Making sense of intransitivity, incompleteness and discontinuity of preferences

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Abstract The starting point of modern social choice theory is the assumption that is the individuals are endowed with complete and transitive preference relations over the set of alternatives. Over the past 60 years a steady flow of experimental results has suggested that people tend to deviate from principles of choice stemming from the utility maximization theory. Especially in choices under risk, this behaviour is quite common. More importantly, this behaviour makes intuitive sense. The usual culprit, i.e. the source of this "deviant" behaviour, is most often found in the violation of transitivity or – under risk – of the monotonicity in prizes principle. We show that there are grounds for arguing that even the completeness principle as well as continuity of preferences may, quite plausibly, be violated.

1 Introduction

When faced with a choice between two options, say x and y, it is in a way natural to choose x if one prefers x to y. If the preference is known not only to the chooser but also to another person observing the choice, it is unlikely that the latter person is puzzled by the choice. Once the preference is known no further information is needed to explain the choice behaviour or to make it intelligible. Choosing the preferred option can be viewed as utility maximization in a straight-forward sense: since the preferred option possesses higher value to the chooser (by definition), then the observed behaviour clearly amounts to maximizing the value (utility) to the chooser. Extending this principle to situations involving more than two options requires more conditions on preference relations than completeness that is implicitly assumed above: for any two options, either one is preferred to the other or the other way around.

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Obviously, if there is no preference, the observed choice behaviour cannot be seen as utility maximization. With three or more options, the assumption of complete preference relations is not enough to characterize choice behaviour as utility maximization: it may well be that x is preferred to y, y preferred to z and z preferred to x. Hence, whichever option is chosen, there is an option that is preferred to the chosen one. Hence, the utility value of the chosen option is not maximal. A way to salvage the maximization principle is to impose the condition of transitivity on preference relations: for any three alternatives x, y and z, if x is preferred to y and y is preferred to z, then it must be that x is preferred to z.

Completeness and transitivity of individual preference relations have become the standard assumptions in decision theory (von Neumann and Morgenstern 2007; Savage 1954). Indeed, under certainty they guarantee the existence of a utility function that represents individual preferences and render preference-consistent behaviour equivalent to utility maximization. Under risk and uncertainty similar representation theorems have been proven, each including completeness and transitivity among the conditions guaranteeing the utility maximization (see e.g. Harsanyi 1977).

From its early days the utility maximization view (UM view, for short) has been challenged by experimental and other empirical evidence suggesting that choice behaviour often deviates from the principles of UM view. Since the representation theorems are not empirical findings but mathematical truths, the source of UM violations has been sought in the principles imposed on preference relations. The earliest violations were observed in choice behaviour under risk, i.e. situations where the experimental subjects make choices among lotteries or risky prospects involving probability mixtures of payoffs. Allais conducted experiments in the 1950's showing that not only do the subjects often deviate from the principles of UM view, but they do it in a systematic manner (see Allais 1979). Somewhat later Kahneman and Tversky built a theory of choice, prospect theory, on the foundations of what they saw as systematic deviations from UM view. They were followed by other similar constructs that aim at making sense of UM deviant regularities in empirical choice behaviour (e.g. Gilboa and Schmeidler 2001; Machina 1982).

In the following we first give a brief overview of the main types of UM violations discussed in the literature. It turns out that most of them are related to choices under risk or uncertainty. Moreover, the explanation of these types of violations is usually sought in the violation of monotonicity in prizes of risky prospects. Our aim to show that violations make sense in simpler settings, viz. under certainty, where cyclic preferences can be expected to emerge in multi-criterion settings. Our main aim, however, is to show that UM view may fail in even simpler situations, viz. those involving only two alternatives. Since transitivity is not relevant in these circumstances, the culprit must be the completeness condition. We show by way of toy examples that under some circumstances it is plausible to expect that individual preference relations are not complete in the sense that an individual may quite plausible strictly prefer x to y and y to x. This could be viewed as a sort of explanation of the well-known preference reversal phenomenon (Lichtenstein and Slovic 1971).

2 A review of some UM violations under risk

The first serious attack on the UM theory was launched by Maurice Allais and carries nowadays the title of the Allais paradox. In his early experiments Allais confronted his subjects with the following pair of choices: (i) choose either r_1 or r_2, and (ii) choose either r_3 or r_4. All options except r_1 are risky. For example, r_2 is an option that results in payoff $5,000,000$ monetary units with probability 0.1, in payoff $1,000,000$ with probability 0.89 and in payoff 0 with probability 0.01.

$$r_1 = (1,000,000, 1.0)$$
$$r_2 = (5,000,000, 0.10; 1,000,000, 0.89; 0, 0.01)$$

$$r_3 = (5,000,000, 0.10; 0, 0.90)$$
$$r_4 = (1,000,000, 0.11; 0, 0.89)$$

Allais found that the majority of his subjects chose r_1 in (i) and r_3 in (ii). The majority choices contradict the UM theory regardless of the utility value assigned to the monetary values. To be more precise, the majority choice behaviour shows that they do not maximize the expected utility when choosing from risky prospects.

Some years later Ellsberg (1961) made somewhat similar observations. His setting, however, involves uncertainty, i.e. partially unknown probabilities of outcomes. The experimental subjects again make choices from two pairs of options: (i) either 1 or 2, and (ii) between 3 and 4. There are 90 balls in an urn. It is known that 30 of them are red, while the remaining 60 are either white or blue in unknown proportion. Option 1 gives the chooser \$100 if he draws a red ball from the urn, and nothing if the ball is either white or blue. Similarly for other options.

	colour (and number) of balls		
	red	white or blue (60)	
options	(30)	white	blue
1	\$100	\$0	\$0
2	\$0	\$0	\$100
3	\$100	\$100	\$0
4	\$0	\$100	\$100

Now, Ellsberg found that "[m]any people would choose 1 over 2, but 4 over 3. . . . [this] choice behaviour is clearly inconsistent with EU [expected utility] theory". Indeed, regardless of which utility values one assigns to payoffs, the type of behaviour cannot be of UM nature.

Strictly speaking, the experiments of Allais and Ellsberg do not address directly the completeness or transitivity assumptions of UM theory. Rather they purport to show – and, indeed, succeed in doing so – that the behaviour reported cannot be reconciled with one that ensues from EM *and* the assumption that people assign risky prospects utility values that are weighted averages of the utility values of the possible outcomes with weights equal to the probabilities of those outcomes. So, in principle it is possible that people do engage in UM, but resort to different utility calculus than the one envisaged in EU theory. Since in addition to completeness and transitivity only monotonicity in prizes is needed to render choice behaviour that follows preferences representable as EU maximizing (Harsanyi 1977), one of the three "axioms" (completeness, transitivity, monotonicity in prizes) has to the source of EU deviant behaviour. Most of the time since Allais' and Ellberg's experiments, the primary suspect has been the monotonicity in prizes condition, but transitivity was questioned as well.

A more direct way to assess the transitivity assumption is to ask the experimental subjects to make pairwise choices from a sequence of risky prospects. Tversky (1969) did just that. He confronted his subjects with the following sequence:

1. $(\$5.00, 7/24; \$0, 17/24)$
2. $(\$4.75, 8/24; \$0, 16/24)$
3. $(\$4.50, 9/24; \$0, 15/24)$
4. $(\$4.25, 10/24; \$0, 14/24)$
5. $(\$4.00, 11/24; \$0, 13/24)$

The expected values of payoffs increase from top to bottom (from value \$1.46 to \$1.83). The same is true for the probability of a non-zero payoff. Tversky found in his experiments that a sizable subgroup of his experimental subjects exhibited behavior whereby in adjacent pairwise choices, they preferred the prospect associated with higher maximum payoff (and smaller expected payoff), but in the comparison between the extreme prospects they preferred the one with the higher winning probability (and expected value). In other words, this group of individuals had a cyclic preference relation over risky prospects.

The preceding examples are but a (biased) sample of the vast literature that stemmed from comparing experimental observations with the theory of individual decision making. These examples have been chosen because in their context the term "paradox" has often been used. And for a good reason: not only do the observations deviate from the dictates of the theory, but those deviations seem to make intuitive sense. Hence, to the extent theory purports to portray rational behaviour, it seems that at least sometimes deviation from rationality makes sense. In what follows we argue that we do not need the risk or uncertainty modalities – as in the preceding examples – to end up in paradoxical choice situations. Consequently, we do not need to consider the specific conditions that pertain to risk and uncertainty modalities to end up with paradoxical yet plausible choice behaviour. Instead we may focus directly on transitivity and completeness conditions.

3 Intransitivity of preferences

Three universities A, B and C are being compared along three criteria: (i) research output (scholarly publications), (ii) teaching output (degrees), (iii) external impact (expert assignments, media visibility, R& D projects, etc.)

crit. (i)	crit. (ii)	crit. (iii)
A	B	C
B	C	A
C	A	B

Assuming that each criterion is a roughly equal importance, it is natural to form the overall preference relation between the universities on the basis of the majority rule: which one of any two universities is ranked higher than the other is preferred to the latter. In the present example this leads to a cycle: $A \succ B \succ C \succ A \succ \ldots$. Hence, an intransitive individual preference relations can be made intelligible by multiple criterion setting and majority principle (cf. Fishburn 1970;Bar-Hillel and Margalit 1988).[1]

4 Incompleteness of preferences

It is sometimes said that in social choice everything works nicely as long as the number of options is strictly less than three. The underlying idea then seems to be that the paradoxes begin with cyclic majorities. It can, however, be shown that voting paradoxes may be encountered in situations involving just two options. In what follows we consider two such paradoxes and provide a reinterpretation of them to show that in some situations it is entire plausible

[1] Nothing new is asserted here: the point has been made some 60 years ago by May (1954). In fact, already in 1930's some authors doubted the general plausibility of preference transitivity on the basis of its symmetric part, viz. the indifference relation. Aleskerov and Monjardet (2002, 4) and Mongin (2000) provide more extensive discussions and further references on this point.

to encounter incomplete preferences.[2] Thereafter we take another look at an important theorem of Baigent (1987) to show that under a wide class of choice situations using nearly any plausible choice rule leads to "unstable" choices (see also Baigent and Eckert 2004; Baigent and Klamler 2004; Eckert and Lane 2002).

4.1 Ostrogorski's paradox

A phenomenon known as Ostrogorski's paradox refers to the ambiguity in determining the popular preference among two alternatives (Daudt and Rae 1978). In the following we recast this paradox in an individual decision-making setting. The individual is to make a choice between two alternatives X and Y, e.g. candidates to a political office. There are three issues that are of primary importance for the office, say, foreign policy, social policy and educational policy. The individual uses 5 criteria in determining his/her favourite: relevant education (marked A, in the table), political experience on the issue (B), negotiation skills in the issue (C), substance expertise (D) and relevant political collaboration network (E). The following table indicates which candidate is preferable to the individual on each issue in terms of each criterion. Thus, e.g. candidate X has preferable (longer) experience in foreign policy than candidate Y.

issue	*issue 1*	*issue 2*	*issue 3*	*the criterion chooses*
crit. A	X	X	Y	X
crit. B	X	Y	X	X
crit. C	Y	X	X	X
crit. D	Y	Y	Y	Y
crit. E	Y	Y	Y	Y
issue-wise choice	Y	Y	Y	*overall choice* ?

Suppose now that the criterion-wise preference is formed on the basis of which alternative is better on more issues than the other. If all issues and criteria are deemed importance, the decision of which candidate the individual should vote is ambiguous: the row-column aggregation with the majority principle suggests X, but the column-row aggregation with the same principle yields Y. Thus, the preference over X and Y appears to exhibit incompleteness: on the basis of row-column aggregation Y cannot be preferred to X and on the basis of column-row aggregation X cannot be preferred to Y. Hence, there is no preference relation between X and Y.

[2] Again, no claim for novelty is made is here. In fact, Aumann (1962) not only suggests the possibility of incomplete preferences, but builds a theory of utility maximization without the completeness condition.

4.2 The exam paradox

The crux of Ostrogorski's paradox is the majority rule used in determining the "winners" of aggregation. A different type of rule is resorted to in a paradox, the exam paradox, that was introduced by Nermuth (1992). In the following we give it a somewhat different interpretation. Consider again an individual making a choice between two candidates of policy options, X and Y. The individual aims to pick the one that is closer to his/her ideal in issues $1, \ldots, 4$. X is located at the following distance from the voter's ideal point in a multi-dimensional space. the individual defines a total score of each alternative as the arithmetic mean of the issue-wise distances rounded to the to the nearest integer with values 0.5 rounded down to 0.

issue	1	2	3	4	average	score
criterion 1	1	1	2	2	1.5	1
criterion 2	1	1	2	2	1.5	1
criterion 3	1	1	2	2	1.5	1
criterion 4	2	2	3	3	2.5	2
criterion 5	2	2	3	3	2.5	2

X's competitor Y, in turn, is located in the space as follows.

issue	1	2	3	4	average	score
criterion 1	1	1	1	1	1.0	1
criterion 2	1	1	1	1	1.0	1
criterion 3	1	1	2	3	1.75	2
criterion 4	1	1	2	3	1.75	2
criterion 5	1	2	1	2	1.75	2

The score of X is smaller than that of Y suggesting that it is closer to the individual's ideal point. And yet, on 4 criteria out of 5 Y is closer to the individual's ideal point. As in Ostrogorski's paradox, there are good for arguing that incomplete preference relations can be quite plausible.

4.3 A reinterpretation of Baigent's theorem

Consider an individual making a choice from a set of alternatives using some criteria (cost, performance, ...). Suppose that the individual occasionally makes mistakes in applying the criteria. A plausible desideratum for an individual choice rule is that *mistakes involving a small number of criteria should not result in larger changes in chosen alternatives than mistakes involving larger number of criteria*. This desideratum rules out instances where decision situations that are very close to each other result in choice outcomes that are further apart than instances where the situations differ substantially. The desideratum is called proximity preservation.

Theorem 1. *(Baigent 1987; Eckert and Lane 2002; Baigent and Eckert 2004; Baigent and Klamler 2004): anonymity and respect for unanimity cannot be reconciled with proximity preservation.*

In other words:

- No matter what rule one uses in combining criterion values into choices (as long as it is anonymous and satisfies Pareto), the choices made in "very similar" circumstances can be further apart than those made in different circumstances.
- The choices – given criterion measurements – may occasionally appear "chaotic".
- The result holds under metric representations of distances between "profiles"
- It also holds under considerably weaker assumptions concerning distance measures (Eckert and Lane)

The theorem – when interpreted in the multiple-criterion choice context – does not challenge completeness or transitivity of individual preferences, but calls into question the continuity of preferences, i.e. their representation by smooth utility functions. In other words, whenever the labeling of criteria does not matter for determining the choice and the Pareto principle is adhered to, there are situations in which the continuity is violated.

5 Conclusion

In the title of his article, Mongin (2000) asks whether optimization implies rationality. My aim here has been related, but more modest, viz. to find out whether reasonable choices can be made – and defended – when the formal preconditions of optimization are absent. The preceding discussion is also somewhat related to reason-based rationality as understood by Dietrich and List (2013). The message of this paper is that even though the standard assumptions of the UM theory are often quite natural, it is not at all irrational to have intransitive, incomplete and/or discontinuous preference relations. In fact, it may be quite reasonable to have them. All that is called for is that the choice involves several criteria and that the alternatives are multidimensional. Under these circumstances incomplete and intransitive preference relations may emerge is a systematic manner that is consistent with the maximization principle that underlies rationality in the standard theory of choice. In fact, intransitive and discontinuous preferences may emerge in a single-dimensional setting as was shown above.

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