



# Formulas for the Visual Angle Metric

Masayo Fujimura<sup>1</sup> · Rahim Kargar<sup>2</sup> · Matti Vuorinen<sup>2</sup>

Received: 15 April 2023 / Accepted: 30 July 2025  
© The Author(s) 2025

## Abstract

We prove several new formulas for the visual angle metric of the unit disk in terms of the hyperbolic metric and apply these to prove a sharp Schwarz lemma for the visual angle metric under quasiregular mappings.

**Keywords** Hyperbolic metric · Visual angle metric · Conformal mapping · Quasiconformal mapping · Schwarz lemma

**Mathematics Subject Classification** 30C62 · 51M09 · 51M15

## 1 Introduction

During the past few decades, various intrinsic metrics of planar domains have become important tools in geometric function theory, for instance in the study of quasiconformal mappings [3, 4, 7]. These metrics, defined in a general domain, on the one hand, share some of the properties of the hyperbolic metric of the unit disk and on the other hand, they are simpler than the hyperbolic metric. Intrinsic metrics are usually not conformally invariant, but have some kind of quasi-invariance properties under subclasses of conformal maps, e.g., under translations or Möbius transformations.

We study here one such intrinsic metric, the visual angle metric, introduced in [8] and further studied in [4, 5, 11]. Let  $G$  be a proper subdomain of  $\mathbb{R}^n$  such that  $\partial G$  is not a proper subset of a line. The *visual angle metric* for  $a, b \in G$  is defined by

$$v_G(a, b) = \sup\{\alpha : \alpha = \angle(a, z, b), z \in \partial G\}.$$

---

✉ Rahim Kargar  
rakarg@utu.fi

Masayo Fujimura  
masayo@nda.ac.jp

Matti Vuorinen  
vuorinen@utu.fi

<sup>1</sup> Department of Mathematics, National Defense Academy of Japan, Yokosuka, Japan

<sup>2</sup> Department of Mathematics and Statistics, University of Turku, Turku, Finland

Finding concrete values for  $v_G$  leads to minimization algorithms even in the simplest case when  $G$  is the unit disk  $\mathbb{B}^2$  because no formulas are known. Our main results are the following three theorems, which give explicit formulas for  $v_G$  when  $G = \mathbb{B}^2$ . Let the line through  $a, b \in \mathbb{C}$  be denoted by  $L[a, b]$ . Let  $S(a, r) = \{b \in \mathbb{R}^n : |a - b| = r\}$  be the circle centered at  $a \in \mathbb{R}^n$  with radius  $r > 0$ . The unit circle is defined by  $S(0, 1)$ .

The first theorem provides a geometric construction for the extremal point  $z$ .

**Theorem 1.1** *Let  $a, b \in \mathbb{B}^2$  with  $|a| \neq |b|$  and  $0 \notin L[a, b]$ . Then*

$$v_{\mathbb{B}^2}(a, b) = \max\{\angle(a, z_1, b), \angle(a, z_2, b)\},$$

where  $z_1$  and  $z_2$  are the points of intersection of the unit circle and an orthogonal circle

$$S(0, 1) \cap S(c, \sqrt{|c|^2 - 1}), \quad c = \frac{a(1 - |b|^2) - b(1 - |a|^2)}{|a|^2 - |b|^2}.$$

Moreover,  $\{z_1, z_2\} = (1 \pm i\sqrt{|c|^2 - 1})/\bar{c}$ . In the case  $|a| = |b|$

$$v_{\mathbb{B}^2}(a, b) = 2 \arctan\left(\frac{|a - b|}{2 - |a + b|}\right).$$

It is easily seen that the visual angle metric  $v_{\mathbb{B}^2}$  is not invariant under Möbius automorphisms of the unit disk. Nevertheless, we prove another formula for the visual angle metric involving the Möbius invariant hyperbolic metric  $\rho_{\mathbb{B}^2}$ . This is possible because for given  $a, b \in \mathbb{B}^2$ , we have  $v_{\mathbb{B}^2}(a, b) = v_{\mathbb{B}^2}(h(a), h(b))$ , whenever  $h$  is an inversion with  $h(\mathbb{B}^2) = \mathbb{B}^2$  and with  $h(\mathbb{B}^2 \cap L[a, b]) = \mathbb{B}^2 \cap L[a, b]$ .

**Theorem 1.2** *For  $a, b \in \mathbb{B}^2$  let  $L[a, b]$  be the line through  $a$  and  $b$ . We have*

$$\tan \frac{v_{\mathbb{B}^2}(a, b)}{2} = \frac{(1 + |m|)u}{1 + \sqrt{1 + (1 - |m|^2)u^2}}, \quad u = \text{sh} \frac{\rho_{\mathbb{B}^2}(a, b)}{2},$$

where  $m = (\bar{a}b - a\bar{b})/(2(\bar{a} - \bar{b}))$  is the midpoint of the chord of the unit disk containing the two points  $a$  and  $b$ . Hence  $|m| = d(L[a, b], \{0\})$ .

Our third main result yields a sharp quasiregular version of the Schwarz lemma for the visual angle metric. This result seems to be new in the case of analytic functions.

**Theorem 1.3** *Let  $f : \mathbb{B}^2 \rightarrow \mathbb{B}^2 = f(\mathbb{B}^2)$  be a non-constant  $K$ -quasiregular mapping, where  $K \geq 1$ . For  $a, b \in \mathbb{B}^2$ , let  $m_1$  and  $m_2$  be the midpoints of the chords of the unit disk containing  $f(a), f(b)$  and  $a, b$ , respectively. Then we have*

$$\tan \frac{v_{\mathbb{B}^2}(f(a), f(b))}{2} \leq 2^{1-1/K} c \left( \tan \frac{v_{\mathbb{B}^2}(a, b)}{2} \right)^{1/K}, \quad c = \sqrt{\frac{1 + |m_1|}{1 - |m_1|} \frac{1}{(1 + |m_2|)^{1/K}}},$$

with equality for  $K = 1$ , and  $m_1 = m_2 = 0$ .

In a later work, these theorems will be applied to prove inequalities between the visual angle metric and the ubiquitous quasihyperbolic and distance ratio metrics [3, 4] and some other intrinsic metrics [6] such as the Hilbert metric [9, 10].

## 2 Preliminary Results

We will give here some formulas about the geometry of lines and triangles, on which our later work is based.

### 2.1 Geometry and Complex Numbers

The extended complex plane  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is identified with the Riemann sphere via the stereographic projection. Let  $L[a, b]$  stand for the line through  $a$  and  $b$  ( $a \neq b$ ). For distinct points  $a, b, c, d \in \mathbb{C}$  such that the lines  $L[a, b]$  and  $L[c, d]$  have a unique point  $w$  of intersection, let

$$w = LIS[a, b, c, d] = L[a, b] \cap L[c, d].$$

This point is given by

$$w = LIS[a, b, c, d] = \frac{u}{v}, \tag{2.2}$$

with (see e.g. [4, Ex. 4.3(1), p. 57 and p. 373])

$$\begin{cases} u = (\bar{a}b - a\bar{b})(c - d) - (a - b)(\bar{c}d - c\bar{d}); \\ v = (\bar{a} - \bar{b})(c - d) - (a - b)(\bar{c} - \bar{d}). \end{cases} \tag{2.3}$$

Let  $C[a, b, c]$  be the circle through distinct noncollinear points  $a, b$ , and  $c$ . The formula (2.2) gives easily the formula for the center  $m(a, b, c)$  of  $C[a, b, c]$ . For instance, we can find two points on the bisecting normal to the side  $[a, b]$  and another two points on the bisecting normal to the side  $[a, c]$  and then apply (2.2) to get  $m(a, b, c)$ . In this way we see that the center  $m(a, b, c)$  of  $C[a, b, c]$  is

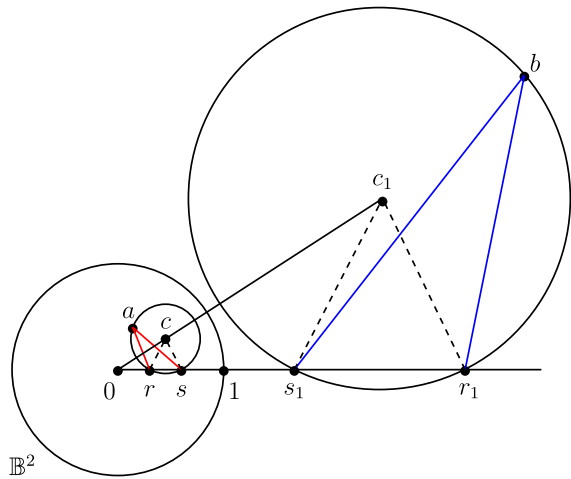
$$m(a, b, c) = \frac{|a|^2(b - c) + |b|^2(c - a) + |c|^2(a - b)}{a(\bar{c} - \bar{b}) + b(\bar{a} - \bar{c}) + c(\bar{b} - \bar{a})}.$$

We sometimes use the notation  $a^* = a/|a|^2 = 1/\bar{a}$  for  $a \in \mathbb{C} \setminus \{0\}$ . The reflection of a point  $z$  in the line through two distinct points  $a$ , and  $b$  is given by

$$w(z) = \frac{a - b}{\bar{a} - \bar{b}} \bar{z} - \frac{a\bar{b} - \bar{a}b}{\bar{a} - \bar{b}}. \tag{2.4}$$

**Proposition 2.5** *Let  $0 < r < s < 1$ ,  $r_1 = 1/r$ , and  $s_1 = 1/s$ . Also, let  $c \in \mathbb{B}^2$  with  $\operatorname{Re} c = (s + r)/2$ ,  $c_1 \in \mathbb{C}$  with  $\operatorname{Re} c_1 = (s_1 + r_1)/2$ , and  $\arg c = \arg c_1$ . If  $a \in S(c, |c - r|)$  and  $b \in S(c_1, |c_1 - r_1|)$ , then  $2\angle(s, a, r) = 2\angle(r_1, b, s_1) = \angle(r, c, s)$ .*

**Fig. 1** Here  $s_1 = 1/s, r_1 = 1/r$ . The key points are that the triangles  $\triangle(r, s, c)$  and  $\triangle(s_1, r_1, c_1)$  are similar, and hence the angles  $\angle(r, a, s)$  and  $\angle(s_1, b, r_1)$  are equal



We omit the simple proof with a reference to Figure 1.

### 2.6 Möbius Transformations

A Möbius transformation is a mapping of the form

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d, z \in \mathbb{C}, ad - bc \neq 0.$$

The special Möbius transformation

$$T_a(z) = \frac{z - a}{1 - \bar{a}z}, \quad a \in \mathbb{B}^2 \setminus \{0\}, \tag{2.7}$$

maps the unit disk  $\mathbb{B}^2$  onto itself with  $T_a(a) = 0$ , and  $T_a(\pm a/|a|) = \pm a/|a|$ . In complex analysis, quadruples of points have a very special role: the absolute cross-ratio of four points  $a, b, c$ , and  $d$  in the complex plane  $\mathbb{C}$ ,

$$|a, b, c, d| = \frac{|a - c||b - d|}{|a - b||c - d|},$$

is invariant under Möbius transformations.

### 2.8 Hyperbolic Geometry

We recall some basic formulas and notation for hyperbolic geometry from [2]. The hyperbolic metric  $\rho_{\mathbb{B}^2}$  is defined by

$$\text{sh} \frac{\rho_{\mathbb{B}^2}(a, b)}{2} = \frac{|a - b|}{\sqrt{(1 - |a|^2)(1 - |b|^2)}}, \quad a, b \in \mathbb{B}^2.$$

For  $a, b \in \mathbb{B}^2 \setminus \{0\}$  let

$$\text{ep}(a, b) = T_{-b}(T_b(a)/|T_b(a)|).$$

This formula defines the endpoints  $\text{ep}(a, b)$  and  $\text{ep}(b, a)$  on the unit circle of the hyperbolic line through  $a$  and  $b$ . The hyperbolic metric also satisfies

$$\rho_{\mathbb{B}^2}(a, b) = \log |\text{ep}(a, b), a, b, \text{ep}(b, a)|.$$

The circle which is orthogonal to the unit circle and contains two distinct points  $a, b \in \mathbb{C}$  is denoted by  $C[a, b]$ . If  $a, b \in \mathbb{B}^2$  are distinct points, then  $C[a, b] \cap \partial\mathbb{B}^2 = \{a_*, b_*\}$  where the points are labelled in such a way that  $a_*, a, b$ , and  $b_*$  occur in this order on  $C[a, b]$ . Note that above we have used the notation  $\text{ep}(a, b) = a_*$ , and  $\text{ep}(b, a) = b_*$ . We denote by  $J[a, b]$  the hyperbolic geodesic segment joining two distinct points  $a, b \in \mathbb{B}^2$ . Then  $J[a, b]$  is a subarc of  $C[a, b] \cap \mathbb{B}^2$  and the hyperbolic line is  $J^*[a_*, b_*] = C[a, b] \cap \mathbb{B}^2$ .

**Lemma 2.9** *Suppose that  $a, b \in \mathbb{B}^2 \setminus \{0\}$  are two points non-collinear with 0 and  $|a| \neq |b|$ . Then the inversion  $h : \mathbb{B}^2 \rightarrow \mathbb{B}^2$  with  $h(a) = b$  is given by*

$$h(z) = \frac{c\bar{z} - 1}{\bar{z} - \bar{c}}, \quad c = LIS[a, b, a^*, b^*] = \frac{a - b + ab(\bar{a} - \bar{b})}{|a|^2 - |b|^2}, \tag{2.10}$$

and  $h$  maps the chord  $L[a, b] \cap \mathbb{B}^2$  onto itself. Moreover, the  $S(c, \sqrt{|c|^2 - 1})$  orthogonal to the unit circle  $S(0, 1)$  intersects the unit circle at the points  $(1 \pm i\sqrt{|c|^2 - 1})/\bar{c}$ .

**Proof** The above simple formula for  $c$  follows from the formulas (2.2) and (2.3) for the intersection of two lines as  $c = LIS[a, b, a^*, b^*]$  and checking  $h(a) = b$  is a simple verification. The points  $z_1, z_2$  are found by solving the equations  $|z|^2 = 1$  and  $|z - c|^2 = |c|^2 - 1$ . □

We start our discussion of the visual angle metric in the simple case of points on the same radius of the unit disk (See Fig. 2).

**2.11 The Case of Radial Points  $v_{\mathbb{B}^2}(r, s)$ , where  $0 < r < s < 1$**

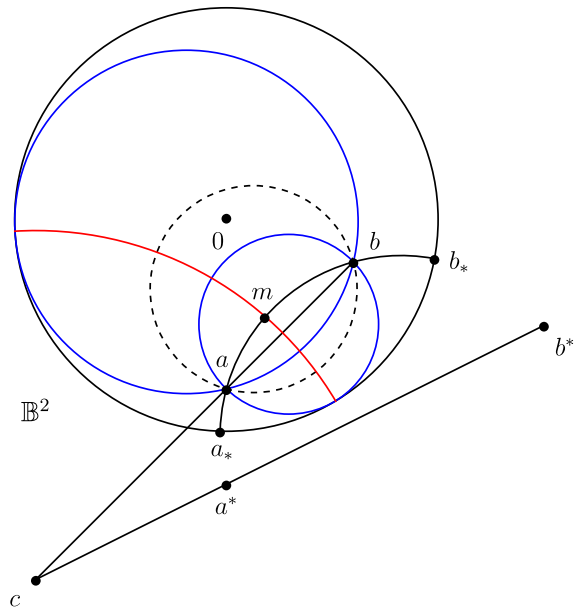
Writing,

$$c = \frac{r + s}{2}, \quad d = \frac{\sqrt{(1 - r^2)(1 - s^2)}}{2}, \quad \text{and} \quad c^2 + d^2 = \left(\frac{1 + rs}{2}\right)^2,$$

we easily see that the circle  $S(c + id, (1 - rs)/2)$  passes through the points  $r$  and  $s$  and is internally tangent to  $S(0, 1)$  at the point

$$p = \left( \frac{r + s}{1 + rs}, \frac{\sqrt{(1 - r^2)(1 - s^2)}}{1 + rs} \right).$$

**Fig. 2** The inversion  $h(z) = (c\bar{z} - 1)/(\bar{z} - \bar{c})$  with  $c = LIS[a, b, a^*, b^*]$  maps the unit disk onto itself and the chord containing  $a$  and  $b$  onto itself. The point  $m$  is the hyperbolic midpoint of  $a$  and  $b$ , and the hyperbolic circle through  $a$  and  $b$  centered at  $m$  is drawn with a dashed line



Therefore

$$v_{\mathbb{B}^2}(r, s) = \arcsin \frac{s - r}{1 - rs}.$$

Moreover, the segment  $[-i, p]$  bisects the angle  $\angle(r, p, s)$  and the circle orthogonal to the unit circle at the points  $p$  and  $\bar{p}$  passes through the hyperbolic midpoint (See Fig. 3)

$$\frac{r + s}{1 + rs + \sqrt{(1 - r^2)(1 - s^2)}},$$

of the segment  $[r, s]$ .

**Lemma 2.12** ([8, Lemma 3.10]) *For  $a, b \in \mathbb{B}^2$  collinear with 0 we have*

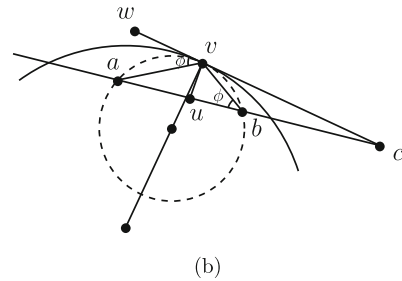
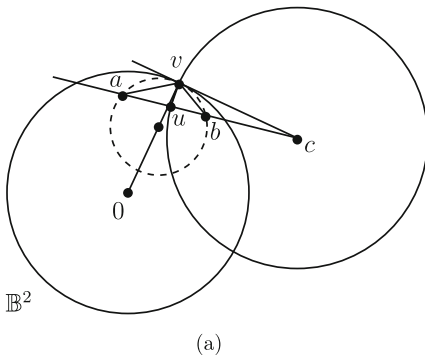
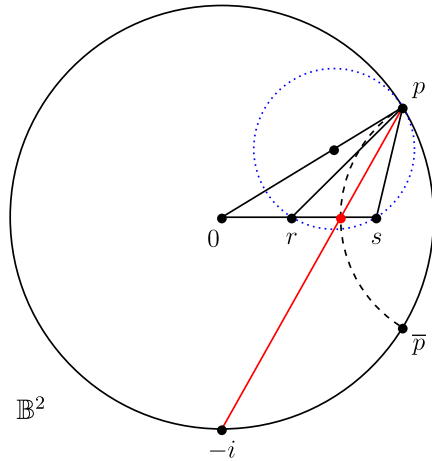
$$\tan v_{\mathbb{B}^2}(a, b) = \text{sh} \frac{\rho_{\mathbb{B}^2}(a, b)}{2}.$$

### 3 A Formula for the Visual Angle Metric

#### 3.1 Angle Bisection Property

The next theorem, on the other hand, generalizes the above observations connected with the visual metric of two points on the same radius. On the other hand, it also yields a proof of Theorem 1.1.

**Fig. 3** The segment  $[-i, p]$  bisects the angle  $\angle(r, p, s)$ . The dashed circle orthogonal to the unit circle at the points  $p$  and  $\bar{p}$  passes through the hyperbolic midpoint of the segment  $[r, s]$ . Note that  $\operatorname{Re}\{p\} = (r + s)/(1 + rs) = \operatorname{th}((\rho_{\mathbb{B}^2}(0, r) + \rho_{\mathbb{B}^2}(0, s))/2)$



**Fig. 4 (a):** Angle bisection visualized. **(b):** Angle bisection visualized, detail. Here  $\angle(w, v, a) = \angle(v, b, a) = \phi$

**Theorem 3.2** For  $a, b \in \mathbb{B}^2$  non-collinear with 0 and with  $|a| \neq |b|$ , the circle  $S(c, \sqrt{|c|^2 - 1})$  centered at  $c = LIS[a, b, a^*, b^*]$  is orthogonal to  $\partial\mathbb{B}^2$ . Let  $u = S(c, \sqrt{|c|^2 - 1}) \cap L[a, b]$  and let  $v \in S(c, \sqrt{|c|^2 - 1}) \cap \partial\mathbb{B}^2$  be the point maximizing the angle  $\angle(a, z, b)$  with  $z \in S(0, 1)$ . Then

$$v_{\mathbb{B}^2}(a, b) = \angle(a, v, b) \quad \text{and} \quad \angle(a, v, u) = \angle(u, v, b).$$

**Proof** Let  $\ell = L[a, b]$ ,  $u$ , and  $v$  be as above and  $w = c + 1.5(v - c)$ , see Figure 4b. It is clear that the triangle  $\Delta(u, c, v)$  is an isosceles triangle. Therefore,  $\angle(c, v, u) = \angle(c, u, v)$  holds. It also follows from the Alternate Segment Theorem that  $\angle(w, v, a) = \angle(v, b, a) = \phi$ . Considering the sum of the inner angles of  $\Delta(u, b, v)$ , we find that  $\angle(b, v, u) = \pi - \angle(u, b, v) - \angle(v, u, b)$ . Considering also the line  $L[w, c]$ , we have  $\angle(a, v, u) = \pi - \angle(u, b, v) - \angle(v, u, b)$ . Hence, we see that  $\angle(a, v, u) = \angle(b, v, u)$ .  $\square$

It should be noticed that in the above proof,  $v_{\mathbb{B}^2}(a, b) \neq 2v_{\mathbb{B}^2}(a, u)$ .

### 3.3 A Functional Identity Between $\rho_{\mathbb{B}^2}(a, b)$ and $v_{\mathbb{B}^2}(a, b)$

We will next prove a new formula for  $v_{\mathbb{B}^2}(a, b)$  and give first an auxiliary lemma.

**Lemma 3.4** *Let  $m_1, m_2 \in \mathbb{B}^2$  be non-collinear with 0, and let  $|m_1| \neq |m_2|$ . Then, there exists an inversion  $h : \mathbb{B}^2 \rightarrow \mathbb{B}^2 = h(\mathbb{B}^2)$  with the following properties:*

$$i) \ h(m_1) = m_2; \quad ii) \ h(L[m_1, m_2] \cap \mathbb{B}^2) = L[m_1, m_2] \cap \mathbb{B}^2.$$

**Proof** The proof follows from Lemma 2.9. □

### 3.5 Proof of Theorem 1.1

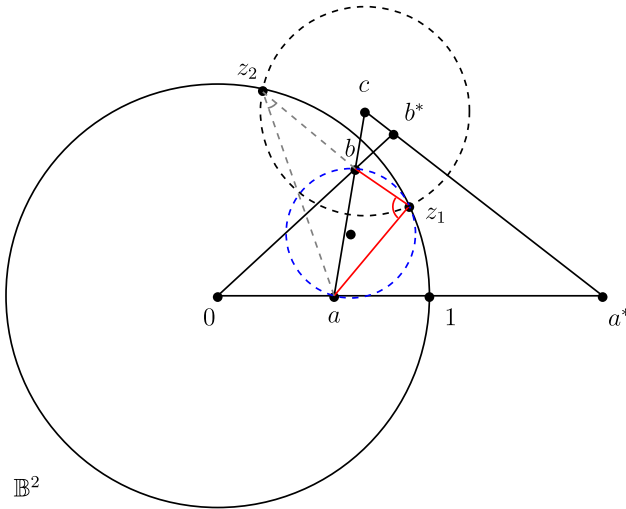
Consider first the case  $|a| \neq |b|$  with  $0 \notin L[a, b]$ . The inversion  $h$  in Lemma 2.9 satisfies  $h(a) = b$  and hence the triangles  $\triangle(c, a, d)$  and  $\triangle(c, b, d)$  are similar where  $d$  refers to  $z_1$  in Figure 5. By the proof of Theorem 3.2 the circle through  $a, d, b$  is internally tangent to the unit circle and  $v_{\mathbb{B}^2}(a, b) = \angle(a, d, b)$ . The formula for the points  $z_1, z_2$  is given in Lemma 2.9. In the case  $|a| = |b|$ , the formula follows easily by symmetry. □

### 3.6 Remark

The two points  $z_1$  and  $z_2$  in Theorem 1.1 are given as solutions to the equation:

$$(\bar{a} \cdot (1 - |b|^2) - \bar{b} \cdot (1 - |a|^2)) \cdot z^2 - 2 \cdot (|a|^2 - |b|^2) \cdot z + a \cdot (1 - |b|^2) - b \cdot (1 - |a|^2) = 0.$$

From Theorem 1.1, substituting  $c = (a(1 - |b|^2) - b(1 - |a|^2))/(|a|^2 - |b|^2)$ ,  $r^2 = |c|^2 - 1$  into the equation  $|z - c| = r$  of the circle with center  $c$  and radius  $r$ , we have



**Fig. 5** If  $c = LIS[a, b, a^*, b^*]$  and  $z_1 = S(c, \sqrt{|c|^2 - 1}) \cap S(0, 1)$  is a point in the sector with vertex  $c$  and sides  $L[c, a]$  and  $L[c, a^*]$ , then  $v_{\mathbb{B}^2}(a, b) = \angle(a, z_1, b)$

$$\begin{aligned} & (z - c)(\bar{z} - \bar{c}) - r^2 \\ &= \frac{(|a|^2 - |b|^2)z\bar{z} - (\bar{a}(1 - |b|^2) - \bar{b}(1 - |a|^2))z - (a(1 - |b|^2) - b(1 - |a|^2))\bar{z} + |a|^2 - |b|^2}{|a|^2 - |b|^2} \\ &= 0. \end{aligned}$$

Therefore, if  $|a| \neq |b|$ , the following holds:

$$(|a|^2 - |b|^2)z\bar{z} - (\bar{a}(1 - |b|^2) - \bar{b}(1 - |a|^2))z - (a(1 - |b|^2) - b(1 - |a|^2))\bar{z} + |a|^2 - |b|^2 = 0.$$

Substituting  $\bar{z} = 1/z$  into the above equality, the intersection of the unit circle  $S(0, 1)$  and the circle  $S(c, \sqrt{|c|^2 - 1})$  is obtained as the solution of the following equation

$$(\bar{a}(1 - |b|^2) - \bar{b}(1 - |a|^2))z^2 - 2(|a|^2 - |b|^2)z + a(1 - |b|^2) - b(1 - |a|^2) = 0. \tag{3.7}$$

On the other hand, we consider the line  $L$  and a circle  $C$  both passing through points  $a$  and  $b$ . The circle  $C$  is divided into two arcs by  $L$ . On each arc, as  $z$  ranges over each arc, the angle  $\angle(a, z, b)$  takes a constant value from the inscribed angle theorem. The larger the radius of the circle  $C$ , the smaller the angle  $\angle(a, z, b)$ . There exist two circles that have chord  $[a, b]$  and are inscribed in the unit circle. Let  $\tilde{C}$  be the circle with smaller radius of them. The intersection point  $q$  of  $\tilde{C}$  and the unit circle gives the visual angle metric for  $a, b$ , i.e.  $v_{\mathbb{B}^2}(a, b) = \angle(a, q, b)$ . In fact, for  $c_0 = m(a, b, p)$ , the circle  $\tilde{C}$  is written as  $|z - c_0| = |q - c_0|$ . Since  $q$  is a point on the unit circle,  $q\bar{q} = 1$  holds. Therefore, we have  $|z - c_0|^2 - |q - c_0|^2 = P/Q = 0$ , where

$$\begin{aligned}
 P = & ((\bar{a} - \bar{b})q^2 + (\bar{b}a - \bar{a}b)q - a + b)z\bar{z} + (-\bar{a}\bar{b}(a - b) + \bar{a} - \bar{b})q + \bar{a}a - \bar{b}b)z \\
 & + ((-\bar{a}a + \bar{b}b)q^2 + ((\bar{a} - \bar{b})ab + a - b)q)\bar{z} + \bar{b}\bar{a}(a - b)q^2 \\
 & + (-\bar{b}a + \bar{a}b)q + (-\bar{a} + \bar{b})ba,
 \end{aligned}$$

and

$$Q = (\bar{a} - \bar{b})q^2 + (\bar{b}a - \bar{a}b)q - a + b.$$

The intersection points of  $\tilde{C}$  and the unit circle are obtained as solutions to the equation substituting  $\bar{z} = 1/z$  into  $P = 0$ . Then, we have,

$$(z - q)((\bar{a}\bar{b}(a - b) + \bar{a} - \bar{b})q - |a|^2 + |b|^2)z + (-|a|^2 + |b|^2)q + (\bar{a} - \bar{b})ab + a - b) = 0.$$

As  $q$  is the point of tangency, the above equation has a double root  $z = q$ . Hence, we have

$$(\bar{a}(1 - |b|^2) - \bar{b}(1 - |a|^2))q^2 - 2(|a|^2 - |b|^2)q + a(1 - |b|^2) - b(1 - |a|^2) = 0.$$

This equation is equivalent to (3.7).

### 3.8 Proof of Theorem 1.2

If  $a, b \in \mathbb{B}^2$  are collinear with 0, the proof follows from Lemma 2.12. First, we consider the case  $|a| = |b|$ , and denote  $t = |a - b|/2$ . Then  $m = (a + b)/2$ ,  $s = |m| > 0$  and

$$v_{\mathbb{B}^2}(a, b) = 2 \arctan \frac{t}{1 - s} \Leftrightarrow t = (1 - s) \tan \frac{v_{\mathbb{B}^2}(a, b)}{2}. \tag{3.9}$$

Therefore, by [2, p. 40] and (3.9) we obtain

$$\operatorname{sh} \frac{\rho_{\mathbb{B}^2}(a, b)}{2} = \frac{2t}{1 - s^2 - t^2} = \frac{2(1 - s) \tan(v_{\mathbb{B}^2}(a, b)/2)}{1 - s^2 - (1 - s)^2 \tan^2(v_{\mathbb{B}^2}(a, b)/2)}.$$

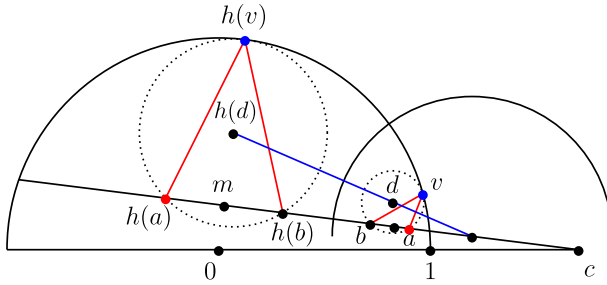
Solving this equation for  $\tan(v_{\mathbb{B}^2}(a, b)/2)$  yields the desired result in this case.

Consider now the case  $|a| \neq |b|$ . Observe that the formula for  $m$  follows from (2.4). Let  $\operatorname{hmid}$  be a point on  $[a, b]$  with

$$\rho_{\mathbb{B}^2}(a, \operatorname{hmid}) = \rho_{\mathbb{B}^2}(\operatorname{hmid}, b).$$

We shall apply Lemma 3.4 with  $m_1 = \operatorname{hmid}$  and  $m_2 = m$  to find an inversion  $h : \mathbb{B}^2 \rightarrow \mathbb{B}^2 = h(\mathbb{B}^2)$ . By Proposition 2.5 we also know that  $v_{\mathbb{B}^2}(a, b) = v_{\mathbb{B}^2}(h(a), h(b))$ . In conclusion, because  $|h(a)| = |h(b)|$ , this inversion reduces the general case to the special case proved above. The proof is now complete.  $\square$

Observe that in the case when  $a, b$ , and 0 are collinear, Theorem 1.2 reduces to Lemma 2.12 and that Corollary 3.10 (2) gives an equivalent form of Lemma 2.12 (See Fig. 6).



**Fig. 6** The idea of the proof of Theorem 1.2 visualized. The key points are that the triangles  $\Delta(b, a, d)$  and  $\Delta(h(a), h(b), h(d))$  are similar, and hence the angles  $\angle(h(b), h(v), h(a))$  and  $\angle(a, v, b)$  are equal.

**Corollary 3.10** (1) For  $a, b \in \mathbb{B}^2$  and  $m$  as in Theorem 1.2, we have

$$\sin v_{\mathbb{B}^2}(a, b) = \frac{(1 + |m|)(1 + \sqrt{1 + (1 - |m|^2)u^2})u}{1 + \sqrt{1 + (1 - |m|^2)u^2} + (1 + |m|)u^2}, \quad u = \operatorname{sh} \frac{\rho_{\mathbb{B}^2}(a, b)}{2}.$$

(2) For  $m = 0$  we have

$$\sin v_{\mathbb{B}^2}(a, b) = \operatorname{th} \frac{\rho_{\mathbb{B}^2}(a, b)}{2} = \frac{|a - b|}{\sqrt{|a - b|^2 + (1 - |a|^2)(1 - |b|^2)}}.$$

**Proof** Both (1) and (2) follow from Theorem 1.2 by simple manipulations. □

### 3.11 Proposition

For  $m \in (0, 1)$ ,  $r > 0$ , and  $u = \operatorname{sh}(r/2)$

$$(1 + m) \operatorname{th} \frac{r}{4} \leq \frac{(1 + m)u}{1 + \sqrt{1 + (1 - m^2)u^2}} \leq \min \left\{ \frac{(1 + m)u}{2}, \sqrt{\frac{1 + m}{1 - m}} \operatorname{th} \frac{r}{4} \right\}.$$

**Proof** Observing  $\operatorname{th}(r/4) = u/(1 + \sqrt{1 + u^2})$  and writing  $v$  for the middle term we have

$$(1 + m) \operatorname{th} \frac{r}{4} = \frac{(1 + m)u}{1 + \sqrt{1 + u^2}} \leq v \leq \frac{(1 + m)u}{\sqrt{1 - m^2}(1 + \sqrt{1 + u^2})} = \sqrt{\frac{1 + m}{1 - m}} \operatorname{th} \frac{r}{4},$$

and trivially  $v \leq (1 + m)u/2$ . □

The next corollary yields, as a special case, Theorem 3.11 of [8].

**Corollary 3.12** For  $a, b \in \mathbb{B}^2$  and,  $m$  as in Theorem 1.2, we have

$$(1 + |m|) \operatorname{th} \frac{\rho_{\mathbb{B}^2}(a, b)}{4} \leq \tan \frac{v_{\mathbb{B}^2}(a, b)}{2} \leq \min \left\{ \frac{1 + |m|}{2} \operatorname{sh} \frac{\rho_{\mathbb{B}^2}(a, b)}{2}, \sqrt{\frac{1 + |m|}{1 - |m|}} \operatorname{th} \frac{\rho_{\mathbb{B}^2}(a, b)}{4} \right\}.$$

**Proof** The proof follows from Theorem 1.2 and Proposition 3.11. □

For the proof of Theorem 1.3 we need some basic facts about quasiregular mappings, see [1, 4]. In particular, we use the quasiregular Schwarz lemma in the following form with detailed information about the distortion function  $\varphi_K$ . For  $r \in (0, 1)$  and  $K \in [1, \infty)$  the function  $\varphi : [0, 1] \rightarrow [0, 1]$  is defined by

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K), \quad (\varphi_K(0) = 0, \varphi_K(1) = 1),$$

where  $\mu : (0, 1) \rightarrow (0, \infty)$  is the decreasing homeomorphism defined by

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}; \quad \mathcal{K}(r) = \frac{\pi}{2} F(1/2, 1/2; 1; r^2),$$

and  $F$  is the Gaussian hypergeometric function.

**Lemma 3.13** (1) Let  $f : \mathbb{B}^2 \rightarrow \mathbb{B}^2$  be a  $K$ -quasiregular mapping, where  $K \geq 1$ , and  $a, b \in \mathbb{B}^2$ . Then

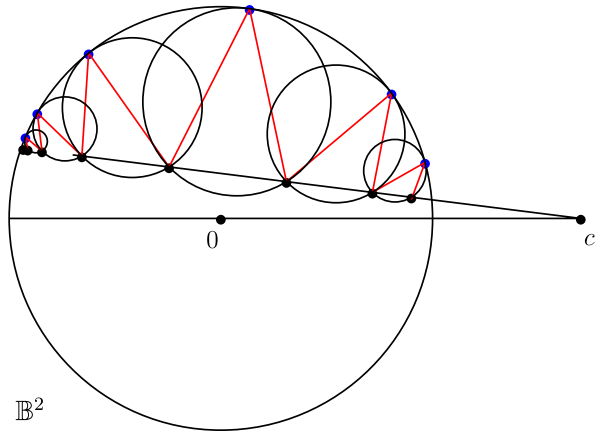
$$\operatorname{th} \frac{\rho_{\mathbb{B}^2}(f(a), f(b))}{2} \leq \varphi_K \left( \operatorname{th} \frac{\rho_{\mathbb{B}^2}(a, b)}{2} \right) \leq 4^{1-1/K} \left( \operatorname{th} \frac{\rho_{\mathbb{B}^2}(a, b)}{2} \right)^{1/K}.$$

(2) The function  $\varphi_K, K \geq 1$ , satisfies for  $0 < r < 1$

$$\frac{\varphi_K(r)}{1 + \sqrt{1 - \varphi_K(r)^2}} = \sqrt{\varphi_K \left( \left( \frac{r}{1 + \sqrt{1 - r^2}} \right)^2 \right)}.$$

**Proof** (1) See [4, Thm 16.2(1)]. (2) See [1, Theorem 10.5]. □

**Fig. 7** Here  $\rho_{\mathbb{B}^2}(a_j, a_{j+1}) = \text{const}$  and also  $v_{\mathbb{B}^2}(a_j, a_{j+1})$  is a constant. It follows from Proposition 2.5 that the triangles with vertices on the unit circle have equal angles



**3.14 Proof of Theorem 1.3**

Write  $\rho' = \rho_{\mathbb{B}^2}(f(a), f(b))$ , and  $\rho = \rho_{\mathbb{B}^2}(a, b)$ . By Theorem 1.2, Proposition 3.11 with  $M = \sqrt{(1 + |m_1|)/(1 - |m_1|)}$  and Lemma 3.13

$$\begin{aligned} \tan \frac{v_{\mathbb{B}^2}(f(a), f(b))}{2} &\leq M \operatorname{th} \frac{\rho'}{4} = \frac{M \operatorname{th} \frac{\rho'}{2}}{1 + \sqrt{1 - \operatorname{th}^2 \frac{\rho'}{2}}} \\ &\leq \frac{M \varphi_K(\operatorname{th} \frac{\rho'}{2})}{1 + \sqrt{1 - \varphi_K(\operatorname{th} \frac{\rho'}{2})^2}} = M \sqrt{\varphi_K\left(\operatorname{th}^2 \frac{\rho'}{4}\right)} \\ &\leq M \sqrt{\varphi_K\left(\left(\frac{1}{1 + |m_2|} \tan \frac{v_{\mathbb{B}^2}(a, b)}{2}\right)^2\right)} \leq \frac{M 2^{1-1/K}}{(1 + |m_2|)^{1/K}} \left(\tan \frac{v_{\mathbb{B}^2}(a, b)}{2}\right)^{1/K}, \end{aligned}$$

where in the second equality we have used this fact that if  $r = \operatorname{th}(\rho/2)$ , then  $r/(1 + \sqrt{1 - r^2}) = \operatorname{th}(\rho/4)$ . □

**Corollary 3.15** Let  $T_w : \mathbb{B}^2 \rightarrow \mathbb{B}^2$  be a Möbius transformation as in (2.7). Let  $a, b, w \in \mathbb{B}^2$ ,  $|m_2| = d(L[a, b], 0)$ , and  $|m_1| = d(L[T_w(a), T_w(b)], 0)$ . Then

$$\tan \frac{v_{\mathbb{B}^2}(T_w(a), T_w(b))}{2} \leq c(m_1, m_2) \tan \frac{v_{\mathbb{B}^2}(a, b)}{2}, \quad c(m_1, m_2) = \frac{1}{1 + |m_2|} \sqrt{\frac{1 + |m_1|}{1 - |m_1|}},$$

with equality for  $m_1 = m_2 = 0$ .

**Proof** The proof follows from Theorem 1.3. □

We next consider an evenly separated sequence of collinear points.

### 3.16 Evenly Separated Collinear Points

Consider a collinear sequence of points  $a_j, j = 1, 2, 3, \dots$  in the unit disk with a constant hyperbolic distance  $\rho_{\mathbb{B}^2}(a_j, a_{j+1}) = \text{const}$ . It turns out that also  $v_{\mathbb{B}^2}(a_j, a_{j+1})$  is a constant—the special case when the points are on the diameter  $(-1, 1)$  was discussed in greater detail in Section 2 (See Fig. 7).

### 4 Additional Identities for $v_{\mathbb{B}^2}(a, b)$

We give here another two analytic formulas, for  $v_{\mathbb{B}^2}(a, b)$ . The first formula is based on the recent explicit formula for the hyperbolic midpoint of two points, whereas for the second formula we have used symbolic computation. Both formulas are based on geometric ideas and are best used for computer work because some lengthy expressions will be needed.

We use the Ahlfors bracket notation  $A[a, b]$  [4, p. 38], for  $a, b \in \mathbb{B}^2$

$$A[a, b] = \sqrt{|a - b|^2 + (1 - |a|^2)(1 - |b|^2)} = |1 - a\bar{b}|. \tag{4.1}$$

The formula for the hyperbolic midpoint is given in the following theorem.

**Theorem 4.2** [12] *For given  $a, b \in \mathbb{B}^2$ , the hyperbolic midpoint  $z \in \mathbb{B}^2$  with  $\rho_{\mathbb{B}^2}(a, z) = \rho_{\mathbb{B}^2}(b, z) = \rho_{\mathbb{B}^2}(a, b)/2$  is given by*

$$z = \frac{b(1 - |a|^2) + a(1 - |b|^2)}{1 - |a|^2|b|^2 + A[a, b]\sqrt{(1 - |a|^2)(1 - |b|^2)}},$$

where  $A[a, b]$  is the Ahlfors bracket defined as (4.1).

Theorem 4.3 is essentially the same as [5, Theorem 3.2]. Note, however, that by Theorem 1.1 we have now explicit formulas for the points  $q_1$  and  $q_2$  (See Fig. 8).

**Theorem 4.3** *For given  $a, b \in \mathbb{B}^2$ , let  $m \in \mathbb{B}^2$  be their hyperbolic midpoint. Then*

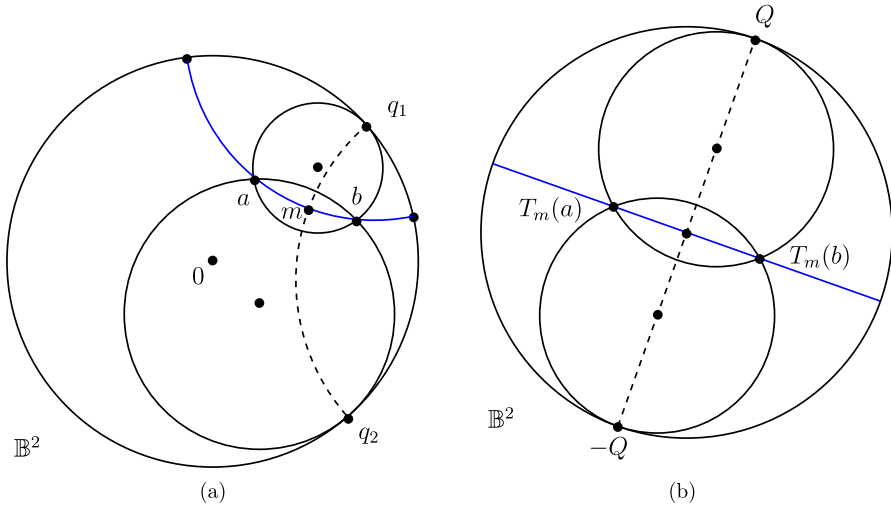
$$v_{\mathbb{B}^2}(a, b) = \max\{\angle(a, q_1, b), \angle(a, q_2, b)\},$$

where

$$q_1 = T_m^{-1}(Q), \quad q_2 = T_m^{-1}(-Q),$$

and  $Q = i(T_m(a) - T_m(b))/|T_m(a) - T_m(b)|$ .

**Proof** Because  $|T_m(a)| = |T_m(b)|$  it is evident that the circles through  $T_m(a), T_m(b), Q$  and  $T_m(a), T_m(b), -Q$  are the maximal circles through  $T_m(a)$  and  $T_m(b)$  contained in the unit disk. Therefore, one of the circles through  $a, b$ , and  $q_1$  or  $a, b$ , and  $q_2$  must be the maximal circle in  $\mathbb{B}^2$  through  $a$  and  $b$ . □



**Fig. 8** (a) Before Möbius transformation  $T_m$ . (b) After Möbius transformation  $T_m$

**Theorem 4.4** For given  $a, b \in \mathbb{B}^2$ , let  $q \in S(0, 1)$  be the point that gives  $v_{\mathbb{B}^2}(a, b) = \angle(a, q, b)$  and set  $p = a + t(b - a)i$ . If a right triangle  $\Delta(a, b, p)$  is inscribed in the circle passing through three points  $a, b$ , and  $p$ , then  $t$  is given as the solution with the smaller absolute value of the following equation

$$((\bar{a}b - a\bar{b})^2 + 4(\bar{a} - \bar{b})(a - b))t^2 + 2(\bar{a}b - a\bar{b})(\bar{a}b + \bar{a}b - 2)it - ((\bar{a}b + a\bar{b})^2 - 4(a\bar{a} + b\bar{b} - 1)) = 0.$$

**Proof** Let  $r$  be the radius of the inscribed maximal circle through  $a$  and  $b$ . Then,  $|c| + r = 1$  and  $c = (p + b)/2$  hold. Eliminating  $c$  from the above equations, we have  $f_1(r) = 0$ , where

$$f_1(r) = (p + b)(\bar{p} + \bar{b}) - 4(1 - r)^2.$$

Since the triangle  $\Delta(a, b, p)$  is a right triangle inscribed in the circle with center  $c$ , we have  $f_2(r) = 0$ , where

$$f_2(r) = (a - b)(\bar{a} - \bar{b}) + (a - p)(\bar{a} - \bar{p}) - 4r^2.$$

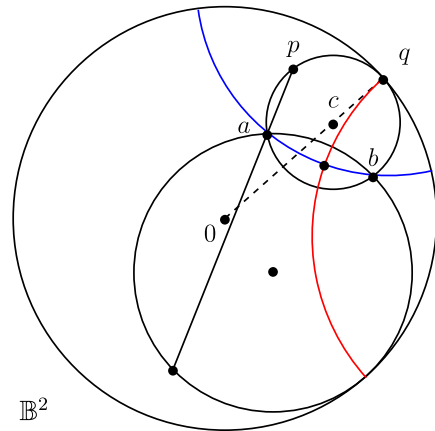
Eliminating  $r$  from equations  $f_1 = 0$  and  $f_2 = 0$  by computing the following resultant (See Fig. 9)

$$\text{result}_r(f_1, f_2) = 0,$$

we have

$$(\bar{a} + \bar{b})^2 p^2 + 2((a + b)(\bar{a} + \bar{b}) - 8)p\bar{p} - 2((\bar{a} - \bar{b})(\bar{a}b + \bar{a}b - 4) + 2\bar{a}^2(a - b))p + (a + b)^2 \bar{p}^2 - 2((a - b)(\bar{a}b + \bar{a}b - 4) + 2a^2(\bar{a} - \bar{b}))\bar{p} + (\bar{a}b + \bar{a}b - 2a\bar{a})^2 + 8(\bar{a}b + \bar{a}b - 2(\bar{a}a + \bar{b}b - 1)) = 0. \tag{4.5}$$

**Fig. 9**  $|c| + |p - c| = 1$  yields an equation to find  $p$  using RISA. Then  $v_{\mathbb{B}^2}(a, b) = \angle(a, q, b) = \angle(a, p, b)$



(Here, we use Risa/Asir, a symbolic computation system, for computing the above resultant.) Substituting  $p = a + t(b - a)i$  into (4.5) gives

$$\begin{aligned} & ((a\bar{b} - \bar{a}b)^2 + 4(\bar{a} - \bar{b})(a - b)t^2 + 2(a\bar{b} - \bar{a}b)(a\bar{b} + \bar{a}b - 2)it \\ & - ((a\bar{b} + \bar{a}b)^2 - 4(a\bar{a} + b\bar{b} - 1)) = 0. \end{aligned} \tag{4.6}$$

Since  $(a\bar{b} - \bar{a}b)$  equals  $2 \operatorname{Im}(a\bar{b})i$ , the number  $(a\bar{b} - \bar{a}b)$  is purely imaginary. Therefore, equation (4.6) is the quadratic equation with the real coefficients whose discriminant  $D$  satisfies  $D = 64|a - b|^2(1 - |a|^2)(1 - |b|^2) > 0$  and hence (4.6) has two real solutions. Here, we remark that the smaller the inradius  $\sqrt{t^2 + |ab|^2}$  of the circle centered at  $c$ , the larger the inscribed angle with respect to the common chord  $[a, b]$ . Hence, the solution  $t$  of the smaller absolute value is the one that yields  $v_{\mathbb{B}^2}(a, b)$  as the maximal inscribed angle.  $\square$

**Acknowledgements** The first author was partially supported by JSPS KAKENHI Grant Numbers JP19K03531, JP25K07039. The research of the second author was supported by a grant from Vilho, Yrjö and Kalle Väisälä fund.

**Funding** Open Access funding provided by University of Turku (including Turku University Central Hospital).

## Declarations

**Conflicts of Interest** The authors declare that they have no conflict of interest.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

## References

1. Anderson, G.D., Vamanamurthy, M.K., Vuorinen, M.: Conformal Invariants, Inequalities, and Quasi-conformal Maps. John Wiley & Sons, New York (1997)
2. Beardon, A.F.: The Geometry of Discrete Groups. Springer-Verlag, New York (1983)
3. Gehring F.W.: Hag K., The Ubiquitous Quasidisk. With contributions by Ole Jacob Broch. American Mathematical Society, Providence, RI, 2012. xii+171 pp
4. Hariri, P., Klén, R., Vuorinen, M.: Conformally Invariant Metrics and Quasiconformal Mappings. Springer Monographs in Mathematics, Springer, Berlin (2020)
5. Hariri, P., Vuorinen, M., Wang, G.: Some remarks on the visual angle metric. *Comput. Methods Funct. Theory* **16**, 187–201 (2016)
6. Hästö, P.A. Ibragimov, Z., Minda, D., Ponnusamy, S., Sahoo, S.: Isometries of some hyperbolic-type path metrics, and the hyperbolic medial axis. (English summary) In the tradition of Ahlfors-Bers. IV, 63–74, *Contemp. Math.*, 432, Amer. Math. Soc., Providence, RI, 2007
7. Heinonen, J.: Lectures on Analysis on Metric Spaces. Springer-Verlag, New York (2001)
8. Klén, R., Lindén, H., Vuorinen, M., Wang, G.: The visual angle metric and Möbius transformations. *Comput. Methods Funct. Theory* **14**, 577–608 (2014)
9. Papadopoulos, A.: Metric spaces, convexity and non-positive curvature. Second edition. IRMA Lectures in Mathematics and Theoretical Physics, 6. European Mathematical Society (EMS), Zürich, 2014. xii+309 pp
10. Rainio, O., Vuorinen, M.: Hilbert metric in the unit ball. *Studia Sci. Math. Hungar.* **60**, 175–191 (2023)
11. Wang, G., Vuorinen, M.: The visual angle metric and quasiregular maps. *Proc. Amer. Math. Soc.* **144**, 4899–4912 (2016)
12. Wang, G., Vuorinen, M., Zhang, X.: On cyclic quadrilaterals in Euclidean and hyperbolic geometries. *Publ. Math. Debrecen* **99**(1–2), 123–140 (2021)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.