Received: 16 September 2022 | Accepted: 27 December 2022

DOI: 10.1002/mma.9042

RESEARCH ARTICLE

WILEY

Minimizers of abstract generalized Orlicz-bounded variation energy

Michela Eleuteri[1] | **Petteri Harjulehto**[2] | **Peter Hästö**[3]

[1]Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università degli Studi di Modena e Reggio Emilia, Italy

[2]Department of Mathematics and Statistics, FI-00014 University of Helsinki, Helsinki, Finland

[3]Department of Mathematics and Statistics, FI-20014 University of Turku, Turku, Finland

Correspondence
Peter Hästö, Department of Mathematics and Statistics, FI-20014 University of Turku, Turku, Finland.
Email: peter.hasto@utu.fi

Communicated by: C. Ji

Funding information
Jenny ja Antti Wihurin Rahasto; Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni(GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM); PRIN 2020 "Mathematics for industry 4.0 (Math4I4)"

A way to measure the lower growth rate of $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ is to require $t \mapsto \varphi(x, t)t^{-r}$ to be increasing in $(0, \infty)$. If this condition holds with $r = 1$, then

$$\inf_{u \in f + W_0^{1,\varphi}(\Omega)} \int_\Omega \varphi(x, |\nabla u|)\, dx$$

with boundary values $f \in W^{1,\varphi}(\Omega)$ does not necessarily have a minimizer. However, if φ is replaced by φ^p, then the growth condition holds with $r = p > 1$ and thus (under some additional conditions) the corresponding energy integral has a minimizer. We show that a sequence (u_p) of such minimizers converges when $p \to 1^+$ in a suitable BV-type space involving generalized Orlicz growth and obtain the Γ-convergence of functionals with fixed boundary values and of functionals with fidelity terms.

KEYWORDS

generalized bounded variation, generalized Orlicz space, minimizer, Musielak–Orlicz space, Γ-convergence

MSC CLASSIFICATION

35J60, 26B30, 35B40, 35J25, 46E35, 49J27, 49J45

1 | INTRODUCTION

Since its introduction in the early 1970s, Γ-convergence has gained more and more importance, being a very flexible instrument and the most natural notion of convergence for variational problems. Therefore, in the last decades, much literature has been devoted to the description of the asymptotic behavior of families of minimum problems, usually depending on some parameters, appearing in various contexts.

Γ-convergence is mostly applied to families of integral functionals, among which a key role is played by the p-Dirichlet integral,

$$\int_\Omega |\nabla u|^p\, dx,$$

as a prototype for integral functionals with standard growth. The case $p = 1$ deserves particular attention: Questions like semicontinuity and relaxation require the use of functions of bounded variation, BV(Ω), instead of the space $W^{1,1}(\Omega)$[1] unless some additional assumptions are made.[2] Recall that BV(Ω) is the Banach space of all $L^1(\Omega)$-functions whose first-order distributional derivatives are bounded Radon measures.

It is natural to ask if the functionals for $p \to 1^+$ are connected to the case $p = 1$. However, few results are presented in literature, at least to our knowledge. Juutinen[3] studied the problem for so-called *functions of least gradient* in $\mathrm{BV}(\Omega) \cap C(\bar\Omega)$ with given boundary values and continuous minimizers of the p-Dirichlet energy. More precisely, for suitable domains Ω, the sequence of the unique p-minimizers $u_p \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\bar\Omega)$ with boundary value $w \in C(\partial\Omega)$ converges uniformly, as $p \to 1^+$, to a function $u \in \mathrm{BV}(\Omega) \cap C(\bar\Omega)$ that is the unique function of least gradient with boundary value w. Apart from the continuity requirement, these variational problems represent the fundamental minimization problems in their respective spaces BV and $W^{1,p}$.

The concept of Γ-convergence, introduced by De Giorgi and Franzoni,[4] has been systematically presented in previous works.[5,6] Γ-convergence was not used by Juutinen,[3] but it seems reasonable to use it in this context, as well. A family of functionals $\mathcal{I}_\varepsilon : X \to \overline{\mathbb{R}}$ is said to Γ-converge (in topology τ) to $\mathcal{I} : X \to \overline{\mathbb{R}}$ if the following hold for every positive sequence (ε_i) converging to zero:

a. $\mathcal{I}(u) \leqslant \liminf_{i\to\infty} \mathcal{I}_{\varepsilon_i}(u_i)$ for every $u \in X$ and every $(u_i) \subset X$ τ-converging to u;
b. $\mathcal{I}(u) \geqslant \limsup_{i\to\infty} \mathcal{I}_{\varepsilon_i}(u_i)$ for every $u \in X$ and some $(u_i) \subset X$ τ-converging to u.

We will use this definition for $\mathcal{I}_{\varepsilon_i} = E_{1+\varepsilon_i}$; for simplicity, we denote $p_i = 1 + \varepsilon_i$ and talk about the Γ-convergence of E_{p_i} as $p_i \to 1^+$.

In this paper, we consider the following functionals with generalized Orlicz growth. This is a very active field recently, boosted by work on the double phase problem by Baroni et al.[7,8] The generalized Orlicz (also known as Musielak–Orlicz) case unifies the study of the double phase problem and the variable exponent growth, which has been intensively studied in the last 20 years.[9] We mention as examples the following very recent studies.[10–18] One motivating factor is applications to image processing initially proposed by Chen et al.[19]; see also previous studies.[20,21] In this context, one is especially interested in the minimization problem with lower growth equal to 1. Here, we study such functionals in an abstract and general setting.

To state our results, we use the notation from Section 2 and refer to Harjulehto and Hästö[22] for more background. Let φ be a weak Φ-function and $u_0 \in W^{1,\varphi}(\Omega)$ be the boundary value function. In Section 4, we study the functional $E_p : L^1(\Omega) \to [0,\infty]$ with fixed boundary values $u_0 \in W^{1,\varphi^r}(\Omega)$, where $r > 1$, for $p \in (1, r)$ defined by

$$E_p(u) := \begin{cases} \int_\Omega \varphi(x, |\nabla u|)^p \, dx & \text{when } u - u_0 \in L_0^{1,\varphi^p}(\Omega); \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

We define the limit functional $E : L^1(\Omega) \to [0,\infty]$ by

$$E(u) := \inf\left\{ \liminf_{i\to\infty} \int_\Omega \varphi(x, |\nabla u_i|)\, dx : u_i - u_0 \in L_0^{1,\varphi}(\Omega) \text{ and } u_i \to u \text{ in } L^1(\Omega) \right\}.$$

In Section 5, we consider corresponding functionals with a fidelity term. We assume this time that $f \in L^2(\Omega)$. For $p > 1$, we define $F_p : L^2(\Omega) \to [0,\infty]$ by

$$F_p(u) := \begin{cases} \int_\Omega \varphi(x, |\nabla u|)^p + |u - f|^2 \, dx & \text{when } u \in L^{1,\varphi^p}(\Omega) \cap L^2(\Omega); \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Then we define the limit functional $F : L^2(\Omega) \to [0,\infty]$ by

$$F(u) := \inf\left\{ \liminf_{i\to\infty} \int_\Omega \varphi(x, |\nabla u_i|) + |u_i - f|^2 \, dx : u_i \in L^{1,\varphi}(\Omega) \cap L^2(\Omega) \text{ and } u_i \to u \text{ in } L^2(\Omega) \right\}.$$

Our main result therefore is the following:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfy (A0) and (Dec).*

(1) E_p Γ-converges to E in the topology of $L^1(\Omega)$ as $p \to 1^+$.
(2) F_p Γ-converges to F in the topology of $L^2(\Omega)$ as $p \to 1^+$ if $C^\infty(\overline{\Omega})$ is dense in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$.

Due to the fidelity term in L^2, it seems natural to consider the latter convergence in $L^2(\Omega)$, although other choices are also possible. From the proofs, we see that assumption (A0) is only needed for the case $E(u) = \infty$ or $F(u) = \infty$; if we consider Γ-convergence on the space $X := \{u \in L^1(\Omega) : E(u) < \infty\}$ or $X := \{u \in L^2(\Omega) : F(u) < \infty\}$, then this assumption can be omitted.

The proof of Theorem 1.3(1) is presented in Section 4 along with results for minimizers of the energy E. The companion Section 5 covers Theorem 1.3(2) and minimizers of the energy F. The results for minimizers of the energies E and F generalize the corresponding ones in Juutinen.[3] Finally, in Section 6, we give explicit formulas for F in the important special cases $\varphi(x,t) = t + a(x)t^2$ (the double phase case) and $\varphi(x,t) = t^{p(x)}$ (the variable exponent case) based on previous works.[20,21,23]

We would like to remark that the limit energies E and F turn to be particular cases of the more general modular we introduce and study in Section 3; this new definition is inspired by the work of Miranda[24] and covers classical bounded variation spaces and Sobolev spaces.

2 | BACKGROUND

Throughout the paper we always consider a **bounded domain** $\Omega \subset \mathbb{R}^n$, that is, a bounded, open, and connected set. By $p' := \frac{p}{p-1}$, we denote the Hölder conjugate exponent of $p \in [1,\infty]$. The notation $f \lesssim g$ means that there exists a constant $c > 0$ such that $f \leqslant cg$. The notation $f \approx g$ means that $f \lesssim g \lesssim f$. By c, we denote a generic constant whose value may change between appearances. A function f is *almost increasing* if there exists $L \geqslant 1$ such that $f(s) \leqslant Lf(t)$ for all $s \leqslant t$ (more precisely, L-almost increasing). *Almost decreasing* is defined analogously. By *increasing*, we mean that this inequality holds for $L = 1$ (some call this non-decreasing), similarly for *decreasing*.

Definition 2.1. We say that $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ is a *weak Φ-function*, and write $\varphi \in \Phi_w(\Omega)$, if the following conditions hold for almost every $x \in \Omega$:

- For every measurable function $f : \Omega \to \mathbb{R}$, the function $y \mapsto \varphi(y, f(y))$ is measurable.
- The function $t \mapsto \varphi(x,t)$ is nondecreasing.
- $\varphi(x,0) = \lim_{t\to 0^+} \varphi(x,t) = 0$ and $\lim_{t\to\infty} \varphi(x,t) = \infty$.
- The function $t \mapsto \frac{\varphi(x,t)}{t}$ is L-almost increasing on $(0,\infty)$ with L independent of x.

If $\varphi \in \Phi_w(\Omega)$ is additionally convex and left-continuous with respect to t for almost every x, then φ is a *convex Φ-function*, and we write $\varphi \in \Phi_c(\Omega)$.

For $\varphi : \Omega \times [0,\infty) \to [0,\infty)$ and $p, q > 0$, we define some conditions:

(A0) There exists $\beta \in (0,1]$ such that $\varphi(x,\beta) \leqslant 1 \leqslant \varphi\left(x, \frac{1}{\beta}\right)$ for a.e. $x \in \Omega$.

$(\text{aInc})_p$ $t \mapsto \frac{\varphi(x,t)}{t^p}$ is L_p-almost increasing in $(0,\infty)$ for some $L_p \geqslant 1$ and a.e. $x \in \Omega$.

$(\text{aDec})_q$ $t \mapsto \frac{\varphi(x,t)}{t^q}$ is L_q-almost decreasing in $(0,\infty)$ for some $L_q \geqslant 1$ and a.e. $x \in \Omega$.

We say that (aInc) holds if $(\text{aInc})_p$ holds for some $p > 1$, and similarly for (aDec). We say that φ satisfies $(\text{Inc})_p$ if $(\text{aInc})_p$ holds with $L_p = 1$, similarly for $(\text{Dec})_q$.

If $\varphi \in \Phi_w(\Omega)$, then $\varphi(\cdot,1) \approx 1$ implies (A0), and if φ satisfies (aDec), then (A0) and $\varphi(\cdot,1) \approx 1$ are equivalent. For instance, $\varphi(x,t) = t^p$ always satisfies (A0), since $\varphi(x,1) \equiv 1$. Note that $\varphi \in \Phi_w(\Omega)$ includes the assumption that φ satisfies $(\text{aInc})_1$. Finally, (aInc) and (aDec) measure the lower and upper growth rates.

We recall some basic notions of generalized Orlicz spaces.[22]

Definition 2.2. Let $\varphi \in \Phi_w(\Omega)$ and $\varrho_\varphi(u) := \int_\Omega \varphi(x,|u|)\,dx$ for all measurable functions $u \in L^0(\Omega)$. The set

$$L^\varphi(\Omega) := \left\{u \in L^0(\Omega) : \varrho_\varphi(\lambda u) < \infty \text{ for some } \lambda > 0\right\}$$

is called a *generalized Orlicz space*. We define a (quasi-)norm on this space by

$$\|u\|_\varphi := \inf\left\{\lambda > 0 : \ \varrho_\varphi\left(\frac{u}{\lambda}\right) \leqslant 1\right\}.$$

We use the abbreviation $\|v\|_\varphi := \||v|\|_\varphi$ for vector-valued functions.

We will often use the following lower semicontinuity result, for which we require the stronger, convex Φ-function assumption.

Lemma 2.3 (Theorem 2.2.8[9]). *Let $\varphi \in \Phi_c(\Omega)$. If $u_i \rightharpoonup u$ in $L^\varphi(\Omega)$, then*

$$\int_\Omega \varphi(x, |u|)\,dx \leqslant \liminf_{i\to\infty} \int_\Omega \varphi(x, |u_i|)\,dx,$$

that is, the modular ϱ_φ is weakly (sequentially) lower semicontinuous.

3 | ABSTRACT BV-TYPE SPACES

We define a Sobolev-type space with function in L^1 and gradient in L^φ:

$$L^{1,\varphi}(\Omega) := \{u \in W^{1,1}(\Omega) : \|u\|_{L^{1,\varphi}(\Omega)} < \infty\} \text{ where } \|u\|_{L^{1,\varphi}(\Omega)} := \|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^\varphi(\Omega)}.$$

Note that $L^{1,\varphi}(\Omega) \subset W^{1,1}(\Omega)$, and $W^{1,\varphi}(\Omega) = L^{1,\varphi}(\Omega) \cap L^\varphi(\Omega)$. In the next definition and the rest of the article, we use the $^\circ$ symbol to indicate only the gradient part of a Sobolev-type norm and the $^-$ symbol to indicate the lim inf-approximation process.

Definition 3.1. For $\varphi \in \Phi_w(\Omega)$ and $u \in L^1(\Omega)$ and define

$$\|u\|_{\overline{\overset{\circ}{1,\varphi}}} := \inf\left\{\liminf_{i\to\infty} \|\nabla u_i\|_\varphi : u_i \in L^{1,\varphi}(\Omega) \text{ and } u_i \to u \text{ in } L^1(\Omega)\right\},$$

$$\bar\varrho^\circ_{1,\varphi}(u) := \inf\left\{\liminf_{i\to\infty} \varrho_\varphi(|\nabla u_i|) : u_i \in L^{1,\varphi}(\Omega) \text{ and } u_i \to u \text{ in } L^1(\Omega)\right\},$$

$$\bar\varrho_{1,\varphi}(u) := \varrho_1(u) + \bar\varrho^\circ_{1,\varphi}(u) \qquad \text{and} \qquad \bar L^{1,\varphi}(\Omega) := \{u \in L^1(\Omega) : \bar\varrho_{1,\varphi}(\lambda u) < \infty \text{ for some } \lambda > 0\}.$$

By a diagonal argument, we find, for every $u \in L^1(\Omega)$, functions $u_i \in L^{1,\varphi}(\Omega)$ with $u_i \to u$ in $L^1(\Omega)$ and

$$\bar\varrho^\circ_{1,\varphi}(u) = \lim_{i\to\infty} \varrho_\varphi(|\nabla u_i|).$$

We call this a test sequence or a sequence that gives $\bar\varrho^\circ_{1,\varphi}$. The same idea works for the norm $\|u\|_{\overline{\overset{\circ}{1,\varphi}}}$. Let us first motivate our definitions with two examples which show that we cover the spaces $W^{1,\varphi}(\Omega)$ and $\mathrm{BV}(\Omega)$.

Example 3.2 (Bounded variation spaces). If $\varphi(x,t) \equiv t$, then $\bar\varrho^\circ_{1,\varphi}(u) = V(u,\Omega)$, the total variation of u, for every $u \in \mathrm{BV}(\Omega)$. Indeed, by Evans[25, Theorem 5.3] or Ambrosio et al.,[26, Theorem 3.9] there exist $u_i \in C^\infty(\Omega)$ such that $u_i \to u$ in $L^1(\Omega)$ and

$$V(u,\Omega) = \lim_{i\to\infty} \int_\Omega |\nabla u_i|\,dx.$$

Thus, by the definition of $\bar\varrho^\circ_{1,\varphi}(u)$, we obtain that

$$\bar\varrho^\circ_{1,\varphi}(u) \leqslant \lim_{i\to\infty} \int_\Omega |\nabla u_i|\,dx = V(u,\Omega).$$

On the other hand, let (u_i) give $\bar\varrho^\circ_{1,\varphi}(u)$. Since $u_i \to u$ in $L^1(\Omega)$, we obtain by the lower semicontinuity of the variational measure[25, Theorem 5.2] that

$$V(u,\Omega) \leqslant \liminf_{i\to\infty} V(u_i,\Omega) = \liminf_{i\to\infty} \int_\Omega |\nabla u_i|\,dx = \bar\varrho^\circ_{1,\varphi}(u).$$

Example 3.3 (Sobolev spaces). If $\varphi \in \Phi_c(\Omega)$, $C_0^\infty(\Omega)$ is dense in the dual space $(L^\varphi(\Omega))^*$ and $u \in L^{1,\varphi}(\Omega)$, then $\bar\varrho^\circ_{1,\varphi}(u) = \varrho_\varphi(|\nabla u|)$. First of all, since $u \in L^{1,\varphi}(\Omega)$, we can choose the constant sequence $u_i = u$ so that $\bar\varrho^\circ_{1,\varphi}(u) \leqslant \varrho_\varphi(|\nabla u|)$ by the definition.

For the opposite inequality, we choose $u_i \in L^{1,\varphi}(\Omega)$ with $u_i \to u$ in $L^1(\Omega)$. Let $g \in C_0^\infty(\Omega;\mathbb{R}^n)$. By the definition of the weak derivative and the L^1-convergence,

$$\int_\Omega \nabla u \cdot g\, dx = -\int_\Omega u \operatorname{div} g\, dx = -\int_\Omega u_i \operatorname{div} g\, dx = \int_\Omega \nabla u_i \cdot g\, dx.$$

Since $C_0^\infty(\Omega)$ is dense in $(L^\varphi(\Omega))^*$, we obtain the claim for any $g \in (L^\varphi(\Omega;\mathbb{R}^n))^*$ (with component-wise approximation), so $\nabla u_i \rightharpoonup \nabla u$ and also $|\nabla u_i| \rightharpoonup |\nabla u|$ in L^φ. Thus, it follows from Lemma 2.3 that

$$\varrho_\varphi(|\nabla u|) \leqslant \liminf \varrho_\varphi(|\nabla u_i|).$$

Taking the infimum over such sequences (u_i), we obtain the opposite inequality, $\varrho_\varphi(|\nabla u|) \leqslant \bar{\varrho}^\circ_{1,\varphi}(u)$.

Let us then consider $\bar{\varrho}^\circ_{1,\varphi}$. Note that $u \mapsto \bar{\varrho}^\circ_{1,\varphi}(u)$ is not a semimodular on $L^1(\Omega)$ in the sense of Diening et al.[9, Definition 2.1.1] since $\bar{\varrho}^\circ_{1,\varphi}(c) = 0$ for every constant $c \in \mathbb{R}$. Indeed, as we saw in the example, $\bar{\varrho}^\circ_{1,\varphi}$ is like $\varrho_\varphi(|\nabla u|)$, so it is natural to add the modular of the function to it, otherwise we can only expect it to generate a seminorm, not a norm.

Lemma 3.4. *Let $\varphi \in \Phi_w(\Omega)$. Then $\bar{\varrho}_{1,\varphi}$ satisfies the following properties:*

(a) $\lambda \mapsto \bar{\varrho}_{1,\varphi}(\lambda u)$ *is nondecreasing for every* $u \in L^1(\Omega)$*;*
(b) $\bar{\varrho}_{1,\varphi}(0) = 0$*;*
(c) $\bar{\varrho}_{1,\varphi}(-u) = \bar{\varrho}_{1,\varphi}(u)$*;*
(d) there exists a constant $\beta > 0$ *such that*

$$\bar{\varrho}_{1,\varphi}(\beta(\theta u + (1-\theta)v)) \leqslant \theta\bar{\varrho}_{1,\varphi}(u) + (1-\theta)\bar{\varrho}_{1,\varphi}(v)$$

for every $u, v \in L^1(\Omega)$ *and for every* $\theta \in (0,1)$*;*
(e1) $\bar{\varrho}_{1,\varphi}(\lambda u) = 0$ *for all* $\lambda > 0$ *implies* $u = 0$.
If additionally φ *satisfies (aDec), then*
(e2) $\bar{\varrho}_{1,\varphi}(u) = 0$ *if and only only if* $u = 0$.

Moreover, if φ *is convex or left-continuous, then so is* $\bar{\varrho}_{1,\varphi}$.

Proof. We consider only $\bar{\varrho}^\circ_{1,\varphi}$ for the first four properties, since these properties are known for ϱ_φ and thus follow for the sum $\varrho_\varphi + \bar{\varrho}^\circ_{1,\varphi}$.

(a) Let $0 \leqslant \lambda_1 \leqslant \lambda_2$. It is enough to show that $\bar{\varrho}^\circ_{1,\varphi}(\lambda_1 u) \leqslant \bar{\varrho}^\circ_{1,\varphi}(\lambda_2 u)$. Let (v_i) be a sequence that gives $\bar{\varrho}^\circ_{1,\varphi}(\lambda_2 u)$. Then $\frac{\lambda_1}{\lambda_2} v_i \to \lambda_1 u$, and thus

$$\bar{\varrho}^\circ_{1,\varphi}(\lambda_1 u) \leqslant \lim_{i\to\infty} \varrho_\varphi(v_i) = \bar{\varrho}^\circ_{1,\varphi}(\lambda_2 u).$$

(b) If $u = 0$ a.e., then it can be approximated by the constant sequence (u), and hence $\bar{\varrho}^\circ_{1,\varphi}(u) = 0$.
(c) This follows directly from the definition of $\bar{\varrho}^\circ_{1,\varphi}$.
(d) Note that by Harjulehto and Hästö,[22, Corollary 2.2.2] there exists $\beta > 0$ such that $\varphi(\beta(\theta u+(1-\theta)v)) \leqslant \theta\varphi(u)+(1-\theta)\varphi(v)$. Let $u, v \in L^1(\Omega)$ and $\theta \in (0,1)$. Choose (u_i) and (v_i), which give $\bar{\varrho}^\circ_{1,\varphi}(u)$ and $\bar{\varrho}^\circ_{1,\varphi}(v)$. Then $\theta u_i+(1-\theta)v_i \to \theta u+(1-\theta)v$ in $L^1(\Omega)$. Hence, we obtain

$$\begin{aligned}
\bar{\varrho}^\circ_{1,\varphi}(\beta(\theta u + (1-\theta)v)) &\leqslant \liminf_{i\to\infty} \varrho_\varphi(\beta|\nabla(\theta u_i + (1-\theta)v_i)|) \\
&\leqslant \liminf_{i\to\infty} \int_\Omega \varphi(x, \beta\theta|\nabla u_i| + \beta(1-\theta)|\nabla v_i|)\, dx \\
&\leqslant \liminf_{i\to\infty} \int_\Omega \theta\varphi(x, |\nabla u_i|) + (1-\theta)\varphi(x, |\nabla v_i|)\, dx \\
&= \theta \lim_{i\to\infty} \varrho_\varphi(|\nabla u_i|) + (1-\theta) \lim_{i\to\infty} \varrho_\varphi(|\nabla v_i|) \\
&= \theta\bar{\varrho}^\circ_{1,\varphi}(u) + (1-\theta)\bar{\varrho}^\circ_{1,\varphi}(v),
\end{aligned}$$

where in the third inequality we used the quasiconvexity of φ.

(e1) If $\bar{\varrho}_{1,\varphi}(\lambda u) = 0$ for all $\lambda > 0$, then $\varrho_\varphi(\lambda u) = 0$ for all $\lambda > 0$. Since ϱ_φ is a semimodular, it follows that $u = 0$ a.e.; see Harjulehto and Hästö.[22, Lemma 3.2.2]

(e2) Assume that (aDec) holds. If $\bar{\varrho}_{1,\varphi}(u) = 0$, then $\varrho_\varphi(u) = 0$. Thus $\varphi(x, |u(x)|) = 0$ a.e., and (aDec) implies that $u = 0$ a.e.

If φ is convex, then we may choose $\beta = 1$ in (d), and hence $\bar{\varrho}_{1,\varphi}$ is convex.

Assume that φ is left-continuous. If $\lambda \in (0, 1)$, then $\bar{\varrho}^\circ_{1,\varphi}(\lambda u) \leqslant \bar{\varrho}^\circ_{1,\varphi}(u)$ by property (a). We next consider the opposite inequality. Let (λ_i) be a sequence converging to 1 from below. For every i we choose $u_i \in L^{1,\varphi}(\Omega)$ such that $\varrho_\varphi(|\nabla u_i|) \leqslant \bar{\varrho}^\circ_{1,\varphi}(\lambda_i u) + \frac{1}{i}$ and $\|\lambda_i u - u_i\|_1 < \frac{1}{i}$. Further, $\lambda_i u \to u$ in $L^1(\Omega)$. Let $\varepsilon > 0$ and choose $i_0 > \frac{2}{\epsilon}$ such that $\|u - \lambda_i u\|_1 < \frac{\varepsilon}{2}$ for all $i \geqslant i_0$. Then

$$\|u - u_i\|_1 \leqslant \|u - \lambda_i u\|_1 + \|\lambda_i u - u_i\|_1 < \varepsilon$$

for all $i \geqslant i_0$, and thus $u_i \to u$ in $L^1(\Omega)$. We obtain

$$\bar{\varrho}^\circ_{1,\varphi}(u) \leqslant \liminf_{i\to\infty} \varrho_\varphi(|\nabla u_i|) \leqslant \liminf_{i\to\infty} \left(\bar{\varrho}^\circ_{1,\varphi}(\lambda_i u) + \frac{1}{i} \right) = \liminf_{i\to\infty} \bar{\varrho}^\circ_{1,\varphi}(\lambda_i u).$$

Thus, $\bar{\varrho}^\circ_{1,\varphi}(u) = \lim_{i\to\infty} \bar{\varrho}^\circ_{1,\varphi}(\lambda_i u)$. □

Remark 3.5. Note that our conditions differ what Musielak[27, Definition 1.1, p. 1] calls a pseudomodular. He requires (b), (c), and

$$\varrho(\theta u + (1 - \theta)v) \leqslant \varrho(u) + \varrho(v)$$

for every u, v and for every $\theta \in (0, 1)$. However, our condition (d) is more useful when dealing with the Luxemburg norm.

We prove the lower semicontinuity of $\bar{\varrho}^\circ_{1,\varphi}$; note that we can handle Φ_w, since we assume strong rather than weak convergence, in contrast to Lemma 2.3

Lemma 3.6 (Lower semicontinuity of the modular). *Let $\varphi \in \Phi_w(\Omega)$, $u_i \in L^{1,\varphi}(\Omega)$ for $i = 1, 2, \dots$, $u \in L^1(\Omega)$, and $u_i \to u$ in $L^1(\Omega)$. Then*

$$\bar{\varrho}^\circ_{1,\varphi}(u) \leqslant \liminf_{i\to\infty} \bar{\varrho}^\circ_{1,\varphi}(u_i).$$

Moreover, if the limit inferior is finite, then $u \in \bar{L}^{1,\varphi}(\Omega)$.

Proof. By the definition of $\bar{\varrho}^\circ_{1,\varphi}(u_i)$, we can find $u_i' \in L^{1,\varphi}(\Omega)$ with

$$\varrho_\varphi(|\nabla u_i'|) \leqslant \bar{\varrho}^\circ_{1,\varphi}(u_i) + \frac{1}{i} \qquad \text{and} \qquad \|u_i - u_i'\|_1 \leqslant \frac{1}{i}.$$

If follows that $u_i' \to u$ in $L^1(\Omega)$. By the definition of $\bar{\varrho}^\circ_{1,\varphi}(u)$, we conclude that

$$\bar{\varrho}^\circ_{1,\varphi}(u) \leqslant \liminf_{i\to\infty} \varrho_\varphi(|\nabla u_i'|) \leqslant \liminf_{i\to\infty} \left(\bar{\varrho}^\circ_{1,\varphi}(u_i) + \frac{1}{i} \right) = \liminf_{i\to\infty} \bar{\varrho}^\circ_{1,\varphi}(u_i).$$

□

We define the Luxemburg (quasi-)norm in the usually way:

$$\|u\|_\varrho := \inf \left\{ \lambda > 0 : \varrho\left(\frac{u}{\lambda}\right) \leqslant 1 \right\}.$$

If φ is a convex semimodular then $u \mapsto \|u\|_\varrho$ is a norm by Musielak.[27, Theorem 1.5, p. 2] As in Harjulehto and Hästö,[28, Lemma 3.2.2] one can check that (a)–(d) from Lemma 3.4 imply that $u \mapsto \|u\|_\varrho$ is a quasi-seminorm.

We next compare $\|\cdot\|_{\bar{\varrho}^\circ_{1,\varphi}}$ induced by the modular via the preceding formula with $\|\cdot\|_{\overline{1,\varphi}}^\circ$ from Definition 3.1.

Proposition 3.7. *If $\varphi \in \Phi_w(\Omega)$, then $\|\cdot\|_{\overline{1,\varphi}}^\circ = \|\cdot\|_{\bar{\varrho}^\circ_{1,\varphi}}$.*

Proof. Let us first show that $\|\cdot\|_{1,\varphi}^{\overline{\circ}} \geqslant \|\cdot\|_{\bar{\varrho}_{1,\varphi}^{\circ}}$. If the left-hand side equals ∞, there is nothing to prove, and thus we may assume without loss of generality that $\|u\|_{1,\varphi}^{\overline{\circ}} < \infty$. Choose $u_i \in L^{1,\varphi}(\Omega)$ with $u_i \to u$ in $L^1(\Omega)$ and $\|\nabla u_i\|_{\varphi} \to \|u\|_{1,\varphi}^{\overline{\circ}}$. Define $\lambda_i := \|\nabla u_i\|_\varphi + \varepsilon$ and $\lambda := \|u\|_{1,\varphi}^{\overline{\circ}} + \varepsilon$ for $\varepsilon > 0$. Then $\|\nabla \frac{u_i}{\lambda_i}\|_\varphi < 1$ and so $\varrho_\varphi\left(|\nabla \frac{u_i}{\lambda_i}|\right) \leqslant 1$. Furthermore,

$$\left\|\frac{u}{\lambda} - \frac{u_i}{\lambda_i}\right\|_1 = \left\|\frac{u-u_i}{\lambda_i} + \left(\frac{1}{\lambda} - \frac{1}{\lambda_i}\right)u\right\|_1 \leqslant \frac{1}{\lambda_i}\|u-u_i\|_1 + \left(\frac{1}{\lambda} - \frac{1}{\lambda_i}\right)\|u\|_1 \to 0$$

so that $\frac{u_i}{\lambda_i} \to \frac{u}{\lambda}$ in $L^1(\Omega)$. Since $\left(\frac{u_i}{\lambda_i}\right)$ is a valid test sequence for $\bar{\varrho}_{1,\varphi}^{\circ}$, we conclude that

$$\bar{\varrho}_{1,\varphi}^{\circ}\left(\frac{u}{\lambda}\right) \leqslant \liminf_{i\to\infty} \varrho_\varphi\left(\left|\nabla \frac{u_i}{\lambda_i}\right|\right) \leqslant 1.$$

Hence, $\|u\|_{\bar{\varrho}_{1,\varphi}^{\circ}} \leqslant \lambda = \|u\|_{1,\varphi}^{\overline{\circ}} + \varepsilon$; the inequality follows as $\varepsilon \to 0^+$.

To prove $\|\cdot\|_{1,\varphi}^{\overline{\circ}} \leqslant \|\cdot\|_{\bar{\varrho}_{1,\varphi}^{\circ}}$, we may assume that $\|u\|_{\bar{\varrho}_{1,\varphi}^{\circ}} < \infty$. Since $\bar{\varrho}_{1,\varphi}^{\circ}\left(\frac{u}{\|u\|_{\bar{\varrho}_{1,\varphi}^{\circ}} + \varepsilon}\right) < 1$, we can choose $u_i \in L^{1,\varphi}(\Omega)$ with $\varrho_\varphi(|\nabla u_i|) \leqslant 1$ and $u_i \to \frac{u}{\|u\|_{\bar{\varrho}_{1,\varphi}^{\circ}} + \varepsilon}$ in L^1. Thus,

$$\left\|\frac{u}{\|u\|_{\bar{\varrho}_{1,\varphi}^{\circ}} + \varepsilon}\right\|_{1,\varphi}^{\overline{\circ}} \leqslant \liminf_{i\to\infty} \|\nabla u_i\|_\varphi \leqslant 1$$

and so $\|u\|_{1,\varphi}^{\overline{\circ}} \leqslant \|u\|_{\bar{\varrho}_{1,\varphi}^{\circ}} + \varepsilon$. The claim again follows as $\varepsilon \to 0^+$. □

4 | FUNCTIONALS WITH FIXED BOUNDARY VALUES

Let $\varphi \in \Phi_{\mathrm{w}}(\Omega)$. We say that $u \in L^{1,\varphi}(\Omega)$ belongs to $L_0^{1,\varphi}(\Omega)$ if there exists a $C_0^\infty(\Omega)$-sequence (ξ_i) such that $\xi_i \to u$ in $L^{1,\varphi}(\Omega)$. Note that $L_0^{1,\varphi}(\Omega) \subset W_0^{1,1}(\Omega)$.

Lemma 4.1. *Let $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfy (aInc) and (aDec). Then $L^{1,\varphi}(\Omega)$ and $L_0^{1,\varphi}(\Omega)$ are reflexive.*

Proof. Let $(u_i) \subset L^{1,\varphi}(\Omega)$ be a bounded sequence. Since $L^{1,\varphi}(\Omega) \subset \mathrm{BV}(\Omega)$, a subsequence, denoted again (u_i), converges in $L^1(\Omega)$ to some function $u \in \mathrm{BV}(\Omega)$. On the other hand, $(\partial_{x_k} u_i)$ is a bounded sequence in $L^\varphi(\Omega)$, which is a reflexive space.[22, Theorem 3.6.6] Hence, we can find $v_1, \dots, v_n \in L^\varphi(\Omega)$ such that $\partial_{x_k} u_i \rightharpoonup v_k$ in $L^\varphi(\Omega)$, up to a subsequence. So we need only prove that $\nabla u = (v_1, \dots, v_n)$. Let $k \in \{1, \dots, n\}$ and let $h \in C_0^\infty(\Omega)$ be a test function. We obtain by the previous convergence that

$$\int_\Omega u\,\partial_{x_k} h\, dx = \lim_{i\to\infty} \int_\Omega u_i\,\partial_{x_k} h\, dx = -\lim_{i\to\infty} \int_\Omega h\,\partial_{x_k} u_i\, dx = -\int_\Omega h v_k\, dx,$$

and thus $v_k = \partial_{x_k} u$.

The usual diagonal argument shows that $L_0^{1,\varphi}(\Omega)$ is a closed subspace of $L^{1,\varphi}(\Omega)$. Hence, it is also reflexive. □

We will study φ^p-energies when $p \geqslant 1$ is a constant. For that purpose, we note some properties of φ^p when $\varphi \in \Phi_{\mathrm{w}}(\Omega)$.

Lemma 4.2. *Let $p \geqslant 1$ and $\varphi \in \Phi_{\mathrm{w}}(\Omega)$. Then $\varphi^p \in \Phi_{\mathrm{w}}(\Omega)$ satisfies $(\mathrm{aInc})_p$ and, moreover,*

(1) if φ satisfies $(\mathrm{aDec})_q$, then φ^p satisfies $(\mathrm{aDec})_{qp}$;
(2) if φ is left-continuous, then so is φ^p;
(3) if φ satisfies (A0), then so does φ^p.

Proof. Since $t \mapsto t^p$ is continuous and increasing in $[0, \infty)$, the first three requirements of Definition 2.1 hold for φ^p. Since φ satisfies $(\mathrm{aInc})_1$ and $t \mapsto t^p$ is increasing, we obtain by $\frac{\varphi^p(x,t)}{t^p} = \left(\frac{\varphi(x,t)}{t}\right)^p$ that φ^p satisfies $(\mathrm{aInc})_p$.

(1) Assume that φ satisfies (aDec)$_q$. Then by $\frac{\varphi^p(x,t)}{t^{pq}} = \left(\frac{\varphi(x,t)}{t^q}\right)^p$, we see that φ^p satisfies (aDec)$_{pq}$.
(2) This claim follows since $t \mapsto t^p$ is continuous.
(3) It follows from (A0) that $\varphi(x,\beta)^p \leqslant 1 \leqslant \varphi\left(x, \frac{1}{\beta}\right)^p$, which is (A0) of φ^p.

□

Γ-convergence

Assume that $u_0 \in L^{1,\varphi^r}(\Omega) \cap L^1(\Omega)$ for some $r > 1$. For $p \in (1, r)$, we consider $E_p : L^1(\Omega) \to [0, \infty]$ and $E : L^1(\Omega) \to [0, \infty]$ from the introduction; see (1.1). Note that E is a variant of $\bar{\varrho}^{\circ}_{1,\varphi}$ where we have fixed the boundary values.

As we mentioned in the introduction, we want to show that E_{p_i} Γ-converges to E in the topology of $L^1(\Omega)$ as $p_i \to 1^+$. We do so by first considering the Γ-liminf inequality and then finding the recovery sequence in Lemma 4.4. This establishes the two properties of Γ-convergence and completes the proof of Theorem 1.3(1).

Lemma 4.3. *Let $\varphi \in \Phi_w(\Omega)$. Assume that $u_0 \in L^{1,\varphi^r}(\Omega)$ for some $r > 1$. Let $u \in L^1(\Omega)$, $u_i \to u$ in $L^1(\Omega)$ and $p_i \to 1^+$. Then*

$$E(u) \leqslant \liminf_{i\to\infty} E_{p_i}(u_i).$$

Proof. If $K := \liminf_{i\to\infty} E_{p_i}(u_i) = \infty$, then there is nothing to prove, so we assume that $K < \infty$. We restrict our attention to a subsequence with $\lim_{i\to\infty} E_{p_i}(u_i) = K$ and $u_i - u_0 \in L_0^{1,\varphi^{p_i}}(\Omega)$. Since $u_i \to u$ in $L^1(\Omega)$, we obtain by the definition of $E(u)$ that

$$E(u) \leqslant \liminf_{i\to\infty} \int_\Omega \varphi(x, |\nabla u_i|)\,dx.$$

From Young's inequality, we conclude that $ab \leqslant a^p + (p-1)\left(\frac{b}{p}\right)^{p'}$. Using this with $a = \varphi(x, |\nabla u_i|)$, $b = 1$ and $p = p_i$, we continue the previous estimate by

$$E(u) \leqslant \liminf_{i\to\infty} \left(\int_\Omega \varphi^{p_i}(x, |\nabla u_i|)\,dx + |\Omega|(p_i - 1)p_i^{-p_i'} \right) = \liminf_{i\to\infty} \int_\Omega \varphi^{p_i}(x, |\nabla u_i|)\,dx,$$

where the equality holds since $(p_i - 1)p_i^{-p_i'} \to 0 \cdot \frac{1}{e} = 0$ as $p_i \to 1^+$. □

We now turn our attention to the Γ-limsup property. For this part, we need to assume (Dec). Recall that (aDec) implies (Dec) if φ is convex.[28, Lemma 2.2.6]

Lemma 4.4. *Assume that $\varphi \in \Phi_w(\Omega)$ satisfies (A0) and (Dec). Let $u \in L^1(\Omega)$ and $u_0 \in L^{1,\varphi^r}(\Omega)$ for some $r > 1$. For every $p_i \to 1^+$, there exist $u_i \in L^{1,\varphi}(\Omega)$ such that $u_i - u_0 \in L_0^{1,\varphi}(\Omega)$, $u_i \to u$ in $L^1(\Omega)$ and*

$$E(u) \geqslant \limsup_{i\to\infty} E_{p_i}(u_i).$$

Proof. If $E(u) = \infty$, then any approximating sequence in $u_0 + C_0^\infty(\Omega)$ works,[22, Corollary 3.7.10] so we assume that $E(u) < \infty$. By the definition of $E(u)$, there exists a sequence (v_k) from $u_0 + L_0^{1,\varphi}(\Omega)$ such that $v_k \to u$ in $L^1(\Omega)$ and

$$E(u) = \lim_{k\to\infty} \int_\Omega \varphi(x, |\nabla v_k|)\,dx.$$

Since $v_k - u_0 \in L_0^{1,\varphi}(\Omega)$, there exists a sequence (ζ_j^k) from $C_0^\infty(\Omega)$ with $u_0 + \zeta_j^k \to v_k$ in $L^{1,\varphi}(\Omega)$. Since φ satisfies (Dec), this yields by Lemma 3.1.6 of Harjulehto and Hästö[22] that

$$\int_\Omega \varphi(x, |\nabla v_k|)\,dx = \lim_{j\to\infty} \int_\Omega \varphi(x, |\nabla(u_0 + \zeta_j^k)|)\,dx.$$

Thus, by the usual diagonal argument, we can choose $\xi_j := \zeta^j_{k_j}$ such that $u_0 + \xi_j \to u$ in $L^1(\Omega)$ and

$$E(u) = \lim_{j\to\infty} \int_\Omega \varphi(x, |\nabla(u_0 + \xi_j)|)\, dx.$$

By Hölder's inequality with exponent $q > 1$, we obtain

$$\int_\Omega \varphi(x, |\nabla(u_0 + \xi_j)|)^{\frac{1}{q}+\frac{r}{q'}}\, dx \leqslant \left(\int_\Omega \varphi(x, |\nabla(u_0 + \xi_j)|)\, dx\right)^{\frac{1}{q}} \left(\int_\Omega \varphi(x, |\nabla(u_0 + \xi_j)|)^r\, dx\right)^{\frac{1}{q'}}.$$

The second integral is finite since $u_0 + \xi_j \in W^{\varphi^r}(\Omega)$. Thus, we may choose $\overline{q_j} \in \left(1, 1 + \frac{1}{j}\right)$ close enough to 1 that the second factor on the right-hand side is at most $1 + \frac{1}{j}$ for all $q \in [1, \overline{q_j}]$. Since $p_i \to 1^+$, there exists $i_j \geqslant j$ such that $q_i := \frac{r-1}{r-p_i} \leqslant \overline{q_j}$ for all $i \geqslant i_j$. Note that $\frac{1}{q_i} + \frac{r}{q_i'} = p_i$. We now choose $u_i := u_0 + \xi_j$ for all $i \in [\max\{i_1, \dots, i_j\}, i_{j+1})$. Then

$$\int_\Omega \varphi(x, |\nabla u_i|)^{p_i}\, dx \leqslant \left(1 + \frac{1}{j(i)}\right) \left(\int_\Omega \varphi(x, |\nabla u_i|)\, dx\right)^{\frac{1}{q_i}},$$

where $j(i) \to \infty$ as $i \to \infty$ since $\overline{q_j} > 1$ so only finitely many values of i can have $q_i > \overline{q_j}$.

Next, we combine all the above estimates and obtain

$$\begin{aligned} E(u) = \lim_{i\to\infty} \int_\Omega \varphi(x, |\nabla u_i|)\, dx &\geqslant \limsup_{i\to\infty} \left(1 + \frac{1}{j(i)}\right)^{-q_i} \left(\int_\Omega \varphi(x, |\nabla u_i|)^{p_i}\, dx\right)^{q_i} \\ &= \limsup_{i\to\infty} \int_\Omega \varphi(x, |\nabla u_i|)^{p_i}\, dx = \limsup_{i\to\infty} E_{p_i}(u_i), \end{aligned}$$

where the last equality follows since $u_i - u_0 = \xi_j \in L^{1,\varphi}_0(\Omega)$. □

A sequence of minimizers

We continue by establishing some results about sequences of minimizers of the functionals E_p from (1.1). These results are in the spirit of Juutinen.[3]

Lemma 4.5. *Let $\varphi \in \Phi_c(\Omega)$ satisfy (A0) and (aDec), $p > 1$ and $u_0 \in L^{1,\varphi^p}(\Omega)$. Then there exists a unique function $u_p \in L^{1,\varphi^p}(\Omega)$ which satisfies $u_p - u_0 \in L^{1,\varphi^p}_0(\Omega)$ and*

$$E_p(u_p) = \inf_{u\in L^1(\Omega)} E_p(u).$$

Proof. Let (u_j) be a minimizing sequence of E_p. Since

$$\inf_{u\in L^1(\Omega)} E_p(u) \leqslant E_p(u_0) = \int_\Omega \varphi(x, |\nabla u_0|)^p\, dx < \infty,$$

we may assume that $(|\nabla u_j|)$ is bounded in $L^{\varphi^p}(\Omega)$. By the Poincaré inequality in $W^{1,1}_0(\Omega)$, we obtain that $(u_j - u_0)$ is bounded in $L^1(\Omega)$:

$$\|u_j - u_0\|_1 \lesssim \|\nabla(u_j - u_0)\|_1 \lesssim \|\nabla u_j\|_{\varphi^p} + \|\nabla u_0\|_{\varphi^p},$$

where the embedding in the last inequality holds since φ^p satisfies (A0) by Lemma 4.2. Since φ^p satisfies (aInc) and (aDec), $L^{1,\varphi^p}_0(\Omega)$ is reflexive, by Lemma 4.1. Thus there exists a subsequence $(u_{j_k} - u_0)$ and $u \in L^{1,\varphi}_0(\Omega)$ such that $\nabla(u_{j_k} - u_0) \rightharpoonup \nabla u$ in $L^{\varphi^p}(\Omega; \mathbb{R}^n)$. Set $u_p := u + u_0$.Since $\varphi \in \Phi_c(\Omega)$, Lemma 2.3 implies that

$$E_p(u_p) \leqslant \liminf_{k\to\infty} E_p(u_{j_k}).$$

Thus, u_p is a minimizer of E_p.

Since φ is convex and $t \mapsto t^p$ strictly convex, we obtain that $t \mapsto \varphi(x,t)^p$ is strictly convex. The usual argument yields uniqueness, namely, if u and v are distinct minimizers, then we obtain a contradiction from $E_p\left(\frac{u+v}{2}\right) < \frac{1}{2}(E_p(u) + E_p(v))$. □

Take a sequence (p_j) such that $p_j \to 1^+$ and a sequence of minimizers (u_j), where u_j is the minimizer of E_{p_j}. Let us also assume that the φ^{p_j}-modular of u_0 is finite for every j. Then the limit function exists in $\bar{L}^{1,\varphi}(\Omega)$.

Lemma 4.6. *Let $p_j \to 1^+$ and $\varphi \in \Phi_w(\Omega)$. Assume that $\varrho_{\varphi^{p_j}}(u_0) \leqslant K$ for every j. Let (u_j) be a sequence of minimizers of E_{p_j} with boundary value u_0. Then there exist $u \in \bar{L}^{1,\varphi}(\Omega)$ and a subsequence (u_{j_i}) converging to u in $L^1(\Omega)$ with*

$$E(u) \leqslant \liminf_{i\to\infty} E_{p_{j_i}}(u_{j_i}).$$

Proof. Since u_j is a minimizer, we see as in the previous lemma that $E_{p_j}(u_j) \leqslant K$. By $L^{\varphi^{p_1}}(\Omega) \hookrightarrow L^{\varphi}(\Omega)$ and $\|u\|_{\varphi} \leqslant \varrho_{\varphi}(u) + 1$, we conclude that $\|\nabla u_j\|_{\varphi}$ is bounded. Since $u_j - u_0 \in W_0^{1,1}(\Omega)$, we obtain as in the previous lemma by the Poincaré inequality that $\|u_j - u_0\|_1 \lesssim 1 + \|\nabla u_0\|_{\varphi^{p_1}}$. Thus, $(u_j - u_0)$ is bounded in $W_0^{1,1}(\Omega)$. Since $W_0^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$, there exists a subsequence (u_{j_i}) converging to u in $L^1(\Omega)$. The claim follows from the first property of Γ-convergence by Theorem 1.3(1). Finally, Lemma 3.6 yields that $u \in \bar{L}^{1,\varphi}(\Omega)$. □

Proposition 4.7. *Let $\varphi \in \Phi_w(\mathbb{R}^n)$ and $\varrho_{\varphi^{p_j}}(u_0) \leqslant K$ for every j. The function $u \in \bar{L}^{1,\varphi}(\Omega)$ from Lemma 4.6 minimizes the E-energy, that is,*

$$E(u) = \inf_{v\in L^1(\Omega)} E(v).$$

Proof. Assume by contradiction that there exists $v \in L^1(\Omega)$ such that $E(v) < E(u)$. Let $p_i \to 1^+$. By the second property of Γ-convergence from Theorem 1.3(1), there exists a sequence (v_i) such that $v_i - u_0 \in L_0^{1,\varphi}(\Omega)$, $v_i \to v$ in $L^1(\Omega)$ and

$$E(v) \geqslant \limsup_{i\to\infty} E_{p_i}(v_i).$$

By the assumption and Lemma 4.6, we have

$$\limsup_{i\to\infty} E_{p_i}(v_i) \leqslant E(v) < E(u) \leqslant \liminf_{j\to\infty} E_{p_{i_j}}(u_{i_j}).$$

Let $\varepsilon := \liminf_{j\to\infty} E_{p_{i_j}}(u_{i_j}) - \limsup_{i\to\infty} E_{p_i}$, and choose M so large that

$$E_{p_i}(v_i) < \limsup_{i\to\infty} E_{p_i}(v_i) + \frac{\varepsilon}{2} \text{ and } E_{p_{i_j}}(v_{i_j}) > \liminf_{j\to\infty} E_{p_{i_j}}(u_{i_j}) - \frac{\varepsilon}{2}$$

as $i, j \geqslant M$. Thus, for all $i, j \geqslant M$, we have

$$E_{p_i}(v_i) < E_{p_{i_j}}(u_{i_j}).$$

Then we just choose $j \geqslant M$ with $i_j \geqslant M$ and obtain

$$E_{p_{i_j}}(v_{i_j}) < E_{p_{i_j}}(u_{i_j}).$$

But this contradicts that u_{i_j} minimizes the $E_{p_{i_j}}$-energy with the boundary values u_0, so the assumption $E(v) < E(u)$ was wrong and the claim is proved. □

5 | FUNCTIONALS WITH THE FIDELITY TERM

Assume then that $f \in L^2(\Omega)$. For $p > 1$, we defined $F_p : L^2(\Omega) \to [0,\infty]$ and $F : L^2(\Omega) \to [0,\infty]$ in the introduction; see (1.2). Note that F is another a variant of $\bar{\varrho}^{\circ}_{1,\varphi}$, this time including a "fidelity term" in L^2. Since this implies that $u \in L^2(\Omega)$, it seems natural to consider the convergence in $L^2(\Omega)$, although convergence in $L^1(\Omega)$ is also a possibility.

Γ-convergence

We want to show that F_{p_i} Γ-converges to F in the topology of $L^2(\Omega)$ as $p_i \to 1^+$, i.e. to prove Theorem 1.3(2). We do so by first considering the Γ-liminf inequality, and then finding the recovery sequence in Lemma 5.2.

Lemma 5.1. *Let $u \in L^2(\Omega)$ and assume that (u_i) is a sequence of L^2-functions converging to u in $L^2(\Omega)$ and that $p_i \to 1^+$. Then*

$$F(u) \leqslant \liminf_{i\to\infty} F_{p_i}(u_i).$$

Proof. If $K := \liminf_{i\to\infty} F_{p_i}(u_i) = \infty$, then there is nothing to prove, so we assume that $K < \infty$. We restrict our attention to a subsequence with $\lim_{i\to\infty} F_{p_i}(u_i) = K$ and $u_i \in L^{1,\varphi^{p_i}}(\Omega) \cap L^2(\Omega)$. Since $u_i \to u$ in $L^2(\Omega)$, we obtain by the definition of $F(u)$ that

$$F(u) \leqslant \liminf_{i\to\infty} \left(\int_\Omega \varphi(x, |\nabla u_i|) + |u_i - f|^2 \, dx \right).$$

From Young's inequality, we conclude that $ab \leqslant a^p + (p-1)\left(\frac{b}{p}\right)^{p'}$. Using this with $a = \varphi(x, |\nabla u_i|)$, $b = 1$ and $p = p_i$, we continue the previous estimate by

$$\begin{aligned} F(u) &\leqslant \liminf_{i\to\infty} \left(\int_\Omega \varphi^{p_i}(x, |\nabla u_i|) \, dx + |\Omega|(p_i - 1)p_i^{-p_i'} + \int_\Omega |u_i - f|^2 \, dx \right) \\ &= \liminf_{i\to\infty} \left(\int_\Omega \varphi^{p_i}(x, |\nabla u_i|) \, dx + \int_\Omega |u_i - f|^2 \, dx \right), \end{aligned}$$

where the equality holds since $(p_i - 1)p_i^{-p_i'} \to 0 \cdot \frac{1}{e} = 0$ as $p_i \to 1^+$. □

We now turn our attention to the Γ-limsup property. Note that in contrast to Lemma 4.4, we now have an assumption regarding density of smooth functions. The difference comes from the fact that $L_0^{1,\varphi}$ is defined as the completion of C_0^∞-functions, so smooth functions are automatically dense in that setting. We refer to Juusti[29, Corollary 4.4] for a sufficient condition for the density of $C^\infty(\overline{\Omega})$ in $W^{1,\varphi}(\Omega)$ where Ω is an (ε, ∞)-domain.

Lemma 5.2. *Assume that $\varphi \in \Phi_w(\Omega)$ satisfies (A0) and (Dec) and that $C^\infty(\overline{\Omega})$ is dense in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$. For every $u \in L^2(\Omega)$ and $p_i \to 1^+$, there exist $u_i \in L^{1,\varphi}(\Omega) \cap L^2(\Omega)$ such that $u_i \to u$ in $L^2(\Omega)$ and*

$$F(u) \geqslant \limsup_{i\to\infty} F_{p_i}(u_i).$$

Proof. If $F(u) = \infty$, any approximating sequence in $C_0^\infty(\Omega)$ works,[22, Corollary 3.7.10] so we assume that $F(u) < \infty$. Thus, there exists a sequence (v_i) from $L^{1,\varphi}(\Omega)$ such that $v_i \to u$ in $L^2(\Omega)$ and

$$F(u) = \lim_{i\to\infty} \int_\Omega \varphi(x, |\nabla v_i|) + |v_i - f|^2 \, dx.$$

Using the density assumption, we can find $\xi_j^i \in C^\infty(\overline{\Omega})$ with $\xi_j^i \to v_i$ in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$. Then by a diagonal argument and Harjulehto and Hästö[22, Lemma 3.1.6] we find $u_i := \xi_{j_i}^i \in C^\infty(\overline{\Omega})$ with $u_i \to u$ in $L^2(\Omega)$ and

$$F(u) = \lim_{i\to\infty} \int_\Omega \varphi(x, |\nabla u_i|) + |u_i - f|^2 \, dx.$$

The proof is concluded in the same way as Lemma 4.4. □

A sequence of minimizers

Lemma 5.3. *Let $\varphi \in \Phi_c(\Omega)$ satisfy (aDec), $p > 1$ and $f \in L^2(\Omega)$. Then there exists a unique function $u_p \in L^{1,\varphi^p}(\Omega) \cap L^2(\Omega)$ which satisfies*

$$F_p(u_p) = \inf_{u \in L^2(\Omega)} F_p(u).$$

Proof. Let (u_j) be a minimizing sequence of F_p. Since

$$\inf_{u \in L^{1,\varphi^p}(\Omega) \cap L^2(\Omega)} F_p(u) \leqslant F_p(0) = \int_\Omega \varphi(x,0)^p + |0 - f|^2 \, dx < \infty,$$

we may assume that $(|\nabla u_j|)$ is bounded in $L^{\varphi^p}(\Omega)$ and (u_j) is bounded in $L^2(\Omega)$. Note that φ^p satisfies (aInc) and (aDec) by Lemma 4.2. Thus, $L^2(\Omega)$ and $L^{1,\varphi}(\Omega)$ are reflexive by Lemma 4.1, and hence, there exists a subsequence (u_{j_k}) and $u \in L^{1,\varphi^p}(\Omega) \cap L^2(\Omega)$ such that $\nabla u_{j_k} \rightharpoonup \nabla u$ in $L^{\varphi^p}(\Omega; \mathbb{R}^n)$ and $u_{j_k} \rightharpoonup u$ in $L^2(\Omega)$. Thus, by weak lower semicontinuity (Lemma 2.3), we have

$$F_p(u) \leqslant \liminf_{k \to \infty} F_p(u_{j_k}).$$

Thus, u is a minimizer. Uniqueness is proved as in Lemma 4.5. □

Let us take a sequence (p_j) such that $p_j \to 1^+$ and a sequence of minimizers (u_j), where u_j is the minimizer of F_{p_j}. Then the limit function exists in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$.

Lemma 5.4. *Let $\varphi \in \Phi_c(\Omega)$. Assume that $f \in L^2(\Omega)$. Let $p_j \to 1^+$ and let (u_j) be a sequence of minimizers of F_{p_j}. Then there exist $u \in \bar{L}^{1,\varphi}(\Omega) \cap L^2(\Omega)$ and a subsequence (u_{j_i}) converging to u in $L^2(\Omega)$ with*

$$F(u) \leqslant \liminf_{i \to \infty} F_{p_{j_i}}(u_{j_i}).$$

Proof. We take a subsequence that gives $\liminf_{i\to\infty} F_{p_{j_i}}(u_{j_i})$ and denote this subsequence by (u_j). Since u_j is a minimizer of F_{p_j},

$$F_{p_j}(u_j) \leqslant \int_\Omega \varphi(x,0)^{p_j} + |0 - f|^2 \, dx = \int_\Omega |f|^2 \, dx < \infty.$$

Thus, by $L^{\varphi^p}(\Omega) \hookrightarrow L^{\varphi}(\Omega)$, we have that $\|\nabla u_j\|_\varphi + \|u_j - f\|_2$ is bounded, and by $f \in L^2(\Omega)$, we obtain that $\|u_j\|_2$ is bounded. Since $L^2(\Omega)$ is reflexive, there exists a subsequence, denoted again by (u_j), and $u \in L^2(\Omega)$ such that $u_j \rightharpoonup u$ in $L^2(\Omega)$.

By the Banach–Saks theorem, $\frac{1}{k}\sum_{j=1}^k u_j \to u$ in $L^2(\Omega)$. We then use the definition of F and the convexity of φ and $t \mapsto t^2$ and obtain that

$$\begin{aligned} F(u) &\leqslant \liminf_{k\to\infty} F\left(\frac{1}{k}\sum_{j=1}^k u_j\right) = \liminf_{k\to\infty} \int_\Omega \varphi\left(x, \frac{1}{k}\sum_{j=1}^k |\nabla u_j|\right) + \left|\frac{1}{k}\sum_{j=1}^k u_j - f\right|^2 dx \\ &\leqslant \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^k \int_\Omega \varphi(x, |\nabla u_j|) + |u_j - f|^2 \, dx. \end{aligned}$$

Arguing as in Lemma 5.1, we continue

$$F(u) \leqslant \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^k \int_\Omega \varphi(x, |\nabla u_j|)^{p_j} + |u_j - f|^2 + (p_j - 1)p_j^{-p_j'} \, dx = \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^k F_{p_j}(u_j).$$

It is well known that if a sequence (z_j) converges to z, so does $\left(\frac{1}{k}\sum_{j=1}^k z_j\right)$. Thus, we obtain $F(u) \leqslant \lim_{j\to\infty} F_{p_j}(u_j)$. Then Lemma 3.6 yields that $u \in \bar{L}^{1,\varphi}(\Omega)$. □

The next result is proved like Proposition 4.7, with the lemmas from this section replacing their counterparts from the previous section.

Proposition 5.5. *Let $\varphi \in \Phi_c(\Omega)$ and $C^\infty(\overline{\Omega})$ be dense in $L^{1,\varphi}(\Omega) \cap L^2(\Omega)$. Assume that $f \in L^2(\Omega)$. The function $u \in L^{1,\varphi}(\Omega) \cap L^2(\Omega)$ from Lemma 5.4 minimizes the F-energy, that is,*

$$F(u) = \inf_{v \in L^2(\Omega)} F(v).$$

6 | DOUBLE PHASE AND VARIABLE EXPONENT CASES

We considered the Γ-convergence of the double phase functional

$$I_\varepsilon(u) := \int_\Omega |\nabla u|^{1+\varepsilon} + a(x)|\nabla u|^2 + |u - f|^2\,dx$$

in Harjulehto and Hästö.[20] Specifically, we proved the following, where $\nabla_a u$ is the absolutely continuous part of the BV-gradient and $V(u,\Omega)$ is the total variation of u. (We refer to Ambrosio et al[26] for more information about BV-spaces.)

Theorem 6.1 (Theorem 4.1[20]). *Suppose that Ω is an open rectangular cuboid, $a \in C^{0,1}(\overline{\Omega})$, and assume that $a > 0$ H^{n-1}-a.e. on the boundary $\partial\Omega$. Then I_ε Γ-converges to $I : BV(\Omega) \to [0,\infty]$,*

$$I(u) := V(u,\Omega) + \int_\Omega a(x)|\nabla_a u|^2 + |u - f|^2 dx,$$

in $L^1(\Omega)$-topology.

By the uniqueness of the Γ-limit, we obtain the following explicit formula for F:

Corollary 6.2. *Suppose that Ω is an open rectangular cuboid, $a \in C^{0,1}(\overline{\Omega})$, and assume that $a > 0$ $\mathcal{H}^{n-1}$-a.e. on the boundary $\partial\Omega$. Then*

$$F(u) = V(u,\Omega) + \int_\Omega a(x)|\nabla_a u|^2 + |u - f|^2 dx,$$

when F is defined with the double phase function $\varphi(x,t) := t + a(x)t^2$.

Proof. Let $u \in L^2(\Omega)$ and $p_i \to 1^+$. We use the elementary estimate $(s+t)^p - s^p \geqslant t^p \geqslant t - (p-1)$. By the second property of Γ-convergence (Theorem 1.3(2)), there exist $u_i \in L^{1,\varphi}(\Omega) \cap L^2(\Omega)$ and $p_i \in (1,2)$ with $p_i \to 1^+$ such that $u_i \to u$ in $L^2(\Omega)$ and

$$\begin{aligned}
F(u) &\geqslant \limsup_{i\to\infty} \int_\Omega (|\nabla u_i| + a(x)|\nabla u_i|^2)^{p_i} + |u_i - f|^2\,dx \\
&\geqslant \limsup_{i\to\infty} \int_\Omega |\nabla u_i|^{p_i} + a(x)|\nabla u_i|^2 + |u_i - f|^2 - (p_i - 1)\,dx \\
&\geqslant V(u,\Omega) + \int_\Omega a(x)|\nabla_a u|^2 + |u - f|^2 dx,
\end{aligned}$$

where the last step is the Γ-convergence of I_ε (Theorem 6.1).

For the opposite inequality, we obtain from the second property of Γ-convergence (Theorem 6.1) functions $(u_i) \subset L^{1,\varphi}(\Omega)$ with $u_i \to u$ in $L^1(\Omega)$ and

$$\begin{aligned}
I(u) &\geqslant \limsup_{i\to\infty} \int_\Omega |\nabla u_i|^{p_i} + a(x)|\nabla u_i|^2 + |u_i - f|^2\,dx \\
&\geqslant \limsup_{i\to\infty} \int_\Omega |\nabla u_i| + a(x)|\nabla u_i|^2 + |u_i - f|^2 - (p_i - 1)\,dx \geqslant F(u),
\end{aligned}$$

where we used the definition of F in the last step. □

In the case of bounded functions, we were able to prove Γ-convergence under a weaker condition on a, namely $\alpha > \frac{1}{2}$ instead of $\alpha = 1$.

Theorem 6.3 (Theorem 4.2[20]). *Suppose that Ω is a bounded Lipschitz domain, $a \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > \frac{1}{2}$, and assume that $a > 0$ H^{n-1}-a.e. on the boundary $\partial\Omega$. Then $I_\varepsilon|_{L^\infty}$ Γ-converges to $I|_{L^\infty}$ in $L^1(\Omega)$-topology.*

This gives another corollary. The proof is similar to the previous corollary, so we skip it.

Corollary 6.4. *Suppose that Ω is a bounded Lipschitz domain, $a \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha > \frac{1}{2}$, and assume that $a > 0$ $\mathcal{H}^{n-1}$-a.e. on the boundary $\partial\Omega$. For bounded u,*

$$F(u) = V(u,\Omega) + \int_\Omega a(x)|\nabla_a u|^2 + |u-f|^2\,dx,$$

when F is defined with the double phase function $\varphi(x,t) := t + a(x)t^2$.

Next, we consider the variable exponent case $\varphi(x,t) := t^{p(x)}$, where $p : \Omega \to [1,\infty)$. We denote $Y := \{x \in \Omega : p = 1\}$ and define $p_\delta(x) := \max\{\delta, p(x)\}$. In Harjulehto et al.,[23, Theorem 1.5] we proved Γ-convergence of the p_δ-energy.

Theorem 6.5. *Let Ω be an open rectangular cuboid and let p be strongly log-Hölder continuous, that is,*

$$|p(x)-p(z)|\log\left(e+\frac{1}{|x-z|}\right) \leqslant c \qquad \text{and} \qquad \lim_{x\to y}|p(x)-1|\log\frac{1}{|x-y|} = 0$$

for every $x,z \in \Omega$ and $y \in Y$. Then

$$D_\varepsilon(u) := \int_\Omega |\nabla u|^{p_{1+\varepsilon}(x)} + |u-f|^2\,dx$$

Γ-converges in $L^1(\Omega)$-topology to

$$D(u) := V(u,Y) + \int_{\Omega\setminus Y} |\nabla u|^{p(x)} + |u-f|^2\,dx.$$

Again, the uniqueness of the Γ-limit allows us to obtain a concrete form of E. The proof is again similar to Lemma 6.2 and thus skipped.

Corollary 6.6. *Suppose that Ω is a rectangular cuboid and p is strongly* log-*Hölder continuous. Then*

$$F(u) = V(u,Y) + \int_{\Omega\setminus Y} |\nabla u|^{p(x)} + |u-f|^2\,dx,$$

where F is defined with the variable exponent function $\varphi(x,t) := t^{p(x)}$.

ACKNOWLEDGEMENTS

M. Eleuteri was partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni(GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by PRIN 2020 “Mathematics for industry 4.0 (Math4I4)” (coordinator P. Ciarletta); P. Hästö was funded in part by a grant from the Wihuri Foundation.

CONFLICT OF INTEREST

The authors declare that they have no competing interests.

ORCID

Peter Hästö https://orcid.org/0000-0002-3507-3424

REFERENCES

1. Dal Maso G. Integral representation on BV(ω) of Γ-limits of variational integrals. *Manuscripta Math*. 1980;30:387-416.

2. Fonseca I, Müller S. Quasiconvex integrands and lower semicontinuity in L^1. *SIAM J Math Anal*. 1992;23(5):1081-1098.
3. Juutinen P. p-Harmonic approximation of functions of least gradient. *Indiana Univ Math J*. 2005;54(4):1015-1029.
4. De Giorgi E, Franzoni T. Su un tipo di convergenza variazionale. *Atti Accad Naz Lincei, VIII Ser, Rend, Cl, Sci Fis Mat Nat*. 1975;58:842-850.
5. Braides A. *Γ-Convergence for Beginners*, Oxford Lecture Series in Mathematics and Its Applications, vol. 22. Oxford: Oxford University Press; 2002.
6. Dal Maso G. *An Introduction to Γ-convergence*, Progress in Nonlinear Differential Equations and Their Applications, vol. 8. Boston: Birkhäuser; 1993.
7. Baroni P, Colombo M, Mingione G. Regularity for general functionals with double phase. *Calc Var Partial Differ Equ*. 2018;57:62.
8. Colombo M, Mingione G. Regularity for double phase variational problems. *Arch Ration Mech Anal*. 2015;215(2):443-496.
9. Diening L, Harjulehto P, Hästö P, Růžička M. *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, vol. 2017. Heidelberg: Springer; 2011.
10. Baasandorj S, Byun S-S. Irregular obstacle problems for Orlicz double phase. *J Math Anal Appl*. 2022;507(1):125791.
11. Benyaiche A, Harjulehto P, Hästö P, Karppinen A. The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth. *J Differ Equ*. 2021;275:790-814.
12. Chlebicka I, Karppinen A. Removable sets in elliptic equations with Musielak-Orlicz growth. *J Math Anal Appl*. 2021;501(1):124073.
13. De Filippis C, Mingione G. Lipschitz bounds and nonautonomous integrals. *Arch Ration Mech Anal*. 2021;242(2):973-1057.
14. Maeda FY, Mizuta Y, Ohno T, Shimomura T. Trudinger's inequality for double phase functionals with variable exponents. *Czechoslovak Math J*. 2021;71:511-528.
15. Mizuta Y, Shimomura T. Hardy-Sobolev inequalities in the half space for double phase functionals. *Rocky Mountain J Math*. 2021;51(6):2159-2169.
16. Papageorgiou NS, Rădulescu VD, Zhang Y. Resonant double phase equations. *Nonlinear Anal Real World Appl*. 2022;64:103454.
17. Skrypnik II, Voitovych MV. On the continuity of solutions of quasilinear parabolic equations with generalized Orlicz growth under non-logarithmic conditions. *Ann Pura Appl Mat*. 2022;201(3):1381-1416.
18. Stegliński R. Infinitely many solutions for double phase problem with unbounded potential in $\mathbb{R}^n$. *Nonlinear Anal*. 2022;214:112580.
19. Chen Y, Levine S, Rao M. Variable exponent, linear growth functionals in image restoration. *SIAM J Appl Math*. 2006;66(4):1383-1406.
20. Harjulehto P, Hästö P. Double phase image restoration. *J Math Anal Appl*. 2021;501(1):123832.
21. Harjulehto P, Hästö P, Latvala V. Minimizers of the variable exponent, non-uniformly convex Dirichlet energy. *J Math Pures Appl*. 2008;89(2):174-197.
22. Harjulehto P, Hästö P. *Orlicz Spaces and Generalized Orlicz Spaces*, Lecture Notes in Mathematics, vol. 2236. Cham: Springer; 2019.
23. Harjulehto P, Hästö P, Latvala V, Toivanen O. Critical variable exponent functionals in image restoration. *Appl Math Letters*. 2013;26:56-60.
24. Miranda M. Functions of bounded variation on "good" metric spaces. *J Math Pures Appl*. 2003;82:975-1004.
25. Evans LC, Gariepy RF. *Measure Theory and Fine Properties of Functions*. Boca Raton: CRC Press; 1992.
26. Ambrosio L, Fusco N, Pallara D. *Functions of Bounded Variations and Free Discontinuity Problems*. Oxford: Oxford University Press; 2000.
27. Musielak J. *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, vol. 1034. Berlin: Springer; 1983.
28. Harjulehto P, Hästö P. Extension in generalized Orlicz spaces. *Nonlin Studies*. 2019;26(4):861-868.
29. Juusti J. Extension in generalized Orlicz–Sobolev spaces. *J Math Anal Appl*. 2023;522(1):126941.

How to cite this article: Eleuteri M, Harjulehto P, Hästö P. Minimizers of abstract generalized Orlicz-bounded variation energy. *Math Meth Appl Sci*. 2023;1-15. doi:10.1002/mma.9042