, C)$ is a condenser, then for $p \geq 1$,*

$$p - \mathrm{cap}(A, C) \geq p - \mathrm{cap}(A^*, C^*).$$

Next we give two important applications of Theorem 2.36. The first one is an extremal property of a Grötzsch ring $R_{T,n}(t), s \in (1, \infty)$, and the second is an extremal property of a Teichmüller ring $R_{T,n}(t), t \in (0, \infty)$.

2.37. **Theorem.** *Let $R = R(C_0, C_1)$ be a ring in $\overline{\mathbf{R}}^n$ such that $\overline{\mathbf{B}}^n \subset C_0$ and $x, \infty \in C_1$, where $|x| = s$, $1 < s < \infty$. Then*

$$\mathrm{cap} R \geq \mathrm{cap} R_{G,n}(s) = \gamma_n(s).$$

2.38. **Theorem.** *Let $R = R(C_0, C_1)$ be a ring in $\overline{\mathbf{R}}^n$, and let $a, b \in C_0$,$x, \infty \in C_1$ be distinct points. Then*

$$\mathrm{cap} R \geq \mathrm{cap} R_{T,n}(t) = \tau_n(t),\ t = \frac{|a-c|}{|a-b|}.$$

*Equality holds for the Teichmüller ring $R_{T,n}(t)$, with $a = 0, b = -e_1$, and $c = te_1$, $t > 0$.*

*Proof.* Since the inequality remains invariant under similarity transformations, let $f(x) = (x-a)/|a-b|$. Then $|f(c)| = t$ and $f(a) = 0$.
The spherical symmetrizations of $f(C_0)$ and $f(C_1)$ in the negative and positive $x_1$-axis, respectively, contain the complementary components of $R_{T,n}(t)$. Hence the result follows from Theorem 2.36. □

2.39. **Lemma.** *For $t > 1$ let*

$$R_1 = R([0, e_1], [te_1, \infty]) \qquad \text{and} \qquad R_2 = R([0, e], [te_1, \infty]),$$

*where $e \in S^{n-1}$. Then*

$$\mathrm{cap} R_2 \leq \mathrm{cap} R_1 = \tau_n(t-1).$$

2.40. **Lemma.** *Let $x, y \in \mathbf{B}^n$ and $E$ a continuum with $x, y \in E$. Then*

$$\mathrm{cap}(\mathbf{B}^n, E) \geq \mathrm{cap}(\mathbf{B}^n, J[x, y]) = \gamma \left(1/\tanh \frac{\rho(x, y)}{2}\right).$$

*Proof.* As in (2.33) we get

$$\mathrm{cap}(\mathbf{B}^n, E) = \mathrm{cap}(\mathbf{B}^n, T_x E) \geq \mathrm{cap}(\mathbf{B}^n, (T_x E)^*)$$

symmetrization in Theorem 2.36 was applied and $*$ stands for the spherical symmetrization in $x_1$-axis. As [Vu, 2.25] implies that $\left[0, \left(\tanh \frac{\rho(x, y)}{2}\right) e_1\right] \subset (T_x E)^*$.
The claim follows from (2.33) □

2.41**. The spherical (circular) symmetrization for n=2.** In the plane one can modify the spherical symmetrization results (cf. Theorem 2.38) by using an auxiliary conformal mapping. In certain cases this method leads to sharper inequalities than does a direct use of spherical symmetrization.

2.42**. Special function $\varphi_{\mathbf{K}}$.** The functions $\gamma_n$ and $\tau_n$ as well as their inverses and various combinations of these will occur often in next results. Of particular importance is the function $\varphi_K : [0,1] \to [0,1]$, which will occur in the quasiregular version of the Schwarz lemma as well as in its many applications. This function is defined as follows, for $0 < r < 1$ and $K > 0$

$$\varphi_K(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))} = \varphi_{K,n}(r) \tag{2.43}$$

and set $\varphi_K(0) = 0$, $\varphi_K(1) = 1$. This function also occurs in the following expressions:

$$\begin{cases} \eta_K(t) = \eta_{K,n}(t) \equiv \dfrac{1 - \varphi_{1/K}(1/\sqrt{1+t})^2}{\varphi_{1/K}(1/\sqrt{1+t})^2} = \tau_n^{-1}\left(\dfrac{1}{K}\tau_n(t)\right) \\ \eta_{K,2}(t) = \left(\dfrac{\varphi_K(\sqrt{t/(1+t)})}{\varphi_{1/K}(1/\sqrt{1+t})}\right)^2, \end{cases} \tag{2.44}$$

where for $\eta_{K,2}$, the identity $\varphi_K(r)^2 + \varphi_{1/K}(\sqrt{1-r^2})^2 = 1$ from [AVV, 10.5(1)] is used. In particular,

$$\lambda(K) \equiv \left(\frac{\varphi_K(1/\sqrt{2})}{\varphi_{1/K}(1/\sqrt{2})}\right)^2 = \eta_K(1) = \left(\frac{\tau_n^{-1}(\pi/(2K))}{\tau_n^{-1}(\pi K/2)}\right)^2, \tag{2.45}$$

It is easy to see that $\varphi_K : [0,1] \to [0,1]$ is a homeomorphism. Next we shall derive from Lemma 2.28(1) some explicit estimates for $\varphi_K$,

$$\omega_{n-1}(\log \lambda_n s)^{1-n} \le \gamma_n(s) \le \omega_{n-1}(\log s)^{1-n} \tag{2.46}$$

for $s > 1$. From Lemma 2.28(1) it follows that for $s > 1$

$$\log s \le \log \Phi(s) \le \log(\lambda_n s)$$

and therefore

$$\alpha \log \Phi(s) \le \alpha \log(\lambda_n s) = \log(\lambda_n^\alpha s^\alpha) \le \log \Phi(\lambda_n^\alpha s^\alpha). \tag{2.47}$$

Since (2.47) implies

$$K\omega_{n-1}\left(\log \Phi(s)\right)^{1-n} \le \omega_{n-1}\left(\log \Phi(\lambda_n^\alpha s^\alpha)\right)^{1-n}$$

we have

$$K\gamma_n(s) \le \gamma_n(\lambda_n^\alpha s^\alpha),$$

which is equivalent to

$$\gamma_n^{-1}\left(K\gamma_n(s)\right) \leq \lambda_n^{\alpha} s^{\alpha}.$$

Similarily we can obtain

$$t^{\alpha}/\lambda_n \leq \gamma_n^{-1}(K\gamma_n(s))$$

and therefore

$$t^{\alpha}/\lambda_n \leq \gamma_n^{-1}(K\gamma_n(t)) \leq \lambda_n^{\alpha} t^{\alpha} \tag{2.48}$$

for all $t > 1$ and $K > 0$, where $\alpha = K^{1/(1-n)}$. From (2.48) it follows that

$$r^{\alpha}/\lambda_n^{-\alpha} \leq \varphi_K(r) \leq \lambda_n r^{\alpha} \tag{2.49}$$

holds for all $K > 0$ and $r \in (0,1)$.
It is easy to see that $0 < A \leq B < \infty$ implies $\varphi_A(r) \leq \varphi_B(r)$. In particular, $\varphi_{1/K}(r) \leq r = \varphi_1(r) \leq \varphi_K(r)$ for $K \geq 1$.

2.50. **Definition.** Let $D$ and $D^{'}$ be a domain in $\mathbf{R}^n$ and let $f : D \to D^{'}$ be a homeomorphism. We call $f$ *conformal* if
(1) $f \in C^1$
(2) $J_f(x) \neq 0$ for all $x \in D$ and
(3) $|f^{'}(x)| = |f^{'}(x)||h|$ for all $x \in D$ and $h \in \mathbf{R}^n$. If $D$ and $D^{'}$ are domains in $\overline{\mathbf{R}}^n$, we call a homeomorphism $f : D \to D^{'}$ *conformal* if the restriction of $f$ to $D \setminus \{\infty, f^{-1}(\infty)\}$ is conformal.

2.51. **Example.** Some basic examples of conformal mappings are the following elementary transformations.

(1) A reflection in hyperplane $P(a,t) = \{x \in \mathbf{R}^n : x \cdot a = t\} \cup \{\infty\}$ where $t \in \mathbf{R}$ and $a \in \mathbf{R} \setminus \{0\}$:

$$f_1(x) = x - 2(x \cdot a - t)\frac{a}{|a|^2}, f_1(\infty) = \infty.$$

(2) An inversion (Reflection) in $S^{n-1}(a,r)$:

$$f_2(x) = a + \frac{r^2(x-a)}{|x-a|^2}, f_2(a) = \infty, f_2(\infty) = a.$$

(3) A translation $f_3(x) = x + a, a \in \mathbf{R}^n, f_3(\infty) = \infty$.
(4) A stretching by a factor $k > 0 : f_4(x) = k \cdot x, f_4(\infty) = \infty$.
(5) An orthogonal mapping, i.e. a linear mapping $f_5$ with

$$|f_5(x)| = |x|, f_5(\infty) = \infty.$$

2.52. **Remark.** The translation $x \mapsto x + a$ can be written as a compostion of reflection in $P(a,0)$ and $P(a, \frac{1}{2}|a|^2)$. The stretching $x \mapsto kx,\ k > 0$, can be written as a composition of inversion in $S^{n-1}(0,1)$, and $S^{n-1}(0,\sqrt{k})$. It can be proved, that an orthogonal mapping can be composed of at most $n+1$ reflections in planes.

2.53. **Möbius transformation.** A homeomorphism $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is called a *Möbius transformation* if $f = g_1 \circ \ldots \circ g_p$ where $g_i$ is one of the elementary transformations in Example 2.51(1)-(5) and $p$ is a positive integer. Equivalently (see Remark 2.52) $f$ is a Möbius transformation if $f = g_1 \circ \ldots \circ g_m$ where each $h_j$ is a reflection in a sphere or in a hyperplane and $m$ is a positive integer.
It follows from the inverse function theorem and chain rule that the set of all conformal mappings of $\overline{\mathbf{R}}^n$ is a group. Also the set of all Möbius transformations constitutes a subgroup of the group of conformal mappings, and we denote it by $GM(\overline{\mathbf{R}}^n)$ or $GM$. Further we shall write

$$GM(D) = \{f \in GM(\overline{\mathbf{R}}^n) : fD = D\}$$

for $D \subset \overline{\mathbf{R}}^n$. We denote by $O(n)$ the set of all orthogonal maps in $\mathbf{R}^n$. A map $f$ in $GM$ with $f(\infty) = \infty$ is called a *similarity* transformation if $|f(x) - f(y)| = c|x - y|$ for all $x, y \in \mathbf{R}^n$ where $c$ is a positive number.

2.54. **Stereographic projection.** The stereographic projection $\pi : \overline{\mathbf{R}}^n \to S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ is defined by

$$\pi(x) = e_{n+1} + \frac{x - e_{n+1}}{|x - e_{n+1}|^2}, \ x \in \mathbf{R}^n, \pi(\infty) = e_{n+1}. \tag{2.55}$$

Thus $\pi$ is precisely the restriction to $\overline{\mathbf{R}}^n$ of an inversion in the $n$-sphere $S^n(e_{n+1}, 1)$ in $\overline{\mathbf{R}}^{n+1}$. In fact, we can identify $\pi$ with this inversion. Since $\pi^{-1} = \pi$, it follows that $\pi$ maps the Riemann sphere $S^n(\frac{1}{2}e_{n+1}, \frac{1}{2})$ onto $\overline{\mathbf{R}}^n$.
The *spherical* (*chordal*) *metric* $q$ in $\overline{\mathbf{R}}^n$ is defined by

$$q(x, y) = |\pi(x) - \pi(y)|, \ x, y \in \overline{\mathbf{R}}^n, \tag{2.56}$$

where $\pi$ is in (2.55). From [Vu, (1.5)] and (2.55) we obtain

$$\begin{cases} q(x, y) = \dfrac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \ x, y \in \mathbf{R}^n \text{and } x \neq \infty \neq y, \\ q(x, \infty) = \dfrac{1}{\sqrt{1 + |x|^2}}, \ x \in \mathbf{R}^n. \end{cases} \tag{2.57}$$

2.58. **Absolute ratio.** For an ordered quadruple $a, b, c, d$ of distinct points in $\overline{\mathbf{R}}^n$, we define the *absolute cross* ratio by

$$|a, b, c, d| = \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)}. \tag{2.59}$$

It follows from (2.57) that

$$|a, b, c, d| = \frac{|a - c||b - d|}{|a - b||c - d|},$$

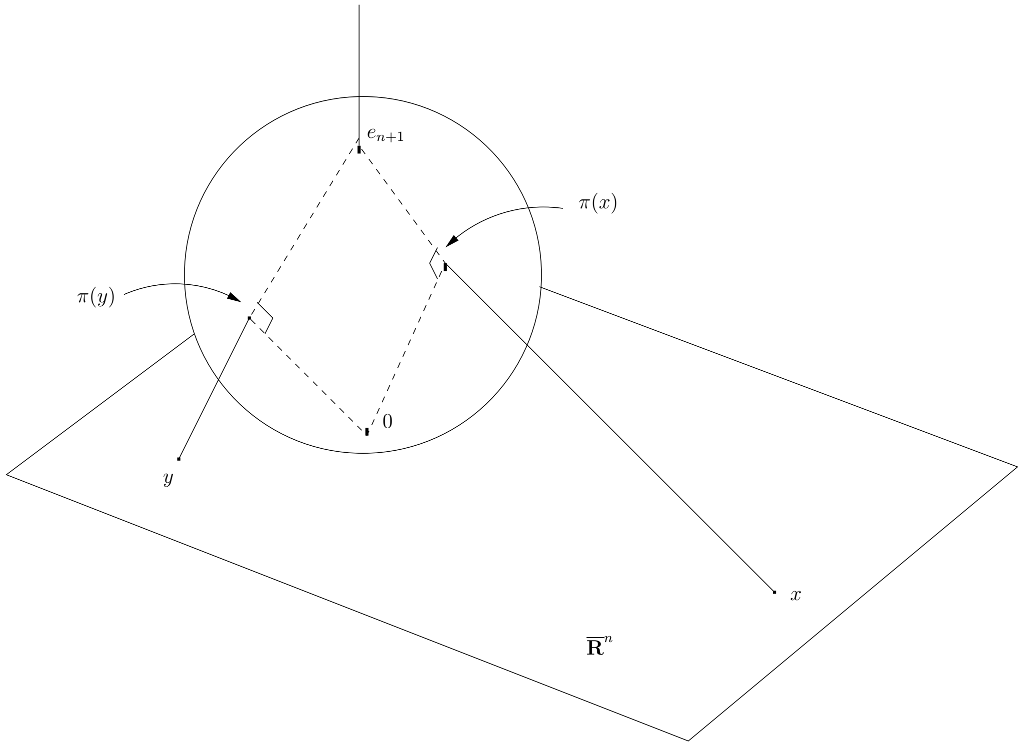


FIGURE 4. Visualization of formulae (2.52) and (2.53).

where the limiting value is taken if one of the points is $\infty$. One of the most important properties of Möbius transformations is that they *preserve absolute ratios*; namely, if $f \in GM$

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d| \tag{2.60}$$

for all distinct $a, b, c, d$ in $\overline{\mathbf{R}}^n$. In fact, the preservation of absolute ratios is a characteristic property of Möbius transformations.

## 3. **Conformal invariants**

In this section we shall introduce two conformal invariants, the modulus metric $\mu_G(x, y)$ and its "dual" quantity $\lambda_G(x, y)$, where $G$ is the domain in $\overline{\mathbf{R}}^n$ and $x, y \in G$. The modulus metric $\mu_G$ is functionally related to the hyperbolic metric $\rho_G$ if $G = \mathbf{B}^n$, while in the general case $\mu_G$ reflects the "capacitary geometry" of $G$ in a delicate fashion. The dual quantity $\mu_G(x, y)$ is also functionally related to $\rho_G$ if $G = \mathbf{B}^n$. For a wide class of domain in $\mathbf{R}^n$, the so called QED-domains, we shall find two sided estimates for $\lambda_G(x, y)$ in terms of

$$r_G(x, y) = \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}}.$$

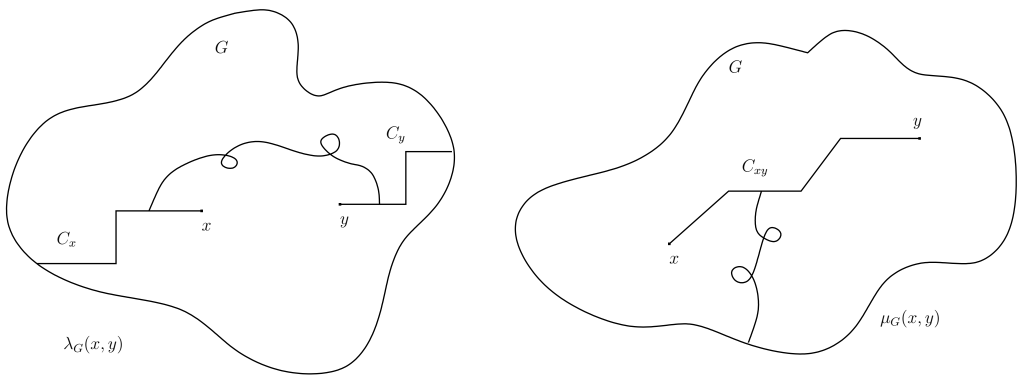


FIGURE 5. Conformal invariants $\lambda_G$ and $\mu_G$.

3.1. **The conformal invariants $\lambda_G$ and $\mu_G$.** If $G$ is a proper subdomain of $\overline{\mathbf{R}}^n$, then for $x, y \in G$ with $x \neq y$ we define

$$\lambda_G(x,y) = \inf_{C_x, C_y} M(\Delta(C_x, C_y; G)). \tag{3.2}$$

Where $C_z = \gamma_z[0,1)$ and $\gamma_z : [0,1) \to G$ is a curve such that $z \in |\gamma_z|$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$. It follows from Lemma 2.16 that $\lambda_G$ is invariant under conformal mappings of $G$. That is, $\lambda_{fG}(f(x), f(y)) = \lambda_G(x,y)$, if $f : G \to fG$ is conformal and $x, y \in G$ are distinct.

If $\mathrm{card}(\overline{\mathbf{R}}^n \setminus G) = 1$, then $\lambda_G(x,y) \equiv \infty$. Therefore $\lambda_G$ is of interest only in the case of $\mathrm{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$. For $\mathrm{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$ and $x, y \in G$, $x \neq y$, there are continua $C_x$ and $C_y$ in (3.2) with $\overline{C}_x \cap \overline{C_y} = \emptyset$ and thus $M(\Delta(C_x, C_y; G)) < \infty$ by [Vu, 5.23]. Thus if $\mathrm{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$, we may assume that the infimum in (3.2) is taken over continua $C_x$ and $C_y$ with $\overline{C}_x \cap \overline{C_y} = \emptyset$.

For a proper subdomain $G$ of $\overline{\mathbf{R}}^n$ and for all $x, y \in G$ define

$$\mu_G(x,y) = \inf_{C_{xy}} M(\Delta(C_{xy}, \partial G; G)), \tag{3.3}$$

where the infimum is taken over all continua $C_{xy}$ such that $C_{xy} = \gamma[0,1]$ and $\gamma$ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$. It is clear that $\mu_G$ is also a conformal invariant in the same sense as $\lambda_G$, $\mu_G$ is a metric if $\mathrm{cap}\partial G > 0$, we call $\mu_G$ the *modulus metric* or *conformal metric of $G$*.

3.4. **Remark.** Let $D$ be a sub domain of $G$. It follows from Remark 2.7 and (2.8) that $\mu_G(a,b) \leq \mu_D(a,b)$ for all $a, b \in D$ and $\lambda_G(a,b) \geq \lambda_D(a,b)$ for all distinct $a, b \in D$. In what follows we are interested only in the non-trivial case $\mathrm{card}(\overline{\mathbf{R}}^n \setminus G) \geq 2$. Moreover, by performing an auxiliary Möbius transformation, we may and shall assume that $\infty \in \overline{\mathbf{R}}^n \setminus G$ throughout this section. Hence $G$ will have at least one boundary point.

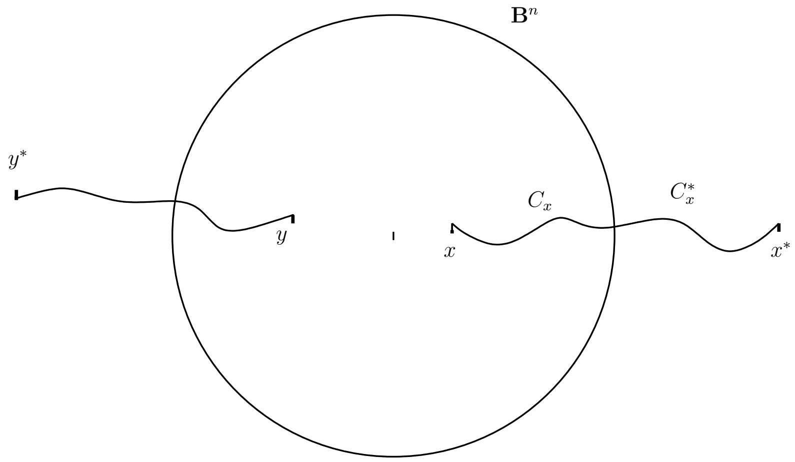


FIGURE 6. Graph for the proof of Theorem 3.5 .

In a general domain $G$, the value of $\lambda_G(x,y)$ and $\mu_G(x,y)$ can not be expressed in terms of well-known simple functions. For $G = \mathbf{B}^n$ they can be given in terms of $\rho(x,y)$ and the capacity of Teichmüller condenser.

3.5. **Theorem.** *For $x, y \in \mathbf{B}^n$, $x \neq y$*

(1) $\mu_{\mathbf{B}^n}(x,y) = \gamma(1/\tanh(\rho(x,y)/2)) = 2^{n-1}\tau(1/\sinh^2(\rho(x,y)/2))$,

(2) $\lambda_{\mathbf{B}^n}(x,y) = \frac{1}{2}\tau(1/\sinh^2(\rho(x,y)/2)) = 2^{-n}\gamma(\cosh(\rho(x,y)/2))$.

*Proof.* (1) follows from Lemma 2.40 and Theorem 2.31.
(2) The assertion is $GM(\mathbf{B}^n)$-invariant, hence we may assume that $x = re_1 = -y$ and $r = \tanh(\rho(x,y)/4)$. Now

$$\begin{aligned} \lambda_{\mathbf{B}^n}(x,y) &\leq M(\Delta(E,-E;\mathbf{B}^n)) \quad (E = [re_1, e_1]) \\ &\leq \frac{1}{2}M(\Delta(E_2,-E_2;\mathbf{B}^n)) \quad \left(E_2 = [re_1, \frac{1}{r}e_1]\right) \\ &= \frac{1}{2}\tau\left(\frac{4r^2}{(1-r^2)^2}\right) = \frac{1}{2}\tau\left(\sinh^2\frac{\rho(x,y)}{2}\right), \end{aligned}$$

because

$$\frac{4r^2}{(1-r^2)^2} = \left(\frac{2r}{1-r^2}\right)^2 = \left(\frac{2\tanh\frac{\rho}{4}}{1-\tanh^2\frac{\rho}{4}}\right)^2 = \left(2\sinh\frac{\rho}{4}\cosh\frac{\rho}{4}\right)^2 = \sinh^2\frac{\rho(x,y)}{2}.$$

Therefore it is enough to prove $\lambda_{\mathbf{B}^n} \geq \frac{1}{2}\tau(\sinh^2\frac{\rho}{2})$, let $C_x, C_y$ be as in (3.2) and $C_x^*, C_y^*$ their images under the inversion $x \mapsto x/|x|^2$, $C_x^s = C_x \cup C_x^*$, $C_y^s = C_y \cup C_y^*$ see Figure 6. May assume $0 \notin C_y$. Choose compact connected sets $E \subset C_x, x \in E, F \subset C_y, y \in F$. Let $\mathrm{Sym}(E^s)$ : $E^s$ symmetrized in the positive $x_1$-axis and $\mathrm{Sym}(F^s)$ : $F^s$ symmetrized in the negative $x_1$-axis Theorem 2.36 implies that

$$(3.6) \qquad \mathrm{cap}\,(\mathbf{R}^n \setminus E^s, F^s) \geq \mathrm{cap}\,(\mathbf{R}^n \setminus \mathrm{sym}(E^s), \mathrm{sym}(F^s))$$

then (3.6) implies

$$(3.7) \qquad \begin{cases} M\,(\Delta([-1/re_1, -re_1], [re_1, 1/re_1])) \\ = \tau\left(\dfrac{4r^2}{(1-r^2)^2}\right) = \tau\left(\sinh^2 \dfrac{\rho(x,y)}{2}\right). \end{cases}$$

The above convergence means: When $d(E, \partial\mathbf{B}^n) \to 0$ and $d(E, \partial\mathbf{B}^n) \to 0$, then $\mathrm{Sym}(E^s) \to [re_1, \frac{1}{2}e_1]$, $\mathrm{Sym}(E^s) \to [-\frac{1}{r}e_1, -re_1]$ and (3.7) holds. On the other hand by

$$\begin{aligned} M(\Delta(C_x, C_y; \mathbf{B}^n)) &\geq M(\Delta(E, F; \mathbf{B}^n)) \\ &\geq \frac{1}{2}\mathrm{cap}\,(\mathbf{R}^n \setminus E^s, F^s) \to \frac{1}{2}\tau\left(\sinh^2 \frac{\rho(x,y)}{2}\right) \end{aligned}$$

when $d(E, \partial\mathbf{B}^n) \to 0$, $d(F, \partial\mathbf{B}^n) \to 0$ and $x \in E$, $E$ continuum $y \in F, F$ continuum. Because $C_x, C_y$ were arbitrary sets in (3.2), the assertion $\lambda_{\mathbf{B}^n} \geq \frac{1}{2}\tau(\sinh^2 \frac{\rho}{2})$ follows. □

3.8. **Remark.** From Theoerm 2.30(3) we obtain the following inequalities for $x, y \in \mathbf{B}^n$

$$\begin{aligned} \frac{1}{2}\tau\left(\sinh^2 \frac{1}{2}\rho(x,y)\right) &\geq c_n \log \tanh \frac{1}{4}\rho(x,y) \\ &= 2c_n \mathrm{arth}\left(e^{-\frac{1}{2}\rho(x,y)}\right) \geq 2c_n e^{-\frac{1}{2}\rho(x,y)}. \end{aligned}$$

Here the identities $2\cosh^2 A = 1 + \cosh 2A$ and $\sinh 2A = 2\cosh A \sinh A$ were applied. Recall that

$$\sinh^2 \frac{1}{2}\rho(x,y) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

by [Vu, 2.19]. Similarily, by Theorem 2.30(3) we obtain also

$$\begin{aligned} \frac{1}{2}\tau\left(\sinh^2 \frac{1}{2}\rho(x,y)\right) &\leq \frac{1}{2}c_n\mu\left(\tanh^2(\frac{1}{4}\rho(x,y))\right) \\ &< \frac{1}{2}c_n \log \frac{4}{\tanh^2 \frac{1}{4}\rho(x,y)} = c_n \log \frac{2}{\tanh \frac{1}{4}\rho(x,y)}. \end{aligned}$$

3.9. **Lemma.** *Let $G$ be a proper subdomain of $\mathbf{R}^n$, $d(x) = d(x, \partial G)$, $B_x = B^n(x, d(x))$, let $y \in B_x$ with $y \neq x$, and $r = |x-y|/d(x)$. Then the following two inequalities hold:*

(1) $\lambda_G(x,y) \geq \lambda_{B_x}(x,y) = \dfrac{1}{2}\tau\left(\dfrac{r^2}{1-r^2}\right) > c_n \log \dfrac{1}{r}$,

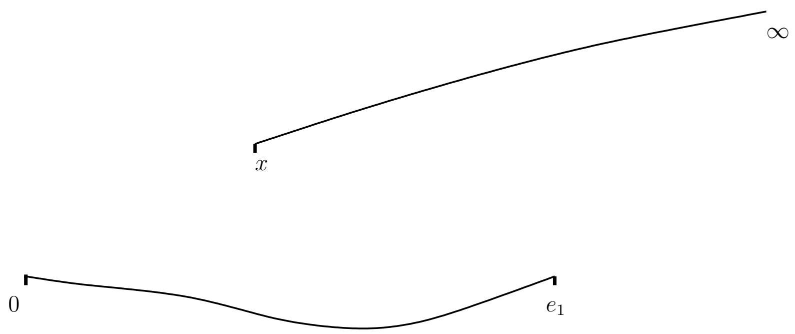


FIGURE 7. The function $p(x)$.

(2) $\mu_G(x,y) \geq \mu_{B_x}(x,y) = \gamma\left(\dfrac{1}{r}\right)\omega_{n-1}\left(\log\dfrac{1}{r}\right)^{1-n}$.

*Proof.* (1) By Remark 3.4, Theorem 3.5(2) and Remark 3.8 we obtain

$$\lambda_G(x,y) \geq \lambda_{B_x}(x,y) = \frac{1}{2}\tau\left(\frac{r^2}{1-r^2}\right) \geq -c_n \log\tanh\frac{1}{4}(2\mathrm{arth}\, r)$$

$$= c_n \log\frac{1+\sqrt{1-r^2}}{r} > c_n \log\frac{1}{r}$$

(2) The desired inequalities follow from Remark 3.4 and [Vu, 7.24]. □

3.10. **The function p(x).** Fix $x \in \mathbf{R}^n \setminus \{0, e_1\}$ and set

$$p(x) = \inf_{E,F} M(\Delta(E,F)) \tag{3.11}$$

where the infimum is taken over all the pairs of continua $E, F$ such that $0, e_1 \in E$, $x, \infty \in F$. If we carry out two spherical symmetrization with centres 0 and $e_1$, respectively, we see by Theorem 2.38, that

$$p(x) \geq \tau_n(\min\{|x|, |x-e_1|\}). \tag{3.12}$$

It is easy to see that equality holds here for $x = se_1$ with $s \in (-\infty, 0) \cup (1, \infty)$.

3.13. **Teichmüller problem:** Find $p(x)$ in terms of well known functions.
This problem was presented in 1938 and solved by Schiffer in 1948 for $n = 2$. We consider here the $n$-dimensional case.

3.14. **Lemma.** *For $x \in \mathbf{R}^n \setminus \{0, e_1\}$*

$$p(x) \geq \max\{\tau(|x|), \tau(|x-e_1|)\}$$

*with equality if $x = se_1$ and $s \in (-\infty, 0) \cup (1, \infty)$.*

*Proof.* Spherical symmetrization with centre at 0 implies that $p(x) \geq \tau(|x|)$.
Spherical symmetrization with centre at $e_1$ implies that $p(x) \geq \tau(|x-e_1|)$.

If $x = te_1, t > 1$ and $E_0 = [0, e_1], F_0 = [te_1, \infty)$ $\quad p(x) = \tau(t-1)$.

For the case $x = te_1, t > 0$ the choice $E_1 = [0, e_1], F_1 = [-te_1, \infty)$ yields

$$p(x) = M(\Delta(E_1, F_1)) = \tau(t).$$

$\square$

We next map the quadruple $(0, e_1, x, \infty)$ to $(-e_1, y, -y, e_1)$.

3.15. **Lemma.** *Let $f \in GM$ with $(0, e_1, x, \infty) \xrightarrow{f} (-e_1, y, -y, e_1), |y| \le 1$. Then*

$$|y| = \frac{|x - e_1|}{1 + |x| + t} \quad \textit{and} \quad |y + e_1|^2 = \frac{|y - e_1|^2}{|x|} = \frac{4}{1 + |x| + t}$$

*where $t = ((1 + |x|)^2 - |x - e_1|^2)^{1/2}$.*

*Proof.* The Möbius invariance yields

$$|0, e_1, x, \infty| = | - e_1, y, -y, e_1| \text{ and } |0, e_1, \infty, x| = | - e_1, y, e_1, -y|$$

and, equivalently, $|y - e_1|^2 = |x||y + e_1|^2$ and $4|y| = |x - e_1||y + e_1|^2$.
The first equation implies

$$2ye_1 = \frac{1 - |x|}{1 + |x|}(1 + |y|^2)$$

substitution of this into the second equation yields

$$4|y| = |x - e_1| \left( |y|^2 + \frac{1 - |x|}{1 + |x|}(1 + |y|^2) + 1 \right) \Leftrightarrow$$

$$|y|^2 - 2|y|\frac{1 + |x|}{|x - e_1|} + 1 = 0 \Rightarrow |y| = \frac{1 + |x| \pm t}{|x - e_1|}.$$

The minus sign yields $|y| \le 1$ and the desired formula follows. This computation also yields the desired formula for $|y + e_1|$. $\square$

3.16. **Corollary.** *Let $f \in GM$ with $(a, b, c, d) \longmapsto (-e_1, y, -y, e_1)$,$|y| \le 1$.*
*If $r = |b, a, c, d|, s = |a, b, c, d|, t = \sqrt{(1 + s)^2 - t^2}$ then*

$$|y| = \frac{r}{1 + s + t} \quad \textit{and} \quad |y + e_1|^2 = \frac{|y - e_1|^2}{s} = \frac{4}{1 + s + t}.$$

3.17. **Lemma.** *For $a \in (0, 1)$ let $b = \dfrac{2a}{1 + a^2}$. Then for $r > 0$*

$$M(\Delta([-are_1, are_1],\ S^{n-1}(r))) = M(\Delta([0, bre_1],\ S^{n-1}(r))) = \gamma\left(\frac{1 + a^2}{2a}\right).$$

*Proof.* Choose $h \in GM(\mathbf{B}^n(r))$ such that $(-re_1, -are_1, are_1, re_1) \longmapsto (-re_1, 0, bre_1, re_1)$. Then

$$|-re_1, -are_1, are_1, re_1| = |-re_1, 0, bre_1, re_1|.$$

This implies $b = 2a/(1+a^2)$. This equality follows from the conformal invariance of modulus and the definition of $\gamma$. □

3.18. **Lemma.** *Let $y \in \mathbf{B}^n \setminus \{0\}$, $E = [-y, y]$, $F = [-e_1, \infty) \cup [e_1, \infty)$ and $E_1 = [-|y|e_1, |y|e_1]$. Then*

$$M(\Delta(E,F)) \le M(\Delta(E_1,F)) = \tau_n\left(\frac{(1-|y|)^2}{4|y|}\right).$$

*Proof.* By the definition of $\tau$ and conformal invariance of the modulus we have

$$M(\Delta(E,F)) = \tau_n(|y|e_1, -|y|e_1, e_1 - e_1|) = \tau_n\left(\frac{(1-|y|)^2}{4|y|}\right).$$

To prove the inequality write $y = t^2 e, |e| = 1, t \in (0,1), S = S^{n-1}(t)$
$\Gamma_1 = \Delta(E,S)$, $\Gamma_2 = \Delta(F,S)$. Then by the Lemma 3.17

$$M(\Gamma_1) = M(\Gamma_2) = \gamma_n\left(\frac{1+t^2}{2t}\right) = 2^{n-1}\tau_n\left(\frac{1+|y|}{2\sqrt{|y|}}\right)^2 - 1 = 2^{n-1}\tau_n\left(\frac{(1-|y|)^2}{4|y|}\right).$$

By (2.15)

$$M(\Delta(E,F))^{1/(1-n)} \ge M(\Gamma_1)^{1/(1-n)} + M(\Gamma_2)^{1/(1-n)} = 2\frac{1}{2}\tau_n\left(\frac{(1-|y|)^2}{4|y|}\right)^{1/(1-n)},$$

which yields the inequality. □

3.19**. The Ahlfors brackets.** For $x, y \in \mathbf{R}^n$, we define the *Ahlfors brackets* $A[x,y]$,

$$\begin{aligned} A[x,y]^2 &= 1 + |x|^2|y|^2 - 2xy \\ &= (1-|x|^2)(1-|y|^2) + |x-y|^2. \end{aligned}$$

It is easy to show that $A[x,y] = A[y,x]$, and $A[x,y] = |x||y - x^*|$ for $x \in \mathbf{R}^n \setminus \{0\}$. This notation is convenient in the study of Möbius transformations of $\mathbf{B}^n$ onto itself.

3.20. **Remark.** The Teichmüller ring $R_{T,n}(t)$ can be mapped by the Möbius transformations onto each of the following ring domains:
(a) $R = R(E,F)$, where $E$ and $F$ are circular arcs in

$$S = S^{n-1} \cap \{x_1 = \cdots = x_{n-2} = 0\},$$

joining $x, y \in S$ and $-x, -y$ respectively, with

$$|x+y| = |x-y|\sqrt{t}.$$

(b) $R = R([-ae_1, -e_1], [e_1, ae_1]), a = 1 + 2(1 + \sqrt{1+t})/t$.
(c) $R = R([-e_1, e_1], [-be_1, \infty] \cup [be_1, \infty])$, $b = 1 + 2t(1 + \sqrt{1 + 1/t})$.

3.21. **Lemma.** *For $x, y \in \mathbf{B}^n$ let $r = |x|, s = |y|, d = |x - y|$. Then*

(1) $d + (1-r)(1-s) \le A[x,y] \le d + r's'$, *with equality in the first iff $y = -tx$ or $x = -ty$ for some $t \in [0,1]$ and in the second iff $x = y$,*

(2) $A[x,y] \le rd + 1 - r^2 \le d + (1-r)(1+s)$, *where the first inequality reduces to equality iff $x = -ty$ or $y = tx$ for some $t \in [0,1]$ and the second inequality reduces to equality iff $y = tx$ for some $t \in [0,1]$,*

(3) $(1-r^2)(1-s^2) \le (1-(r+s)^2/4)^2 \le (1-d^2/4)^2$, *with equality in the first iff $r = s$ and in the second iff $y = -tx$ or $x = -ty, t \in [0,1]$,*

(4) $(1-r^2)(1-s^2) \le (1-rs)^2 \le A[x,y]^2$. *There is equality on the left if and only if $r = s$ and on the right if and only if $x = ty$ or $y = tx, t \in [0,1]$.*

3.22. **Theorem.** *For $x \in \mathbf{R}^n \setminus [0, e_1]$,*

$$p(x) \le M(\Delta(E, F; \mathbf{R}^n)) \le \tau_n \left( \frac{|x| + |x - e_1| - 1}{2} \right),$$

*where $E$ is a circular arc with $0, e_1 \in E$, and $F = [xt : t \ge 1]$ is a ray with $x \in F$. Both inequalities reduce to equality if $x = se_1$ and $s \in (-\infty, 0) \cup (1, \infty)$.*

*Proof.* Let $h : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be the Möbius transformation taking $x, 0, e_1, \infty$ onto $-e_1, -y, y, e_1$, respectively, where $|y| < 1$. With $E_1 = [-|y|e_1, |y|e_1], E' = [-y, y]$ and $F' = [-e_1, \infty] \cup [e_1, \infty]$, by Lemma 3.18 we have

$$M(\Delta(E_1, F')) = \tau_n \left( \frac{(1-|y|)^2}{4|y|} \right).$$

Next, by invariance of cross ratio, we get

$$|x, \infty, 0, e_1| = |-e_1, e_1, -y, y| \text{ and } |x, \infty, e_1, 0| = |-e_1, e_1, y, -y|,$$

which give

$$|x| = \frac{|y - e_1|^2}{4|y|} \text{ and } |x - e_1| = \frac{|y + e_1|^2}{4|y|}.$$

Hence

$$|x| + |x - e_1| - 1 = \frac{(1-|y|)^2}{2|y|}.$$

Now, with $E = h^{-1}(E'),\ F = h^{-1}(F')$, from 4.15 and [AVV, 8.17(1)]

$$\begin{aligned} p(x) \le M(\Delta(E, F)) &= M(\Delta(E', F')) \\ &\le M(\Delta(E_1, F')) = \tau_n \left( \frac{1}{2}(|x| + |x - e_1| - 1) \right). \end{aligned}$$

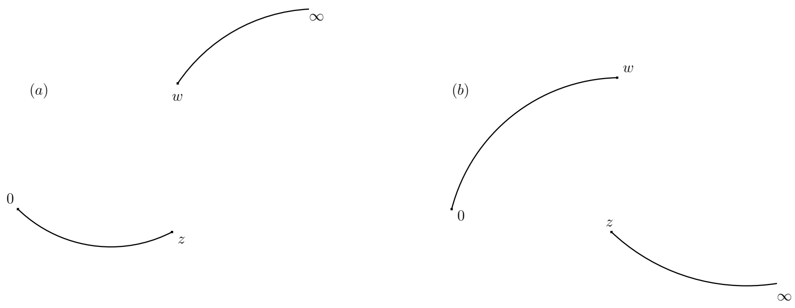


FIGURE 8. Graph for the proof of Theorem 3.25.

□

3.23. **Corollary.** *For $x \in \mathbf{R}^n \setminus \overline{\mathbf{B}}^n$, the following inequalities hold:*

$$\tau_n(|x-e_1|) \le p(x) \le \tau_n(|x-e_1|/2) \le \sqrt{2}\tau_n(|x-e_1|).$$

*Proof.* The first inequality follows from (3.12) and second one from Theorem 3.22. The third one follows from $\tau_n(ct)/\tau_n(t) \in (1, 1/\sqrt{c})$ for $t > 0$ and $c \in (0,1)$ [AVV, 11.25(1)]. □

3.24. **Corollary.** *Let $G = \mathbf{R}^n \setminus \{0\}$ and $x, y \in G$ with $x \neq y$. Then*

$$\lambda_G(x,y) \le \tau_n\left(\frac{|x-y| + ||x|-|y||}{2\min\{|x|,|y|\}}\right) \le \tau_n\left(\frac{|x-y|}{2\min\{|x|,|y|\}}\right).$$

*Proof.* By invariance under homotheties we may assume that $y = e_1$ and $|x| \ge 1$. Since $\min\{|x|,|y|\} = 1$ the result follows from Theorem 3.22 and (3.2). □

3.25. **Theorem.** *Let $G = \mathbf{C} \setminus \{0\}$ and $z, w \in G$, $z \neq w$, then*

$$\lambda_G(z,w) = \min\{p(z/w), p(w/z)\}.$$

*Proof.* In view of the definition of $\lambda_G$ we have two possible choices of continua (a) and (b) see Figure 8.

The choice (a) leads to $p(w/z)$ whereas (b) leads to $p(z/w)$. □

3.26. **Theorem.** *Let $G = \mathbf{R}^n \setminus \{0\}$ and $x, y \in G$, $x \neq y$, and let $r_z$ be the similarity mapping with $r_z(0) = 0$ and $r_z(e) = e$. Then*

$$\lambda_G(x,y) = \min\{p(r_x(y)), p(r_y(x))\}.$$

*Proof.* We see that

$$|r_x(y) - e_1| = |x-y|/|x|$$

and that $r_z(x)$ takes the role of $y/z$. □

3.27. **Theorem.** *Let $G \subset \mathbf{R}^n$ be a domain, $x, y \in G$, $x \neq y$, and*

$$m(x,y) = \min\{d(x), d(y)\}\,.$$

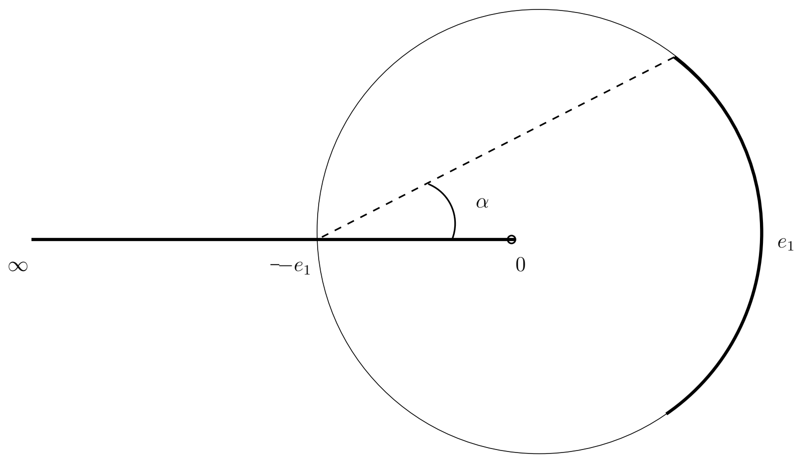


FIGURE 9. Mori's extremal ring domain.

*Then*

$$\lambda_G(x,y) \le \inf_{z\in\partial G} \lambda_{\mathbf{R}^n\setminus\{z\}}(x,y) \le \tau_n\left(\frac{|x-y|}{2m(x,y)}\right) \le \sqrt{2}\tau\left(\frac{|x-y|}{m(x,y)}\right).$$

*Proof.* The claim follows from Corollaries 3.23 and 3.24. □

3.28. **Corollary.** *For $x, y \in G = \mathbf{R}^n \setminus \{0\}$, $x \neq y$,*

$$\tau_n\left(\frac{|x-y|}{m}\right) \le \lambda_G(x,y) \le \tau_n\left(\frac{|x-y|}{2m}\right) \le \sqrt{2}\tau_n\left(\frac{|x-y|}{m}\right),$$

*where $m = \min\{|x|, |y|\}$.*

*Proof.* The first inequality follows from Theorem 2.38, second follows from Corollary 3.24 and third follows from [AVV, 11.25(1)]. □

We next use Theorem 3.22 to estimate the capacity of a particular ring domain, the $n$-dimensional analogue of the so-called Mori's ring, well known in the theory of plane quasiconformal mappings [LV, p.58].

3.29**. Mori's ring domain in $\mathbf{R}^n$.** For $0 < \alpha < \pi/4$, Mori's ring domain in $\mathbf{R}^n$, denoted by $R_{M,n}(\alpha)$, has a boundary components the ray $\{te_1 : t \le 0\}$ and the circular arc

$$\{x \in S^{n-1} : x_3 = \ldots = x_n = 0, |x + e_1| \ge 2\cos\alpha\}.$$

Mori's extremal ring is sketched in Figure 9.
set

$$\nu_n(\alpha) = \mathrm{cap} R_{M,n}(\alpha).$$

The sets $E$ and $F$ in Theorem 3.22 are the boundary components of a generalized Mori's ring domain. Indeed, while such a ring domain is not, in general, conformally equivalent to the Mori's ring domain defined above, its boundary components are still a circular arc and a half line. Note that for $n = 2$ and $x = (\frac{1}{2}, t)$ for some real $t$, Mori's ring is extremal for $p(x)$ (cf.[LV]). The explicit formula $\nu_2(\alpha) = 2\pi/\mu(\sin\alpha)$, where $\mu$ is as in (2.29) and [LV, p.59].

3.30. **Corollary.** *For* $0 < \alpha < \pi/4$,

$$\tau_n\left(\frac{1}{2\sin(2\alpha)}\right) \le \nu_n(\alpha) \le \tau_n\left(\frac{1}{2\tan(2\alpha)}\right).$$

*Proof.* The lower bound follows from Theorem 2.38. For the upper bound, clearly the ring $R_{M,n}(\alpha)$ is equivalent, under the Möbius transformation, to the ring $R$ whose boundary components are $C_0 = [-e_n\tan\alpha, e_n\tan\alpha]$ and $C_1 = [-e_1,\infty] \cup [e_1,\infty]$. Hence, by [AVV, 8.17(1)(c),7.52(1)(c)],
$\nu_n(\alpha) = \mathrm{cap} R \le \tau_n((1-\tan\alpha)^2/(4\tan\alpha)) = \tau_n(1/(2\tan(2\alpha)))$. □

## 4. **Quasiconformal mapping**

The study of quasiconformal and quasiregular mappings in this section will be based on the trasformation formulae for the moduli of curve families under these mappings. In most cases it will be enough to make use of these transformation formulae specialized to the conformal invariants $\mu_G$ and $\lambda_G$.

4.1. **Definition.** The set $\mathbf{T}^n$ consists of all the triples $(y, f, D)$, where $f : G \to \overline{\mathbf{R}}^n$ is a continuous mapping, $G \subset \overline{\mathbf{R}}^n$ is a domain, $D$ is a domain with $\overline{D} \subset G$ and $y \in \overline{\mathbf{R}}^n \setminus f\partial D$.

4.2. **Lemma.** *There exists a unique function* $\mu : \mathbf{T}^n \to \mathbf{Z}$*, the topological degree, such that*

(1) $y \mapsto \mu(y, f, D)$ *is a constant in each component of* $\overline{\mathbf{R}}^n \setminus f\partial D$.
(2) $|\mu(y, f, D)| = 1$ *if* $y \in fD$ *and* $f|\overline{D}$ *is one-to-one.*
(3) $\mu(y, \mathrm{id}, D) = 1$ *if* $y \in D$ *and* id *is the identity mapping.*
(4) *Let* $(y, f, D) \in \mathbf{T}^n$ *and* $D_1, \ldots, D_k$ *be disjoint domains such that* $(y, f, D_i) \in \mathbf{T}^n$ *and* $f^{-1}(y) \cap D \subset \bigcup_{i=1}^k D_i$*. Then*

$$\mu(y, f, D) = \sum_{i=1}^k \mu(y, f, D_i).$$

(5) *Let* $(y, f, D), (y, g, D) \in \mathbf{T}^n$ *be such that there exists a homotopy* $h_t : \overline{D} \to \overline{\mathbf{R}}^n$, $t \in [0,1]$, *with* $h_o = f|\overline{D}$, $h_1 = g|\overline{D}$, *and* $(y, h_t, D) \in \mathbf{T}^n$ *for all* $t \in [0,1]$*. Then* $\mu(y, f, D) = \mu(y, g, D)$.

4.3. **Definition.** A mapping $f : G \to \overline{\mathbf{R}}^n$ is called *sense-preserving (orientation-preserving)* if $\mu(y, f, D) > 0$ whenever $D$ is a domain with $\overline{D} \subset G$ and $y \in fD \setminus f\partial D$. If $\mu(y, f, D) < 0$ for all such $y$ and $D$, then $f$ is called *sense-reversing (orientation-reversing)*.

The *branch set* $B_f$ of a mapping $f : G \to \overline{\mathbf{R}}^n$ is defined to be the set of all points $x \in G$ such that $f$ is not a local homeomorphism at $x$. It is easily seen that $B_f$

is a closed subset of $G$. We call $f$ *open* if $fA$ is open in $\overline{\mathbf{R}}^n$ whenever $A \subset G$ is open, *light* if $f^{-1}(y)$ is totally disconnected for all $y \in fG$, and *discrete* if $f^{-1}(y)$ is isolated for all $y \in fG$.

Let $G \subset \overline{\mathbf{R}}^n$ be a domain. We denote by $J(G)$ the collection of all subdomains $D$ of $G$ with $\overline{D} \subset G$.

4.4. **Definition.** Let $f : G \to \overline{\mathbf{R}}^n$ be discrete. Fix $x \in G$ and a neighborhood $U \in J(G)$ of $x$ such that $x = \overline{U} \cap f^{-1}(f(x))$. The number $\mu(f(x), f, U)$ is denoted by $i(x, f)$ and called the *local* (topological) *index* of $f$ at $x$.

4.5. **Lemma.** *Suppose that $f : G \to \overline{\mathbf{R}}^n$ is open, that $U \subset \mathbf{R}^n$ is a domain, and that $D$ is a component of $f^{-1}U$ such that $D \in J(G)$. Then $D$ is a normal domain, $fD = U$, and $U \in J(fG)$.*

4.6. **Lemma.** *Suppose that $f : G \to \overline{\mathbf{R}}^n$ is a discrete and open mapping. Then $\lim_{r\to 0} d(U(x, f, r)) = 0$ for every $x \in G$. If $U(x, f, r) \in J(G)$, then $U(x, f, r)$ is a normal domain and $fU(x, f, r) = B^n(f(x), r) \in J(fG)$. Furthermore, for every point $x \in G$ there is a positive number $\sigma_x$ such that the following conditions are satisfied for $0 \le r \le \sigma_x$ :*

(1) *$U(x, f, r)$ is a normal neighborhood of $x$.*
(2) *$U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}B^n(f(x), r)$.*
(3) *$\partial U(x, f, r) = U(x, f, \sigma_x) \cap f^{-1}S^{n-1}(f(x), r)$ if $r < \sigma_x$.*
(4) *$\overline{\mathbf{R}}^n \setminus \overline{U}(x, f, r)$ is connected.*
(5) *$\overline{\mathbf{R}}^n \setminus U(x, f, r)$ is connected.*
(6) *If $0 < r < s < \sigma_x$, then $\overline{U}(x, f, r) \subset U(x, f, s)$, and $U(x, f, s) \setminus \overline{U}(x, f, r)$ is a ring, i.e. its complement has exactly two components.*

If $f : G \to \overline{\mathbf{R}}^n$, $A \subset \mathbf{R}^n$ and $y \in \mathbf{R}^n$, denote

$$N(y, f, A) = \mathrm{card}(A \cap f^{-1}(y)),$$

$$N(f, A) = \sup\{N(y, f, A) : y \in \mathbf{R}^n\},$$

$$N(f) = N(f, G).$$

Here $N(y, f, A)$ is called the *multiplicity* of $y$ in $A$ and $N(f, A)$ the *maximal multiplicity* of $f$ in $A$.

4.7**. Quasiregular mappings.** Let $G \subset \mathbf{R}^n$ be a domain. A mapping $f : G \to \mathbf{R}^n$ is said to be *quasiregular* if $f$ is ACL$^n$ (*absolutely continuous on almost all lines*) and there exists a constant $K \ge 1$ such that

$$|f'(x)|^n \le K J_f(x), \quad |f'(x)| = \max_{|h|=1} |f'(x)h|, \tag{4.8}$$

almost every where in $G$. Here $f'(x)$ denotes the formal derivative of $f$ at $x$. The smallest $K \ge 1$ for which this inequality is true is called the *outer dilatation* of $f$ and denoted by $K_o(f)$. If $f$ is quasiregular, then the smallest $K \ge 1$ for which the inequality

$$(4.9) \qquad J_f(x) \le K l(f^{'}(x)), \quad l(f^{'}(x)) = \max_{|h|=1} |f^{'}(x)h|,$$

holds almost every where in $G$ is called the *inner dilatation* of $f$ and denoted by $K_I(f)$. The *maximal dilatation* of $f$ is the number $K(f) = \max\{K_o(f), K_I(f)\}$. If $K(f) \le K$, $f$ is said to be *$K$-quasiregular*. If $f$ is not quasiregular, we set $K_o(f) = K_I(f) = K(f) = \infty$.
It follows from linear algebra (see [Va, p.44] and [R, p.22]) that

$$K_o(f) \le K_I(f)^{n-1}, \quad K_I(f) \le K_o(f)^{n-1}.$$

4.10**. Quasiconformal mapping.** If $f$ is homeomorphism satisfying (4.8) and (4.9) with $|J_f(x)|$ in place of $J_f(x)$ then $f$ is called *quasiconformal*.
We now give an analytic definition of quasiconformal mapping. Let $G, G^{'}$ be a domain in $\overline{\mathbf{R}}^n$ $f : G \to G^{'}$ be a homeomorphism. Then $f$ is *$K$-quasiconformal* if

$$(4.11) \qquad M(\Gamma)/K \le M(f\Gamma) \le K M(\Gamma)$$

for every curve family $\Gamma$ in $G$. Moreover, the dilatations of $f$ are defined as

$$(4.12) \qquad K_I(f) = \sup \frac{M(f\Gamma)}{M(\Gamma)}, \quad K_o(f) = \sup \frac{M(\Gamma)}{M(f\Gamma)},$$

where the superema are taken over all curve families $\Gamma$ and $G$ such that $M(\Gamma)$ and $M(f\Gamma)$ are not simultaneously 0 or $\infty$. Thus

$$M(\Gamma)/K_o(f) \le M(f\Gamma) \le K_I(f) M(\Gamma)$$

for every curve family $\Gamma$ in $G$.

4.13. **Notation.** For domains $D, D^{'}$ in $\overline{\mathbf{R}}^n$ we let $QC_K(D, D^{'})$ denote the class of all $K$-quasiconformal mappings of $D$ into $D^{'}$. We also let $QC_K(D, D) = QC_K(D)$.

4.14. **Theorem.** *Let $f \in QC_K(\mathbf{R}^n)$ with $f(0) = 0$ and $f(e_1) = e_1$. Then*

$$|f(x)| + |f(x) - e_1| \le 1 + 2\tau_n^{-1}\left(\frac{1}{K}\tau_n(m)\right)$$

*for $x \in \mathbf{R}^n$, where $m = \min\{|x|, |x - e_1|\}$. Equality holds here if $K = 1$ and $x = te_1$, where $t \in (-\infty, 0) \cup (1, \infty)$.*

*Proof.* If $K = 1$, then $f$ is a Möbius transformation (cf.[G1]). Further, since a Möbius transformation fixing $0, e_1$ and $\infty$ must be the identity map, we must have $f(te_1) = te_1$, and equality follows.
From (3.12), Theorem 3.22 and Definition 4.10 we obtain

$$\tau_n(m) \le p(x) \le K p(f(x)) \le K \tau_n\left(\frac{|f(x)| + |f(x) - e_1| - 1}{2}\right).$$

$\square$

4.15. **Theorem.** *Let $f \in QC_K(\mathbf{R}^n)$ with $f(0) = 0$. Then for $x, y \in \mathbf{R}^n \setminus \{0\} \equiv G$,*

$$\eta_{1/K,n}(r(x,y)/2) \le r(f(x), f(y)) \le 2\eta_{K,n}(r(x,y)),$$

*where $\eta_{C,n}(t) = \tau_n^{-1}(\tau_n(t)/C), C > 0$, and $r(x,y) = \dfrac{|x-y|}{\min\{|x|,|y|\}}$.*

*Proof.* From Theorem 3.28 and quasi-invariance of $f$, we have

$$\tau_n(r(x,y)) \le \lambda_G(x,y) \le K\lambda_G(f(x), f(y)) \le K\tau_n(r(f(x), f(y))/2).$$

Solving this for $r(f(x), f(y))$, we get the second inequality. The first inequality is proved similarly, or may be obtained by applying the second inequality to $f^{-1}$. □

4.16. **Corollary.** *Let $f \in QC_K(\mathbf{R}^n)$ with $f(\mathbf{B}^n) \subset \mathbf{B}^n$, $f(0) = 0$. Then*

$$|f(x) - f(x)| \le 1 + 2\tau_n^{-1}\left(\frac{1}{K}\tau_n\left(\frac{|x-y|}{\min\{|x|,|y|\}}\right)\right)$$

*for $x, y \in \mathbf{B}^n$.*

*Proof.* This follows immediately from Theorem 4.15, since $\min\{|x|,|y|\} < 1$ when $x, y \in \mathbf{B}^n$. □

4.17. **Theorem.** *Let $f \in QC_K(\mathbf{R}^n)$ with $f(0) = 0$ and $f(\mathbf{B}^n) \subset \mathbf{B}^n$. Then*

$$|f(x) - f(x)| \le 128|x-y|^{1/K}$$

*for all $x, y \in \mathbf{B}^n$ with equality if and only if $x = y$.*

*Proof.* Clearly we may assume that $x \neq y$. For $|x-y| \ge 2^{1-3K}$ we have the inequality

$$|f(x) - f(y)| \le 2 \le 2\left(\frac{|x-y|}{2^{1-3K}}\right)^{1/K} = 2^{4-1/K}|x-y|^{1/K}.$$

Thus in the rest of proof we may assume that $0 < |x-y| < 2^{1-3K}$. Let $m = \min\{|x|,|y|\}, r = r(x,y) = |x-y|/m$, and $\rho' = \rho(f(x), f(y))$.

Case 1. $0 < |x-y| < 2^{1-3K}$ and $m \le 15/16$.
By Lemma 3.21 and [AVV, 7.64,13.20(2)] we obtain

$$\begin{aligned}
|f(x) - f(x)| &\le 2\tanh\frac{\rho'}{4} \le 2\min\{2,K\}\left(\tanh\frac{\rho}{4}\right)^{1/K} \\
&\le \min\{2,K\}\left(\frac{|x-y|}{A[x,y] + \sqrt{(1-|x|^2)(1-|y|^2)}}\right)^{1/K} \\
&\le \min\{2,K\}\left(\frac{|x-y|}{1-|x||y|}\right)^{1/K} \\
&\le \min\{2,K\}16^{1/K}|x-y|^{1/K} < 64|x-y|^{1/K}.
\end{aligned}$$

Case 2. $0 < |x-y| < 2^{1-3K}$ and $m > 15/16$ by Corollary 3.28 and [AVV, 13.39]

$$\begin{aligned}|f(x)-f(x)| &\leq 2\tau_n^{-1}\left(\frac{1}{K}\tau_n\left(\frac{|x-y|}{m}\right)\right) \leq 2\cdot 4^{3-(1/K)}\left(\frac{16}{15}\right)^{1/K}|x-y|^{1/K} \\ &= \left(\frac{4}{5}\right)^{1/K} 128|x-y|^{1/K}.\end{aligned}$$

□

4.18. **Theorem.** *Let $G$ be a proper subdomain of $\mathbf{R}^n$ and let $f : G \to G'$ be a $K$-quasiconformal mapping of $G$ onto a subdomain $G'$ of $\overline{\mathbf{R}}^n$. Then, for $x, y \in G$*

$$q(f(x), f(y))q(\partial G') \leq 128\left(\frac{|x-y|}{d(x,\partial G)}\right)^{1/K}.$$

*Here $q$ denotes the chordal metric as in* (2.56).

*Proof.* We may assume that $x \neq y$. Fix $a, c \in \partial G'$ with $q(a,c) = q(\partial G') > 0$. Let $r = |x-y|/d(x,\partial G)$. If $r \geq 2^{-7K}$, then the right side of the inequality is at least unity, and there is nothing to prove. For $r \in (0, 2^{-7K})$ we argue as follows.

With $\mathbf{B} = \mathbf{B}^n(x, d(\partial G))$ we have by Theorem 3.9

$$\lambda_G(x,y) \geq \lambda_B(x,y) = \frac{1}{2}\tau_n\left(\frac{r^2}{1-r^2}\right).$$

Let $D = \overline{\mathbf{R}}^n \setminus \{a, c\}$. By (3.4), (3.24), (2.59) and [AVV, 8.48] we have

$$\begin{aligned}\lambda_{G'}(f(x), f(y)) &\leq \lambda_D(f(x), f(y)) \\ &\leq \tau_n(\max\{|f(x), a, f(y), c|, |f(y), a, f(x), c|\}/2) \\ &\leq \tau_n(q(f(x), f(y))q(\partial G')/2).\end{aligned}$$

Since

$$\lambda_G(x,y) \leq K\lambda_{G'}(f(x), f(y)),$$

the above inequality yields

$$q(f(x), f(y))q(\partial f(G)) \leq 2\tau_n^{-1}\left(\frac{1}{2K}\tau_n\left(\frac{r^2}{1-r^2}\right)\right).$$

Now $r < 2^{-7K}$ implies that $r^2/(1-r^2) < 2^{-7K}$, and by [AVV, 13.44(11b)] we get

$$\begin{aligned}q(f(x), f(y))q(\partial f(G)) &\leq 2\cdot 4^{2-1/2K}\lambda_n^{2(2K)^{1/(n-1)}-2}\left(\frac{r}{r'}\right)^{1/K} r'^{(1/K)-2(2K)^{1/(n-1)}} \\ &\leq 128\left(\frac{r}{2r'}\right)^{1/K} \leq 128 r^{1/K},\end{aligned}$$

as desired. □

4.19. **Theorem.** *Let $f : \overline{\mathbf{R}}^2 \setminus E \to \overline{\mathbf{R}}^2 \setminus \mathbf{B}^2$ be a $K$-quasiconformal mapping, where $E$ is a continuum with $0, 1 \in E$ and let $f(\infty) = \infty$. Then for $x \in \mathbf{R}^2 \setminus E$,*

$$|f(x)| \le \gamma_2^{-1}\left(\frac{1}{2K}\gamma_2(\sqrt{1+m})\right) \le 4^{2K-1}(1+m)^K,$$

*where $m = \min\{|x|, |x-1|\}$.*

*Proof.* Let $f = f^{-1}([f(x), \infty])$ so that $x, \infty \in F$. By circular symmetrization see (2.31), (2.35), (2.41) and (3.12),

$$M(\Delta(E,F)) \ge \tau_2(\min\{|x|, |x-e_1|\}) = \frac{1}{2}\gamma_2(\sqrt{1+m}),$$

where $m = \min\{|x|, |x-e_1|\}$, and further

$$M(\Delta(f(x), [f(x), \infty])) = \gamma_2(|f(x)|) \ge \frac{M(\Delta(E,F))}{K},$$

so

$$\gamma_2(|f(x)|) \ge \frac{1}{2K}\gamma_2(\sqrt{1+m}).$$

Hence by [AVV, 8.74(2),8.69],

$$|f(x)| \le \gamma_2^{-1}\left(\frac{1}{2K}\gamma_2(\sqrt{1+m})\right) = \frac{1}{\varphi_{1/(2K),2}(1/\sqrt{1+m})} \le 4^{2K-1}(1+m)^K.$$

$\square$

For plane conformal mappings we have the following sharper result.

4.20. **Theorem.** *If the mapping in Theorem 4.19 is conformal, that is, $K = 1$, then*

$$|f(x)| \le (\sqrt{1+m} + \sqrt{m})^2,$$

*where $m = \min\{|x|, |x-1|\}$.*

*Proof.* By the Theorem [AVV, 10.5(4)] we have

$$\varphi_{1/2,2}(r) = \left(\frac{r}{1+r'}\right)^2.$$

From the proof of Theorem 4.19, with $K = 1$, we then have

$$|f(x)| \le \frac{1}{\varphi_{1/2,2}(1/\sqrt{1+m})} = (\sqrt{1+m} + \sqrt{m})^2,$$

as desired. $\square$

The result of Theorem 4.20 is sharper than the result of Theorem 4.19 because

$$4^{2K-1}(1+m)^K \ge 4(1+m) \ge (\sqrt{1+m} + \sqrt{m})^2.$$

4.21. **Theorem.** *For $x \in \mathbf{R}^2 \setminus [0,1]$ there exist a circular arc $E$ with $0, 1 \in E, x \in \mathbf{R}^2 \setminus E$, and a conformal mapping $f$ of $\overline{\mathbf{R}}^2 \setminus E$ onto $\overline{\mathbf{R}}^2 \setminus \overline{\mathbf{B}}^2$ with $f(\infty) = \infty$ such that*

$$|f(x)| \geq s + \sqrt{s^2 - 1}, \; s = |x| + |x - 1|.$$

*Proof.* Choose $E$ and $F$ as in the proof of Theorem 3.22. Let $f$ be given by the Riemann Mapping Theorem with $f(\infty) = \infty$. Then

$$M(\Delta(f(E), f(F))) = M(\Delta(E, F)) \leq \tau_2 \left( \frac{|x| + |x - 1| - 1}{2} \right),$$

while by 2.36,

$$M(\Delta(f(E), f(F))) \geq \gamma_2(|f(x)|).$$

Then by Theorem 2.31 we get, as in the proof of Theorem 4.20,

$$\begin{aligned} |f(x)| \geq \; & \gamma_2^{-1} \left( \frac{1}{2} \gamma_2 \left( \sqrt{\frac{s+1}{2}} \right) \right) = \frac{1}{\varphi_{1/2,2}(\sqrt{2/(s+1)})} \\ & = \left( \frac{\sqrt{s+1} + \sqrt{s-1}}{\sqrt{2}} \right)^2 = s + \sqrt{s^2 - 1}, \end{aligned}$$

where $s = |x| + |x - e_1|$. $\square$

4.22. **Remark.** The fact that the bounds in Theorems 4.20 and 4.21 concide for $x = te_1, t > 1$, shows that these estimates are sharp.
When $n = 2$, the next theorem shows that the upper bound in Theorem 3.22 is quite sharp, by exhibiting it as the minimal modulus of a class of curve families in $\mathbf{R}^2 \setminus \{0, 1\}$.

4.23**. The Joukowski map.** The conformal mapping $z \mapsto \frac{1}{2}(z + 1/z)$ of the exterior of the unit disk on to $\mathbf{R}^2 \setminus [-1, 1]$ is called the *Joukowski map.* As well-known property of this map is that it transforms circles centered at the origin, e.g. $|z| = r > 1$, onto ellipses with the foci -1 and 1 : $\{w = (u, v : |w - 1| + |w + 1| = r + 1/r)\}$. This property of the Joukowski map is similiar to the property of the conformal map of the disk minus radial segment onto annulus, given in [AVV, 4.61].

4.24. **Theorem.** *For $x \in \mathbf{R}^2 \setminus [0,1]$, let $F$ be a any continuum in $\overline{\mathbf{R}}^2$ containing $x$ and $\infty$. Then*

$$M(\Delta([0,1], F)) \geq \tau_2 \left( \frac{|x| + |x - 1| - 1}{2} \right).$$

*Moreover there exists an extremal continuum $F$ for which equality holds.*

*Proof.* Let $f$ be the conformal mapping of $\overline{\mathbf{R}}^2 \setminus \overline{\mathbf{B}}^2$ onto $\overline{\mathbf{R}}^2 \setminus [0,1]$ given by $f(z) = (z+1)^2/(4z), f(\infty) = \infty$, and let $g = f^{-1}$. We note that $f$ is the Joukowski map in 4.23 followed by similarity transformation $z \mapsto (z+1)/2$. Then by Theorem 2.37

$$\begin{aligned} M(\Delta([0,1],F)) &= M(\Delta(g[0,1],g(F))) \\ &\geq \gamma_2(|g(x)|) = \tau_2\left(\frac{(|g(x)|)-1)^2}{4|g(x)|}\right) \\ &= \tau_2\left(\frac{|x|+|x-1|-1}{2}\right). \end{aligned}$$

Finally, the choice $F = f([g(x),\infty])$ clearly gives equality. $\square$

## 5. **Distortion theory**

The goal of this section is to study the distortion functions $\varphi_{K,n}, \varphi^*_{K,n}$ and $\psi_{K,n}$. We will obtain estimates, functional inequalities, and asymptotic limits for them, as well as relations among them. We begin with the following definition.

5.1. **Definition.** For $n \geq 2,\ 0 < K < \infty,\ 0 < r < 1,\ r' = \sqrt{1-r^2}$, $\alpha = K^{1/(1-n)}$,

(1) $\varphi_{K,n}(r) = \dfrac{1}{\gamma_n^{-1}(K\gamma_n(1/r))} = M_n^{-1}(\alpha M_n(r))$,
$\varphi_{K,n}(0) = 0,\ \varphi_{K,n}(1) = 1$,

(2) $\psi_{K,n}(r) = \sqrt{1-\psi_{1/K,n}(r')^2}$,

(3) $$\varphi^*_{K,n}(r) = \begin{cases} \sup\{|f(x)| : |x| = r, \quad f \in QC_K(\mathbf{B}^n), f(0) = 0\}, \\ \text{if} \quad 1 \leq K \leq \infty, \\ \\ \inf\{|f(x)| : |x| = r, \quad f \in QC_{1/K}(\mathbf{B}^n), f(0) = 0 \\ (\mathbf{B}^n) = \mathbf{B}^n\}, \quad \text{if} \quad 0 < K \leq 1, \end{cases}$$

where for $1 \leq K < \infty$, $QC_K(\mathbf{B}^n)$ denotes the the set of all $K$-quasiconformal mappings of $\mathbf{B}^n$ into itself.

5.2. **Lemma.** *For $n \geq 2$ and $r \in (0,1)$ we have*

(1) $\varphi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^\alpha, \quad for \quad K \geq 1, \quad \alpha = K^{1/(1-n)}$,
(2) $\varphi_{1/K,n}(r) \geq \lambda_n^{1-\beta} r^\beta, \quad for \quad K \geq 1, \quad \beta = 1/\alpha$.

*Proof.* (1) By the proof of Lemma 2.26 we see that $M(r) + \log r$ is decreasing on $(0,1)$. Set $s = \varphi_{K,n}(r) \geq r$, then

$$\begin{aligned} M(r) + \log r &\geq M(s) + \log s \\ M(r) + \log \frac{r}{\lambda_n} &\geq M(s) + \log \frac{s}{\lambda_n} = \alpha M(r) + \log \frac{s}{\lambda_n}. \end{aligned}$$

Note that by (2.27) $\log \lambda_n \geq M(r) + \log r$ implies $0 \leq -M(r) + \log \lambda_n/r$.
Therefore

$$-\alpha M(r) + \alpha \log \lambda_n/r \leq -\alpha M(r) + \log \lambda_n/s,$$

which is equivalent to

$$\alpha \log \lambda_n / r \le \log \lambda_n / s. \tag{5.3}$$

Inequality (5.3) is equivalent to

$$s \le \lambda_n^{1-\alpha} r^{\alpha}.$$

The part (2) is proved in the same way. $\square$

5.4. **Corollary.** *For $K \ge 1$ we have*

$$\varphi_{K,n}(r) \le 2^{1-1/K} K r^{\alpha}, \quad \alpha = K^{1/(1-n)}.$$

*Proof.* Lemma 2.28 implies $\log \lambda_n \le n - 1 + \log 2 \quad$ for all $n \ge 2$. Clearly $1 - \alpha \le 1 - 1/K$ and $(1-\alpha)(n-1) = (1 - K^{1/(1-n)})(n-1) \le \log K$ where in the last step the inequality $1 - e^{-x} \le x, \quad x > -1$, was used. Now

$$(1-\alpha)\log \lambda_n \le (n - 1 + \log 2)(1-\alpha) \le \log K + (1 - 1/K)\log 2.$$

This last inequality together with Lemma 5.2 yields

$$\varphi_{K,n}(r) \le \lambda_n^{1-\alpha} r^{\alpha} \le 2^{1-1/K} K r^{\alpha}.$$

$\square$

5.5. **Theorem.** *Let $f : G \to \mathbf{R}^n$ is a non-constant quasiregular mapping, then*

(1) $\mu_{fG}(f(a), f(b)) \le K_I(f)\mu_G(a,b) \quad \forall a, b \in G$ *in particular,* $f : (G, \mu_G) \to (fG, \mu_{fG})$ *is Lipschitz-constant. If* $N(f,G) < \infty$, *then for all* $a, b \in G$, $f(a) \ne (b)$,

(2) $\lambda_G(a,b) \le K_o(f) N(f,G) \lambda_{fG}(f(a), f(b))$.

5.6. **Theorem.** *Let $f : \mathbf{B}^n \to \mathbf{B}^n$ be $K$-quasiregular and $\alpha = K^{1/(1-n)}$. Then for all $x, y \in \mathbf{B}^n$*

(1) $\tanh \frac{\rho(f(x),f(y))}{2} \le \varphi_{K,n}\left(\tanh \frac{\rho(x,y)}{2}\right) \le \lambda_n^{1-\alpha}\left(\tanh \frac{\rho(x,y)}{2}\right)^{\alpha}$.

(2) $\rho(f(x), f(y)) \le K\mu(e^{-\rho(x,y)}) \le K(\rho(x,y) + \log 4)$.

*Proof.* Fix $x, y \in \mathbf{B}^n$. Because $f\mathbf{B}^n \subset \mathbf{B}^n$, Theorem 3.4 and Theorem 3.5(1) imply that

$$\mu_{f\mathbf{B}^n}(fx, fy) \ge \mu_{\mathbf{B}^n}(fx, fy) = \gamma_n(1/\tanh b)$$

where $b = \rho(fx, fy)/2$. By Theorem 5.5(1) and Theorem 3.5(1)

$$\mu_{f\mathbf{B}^n}(fx, fy) \le K\mu_{\mathbf{B}^n}(x,y) = K\gamma_n(1/\tanh a)$$

where $a = \rho(x,y)/2$. Theorem 5.2(1) implies (1). Part (2) follows from the above inequalities (cf. (2.34) and (2.40))

$$\begin{aligned} 2^{n-1}c_n\rho(fx, fy) &\le \gamma_n(1/\tanh b) \\ \text{and } \gamma_n(1/\tanh a) &\le 2^{n-1}c_n\mu(e^{-\rho(x,y)}) \le 2^{n-1}c_n(\rho(x,y) + \log 4). \end{aligned}$$

$\square$

5.7. **Corollary.** *Let* $f : \mathbf{B}^n \to \mathbf{B}^n$ *K-quasiconformal,* $f(0) = 0$ *and* $f\mathbf{B}^n = \mathbf{B}^n$. *Then*

$$\varphi_{1/K,n}(|x|) \le |f(x)| \le \varphi_{K,n}(|x|).$$

*Proof.* [AVV, 2.16] implies that $\tanh \dfrac{\rho(0, f(x))}{2} = |f(x)|$.
The second inequality follows from Theorem 5.6(1). Because $f\mathbf{B}^n = \mathbf{B}^n$ and $f$ is injection we may apply Theorem 5.6(1) also to the K-quasiconformal map $f^{-1} : \mathbf{B}^n \to \mathbf{B}^n$, obtaining

$$|x| = |f^{-1}(f(x))| \le \varphi_{K,n}(|f(x)|)$$

observing that $\varphi_{K,n}^{-1} = \varphi_{1/K,n}$ this yields the first inequality. □

Theorem 5.6 shows that a K-quasiconformal map $f : \mathbf{B}^n \to \mathbf{B}^n$ is uniformly continuous as a map $f : (\mathbf{B}^n, \rho) \to (\mathbf{B}^n, \rho)$, with the modulus of continuity

$$\omega_f = 2\text{arctanh}\varphi_{K,n}(\tanh \frac{t}{2}). \tag{5.8}$$

In the case $f\mathbf{B}^n \ne \mathbf{B}^n$ it is natural to expect that here the target space $(\mathbf{B}^n, \rho)$ could be replaced by $(f\mathbf{B}^n, k_{f\mathbf{B}^n})$.

5.9. **Example.** We show that the analytic function $f : \mathbf{B}^2 \to \mathbf{B}^2 \setminus \{0\} = f\mathbf{B}^2$, $f(z) = \exp\left(\dfrac{z+1}{z-1}\right)$, $z \in \mathbf{B}^2$, is not uniformly continuous as a map $f : (\mathbf{B}^2, \rho) \to (f\mathbf{B}^2, k_{f\mathbf{B}^2})$. Let $x_j = (e^j - 1)/(e^j + 1)$, $j = 1, 2, 3, \ldots$ then [Vu, 2.16] implies $\rho(0, x_j) = j$ and thus $\rho(x_j, x_{j+1}) = 1$. Write $Y = \mathbf{B}^2 \setminus \{0\}$. Because $f(x_j) = \exp(-e^j)$ we obtain by [Vu, 3.5]

$$\begin{aligned} k_Y(fx_j, fx_{j+1}) &\ge j_Y(fx_j, fx_{j+1}) = \log(1 + (\exp e^{j+1}) - \exp(-e^{j+1})) \\ &= e^{j+1} - e^j \to \infty, \quad j \to \infty. \end{aligned}$$

Because $\rho(x_j, x_{j+1}) = 1$, we see that $f : (\mathbf{B}^2, \rho) \to (Y, k_Y)$ is not uniformly continuous.

5.10. **Theorem.** *Let* $f : \mathbf{B}^n \to \mathbf{B}^n$ *be K-quasiregular,* $N = N(f, \mathbf{B}^n) < \infty$, $x, y \in \mathbf{B}^n$, $f(x) \ne f(y)$, $\rho = \rho(x, y)$, $\rho' = \rho(fx, fy)$. *Then*

(1) $\sinh^2 \frac{\rho'}{2} \le \tau^{-1}\left(\frac{1}{NK}\tau(\sinh^2 \frac{\rho}{2})\right)$,

(2) $\tanh \frac{\rho'}{4} \le 2\left(\tanh \frac{\rho}{4}\right)^{1/(NK)}$.

5.11. **Theorem.** *Let* $f : \mathbf{B}^n \to \mathbf{B}^n$ *be K-quasiconformal and* $f(0) = 0$. *Then* $\forall x \in \mathbf{B}^n$

$$A(|f(x)|) \le 2A(|x|)^{1/K} : \quad A(t) = t/(1 + \sqrt{1 - t^2}).$$

*Proof.* Use Theorem 5.10(2) and

$$\tanh\left(\frac{1}{4}\log\frac{1+s}{1-s}\right) = \tanh\left(\frac{1}{2}\text{arctanh}\right) = A(s).$$

□

5.12. **Remark.** Putting together Lemma 5.2, Theorem 5.7 and Theorem 5.11 one can prove: If $f:\mathbf{B}^n\to\mathbf{B}^n=f\mathbf{B}^n$, $f(0)=0$ is K-quasiconformal then

$$4^{1-K^2}|x|^K\le|f(x)|\le 4^{1-1/K^2}|x|^{1/K}.$$

5.13. **Theorem.** *Let $f:\mathbf{R}^n\to\mathbf{R}^n$ be a K-quasiconformal map with $f(0)=0$. Then*

$$\frac{|f(x)-f(y)|}{\min\{|f(x)|,|f(y)|\}}\le\tau_n^{-1}\left(\frac{1}{K\sqrt{2}}\tau_n\left(\frac{|x-y|}{\min\{|x|,|y|\}}\right)\right)\qquad\forall x,y\in\mathbf{R}^n\setminus\{0\}.$$

*Proof.* Let $D=\mathbf{R}^n\setminus\{0\}$, $D'=\mathbf{R}^n\setminus\{0\}$. Then $fD=D'$ and $\forall x,y\in D$. Corollary 3.24 and Theorem 3.27 implies that

$$\begin{aligned}\lambda_D(x,y)&\ge\tau_n(r_D(x,y))\quad\left[\text{because}\quad\lambda_D(x,y)\ge\tau_n\left(\frac{|x-y|}{\min\{|x|,|y|\}}\right)\right]\\ \lambda_{D'}(fx,fy)&\le\sqrt{2}\tau_n(r_{D'}(fx,fy))\\ \lambda_D(x,y)&\le K\lambda_{D'}(fx,fy).\end{aligned}$$

These imply the assertion

$$r_{D'}(fx,fy)\le\tau_n^{-1}\left(\frac{1}{K\sqrt{2}}\tau_n(r_D(x,y))\right).$$

□

5.14. **Theorem.** *Let $G\subset\mathbf{R}^n$ be c-QED and $f:G\to fG\subset\mathbf{R}^n$ be K-quasiconformal. Then for all $x,y\in G$, $f(x)\ne f(y)$*

$$r_{fG}(fx,fy)\le\tau_n^{-1}(d\tau_n(r_G(x,y))),\quad d=\frac{c}{2^nK}.$$

*Proof.* Write $r=r_G(x,y)$
it follows from Lemma 2.23(1) that

$$\lambda_G(x,y)\ge c\tau_n(4r^2+4r)\ge 2^{1-n}c\tau_n(r),\tag{5.15}$$

Theorem 3.27 yields

$$\lambda_{fG}(fx,fy)\le\sqrt{2}\tau_n(r_{fG}(fx,fy)).\tag{5.16}$$

Inequalities (5.15) and (5.16) together imply the assertion. □

5.17. **Example.** We show that the c-QED hypothesis in Theorem 5.14 can not be left out. Let $G=\mathbf{B}^2\setminus[0,1)$ and $f:G\to fG=\mathbf{B}^2\cap H^2$ be the conformal map $f(z)=\sqrt{z}$ , $z\in G$. Write $x_j=(1/2,1/j), y_j=(1/2,-1/j)$ $j=4,5,\ldots$. Then $r_G(x_j,y_j)=2$ but it is easy to see that $r_{fG}(f(z_j),f(y_j))\to\infty, j\to\infty$. (Note: It is easy to show that G is not c-QED for any $c>0$).

5.18. **Remark.** Let $f : \mathbf{B}^n \to \mathbf{B}^n$, $f(0) = 0$, be K-quasiconformal. Then $\sinh^2(\rho(x,0)/2) = |x|^2/(1-|x|^2) = A(|x|)$ and Theorem 5.10(1) yields

$$A(|f(x)|) \le \tau_n^{-1}\left(\tau_n(A(|x|))/K\right). \tag{5.19}$$

Because $2^{n-1}\tau_n(s) = \gamma(\sqrt{1+s})$ we see that $2^{n-1}\tau_n(A(t)) = \gamma(1/\sqrt{1-t^2})$. Therefore (5.19) is equivalent to

$$\gamma(1/\sqrt{1-|fx|^2}) \ge \gamma(1/\sqrt{1-|x|^2})\frac{1}{K},$$

which implies

$$1/(1-|fx|^2) \le \gamma^{-1}(\gamma(1/\sqrt{1-|x|^2})/K)^2 = \varphi_{1/K,n}(\sqrt{1-|x|^2})^{-2}. \tag{5.20}$$

The inequality (5.20) implies that

$$|f(x)|^2 \le 1 - \varphi_{1/K,n}(\sqrt{1-|x|^2})^2. \tag{5.21}$$

For $n=2$, $1-\varphi_{1/K,2}(\sqrt{1-r^2})^2 \equiv \varphi_{K,2}(r)^2$, hence (5.21) is the same as the bound of Corollary 5.7. For $n \ge 3$, (5.21) improves the Schwarz lemma.

## 6. Quadruples and quasiconformal maps

The absolute ratio (see Definition 2.58) is invariant under Möbius transformations of $\overline{\mathbf{R}}^n$. In this section we will prove a corresponding result for $K$-quasiconformal mappings. As $K$ tends to 1, this result reduces to the aforementioned property of Möbius transformations.
We begin with the following lemma.

6.1. **Lemma.** *Let $f : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be $K$-quasiconformal with $0$ and $\infty$ as fixed points. Then for $0 < |z| < |x|$,*

$$B_{K,n}\left(\frac{|z|}{|x|}\right) \le \frac{|f(z)|}{|f(x)|} \le C_{K,n}\left(\frac{|z|}{|x|}\right),$$

*where*

$$\begin{cases} C_{K,n}(t) = \dfrac{\varphi_{K,n}(\sqrt{t})^2}{1-\varphi_{K,n}(\sqrt{t})^2}, \\ B_{K,n}(t) = \varphi_{1/K,n}\left(\sqrt{\frac{t}{1+t}}\right)^2 = C_{K,n}^{-1}(t), \end{cases} \tag{6.2}$$

*for $0 < t < 1$. In particular,*

$$\frac{|f(x)|}{|f(y)|} \le \frac{C_{K,n}(t)}{B_{K,n}(t)}$$

*whenever $t \in (0,1)$ and $|x| = |y| > 0$.*

*Proof.* We prove only upper bound for $|f(z)|/|f(x)|$, since the lower bound is obtained from it by inversion. Let $\Delta = \Delta([0,z],[x,\infty])$. Then by Lemma 2.39 and Theorem 2.31,

$$M(\Delta) \le \tau_n((|x|/|z|)-1) = 2^{1-n}\gamma_n\left(\sqrt{|x|/|z|}\right).$$

Since $f(0) = 0, f(\infty) = \infty$ we obtain, by spherical symmetrization (see Theorem 2.37 and Theorem 2.31),

$$M(f(\Delta)) \geq \tau_n \left( \frac{|f(x)|}{|f(z)|} \right) = 2^{1-n} \gamma_n \left( \sqrt{1 + \frac{|f(x)|}{|f(z)|}} \right).$$

Those two inequalities together with the estimate $M(f(\Delta)) \leq KM(\Delta)$ (see 4.12), yield

$$\gamma_n \left( \sqrt{1 + \frac{|f(x)|}{|f(z)|}} \right) \leq K \gamma_n \left( \sqrt{\frac{|(x)|}{|(z)|}} \right).$$

Since $\gamma_n$ is strictly decreasing, the result follows from Definition 5.1(1). $\square$

For $n \geq 2,\ 1 \leq K < \infty, t \in [0, \infty)$, as in [AVV, 9.13(3)], we let

(6.3)
$$\begin{cases} \eta^*_{K,n}(t) = \sup\{|f(x)| : |x| \leq t, f \in QC_K(\overline{\mathbf{R}}^n)\}, \\ \\ f(0) = 0, f(e_1) = e_1, f(\infty) = \infty. \end{cases}$$

It is well known ([A],[LV, pp.80-82, 105-108], and [AVV, 9.33]) that

(6.4)
$$\eta^*_{K,2}(t) = \eta_{K,n}(t) \leq \eta^*_{K^{n-1},n}(t)$$

for all $n \geq 2,\ k \geq 1, t \in (0, \infty)$ where $\eta_{K,2}$ is as in [AVV, 10.3].
More precisely, following [LVV, pp.9.10], we will find the extremal $K$-quasiconformal mapping for (6.3) with $t = 1$, which was used by Agard [A] to prove the identitiy in (6.4). The map takes the upper half plane into itself (hence, by reflection, $\mathbf{R}^2$ onto itself), with the quadruple $-1, 0, 1, \infty$ going onto the quadruple $-\lambda(K), 0, 1, \infty$. The construction is as follows: First, an elliptic function maps the quadrilateral $H(-1, 0, 1, \infty)$ conformally onto a square with vertices corresponding. Next, an affine mapping with constant dilatation $K$ takes this square onto a rectangle that is conformally equivalent (by an elliptic function) to the quadrilateral $H(\lambda(K), 0, 1, \infty)$. The composed mapping ot $H$ onto itself is, after reflection in the real axes, the desired extremal mapping of $\mathbf{R}^2$. Hence, by Theorem [AVV, 9.33], the rotation $F$ of the mapping $f$ about the real axes is a $K^{n-1}$-quasiconforaml self-mapping of $\mathbf{R}^n$, so that the inequality in (6.4) holds.

6.5. **Lemma.** *The following inequalities hold for $n \geq 2,\ K \geq 1$:*

(1) $\eta^*_{K,n}(1) = \sup \left\{ \dfrac{|f(x)|}{|f(y)|} : |x| = |y| > 0, f \in \mathcal{F}(n, K) \right\}$,

(2) $\eta^*_{K,n}(1) = \sup \left\{ \dfrac{|f(x)|}{|f(y)|} : |x| = |y| = t, f \in \mathcal{F}(n, K) \right\}$,

*for each $t > 0$, where $\mathcal{F}(n, K) = \{f \in QC_K(\overline{\mathbf{R}}^n) :\ f(0) = 0,\ f(\infty) = \infty\}$.*

*Proof.* First, if we denote the right side of (1) by $a$, it is clear that $\eta^*_{K,n}(1) \leq a$. Now fix $b < a$ and choose $f \in \mathcal{F}(n, K)$, and $x_o, y_o \in \mathbf{R}^n \setminus \{0\}$ with $|x_o| = |y_o|$ and $|f(x_o)|/|f(y_o)| > b$. Let $h_j : \mathbf{R}^n \to \mathbf{R}^n$ be similarities with $h_j(0) = 0,\ j = 1, 2,$

and $h_1(e_1) = y_o, h_2(f(y_o)) = e_1$. Then $g \equiv h_2 \circ f \circ h_1$ satisfies $g(0) = 0, g(e_1) = e_1, g(\infty) = \infty$, and

$$|g(h_1^{-1}(x_o))| = |h_2(f(x_o))| = |f(x_o)|/|f(y_o)| > b.$$

Since $b < a$ was arbitrary, we have proved that $\eta^*_{K,n}(1) \ge a$, and (1) follows. Part (2) follows from (1) by Möbius invariance of the absolute ratio (see (2.60)). $\square$

6.6. **Theorem.** *With notation as in Definitions* 5.1(1), (6.2) *and* (6.3), *the following inequalities hold for $n \ge 2$ and $K \ge 1$:*

(1) $\eta^*_{K,n}(t) \le \eta^*_{K,n}(1)\varphi^*_{K,n}(t),\ 0 \le t \le 1,$

(2) $\eta^*_{K,n}(t) \le \eta^*_{K,n}(1)/\varphi_{1/K,n}(1/t), t \le 1,$

(3) $\eta^*_{K,n}(1) \le \inf_{0<t<1} \dfrac{C_{K,n}(t)}{B_{K,n}(t)}.$

*Proof.* For part(1) let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a $K$-quasiconformal mapping with $f(0) = 0$ and $f(e_1) = e_1$. Then the function $g : \mathbf{B}^n \to g(\mathbf{B}^n) \subset \mathbf{B}^n$ defined by $g(x) = f(x)/\eta^*_{K,n}(1)$, satisfies $|g(x)| \le \varphi^*_{K,n}(|x|)$, by Definition 5.1(3), so (1) follows.

In part (2) we may assume that $t > 1$. Fix $x_o$ with $|x_o| = t, y_o = f(x_o)$ so that $|y_o| = |f(x_o)| = \max\{|f(x)| : |x| = t\}$. We may assume that $|y_o| > \eta^*_{K,n}(1) = c$, say. Let $g = f^{-1}$. Then $g(\mathbf{B}^n(|y_o|)) \supset \mathbf{B}^n(t)$, and $g(\mathbf{B}^n(c)) \supset \mathbf{B}^n$. Let $h(x) = x^* = x/|x|^2$ be reflection in $S^{n-1}$, and let $F = h \circ g \circ h$ ( note that $h = h^{-1}$). Then $F \in QC_K(\overline{\mathbf{R}}^n), f(0) = 0, f(e_1) = e_1, F(\infty) = \infty$, and $F(\mathbf{B}^n(1/c)) \subset \mathbf{B}^n$, $F(\mathbf{B}^n(1/y_o)) \subset \mathbf{B}^n(1/t)$. Hence, by Definition 5.1(3) and Theorem [AVV, 13.2(1)],

$$1/t = 1/|g(y_o)| = |F(y_o^*)| \le \varphi^*_{K,n}(c/|y_o|) \le \varphi_{K,n}(c/|y_o|),$$

so that $|y_o| = |f(x_o)| \le c/\varphi_{1/K,n}(1/t)$. The result follows when we take the supremum over all $f$.

Part (3) follows from Lemmas 6.1 and 6.5(2). $\square$

The next result improves Theorem 6.6 for $n = 2$.

6.7. **Theorem.** *For $K \ge 1$ let $f$ be a $K$- quasiconformal automorphism of the plane $\overline{\mathbf{R}}^2$. Then*

$$\frac{1}{\lambda(K)} \min\{t^{1/K}, t^K\} \le |f(a), f(b), f(c), f(d)| \le \lambda(K) \max\{t^{1/K}, t^K\}$$

*for each ordered quadruple of distinct points $a, b, c, d$ in the plane, where $t = |a, b, c, d|$. Moreover the inequalities are sharp for each $K \ge 1$.*

*Proof.* By symmetry it is sufficient to prove the upper bound. By Möbius invariance of (absolute) cross ratio (2.60), we may assume that $a = f(a) = 0, b = f(b) = 1$, and $d = f(d) = \infty$. Then, by (6.3), (2.44), (6.4), and Theorem [AVV, 10.24],

$$\begin{aligned} |f(a), f(b), f(c), f(d)| &= |f(c)| \le \lambda(K) \max\{|c|^{1/K}, |c|^K\} \\ &= \lambda(K) \max\{t^{1/K}, t^K\}. \end{aligned}$$

Finally, let $g$ be the extremal $K$-quasiconforaml automrphism (see [AVV, 9.25]) of the plane that maps the ordered quadruple $-1, \infty, 0, 1$ onto $-\lambda(K), \infty, 0, 1$. Since $|-1, \infty, 0, 1| = 1$ and $|-\lambda(K), \infty, 0, 1| = \lambda(K)$, the sharpness assertion follows. $\square$

Next, we derive an explicit upper bound for $\eta^*_{K,n}(1)$.

6.8. **Theorem.** *For $n \geq 2$ and $K \geq 1$,*

$$\eta^*_{K,n}(1) \leq \exp(4K(K+1) \leq \sqrt{K-1}).$$

*Proof.* By Corollary [AVV, 8.74(1)] we have $1 - \varphi_{K,n}(\sqrt{t})^2 \geq 1 - \lambda_n^{2(1-\alpha)} t^\alpha$. If $1 - \lambda_n^{2(1-\alpha)} t_o^\alpha = 1/K$, then $t_o = (\lambda_n^{2(\alpha-1)}(K-1)/K)^\beta$, so $t_o \leq (K-1)/K$. Thus $1 - \varphi_{K,n}(\sqrt{t})^2 \geq 1/K$ for $0 < t \leq t_o$. Hence, by Theorem 6.6(3) and Corollary [AVV, 8.74(1),(2)], we have

$$\begin{aligned}
\eta^*_{K,n}(1) &\leq \frac{C_{K,n}(t_o)}{B_{K,n}(t_o)} \leq K\lambda_n^{2(\beta-\alpha)} t_o^{\alpha-\beta}(1+t_o)^\beta \\
&\leq K\lambda_n^{2(\beta-\alpha)} t_o^{(\alpha-\beta)} \left(2 - \frac{1}{K}\right)^\beta \\
&= K^{\beta(\beta-1)}(K-1)^{1-\beta^2} \lambda_n^{2(\beta^2-1)}(2K-1)^\beta \equiv E.
\end{aligned}$$

Since $\max_{x>0} x^{-x} = e^{1/e}$ by elementary calculus, it follows that

$$(K-1)^{1-K} \leq \exp\left((2/e)\sqrt{K-1}\right),$$

where we have used $x = \sqrt{K-1}$. We will also need the estimate $\lambda_n^{\beta-1} \leq 2^{K-1}K^K$ from Lemma [AVV, 8.74(2)]. The rest of the proof is divided into two cases.

Case 1. If $K \geq 2$, then $(K-1)^{1-\beta^2} \leq 1$ and

$$\begin{aligned}
E &\leq K^{K(K-1)} \lambda_n^{2(\beta^2-1)}(2K-1)^K \\
&\leq K^{K(K-1)} \left(2^{K-1}K^K\right)^{2(K+1)} (2K-1)^K \\
&= K^{K(K-1)} 2^{2(K^2-1)} K^{2K(K+1)}(2K-1)^K \\
&= \exp[(3K^2+K)\log K + K\log(2K-1) + 2(K^2-1)\log 2] \\
&< \exp[(3K^2+K)\sqrt{K-1} + K\sqrt{2}\sqrt{K-1} + \sqrt{2}(K^2-1)] \\
&< \exp[(3K^2+3K)\sqrt{K-1} + 2(K^2-1)] \\
&= \exp[\sqrt{K-1}(K+1)(3K+2\sqrt{K-1})] \\
&\leq \exp\left(4K(K+1)\sqrt{K-1}\right),
\end{aligned}$$

where we have used the inequality $\log x \leq \sqrt{x-1},\ x > 1$.

Case 2. Next, if $1 < K \le 2$, we have

$$(K-1)^{1-\beta} \le (K-1)^{1-K} \le \exp\left(\frac{2}{e}\sqrt{K-1}\right),$$

so

$$\begin{aligned} E &\le K^{K(K-1)} \left(2^{K-1}K^K\right)^{2(K+1)} (2K-1)^K \exp\left(\frac{2}{e}(K+1)\sqrt{K-1}\right) \\ &= K^{K(3K+1)} 4^{(K-1)(K+1)} (2K-1)^K \exp\left(\frac{2}{e}(K+1)\sqrt{K-1}\right) \\ &= \exp[(3K^2+K)\log K + (K-1)(K+1)\log 4 + \frac{2}{e}(K+1)\sqrt{K-1} + K\log(2K-1)] \\ &\le \exp[(3K^2+K)\frac{K-1}{\sqrt{K}} + \frac{2K(K-1)}{\sqrt{K}} + (K-1)(K+1)\log 4 + \frac{2}{e}(K+1)\sqrt{K-1}] \\ &= \exp[3(K-1)(K+1)\sqrt{K} + (K-1)(K+1)\log 4 + \frac{2}{e}(K+1)\sqrt{K-1}] \\ &= \exp\left\{(K+1)\sqrt{K-1}\left[\left(3\sqrt{K}+\log 4\right)\sqrt{K-1} + \frac{2}{e}\right]\right\}. \end{aligned}$$

Then, by the arithmetic-geometric mean inequality [AVV, (1.42)], and the fact that $0 < K-1 \le 1$ and $2 < e$, we have

$$3\sqrt{K(K-1)} + (\log 4)\sqrt{K-1} + \frac{2}{e} \le \frac{3}{2}(2K-1) + \log 4 + 1 < 3K + 1 < 4K,$$

so $E \le \exp(4K(K+1))\sqrt{K-1}$, as desired. □


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## References


[A] S. Agard: *Distortion theorems for quasiconfrmal mappings*, Ann. Acad. Sci. Fenn. Ser. A I 413 (1968), 1-12.

[AVV] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: *Conformal invariants, inequalities and quasiconformal maps.* J. Wiley, 1997, 505 pp,

[G1] F. W. Gehring: *Rings and quasiconformal mappings in space.* Trans. Amer. Math. Soc. 103(1962), 353-393.

[G2] F. W. Gehring: *Quasiconformal mappings in Euclidean spaces.* Handbook of complex analysis: geometric function theory. Vol. 2, ed. by R. Kühnau, 1-29, Elsevier, Amsterdam, 2005.

[LV] O. Lehto and K. I. Virtanen: *Quasiconformal mappings in the plane*, 2nd edition, Grundlehren Math., Wiss., Band 126, Spring-Veflag, New York, 1973.

[LVV] O. Lehto and K. I. Virtanen and J. Väisälä: *Contribution to the distortion theory of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I 273 (1959), 1-14.

[R] S. Rickman: *A path lifting construction for discrete open mappings with application to quasimeromorphic mappings*. Duke Math. J. 42 (1975), 797-809.

[S] J.Sarvas: *Symmetrization of condensers in n-space*, Ann. Acad. Sci. Fenn. Ser. A I 522 (1972), 1-44.

[Va] J. Väisälä: *Lectures on n-dimensional quasiconformal Mappings*, Lecture Notes in Mathematics 229, Springer-Verlag, Berlin, 1971.

[Vu] M. Vuorinen: *Conformal geometry and quasiregular mappings*, Lecture Notes in Mathematics 1319, Springer, Berlin, 1988.