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MARKET ANOMALIES AND TACTICAL ASSET ALLOCATION

Utilising market anomalies in multiple asset class portfolios with the Black–Litterman model

Master´s Thesis
in Accounting and Finance

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1 INTRODUCTION

1.1 Background

In efficient markets, all relevant information should already be included in asset prices and thus investors should not be able to gain excess returns with respect to common risk factors by analysing the past. However, occasionally some consistent patterns appear to violate the efficient market hypothesis. The hypothesis argues that the current asset prices reflect all relevant information, thus, making it impossible to outperform the market with timing or excellent asset picking skills (see, e.g., Fama 1970). Some of these patterns, also known as anomalies, tend to persist from year to another even though they should disappear after becoming publicly known. If such anomalies exist, utilising them within investment strategies could provide superior returns to those of a benchmark, e.g., the market portfolio.

Asness, Moskowitz, and Pedersen (2013) study the existence of the *value* and *momentum* effects across different markets and asset classes, i.e., the relation between asset returns and *book-to-market* ratio and the one between asset returns and their recent cumulative returns, respectively. Their results show that value and momentum strategies are not only profitable when considered separately but also when combined together. In addition, these phenomena are significant across different asset classes. Significance of value factor acquires strong empirical evidence in many different studies (see e.g. Fama and French 1993). Accordingly, the momentum strategy, first analysed by Jagadeesh and Titman (1993), is shown to realise significant abnormal returns. However, there seems to be contradictory results when it comes to applying these factors in an international context. While Asness et al. (2013) report significant value and momentum premia across different markets, it is also claimed that the Fama–French three-factor model is not applicable for international portfolios (Griffin 2002; Mirza & Afzal 2011; Fama & French 2012).

Momentum effect is shown to be strong with the past 3–12 month returns but with a longer time period the phenomenon is shown to revert, i.e., the past winners become losers and vice versa. This anomaly is known as a long-term reversal. One of the first who discovered the long-run reversal effect were DeBondt and Thaler (1985) showing that the long term losers change their course becoming winners in the future. These two effects rest on performance of assets, however, there is another phenomenon that leads to mispricing of assets, which is based on unsystematic or idiosyncratic risk. There are number of studies on idiosyncratic volatility effect claiming that either high or low volatility of the error terms of an underlying asset pricing model results in a higher return-risk ratio in the future. The effect is interesting because theory and empirical studies argue how idiosyncratic volatility should affect asset returns (see, e.g., Merton 1987 or Ang, Hodrick, Xing, & Zhang 2009). All things considered, there is plenty of debate on these different anomalies but it would be appealing to observe them together from the perspective of the modern portfolio theory.

The common idea behind successful investing is diversification. Allocating investor's wealth over different assets results in lower risk based on the low correlation between asset returns. This means that an investor is not dependent only on the performance of one asset. However, a relevant question is, how should this diversification be done in an optimal way. Markowitz is generally considered as the founding father of modern portfolio theory. Markowitz (1952) argues that one can construct a portfolio that minimises variance—considered as the risk of the investment—with respect to the desired expected return. The *mean-variance* optimisation framework provided by modern portfolio theory is widely used by practitioners as it addresses the main question of how an investment portfolio should be diversified. However, the mean-variance optimisation tends to be very sensitive to the expected returns, which raises concerns about the accuracy of the predicted returns. This means that investors who are willing to utilise strategies based on anomalies in this framework are likely to face the problem of corner solutions and instability of the optimal weights.

As a promising solution to the problem, investors could express their views with respect to a benchmark, namely the market portfolio. One method that allows investors to approach the asset allocation problem from this viewpoint is the Black–Litterman model. This promising approach to enhance mean-variance optimisation was introduced by Black and Litterman (1992). The fundamental advantage of the model is that investors are able to include their views on asset returns in the optimisation problem, thus making the investment decision more flexible. In addition, a tempting feature of the model is that the output, i.e., expected returns, can be adapted to the familiar mean-variance framework. In this thesis, the Black–Litterman model is used as a tool in order to include the views based on market anomalies in the portfolio optimisation. There are several reasons that make the model superior to the traditional mean-variance method or volatility weighted long-short portfolios (see, e.g., Asness et al. 2013). Firstly, the portfolios given by the Black–Litterman model are realistic in terms of weights. Secondly, it provides a flexible way to translate the observed phenomenon into a relative view without a need to express an absolute forecast to each asset. Additionally, the method makes it possible to incorporate more than one anomaly simultaneously in the analysis still allowing the familiar optimisation procedure.

The method used here is inspired by Fabozzi, Focardi, and Kolm (2010, 385–393) although, within this thesis, the method is taken to the next level. In other words, investors will be allowed to add different views on different asset classes. Additionally, different weighting methods are allowed when translating anomalies into model specific views. At first, the assets under each asset class are ranked according to the factor in question. For example, with momentum, the ranking is based on 12-month cumulative returns. Unlike in Fabozzi et al., the ranked assets are divided into groups (see, e.g., Asness et al. 2013) in order to more clearly separate the two extremes. Then, a long-short portfolio is constructed in which the weight of each asset depends on the group it belongs to. Using historical returns of this portfolio the expected return can be estimated, e.g., as a simple mean. This scalar is then fed into the Black–Litterman framework as a view, i.e., this would be the relative return by which the group of the 'winners' outperforms that of the 'losers'. Finally, when all the parameters for the model are estimated, rest of the analysis is conducted as a conventional out-of-sample study where the performance of such a strategy is analysed through different portfolio performance measures.

1.2 Purpose of the study and research questions

The primary goal of this research is to examine whether market anomalies can be profitably utilised within portfolio strategies for multiple asset classes. The anomalies used in the study, are the value, momentum, long-term reversal, and the idiosyncratic volatility anomaly. Based on these market anomalies, a portfolio of stocks, government bonds, currencies, and commodities is optimised and studied, out of sample. An interesting question to answer is how the portfolio performance is influenced by information given by a simple long-short portfolio.

Different anomalies have been studied a lot and some of them have become commonly used risk factors in finance literature, value and momentum being examples. However, most of the studies concentrate purely on the stock market, which does not benefit those investors who are seeking returns over different asset classes, e.g., institutional investors. On the other hand, finance theory provides sophisticated tools for efficient portfolio allocation. The Black–Litterman model is a good example of such an intuitive tool but yet, practical solutions to utilise theoretical findings are quite rare. Within this thesis, the purpose is to show that market anomalies can be incorporated into portfolio optimisation as views.

Going into the details, the actual research question is, can investors enhance their portfolio performance by including anomaly-based views into the optimisation process. More specifically, if an anomaly-based zero-cost portfolio yields a superior risk-return ratio, is it profitable to consider a historical mean return as a source of information, and thus, as a view in portfolio optimisation. If so, then such a strategy should provide significant returns and risk-return ratio compared to benchmarks. Additionally, it is essential to investigate whether there is a persistent return premium offered by the market anomalies. Finally, it is worth studying whether it is effective to simultaneously incorporate different views from all asset classes into a cross-asset context.

The comparison of each portfolio strategy is done with several measures. The return-risk relation is measured with the *Sharpe ratio* and also its components mean return (with statistical significance level) and standard deviation. In order to show whether the strategies truly provide superior returns, it is crucial to compare the returns to those explained by the Fama–French three-factor model. In other words, it is studied whether the test portfolios create statistically significant positive alpha. Understanding the challenges of a multifactor regression in a world of multiple asset classes, the Sharpe ratio will be given great importance. That is why robustness of the reported Sharpe ratios is assessed with respect to underlying parameters. Observing out-of-sample performance ensures a realistic analysis. This means that return of the next period purely depends on decisions based on information prior to each portfolio rebalancing.

1.3 Scope and limitations

The theoretical framework in this thesis is the modern portfolio theory (MPT), introduced by Markowitz (1952). It provides the concept of mean-variance optimisation and supports the general understanding of the benefits of diversification. In this thesis MPT is approached from a more practical perspective, namely, the theory provides a conceptual framework and tools for dealing with the actual subject. Especially concepts, such as, *efficient frontier, minimum variance portfolio*, *tangency portfolio* and *capital market line* will be central. There are number of extensions to the basic mean-variance framework. One of these is the Black–Litterman model, the target of which is to provide an intuitive way to set expected returns for the optimisation process.

The scope of the empirical study is to cover a wide range of investable assets from different asset classes with a geographical focus. Here, the analysed asset classes are stocks, government bonds, currencies, and commodities. Country indices are chosen for stocks and government bonds for the interest rate. In order to improve reliability of the results, it is essential to investigate a long time period. Hence, a 20-year long time series is examined to conduct the empirical study. This requirement naturally sets limitations to the data sets. For all assets this long data was not available. This also raises a question about selection bias. For example, collecting stock prices of individual companies over 20 years would automatically leave out those companies that have gone bankruptcy. Only the strong old enterprises existing over such a long period of time would be included in the sample. One important reason for using country level indices has been to avoid this bias.

There are several theories that are linked to MPT. On the other hand the MPT is based on *utility theory* and *utility function.* It is assumed, however, that the reader is familiar with the basics of this theory, and therefore certain topics are not discussed. In addition, there are some (not that obvious) theoretical concepts that require a brief discussion before stepping into certain essential topics. For example, *market equilibrium return* will be discussed within the Black–Litterman model, which is tightly linked to the *Capital Asset Pricing* model of Sharpe (1964). Another important theory is associated with the desire of making future predictions based on historical data. According to the *Efficient Market Hypothesis*, future returns cannot be predicted from historical prices thus making it impossible to earn excessive returns with solutions discussed in this paper (Fama 1970). Hence, this theory must not be left without consideration, and moreover, some contrary evidence should be presented.

Another important note, considering this thesis, is related to the concept of the market portfolio. Market portfolio is a wide concept that may include many kinds of investments, e.g., stocks, real estates, bonds, currencies, commodities, derivatives, etc. In this thesis, different asset classes, such as, stocks, bonds, currencies, and commodities are to be covered but obviously not the whole investable universe. Even this wide selection of asset classes sets some major challenges when it comes to defining the market portfolio. It is not realistic to have reliable time-series data on relative market values of these market portfolio constituents. This means that even a limited estimate of the market portfolio structure is hard to retrieve. Due to this limitation, the market portfolio will be considered as an equal weighted portfolio of all test assets.

When discussing the anomalies, it should be remembered that there are plenty of different anomalies related to, e.g., fundamentals, calendar events, and technical trading rules. The value effect is an example of the first category and the momentum effect belongs to the latter one. In addition to these two, long-term reversals as close cousins of the momentum and idiosyncratic volatility anomaly are discussed within this thesis. Only cross-sectional anomalies are chosen because they are easy to utilise in portfolio strategies. For example, calendar anomalies are ignored because they do not tell which assets should be preferred to others.

1.4 Structure of the thesis

This thesis is divided into three entities: Theoretical foundation and literature review, description of the data and empirical methods, and results of the empirical study. Theoretical discussion covers essentials of the modern portfolio theory and leads then to a more important question of how to incorporate investor's views into efficient portfolio optimisation. More precisely, the Black–Litterman model and its parameters will be discussed thoroughly from both practical and theoretical point of view. The other half of the theoretical discussion covers four different anomalies, namely, value, momentum, reversal, and idiosyncratic volatility. These effects will be defined and their theoretical importance is justified with the existing literature. Here, the currently known empirical findings will be compared and assessed in the light of different theories that explain the existence of these anomalies.

In the empirical part, the research data is studied and analysed. In addition, the methods used in measuring different anomalies will be discussed and justified. More importantly, it is explained how to construct simple zero-cost portfolios based on these anomalies, and moreover, how to turn them into investor's views. In addition, it will be shown how to incorporate these views into the Black–Litterman model and how the model parameters are calibrated in this thesis. For those readers who are interested in how the study was actually conducted, a programming code for *R* is provided in the appendix of this thesis.

Finally, results of the empirical study are reported and analysed. This covers analysis of both long-only and long-short portfolios and, of course, performance of the optimised portfolios. Based on the findings, robustness of the test portfolios will be assessed which is highly important in terms of implementing a strategy in real life.

2 THEORETICAL FOUNDATION

2.1 Modern portfolio theory

2.1.1 Return and risk of the portfolio

When discussing investing it is necessary to be familiar with two key-terms: return and risk. Return on a stock can be calculated by using either percentage (simple) returns or continuously compounded (log) returns. Continuously compounded return at time t can be denoted by r_t and percentage returns by R_t. The link between these two can be shown as

$$r_t = \ln(1 + R_t) = \ln\left(\frac{P_t + D_t}{P_{t-1}}\right), \tag{1}$$

where r_t and R_t are continuously compounded and percentage returns (respectively) at time t, and P and D are price and dividend at time t or at the previous period, $t - 1$. The natural logarithm of the price (including a possible dividend) at time t is denoted by p_t. Continuously compounded returns (or log-returns) are commonly used by researchers because of their statistical properties. As can be seen from the Equation 1, the logarithmic return of one period is the remainder of two serial logarithmic prices. This procedure enables the returns of different periods to be summed together and thus, makes it simple to compute cumulative returns of longer periods of time.

The advantage of continuously compounded returns over the simple ones is illustrated in the following two equations:

$$r_{i,t} = \sum_{t=1}^{T} r_{it}$$

$$R_{i,t} = \prod_{t=1}^{T} (R_{it} + 1) - 1 \tag{2}$$

While percentage cumulative return is a sum of products of simple returns, the corresponding log-return can be calculated as a sum of shorter period log-returns. Another advantage of continuously compounded return is in its simplicity to calculate mean returns. With percentage returns, this is given by a geometric mean, whereas logarithmic returns allow the use of an arithmetic average. It is obvious that log-returns provide an easier approach to return computations, but it should be noticed that the case is slightly different when dealing with portfolios. The return on a portfolio is the weighted average of the simple return on each asset. However, the same does not apply to log-returns. To be precise, they need to be transformed back to simple returns first in order to compute weighted sums. (Tsay 2005, 2–5; Rasmussen 2003, 9–11.)

As a concept, risk is not as unambiguous as return and thus, it needs to be explained. It could be easily assumed that investors prefer higher returns. Since returns are often assumed symmetrically distributed, the return is most likely equal to the mean return. Naturally, only negative changes should be considered as risk (downside risk) from the investor's point of view, but because of the symmetrical distribution both gains and losses have equal probability. Hence variance or equivalently its square root, standard deviation (or volatility) is generally used as a measure of risk. (Elton et al. 2011, 46–49.)

When adding more securities to a portfolio, computing variance becomes less straightforward. In addition to individual variances, the correlation between the returns also becomes essential. Since returns are dependent on each other, it affects their co-movement, namely, covariance. If, for example, positive returns occur simultaneously, the correlation between these returns is high (maximum $+1$). In the opposite case where two return series are the mirror image of each other, the correlation is -1, which is the minimum. As mentioned, covariance between two individual assets describes how their returns are moving together. When computing covariance of two assets, the correlation is multiplied with the product of standard deviations of the both return series. The following formula presents the role of correlation in covariance:

$$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j, \tag{3}$$

where σ_{ij} is the covariance between assets i and j, ρ_{ij} is the correlation and σ denotes the standard deviation. (Elton et al. 2011, 54.)

In order to analyse the risk of all the securities together, a formula for the variance of the portfolio needs to be constructed. To conclude what is already discussed above, it can be said that the variance of the portfolio consists of both variance of each asset and covariance terms. However, it can be shown that covariance of an asset with itself equals the variance of the asset. Hence, variance of the portfolio can be expressed as

$$\sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j \sigma_{ij}\,, \tag{4}$$

where the variance of the portfolio σ_P^2 is the weighted sum of the covariance (and variance) coefficients. (Markowitz 1952, 81.)

2.1.2 Asset diversification and the efficient frontier

The concept of efficient portfolios is based on the idea that investors are trying to maximise the expected return on a portfolio, while avoiding variance, which can be seen as a risk that should be minimised (Markowitz 1952, 77). When more assets are added to the portfolio the variance decreases as a consequence of the smaller correlation. As discussed above, the variance of the portfolio can be divided into two components: variances of individual assets and covariance terms. The first, the individual variances, can be diversified away but part of the risk remains; this is the average covariance, and eventually the market's volatility. (Elton et al. 2011, 59–60; Rasmussen 2005, 90–91.)

The essential outcome of Markowitz's (1952) portfolio theory is that it gives a tool for constructing portfolios in an efficient way, i.e., it provides a way to determine *mean-variance* efficient portfolios. This theory allows investors to utilise two different approaches (Markowitz 1952, 82):

1. Minimum risk with desired level of expected return, or
2. the highest possible expected return with minimum risk.

For each level of risk (volatility) given there is a maximum expected return. Alternatively, for each desired level of return there is a minimum achievable volatility. Finally, this results in the *efficient frontier*, on which each point is an efficient portfolio.

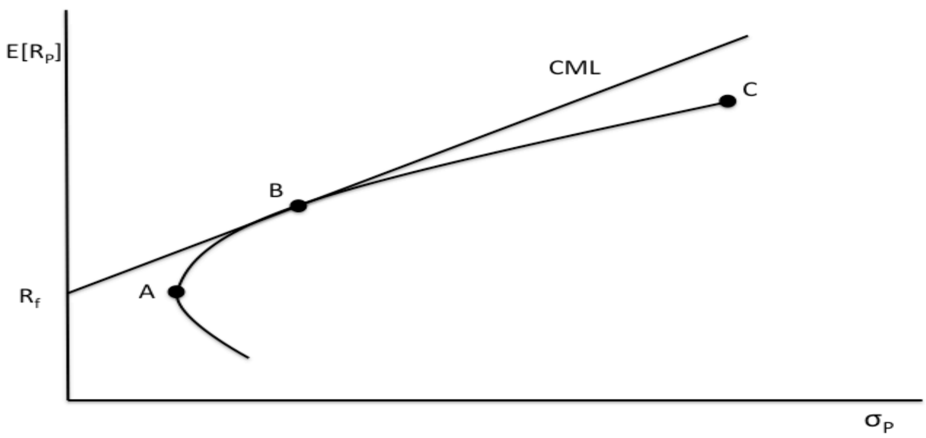

Figure 1 The Efficient Frontier

Each different asset combination produces a different result in the return-volatility space. However, no portfolio can sit on the left side of the frontier shown in Figure 1[1]. Even if the final placement of a portfolio is on the frontier line, this does not necessarily

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[1] There are also other important terms related to the efficient frontier, such as a tangency portfolio, risk-free return, and a capital market line. These concepts will be discussed later in this chapter.

mean that it is efficient. By definition, the efficient frontier is a range from the *minimum-variance portfolio* A to the portfolio C with the maximum-variance. Once these two portfolios are found the entire efficient frontier can be derived as their linear combination. (Rasmussen 2003, 111−114.)

The shape of the efficient frontier indicates how much one can benefit from diversification. This means that correlation between the assets determines the shape of the curve of the portfolio possibilities. This can be understood by observing a portfolio of two (risky) assets, where short sales are not allowed. In this case, the curve will be drawn between two extreme portfolios, both of which consist of either asset A or B. If the correlation between these two assets equals one, the assets are alike, and thus, the efficient frontier is a straight line between the assets A and B. Accordingly, one cannot benefit from diversification. Another extreme case is when the correlation is perfectly negative ($\rho = -1$). In this theoretical situation, investors would be able to find the optimal weights for both assets, which would result in a portfolio of zero risk. In this case, the curve would actually be a line from A to the expected returns axis and from there to asset B. In real life, however, the correlation is almost always something between zero and one. This leads to the portfolio possibilities curve being more likely to resemble Figure 1. (Elton et al. 2010, 68−77.)

When adding more securities to the portfolio, investors obtain more diversification opportunities. The efficient frontier still looks the same but it shifts to the left. This is very understandable. As mentioned before, the risk becomes lower when more securities are added to the portfolio. Thus, any combination of a few assets cannot be as efficient as the optimal combination of a larger number of assets. Most likely none of the securities alone lie on the efficient frontier but the optimal combination of them offers the mean-variance efficient portfolio. Again, no higher expected return can be obtained for a certain risk level and, respectively, no lower risk can be obtained for the desired expected return. (Rasmussen 2003, 113, 125−126.)

2.1.3 Risk-free asset and market portfolio

In the above, the discussion was limited to cases of portfolios with risky assets only. Hence, their returns were uncertain, and in order to compute the riskiness of the entire portfolio, one had to be aware of the correlation among the returns. Usually, however, there is a risk-free asset available. Basically the idea is that one can borrow and lend at a risk-free interest rate. Lending here means that investors obtain, e.g., by buying a short-term government bond, a risk-free rate of return. On the other hand, there is the possibility to borrow at a risk-free rate, which gives investors an opportunity to purchase more securities than what they can afford. Since the asset is risk-free, it is clear that the return is certain and thus, the standard deviation is zero. (Elton et al. 2011, 84–85.)

The risk-free asset changes the problem of constructing efficient portfolios. In this case the task is to find the optimal allocation of two assets, namely, the risk-free asset and a mean-variance efficient portfolio. As the risk-free return must have a standard deviation equal to zero, it can be proven that the covariance coefficient (see *equation 2.3*) will equal zero. Take for example, a given portfolio B (see Figure 1) from the efficient frontier. If there is a risk-free asset available, the risk of the whole portfolio results only from portfolio B:

$$\sigma_P = \mathrm{X}\sigma_B, \tag{5}$$

where X denotes the proportion of portfolio B in the mixed portfolio and σ_B is the standard deviation of portfolio B.

As highlighted previously, the return on a portfolio is the weighted average of returns on individual assets. The weights in the portfolio are assumed to sum up to one, which implies that the proportion invested in the risk-free asset, must be $1 - X$. By rearranging the previous equation, the weight for portfolio B can be solved as $\mathrm{X} = \sigma_P/\sigma_B$. Now, the expected return on the portfolio can be expressed as follows:

$$E[R_P] = R_f + \left(\frac{E[R_B] - R_f}{\sigma_B}\right)\sigma_P\,, \tag{6}$$

where R_f is the risk-free rate, $E[R_B]$ is the expected return on portfolio B and σ_P is the standard deviation of the whole portfolio. (Elton et al. 2011, 85–86.)

Analysis of the equation above reveals that return on the portfolio is a linear function of standard deviation of the portfolio. The risk-free rate is a constant and the ratio of expected excess returns and standard deviation of portfolio B is the slope. This slope is also known as the Sharpe ratio, which changes along the efficient frontier and reaches the maximum above the minimum-variance portfolio (Rasmussen 2005, 198). Earlier efficiency was defined as a relation between return and risk. In this sense maximising the Sharpe ratio seems rational. If riskless lending and borrowing is possible, a rational investor should maximise this slope and finally find the tangent drawn from R_f to the efficient frontier. The portfolio, at the point where the tangent touches the frontier, is called the *tangency portfolio* or *maximum Sharpe portfolio* (see portfolio B in Figure 1). (Elton et al. 2011, 86–87, 100–101.)

If the whole asset universe is considered, the tangency portfolio is actually the same as the market portfolio. Now the weight for each asset is based on their market capitalisation. In fact, this portfolio will serve as a reference point offering the market equilibrium returns when introducing the Black–Litterman model. The tangent between the risk-free asset and the market portfolio is known as the *capital market line* (*CML*). Eventually, investors are able to choose any point on the CML by mixing the risk-free asset and the tangent portfolio. It is clear that the other portfolios on the efficient frontier are not efficient anymore because investors are now able to obtain higher expected return without taking any additional risk. (Elton et al. 2011, 282–283; Sharpe 1964; Lintner 1965.)

2.2 Mean-variance optimisation in practice

2.2.1 Description of the optimisation process

In the beginning of this chapter, the main features of modern portfolio theory were introduced. When implementing the theory in real world portfolio management there are several phases that are essential in the process. First of all, the basic inputs for the mean-variance model need to be well determined. Practitioners need forecasts for the expected returns on each asset in the portfolio. There are a number of different predicting models for the returns available, some of which will be discussed in this thesis. Estimates for volatility and correlation terms are also crucial in order to obtain accurate results from the optimisation. In addition, usually the portfolio manager has some limitations in terms of making investment decisions. These limitations need to be added as constraints to the optimisation software. (Fabozzi, Focardi & Kolm 2010, 316.) The investment process in its entirety can be seen in following figure:

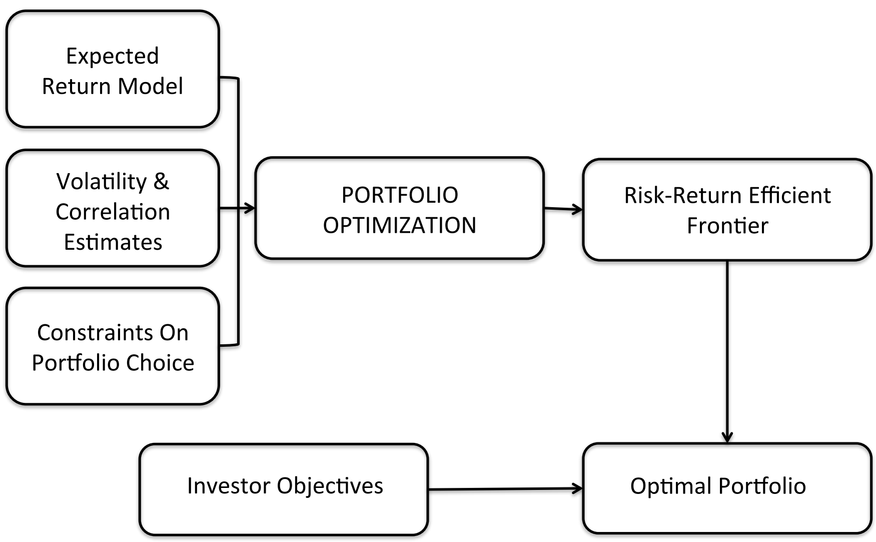

Figure 2 The MPT Investment Process (Fabozzi, Gupta, Markowitz 2002, 8)

One approach to expected returns is to consider historical mean returns as a prediction for the future returns. If the returns are assumed to be distributed normally, the prediction holds. Normality of returns is a good approximation indeed, but does not really reflect the real world. From real life observations, it can be seen that return distribution often has excess kurtosis and positive skewness. The first means that the distribution has fat tails, which implies that extreme values have higher probability than in the normal distribution. The latter means that there is more probability mass on the right tail of the distribution. These statistics can vary depending on whether daily or monthly returns are used, or whether the analysis includes returns on individual securities or, e.g., on indices. (Tsay 2005, 14–15, 17–19.) Another important point related to the normality of returns has to do with the length of the return series. If returns from a long period of time are observed, the historical mean is a more precise predictor of the future return. However, the more years taken into the analysis, the more changes in economic conditions become involved. Including these in the calculations may be irrelevant. (Elton et al. 2011, 238.)

Due to the facts deliberated above, it can be argued that the historical mean is a fairly poor predictor of the future return. Financial literature offers many different models for return forecasting, e.g., the capital asset pricing model, multifactor models, and different time series models. This thesis proposes the Black–Litterman approach, in which the expected returns are calculated with the market equilibrium returns and the views of analysts. The market equilibrium returns are derived from capital asset pricing theory, and the views are dependent on the analyst. In this paper, a simple long-short trading strategy is introduced to create factors that can be further used to impose analysts' views.

The variance of the portfolio is another key component in the mean-variance optimisation. In this analysis the portfolio manager has to deal with both variance of individual stocks and correlation between the assets or asset classes. Again, one approach is to compute historical sample variance and correlation. From statistical perspective, the longer the sample period the more accurate the prediction. Nevertheless, the problem is that the economic conditions change over time, which changes the characteristics of the returns. (Elton et al. 2011, 90.)

Before running the optimisation, investors may have to add constraints. In the simplest case, the optimisation is subject to a constraint that the portfolio should be fully invested. In addition, they may want to achieve a certain expected return on the portfolio, or they may not be able to sell the securities short. Restrictions may be comprised of institutional requirements or investors' objectives. Fabozzi et al. (2010) divide the possible constraints into linear and quadratic, nonlinear, combinatorial, or integer constraints. Linear and quadratic constraints are the most common ones and they can be efficiently solved by using straightforward quadratic optimisation algorithms. More complex constraints require more sophisticated algorithms, which obviously makes the optimisation less practical. (Fabozzi et al. 2010, 327–333.)

2.2.2 Computing mean-variance efficient portfolios

Up to this point, the different inputs concerning *mean-variance* framework have been discussed, but not so much the quadratic optimisation itself; therefore this problem will be approached using matrix notations. In addition, general solutions for minimum variance portfolio and tangency portfolio will be given. In the portfolio, there are *N* securities, the optimal weights of which are not yet known. Weights are denoted by $N\times 1$ vector $\mathbf{w}$,[2] the elements of which are the weights of individual securities: $\mathbf{w}' = (w_1, w_2, \dots, w_N)$. By default, portfolios are always assumed to be fully invested. This can be expressed as follows:

$$\mathbf{w}'\mathbf{1} = 1 \tag{7}$$

In the formula, the transposed vector of weights, $\mathbf{w}'$ is multiplied by $N\times 1$ unit vector.

In the beginning of this thesis, covariance was presented as an important part of computing variance of the portfolio. When covariance for each security pair and variance for each security is computed, an $N\times N$ covariance matrix can be constructed:

$$\Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{pmatrix} \tag{8}$$

It is already known that the covariance of a security with itself is equal to its variance, i.e., $\sigma_{ii} = \sigma_i^2$. Hence, the values on the diagonal are variances. By utilising the covariance matrix, the variance of the whole portfolio can finally be computed as:

$$\sigma_P^2 = \mathbf{w}'\Sigma\mathbf{w} \tag{9}$$

It was already pointed out that the expected return on a portfolio is the weighted average of simple returns on individual assets. This, denoted by μ_P, can be calculated as a product of the transposed vector of weights, $\mathbf{w}'$ and the vector of expected returns, $\boldsymbol{\mu}$ as follows:

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[2] To avoid misunderstandings vectors are denoted by highlighted and scalars by normal letters (or numbers). By default, all vectors are written in a horizontal direction, i.e., as $nx1$. Transpose of a vector would be indicated with an apostrophe on the right corner.

$$\mu_P = \mathbf{w}'\boldsymbol{\mu} \tag{10}$$

(Fabozzi et al. 2010, 317.)

A usual setup in the mean-variance problem is that the variance is minimised, subject to some constraints. As already discussed, there is always at least one constraint, the budget restriction, according to which the weights must sum up to one. If the optimal allocation should reach a certain expected return on the portfolio, it is common to add this as a constraint as well. Now the optimisation problem can be expressed as follows:

$$\min_{x} \frac{1}{2}\mathbf{w}'\Sigma \boldsymbol{w}$$

$$\text{s.t. } \mathbf{w}'\boldsymbol{\mu} = \mu_0$$

$$\mathbf{w}'\mathbf{1} = 1 \tag{11}$$

Finally, by using Lagrange multipliers, the problem can be interpreted in the following linear form:

$$\mathrm{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = \frac{1}{2}\mathbf{w}'\Sigma\mathbf{w} + \boldsymbol{\lambda}(\mathbf{w}'\boldsymbol{\mu} - \mu_0) + \boldsymbol{\gamma}(\mathbf{w}'\mathbf{1} - 1)\,, \tag{12}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ are the Lagrange multipliers that allow the constraints to be added to the same equation with the objective function. The problem can be solved by taking partial derivatives with respect to the variables, and then by solving the first-order conditions. This kind of analytical solution is possible only when using equality constraints. In the previous, equation the weights can result in negative values since the individual weights are not constrained. In practice this means that short selling is allowed. If they need to be ruled out, inequality constraints come into question. For example, the long-only rule ($w_i \geq 0,\ i = 1, \dots, N$) is an inequality constraint that requires numerical optimisation.

It is not relevant to go any deeper in the mathematics but instead the general solutions for the global minimum-variance and tangency portfolio will be given. Both of these solutions can be derived from the function presented above. One should note, that in tangency portfolio the expected excess returns ($\boldsymbol{\mu} - R_F\mathbf{1}$) are taken in consideration. The solution for the global minimum variance portfolio is:

$$\mathbf{w} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}, \tag{13}$$

and correspondingly, tangency portfolio acquires the following expression:

$$\mathbf{w} = \frac{\Sigma^{-1}(\boldsymbol{\mu} - R_F\mathbf{1})}{\mathbf{1}'\Sigma^{-1}(\boldsymbol{\mu} - R_F\mathbf{1})} \,, \tag{14}$$

where Σ^{-1} is an inversed covariance matrix. (Fabozzi 2010, 318–319, 323.)

2.2.3 Issues on the inputs of mean-variance framework

Jobson and Korkie (1980) have studied the key inputs, mean return, and variance, in Markowitz's mean-variance framework. They tested the normality assumption of returns with different sample sizes by comparing theoretical values and simulated (Monte Carlo) values with each other. The results prove that the estimation bias is the ratio of the number of assets and sample size, i.e., N/T. Variance especially, in the portfolio did not seem to be comparable with the simulated value even when the sample size grew to 1,000. As already mentioned, such a large number of returns is not relevant, and therefore, it will make the use of the sample values less accurate.

Another interesting finding was that quality of the estimators clearly improved when mean excess returns were relatively high compared to the covariance terms among the assets. Correspondingly, small element values in the vector of expected (mean) returns together with relatively high off-diagonal elements in the sample covariance matrix resulted in weak estimates for the mean-variance analysis. However, t securities with these "good" properties tend to acquire unreasonably large weights in the optimisation process, which leads to errors in portfolio weights (Michaud 1989).

Another problem is that the optimal portfolio weights are very unstable when the inputs change. The main reason for this is the ill-conditioned covariance matrix, which appears when using historical data. As an inversed covariance matrix is used in the optimisation process, ill-conditioning[3] of the matrix makes the results extremely unstable. In some cases, when the number of assets exceeds the sample size, the covariance matrix is not even invertible (Ledoit & Wolf 2004). In the worst case, the mean-variance optimiser becomes an "estimation-error maximiser" rather than anything else. Actually, it often emerges that equal allocation outperforms mean-variance efficient portfolios. (Michaud 1989.)

[3] Ill-conditioning of a matrix is related to its condition number. Condition number determines how accurate and stable results an inversed matrix produces in linear problem solving.

2.3 The Black–Litterman Model

2.3.1 Role in the asset allocation problem

Black and Litterman were working at Goldman Sachs at the time when they published the first paper on the model in the company's internal issue in 1990. In 1992 the method entered the academic world as their paper, *Global Portfolio Optimization*, was published in *Financial Analysts Journal*. In this context, the importance of Black's academic publications is worth mentioning. Among his other work with asset pricing theory, the most famous achievement is the Black–Scholes option pricing equation published with Scholes (1973). Motivation of Black and Litterman when creating the model was driven by the problem that quantitative asset allocation models were very complex, and furthermore, they tended to result in unstable portfolios with remarkable short positions. A reason behind this is that portfolio optimisation models are highly sensitive to expected returns. Even long-only rule does not solve the problem since it leads to solutions where small-cap markets have extremely large weights, and on the other hand, many securities result in almost zero weight. Accordingly, this is an issue that would make such a model practically useless.

The Black–Litterman model approaches the allocation problem from the perspective of expected returns suggesting that the global CAPM equilibrium (see e.g., Sharpe 1964; 1974; Lintner 1965) should be used as a reference point when estimating the expected returns. The market portfolio represents the equilibrium where all investors have neutral views and which equilibrates the global demand and supply. This equilibrium consists only of long positions, which makes it appealing as a benchmark. Eventually, investors are allowed to tilt these market equilibrium weights towards their views controlled by the level of confidence. Hence, the excess returns are expected to be provided by the public information of the market and an insight into the companies or markets, which investors may believe they have. In other words, the model combines two different sources of information. Finally, the familiar mean-variance model is fed by the output of the model, namely, the vector of expected returns conditional on investors' views. (Black & Litterman 1992.)

The ultimate purpose of the model is to produce expected returns, which behave well in the optimisation framework. As a process the Black–Litterman model is relatively straightforward despite the mathematical presentation. Shortly described, the model consists of three entities: Vector of implied equilibrium returns, investor's views, and uncertainty of views. The first one requires an estimate of investor's risk aversion, a covariance matrix of the returns, and the market capitalisation weights. Since the approach considers both equilibrium returns and views uncertainty, they are handled with probability distributions (Black & Litterman 1992). The following figure illustrates the process and illustrates the role of different inputs:

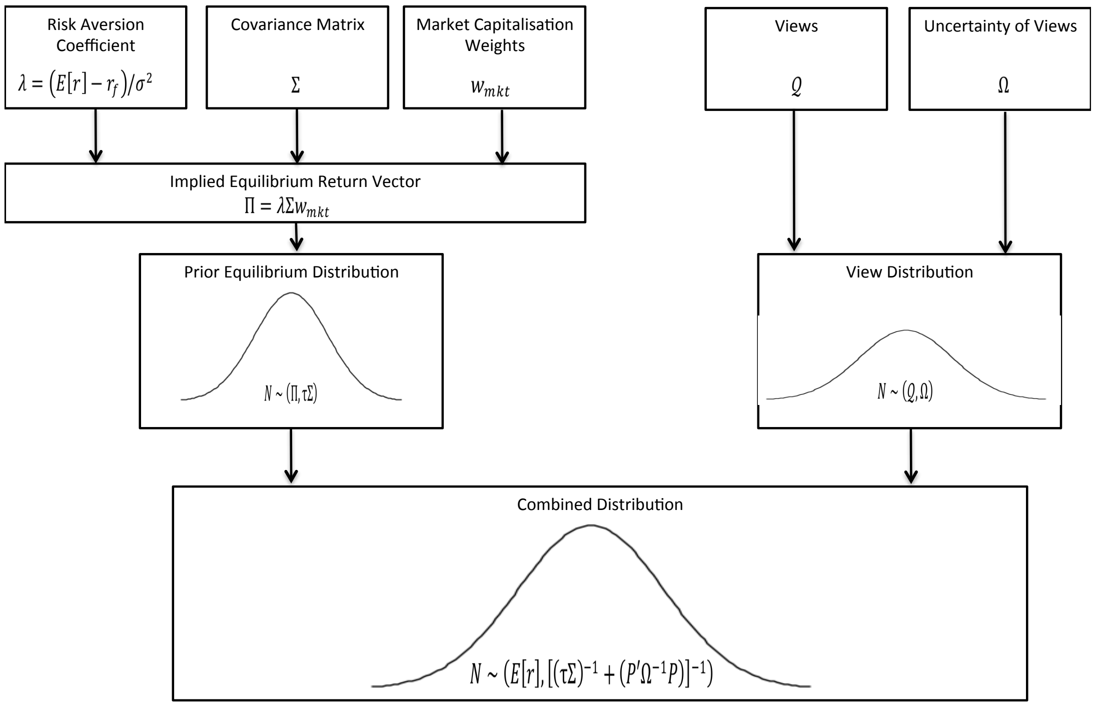

Figure 3 Combining market equilibrium returns and investor's views (Idzorek 2005)

As presented in the figure, the means and variances of the distributions are the inputs shortly introduced above. Some of them are very intuitive and simple to calculate whereas some are more theoretical. For example, defining the market portfolio may not be easy since it is supposed to cover the whole investable universe. Additionally, quantifying uncertainty of views requires more theoretical approach.

After defining all the parameters a complete expression of the mean of the posterior return distribution, i.e., a combined return vector, acquires the following formula:

$$E[\boldsymbol{R}] = [(\tau\Sigma)^{-1} + P'\Omega^{-\mathbf{1}}P]^{-1}[(\tau\Sigma)^{-1}\boldsymbol{\Pi} + P'\Omega^{-\mathbf{1}}\boldsymbol{Q}], \qquad (15)$$

where Σ is a covariance matrix of excess returns, τ is a scaling factor, P is a matrix to weight investor's views, Ω is a diagonal covariance matrix of the views, and $\boldsymbol{Q}$ denotes the actual views (Idzorek 2005). The components and composition of the formula will be discussed in the following sub sections.

2.3.2 The equilibrium portfolio and implied equilibrium returns

The basic principles of the market portfolio, considered as a benchmark in this context, are already discussed within this thesis. It was explained that every investor should pick the portfolio from the efficient frontier, which maximises the Sharpe ratio. Assuming that investors are able to borrow and lend at a risk-free rate of interest, all with equal terms, they should be able to move from this point along the capital market line by mixing the maximum Sharpe ratio portfolio and the risk-free asset in their portfolios. It is also assumed that all investors have similar expectations concerning the expected returns, variance, and correlation structure among the assets in the portfolio. Even though these assumptions are not realistic, they set the frames for the theory. Under these circumstances all investors finds the same market portfolio. (Sharpe 1964.)

When discussing the equilibrium, it can be understood as a portfolio that equilibrates the global market when investors have equal views, i.e., the global market portfolio in which each asset is weighted based on its capitalisation. To be precise, the market portfolio contains the whole investable universe including equities, bonds, currencies, etc., which is why defining the market portfolio may be challenging if not impossible. Additionally, in the original paper (Black & Litterman 1992), the equilibrium portfolio is partly currency hedged based on *universal hedging constant.* According to so-called "Siegel's paradox", investors can benefit from bearing part of the currency risk. This ratio, the universal hedging constant, is discussed in Black (1989).

Regarding the global market portfolio with given market capitalisation weights, the expected returns can be found implicitly utilising Markowitz's mean-variance framework. The idea is introduced in Sharpe (1974) where the essential finding relies on the fact that the market portfolio already holds expectations of the investors, which can be imputed from asset weights. In the context of the Black–Litterman model, the implied market equilibrium returns are calculated using the following equation for a *reverse optimisation*:

$$\mathbf{\Pi} = \lambda \Sigma \boldsymbol{w}_{mkt} \,, \tag{16}$$

where the implied equilibrium returns, $\mathbf{\Pi}$, is calculated as a product of a risk-aversion parameter, λ, the covariance matrix, Σ, and the market capitalisation weights denoted by w_{mkt}. The term 'reversed' here means that the vector of returns given by the formula provides an optimal portfolio when plugged into the Markowitz framework without constraints.

The other multipliers have been discussed earlier within this thesis except the risk-aversion parameter. This can be solved as $\lambda = \left(E[r_m] - r_f\right)/\sigma_m^2$, which defines the amount of additional risk that investors are willing to take when the expected excess return increases. Note that the formula of the risk-aversion is almost similar to the one of Sharpe ratio; instead of variance, the market risk premium is divided by volatility of the market in case of the Sharpe ratio. (Rachev et al. 2008, 96, 143.)

The vector of implied equilibrium returns represents the prior estimate of the mean returns of a multivariate return distribution. To clarify the idea of the prior equilibrium distribution, normally distributed expected returns could be used as a starting point. In the portfolio world, a vector of expected returns, $E[r]$, can be assumed to follow a multivariate normal distribution with mean, $\boldsymbol{\mu}$, and covariance, Σ:

$$E[\boldsymbol{r}] \sim N(\boldsymbol{\mu}, \Sigma) \tag{17}$$

However, the market is not in the equilibrium all the time, which is why an estimate is needed, namely, the vector, $\boldsymbol{\Pi}$, given by the reversed optimisation. This estimate is a combination of the actual means and errors, ε, which are normally distributed with mean, 0, and variance, Σ_ε. Since the estimate is affected by the views given by the market, it should be considered more reliable than simple mean returns and, thus, $\Sigma_\varepsilon < \Sigma$. The proportionality of the covariance matrices of the estimate and the returns ($\Sigma_\varepsilon$and Σ) is indicated by τ. This parameter is also understood as uncertainty of the estimate. Finally, the prior distribution is expressed as

$$\boldsymbol{\mu} \sim N(\boldsymbol{\Pi}, \tau\Sigma) \ . \tag{18}$$

The '*tau*' coefficient is rather theoretical and there are several different approaches suggested in various papers. This problem will be discussed later within this section. (Walters 2011; Cheung 2010.)

2.3.3 Incorporating investors' views

Allocating the funds according to the market portfolio can be interpreted as having no views at all. However, usually investors do have some kind of insight into the assets they consider investing in. The Black–Litterman model introduces two different approaches to incorporate these views, namely, the investors may have either absolute or relative views on the performance of the assets. An absolute view means that the investor has a specific estimate for the return on a particular asset. Accordingly, a relative view stands for a usual case when, e.g., one asset class is expected to outperform/underperform another by X per cent.

In the mathematical expression, the views are added using matrix notations. The assets that the views apply to are defined by matrix P with dimensions, $k \times n$. In the matrix, k represents the number of views, whereas n is the number of assets. Both absolute and relative views are included in the same matrix with a small difference. Assume three assets, A, B, and C, in a portfolio. The investor may believe, for example, that the asset A will yield 3 % and the asset B will outperform the asset C by 1.5 %. In this case, the matrix, P is composed as

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.5 \end{pmatrix}. \tag{19}$$

Since there are three assets and two views, the dimensions are 2×3. An essential rule is that when adding an absolute view, the element equivalent to the particular view and asset has to equal one. Accordingly, in case of relative view, the entire row needs to sum to zero. There are also alternative ways to approach the relative view setting. One possibility is to give equal weights to the outperforming assets, and correspondingly, to the underperforming ones as done in the example above. This means that on each row, the outperforming assets get a value of one divided by the number of the outperforming securities related to a particular view. The same principle applies to the underperforming securities (see Satchell & Scowcroft 2000). However, Idzorek (2005) prefers so-called market capitalisation scheme where the assets related to the view in question are weighted as $market\ cap_i\ /\ market\ cap_{view\ total}$, where '*view total*' means the total market capitalisation of either outperforming or underperforming assets. Again, the weights are positive or negative if the corresponding assets are expected to outperform or underperform, respectively. The intuition behind this approach is that now the final changes in portfolio weights will be proportional to market capitalisations of the assets.

As presented above, the matrix, P weights the views across the assets. The expected excess returns, accordingly, are included in a $k \times 1$ vector $\boldsymbol{Q}$, where k represents the number of views. In the example, given above, it was expected that the asset A will yield 3 % and the asset B will outperform the asset C by 1.5 %. In this case, the views would be presented as

$$\boldsymbol{Q} = \begin{pmatrix} 3\% \\ 1.5\% \end{pmatrix}. \tag{20}$$

It should be noted that each asset is not necessarily included in the $\boldsymbol{Q}$ vector but the entries are given by the views instead, whether they are absolute or relative. In the example there are only two views but, however, assets B and C are both affected by the second view.

In addition to the views themselves, investors are asked to set the uncertainty of the views that is given by a covariance matrix of the views. It is assumed that the views are independent, and therefore, uncorrelated. This means that the covariance matrix is diagonal consisting of the variances of the views:

$$\Omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \tag{21}$$

Presumably, the values for this matrix are difficult to estimate intuitively. Black and Litterman (1992) do not provide any solution for the matrix either, which makes it rather problematic. However, there are several ways to approach this problem, which will be discussed next. The expressed views and uncertainty of the views finally define the distribution of the views. In an asset space, views are distributed as

$$\boldsymbol{V} \sim N(\mathrm{P}^{-1}\boldsymbol{Q}, [P'\Omega^{-1}P]^{-1}) \ . \tag{22}$$

Considering the views only, the distribution would be determined by a mean, $\boldsymbol{Q}$ and variance, Ω. The P matrix, however, translates the views in accordance with the assets. (Walters 2011.)

2.3.4 Uncertainty of the views and tau coefficient

Both an uncertainty matrix Ω and a scalar τ are in the model to cope with confidence issues. The uncertainty matrix reflects variances of the views whereas τ is understood as a confidence of the prior equilibrium. The shrinkage between the prior distribution and views is a function of these two factors, which requires that they should be considered together. They are also highly abstract concepts, which makes them difficult to calculate. In general, the principle is that a higher confidence is related to a greater departure from the market equilibrium portfolio.

As already discussed, an uncertainty matrix of views, Ω is a $k \times k$ diagonal matrix the elements of which consist of variances of the error terms of individual views. This means that value zero can be interpreted as 100 % confidence on a particular view and vice versa. The views can be written as $\boldsymbol{Q} + \boldsymbol{\varepsilon}$, where the error term vector is independent, random, and it follows a normal distribution with mean zero and a covariance matrix, Ω. One way to calculate the diagonal values of Ω is to start with variance of the view portfolios. For each view (k = 1…K), the variance of the view portfolio is $\mathrm{p}_k \Sigma \mathrm{p}'_k$, where p_k is a row (vector) of the P matrix subject to a view k and Σ is a covariance matrix of excess returns. He and Litterman (1999) suggest a link between Ω and τ so that variance of a view portfolio equals ω_k/τ. According to this approach, the elements of Ω acquire the following expression:

$$\Omega = \begin{pmatrix} (\boldsymbol{p}_1'\Sigma\mathbf{p}_1)\tau & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\boldsymbol{p}_K'\Sigma\mathbf{p}_K)\tau \end{pmatrix} \tag{23}$$

The same expression can be also written as $\Omega = diag(P(\tau\Sigma)P')$. This approach mitigates the estimation process since it makes the scalar τ irrelevant regarding the final combined vector of excess returns. It is also the most used method in literature. (Idzorek 2005.)

Another fairly intuitive way to quantify the variance of the views is to utilise confidence intervals. Assume an investor who has estimated that a mean return of an asset is 5%. The investor is 68% confident that the mean falls onto a range of 4 %–6 %. Now, it is possible to calculate the standard deviation of this view by using an assumption of the normal distribution. In the standardized normal distribution, probability of 68 % corresponds 0.47 standard deviations. Considering both tails of the distribution, this falls within 1 standard deviation of the mean and thus, variance for the view can be translated into (1^2 %).

In addition, there are some other alternative ways to estimate the variance of the error term. When a factor model is used to estimate mean excess returns a natural method is to calculate the variance of the error term of the regression. It is typically assumed that residuals are independent and identically distributed (*i.i.d.)*, which leads to the fact that only diagonal values (variances) will be included in the view uncertainty matrix. On the other hand, Idzorek (2005) suggests that there would be also other factors affecting investors' confidence on their views. He calculated so-called *Implied Confidence Levels* by combining market capitalised weights and unconstrained weights based on 100% confidence and user specified confidence levels, i.e., w_{mkt}, $w_{100\%}$, and $\widehat{w}$, respectively. This approach utilises the weight differences of these three different weight vectors to estimate an implied confidence level. This is done by dividing individual values of $(\widehat{w} - w_{mkt})$ by the corresponding differences from $(\widehat{w}_{100\%} - w_{mkt})$. (Walters 2011.)

In connection with a distribution of the market equilibrium returns, the variance of the prior mean is proportional to the covariance matrix of the asset returns. This proportionality is expressed with the constant, τ, and thus the covariance matrix of the estimate as $\tau\Sigma$ (see equation 18). Hence, τ may be understood to indicate the uncertainty of the prior distribution given by CAPM (He & Litterman 1999). In Black and Litterman (1992), the constant is only defined to be close to zero meaning that the expected excess returns are fairly stable. However, there has been a lot of discussion on a reasonable value of the '*tau'* coefficient. For example, Satchell and Scowcroft (2000) suggest that τ should be set to one because investors' return forecasts and equilibrium excess returns are equal on average. Michaud, Esch, and Michaud (2013) acknowledge the combined effect of τ and Ω by adjusting τ so that the model results in investable portfolios. The smaller the scalar is the more the final portfolio approaches the benchmark portfolio, which is constrained by definition. Following the method of He and Litterman (1999), however, leads to a situation where the constant becomes completely irrelevant because Ω is proportional to τ (see equation 23).

When τ is relevant in the final Black–Litterman equation one commonly used method is to approach the scalar from a statistical viewpoint. Now, the sample size, T (length of return series), and possibly the number of assets, N can be used in the estimation. Thus, the scalar τ can be calculated either as

$$\tau = \frac{1}{T}$$

or

$$\tau = \frac{1}{T - N} , \tag{24}$$

where the former formula is so called *the maximum likelihood estimator*, and the latter one is *the best quadratic unbiased estimator.* This approach is based on statistics and it is commonly used when a covariance matrix is estimated from a sample of return.

Since the variance of the prior mean is assumed to be proportional to the covariance matrix of the returns as $\tau\Sigma$, the scalar can be also calibrated with respect to a confidence interval. With given confidence level, estimated mean, and observed standard deviation, it is possible to define a confidence interval for the most likely mean values. As a consequence of a higher standard deviation, the confidence interval becomes wider as τ increases. This allows setting the scalar so that the interval becomes reasonable and a good prior estimate of the mean. (Walters 2011.)

2.4 Market anomalies

2.4.1 Definition and persistence on the markets

When markets are efficient all relevant information should be included in asset prices, which should make it ineffectual to pursue excess returns consistently. Studies on market efficiency rely strongly on the work of Fama (1970) where he introduces three forms of efficiency: *weak form*, *semi-strong form*, and *strong form.* Weak form of market efficiency states that historical price or return patterns do not provide any relevant information for investors profit wise. Accordingly, semi-strong form realises when publicly available information like fundamental factors cannot be utilised in order to make excess returns. The strictest requirement for efficient markets is set by the strong form claiming both either private and public information, held by any group of investors, should be included in asset prices. Finding evidence against the efficient market hypothesis may be a sign of inefficiency although, usually the underlying asset pricing model is the one that should be put in test.

Generally, financial markets can be considered efficient and nowadays even more so as computerised trading makes arbitrage opportunities to disappear before they are even noticed. However, sometimes there are return patterns appearing on the market that violate the efficient market hypothesis and cannot be explained by asset pricing models, like CAPM. Such anomalous returns can be based on historical performance, fundamental characteristics, or some other technical trading rules. By the nature of the anomalies, Keim (2008) categorises them into cross-sectional and time series anomalies. All the anomalies discussed within this thesis belong to the first group but, of course, momentum and reversal effects are based on time series characteristics.

Even though, anomalies seem to provide superior trading opportunities there is one common problem. Once they are studied and become publicly known they usually cease to exist, i.e., they are quickly arbitraged away. This also supports the efficient market theorem. However, some of the anomalies have been reported to show strong evidence still after a long time since they were found. Rather than a persistent inefficiency, this should most likely be considered as a consequence of an inadequate asset pricing model used as a benchmark. After the basic CAPM some of the anomalies have been included in better explaining asset pricing models. For example, Fama and French (1992; 1993) introduce a three-factor model that, in addition to market, explains asset returns with size and value factors. This means that small cap stocks with high book-to-market ratio (value) outperform the market. Additionally, Carhart (1997) improves the model by adding a fourth factor, momentum to explain the continuing rise of the past winners. In order to investigate the returns provided by the strategies suggested within this thesis, the three-factor model is used as a benchmark. (Keim 2008; Schwert 2003.)

2.4.2 Value effect

A phenomenon where securities with high value outperform those with lower value is called value effect. Stocks may be classified as value and glamorous stocks or value and growth stocks. A theoretical explanation to the value effect is that assets with a higher risk of financial distress also provide a higher expected rate of return (Fama & French 1993). Typically, value is assessed with respect to market valuation, which reveals whether an asset is under- or overvalued. There has been a number of different studies investigating the relation between value, calculated with different measures, and cross-sectional returns. First, Stattman (1980) reports a positive relation between cross-sectional US stock returns and book-to-market (*BE/ME*) ratio. Book-to-market ratio is also considered as value measure in Fama and French (1992), however, they also test if earnings-to-price (*E/P*) ratio explains average cross-sectional returns. The results show that the positive relation between positive *E/P* ratio and average returns is mostly due to the fact that high *E/P* firms often have a high book-to-market equity ratio, i.e., there seem to be multicollinearity issues in the regression. In addition to *BE/ME* and *E/P*, Babameto and Harris (2008) consider also dividend yield and cash-to-price ratio when constructing value portfolios. Nevertheless, book-to-market ratio appears to be the most used measure to characterise value in literature.

All the value indicators, discussed above, are accounting based measures that are applicable with stocks. However, the value effect is reported to exist in other asset classes as well, which means that alternative value measures are required. Asness et al. (2013) estimate value for commodities, currencies, and bonds as a negative five-year return. This method is supported by Gerakos and Linnainmaa (2012) who suggest that *BE/ME* could be replaced by market value when calculating a value measure for stocks. They argue that book and market values covary and that an increment in a book value yields even a higher increment in the market value. As a consequence, the BE/ME ratio starts to decrease. In addition, cross-sectional Fama–MacBeth[4] regressions show that the market value of equity captures both past stock returns and net issuance, which makes a negative five-year return a good estimate for value. Also Fama and French (1996) report positive correlation between book-to-market measure and negative five-year return.

Asness et al. (2013) apply this method to different asset classes with slight nuances. More specifically, they calculate the spot price five years ago as an average of the spot price from 5.5 to 4.5 years ago. Finally, the negative five-year return becomes $ln(S_{t-5} - S_t)$, where index t represents years. For each asset class, the basic principle is the same but the calculation methods differ slightly in practice. The equation, presented above, can be straight used to estimate the value measure for commodities. For currency exchange rates, the return is adjusted with *Consumer Price Index* (*CPI*) and thus, the final measure consists of five-year change in *Purchasing Power Parity* (*PPP*). The calculation is better explained by Accominotti and Chambers (2014)[5] and it can be written as

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[4] Fama–MacBeth regression is used to test risk factors of asset pricing models across multiple assets. The method was introduced by Fama and MacBeth (1973).

[5] Accominotti and Chambers (2014) calculate undervaluation of currencies so that currencies with small values are considered undervalued. Here, the measure should be comparable with book-to-market ratio. Hence, equation (25) is a negative form of that introduced by Accominotti and Chambers.

$$Value = s_{t-5}^{i} - s_{t}^{i} - \left(p_{t-5}^{i} - p_{t-5}^{USD}\right) + \left(p_{t}^{i} - p_{t}^{USD}\right), \tag{25}$$

where s_t^i denotes the natural logarithm of spot exchange, p_t^i and p_t^{USD} denote respectively the natural logarithms of foreign and domestic price indices (measured with *CPI*); again, the index t corresponds to years. Accordingly, for bonds the negative five-year return can be calculated with changes in ten-year bond yields. In addition, Asness et al. suggest three alternative value measures for bonds, namely, five-year change in yields of ten-year bonds, yield of a ten-year bond minus five-year inflation forecast (i.e., the real bond yield), and term spread (difference between a ten-year and short bond yields). Ultimately, they report that even more reliable risk premia arise when these different measures are combined.

In case of stocks, the definition of value is quite intuitive to understand as the market price can be evaluated against the book value. For other asset classes, this might not be as straightforward. At least, the definition is not really discussed in anomaly related studies. However, the measures used here give some guide. For example, fair value for currencies can be measured with PPP, which is assessed with foreign and domestic CPI changes. Additionally, using steepness of the yield curve (term spread) to analyse value for a bond can be well explained. The term spread indicates steepness of the yield curve. When the yield curve is steep, bond investors benefit from positive roll-down effect, assuming that no parallel shifts of the curve are not seen. Now, the price of the bond would increase as the maturity comes closer.

When it comes to portfolio strategies, a typical way to utilise such an anomaly is to construct a zero-cost long-short portfolio where, in this case, undervalued assets are bought and overvalued assets are sold. The same principle is behind the multifactor models, e.g., the value factor, *HML* in the Fama–French three-factor model is a portfolio where returns of small value stocks are deducted from those of high value stocks. However, different methods have been used to construct these portfolios. Firstly, assets can be divided in number of portfolios based on their ranked values. Secondly, these portfolios can be weighted in different ways. For example, Fama and French (1993) create the value factor by dividing both small and big stocks (related to the size factor) into three different value weighted portfolios based on their book-to-market rank. Finally, an *HML* (*high-minus-low*) portfolio is constructed by subtracting the returns of these two extreme portfolios one from another. Value portfolios are usually weighted by market capitalisation (see, e.g., Babameto & Harris 2008; Chaves & Arnott 2012; Asness et al. 2013). In addition to capitalisation-weighting, Asness et al. construct factors where assets of a long-short portfolio are weighted based on their ranks so that assets with highest value (or momentum) acquire the largest weight and vice versa. On the other hand, the ones in the middle have weight close to zero. Moreover, equal weighted returns in high or low value portfolios are used by, e.g., Hjalmarsson (2011), although instead of three portfolios, he divides the assets into deciles, which is to emphasise the effect of the most extreme characteristics. This is obviously technically more simple approach but it surely serves the purpose when it comes to results.

2.4.3 Momentum effect

There is strong evidence that those assets with superior short-term past returns will continue outperforming the past losers in the near future. Such a momentum effect is proven to be significant not only for stocks but also for various different asset classes. In this case, the meaning of 'short-term' is crucial since long-term reversal effect (discussed next) has exactly the opposite effect on future returns. The work of Jagadeesh and Titman (1993) claims that a relative strength trading strategy where recent winner stocks over the past 3 to 12 months are selected provides abnormal returns in the next 12 months following the time of portfolio formation. However, after these 12 months the stocks start losing this return up to the following 24 months. Jagadeesh and Titman report the highest annual return when stocks are selected based on the past six-month return and held for the next six months.

It has been shown that momentum returns cannot be explained by common risk factors. Hence, returns realised by momentum effect cannot be a result of systematic risk. Delayed stock price reactions to the risk factors could possibly result in lead-lag effects causing the profitability of such a momentum strategy. For example, Lo and MacKinlay (1990) report that higher expected returns provided by a contrarian strategy where recent winners are sold and losers bought (related to reversals but theoretically close to momentum) are not solely due to negative serial dependence in individual stocks, i.e., market overreaction. Instead, they document that the phenomenon is a consequence of positive cross-autocorrelation among asset returns. This so-called lead-lag effect means that positive returns of large capitalisation stocks imply good performance of small-cap firms in the future.

However, Jagadeesh and Titman (1993) report significantly negative relation between relative strength portfolio returns (based on momentum effect) and lagged squared returns of a value-weighted portfolio. This strongly supports a hypothesis according to which relative strength returns would not be contributed by lead-lag effect. In addition to Jagadeesh and Titman (1993), also DeLong, Shleifer, Summers, and Waldmann (1990) suggest that momentum effect would be caused by investors who make prices to overreact by buying recent winners and selling the losers. They call the investors following such a strategy *positive feedback* traders. The abnormal returns, however, are only created in short run because investors tend to overestimate the long-term prospects thus eventually causing return reversals. This is not far from the explanation of Vayanos and Woolley (2013) who introduce an institutional theory of momentum. They explain continuingly falling prices so that funds impacted by a negative shock start losing investors gradually. These outflows make the fund managers to sell the underlying assets, which causes a continuing decline in asset prices and therefore, their expected returns.

Momentum effect has proven its significance against common risk factors. Fama and French (1996) study whether different anomalies provide abnormal returns that cannot be explained with their three-factor model. Unlike all the other competing anomalies, the short-term (-12 to -2 months) momentum portfolio generates significant positive alpha, which can be considered anomalous from the viewpoint of the model. Additionally, Carhart (1997) considers one-year momentum as a common risk factor and presents his four-factor model to explain equity returns. He improves the three-factor model of Fama and French by introducing the four-factor model, which includes one-year momentum factor in addition to market, size, and value factors. Carhart uses equal-weighted decile portfolios in constructing the factor-mimicking portfolio based on eleven-month returns lagged one month. The results indicate that the four-factor model results in lower pricing error than CAPM or three-factor model in explaining cross-sectional variation in stock returns. In fact, the results are to a great extent supported by those discussed above. When researching performance of mutual funds, buying the winners of the last year and selling the losers yields higher expected returns in the following year. However, after one year the continuance tends to disappear. After all, Carhart does conclude that transaction costs cut a great deal of the returns when momentum strategy is followed.

In the momentum literature, it is a standard to exclude the most recent month in portfolio formation. The reason for such a procedure is to avoid one-month reversal, which implies an inversion in price movements. This is possibly a consequence of liquidity issues or microstructure related factors making the momentum measures biased (Asness et al. 2013). It has been reported that illiquid stocks often cause return reversals. However, the effect might be also dependent on firm size as well since small firm stocks are more likely to be illiquid. (Tang & Zhang 2014.) A vital microstructure related issue is a bid-ask spread leading to weaker momentum returns. Keim (1989) suggests that systematic trading patterns (such as momentum) lead to wider bid-ask spreads, which may introduce errors when calculating returns.

Despite of the general approach to skip the latest month, Asness et al. (2013) remind that some asset classes suffer little from illiquidity. Therefore, they suggest that the momentum effect could be even stronger when the most recent observations are included in the estimation window. However, this does not remove the fact that bid-ask spread leads to biased estimates if the most recent prices are taken into account. Additionally, Tang and Zhang (2014) find that winner stocks tend to show reversal in a short run but the loser ones tend to continue perform poorly, i.e., these two groups behave in different ways. This may also have an impact on how the recent winners and the losers should be treated.

Explanations to momentum in the finance literature can be divided into three groups: risk and characteristics-based explanations, bias related explanations, and explanations related to transaction costs. The first group of explanations is based on results suggesting that momentum is higher among small firms and those bearing higher financial risks. Investors' underreaction to news is a clear example of behavioural biases causing the momentum effect. Underreaction is often the greater the less analysts are following a firm. Transaction costs, again, are usually seen as a drawback for momentum strategies. (Menkhoff et al. 2012.)

2.4.4 Long-term reversal

If the momentum effect is a continuing pattern in asset returns reversal is the opposite. After a period of continuous decline or growth the market recognises the mispricing and the direction tends to change, i.e., a reversal occurs. Return reversals are often explained with investors' overreaction to fundamentals. After an overreaction, either positive or negative, the prices are ought to reverse back towards their fundamental values. For this reason, the effect is originally discussed as stock market overreaction. In finance literature, two types of reversals are observed: long-term and short-term reversals. The discussion, however, is often related to long-term return reversals since short-term reversals are usually led by biases concerning bid-ask spreads or liquidity issues (see, Keim 1989; Asness et al. 2013). Recognising these patterns could possibly provide appealing trading strategies for investors who make decisions based on historical performance of assets, against the efficient market hypothesis.

From institutional perspective, the (long-term) reversal effect is explained with rational investors who make market prices to deviate from fundamental values. When asset prices drop suddenly due to a negative shock investors tend to withdraw their shares from investment funds. This causes pressure for the portfolio managers to start selling assets. Eventually, these outflows push the prices below fundamental values, which finally increases expected returns thus making the assets tempting. (Vayanos & Woolley 2013.) Such an overreaction is an elementary factor explaining the success of reversal strategies. Comparison of past winner and loser portfolios shows that the loser portfolios yield considerably higher returns than the winner portfolios, regardless of the fact that the former one carries less systematic risk. This result is found to be significant with a formation period of 36 months and a test period with the same length. Being that the cumulative abnormal returns (over the market) are significantly positive to the loser portfolios confirms that return reversal is explained by general overreaction. It is also found that the effect is stronger for loser portfolios meaning that overreaction is not symmetric. (DeBondt & Thaler 1985.)

Another factor explaining return reversals is leverage. Chopra, Lakonishok, and Ritter (1992), referring to past literature, explain that negative returns result in higher betas for equity, which increases expected returns. Since the equity beta is a function of asset risk and leverage, continuous negative returns decreases the level of equity thus increasing the beta. In addition, they highlight that the size effect might be behind reversals. However, Tang and Zhang (2014) stress that liquidity issues would be the actual reason. Small firms are only associated with small liquidity and that is why they seem to come to the fore in practical studies.

Despite of the clear evidence, there are also major drawbacks regarding the persistence of the long-term reversal effect. Jordan (2011) stresses that long-term reversals only exist when transaction costs and return-risk adjustments are ignored. Risk adjustments are made with a conditional CAPM, which allows time varying alpha. However, McLean (2010) argues that the mispricing is strongly affected by idiosyncratic risk, i.e., the volatility of the residuals resulting from an asset-pricing model (e.g., Fama–French three-factor model). Testing with stock return data shows that securities with higher idiosyncratic risk demonstrate stronger reversal effect. However, results of this study do not support such a proposition. Moreover, the findings remain significant when transaction costs are taken into account.

2.4.5 Idiosyncratic volatility

The standard Capital Asset Pricing model emphasizes that diversification results in portfolios where only systematic risk is carried and idiosyncratic risk is eliminated. Technically speaking, idiosyncratic risk yields when an asset-pricing model fails in explaining asset returns, i.e., the regression returns error terms. Idiosyncratic volatility, i.e., the volatility of the error terms, and its importance in the asset pricing has been discussed in finance literature. It is often argued that idiosyncratic risk is highly relevant factor when it comes to explaining stock returns. Many empirical studies report a high idiosyncratic risk associated with low returns. This effect has been reported, not only in country level, but also in an international context for stock returns (Ang, Hodrick, Xing, & Zhang 2009.)

From theoretical point of view, Merton (1987) shows that idiosyncratic risk cannot be fully eliminated due to market frictions, such as, limited access to market information. In a world of perfect information, investors would hold mean-variance efficient portfolios, which would yield a mean-variance efficient market portfolio. In reality, however, market frictions (e.g., costly information, short selling restrictions etc.) make it unrealistic to diversify firm specific risk completely away. For this purpose, Merton (1987) derives the capital market equilibrium with incomplete information. The model predicts that alpha is a function of different characteristics, such as idiosyncratic volatility, which has a positive impact on alpha. This would be intuitive, as taking additional risk should indeed be rewarded with higher expected return. More recent support for this view is given by Goyal and Santa-Clara (2003) who find a significantly positive relation between average firm specific variance (lagged for one month) and market returns. They suggest that background risk would partly explain their results as holding non-traded assets (i.e., human capital or private businesses) increases background risk to investors' portfolio decisions. This means that increasing risk in non-traded assets makes investors to demand higher premium in order to hold traded assets.

All in all, a positive relation between idiosyncratic risk and expected returns seems to have a link to expected returns. However, empirical studies often suggest that this relation is negative. For example, Ang et al. (2006) find that monthly stock returns are negatively associated with one month lagged idiosyncratic volatilities. It is known that periods of high market volatility are strongly related to increasing downside risk. Thus a logical explanation to their finding would be that stocks with high idiosyncratic volatility would have a high exposure to market aggregate volatility. However, Ang et al. (2006) show that this does not completely explain the phenomenon. Moreover, the results are robust to other known factors such as size, book-to-market, leverage, liquidity, volume, turnover, bid-ask spreads, coskewness as well as dispersion in the characteristics of analysts' forecasts. As a counterargument, Fu (2009) addresses that the results can be explained by return reversal and small size of the firms. The negative association, however, is given more support in Ang et al. (2009). In addition to U.S. stock data, they include stock markets worldwide and manage to repeat the results, out-of-sample.

Fink, Fink and He (2012) approach the effect of idiosyncratic risk from both theoretical and practical viewpoints. They estimate idiosyncratic risk using exponential general autoregressive conditional heteroskedasticity model (EGARCH) and test whether idiosyncratic volatility is associated with expected returns, in and out-of-sample[6]. The in-sample results prove that accurate estimates of idiosyncratic volatility do have significantly positive relation with expected returns, which supports the theory discussed in

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6 In-sample and out-of-sample data are important concepts when back-testing with historical data. In-sample data are the initial or already "known" information used as an input of a test. In this context, out-of-sample data are considered "unknown" at the time of fitting the model. As more iteration is done, the out-of-sample data become in-sample, and thus input of further optimisations.

Merton (1987). However, out-of-sample tests with EGARCH forecasts of idiosyncratic volatility show that such a relation cannot be approved.

Finding the true effects of volatility premium in asset pricing is still under investigation. A number of empirical studies have been conducted during the recent years and the results remain somewhat contradictory. Some report a positive relation between idiosyncratic risk and expected returns and some findings support the opposite view. In many studies, the relation has been shown to be flat as well. In relating discussion, however, it is commonly agreed that idiosyncratic volatility is a risk factor that should be priced. Although, some argue that the effect is driven by other factors.

3 DATA AND METHODOLOGY

3.1 Test assets

The main source of data for this thesis is *Thomson Reuters Datastream* and the empirical study is conducted by using *R* programming language. In order to conduct a reliable empirical study, data are collected from the year 1989 to 2014, i.e. over 25 years. Overall, daily observations of stock prices from different markets, government bonds, and spot rates of currencies and commodities are used covering a time period of 31.3.1989–1.4.2014.

In order to avoid selection bias, indices from different markets and asset classes are used. Stocks are presented by MSCI country total return indices covering USA, UK, Finland, Sweden, Norway, Denmark, Germany, France, Italy, Spain, Japan, Hong Kong, Singapore, Korea, Australia, Chile, Mexico, Brazil, Canada, and New Zealand, quoted in US Dollars. Bonds include ten-year government bond indices of the following 11 countries: Australia, Austria, Canada, France, Germany, Ireland, Japan, Netherlands, Sweden, UK, and USA. Currencies form a group of six spot exchange rates with respect to US Dollar: Australia, Denmark, Germany, Singapore, Switzerland, and UK. The last group, commodities, includes spot prices of the following commodities: wheat, cocoa, corn, cotton, crude oil, nickel, aluminium, copper, gold, and silver. In addition, the S&P 500 index is considered as a proxy of the market portfolio in multifactor regressions.

For some asset, there are some additional data required in determining under or over valuation. Term spread is used when calculating value for bonds, which means that yields of long- and short-term government bonds, 10 and 3 years respectively, need to be downloaded. Also currencies require economic variables when defining value, namely, *Consumer Price Index* (CPI) data. Generally, CPI is available for each country on monthly basis except for Australia the CPI of which is announced quarterly. However, non-daily data are matched with daily time series by filling the gaps with previous values available. A key inspiration in terms of data and methodology is the work of Asness et al. (2013), which certainly has an impact on the asset classes and markets chosen for this thesis.

3.2 Descriptive statistics

Descriptive statistics are calculated for all daily total return series covering the time period of 3.1.2000–1.4.2014 (see, Table 1). In total, this results in 3717 daily observations per asset, which is a comprehensive sample of the capital market returns over the past two decades. There are two reasons not to report the statistics for the entire downloaded dataset. Firstly, trading period starts from 11th July 1994, which is 5.5 years from the beginning of the data. This is because estimating the value anomaly requires substantial sample size. Secondly, even though, all the time series are daily observations the quality of the data is still not perfect for all asset classes. In order to make the results more comparable, the sample is limited to start from the beginning of year 2000.

The data is described with the following measures: mean, standard deviation, Sharpe ratio, skewness, and kurtosis. Mean and standard deviation are annualised with an assumption of 250 trading days per year. This procedure is made to provide a better understanding of the real performance of the assets. Kurtosis is reported as excess kurtosis in relation to the normal distribution. More specifically, the normal distribution has a kurtosis of three and thus the excess kurtosis in this case would be zero.

Table 1 Equities: Descriptive statistics

This table reports the basic statistics to describe the distributions of the underlying MSCI country total return indices. Mean, standard deviation, Sharpe ratio, skewness, and kurtosis are reported on each benchmark index. Also the first-order autocorrelation AR(1) is reported in order to assess if the returns follow a random walk. Daily continuously compounded returns are used in calculating the values covering data over 3.4.2000–1.4.2014. Each time series thus equals 3717 daily observations. Mean, standard deviation, and Sharpe ratio are annualised assuming 250 trading days per year. Excess kurtosis is calculated to measure kurtosis over the normal distribution, where kurtosis equals. At the bottom, average of all statistics are calculated to give a better overview of the asset class.

Country	Mean (%)	Stdev (%)	Sharpe	Skewness	Kurtosis	AR(1)
		Equities				
USA	1.566	20.706	0.076	-0.177	7.734	-0.079
UK	3.859	22.534	0.171	-0.163	9.151	-0.024
Finland	0.143	34.404	0.004	0.196	11.505	0.008
Sweden	7.009	31.633	0.222	0.414	8.221	0.014
Norway	10.558	30.705	0.344	-0.360	7.598	-0.002
Denmark	11.408	23.534	0.485	-0.059	7.748	0.023
Germany	5.515	27.092	0.204	0.220	6.962	0.002
France	4.431	26.187	0.169	0.127	6.489	-0.015
Italy	2.793	27.054	0.103	0.225	7.642	0.007
Spain	6.136	27.943	0.220	0.164	6.634	0.024
Japan	-0.312	22.637	-0.014	-0.261	4.693	-0.071
Hong Kong	6.060	21.804	0.278	-0.309	10.430	0.013
Singapore	5.976	22.352	0.267	-0.731	17.329	0.019
Korea	9.451	34.910	0.271	-0.544	13.521	0.029
Australia	10.531	24.547	0.429	-0.719	9.831	0.021
Chile	9.112	21.397	0.426	-0.229	12.836	0.117
Mexico	11.415	27.001	0.423	0.102	10.383	0.079
Brazil	12.108	36.491	0.332	-0.075	7.791	0.087
Canada	8.432	23.254	0.363	-0.416	12.417	0.043
New Zealand	9.376	21.876	0.429	-0.513	5.489	0.044
Average	*6.778*	*26.403*	*0.260*	*-0.155*	*9.220*	*0.017*

Starting from stock returns, the first and an expected observation is that during the sample period the annual return has been positive in almost all countries. The only country making an exception is Japan, where the annual mean was -0.312 %. The best performing market was Brazil with 12.108 % annual rate of return. While the average annual mean was roughly half of that in in Brazil, this could be considered quite an outstanding result over fourteen years of time. However, high returns usually come with a great risk. The annual standard deviation in Brazilian stock market was 36.491 %, which is the highest value in the group, the average being 26.403 %.

Risk-wise there are no outliers to mention but Sharpe ratio gives a better comprehension on how the risk has been compensated. For example, the European and the US stock markets have not been much of a competition for most of the developing countries like Chile, Mexico, and Brazil when comparing the risk-return relationship. Nevertheless, Denmark manages to outperform all the other countries with a Sharpe ratio of 0.485. Compared to Brazil, for instance, the annual mean return is not quite as high but a substantially lower volatility makes the return a better compensation on the risk taken.

Stock returns are often reported to show negative skewness and excess kurtosis. The former one means that the left tail of the distribution is longer and there is more probability mass on the right side of the mean. Accordingly, excess kurtosis can be seen as fat tails of the distribution. A good interpretation to this combination is that the returns have a positive tendency but, from time to time, considerably higher wins or losses are reported than expected by the normal distribution. The stock returns used in this study seem to follow the same rule. The average skewness being -0.155 and excess kurtosis 9.220 the shape of stock returns can be quite well pictured. Regardless of the average, not all of the markets are negatively skewed. An interesting observation is that big European countries, such as, Germany, France, Italy, and Spain all appear with positive skewness. In addition, among stock returns, Singapore with the most negative skewness also has the fattest tails. As will be noticed below, this seems to be a general feature for some other asset classes as well.

In addition to the basic descriptive statistics, the Jarque–Bera test was run for all markets. As expected, the normal distribution was rejected in almost all cases at a significance level less than one per cent. This means that none of the return series are normally distributed. Even though, the normal distribution is assumed in many economic models (also in the MPT), it is commonly known that this is not the case.

Descriptive statistics on government bond returns show that they are less volatile than equity markets. Even if any superior returns are not seen, the average annualised mean return exceeds that of reported on equities over the same time period. The table below describes the return distributions of the eleven government bond indices studied within this thesis.

Table 2 Government bonds: Descriptive statistics

This table reports the descriptive statistics on government bond returns. For each country, total returns of 10-year benchmark bond index are used. The reported statistics are same for all the asset classes.

Country	Mean (%)	Stdev (%)	Sharpe	Skewness	Kurtosis	AR(1)
			Bonds			
Germany	8.082	11.351	0.712	0.315	3.498	0.008
France	8.200	11.744	0.698	0.234	2.915	0.014
Australia	8.793	14.047	0.626	-0.394	9.966	-0.055
Japan	2.616	11.385	0.230	0.326	3.330	0.008
UK	5.905	10.572	0.558	0.091	3.146	0.033
USA	5.511	7.780	0.708	0.084	2.674	-0.003
Sweden	7.655	12.926	0.592	0.249	2.937	-0.018
Netherlands	8.435	11.454	0.736	0.290	3.255	0.005
Ireland	8.382	14.051	0.597	0.310	8.917	0.095
Canada	8.049	10.355	0.777	0.111	3.473	-0.060
Austria	8.358	11.618	0.719	0.223	2.963	0.016
Average	*7.271*	*11.571*	*0.632*	*0.167*	*4.279*	0.004

As reported in Table 2, government bond returns seem to have much lower standard deviation than equity returns. All the means are relatively close to the average and there does not seem to be significant variation over the standard deviations either. Due to much lower average standard deviation compared to equities (26.403 %), the Sharpe ratio is higher.

If stock returns were mostly negatively skewed, the tails of bond returns, on average, are tilted to the positive side. The only difference is Australia, the skewness of which was -0.394. The distributions are not either as leptokurtic as those of equity returns. The average excess kurtosis of government bond returns was 4.279 and there are no real outliers within this sample. The most leptokurtic distributions were those of Australia and Ireland with values 9.666 and 8.917, respectively. Additionally, as mentioned among stock returns, the most leptokurtic deviation comes with the one having the most negative skewness. Finally, to make the comparison with the normal distribution, Jargue-Bera test was run for all the bond return series as well. The results prove that in all cases, the normal distribution is rejected at a significance level less than one per cent.

Currency returns of six different currency pairs are described in Table 3. Common features of the distributions are not as obvious as for those reported earlier. The average mean and standard deviation are lower than those of equities or government bonds but deviation among the figures is more prominent. Also the shapes of the distributions differ a lot from the asset classes described earlier as any common features do not seem to exist. However, six currency pairs would most probably not represent the whole population very well.

Table 3 Currencies: Descriptive statistics

This table reports the descriptive statistics on currency returns. For each currency, daily spot exchange rates have been used to calculate the returns. All the spot price series are quoted per USD to show the returns from a dollar investor's point of view. The reported statistics are same for all the asset classes.

Country	Mean (%)	Stdev (%)	Sharpe	Skewness	Kurtosis	AR(1)
Currencies						
Germany	3.671	14.365	0.256	0.007	-0.230	-0.080
Australia	2.526	13.581	0.186	-0.282	6.965	-0.009
UK	0.221	9.532	0.023	-0.050	3.900	0.029
Denmark	2.212	10.307	0.215	-0.057	2.305	0.009
Switzerland	4.139	11.116	0.372	-0.209	6.561	0.011
Singapore	1.941	5.282	0.367	0.057	4.897	-0.025
Average	*2.452*	*10.697*	*0.237*	*-0.089*	*4.066*	*-0.011*

Currencies differ from the other asset classes as exchange rates are affected by the riskless interest rates of both countries of the currency pair. US Dollar, as a quote currency, is paid USD Libor interest rate, whereas foreign interbank offered rate is paid on the base currency. In real life, currency returns should be adjusted with these rates. However, the effect is not likely to be significant, which is why currency returns are simply calculated from the spot prices, just like for any other asset within this study.

The average mean return per annum is 2.452 %. which clearly below the average among the other asset classes. However, a relatively low standard deviation in average results a risk-return ratio close to that of equities. Skewness and kurtosis among exchange rates vary quite a lot, which makes it challenging to find similarities. Nevertheless, the sign seems somewhat clear.

According to the sample used here, currency returns are usually negatively skew and the tails are heavy. This finding is also documented in other FX-related studies (see, e.g., Cotter 2005). It should be noted, however, that the sign of skewness depends on how the exchange rate is presented. Here, the currency pairs are reported as US dollars per one unit of the underlying currencies. For example, Cotter (2005) uses the other expression. In any case, normality of currency returns can be ruled out quite unambiguously. The only currency close to normal is the German Mark/Euro. In Jargue-Bera test Germany fails in rejecting the normal distribution at 5 % significance level (*p-value* = 0.071).

Commodity returns, calculated from daily spot prices, result in the highest annual mean returns in this paper. There is some dispersion across the commodities but the means are rather high indeed. Aluminium has the lowest mean, 0.527 % per annum, but all the others have means at least 4.530 % (nickel) while the maximum is 42.717 % (cotton). However, these returns do not come without cost. The average standard annualised standard deviation is 57.721 %, and even if cotton were excluded the average would be 36.563 %. Due to volatile prices, the compensation is rather poor risk-return wise. The average Sharpe ratio across the commodities is 0.248, which is somewhat close to the average Sharpe ratio of stocks and currencies. As shown in Table 4, all individual Sharpe ratios are close to the average.

Table 4 Commodities: Descriptive statistics

This table reports the descriptive statistics on commodity returns. For each commodity, daily spot prices have been used to calculate the returns. The reported statistics are same for all the asset classes.

Country	Mean (%)	Stdev (%)	SR	Skewness	Kurtosis	AR(1)
			Commodities			
Wheat	13.686	59.128	0.231	-0.111	0.643	-0.156
Cocoa	9.773	33.407	0.293	-0.221	11.146	-0.077
Corn	15.832	59.307	0.267	-0.043	-1.021	-0.212
Cotton	42.717	248.137	0.172	0.010	-1.479	-0.746
Crude oil	10.290	35.914	0.287	-0.124	3.470	0.003
Nickel	4.530	38.193	0.119	-0.130	3.384	-0.010
Aluminium	0.527	22.743	0.023	-0.245	2.160	-0.046
Copper	8.939	28.479	0.314	-0.078	3.949	-0.061
Gold	9.971	19.039	0.524	-0.178	5.380	-0.005
Silver	8.348	32.859	0.254	-0.913	7.402	-0.016
Average	*12.461*	*57.721*	*0.248*	*-0.203*	*3.503*	*-0.133*

For all commodities, the return distributions are negatively skewed, which means that observations are packed more to the right side of the mean. This can be a good thing from an investor’s point of view, as the returns tend to be positive. However, excess kurtosis increases the risk of extreme values in most cases. Especially, the return distribution of cocoa is quite unique when it comes to heavy tails. However, corn and cotton represent the other side with kurtosis lower than in the normal distribution. Nevertheless, Jargue-Bera test proves that none of the commodity return distributions are normal. In all cases, the null hypothesis can be easily rejected.

Randomness of the asset returns was tested with first-order autocorrelation denoted by AR(1). As reported, there is little or no serial dependence in daily return. This means that the previous day’s return cannot be used in predicting return of the following day. This shows some evidence that daily returns follow a random process. For each asset class, the results are quite similar and no major outliers are seen within asset classes.

The highest absolute AR(1) was reported for cotton (-0.746) which no doubt challenges the randomness assumption. However, the average first-order autocorrelation among commodities was -0.133, and thus, not very high. Considerably lower autocorrelations among the other asset classes imply that the returns are truly random. The average first-order autocorrelation for stocks, government bonds, currencies, and commodities are 0.017, 0.004, -0.011, and -0.133, respectively. Visual observation also reveals that the asset class matters in terms of the sign of autocorrelation. Among stock returns, autocorrelation is usually positive whereas it negative for each commodity.

In addition to descriptive statistics, further investigation was made by analysing aggregated return histograms of all four asset classes (see, Figure 4). The aggregated return series were created taking the average of individual daily log-returns. Also, the normal distribution is placed over each histogram. This provides a good comprehension of the nature of sample density curves. A highly important detail to note is that the histograms are not directly comparable with the descriptive statistics tables.

The aggregated return series behave like asset baskets containing each underlying asset with equal weight. This means that the correlation structure of these assets will also be captured. When visually analysing the shapes of the curves, it is important to understand that skewness and kurtosis are actually coskewness and cokurtosis of the underlying assets. For example, when correlation across the assets is weak, kurtosis might be much lower than what could be expected by looking at individual distributions (see, e.g., currencies). However, the purpose of the graphs is to shed light on how the asset classes behave risk-return wise in relation to each other.

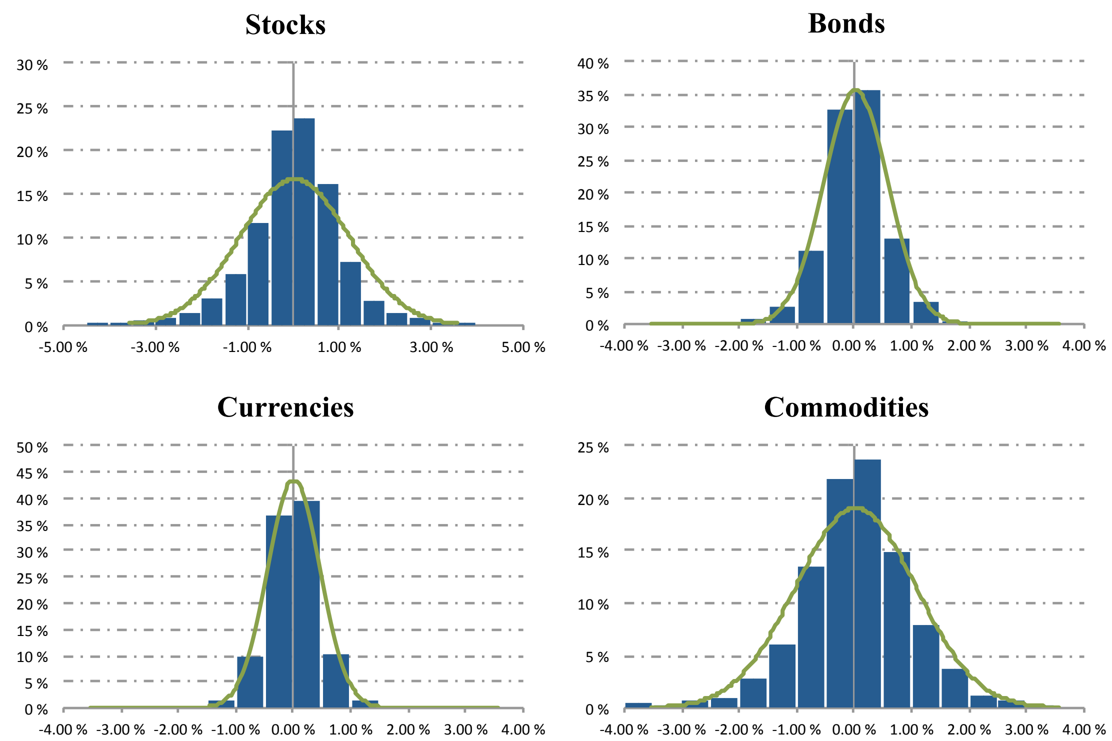

Figure 4 Return distributions of different asset classes against the normal distribution from the time period of 3.1.2000–1.4.2014.

The histograms above reveal the increasing tail risk that leptokurtosis brings. For example, return distributions of both stocks and commodities are spikey and also negatively skewed. In both cases, there are more extreme events than the normal distribution predicts. Moreover, most of these events occur on the left side of the curve. In addition, as mentioned above, the distribution of currency returns is not as spikey as could be expected. One possible reason for this could be a negative correlation across the currency pairs, which is obviously good in terms of portfolio diversification.

3.3 Building anomaly-based factors

3.3.1 Defining the anomalies in different asset classes

As the theoretical background shows, defining anomalies might not always be unambiguous, especially when the observations go beyond stock markets. Some anomalies, such as momentum or long-term reversal, can be easily defined for other asset classes as well. However, the definition of value or idiosyncratic volatility is not as straightforward. All the four anomalies (value, momentum, long-term reversal, and idiosyncratic volatility) will be next given definitions in the contexts of stocks, bonds, currencies, and commodities.

Starting from value anomaly, the common method for almost all asset classes is the negative five years return. The method follows mainly the guideline introduced by Asness et al. (2013). However, the general value measure for stocks, namely, book-to-market ratio, was not used due to data availability. Instead, the negative five-year return is applied to stocks based on the findings of Gerakos and Linnainmaa (2012). As was more specifically explained earlier, the negative of five years return and book-to-market ratio seems to have a strong positive correlation over time. About the exact calculation of this measure, the price five years ago is here calculated as an average of the prices 5.5 and 4.5 years ago, according to Asness et al. (2013). This simple version was also applied to commodities but the two other asset classes need further clarification.

Value measure for government bonds is the simplest of all asset classes. It is calculated as a difference between long and short-term yields of a treasury note, which implies the steepness of the yield curve. This term spread is actually calculated using ten and three years yields. As discussed earlier, Asness et al. (2013) suggest two other value measures where bond yields are associated as well. One of these includes five-year inflation forecast and the other one compares the change in a bond's yield during five years. Also a combination of different methods has been proposed. Nevertheless, a simple term-spread is chosen for this study for sake of simplicity and data availability issues. More complex estimate is required for currencies as the Purchasing Power Parity is considered as the driver of value. The formula 25 demonstrates how it is calculated and this method is also used in this study. The intuition behind this method is that negative five years return is adjusted with the corresponding Consumer Price Index values. In other words, the value is based on the negative five years return and five-year change in the difference of the base and quote currency CPIs. Within this thesis USD is always the quote currency and its pair is the base currency. For example, the exchange rate of GBP/USD quotes how many USDs (quote) is needed for one GBP (base).

Momentum is calculated the same way for all asset classes. Of course, the parameter values can be changed and the same setting might not be optimal for all asset classes. The benchmark in this thesis is the past twelve-month cumulative (logarithmic) return but it can be also longer. Jagadeesh and Titman (1993) suggest that momentum effect (for stocks) is strong when recent winners over the past 3 to 12 months are selected. Ranking the assets based on the cumulative returns is easy but there is an issue that should be considered. In momentum literature, the standard is to exclude the most recent month from the calculations in order to avoid liquidity issues. This is also taken into account within this thesis. Considering 20 trading days per month, the assets are sorted on past t-240 [7] through t-20 days when forming momentum portfolio. Additionally, six and three-month momentum portfolios are analysed.

The principle of selecting assets based on long-term reversal is similar to momentum; only the expected outcome is the opposite. The prices of long-term winners are expected to decline in the future and vice versa. The formation period, however, is quite ambiguous when having both momentum and reversal in the same analysis. For example, McLean (2010) sorts stocks for reversal (momentum) portfolio on past returns from t-36 through t-7 months (t-6 through t-1 months). The reason for this approach is that the same sample should not be included in the same formation period. Hjalmarsson (2010) uses this same principle when comparing momentum and reversal anomalies. Here, such a one-month cap is not necessary as daily logarithmic returns are used. However, since momentum is based on returns from t-240 days this must be the back line for long-term reversal. Within this thesis, the benchmark formation period for long-term reversal portfolios is t-36 through t-12 months. In addition, two other formation periods, t-48 and t-24 through t-12 moths, are compared.

Idiosyncratic volatility is the fourth anomaly discussed within this thesis. The definition of idiosyncratic volatility (or risk) is rather intuitive. It is the unsystematic risk that cannot be explained with the underlying asset pricing model. However, a relevant question is, which asset pricing model to use. Besides, how to cope with the fact, that different asset classes are analysed in the same study? In order to prevent the theoretical framework from expanding too much, some kind of line had to be drawn. Idiosyncratic volatility here is captured by Fama–French three-factor model regressions (see Ang et al. 2009). More specifically, it is calculated as a standard deviation of the residuals of a regression run individually for each asset. For this purpose, daily US stock market factors are downloaded from Kenneth French's website. The decision to use this data in particular is justified by the idea that US stock market is likely to affect the most other markets in the world, even beyond asset classes. In addition, if market, value, and size factors together manage to capture majority of the asset risk, the error terms add value for the analysis. Of course, the main reason for such a compromise was availability of the data.

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[7] A general assumption in this thesis is that there are 250 trading days per year. In average, however, the number of trading days per month is 20. In momentum and reversal calculations, the changing parameter is the number of months, i.e., this yields 240 observations for a year. However, this is not a problem since days per year and days per month are never used in the same context.

3.3.2 Trading algorithm for the factors

The empirical part of this thesis was mainly conducted using R which is a free software environment for statistical computing. For those who are interested, the code is freely available in the appendix of this paper. In order to construct the factors, functions were created to rank the assets based on the underlying anomaly. The basic idea regarding the factors is that two portfolios are formed, one following a long-only and another one a short-only strategy. In fact, the short-only portfolio is treated as the long-only one, although, the final returns are multiplied by minus one to transform them into short position returns. Finally, the factor is considered as a long-short portfolio, which is a combination of these two, i.e., factor returns are calculated as a difference of the two portfolio returns.

Before running a portfolio back-testing procedure, it is essential to define some basic parameters regarding the portfolio construction, and also those behind the anomalies. From the back-testing point of view, the relevant parameters are the estimation window and portfolio rebalancing period. The former parameter is often defined by the anomaly, e.g., the window for a momentum portfolio is usually twelve months. Rebalancing or update period defines how often the portfolio weights are readjusted according to the strategy. The default value for this parameter is 20 days, i.e., portfolios are rebalanced every month. Also the anomalies need further specifications. This is important when different settings are compared. For example, momentum and reversal strategies can be based on different lengths of historical returns. On the other hand, idiosyncratic volatility anomaly has two different approaches. One says that the assets with high idiosyncratic risk should be over weighted, and the other one says just the opposite. That is why this parameter should be left open for researcher's decision. Additionally, there are some general anomaly-linked parameters that affect the portfolio formation. The first decision to make is, in how many percentiles the assets should be sorted based on the anomaly in question. By default, there are three percentiles within this study, i.e., the top third is represents the long-only portfolio and the bottom third is included in the short-only portfolio. This is also the setting that Asness et al. (2013) use. However, different percentiles are used in different papers. The other question is, how should these sub-portfolios be weighted. Often, the portfolios are equal weighted but different approaches are utilised as well, e.g., weighting based on volatility or ranking according to the effect in question.

In most academic papers, the sub-portfolios are equal weighted. However, the original groups can be further divided in smaller groups. For example, Fama and French (1993) use this method when building value and size factors. Take, e.g., the size factor where the stocks are first sorted into three equal weighted groups based on size. Then both small and big size groups are sorted again by value to low, medium and high. Finally, they are left with six portfolios that they use to mimic the risk factor. Asness et al. (2013) build factors by weighting the assets in proportion to the rank based on value or momentum. Then, McLean (2010) tries different weighting schemes, such as, equal, value, idiosyncratic risk and inversed idiosyncratic risk weights. The number of different approaches in the academic literature supports the idea of comparing different weighting schemes for the portfolios. In this study, the portfolios are weighted in proportion to the ranks based on the underlying anomaly or according to either idiosyncratic risk or reversed idiosyncratic risk. Also, equal weights are applied.

The portfolio algorithm is rather simple and takes into account the fact that logarithmic returns are problematic when calculating return on portfolio. Logarithmic returns are used because of their statistical properties, but they must not be weighted. To avoid this, the algorithm calculates the quantity of each asset in the portfolio given the weights, value of the portfolio, and price of each asset. Following this logic, portfolio weights can be calculated when the number of each asset, their prices, and value of the portfolio are known. Logarithmic returns are only used as input for the models. This connection can be illustrated with the following figure:

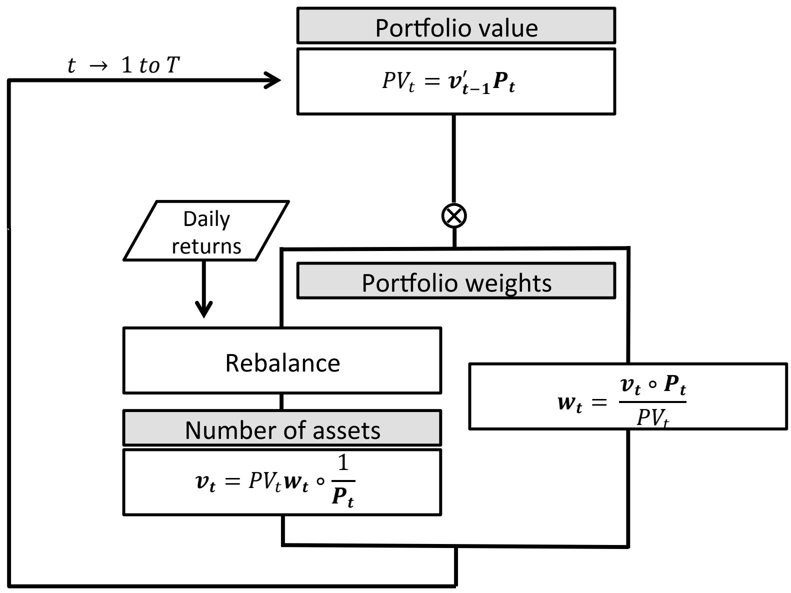

Figure 5 The portfolio construction algorithm

Here, $\boldsymbol{v}_t$ is a vector including the number of assets held in the portfolio at time t. Portfolio weights are denoted by vector $\boldsymbol{w}_t$, portfolio value by scalar PV_t, and price vector at time t by P_t. Obviously, when rebalancing a portfolio, the weights are produced with a model that uses daily returns as an input. However, when the portfolio is not rebalanced, the weights can be calculated from the other known parameters as the unit prices change. Also, the number of each asset in the portfolio remains unchanged when rebalancing is not done. The value of portfolio is updated after each iteration by multiplying the number of each asset by the corresponding price. Mathematically speaking, this element-wise product is denoted by $\circ$ which is known as Hadamard product. In the beginning, the value of portfolio is one hundred and the first rebalancing is done based on data prior to the investment period.

The same principal is followed next as the Black–Litterman model and mean-variance based optimisation are applied. The only changing part is the actual rebalancing process.

3.4 Portfolio optimisation in the Black–Litterman framework

3.4.1 Parameters for the model

The model itself is simple to configure and use as a part of portfolio optimisation. However, some of its parameters are abstract and there are number of different approaches to solve them. Probably the most problematic parts of the model are the market portfolio and, especially, its weights, uncertainty of the views, and the tau parameter, which indicates the uncertainty of the prior mean. In addition, the risk aversion parameter, λ is not straightforward to estimate. In relating literature, different methods are discussed to solve these issues, some more thoroughly than others. Here the methods have been chosen based on the existing research papers and also their importance regarding the main focus of the thesis. Since efficient trading strategies are more of interest, no thorough comparison of minor details will be necessary.

Starting from the market portfolio, the data already defines the accuracy that can be realistically considered. In a stock portfolio study, it would be rather simple to use the weights of a major stock market index, such as, S&P 500. However, in this study not only market indices are used but also different asset classes, which makes it much more difficult to estimate their weights in an unobservable market portfolio. For this reason, an equally weighted portfolio (1/N) is used as a benchmark. Nevertheless, it should be noted that an investor with neutral or no views would eventually consider an equal weighted portfolio optimal. This is due to the inversed optimisation, which is to calculate the implied equilibrium returns.

For the implied equilibrium returns, it is also crucial to estimate the risk aversion parameter, λ. By default, this is the ratio of expected excess return and variance of the market. However, it is often considered constant over the investment horizon. Curtillet and Dieudonne (2007) calculated risk aversion parameters for different asset classes and found that the variation can be relatively high. For commodities, they estimated risk aversion of 1.8 whereas the value for investment grade bonds was 20. In many articles, the parameter is set to 3, which is appropriate for the global stock market. Within this thesis, the benchmark is also 3 but the robustness tests indicate that performance is not really sensitive to risk aversion.

The views themselves are the core of the model, although, it is the investor's responsibility to express them. Nevertheless, weighting them and defining uncertainty of the views is important from theoretical perspective. In the Black–Litterman model, there are two ways of expressing views; they can be either absolute or relative. The nature of this study supports the latter, i.e., the question is, how much the winners will outperform the losers. For this reason, the rows of the P-matrix must sum up to zero. In this view weighting scheme, the expected winners are given positive weights and vice versa. However, the weighting method slightly differs from that usually seen in relating articles. Instead of placing either minus one or one depending on which group the asset belongs to, the weighting follows the same method that is applied to the underlying factor. For example, momentum view is applied to stocks. At the time of rebalancing, the past winners and losers are identified and long-short weights are given accordingly. In this case, the first row of the P-matrix, i.e., the view for stocks, acquires exactly these portfolio weights. All the non-stock elements of the first row are given zero weight. As a result, the expected performance of stocks can be traced from this specific combination. The same principal is followed on all asset classes as indicated below.

$$P = \begin{pmatrix} p_{S1} & \cdots & p_{Sn} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{B1} & \cdots & p_{Bn} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_{FX1} & \cdots & p_{FXn} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{C1} & \cdots & p_{Cn} \end{pmatrix} \tag{26}$$

Here, views are expressed individually on each asset class, stocks, bonds, currencies, and commodities, respectively. Each row of the matrix will sum to zero. The essential benefit of this method is in its ability to include different views in the same portfolio, consisting of many asset classes.

Uncertainty of the views can be solved in different ways as already discussed earlier. This uncertainty is indicated with omega matrix, which can be understood as a covariance of the views. Since the out-of-sample study covers over twenty years, it is clear that a simple and adaptive method is well justifiable. In this light, the method suggested by He and Litterman (1999) serves the purpose better than well. If the variance of a view k can be written as $(\mathrm{p}_k \Sigma \mathrm{p}'_k)\tau$, the result is that the scalar τ, eventually, becomes irrelevant. Moreover, as highlighted by Idzorek (2005), this method is the most used in relating literature.

3.4.2 Optimal portfolios based on market anomalies

The first step with market anomaly based investing was to create portfolios simply based on the properties of the assets. Relying on the anomalies, the assets have been ranked in proportion to past performance, value, or what ever the underlying anomaly might be. As a long-short trading strategy, this yields a factor that can provide relevant information also for a more diversified portfolio strategy. In this study, these asset class specific factors are crucial when it comes to expressing views on portfolios.

The portfolio construction algorithm is discussed in connection to the factors. The logic is well explained by Figure 5, although, rebalancing the portfolio is where the Black–Litterman model and mean-variance optimisation come into picture. When it comes to rebalancing the portfolio the market portfolio should be considered as a benchmark. As already noted, equal weights give a starting point for every rebalancing procedure. Accordingly, implied equilibrium returns are calculated as a product of the benchmark weights, market risk aversion parameter, and the covariance matrix of the underlying asset returns. Since estimation of the covariance matrix is not in the focus of this study, a basic sample covariance is estimated. However, it is to be considered that covariance matrix is highly important when it comes to portfolio optimisation.

Incorporating investors' views is what makes the Black–Litterman model unique. Given the factors built prior to optimisation, it is possible to express views based on their historical performance. The view itself expresses the rate of return on which the winners are expected to outperform the losers, i.e., it is the expected return of the underlying factor. Within this thesis, the expected return of the following investment period equals the mean return of the previous month. The idea of expressing the view as an expected return of the corresponding long-short portfolio is suggested by Fabozzi et al. (2010). However, they do not provide any method to estimate this return. Babameto and Harris (2008), who test Black–Litterman strategy based on value and momentum, suggest an appealing method to estimate the expected return of a long-short portfolio. Their solution to the problem is regression. They predict momentum returns with the US term spread and value with the aggregate book-to-market ratio of the market.

The use of the prior mean as an expected return can be explained with number of reasons. First, this method is not trying to predict the return of any instrument. Instead, the prior mean indicates, how much the winner group can be expected to outperform the losers. The allocation that is expected to achieve this result is solved at the time of rebalancing. Of course, simplification and focus of the study are also relevant reasons. Trying to predict the return on four different anomalies in multiple asset classes would go beyond the scope of this study.

From the perspective of portfolio optimisation, some general parameters and assumptions are worth mentioning. In a simple mean-variance study, the in-sample period is unambiguous since predetermined length of data is used as an input of the optimisation. In this case, however, the concept is not as clear. For the covariance matrix, a sample of 250 trading days is always used as input. However, strategies based on historical performance, such as, momentum and reversal, require samples with different lengths. Moreover, there are a large variety of different combinations with all possible parameters. This is why the comparison is mostly limited to the different ways to utilise the anomalies. When it comes to portfolio optimisation process itself, holding period is probably the one that is still essential to investigate. This is because in the reality transaction costs due to frequent rebalancing tend to eat all the benefits gained by the optimisation.

3.5 Measuring portfolio performance

The algorithm described earlier is a tool to back-test performance of portfolios following different strategies. In the basic setting, all the four asset classes are included in the portfolio. Then different anomalies are applied to the assets depending on which asset class they belong to. In order to attach the views into the portfolio optimisation, the Black−Litterman model is to be utilised. After running the code, the final product is a time series of the portfolio value from 11th Jul 1994 to 1st Apr 2014. The next step is to evaluate whether or not the strategy pays off. Does the strategy provide abnormal returns or what is the return vis-à-vis the taken risk compared to other strategies?

Probably the most intuitive measure to compare performance of different investment strategies is the Sharpe ratio. The ratio indicates how well the risk is compensated by return. Hence, adjusting returns with risk makes it easier to compare different portfolios together. As defined by Sharpe (1994), this *reward-to-variability* ratio can be written as

$$SR_i = \frac{R_i - R_f}{\sigma_i}, \tag{27}$$

Where R_i and R_f denote return of the asset i and the risk-free rate return, respectively. Standard deviation of the asset i, i.e., the risk measure is denoted by σ_i. The original expression characterises the return-risk ratio ex ante, which is why the expected excess return is calculated. As an ex post performance measure, the realised returns and standard deviation will be used. In addition, the risk-free rate of return is ignored within this study.

Since daily results are investigated, the Sharpe ratio will also acquire a daily form. Usually it is more intuitive to study annualised values, which is also done here. A daily Sharpe ratio can be annualised as follows:

$$SR_{p.a} = \sqrt{250}\, SR_{p.d} \tag{28}$$

This expression assumes 250 trading days per year, which is the assumption made in this thesis.

When using the Sharpe ratio as a performance measure, it is important to be aware of its flaws. Given excess kurtosis and skewness often reported on, e.g., stock returns may cause biased measures. The biggest problem relates to standard deviation and its ability to reflect risk. When returns are not normally distributed it may not be unambiguous to tell what is the effect of additional volatility risk-return wise. Especially, assets with different distributions might become less comparable. However, the Sharpe ratio is still often used in measuring the performance of a portfolio both in academic and business world. This is why it is the most central performance measure also in this study.

A more sophisticated model is needed to investigate if a portfolio provides abnormal returns with respect to known risk factors. Abnormal returns or alpha is measured with the Fama-French three-factor model, which tries to explain returns with three common risk factors: market, size, and value. The regression is defined as

$$r_{it} - r_{ft} = \alpha_i + \beta_{im}(r_{mt} - r_{ft}) + \beta_{is}SMB_t + \beta_{iv}HML_t + \varepsilon_{it} , \tag{29}$$

where excess returns are explained with the three risk factors. Market factor is simply the excess return of the market. The size factor, i.e. *small minus big,* captures the exposure to the size premium, which should be gained by investing in small capitalisation stocks. The value factor, i.e. *high minus low,* measures the premium earned by preferring high book-to-market stocks to the low ones. In terms of abnormal return, alpha is the key value. Significantly positive alpha can be interpreted as the investor's ability to gain excess return beyond the risk factors, assuming the factor exposures are fully captured. Finally, the error term of the regression (for a given asset at a given time) is denoted by ε_{it}.

Because of availability of the factors, daily risk factors on the US stock market have been used within this thesis. The factors have been downloaded from French's website. In order to analyse the significance of alpha, also according t-values are reported. Since cumulative returns are used in calculating the t-values, the autocorrelation needs to be controlled by using Newey-West (1987) standard errors.

4 RESULTS OF THE EMPIRICAL STUDY

4.1 Zero-cost portfolios

Before applying the anomalies into the portfolio optimisation, zero-cost portfolios are analysed with respect to individual anomalies. This is to see if they provide significant return premia and thus appealing factors to be utilised in portfolio construction. In addition, long-only strategies are tested because it is important to see whether more sophisticated portfolio optimisation adds value in comparison with a simple anomaly strategy.

As discussed earlier, there are plenty of different combinations when comparing different strategies. In addition to the four asset classes and anomalies, the assets can be sorted to different numbers of groups, there are different weighting methods, and different lengths of data included in estimation. In total 124 portfolios are back-tested and the most relevant ones are taken to the next round when analysing Black–Litterman based portfolio strategy.

The first step in analysing the anomalies is to construct portfolios that follow a simple long-only strategy where assets are selected based on an individual anomaly. For each portfolio, the input parameters are selected from Table 5. In other words, each individual portfolio includes only one asset class, the allocation of which is determined by one anomaly with given parameters. Say, a stock momentum factor is to be built. For each month, the portfolio is rebalanced and assets are divided in, e.g., three groups according to the momentum effect. For momentum, cumulative return over, e.g., twelve months (minus the previous month) is assessed. Within the high momentum group, weights could be given, e.g., by idiosyncratic volatility where either high or low volatility indicates the ranking order. Following this logic, different combinations have been used when constructing factors for different asset classes and anomalies.

Table 5 Properties of zero-cost portfolios

When constructing zero-cost or long-short portfolios based on anomalies, combinations of different parameters can be compared. A single strategy is always for one asset class and anomaly only. When sorting assets based on the anomaly they can be allocated in different numbers of groups or percentiles. For a long-short portfolio, the first and the last percentiles are considered, and for a long-only portfolio, the last or the “winner” one. Within these groups, the assets can be equally weighted, or they can be weighted based on ranking or idiosyncratic volatility. If the latter is used, assets with either high (1) or low (0) idiosyncratic volatility will be given more weight. For momentum and reversal strategies, different combinations of historical returns are used. This determines how many months are selected and how many months is the lag from the present time.

Asset class	Anomaly	Groups	From (months)	Until (months)	Weights	IVOL (high/low)
Stocks	Momentum	2	12	1	Equal	1
Bonds	Reversal	3	6	1	Rank	0
Currencies	Value	5	3	1	IVOL	
Commodities	IVOL		24	12		
			36	12		
			48	12		

Later when the Black–Litterman model is applied, the portfolios will be long only. This is why it is relevant to see if a simple strategy could outperform such a sophisticated model. As will be seen in Table 6 and Table 7, it is not obvious that a beneficial long-only strategy would be a sign of a significant risk factor. For example, each one of the anomalies seems to fit very well for government bonds. However, results of the long-short strategy reveal that the long-only portfolio does not have superior returns compared to an opposite strategy. This might indicate that the anomaly would not add value in the Black–Litterman context.

After constructing the long-only portfolios, returns of long-short portfolios are analysed accordingly. These results should shed more light on the question of the positive premium these anomalies might provide. As already discussed, these portfolios are considered as potential risk factors, which are to be utilised in a Black–Litterman based portfolio strategy.

Given the parameters of Table 5, 124 portfolios are back-tested. However, in order to compare different anomalies across the asset classes, portfolios with the basic settings were analysed in the first place. This initial configuration yields 44 portfolios with the following parameters. In value portfolios, the assets are ranked based on their five-year negative return that, more precisely, comes from the average of the past 5.5 and 4.5-year prices. As discussed earlier, the value measure is slightly different between the asset classes; however, the calculation remains the same in all configurations. In momentum portfolios, assets are ranked according to the past 12-month cumulative returns less the returns of the previous month. Accordingly, past 36 less the most recent 12 months are of interest in the reversal strategy. For idiosyncratic volatility strategy, the assets with lower idiosyncratic volatility are selected in the portfolio. Idiosyncratic volatility is measured against the three-factor model.

For each portfolio, the following weighting schemes are considered in the initial setting. In all cases, the assets are divided into three groups after sorting them on the anomaly in question. The middle group is always ignored and, in long-only portfolios, only the upper third is selected. These assets are then weighted according to three different methods: equal, rank, and idiosyncratic volatility (IVOL). In equal weighting, all assets are given the same weight. Rank based weighting means that those assets having a bigger exposure to a specific anomaly have larger weights in the portfolio. In fact, the ranking procedure is already done when selecting assets in the portfolio. Finally, IVOL weighting is exactly the same as in the IVOL strategy. However, the principle in the basic setting is exactly the opposite: those assets with higher IVOL are given larger weights. The opposite weighting is also tested but this parameter was chosen for the initial setting in order to see how both high and low idiosyncratic volatility affects the returns.

Table 6 Results for long-only portfolios

This table reports annualised mean with t-value, standard deviation, Sharpe ratio, and 3-factor alpha with a respective t-value (Newey-West correction applied) for long-only portfolios. Momentum is based on past 12 to 1 months and reversal on past 36 to 12 months. In IVOL portfolio (weighting), assets with low (high) idiosyncratic volatility are overweighed. Statistical significance is indicated with stars: ***, **, and * referring to p-values <0.001, <0.01, and <0.05, respectively. In this table, only the basic setting is tested. Other combinations are considered in further analysis. Portfolios are constructed with rolling window estimation where, after each month, historical observations are used for rebalancing the portfolio. In total, daily observations cover the time period of 11.7.1994–1.4.2014.

		Value			Momentum			Reversal			IVOL	
		Equal	Rank	IVOL	Equal	Rank	IVOL	Equal	Rank	IVOL	Equal	Rank
Stocks	Mean (%)	8.349	8.522	9.176	13.240	13.390	13.600	8.812	8.233	7.617	7.823	7.847
	(t-stat)	(1.946)	(1.975 *)	(2.006 *)	(3.076 **)	(3.083 **)	(2.895 **)	(2.016 *)	(1.866)	(1.846)	(2.127 *)	(2.156 *)
	Stdev (%)	19.469	19.576	20.755	19.529	19.705	21.313	19.833	20.023	18.724	16.690	16.514
	Sharpe	0.429	0.435	0.442	0.678	0.680	0.638	0.444	0.411	0.407	0.469	0.475
	Alpha (%)	3.097	3.214	4.458	7.804	7.928	8.344	3.162	2.412	0.580	1.451	1.381
	(t-stat)	(0.851)	(0.876)	(1.115)	(2.128 *)	(2.130 *)	(2.014 *)	(0.885)	(0.688)	(0.194)	(0.579)	(0.569)
Bonds	Mean (%)	7.650	7.577	8.122	8.665	8.682	8.837	8.875	8.354	8.548	6.716	6.696
	(t-stat)	(3.98 ***)	(3.94 ***)	(4.009 ***)	(4.29 ***)	(4.285 ***)	(4.279 ***)	(4.515 ***)	(4.154 ***)	(4.459 ***)	(4.351 ***)	(4.406 ***)
	Stdev (%)	8.721	8.724	9.191	9.164	9.193	9.369	8.917	9.124	8.697	7.004	6.895
	Sharpe	0.877	0.868	0.884	0.946	0.944	0.943	0.995	0.916	0.983	0.959	0.971
	Alpha	7.766	7.683	8.073	8.775	8.792	8.890	9.027	8.497	8.780	7.087	7.104
	(t-stat)	(3.857 ***)	(3.816 ***)	(3.783 ***)	(4.209 ***)	(4.203 ***)	(4.197 ***)	(4.383 ***)	(4.072 ***)	(4.361 ***)	(4.250 ***)	(4.322 ***)
Currencies	Mean (%)	1.642	1.676	1.693	1.459	1.431	1.465	2.808	2.951	2.698	0.403	0.454
	(t-stat)	(0.867)	(0.886)	(0.857)	(0.817)	(0.796)	(0.773)	(1.607)	(1.651)	(1.596)	(0.309)	(0.358)
	Stdev (%)	8.594	8.582	8.964	8.101	8.154	8.596	7.927	8.107	7.670	5.931	5.758
	Sharpe	0.191	0.195	0.189	0.180	0.175	0.170	0.354	0.364	0.352	0.068	0.079
	Alpha (%)	0.977	0.997	1.035	0.990	0.972	1.040	2.214	2.360	2.088	-0.344	-0.308
	(t-stat)	(0.502)	(0.515)	(0.514)	(0.541)	(0.528)	(0.538)	(1.258)	(1.324)	(1.228)	(-0.254)	(-0.234)
Commodities	Mean (%)	12.922	13.447	11.906	9.683	9.700	9.677	10.720	8.344	10.833	3.423	3.590
	(t-stat)	(2.212 *)	(2.230 *)	(1.528)	(1.768)	(1.752)	(1.438)	(1.932)	(1.562)	(2.266 *)	(0.994)	(1.069)
	Stdev (%)	26.499	27.363	35.362	24.843	25.125	30.540	25.179	24.241	21.694	15.620	15.234
	Sharpe	0.488	0.491	0.337	0.390	0.386	0.317	0.426	0.344	0.499	0.219	0.236
	Alpha (%)	10.037	10.574	9.088	7.503	7.550	7.249	8.072	5.727	8.216	2.036	2.243
	(t-stat)	(2.156 *)	(2.224 *)	(1.600)	(1.514)	(1.502)	(1.304)	(1.763)	(1.239)	(1.924)	(0.612)	(0.689)

The first observation when analysing the results is that different anomalies behave differently in different asset classes. All strategies seem to provide positive returns in a long run but some strategies leave the others far behind; at least, when observing one asset class only. A good example would be momentum among stocks. Momentum strategy for stocks yields Sharpe ratios above 0.6 regardless of the weighting scheme while the Sharpe ratio of the other strategies vary between 0.4 and 0.5. This is an impressive result considering the average Sharpe ratio among the stock indices, which was 0.26. An interesting and somewhat surprising observation was that an important factor behind this result was lower volatility provided by momentum strategy. In addition, momentum strategy yields superior and statistically significant three-factor alpha of 7.8–8.3 per cent per annum.

Momentum strategy suits well for other asset classes as well, especially for government bonds. However, it is never the best anomaly to be considered for the other asset classes. Another noticeable difference can be found in commodity strategies. Value strategy with either equal or ranking based weighting yields clearly better risk-return ratio than any other strategy for commodities. Additionally, value effect is the only one that is capable of producing significantly positive alpha among commodities.

Government bonds are interesting as an asset class since all anomaly-based long-only strategies seem appealing for them. While the average Sharpe ratio among the bond indices is 0.632 and the maximum is 0.777 (Canada), all strategies provide a Sharpe ratio greater than 0.868. Even more surprising is that alpha is positive and statistically significant for all bond strategies. These findings separate government bonds from all other asset classes. One very likely reason for the results can be found from the bond indices themselves. They have not only a tendency to increase in value but they are also not very volatile. In fact, as the graph below shows, different strategies do not make a big difference between bond returns. Of course, the sample may have an important impact on these findings. First, government bonds are usually not as risky as assets driven by corporate risk. As a result, the volatility is also considerably lower. On the other hand, interest rates have been mostly declining through the 21st century, which has made bonds look superior compared to many other asset classes.

The reversal strategy is profitable in all asset classes but it only provides significantly positive alpha for government bonds. In addition, it usually does not shine next to the other strategies. However, it seems to fit especially well for currencies. Sharpe ratios in currencies are not very high in general but still reversal strategy manages to provide a tempting compensation on risk. Sharpe ratios of individual currencies are in a range of 0.023–0.372 while the average is only 0.237. Reversal strategy, however, yields Sharpe ratios varying between 0.352 and 0.364, depending on weighting. This is clearly a better ratio than what any other strategy provided.

Overweighting assets with low idiosyncratic volatility yields somewhat disappointing results in comparison with the other strategies, especially, when it comes to mean returns or alpha. Risk-return wise, the most potential asset class for IVOL strategy is bonds where the results are at least competitive. The most evident and common result is that the strategy produces the lowest standard deviation in each asset class. That certainly makes it important strategy in terms of portfolio allocation.

Another interesting finding is that weighting methods have little impact on the results within the anomalies. In general, the results are usually in the same ballpark when comparing the strategies among different asset classes. For example, momentum strategy for stocks yields Sharpe ratios of 0.678, 0.680, and 0.638 when the assets in the winner group are weighted by equal weight, ranking, or IVOL, respectively. However, the number of sub groups seems to have a greater effect. Take, e.g., equally weighted stock momentum portfolios with different number or sub group. When the stock indices are sorted into 2, 3, and 5 groups, the most extreme winner portfolio provides Sharpe ratios of 0.662, 0.678, and 0.704, respectively. This obviously indicates that momentum effect is strong among stocks. In other words, the risk adjusted return is higher for those momentum portfolios including recently poorly performed stocks.

In addition to static performance measures, it is relevant to analyse the dynamic behaviour of different strategies throughout the whole investment period. The following four graphs plot the portfolio performance of stocks, bonds, currencies, and commodities when different long-only strategies are applied. For each strategy, weighting method is selected according to the biggest Sharpe ratios showed in Table 6.

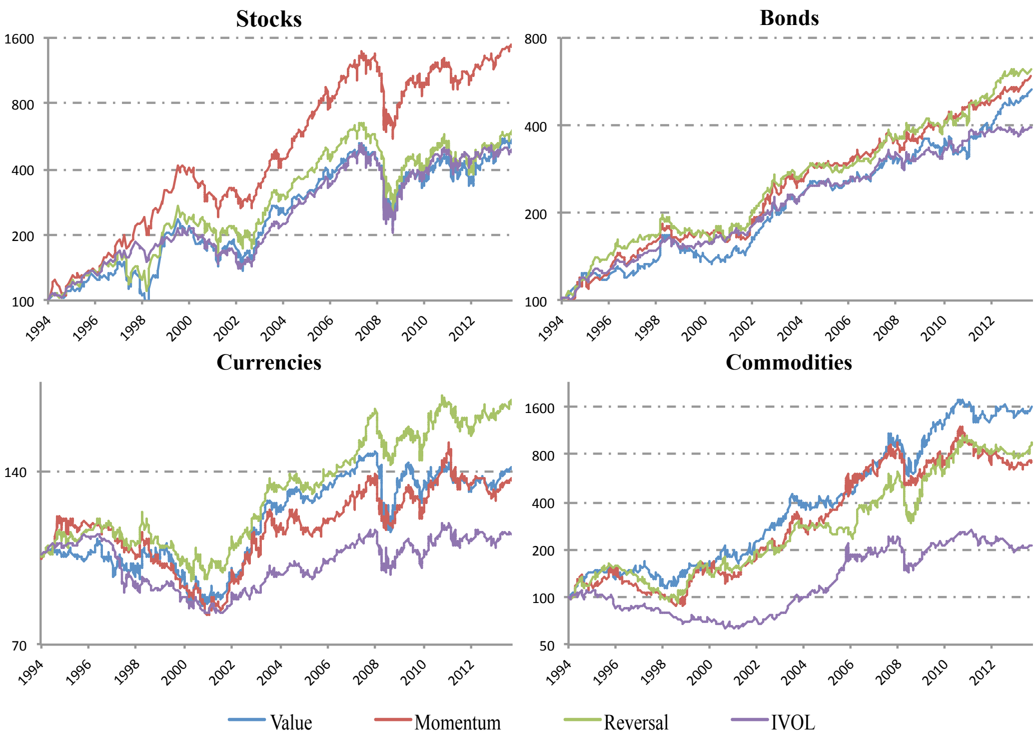

Figure 6 Performance of long-only strategies by asset classes; values presented with a logarithmic scale

The graph strengthens the understanding that different anomalies behave in a different manner in each asset class. While momentum is the most profitable for stocks, it is not as outstanding in the other asset classes. As the values are presented with a logarithmic scale, movements of the lines remain comparable regardless of the initial value of the portfolio. As a consequence, it could be observed that even though, idiosyncratic volatility anomaly is not outstandingly beneficial, the returns are not very volatile either. However, no strategy manages to avoid the collapse caused by the fall of Lehman Brothers. Nevertheless, the most distinct graph is the one of government bonds. It clearly shows that all strategies are highly profitable for the bond indices and that the difference between them is not very strong.

Strong movements emphasize the nature of some anomalies. For example, momentum, despite of its great profitability, is reported to crash from time to time when the market is stressed and highly volatile (Daniel & Moskowitz, 2013). When looking at Figure 6, the strongest declines are seen during 2008, indeed. In fact, the biggest monthly shortfall was realised in September–October 2008 for stock momentum when the portfolio lost 44 % of its value. Knowing the negative skewness of momentum returns would make it important to adjust the model under a certain type of market environment. However, these considerations are left for further investigation. Additionally, remarkable shortfalls are reported for other asset classes and anomalies as well. It also seems that those of momentum strategies do not seem to occur more frequently. For example, within the stock strategies, there are eleven individual months for value, momentum, and reversal strategy when the drop has been over -10 %.

Many of the strategies discussed previously were highly profitable. If the anomalies were strong it would be natural to assume that going short with an opposite strategy would yield even more superior returns. However, making such an assumption will probably cause a disappointment. Short strategies rarely seem to be profitable, which makes long-short strategies to suffer. Anyhow, some convincing results are reported, which become important when applying the anomalies to the Black–Litterman model.

Table 7 presents respective results for long-short portfolios. The findings are important in finding significant risk premia to be utilised in expressing views. A significant return in the table can be shortly interpreted as how many per cent a single long-only strategy outperforms a corresponding short strategy. This is relevant information when deciding to what extent certain assets should outperform some others. Except the zero-cost aspect, all strategies follow the same principles as the long-only portfolios. In addition, the reported values are fully comparable with those in Table 6.

Table 7 Results for long-short portfolios

This table reports annualised mean with t-value, standard deviation, Sharpe ratio, and 3-factor alpha with a respective t-value (Newey-West correction applied) for long-short portfolios. Momentum is based on past 12 to 1 months and reversal on past 36 to 12 months. In IVOL portfolio (weighting), assets with low (high) idiosyncratic volatility are overweighed. Statistical significance is indicated with stars: ***, **, and * referring to p-values <0.001, <0.01, and <0.05, respectively. In this table, only the basic setting is tested. Other combinations are considered in further analysis. Portfolios are constructed with rolling window estimation where, after each month, historical observations are used for rebalancing the portfolio. In total, daily observations cover the time period of 11.7.1994–1.4.2014.

		Value			Momentum			Reversal			IVOL	
		Equal	Rank	IVOL	Equal	Rank	IVOL	Equal	Rank	IVOL	Equal	Rank
Stocks	Mean (%)	-0.928	0.348	0.988	6.114	5.867	6.884	-0.392	-1.179	-1.926	-6.720	-6.440
	(t-stat)	(-0.283)	(0.107)	(0.272)	(1.783)	(1.758)	(1.848)	(-0.131)	(-0.408)	(-0.549)	(-2.093 *)	(-2.021 *)
	Stdev (%)	14.862	14.700	16.458	15.560	15.140	16.899	13.552	13.125	15.929	14.568	14.458
	Sharpe	-0.062	0.024	0.060	0.393	0.388	0.407	-0.029	-0.090	-0.121	-0.461	-0.445
	Alpha (%)	-0.023	0.811	2.336	6.577	6.204	8.004	-0.258	-1.132	-2.985	-7.209	-7.201
	(t-stat)	(-0.008)	(0.268)	(0.711)	(2.076 *)	(2.029 *)	(2.342 *)	(-0.092)	(-0.422)	(-0.911)	(-2.363 *)	(-2.465 *)
Bonds	Mean (%)	-0.315	-0.049	0.640	2.263	2.283	2.649	2.238	1.836	1.714	-2.409	-1.305
	(t-stat)	(-0.182)	(-0.028)	(0.366)	(1.038)	(1.036)	(1.217)	(1.092)	(0.907)	(0.828)	(-1.28)	(-0.675)
	Stdev (%)	7.863	8.037	7.941	9.887	9.992	9.873	9.302	9.179	9.387	8.541	8.773
	Sharpe	-0.040	-0.006	0.081	0.229	0.228	0.268	0.241	0.200	0.183	-0.282	-0.149
	Alpha (%)	0.001	0.248	0.716	2.630	2.615	2.871	2.458	2.055	2.074	-1.233	-0.291
	(t-stat)	(0.000)	(0.152)	(0.423)	(1.178)	(1.186)	(1.294)	(1.256)	(1.073)	(1.065)	(-0.741)	(-0.163)
Currencies	Mean (%)	1.201	1.208	1.389	1.591	1.271	1.460	2.626	2.749	2.416	-1.377	-1.043
	(t-stat)	(0.631)	(0.613)	(0.718)	(0.755)	(0.59)	(0.685)	(1.419)	(1.461)	(1.244)	(-0.686)	(-0.535)
	Stdev (%)	8.636	8.937	8.775	9.560	9.777	9.678	8.397	8.536	8.813	9.109	8.852
	Sharpe	0.139	0.135	0.158	0.166	0.130	0.151	0.313	0.322	0.274	-0.151	-0.118
	Alpha (%)	0.823	0.887	1.063	1.749	1.446	1.682	2.476	2.598	2.204	-1.862	-1.595
	(t-stat)	(0.504)	(0.522)	(0.637)	(0.94)	(0.762)	(0.893)	(1.577)	(1.648)	(1.345)	(-1.129)	(-1.012)
Commodities	Mean (%)	7.800	5.195	6.891	0.719	2.626	-2.830	4.832	1.375	3.595	-11.865	-9.656
	(t-stat)	(1.133)	(0.721)	(0.808)	(0.086)	(0.309)	(-0.37)	(0.655)	(0.175)	(0.49)	(-1.633)	(-1.527)
	Stdev (%)	31.224	32.695	38.710	37.933	38.606	34.724	33.474	35.669	33.320	32.958	28.692
	Sharpe	0.250	0.159	0.178	0.019	0.068	-0.081	0.144	0.039	0.108	-0.360	-0.337
	Alpha (%)	6.743	4.004	5.704	1.412	3.568	-2.594	4.319	0.915	2.988	-10.634	-8.153
	(t-stat)	(1.217)	(0.714)	(0.895)	(0.19)	(0.465)	(-0.42)	(0.687)	(0.131)	(0.499)	(-1.898)	(-1.496)

The first observation when comparing the results with those of long-only portfolios is that the level of statistical significance is extremely lower. While all long-only strategies for bonds provided significantly positive alpha, the zero-cost strategies are never as appealing. The most obvious reason for this is that going short on government bonds is not profitable.

Momentum strategy, in average, appears as the most profitable, especially for stocks. Regardless of weighting method, all momentum strategies provide significantly positive alpha. Of all strategies and asset classes, momentum strategy for stocks is ultimately the most effective zero-cost strategy. Furthermore, the highest mean return and Sharpe ratio is achieved when the stocks with high idiosyncratic volatility are overweighed. In fact, the results show that high IVOL predicts higher returns, at least for stocks. When low IVOL strategy is applied, the stock portfolio provides noticeably negative alpha at a confidence level less than 0.005. However, just an opposite strategy seems well profitable. When high IVOL strategy, with three sub groups and rank weighting, is applied to stocks, the Sharpe ratio becomes 0.391 per annum and alpha is nearly significantly positive with a p-value of 0.064.

As discussed, idiosyncratic volatility divides opinions in finance literature. The results of all IVOL strategies here support more the findings of Merton (1987) or Goyal and Santa-Clara (2003). High idiosyncratic volatility, indeed, seems to have a positive impact on alpha. In fact, the results show that this connection exists within each asset class. For all asset classes, except currencies, the Sharpe ratio of low IVOL strategy is more negative than it is positive for other strategies. This is a strong evidence for existence of high IVOL effect although the alphas are not significant.

The results on value effect are note quite as expected. Considering the work of Fama and French (1992, 1996), the absence of value premium is surprising. Of course, it crucial to note that their value measure is applicable for stocks although, they show (1996) that negative five-year return is linked to book-to-market value. When different anomalies are compared among asset classes, the negative five-year return realises the highest Sharpe ratio for commodities. Yet, significant value premium is not found. However, it seems that value premium whether it is significant or not, remains rather predictable. This is shown in a portfolio context when the value effect is applied alone to the Black–Litterman model.

When different anomalies are compared, it is again obvious that they behave differently in each asset class. For example, value strategy yields appealing risk-return ratio for stocks in comparison to other asset classes. On the other hand, the same strategy for bonds or stocks provides even negative mean return with certain weighting methods. Reversal is another good example. For stocks, it can never be considered profitable as it almost always provides a negative premium. However, all the other asset classes usually benefit from the strategy, especially currencies.

4.2 Black–Litterman portfolios

When utilising different anomalies in a multiple asset class portfolio an important question is, which anomaly should be considered for each asset class. One approach is to make the decision based on the Sharpe ratios realised from the factors analysed earlier. Portfolios containing multiple asset classes are optimised based on these factors with ultimate high Sharpe ratios. If all 124 factors are compared, the best Sharpe ratios are provided by the following factors: *S_MOM_5_12_1_R*, *B_REV_2_36_12_E*, *FX_REV_3_36_12_IV_0*, and *C_REV_3_48_12_E*. In short, only momentum or reversal factors are considered for the Black–Litterman framework. This portfolio will be called *Sharpe* when analysing the results. The marking above defines the parameters selected for each factor, and it always consists of the following parts: Asset class, anomaly, number of groups, past months and those recent months ignored (for momentum and reversal), weighting method, and 1 or 0 depending on whether high or low idiosyncratic volatility is overweighed. When IVOL is the actual anomaly, it is specifically indicated if high IVOL strategy is considered.

Even the most tempting factor is not necessarily a proof of a predictable return premium. In that case a factor might not be beneficial in the Black–Litterman framework. Because of this, one interesting approach is to apply the same anomaly to all asset classes. Here, all the anomalies are considered with the basic settings discussed earlier.

In addition to the Black–Litterman portfolios, three different benchmarks are analysed. Market portfolio is a natural choice for a benchmark, which in this study, is Standard & Poor's 500 total return index. Another relevant benchmark is $1/N$ portfolio since it is often reported to outperform more sophisticated asset allocation strategies. Additionally, this is considered as a benchmark in the Black–Litterman framework because market portfolio is rather abstract when different asset classes are considered. The third competing portfolio is the global minimum variance portfolio with a short selling constraint. Since the mean-variance framework plays an important role behind the Black–Litterman model, it is relevant to see whether a more sophisticated model is worth the effort.

Table 8 Results of multiple asset class portfolios with the Black–Litterman strategy

The table reports annual means, standard deviations, Sharpe ratios, and 3-factor alphas of different investment strategies. The Black–Litterman portfolios are based on anomalies, such as, value, momentum, reversal, and idiosyncratic volatility. The Sharpe portfolio mixes different anomalies depending on asset class specific Sharpe ratios. All BL portfolios are tangency portfolios, and thus, optimised to maximise the Sharpe ratio. The market is represented by S&P 500 index. Equal weight and minimum variance portfolios are the other two benchmarks. All portfolios are long only and they are rebalanced after every 20 trading days. Alpha is estimated with 3-factor model and the t-values are Newey-West corrected to control autocorrelation.

11.7.1994–1.4.2014	Sharpe	VAL	MOM	REV	IVOL	Market	1/N	GMV
mean (%)	9.366	9.200	6.867	7.620	6.951	7.966	7.961	2.916
(t-value)	(4.240***)	(4.241***)	(2.91**)	(3.506***)	(2.856**)	(3.357***)	(3.345***)	(3.502***)
stdev (%)	10.023	9.841	10.708	9.860	11.044	10.764	10.799	3.778
Sharpe	0.934	0.935	0.641	0.773	0.629	0.740	0.737	0.772
alpha (%)	7.916	7.593	4.909	5.923	4.665	5.079	5.067	2.402
(t-value)	(3.575***)	(3.739***)	(2.180*)	(2.863**)	(1.978*)	(2.452*)	(2.438*)	(2.621**)

An impressive result is that all strategies provide significantly positive alpha. Risk-return wise the Sharpe portfolio meets the expectations since it provides a highly competitive Sharpe ratio of 0.934. Additionally, the annual mean and alpha are the highest of all strategies. The Sharpe portfolio yields in average 9.366 % per annum while the abnormal return is 7.916 % p.a. The alpha is also statistically highly significant. However, value anomaly is the most surprising when applied to all asset classes in the Black–Litterman strategy. It yields the highest Sharpe ratio of 0.935, which together with the Sharpe portfolio competes in its own league. For all other strategies, the ratio is considerably lower.

Performance of the value strategy is outstanding considering that it does not quite stand out when comparing the factors. A potential reason for this performance is that the effect remains rather stable and thus predictable. Even if a simple long-short strategy does not perform well, predictability might add value in the Black–Litterman framework. This is because the view is defined based on past month's cumulative return, whether it is positive or negative.

Even though the first two strategies are superior, all the different Black–Litterman portfolios are promising indeed. The remaining three strategies, momentum, reversal, and IVOL, provide somewhat similar results. Sharpe ratios of the strategies vary between 0.629 and 0.773. They also perform well against the competing benchmarks, although, 1/N portfolio is really impressive in its simplicity. It provides very similar results with the three last mentioned strategies and even outperforms some of them in, e.g., Sharpe ratio or alpha. A good comprehension on how the different strategies perform is given by the following graph where out-of-sample performance of the strategies can be compared on a log scale.

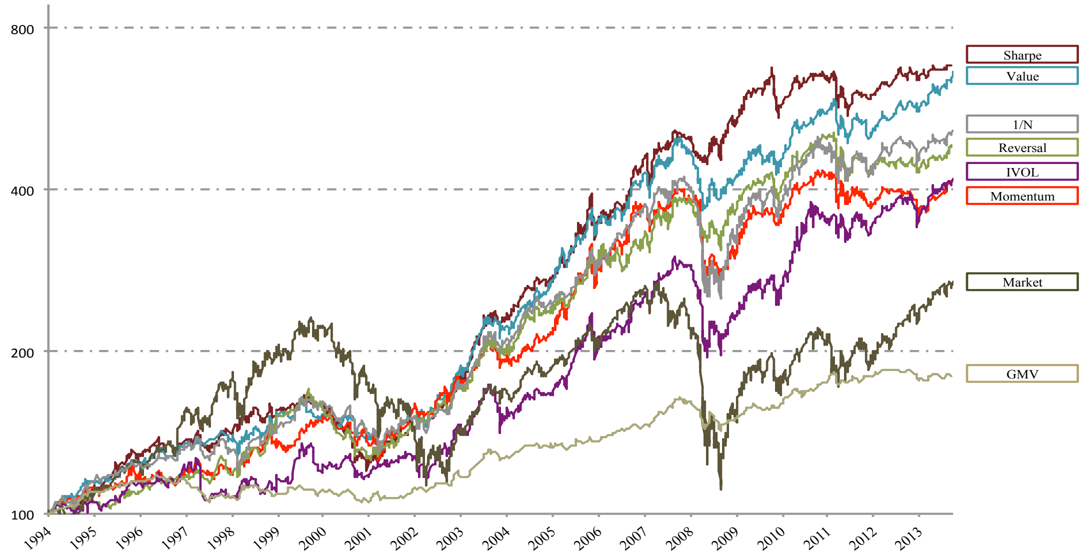

Figure 7 Performance of anomaly based Black–Litterman portfolios presented on a logarithmic scale

Visual observation strengthens the perception given by Table 8. The figure clearly shows the superior out-of-sample performance of Sharpe and value strategies. In addition, sensitivity to macroeconomic events is well demonstrated by the lines. For example, the latest financial distress launched by the fall of Lehman Brothers on 15th September 2008 seems to cause a drastic decline on most investment strategies. However, as the figure reveals, the Sharpe portfolio performs surprisingly well during those chaotic times compared to most other strategies.

4.3 Robustness checks

In terms of real world implementations, it is critical to examine how sensitive the portfolios are to changes in the underlying parameters. This analysis would raise whole new research questions but that goes beyond the scope of this study. However, a relevant question here is, are the results reliable when the setting is not exactly as suggested in the first place. In this analysis, the two best performing portfolios are tested, i.e., the Sharpe and value portfolio. These two are compared to 1/N portfolio, which obviously stands out as hard benchmark to win.

Robustness of the portfolios is tested against different parameters. First, length of the investment period is an important parameter since it is closely related to transaction costs. Second, for some investors selling short is an opportunity, which deserves to be tested. Third, the Black–Litterman model has number of different parameters that might have a strong impact on the final allocation. Here, the effect of the risk aversion parameter, lambda, is assessed as it has a noticeable role in defining implied equilibrium returns. The default value for this parameter is three as discussed earlier.

In order to assess the magnitude of the change, the Sharpe ratios are compared to each other. More specifically, a paired test of equality of Sharpe ratios is conducted to see if the Sharpe ratios differ statistically significantly from that of the 1/N portfolio. In the analysis, the method of Leung and Wong (2008) is followed and executed by R package *SharpeR*.

Table 9 Robustness of test portfolios

This table reports robustness of Sharpe and value portfolios to changing parameters in strategy implementation and over different time periods. Sharpe portfolio is a tangency portfolio where BL views consist of momentum for stocks and different versions of reversal for the rest of the asset classes. For the value portfolio, BL views are formed by value with the basic settings. The two portfolios provided the highest Sharpe ratios in the actual out-of-sample study. These portfolios are represented by the first benchmarks denoted by Basic. The other benchmark portfolio is the 1/N where equal weights are allocated when rebalancing. Robustness is tested with three types of variables: length of rebalancing period, release of short selling restriction, and risk aversion parameter (lambda) of the BL model. Also, pre and post-Lehman periods are investigated separately. The breaking point is 15th Sep 2008. The strategies are back-tested using a rolling estimation window, where the whole data set covers the period of 11.7.1994–1.4.2014.

		Rebalancing		Short	Risk Aversion			Benchmark	
		5 days	60 days	allowed	1	5	10	Basic	1/N
PANEL A: Sharpe portfolio									
Total	mean (%)	8.822	7.750	7.024	9.176	9.190	8.987	9.366	7.961
	stdev (%)	10.337	10.124	24.730	10.569	9.647	9.128	10.023	10.799
	Sharpe	0.853	0.766	0.284	0.868	0.953	0.985	0.934	0.737
Pre-Lehman	mean (%)	10.037	7.437	12.027	10.442	9.925	9.657	10.272	8.588
	stdev (%)	10.020	10.130	25.525	10.575	9.469	8.804	9.912	8.533
	Sharpe	1.002	0.734	0.471	0.987	1.048	1.097	1.036	1.007
Post-Lehman	mean (%)	5.715	8.551	-5.765	5.937	7.312	7.274	7.051	6.357
	stdev (%)	11.216	10.226	22.976	10.677	10.182	9.990	10.404	15.139
	Sharpe	0.510	0.836	-0.251	0.556	0.718	0.728	0.678	0.420
PANEL B: Value portfolio									
Total	mean (%)	9.436	8.942	10.362	8.969	9.167	8.748	9.200	7.961
	stdev (%)	9.801	9.789	23.626	10.222	9.606	9.391	9.841	10.799
	Sharpe	0.963	0.913	0.439	0.877	0.954	0.932	0.935	0.737
Pre-Lehman	mean (%)	9.361	8.830	10.339	9.528	9.667	9.166	9.709	8.588
	stdev (%)	9.455	9.336	24.718	10.061	8.971	8.464	9.410	8.533
	Sharpe	0.990	0.946	0.418	0.947	1.078	1.083	1.032	1.007
Post-Lehman	mean (%)	9.628	9.228	10.422	7.541	7.887	7.679	7.899	6.357
	stdev (%)	10.670	10.923	21.031	10.692	11.101	11.458	10.906	15.139
	Sharpe	0.902	0.845	0.496	0.705	0.710	0.670	0.724	0.420

Profitability of the Sharpe portfolio is evident regardless of the observation period. With the original parameters, the strategy yields highly competitive Sharpe ratio, which is always higher than that of 1/N portfolio and most of the test portfolios. The results are rather stable both before and after the fall of the investment bank Lehman Brothers on

15th September 2008. This shows that the strategy could be profitable in different investing environments.

Rebalancing period becomes an important parameter, especially in terms of transaction costs. Initially, portfolios are rebalanced every 20 trading days, i.e., once a month. More frequent trading has only a limited effect on the risk-return ratio of Sharpe portfolio. In fact, both before and after the fall of Lehman Brothers, weekly rebalancing has even weakened the Sharpe ratio. On the other hand, quarterly rebalancing (60 trading days) provides competitive results as well. Especially after the Lehman bankruptcy, less frequent trading even improves risk-return ratio of the Sharpe portfolio. Nevertheless, the central finding is that returns of Sharpe portfolio are robust to different trading frequencies.

Same observation can be made about the value portfolio. Regardless of the holding period, the Sharpe ratios are always close to that of the original portfolio. Additionally, excluding pre-Lehman period, the Sharpe ratios are always higher than what 1/N portfolio is able to provide. An interesting observation, however, is that the length of rebalancing period has an opposite effect on the results compared to the Sharpe portfolio. Here, more frequent trading provides overall slightly higher Sharpe ratios than quarterly rebalanced portfolios. However, the difference is small.

When short selling is allowed, the results collapse. For both Sharpe and value portfolio, before and after the financial distress started in 2008, the Sharpe ratios are significantly lower in comparison with the initial portfolio setting or the 1/N portfolio. In all cases, this is mostly due to variance which more than doubles in contrast to the original portfolio.

Risk aversion measures the marginal reward an investor requires for additional risk. In the Black–Litterman framework, a higher risk aversion parameter implies greater expected excess returns. Even though, risk aversion would be assumed to affect the out-of-sample returns the results prove otherwise. When the parameter is either increased or decreased from the initial three, mean return and standard deviation remain relatively stable. The same observation applies to both Sharpe and value portfolio, regardless of the investment period in focus.

In order to statistically prove robustness of the portfolios, equality of Sharpe ratios was tested. All test portfolios were compared to the original one and equality of their Sharpe ratios were tested following the method of Leung and Wong (2008). In short, their method tests performance equality of multiple funds assuming independent and identically distributed (*i.i.d.*) returns. In order to cope with autocorrelation and heteroskedasticity issues, covariance matrix of Sharpe ratios is estimated using so-called HAC matrix proposed by Andrews (1991). Table 10 shows the p-values of *Hotelling* T^2 statistics.

Table 10 Equality of Sharpe ratios

This table reports the p-values of Sharpe equality test results. Equality of Sharpe ratios is tested following the method of Leung and Wong (2008). Autocorrelation and heteroskedasticity is controlled by HAC covariance matrix proposed by Andrews (1991). Practical test is conducted using an R package, *SharpeR*. Sharpe and value portfolios are tangency portfolios where portfolio weights are solved by utilising the Black–Litterman model. Performance of the original portfolios is tested against those with modified input parameters, ceteris paribus. To test the robustness of a portfolio, rebalancing periods, short selling constraint, and risk aversion parameter (lambda) in the BL model are readjusted.

	Rebalancing		Short selling	Risk Aversion		
	5 days	60 days	allowed	1	5	10
Sharpe portfolio						
Total	0.418	0.158	0.001	0.174	0.607	0.556
Pre-Lehman	0.758	0.006	0.010	0.198	0.736	0.464
Post-Lehman	0.480	0.613	0.018	0.395	0.665	0.808
Value portfolio						
Total	0.702	0.813	0.005	0.267	0.588	0.966
Pre-Lehman	0.673	0.407	0.003	0.050	0.171	0.566
Post-Lehman	0.230	0.497	0.510	0.891	0.874	0.746

The p-values are in line with the findings made earlier. Both Sharpe and value portfolio are robust to changes in various input parameters. For almost all pairwise tests the p-value is fundamentally above any reasonable rejection level. This means that the Sharpe ratios are not likely to differ from that given by the original portfolio. In addition, the results are mostly insensitive to the observation period.

Even though a vast majority of the p-values are high, a few interesting findings are made where the null can be rejected. After seeing the results of unrestricted portfolios, it is not a surprise that the Sharpe ratio is statistically different from the original long-only portfolio. However, in two cases the difference is significant at 5 % level when observing a specific timeframe. Firstly, the Sharpe portfolio yields 0.302 lower risk-return ratio before the fall of Lehman when rebalanced quarterly. This difference is highly significant with p-value of 0.006. Secondly, when risk aversion of Value portfolio is set to 1, the Sharpe ratio before the financial crisis is 0.085 units lower at 5 % level of significance. When testing equality of multiple Sharpe ratios (excluding those of unconstrained portfolios) within these specific time periods, the results of F-statistics support this same finding. Before September 2008, there are one or more portfolios the Sharpe ratio of which is significantly different from the others. For Sharpe portfolios, the p-value is 0.017. Accordingly, the p-value for Value portfolios is 0.103. Most probably, the portfolios making the significant difference are exactly the two mentioned above. Within other observation periods the p-values were drastically higher.

5 CONCLUSION

The primary objective of this thesis was to combine anomalies of asset pricing theory with applications of the modern portfolio theory. Whenever there is a persistent market anomaly, additional information should be applied to the portfolio level. The link between these two fields was made with the Black–Litterman model, which allows investors to set their own views as a part of the optimisation problem. The results speak for themselves. Market anomalies, such as value, momentum, long-term reversal, and idiosyncratic risk, provide valuable information to optimal asset diversification. Moreover, the findings extend the literature as the results can be utilised across different asset classes.

The subject was studied by investigating out-of-sample performance of different anomaly-based portfolio strategies. The first step was to construct factors on different anomalies among stocks, bonds, currencies, and commodities. These factors were built by creating long-short portfolios, where underlying assets were selected to the portfolio based on a specific anomaly. For value and momentum, the methodology followed Asness et al. (2013) to a great extent. Reversal effect was measured in a similar manner with momentum, based on the past returns (see, e.g., McLean 2010). Idiosyncratic volatility was captured from three-factor regression as suggested by Ang et al. (2009). When it came to creating the portfolios, different weighting schemes were considered to assess the power and robustness of the anomaly. Out-of-sample returns of these portfolios were studied to find if there is a significant positive premium. The second and the most important step of this study was to generate views based on return premium of the factors and apply them to portfolio optimisation. The objective was that the most profitable anomaly for each asset class could be considered in the same optimisation procedure.

The main findings can be divided into two groups. First, market anomalies seem to provide tempting return patterns, however, they seldom result in significantly positive return premium within different asset classes. Second, persistent return premium gives valuable information to investors to be utilised in multi asset class portfolio optimisation. According to the results, stocks demonstrate 12-month momentum with significantly positive three-factor alpha. The findings are strongly supported by the existing literature. Additionally, high idiosyncratic volatility is positively related to stock returns on country level. The effect is even stronger when momentum portfolio is scaled by idiosyncratic volatility. The rest of the anomalies are not as powerful as significantly positive alphas are not found. However, Sharpe ratios indicate that different anomalies might offer valuable information to portfolio optimisation. The second part puts this hypothesis into a test when the best factors are chosen from the 124 test portfolios simply based on Sharpe ratio. The factors are further investigated as a source of information.

The most outstanding finding of this thesis is that anomaly based factors include valuable information to enhance efficient portfolio diversification. When the highest Sharpe ratios for each asset class are picked from the 124 factors and applied to the Black–Litterman model, the final portfolio results in superior risk-return combination. If alphas are ignored, the highest Sharpe ratios are provided by momentum strategy for stocks and long-term reversal for the rest of the asset classes. When historical performance of these factors is used to generate views for different asset classes, the result is a tangency portfolio that outperforms all the benchmarks, i.e., minimum variance and 1/N portfolios. Competing single anomaly strategies are also appealing, especially value strategy, which basically performs as well as the previously mentioned Sharpe strategy. In addition, these results are robust to real life implementation issues, such as, rebalancing frequency, trading period, and investor's risk aversion. However, when short selling is allowed, poorer results are shown, which might actually be good news for most investors in terms of implementation.

The scope of this thesis is wide in order to shed light on understanding the characteristics of market anomalies as a part of portfolio strategy. The current literature, if it covers both market anomalies and different asset classes (see, e.g., Asness et al. 2013), does not suggest how to utilise these characteristics in real life applications. On the other hand, when these effects are investigated in portfolio context, the study is often limited to equities only (see, e.g., Babameto & Harris 2008). However, as a coin has two sides, this thesis has some major drawbacks as well. First of all, a wide angle unfortunately often requires simplifications or that some fields have to be examined superficially. Here, this can be seen in the depth that individual anomalies are studied. When four asset classes and four different anomalies are in focus, it is not of interest to go too deep into specifics. In addition, the Black–Litterman model has various different parameters, the calibration of which would have both practical and theoretical importance. Instead, as the main focus of this thesis was to give a broad comprehension on the benefits of anomaly based portfolio strategies across the asset classes, some of the details had to be treated more or less as given.

This thesis surely answers to many practical needs regarding optimal portfolio diversification. However, there are questions that are worth a more detailed examination. On the other hand, it would be interesting to see how the methods, utilised within this thesis, could be taken to the next level. As an example, developing more exact measures for the anomalies would make the results more reliable. For example, value for corporate bonds, within this thesis, is explained with the shape of the yield curve. An important step would be to study more thoroughly, how should different anomalies be understood within each asset class. This would extend the current literature, which to a great extent relies on stocks.

Additionally, there is still no common opinion on the link between idiosyncratic risk and stock returns. Here idiosyncratic risk results support some theoretical frameworks but are contradictory to some of the latest literature. The question becomes even more attractive in terms of defining idiosyncratic risk on other asset classes. In general, when new risk factors themselves are of focus, more theoretical testing would be recommended, e.g., testing with Fama-MacBeth regression.

When it comes to taking anomaly based portfolio optimisation to the next level, some further investigation on predicting factor returns will be needed. For example, Babameto and Harris (2008) already apply forecasting regressions for value and momentum whereas here the expected future premium is simply calculated from historical returns. If there was a reliable forecast for this premium, investors could dynamically set their views based on the most appealing anomaly whenever it is time to rebalance the portfolio. It is obvious that even the most effective market anomaly does not work consistently in all market environments.

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APPENDIX CODE FOR THE EMPIRICAL STUDY

```
# LOAD DATA FOR EMPIRICAL STUDY
==============================================================
wdpath <- "/directory/path/"
setwd(wdpath)
srcfile<-"Functions.r"
source(srcfile)
file<-"Price_data.csv"
ff3<-"FF3_factors.csv" # Fama-French 3-factor data from Kenneth French's website
values_file<- "Value.csv" # Daily value estimates for each asset

rawdata=read.csv(file, header=T, sep=";", dec=".", na.strings="N/A")
ff3=read.csv(ff3, header=T, sep=";", dec=".", na.strings="N/A")
values=read.csv(values_file, header=T, sep=";", dec=".", na.strings="N/A")
val= values[,2:48]

P<-rawdata[,2:(dim(rawdata)[2]-1)] # Asset prices
PM<-rawdata[,(dim(rawdata)[2])] # Market portfolio prices (MSCI World)
ff3_fact <- ff3[,(2:5)]

T<-dim(P)[1]
N<-dim(P)[2]

RE=rbind(rep(0,N),data.frame(diff(log(data.matrix(P))))) #log return matrix
RM=rbind(rep(0,1),diff(log(data.matrix(PM)))) #log-return vector of market returns
#_____________________________________________________________

# FUNCTIONS FOR EMPIRICAL STUDY
==============================================================
library(quantmod)
library(quadprog)
library(fGarch)
library(tseries)
library(lmtest)
library(sandwich)
library(MASS)
library(SharpeR)

# Function for covariance matrix *******
# {sample} or {SI} "single-index"
covMat = function(
  xx,
  RMark,
  N=N,
  method="sample" ) {
  if(method=="sample"){covMat<- cov(xx)}

  if(method=="SI"){
    betas<-rep(0,N)
```

```
    residuals<-rep(0,N)
      for (k in 1:N){ #run market regressions, pick betas and residual terms
        k.lm= lm(xx[,k]~RMark)
        betas[k]=coeftest(k.lm, vcov=NeweyWest)[2,1] # first coefficient
        residuals[k]=summary(k.lm)$'sigma'^2
      }
    Mvar=as.numeric(var(RM))
    covMat<-Mvar*(betas%*%t(betas))+diag(residuals) #Define sinlge-index matrix
  }


  covMat <- matrix(covMat,nrow=ncol(xx),ncol=ncol(xx))
}
#____________________________________________________________
# Calculate Efficient frontier

effFrontier = function (averet, rcov, nports = 20, shorts=T, wmax=1) {
  mxret = max(abs(averet))*1.8

  mnret = 0
  n.assets = ncol(averet)
  reshigh = rep(wmax,n.assets)
  if( shorts )
  {
    reslow = rep(-wmax,n.assets)
  } else {
    reslow = rep(0,n.assets)
}
  min.rets = seq(mnret, mxret, len = nports)
  vol = rep(NA, nports)
  ret = rep(NA, nports)
  weights = matrix(rep(NA, n.assets*nports),ncol = n.assets, nrow = nports)
  for (k in 1:nports)
  {
    port.sol = NULL
    try(port.sol <- portfolio.optim(x=averet, pm=min.rets[k], covmat=rcov,
                                 reshigh=reshigh, reslow=reslow,shorts=shorts),silent=T)
    if ( !is.null(port.sol) )
    {
      vol[k] = sqrt(as.vector(port.sol$pw %*% rcov %*% port.sol$pw))
      ret[k] = averet %*% port.sol$pw
      weights[k,] = port.sol$pw     }
  }

  return(list(vol = vol, ret = ret, weights=weights))
}
# ____________________________________________________
# Max Sharpe Ratio Portfolio function ******************************

maxSharpe = function (averet, rcov, shorts=T, wmax = 1){
  optim.callback = function(param,averet,rcov,reshigh,reslow,shorts)
```

```
  {
    port.sol = NULL
    try(port.sol <- portfolio.optim(x=averet, pm=param, covmat=rcov,
                    reshigh=reshigh, reslow=reslow, shorts=shorts), silent = T)
    if (is.null(port.sol)) {
      ratio = 10^9
    } else {
      m.return = averet %*% port.sol$pw
      m.risk = sqrt(as.vector(port.sol$pw %*% rcov %*% port.sol$pw))
      ratio = -m.return/m.risk
      assign("w",port.sol$pw,inherits=T)
    }
    return(ratio)
  }
  ef = effFrontier(averet=averet, rcov=rcov, shorts=shorts, wmax=wmax, nports = 200)

  n = ncol(averet)
  reshigh = rep(wmax,n)
  if( shorts ) {
    reslow = -reshigh
  } else {
    reslow = rep(0,n)
  }
  max.sh = which.max(ef$ret/ef$vol)

  w = rep(0,ncol(averet))
  xmin = optimize(f=optim.callback, interval=c(ef$ret[max.sh-1], upper=ef$ret[max.sh+1]),
               averet=averet,rcov=rcov,reshigh=reshigh,reslow=reslow,shorts=shorts)

  m.return= averet %*% w
  m.risk = sqrt(as.vector(w %*% rcov %*% w))
  ratio = m.return/m.risk
  return(list(Sharpe = ratio, ret = m.return, vol=m.risk, weights=w))
}
# _______________________________________________________________

# Global Minimum Variance Portfolio ****************************************************


GMV = function(covar,
      N = ncol(covar),
      shorts=FALSE){

  zeros <- array(0, dim = c(N,1))

  if(shorts==TRUE){
  # Optimization to generate minimum variance portfolio without short selling constraint
    aMat  <- t(array(1, dim = c(1,N)))
    res <- solve.QP(covar, zeros, aMat, bvec=1, meq = 1)} else {

  # Optimization with short selling constraint (i.e. non-negative weights)
    aMat <- cbind(t(array(1, dim = c(1,N))), diag(N))
```

```
    b0 <- as.matrix(c(1, rep.int(0,N)))
    res <- solve.QP(covar, zeros, aMat, bvec=b0, meq = 1)}

  return(res)}
# _______________________________________________________________

# Value function ********************************************************************

value=function(
  xx,
  vals,
  ff3_load=NA,
  percentiles=3,
  weights="equal",     # {equl,rank,IVOL}
  method=method,
  high_IVOL=1,  # 0 if bigger weight for assets with low idiosyncratic risk
  showOutput=TRUE){

  t <- NROW(xx)
  cols <- NCOL(xx)

  sumReturns <- colSums(xx[(t-249):(t),]) # Cumulative return for all assets
  no_rets<- 250 # To calculate mean returns

  rankVec<-as.vector(rank(vals)) # Rank by value factors (Small values first)
  chunk <- function(xx,n) {split(xx, factor(sort(rank(xx)%%n)))} #function to split a vector

  subSets<-chunk(1:cols,percentiles)

  # Create vectors for the high and low value assets
  H_Group<-eval(parse(text=paste(sep="", "subSets$'", percentiles-1, "'"))) # High value ranks
  L_Group<-subSets$'0' # Low value ranks

  L1<-length(H_Group)
  L2<-length(L_Group)
  L<-min(L1,L2) # Vector lengths will be adjusted according to the shorter one

  H_Group <- H_Group[(L1-L+1):L1]
  Hvalue<-match(H_Group,rankVec) # Which assets we pick
  HvalueRets<-rep(NA,L)
  HvalueVals<-rep(NA,L)
  for (c in 1:L) {HvalueRets[c]<-sumReturns[Hvalue[c]]} # Vector of winners returns
  for (c in 1:L) {HvalueVals[c]<-vals[Hvalue[c]]}
  winMeans <- HvalueRets/no_rets # Vector of winner mean returns

  L_Group <- L_Group[(L2-L+1):L2]
  Lvalue<-match(L_Group,rankVec) # Which assets we pick
  LvalueRets<-rep(NA,L)
  LvalueVals<-rep(NA,L)
  for (c in 1:L) {LvalueRets[c]<-sumReturns[Lvalue[c]]} # Vector of losers returns
  for (c in 1:L) {LvalueVals[c]<-vals[Lvalue[c]]} # Vector of losers values
```

```
  losMeans <- LvalueRets/no_rets # Vector of loser mean returns

  # Weighting method
  wHigh<-rep(NA,L)
  wLow<-rep(NA,L)
  if(weights=="equal"){      # EQUAL WEIGHTING
    wHigh<-rep(1/L,L)
    wLow<- -(rep(1/L,L))
  }
  if(weights=="IVOL"){       # WEIGHTING BY IDIOSYNCRATIC RISK
    # Portfolio is weighted based on idiosyncratic risk
    IVOL <- IR(xx, RMark,ff3_load, method=method, high_IVOL=high_IVOL)$"IVOL"
    wHigh <- IVOL[Hvalue]
    wHigh <- wHigh/sum(wHigh)

    wLow <- 1/IVOL[Lvalue]
    wLow <- -wLow/sum(wLow)

  }
  if(weights=="rank"){       # RANK WEIGHTING
    # Weights calculated according to ranks
    # Assets with high value should get the biggest weight

wHigh<-H_Group
    scale <- 1/(sum(wHigh)) # weights need to be scaled to 100 %

    wHigh<-wHigh*scale
    # Assets with low value should get the biggest short position
    wLow <- L_Group
    scale <- 1/(sum(wLow))
    wLow <- -wLow*scale # Low value weights are presented as absolute values
  }

  # Count weights for all assets
  weights <- rep(0,cols)
  for(i in 1:L){weights[Hvalue[i]] <- wHigh[i]} # set High value weights
  for(i in 1:L){weights[Lvalue[i]] <- wLow[i]}  # set Low value weights

  if(showOutput==TRUE){
    return(list(high=Hvalue,winReturns=winMeans,HighValueWeights=wHigh,
            low=Lvalue,losReturns=losMeans,LowValueWeights=wLow,allWeights=weights))
  }
}

#  Momentum ***************************************************************************
momentum=function(
  xx,
  RMark=NA,
  ff3_load=NA,
  percentiles=3,
  pastMonths=12,
```

```
  monthsTill=1,
  weights="equal",     # {equl,rank,IVOL}
  method=method,
  high_IVOL=1, # 0 if big weight for assets with low idiosyncratic risk
  showOutput=TRUE){

  t <- NROW(xx)
  cols <- NCOL(xx)

# Cumulative return for all assets
sumReturns <- colSums(xx[(t-(pastMonths)*20):(t-monthsTill*20),])
  no_rets<- length(xx[(t-(pastMonths)*20):(t-monthsTill*20),]) # To calculate mean returns
  rankVec<-as.vector(rank(sumReturns)) # Rank by past performance (winners --> big values)
  chunk <- function(xx,n) {split(xx, factor(sort(rank(xx)%%n)))} #function to split a vector
  subSets<-chunk(1:cols,percentiles)

  # Create vectors for the winners and losers
  H_Group<-eval(parse(text=paste(sep="", "subSets$'", percentiles-1, "'"))) # Winner ranks
  L_Group<-subSets$'0' # Loser ranks

  L1<-length(H_Group)
  L2<-length(L_Group)
  L<-min(L1,L2) # Vector lengths will be adjusted according to the shorter one

  H_Group <- H_Group[(L1-L+1):L1]
  winners<-match(H_Group,rankVec) # Which assets we pick
  winRets<-rep(NA,L)

  sumReturns2 <- colSums(xx[(t-249):t,]) # Cumulative one year return of all assets
  for (c in 1:L) {winRets[c]<-sumReturns2[winners[c]]} # Vector of winners returns
  winMeans <- winRets/250 # Vector of winner mean returns

  L_Group <- L_Group[(L2-L+1):L2]
  losers<-match(L_Group,rankVec) # Which assets are picked
  losRets<-rep(NA,L)
  for (c in 1:L) {losRets[c]<-sumReturns2[losers[c]]} # Vector of losers returns
  losMeans <- losRets/250 # Vector of loser mean returns

  # Weighting method
  wWin<-rep(NA,L)
  wLos<-rep(NA,L)
  if(weights=="equal"){       # EQUAL WEIGHTING
    wWin<-rep(1/L,L)
    wLos<- -(rep(1/L,L))
  }
  if(weights=="IVOL"){       # WEIGHTING BY IDIOSYNCRATIC RISK
    # Portfolio is weighted based on idiosyncratic risk
    IVOL <- IR(xx, RMark, ff3_load, method=method,  high_IVOL=high_IVOL)$"IVOL"

    wWin <- IVOL[winners] # Idiosyncratic risks of the winner assets
    wWin <- wWin/sum(wWin)
```

```
    wLos <- 1/IVOL[losers] # For losers, biggest short position for the lowest idiosyncratic risk
    wLos <- -wLos/sum(wLos) # scaled to 100%
  }
  if(weights=="rank"){       # RANK WEIGHTING
    # Weights calculated according to ranks
    # Past winners should get the biggest weight

    wWin<-H_Group
    scale <- 1/(sum(wWin)) # weights need to be scaled to 100 %
    wWin<-wWin*scale
    # Assets with low value should get the biggest short position
    wLos <- L_Group
    scale <- 1/(sum(wLos))
    wLos <- -wLos*scale # Low value weights are presented as absolute values
  }

  # Count weights for all assets
  weights <- rep(0,cols)
  for(i in 1:L){weights[winners[i]] <- wWin[i]} # set winner weights
  for(i in 1:L){weights[losers[i]] <- wLos[i]}  # set loser weights

if(showOutput==TRUE){
    return(list(winners=winners,winReturns=winMeans,winWeights=wWin,
              losers=losers,losReturns=losMeans,losWeights=wLos,allWeights=weights))
  }}

# ______________________________________________________________________
# Reversal *************************************************************************

# Works with the same method as momentum and so it is utilised
# The only difference is that the final weight vector will be inversed

reversal = function(
  xx,
  RMark=NA,
  ff3_load = NA,
  percentiles=3,
  pastMonths=36,
  monthsTill=12,
  weights="equal",     # {equl,rank,IVOL}
  method=method,
  high_IVOL=1) {       # 0 if big weight for assets with low idiosyncratic risk

  rev <- momentum(xx=xx, RMark=RMark, ff3_load=ff3_load, percentiles=percentiles,
                pastMonths=pastMonths, monthsTill=monthsTill, weights=weights ,
high_IVOL=high_IVOL,
                method=method)
  # Here winReturns (and losReturns) are 1-year mean returns of the long term losers (winners)
  return(list(winners=rev$winners,winReturns=rev$losReturns,winWeights=rev$winWeights,
            losers=rev$losers,losReturns=rev$winReturns,
```

```
             losWeights=rev$losWeights,allWeights= -1*(rev$allWeights)))
} ______________________________________________________________________

# Idiosyncratic Risk ***********************************************************************

IR <- function(xx,
               RMark=NA,
               ff3_load=NA,
               assets=(1:dim(xx)[2]),
               method = "markMod",  #{markMod, ff3Mod}
               high_IVOL=1)
{

  e <- rep(NA,length(assets))

  for(i in assets){
    if(method=="markMod"){
    # Market model version
         y <- xx[,i]
         x <- RMark
         fit <- lm(y~RMark)
         e[i] <- (summary(fit)$'sigma')^2 # variance of residuals
        }
    if(method=="ff3Mod"){
      # Fama-French 3-factor model
      MKT<-ff3_load[,1]
      SMB<-ff3_load[,2]
      HML<-ff3_load[,3]
      y<-xx[,i]
      fit <- lm(y~MKT+SMB+HML)

      e[i] <- sd(residuals(fit))
    }

if(high_IVOL==0){e<-1/e} # If we want a big weight for assets with low idiosyncratic risk

  return(list("IVOL"=e, "errors"=err))
  }
}
# ______________________________________________________________________

# Idiosyncratic Risk Weights ***************************************************************

IVOL=function(
  xx,
  RMark=RMark,
  ff3_load=ff3_load,
  method=method,
  percentiles=3,
  weights="equal",     # {equal,rank}
  high_IVOL=1, # 0 if big weight for assets with low idiosyncratic risk
```

```
 showOutput=TRUE){

 t <- NROW(xx)
 cols <- NCOL(xx)

 values <- IR(xx=xx,
              RMark=RMark,
              ff3_load=ff3_load,
              method=method,
              assets=(1:dim(xx)[2]),
              high_IVOL=high_IVOL)$"IVOL"

 rankVec<-as.vector(rank(values)) # Rank by past performance (winners --> big values)

 sumReturns <- colSums(xx[(t-249):t,]) # Cumulative return for all assets
 no_rets<- 250 # To calculate mean returns

 chunk <- function(xx,n) {split(xx, factor(sort(rank(xx)%%n)))} #function to split a vector
 subSets<-chunk(1:cols,percentiles)

 # Create vectors for the high and low weights
 H_Group<-eval(parse(text=paste(sep="", "subSets$'", percentiles-1, "'"))) # High ranks
 L_Group<-subSets$'0' # Low ranks

 L1<-length(H_Group)
 L2<-length(L_Group)
 L<-min(L1,L2) # Vector lengths will be adjusted according to the shorter one

 H_Group <- H_Group[(L1-L+1):L1]
 highValues<-match(H_Group,rankVec) # Which assets are picked
 highRets<-rep(NA,L)

 for (c in 1:L) {highRets[c]<-sumReturns[highValues[c]]} # Vector of high IVOL risk returns

# Here high IVOL assets are considered winners/losers depending on settings
winMeans <- highRets/no_rets

 L_Group <- L_Group[(L2-L+1):L2]
 lowValues<-match(L_Group,rankVec) # Which assets are picked
 lowRets<-rep(NA,L)
 for (c in 1:L) {lowRets[c]<-sumReturns[lowValues[c]]} # Vector of low idiosyncratic risk returns
 losMeans <- lowRets/no_rets # Vector of low idiosyncratic risk mean returns

 # Weighting method
 wHigh<-rep(NA,L)
 wLow<-rep(NA,L)
 if(weights=="equal"){        # EQUAL WEIGHTING
   wHigh<-rep(1/L,L)
   wLow<- -(rep(1/L,L))
 }
```

```
  if(weights=="rank"){       # RANK WEIGHTING
    # Weights calculated according to ranks
    # Assets with the highest idiosyncratic risk should get the biggest weight

    wHigh<-H_Group
    scale <- 1/(sum(wHigh)) # weights need to be scaled to 100 %
    wHigh<-wHigh*scale

    # Assets with low value should get the biggest short position
    wLow <- L_Group
    scale <- 1/(sum(wLow))
    wLow <- -wLow*scale # Low value weights are presented as absolute values
  }

  # Count weights for all assets
  weights <- rep(0,cols)
  for(i in 1:L){weights[highValues[i]] <- wHigh[i]} # set high IVOL weights
  for(i in 1:L){weights[lowValues[i]] <- wLow[i]}  # set low IVOL weights


  if(showOutput==TRUE){
    return(list(High_Values=highValues,winReturns=winMeans,highWeights=wHigh,
            Low_Values=lowValues,losReturns=losMeans,Low_Weights=wLow,allWeights=weights))
   }
}
# ________________________________________________________________________
# Function for the Black-Litterman model ****************
B_L = function(
  xx,
  xx_market,
  exp_market,
  mWeights,
  PMat,
  QVec,
  sigma,
  lambda=NA
){
  var_m <- var(xx_market)
  if(lambda==0){lambda <- exp_market/var_m} # Risk aversion parameter

  T <- NROW(xx)
  N <- NCOL(xx)
  tau <- 1/(T-N)
  nviews <- NROW(PMat)


  Phi <- (lambda * sigma) %*% mWeights  # Market equilibrium returns
  omega <- PMat %*% (tau*sigma) %*% t(PMat)

  for(r in 1:nviews){ # Diagonalise the omega matrix
    for(c in 1:nviews){
```

```
    if(r != c){omega[r,c]=0}
  }}

  rets <- ginv(ginv(tau*sigma)+t(PMat)%*%ginv(omega)%*%PMat) %*%
    (ginv(tau*sigma)%*%Phi+t(PMat)%*%ginv(omega)%*%QVec)

  return(list(Returns=rets,OmegaMatrix=omega,Phi=Phi,lambda=lambda,tau=tau))
}
# *******************************************************


# OUT-OF-SAMPLE STUDY
============================================

#### FACTOR BUILDING #####
dates<- rawdata[beg:T,1]
# Choose assets
all <- 1:47
stocks <- 1:20
bonds <- 21:31
FX <- 32:37
commodities <- 38:47
assets <- all # {choose one from above}
anomaly <- "momentum"  # {momentum, reversal, value, IVOL}
weight_method <- "rank" # {equal, rank, IVOL}
IVOL_method <- "ff3Mod" #{markMod, ff3Mod}
high_IVOL <- 1 # {1=bigger weight for high idiosyncratic risk assets, 0=opposite}
perc <- 3 # percentiles
pastMonths <- 12   # Beginning of the estimation window for momentum and reversal
monthsTill <- 1

update<-20  #How often the portfolio is rebalanced
yrDays <- 250
beg <- 1377 # Value data available from 11.7.1994 (row 1377)
fact_win <- beg-1 # Estimation window for factor building
len <- T-beg+1

Hseries<-numeric(0)
Lseries<-numeric(0)
XH<-100 # starting wealth for the 'high' porfolio
XL<-100 # starting wealth for the 'low' porfolio
vH=(rep(0,N))  # Denotes the amount of each asset held in the portfolio
vL=(rep(0,N))

#START ITERATION

for(i in beg:T){

  XH=sum(vH*P[i,])
  XL=sum(vL*P[i,])
```

```
if(XH==0){XH=100}
if(XL==0){XL=100}
wH=vH*P[i,]/XH # weights for both portfolios change due to price changes
wL=vL*P[i,]/XL

if (any(seq(beg,T,update)==i)==TRUE) { #Determines if it is time to rebalance

  fact_rets <- RE[(i-fact_win):i,assets]
  RMark <- RM[(i-fact_win):i]
  ff3_load <- ff3_fact[(i-fact_win):i,]

  # Select anomaly

  if (anomaly=="momentum"){values<-momentum(fact_rets,percentiles=perc,RMark=RMark,
                                            ff3_load=ff3_load,
                                            pastMonths=pastMonths, monthsTill=monthsTill,
                                            method=IVOL_method, weights=weight_method,
                                            high_IVOL=high_IVOL)}
  if (anomaly=="reversal"){values<-reversal(xx=fact_rets, ff3_load=ff3_load, percentiles=perc,
                                            pastMonths=pastMonths, monthsTill=monthsTill,
                                            weights=weight_method,method=IVOL_method,
                                            high_IVOL=high_IVOL)}
  if (anomaly=="value"){values<-value(xx=fact_rets, vals=val[i,assets],ff3_load=ff3_load,
                                      percentiles=perc, weights=weight_method,
                                      method=IVOL_method, high_IVOL=high_IVOL)}
  if (anomaly=="IVOL"){values<-IVOL(fact_rets, RMark=RMark,ff3_load=ff3_load, percentiles=perc,
                                    weights=weight_method, method=IVOL_method,
                                    high_IVOL=high_IVOL)}

  # Create two different weights vectors: 1) High and 2) Low portfolio
  wH<-as.vector(values$'allWeights')
  wL<-(-wH)

  # Only High (Low) group weights are taken into account
  for(c in 1:length(wH)){if(wH[c] < 0){wH[c] <- 0}}
  for(c in 1:length(wL)){if(wL[c] < 0){wL[c] <- 0}}

  startCol <- assets[1]
  endCol <- assets[length(assets)]

  wH <- c(rep(0,startCol-1), wH, rep(0,N-endCol)) # weight vector can be a subset of all assets
  wL <- c(rep(0,startCol-1), wL, rep(0,N-endCol))

  vH <- wH*XH/P[i,]
  vL <- wL*XL/P[i,]
}

Hseries <- rbind(Hseries,XH)
Lseries <- rbind(Lseries,XL)

len<-length(Hseries)
```

```
  if(i%%30==0) print(paste(sep="", round((i-beg+1)/(T-beg)*100,2)," %", " // length= ",len,
                         " // i = ",i))     # report every 30th loop passes
}

# STOP ITERATION

# Translate premium to percentage returns
FactRets <- (Hseries[2:len]/Hseries[1:(len-1)]-1)-
         (Lseries[2:len]/Lseries[1:(len-1)]-1)

# Create factor index
Factor<-c(100,rep(NA,len-1))
  for(i in 2:len){
      Factor[i]<-Factor[i-1]*(1+FactRets[i-1])
  }
# Factors are saved for further analysis (”Factor_Returns.csv”)
__________________________________________________________________

# BACK-TEST BLACK-LITTERMAN PORTFOLIOS
==================================================================

# Load factors as source of views
FR_file <- "Factor_Returns.csv"
FR_raw <- read.csv(FR_file, header=T, sep=";", dec=".", na.strings="N/A")
FR<-FR_raw[,2:(dim(FR_raw)[2])]
fact_var_file <- "Factor_Variables.csv" # Variables/parameters are manually defined for the factors
fact_var <- read.csv(fact_var_file, header=T, sep=";", dec=".", na.strings="N/A")
fact_var <- fact_var[,1:8]

# Asset class columns
# Equity indices: columns: 1-20
# Bond indices: columns: 21-31
# Currencies: columns: 32-37
# Commodities: columns: 38-47
# Market portfolio: column 48

#Variables
strategy<-"BL" # Set strategy {1/N, sample, BL,}
no_views <- FALSE # If true, P-matrix will be changed to zero
shorts <- FALSE     # Constraints {FALSE = long-only, TRUE = short-sales allowed}
update <- 20  # How often the portfolio is rebalanced
X <- 100 # Starting wealth
beg <- 1377 # Beginning of trading period (value data starts from row 1377 )
yrDays <- 250
fact_win <- beg # Estimation window for factor building
window <- 250 # Estimation window for the covariance matrix
lambda <- 10 # Market risk aversion parameter
IVOL_method <- "ff3Mod" #{markMod, ff3Mod}

# Choose assets
```

```
all <- 1:47
stocks <- 1:20
bonds <- 21:31
FX <- 32:37
commodities <- 38:47
  assets_1 <- stocks
  assets_2 <- bonds
  assets_3 <- FX
  assets_4 <- commodities

pfolio_assets <- all # {all or assets} The assets in the portfolio
n <- length(pfolio_assets) # number of assets in the portfolio

# Reference factors for all asset classes
fID_1 <-"S_VAL_3_R"    #Stock factor
fID_2 <-"B_VAL_3_R"    #Bond factor
fID_3 <-"FX_VAL_3_R"   #FX factor
fID_4 <-"C_VAL_3_R"    #Commodity factor

Factor_1 <- c(rep(0,beg),FR[,fID_1])
Factor_2 <- c(rep(0,beg),FR[,fID_2])
Factor_3 <- c(rep(0,beg),FR[,fID_3])
Factor_4 <- c(rep(0,beg),FR[,fID_4])

# Anomaly specific variables
anomaly_1 <- fact_var[match(fID_1,fact_var[,1]),"anomaly"]  # {momentum, reversal, value, IVOL}
weight_method_1 <- fact_var[match(fID_1,fact_var[,1]),"weight_mtd"]
        # {equal, rank, IVOL} Weights for views
IVOL_method_1 <- "ff3Mod" # {markMod, ff3Mod} How idiosyncratic volatility is calculated
high_IVOL_1 <- fact_var[match(fID_1,fact_var[,1]),"high_IVOL"]
        # {1=bigger weight for high idiosyncratic risk assets, 0=opposite}
perc_1 <- fact_var[match(fID_1,fact_var[,1]),"perc"]
pastMonths_1 <- fact_var[match(fID_1,fact_var[,1]),"From"]
monthsTill_1 <- fact_var[match(fID_1,fact_var[,1]),"Till"]

anomaly_2 <- fact_var[match(fID_2,fact_var[,1]),"anomaly"]  # {momentum, reversal, value, IVOL}
weight_method_2 <- fact_var[match(fID_2,fact_var[,1]),"weight_mtd"]
        # {equal, rank, IVOL} Weights for views
IVOL_method_2 <- "ff3Mod" # {markMod, ff3Mod } How idiosyncratic volatility is calculated
high_IVOL_2 <- fact_var[match(fID_2,fact_var[,1]),"high_IVOL"]
        # {1=bigger weight for high idiosyncratic risk assets, 0=opposite}
perc_2 <- fact_var[match(fID_2,fact_var[,1]),"perc"]
pastMonths_2 <- fact_var[match(fID_2,fact_var[,1]),"From"]
monthsTill_2 <- fact_var[match(fID_2,fact_var[,1]),"Till"]

anomaly_3 <- fact_var[match(fID_3,fact_var[,1]),"anomaly"]  # {momentum, reversal, value, IVOL}
weight_method_3 <- fact_var[match(fID_3,fact_var[,1]),"weight_mtd"]
        # {equal, rank, IVOL} Weights for views
IVOL_method_3 <- "ff3Mod" # {markMod, ff3Mod } How idiosyncratic volatility is calculated
high_IVOL_3 <- fact_var[match(fID_3,fact_var[,1]),"high_IVOL"]
        # {1=bigger weight for high idiosyncratic risk assets, 0=opposite}
```

```
perc_3 <- fact_var[match(fID_3,fact_var[,1]),"perc"]
pastMonths_3 <- fact_var[match(fID_3,fact_var[,1]),"From"]
monthsTill_3 <- fact_var[match(fID_3,fact_var[,1]),"Till"]

anomaly_4 <- fact_var[match(fID_4,fact_var[,1]),"anomaly"]  # {momentum, reversal, value, IVOL}
weight_method_4 <- fact_var[match(fID_4,fact_var[,1]),"weight_mtd"]
        # {equal, rank, IVOL} Weights for views
IVOL_method_4 <- "ff3Mod" # {markMod, ff3Mod } How idiosyncratic volatility is calculated
high_IVOL_4 <- fact_var[match(fID_4,fact_var[,1]),"high_IVOL"]
        # {1=bigger weight for high idiosyncratic risk assets, 0=opposite}
perc_4 <- fact_var[match(fID_4,fact_var[,1]),"perc"]
pastMonths_4 <- fact_var[match(fID_4,fact_var[,1]),"From"]
monthsTill_4 <- fact_var[match(fID_4,fact_var[,1]),"Till"]

v=(rep(0,n))  # Denotes the amount of each asset held in the portfolio
Xseries<-numeric(0) # Portfolio value performance
Wseries<-matrix(NA,1,n) # Time series of portfolio weights
lambdas <- numeric(0) # Risk aversion
colnames(Wseries)<-colnames(P[,pfolio_assets])

# START ITERATION ****************************************************************

BL_errors=0 # Count when BL model doesn't work
skips=0 # Count when cannot optimise
k=4 #number of views

for(i in beg:T){
  X=sum(v*P[i,pfolio_assets])
  if(X==0){X=100}
  w=v*P[i,pfolio_assets]/X

  if (any(seq(beg,T,update)==i)==TRUE) { #Determines if it's time to rebalance

    if(strategy=="1/N"){w=rep(1/n,n)} else {
            R <- RE[(i-window+1):i,pfolio_assets]
            RMark <- RM[(i-window+1):i]
            covar <- covMat(R, RMark=RMark,
                      method="sample" ,  # {"sample" or "SI"}
                      N=n) # covariance matrix
            covar <- ginv(ginv(covar))}
                  # Do generalised inverse twise to return a truly inversable matrix

    # Call the portfolio optimisation procedure
    if(strategy=="sample") { # Minimum variance portfolio
         # If error then solve GMV portfolio
         w1 <- tryCatch(as.vector(GMV(covar, shorts=shorts)$'solution'),
                     error=function(err) NULL,
                     warning=function(warn) NULL)

      if(!is.null( w1 )){w<-w1} else {w<-w # If still no solution is found then skip optimising
            skips <- skips +1}
```

```
}

# B-L starts ___________________________________________
if(strategy=="BL") { # Maximum Sharpe portfolio with Black-Litterman model

  rets <- RE[(i-fact_win):i,pfolio_assets]
  RMark <- RM[(i-fact_win):i]
  ff3_load <- ff3_fact[(i-fact_win):i,]

  # Benchmark weights
  mweights <- rep(1/n, n) # Use 1/N as benchmark

  # P-Matrix reset
  PMat <- matrix(ncol=n, nrow=k, rep(0, n*k)) # Create an empty P-matrix

  # Q-Vector reset
  QVec <- rep(0, k)

  for(j in 1:k){ # Define each view

    # Select anomaly
    Factor <- eval(parse(text=toString(paste(sep="","Factor_",j))))
    Factor <- Factor[(i-window+1):i]
    assets <- eval(parse(text=toString(paste(sep="","assets_",j))))
    anomaly <- eval(parse(text=toString(paste(sep="","anomaly_",j))))
    weight_method <- eval(parse(text=toString(paste(sep="","weight_method_",j))))
    high_IVOL <- eval(parse(text=toString(paste(sep="","high_IVOL_",j))))
    perc <- eval(parse(text=toString(paste(sep="","perc_",j))))
    pastMonths <- eval(parse(text=toString(paste(sep="","pastMonths_",j))))
    monthsTill <- eval(parse(text=toString(paste(sep="","monthsTill_",j))))


      if (anomaly=="momentum")
     {values< momentum(rets[,assets],percentiles=perc,RMark=RMark, ff3_load=ff3_load,
                                        pastMonths=pastMonths, monthsTill=monthsTill,
                                        method=IVOL_method, weights=weight_method,
                                        high_IVOL=high_IVOL)}
      if (anomaly=="reversal")
     {values<-reversal(xx=rets[,assets], ff3_load=ff3_load, percentiles=perc,
                                        pastMonths=pastMonths, monthsTill=monthsTill,
                                        weights=weight_method,method=IVOL_method,
                                        high_IVOL=high_IVOL)}
      if (anomaly=="value")
     {values<-value(xx=rets[,assets], vals=val[i,assets],ff3_load=ff3_load,
                                        percentiles=perc, weights=weight_method,
                                        method=IVOL_method, high_IVOL=high_IVOL)}
      if (anomaly=="IVOL")
     {values<-IVOL(rets[,assets], RMark=RMark,ff3_load=ff3_load, percentiles=perc,
                                        weights=weight_method, method=IVOL_method,
                                        high_IVOL=high_IVOL)}
```

```
          # P-Matrix
          allWeights <- values$'allWeights'

          for(c in assets){
            # When an anomaly is applied only to one asset class the weights have to match...
            # ...with the whole portfolio (if all asset classes are considered)
            PMat[j,c] <- allWeights[c-min(assets)+1]
          }

          if(no_views==TRUE){PMat <- matrix(ncol=n, nrow=k, rep(0, n*k))}

          # Q-vector
          QVec[j] <- mean(Factor) # How much the "winners" win the "losers" p.d. on average

      }# BL parametres ready

        # Black-Litterman returns
      BL <- B_L(R, RMark, exp_MRet[i], mweights, PMat, QVec, covar, lambda=lambda)
      BL_Rets <- t(as.vector(BL$'Returns'))
      lam <- BL$"lambda"

      # Max Sharpe portfolio
      SR <- tryCatch(maxSharpe (BL_Rets, covar, shorts=shorts),
                     error=function(err) NULL,
                     warning=function(warn) NULL) # Error handler returns 0 if no solution is found

      if(!is.null(SR)){ w <- as.vector(SR$'weights') }else {
        w1 <- tryCatch(as.vector(GMV(covar, shorts=shorts)$'solution'),
                  # If error then solve GMV portfolio
                       error=function(err) NULL,
                       warning=function(warn) NULL)

        if(!is.null( w1 )){w<-w1} else {w<-w # If still no solution is found then skip optimising
                                         skips <- skips +1}
        BL_errors <- BL_errors+1} # Count errors when Black-Litterman model won't work
    }

    v=w*X/P[i,pfolio_assets]
  }
  Xseries=rbind(Xseries,X)
  Wseries=rbind(Wseries,w)
  lambdas=rbind(lambdas,lam)
  len<-length(Xseries)
  if(i%%10==0) print(paste(sep="", round((i-beg+1)/(T-beg)*100,2)," %", " // length= ",len,
                           " // i = ", i,
                           " // BL_errors = ", BL_errors,
                           " // skips = ", skips))     # report every 10th loop passes
}
#---END ITERATION---#
```