Locally identifying colourings for graphs with maximum degree Δ

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Abstract

A proper vertex-colouring of a graph G is said to be locally identifying if for any pair u,v of adjacent vertices with distinct closed neighbourhoods, the sets of colours in the closed neighbourhoods of u and v are different. We show that any graph G has a locally identifying colouring with $2\Delta^2 - 3\Delta + 3$ colours, where Δ is the maximum degree of G, answering in a positive way a question asked by Esperet *et al.* We also consider locally identifying colourings which have the property that the colours in the neighbourhood of each vertex are all different and apply some results to the class of chordal graphs.

1 Introduction

Let $G = (V, E)$ be a simple undirected finite graph. Let $c : V \to \mathbb{N}$ be a colouring of the vertices of G. For a subset S of V, we denote by $c(S)$ the set of colours that appear in S: $c(S) = \{c(u) \mid u \in S\}$ and we denote by

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$N[u]$ (resp. $N(u)$) the closed (resp. open) neighbourhood of u: $N[u] = \{v \in V \mid d(u,v) \leq 1\} = B_1(u)$ (resp. $N(u) = \{v \in V \mid d(u,v) = 1\}$).

The colouring c is a *locally identifying colouring* (*lid-colouring for short*) if it is a proper colouring (no pair of adjacent vertices has the same colour) such that for each pair of adjacent vertices u,v with $N[u] \neq N[v]$, we have $c(N[u]) \neq c(N[v])$. An edge uv is said to be *bad* if $N[u] \neq N[v]$ and $c(N[u]) = c(N[v])$. So a locally identifying colouring is a proper vertex colouring such that no edge is bad. The *locally identifying chromatic number* of G, $\chi_{lid}(G)$, is the minimum number of colours required in any locally identifying colouring of G.

Locally identifying colourings have been introduced in [7] and are related to identifying codes [8, 10], distinguishing colourings [1, 4, 6] and locating-colourings [5]. An open question asked in [7] was to know whether one can find a locally identifying colouring of a graph G with $O(\Delta^2)$ colours, where Δ is the maximum degree of G. Examples using the projective plane provide graphs G with $\chi_{lid}(G) = \Delta^2 - \Delta + 1$ (see [7]). In this note, we show that we always have $\chi_{lid}(G) \leq 2\Delta^2 - 3\Delta + 3$, answering in a positive way the question asked in [7]. The result is effective: we give a construction for a locally identifying colouring with $2\Delta^2 - 3\Delta + 3$ colours. This construction can be slightly modified, using $2\Delta^2 - \Delta + 1$ colours to provide a locally identifying colouring which has the property that the colours in the neighbourhood of each vertex are all distinct. We finally consider the class of chordal graphs, for which it is conjectured in [7] that $\chi_{lid}(G) \leq 2\chi(G)$, for any chordal graph G. We give a bound for this class in terms of Δ and χ, in the direction of the previous conjecture. For terminology and notations of graph theory, we refer to the book [2].

2 Upper bound in terms of the maximum degree

The following lemma shows that, given a locally identifying colouring, we can change the colour of a single vertex in a number of ways, without sacrificing the property that the colouring is locally identifying.

Lemma 1 (Recolouring Lemma)
Let G be a graph with maximum degree $\Delta \geq 3$. Let v be a vertex of degree d. Assume that G has a locally identifying colouring c with strictly more than $2d(\Delta - 1)$ colours. Then, there is a list L of colours of size at most $2d(\Delta - 1)$ such that if we change the colour of v for a colour not in L, the colouring remains locally identifying.

Proof. Let $v_1, ..., v_d$ be the neighbours of v. For each vertex v_i, let $u_{i,1}, ..., u_{i,s_i}$ be the neighbours of v_i that are not neighbours of v, see Figure 1. For $1 \leq i \leq d$, we construct a list L_i of colours with at most $2(\Delta - 1)$ colours. We first put $c(v_i)$ in L_i.

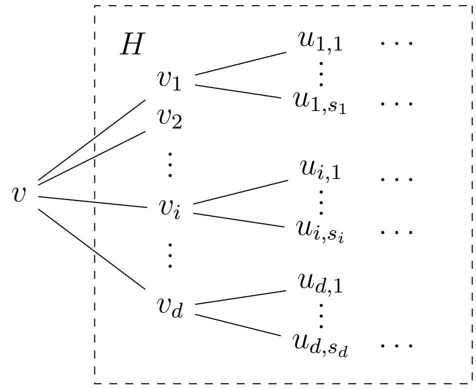

Figure 1: Neighbourhood of a vertex v

If there is a vertex $u_{i,j}$ such that $c(N[u_{i,j}]) = c(N[v_i] \setminus \{v\})$, we say that v_i is *of type A* and we add to L_i all the colours of $c(N[v_i] \setminus \{v\})$. We add at this point at most $\Delta - 1$ colours because $c(v_i)$ is already in L_i. Then, for all vertices $u_{i,j'}$ such that $c(N[u_{i,j'}]) \setminus c(N[v_i] \setminus \{v\})$ is not empty, we add an arbitrary colour of $c(N[u_{i,j'}]) \setminus c(N[v_i] \setminus \{v\})$ to L_i. In this step, we add at most $\Delta - 2$ colours because $j' \neq j$. So at the end $|L_i| \leq 2(\Delta - 1)$.

Otherwise, we say that v_i is *of type B* and for each neighbour $u \neq v$ of v_i, if $c(N[u]) \setminus c(N[v_i] \setminus \{v\})$ is not empty, we add one colour of this set to L_i. Note that u can be some other vertex v_j or some vertex $u_{i,j}$, but there are at most $\Delta - 1$ such vertices. If $c(N[v_i] \setminus c(N(v))$ is not empty, we add one colour of this set to L_i. In the end, $|L_i| \leq \Delta + 1$.

Let $L = \cup_{i=1,...,d} L_i$. Because $\Delta \geq 3$, $|L| \leq 2d(\Delta - 1)$. We define a new colouring c' of G by just changing the colour of v to a colour not in L. We will prove that c' is locally identifying. First, c' is a proper colouring because L contains all the colours $c(v_1)$,...,$c(v_d)$. Let now x, y be a pair of adjacent vertices, with $N[x] \neq N[y]$. We will show that $c'(N[x]) \neq c'(N[y])$. If neither x nor y are in $N[v]$ then the colours in their neighbourhood did not change and we have $c'(N[x]) \neq c'(N[y])$. So we can assume, without loss of generality, that $x = v_1$.

Assume first that $y = v$. If there exists $c_0 \in c(N[v_1]) \setminus c(N(v))$ then either v_1 is of type A and then $c_0 \in L$ and so $c_0 \in c'(N[v_1]) \setminus c'(N[v])$ is still separating v from v_1. Or v_1 is of type B, and then at least one colour of $c(N[v_1]) \setminus c(N(v))$ is in L (not necessarily c_0), and is separating v_1 and v. Otherwise, $c(N[v_1]) \subset c(N(v))$ and one colour of $N(v)$, is still separating v and v_1.

Assume now that $y = v_j$, with $j \neq 1$ and $v_i v_j \in E$. Then without loss of generality we can assume that there exists $c_0 \in c(N[v_1]) \setminus c(N[v_j])$, $c_0 \neq c(v)$. If v_1 is of type A, then c_0 is not the new colour $c'(v)$ of v because $c_0 \in L$. Otherwise v_1 is of type B and then there is one colour of $c(N[v_1]) \setminus c(N[v_j])$ in L (not necessarily c_0) that is separating v_1 and v_j.

Finally, we can assume without loss of generality that $y = u_{1,1}$. If $c(N[u_{1,1}]) = c(N[v_1] \setminus \{v\})$ then v_1 is of type A and so $c(N[u_{1,1}]) \subseteq L$. So the new colour of v, $c'(v)$, is not in $c(N[u_{1,1}])$ and is separating v_1 from $u_{1,1}$. If there is a colour in $c(N[v_1] \setminus \{v\}) \setminus c(N[u_{1,1}])$ then it is still separating v_1 from $u_{1,1}$. Otherwise, we necessarily have $c(N[u_{1,1}]) \setminus c(N[v_1] \setminus \{v\})$ nonempty, and so there is a colour of L that is separating $u_{1,1}$ from v_1. □

Let d be an integer. A graph G is *d-degenerate* if each induced subgraph of G has a vertex of degree at most d (see [9]).

Proposition 1
Let G be a d-degenerate graph with maximum degree $\Delta \geq 3$, with $d < \Delta$, then:

$$\chi_{lid}(G) \leq 2(\Delta - 1)^2 + d.$$

Proof. Let Δ be fixed. We prove the claim by induction on the number of vertices in G. The claim is clearly true for graphs with few vertices. Assume that the claim is true for every d-degenerate graph with fewer than n vertices and maximum degree at most Δ. Let G be a d-degenerate graph with n vertices and maximum degree at most Δ.

Let v be one vertex of minimum possible degree $t \leq d$ in G. Let $H = G \setminus \{v\}$. The graph H is also d-degenerate and by induction hypothesis, there is a locally identifying colouring c of H with $2(\Delta-1)^2 + d$ colours. As in the previous lemma, we denote the t neighbours of v by v_1,...,v_t, (if v has no neighbours the claim is trivial) and for each $i \in \{1, ..., t\}$, we denote by $u_{i,1}$,...,u_{i,s_i} the neighbours of v_i that are not neighbours of v (see Figure 1). We construct a list L' of size at most d containing for each $i \in \{1, .., t\}$, the colour $c(u_{i,1})$ if $u_{i,1}$ exists. Each vertex v_i has degree at most $\Delta - 1$ in H. Using Lemma 1, we can recolour each vertex v_i with a colour that is not in L': indeed, there are $2(\Delta - 1)^2$ forbidden colours from the lemma applied to v_i and d colours in L', but the colours $c(u_{i,1})$, if $u_{i,1}$ exists, is counted twice, so there are at most $2(\Delta - 1)^2 + d - 1$ forbidden colours and at least one colour is free.

We can now assume that c is a locally identifying colouring of H such that no vertex v_i has a colour in L'. We now assign a new colour $c(v)$ never used in c to v. We will prove that this colouring of G is locally identifying. It is clearly a proper colouring, and the only pair of adjacent vertices that are not clearly separated are the pair (v, v_i). If v_i has a vertex $u_{i,1}$, then $c(u_{i,1}) \in L'$ and $c(N[v])$ is not containing $c(u_{i,1})$, so they are separated. Otherwise, we have $N[v_i] \subseteq N[v]$. But v has minimum degree, so necessarily $N[v_i] = N[v]$ and the two vertices do not need to be separated.

Finally, we obtain a lid-colouring of G using $2(\Delta-1)^2+d+1 \geq 2(\Delta-1)^2+2$ colours, but the colour of v is used only once. By the recolouring lemma there are at least two colours that we can use for recolouring v, and at least one of them is not the (new) colour of v and so we can change the colour of v to a colour already used, and we thus obtain a lid-colouring of G using $2(\Delta - 1)^2 + d$ colours. □

Corollary 1
If G is a graph with maximum degree $\Delta \geq 3$, then:

$$\chi_{lid}(G) \leq 2\Delta^2 - 3\Delta + 3.$$

Proof. If G is not connected, we colour the components independently. So we can assume that G is connected.

If G is not Δ-regular, then G is $(\Delta - 1)$-degenerate. Indeed, if we take any proper subset V' of vertices, and consider the subgraph induced by V', then if every vertex in this induced subgraph had degree Δ, there would be no edges between V' and $V(G) \setminus V'$, and therefore G would not be connected. So in this case we can directly apply Proposition 1 and the result is clear.

Assume now that G is Δ-regular. Let v be any vertex of G. As before, the graph $G \setminus \{v\}$ is $(\Delta - 1)$-degenerate and so, by Proposition 1, it has a lid-colouring with $2\Delta^2 - 3\Delta + 1$ colours. As before, we can recolour all the neighbours of v in such a way that the colouring remains locally identifying and such that if a neighbour of v has not the same closed neighbourhood as v, it has a neighbour with a colour different from all the colours of $N(v)$. For this step, there could be $2\Delta^2 - 3\Delta + 1$ forbidden colours, so we finally obtain a colouring of $G \setminus \{v\}$ with $2\Delta^2 - 3\Delta + 2$ colours. Then we assign a new colour to v and as before we can show that the colouring is locally identifying, leading to a locally identifying colouring with $2\Delta^2 - 3\Delta + 3$ colours. □

We now study the case of $\Delta = 2$:

Proposition 2
Let $n \geq 4$ be a positive integer. Let $\mathcal{C}_n$ be the cycle of order n. We have:

- $\chi_{lid}(\mathcal{C}_n) = 3$ *if* $n \equiv 0 \bmod 4$,
- $\chi_{lid}(\mathcal{C}_n) = 5$ *if* $n = 5$ *or* 7,
- $\chi_{lid}(\mathcal{C}_n) = 4$ *otherwise.*

As a consequence, any graph with maximum degree 2 has a locally identifying colouring with 5 *colours.*

Proof. Let v_0,..., v_{n-1} be the vertices of $\mathcal{C}_n$. We clearly have $\chi_{lid}(\mathcal{C}_n) \geq 3$.

We colour $\mathcal{C}_n$ with four colours using the following family of sequences described by the following word:

$$[124341232][42](1232)^*$$

for $n \geq 4$ and $n \neq 5, 7$. A sequence in bracket, $[M]$, means that we can take or not take the sequence M, the sequence $(M)^*$ means that we can repeat sequence M as many times as we need (or not use it at all). One can check that if we colour vertices of $\mathcal{C}_n$ with one of the sequence described by the previous word, we obtain a locally identifying colouring with three colours if $n \equiv 0 \bmod 4$ and with four colours otherwise.

If $n \not\equiv 0 \bmod 4$, then there is no locally identifying colouring with three colours. Indeed, if we try to colour the vertices of $\mathcal{C}_n$ with three colours there is no choice to do it and we must colour, without loss of generality: $c(v_i)$ with colour 1 if $i \equiv 0 \bmod 4$, with colour 2 if $i \equiv 1, 3 \bmod 4$ and with colour 3 if $i \equiv 2 \bmod 4$. But v_{n-1} must have colour 2, and v_{n-2} must have colour 3. Then $n - 2 \equiv 2 \bmod 4$ and so $n \equiv 0 \bmod 4$, a contradiction.

A case analysis shows that $\chi_{lid}(\mathcal{C}_5) = \chi_{lid}(\mathcal{C}_7) = 5$.

For the last part of the proposition, if G has maximum degree 2 then it is composed by connected components that are cycles or paths. One can easily check that a path has always a locally identifying colouring with four colours, and so we can colour each connected components of G independently with at most five colours. □

One can notice that in the locally identifying colouring of the cycle provided in the proof, only three colours are used an unbounded number of times whereas the other colours are used at most three times. In some sense, we can say that $\mathcal{C}_n$ has *almost* a locally identifying colouring with three colours.

3 Strong locally identifying colourings

In this section, we consider a variation of locally identifying colourings. We say that a colouring c is a *strong locally identifying colouring* (*slid-colouring* for short) if it is a locally identifying colouring and if for each vertex u, all the colours in $N[u]$ are different (the colouring is locally injective). In other words, a slid-colouring is a proper distance-two vertex colouring (each pair of vertices at distance at most 2 has different colours) without bad edges. We denote by $\chi_{slid}(G)$ the minimum number of colours required in any slid-colouring of G. Clearly, $\chi_{lid}(G) \leq \chi_{slid}(G)$. In a graph G with maximum degree Δ, each vertex has at most Δ^2 vertices at distance at most 2, and so there is always a proper distance-two vertex colouring with Δ^2+1 colours. We will adapt the proof of the previous section to show that adding the locally-identifying property in this case will not increase the asymptotic order of the bound. One can notice that the following lemma can also be applied in the locally identifying colouring case, leading in some cases to a better bound than the one of Recolouring lemma 1.

Lemma 2 (Recolouring lemma 2)
Let v be a vertex of degree d_1 of a graph G. Assume that v has d_2 vertices at distance exactly 2 and let c be a slid-colouring c with strictly more than $d_1 + 2d_2$ colours. Then, there is a list L of colours of size at most $d_1 + 2d_2$ such that if we change the colour of v for a colour not in L, the colouring remains a slid-colouring.

Proof. In this case we also need to put in L all the colours of vertices at distance 2 of v. We keep the same notations as before and we construct L as following:

1. For each vertex v_i, add to L colour $c(v_i)$ (at most d_1 colours are added).
2. For each vertex $u_{i,j}$, add to L colour $c(u_{i,j})$ (at most d_2 colours are added).
3. For each vertex $u_{i,j}$, if there is some colour in $c(N[u_{i,j}])$, but not already in L, add one of them to L. At this step we add at most d_2 colours.

In the end, L contains at most $d_1 + 2d_2$ colours. If we consider a new colouring c' from c where we change the colour of v to a colour not in L then clearly c' is still a colouring at distance 2. Furthermore, no edge becomes bad. Indeed, the only edges that could become bad would be of the form $v_iu_{i,j}$. There are two cases depending on whether $c(N[u_{i,j}])$ is included in $c(N[v_i])$ or not.

If $c(N[u_{i,j}]) \subset c(N[v_i])$, then there is a colour $c_0 \in c(N[v_i]) \setminus c(N[u_{i,j}])$. If c_0 was the colour $c(v)$, because $c'(v) \notin c(N[u_{i,j}]) \subset c(N[v_i])$, we have $c'(v) \in c'(N[v_i]) \setminus c'(N[u_{i,j}])$. Otherwise, we still have $c_0 \in c'(N[v_i]) \setminus c'(N[u_{i,j}])$.

Otherwise $c(N[u_{i,j}]) \setminus c(N[v_i])$ is not empty and during the construction of L, one colour of $c(N[u_{i,j}]) \setminus c(N[v_i])$ has been added to L, still separating v_i from $u_{i,j}$. □

As before, we can use this lemma to construct a slid-colouring of a graph G by induction. We first colour G when it is d-degenerate:

Proposition 3
Let G be a d-degenerate graph with maximum degree $\Delta \geq 2$, with $d < \Delta$.

$$\chi_{slid}(G) \leq (\Delta - 1)(2\Delta - 1) + 2d - 1.$$

Proof. The idea of the proof is quite similar to the proof of Proposition 1. We construct the colouring by induction. We choose a vertex v with minimum possible degree $t \leq d$. Then $G \setminus \{v\}$ has a slid-colouring with $(\Delta - 1)(2\Delta - 1) + 2d - 1$ colours. For each neighbour v_i of v which has a neighbour $u_{i,1}$ at distance 2 of v, we put the colour of $u_{i,1}$ in a list L'. We recolour each neighbour v_i of v in such a way that all the neighbours of v have different colours and no one of them has a colour in L'. To recolour v_i, there are at most $(\Delta - 1)(2\Delta - 1)$ forbidden colours from the lemma, at most $d - 1$ colours from the other neighbours of v and at most $d - 1$ forbidden colours from L' (if v_i has a neighbour $u_{1,1}$ then the colour of $u_{1,1}$ is already forbidden for v_i in the lemma). So at most $(\Delta - 1)(2\Delta - 1) + 2d - 2$ are forbidden but we have $(\Delta - 1)(2\Delta - 1) + 2d - 1$ colours, so at least one colour is free. After that, we give a completely new colour to v, obtaining a slid-colouring with $(\Delta - 1)(2\Delta - 1) + 2d$ colours, and by Lemma 2, we can change the colour of v to a colour already used and thus obtain a slid-colouring with $(\Delta - 1)(2\Delta - 1) + 2d - 1$ colours (for this last step, at least two colours are free so one of them is not the colour of v and so we can change the colour of v to a colour already used). □

Corollary 2
If a graph G has maximum degree Δ, then

$$\chi_{slid}(G) \leq 2\Delta^2 - \Delta + 1.$$

Proof. As before, we can assume that G is connected. If G has a vertex of degree $d < \Delta$, then it is $(\Delta - 1)$-degenerate and we have $\chi_{slid}(G) \leq 2\Delta^2 - \Delta - 2$. Otherwise, G is Δ-regular. If we remove one vertex v then $G \setminus \{v\}$ is $(\Delta - 1)$-degenerate and there is a slid-colouring with $2\Delta^2 - \Delta - 2$ colours. We recolour the neighbours of v as before, but here there are Δ neighbours so we will need $(\Delta - 1)(2\Delta - 1) + 2\Delta - 2 + 1 = 2\Delta^2 - \Delta$ colours. We complete the colouring by giving a completely new colour to v, thus obtaining a slid-colouring with $2\Delta^2 - \Delta + 1$ colours. □

We then consider the case of the cycle:

Proposition 4
Let $n \geq 4$ be a positive integer. Let $\mathcal{C}_n$ be the cycle of order n. We have:

- $\chi_{slid}(\mathcal{C}_n) = 4$ *if* $n \equiv 0 \bmod 4$,
- $\chi_{slid}(\mathcal{C}_n) = 6$ *if* $n = 6$ *or* 11,
- $\chi_{slid}(\mathcal{C}_7) = 7$,
- $\chi_{slid}(\mathcal{C}_n) = 5$ *otherwise.*

As a consequence, any graph with maximum degree 2 *has a slid-colouring with seven colours.*

Proof. A colouring of $\mathcal{C}_n$ is a slid-colouring if and only if every four consecutive vertices have different colours. Then it is clear that $\chi_{slid}(\mathcal{C}_n) = 4$ if and only if $n \equiv 0 \bmod 4$, $\chi_{slid}(\mathcal{C}_6) = 6$ and $\chi_{slid}(\mathcal{C}_7) = 7$.

If $n \equiv i \bmod 4$ ($i \neq 0$), then $\chi_{slid}(\mathcal{C}_n) = 5$ using the colouring described by the word $(12345)^i(1234)^*$, if $n \geq 5i$ (M^i means that we repeat the pattern M i times). It remains to consider the case $n = 11$. There is no slid-colouring of $\mathcal{C}_{11}$ with five colours, otherwise one colour would appear three times, and so two occurrences of it will be at distance less than 4. On the other side, 12345123456 is a slid-colouring of $\mathcal{C}_{11}$.

Clearly, a path has a slid-colouring with four colours and so any graph with maximum degree 2 has a slid-colouring with seven colours. □

Finally, we consider the class of chordal graphs. Chordal graphs are graphs where each induced cycle has size at most three. They belong to the class of perfect graphs. One of their properties (see [3]) is to admit a simplicial order of elimination for vertices: if G is a chordal graph, there is a vertex v whose neighbourhood is a clique (a *simplicial vertex*), and then $G \setminus \{v\}$ is still a chordal graph. For chordal graphs, we have $\omega(G) = \chi(G)$ where $\omega(G)$ is the *clique number* of G, i.e. the maximum size of a clique of G (see [3]). In [7], it is conjectured that $\chi_{lid}(G) \leq 2\chi(G)$, for any chordal graph G. We give a new bound of χ_{slid}, and so of χ_{lid}, for chordal graphs in terms of Δ and $\omega = \chi$, in the direction of the previous conjecture.

Proposition 5
Let G be a chordal graph and let $\omega = \omega(G)$. If $\omega \leq \frac{\Delta}{2} + 1$ then

$$\chi_{slid}(G) \leq 2\Delta\omega - 2\omega^2 + 5\omega - 2\Delta - 2.$$

Otherwise:

$$\chi_{slid}(G) \leq \frac{\Delta(\Delta+1)}{2} + 1 \leq 2\omega^2 - 7\omega + 7.$$

Proof. Let $\omega = \omega(G)$ and Δ be fixed. One can first notice that $M(\omega, \Delta) = \max_{1 \le d \le \omega - 1}(d(2\Delta - 2d + 1))$ is equal to $2\Delta\omega - 2\omega^2 + 5\omega - 2\Delta - 3$ if $\omega - 1 \le \frac{\Delta}{2}$ and to $\frac{\Delta(\Delta+1)}{2}$ otherwise. If $\omega - 1 > \frac{\Delta}{2}$, we clearly have $\frac{\Delta(\Delta+1)}{2} \le 2\omega^2 - 7\omega + 6$. Hence it is enough to prove that $\chi_{slid}(G) \le M(\omega, \Delta) + 1$.

We prove by induction on the number of vertices that any chordal graph with clique number at most ω and maximum degree at most Δ has a slid-colouring with $M(\omega, \Delta) + 1$ colours. It is clearly true for small graphs. Let G be a chordal graph with clique number at most ω and maximum degree at most Δ. Let v be a simplicial vertex of G. By induction, let c be a slid-colouring of $G \setminus \{v\}$ with $M(\omega, \Delta) + 1$ colours. Necessarily, all the vertices of $N(v)$ have different colours, and all the vertices at distance 2 of v have colours different from colours of $N(v)$ because they are at distance at most 2 of any vertex of $N(v)$. Let c' be the colouring of G extending c and giving to v a completely new colour. Then c' is a slid-colouring of G with $M(\omega, \Delta) + 2$ colours. Let $d \le \omega - 1$ be the degree of v. Then v has at most $d(\Delta - d)$ vertices at distance 2, and $d + 2d(\Delta - d) \le M(\omega, \Delta)$. By Lemma 2, we can recolour the vertex v with a colour already used and thus obtain a slid-colouring with $M(\omega, \Delta) + 1$ colours. □

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