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Forbidden Four Cycle, Star Graphs and Isometric Embeddings

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Abstract. We prove the necessary and sufficient conditions under which ultrametric spaces of arbitrary infinite cardinality admit isometric embeddings into ultrametric spaces generated by labeled star graphs.

Анотація. Ми доводимо необхідні та достатні умови, за яких ультраметричні простори довільної нескінченної потужності допускають ізометричні вкладення в ультраметричні простори, породжені міченими зіркоподібними графами.

Introduction

Over the past several years, ultrametric spaces have been studied using models based on labeled trees. In the finite case, every ultrametric space can be represented up to isometry by tree with non-negative vertex labeling, known as the Gurvich–Vyalyi tree [26]. This representation and its geometric interpretation [34] have led to solutions of various problems in finite ultrametric spaces [19–22,33]. Recently, an analogue of Gurvich–Vyalyi representation was obtained for totally bounded ultrametric spaces [12].

The concept of an ultrametric space generated by an arbitrary non-negative vertex labeling on both finite and infinite trees was introduced in [9] and investigated in [14,16,24,25]. König's Infinity Lemma [30] shows that infinite star-graphs and rays are fundamental to the construction of any infinite graph. A purely metric characterization of ultrametric spaces generated by labeled star graphs is provided in [23]. Compact ultrametric spaces generated by labeled star graphs are studied in [13]. In particular, [13] gives a criterion for when a compact ultrametric space admits an isometric embedding into an ultrametric space generated by a labeled star graph. The same paper [13] contains the following conjecture: "An ultrametric space (X, d) admits an isometric embedding in an ultrametric space generated by labeled star graph if and only if each four-point subspace

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of (X, d) admits such embedding." The main result of the present paper, Theorem 4.6, gives a proof of this conjecture.

The proof of Theorem 4.6 is rather cumbersome, but it is based on the following simple facts. All ultrametric spaces generated by labeled star graphs are complete and each metric space can be isometrically embedded into its completion. To prove Theorem 4.6, we modify the metric of original ultrametric space so that the modified space contains a Cauchy sequence, and show that this space is isometrically embedded into space generated by labeled star graph if and only if the original space has this property. After that, the standard isometric embedding of the modified space into its completion is used to construct the desired isometric embedding of the original space.

The last section of the paper presents several hypotheses that can be probably proven using Theorem 4.6.

1. Preliminaries. Metric spaces

Let us denote by $\mathbb{R}^+$ the set $[0, \infty)$.

A *metric* on a nonempty set X is a function $d\colon X \times X \to \mathbb{R}^+$ satisfying the following conditions for all x, y, $z \in X$:

(i) $d(x, y) = d(y, x)$;

(ii) $(d(x, y) = 0) \iff (x = y)$;

(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$.

If instead of the triangle inequality (iii) the *strong triangle inequality*

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$$

holds for all $x, y, z \in X$, then (X, d) is called an *ultrametric space.*

Definition 1.1. Let (X, d) and (Y, ρ) be metric spaces. A map $\Phi\colon X \to Y$ is called an *isometric embedding* of (X, d) into (Y, ρ) if

$$d(x, y) = \rho(\Phi(x), \Phi(y))$$

holds for all x, $y \in X$. In the case when Φ is bijective, we say that it is an *isometry* of (X, d) and (Y, ρ). The metric spaces are *isometric* if there is an isometry of these spaces.

Let S be a nonempty subset of a metric space (X, d). The quantity

$$\operatorname{diam} S \coloneqq \sup\{d(x, y) : x, y \in S\} \tag{1.1}$$

is called the *diameter* of S.

We define the *distance set* $D(X)$ of a metric space (X, d) as

$$D(X) \coloneqq \{d(x, y) : x, y \in X\},$$

and write

$$D_0(X) := D(X)\backslash\{0\}. \tag{1.2}$$

It is clear that $\operatorname{diam} X = \sup D(X)$. Furthermore, by using (1.1) and the strong triangle inequality, it is easy to show that the equality

$$\operatorname{diam} S = \sup\{d(x_0, y) : y \in S\}$$

holds for each nonempty $S \subseteq X$ and every $x_0 \in S$ if (X, d) is an ultrametric space.

The following simple proposition seems to be new.

Proposition 1.2. *Let (X, d) be an ultrametric space with $|X| \geqslant 2$. If x_1 and x_2 are two distinct points of X such that*

$$d(x_1, x_2) \leqslant d(y_1, y_2) \tag{1.3}$$

for all distinct points $y_1, y_2 \in X$, then the mapping $\Phi \colon X \to X$ defined, for every $x \in X$, as

$$\Phi(x) := \begin{cases} x_2, & \text{if } x = x_1, \\ x_1, & \text{if } x = x_2, \\ x, & \text{otherwise} \end{cases} \tag{1.4}$$

is a self-isometry of the space (X, d).

Proof. It follows from (1.4) that Φ is a self-isometry if and only if the equality

$$d(x, x_1) = d(x, x_2) \tag{1.5}$$

holds whenever

$$x_1 \neq x \neq x_2. \tag{1.6}$$

Suppose contrary that there exists $x \in X$ such that (1.6) holds but

$$d(x, x_1) \neq d(x, x_2).$$

Then, without loss of generality, we may assume

$$d(x, x_1) < d(x, x_2). \tag{1.7}$$

Using (1.3) with $y_1 = x$ and $y_2 = x_1$, we obtain the inequality

$$d(x_1, x_2) \leqslant d(x, x_1).$$

The latter inequality and (1.7) give us the following strict inequality

$$\max\{d(x_2, x_1), d(x_1, x)\} < d(x_2, x),$$

which contradicts the strong triangle inequality

$$d(x, x_2) \leqslant \max\{d(x, x_1), d(x_1, x_2)\}.$$

Equality (1.5) follows and thus Φ is a self-isometry of (X, d). $\square$

Corollary 1.3. *Suppose that an ultrametric space* (X, d) *contains two distinct points* x_1, x_2 *such that* (1.3) *holds for all distinct* $y_1, y_2 \in X$*, and let*

$$X_1 := X\backslash\{x_1\} \qquad\qquad X_2 := X\backslash\{x_2\}.$$

Then the ultrametric spaces $(X_1, d|_{X_1 \times X_1})$ *and* $(X_2, d|_{X_2 \times X_2})$ *are isometric.*

Proof. Let $\Phi\colon X \to X$ be the self-isometry of (X, d) defined by (1.4). Then it directly follows from (1.4) that $\Phi(X_1) = X_2$ holds. Hence the mapping

$$X_1 \ni x \mapsto \Phi(x) \in X_2$$

is an isometry of the spaces $(X_1, d|_{X_1 \times X_1})$ and $(X_2, d|_{X_2 \times X_2})$. □

Proposition 1.4. *Let* (X, d) *be a metric space with* $|X| \geqslant 2$*. Then the following statements are equivalent:*

(i) *There exist two distinct points* $x_1, x_2 \in X$ *such that*

$$d(x_1, x_2) \leqslant d(y_1, y_2)$$

for all distinct y_1, $y_2 \in X$.

(ii) *The set* $D_0(X)$ *contains the smallest element,*

$$\inf D_0(X) \in D_0(X).$$

Proof. The equivalence (i)⇔(ii) follows directly from (1.2). □

Let (X, d) be a metric space. An *open ball* of a *radius* $r > 0$ and a *center* $c \in X$ is the following set

$$B_r(c) := \{x \in X : d(c, x) < r\}.$$

The next lemma follows from [35, Proposition 18.5].

Lemma 1.5. *Let* (X, d) *be a finite ultrametric space and let* B_1, B_2 *be two distinct open balls in* (X, d).

- *If* $B_1 \cap B_2 \neq 0$*, then either* $B_1 \subset B_2$ *or* $B_2 \subset B_1$.
- *If* B_1 *and* B_2 *are disjoint, then*

$$d(x_1, x_2) = \operatorname{diam}(B_1 \cup B_2)$$

for all $x_1 \in B_1$ *and* $x_2 \in B_2$.

Definition 1.6. Let (X, d) be a metric space. If for every $x \in X$ there exists $r > 0$ such that

$$|B_r(x)| = 1,$$

then the metric space (X, d) is called *discrete*.

Following [6, p. 48], we will say that a metric space (X, d) is *metrically discrete* if there exists $t > 0$ such that

$$d(x, y) \geqslant t \tag{1.8}$$

for all distinct $x, y \in X$.

It is clear that every metrically discrete space is discrete, but not conversely in general.

The next proposition will be used in Section 5 below.

Proposition 1.7. *Let (X, d) be a discrete metric space. Then the following statements are equivalent:*

(i) *(X, d) is not metrically discrete.*

(ii) *The equality*

$$\inf D_0(X) = 0$$

holds, where $D_0(X)$ is defined by (1.2).

(iii) *There are a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of X and a strictly decreasing sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers such that*

$$\lim_{n\to\infty} r_n = 0$$

and

$$|B_{r_n}(x_n)| = 1, \qquad |B_{kr_n}(x_n)| \geqslant 1$$

for each $n \in \mathbb{N}$ and every $k \in (1, \infty)$.

Proof. It is easy to see that the inequality

$$\inf D_0(X) > 0$$

implies

$$d(x, y) \geqslant \inf D_0(X) > 0$$

for all distinct $x, y \in X$. Hence (1.8) holds with $t = \inf D_0(X)$. Conversely, if (1.8) holds for all distinct $x, y \in X$ and $t > 0$, then we have the double inequality

$$\inf D_0(X) \geqslant t > 0.$$

This proves the equivalence (i)⇔(ii).

Simple arguments show that (iii) implies (i).

For the proof of the implication (i)⇒(iii) we only note that for each $x \in X$, the inequality

$$\inf\{d(x, y) : y \in X\backslash\{x\}\} > 0$$

holds, and we have

$$|B_{kr}(x)| \geqslant 2$$

if $r = \inf\{d(x, y) : y \in X\backslash\{x\}\}$ and $k \in (1, \infty)$. □

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of points of a metric space (X,d) is said to *converge* to a point $a \in X$ if

$$\lim_{n\to\infty} d(x_n, a) = 0.$$

A point $x \in X$ is a *limit point* of a set $A \subseteq X$ if there is a sequence $(a_n)_{n\in\mathbb{N}}$ of mutually distinct points of A such that $(a_n)_{n\in\mathbb{N}}$ converges to the point x. A set A is said to be *dense* in (X,d) if each $x \in X\backslash A$ is a limit point of A.

A sequence $(x_n)_{n\in\mathbb{N}}$ of points of a metric space (X,d) is called a *Cauchy sequence* iff

$$\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0.$$

We will also use the following "ultrametric" form of the concept of Cauchy sequences (see, for example, [31, p. 4] or [1, Theorem 1.6]).

Proposition 1.8. *Let (X,d) be an ultrametric space. A sequence $(x_n)_{n\in\mathbb{N}}$ of points of X is a Cauchy sequence if and only if the following limit relation*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

holds.

A metric space (X,d) is *complete* if every Cauchy sequence of points of X converges to some point of X.

Definition 1.9. Let (X,d) be a metric space. A complete metric space (Y,ρ) is called a *completion* of (X,d) if (X,d) is isometric to a dense subspace of (Y,ρ).

The next proposition directly follows from Definition 2.3 and [36, Theorem 10.12.5].

Proposition 1.10. *Let (X,d) be a metric space, and let (Y^1,ρ^1), (Y^2,ρ^2) be completions of (X,d). Then (Y^1,ρ^1) and (Y^2,ρ^2) are isometric.*

The following definition gives a generalization of the concept of isometry.

Definition 1.11. Let (X,d) and (Y,δ) be metric spaces. A bijective mapping $\Phi\colon X \to Y$ is called a *weak similarity* if there is a strictly increasing bijection $f\colon D(Y) \to D(X)$ such that the equality

$$d(x,y) = f\big(\delta\big(\Phi(x), \Phi(y)\big)\big) \tag{1.9}$$

holds for all $x, y \in X$.

Remark 1.12. The notion of weak similarity was introduced in [18]. Papers [2, 8, 17, 32] contain some interconnections between weak similarities and other generalizations of the concept of isometry.

2. Preliminaries. Graphs

A *graph* is a pair (V, E), where V is a nonempty set and E is a set of unordered pairs $\{u, v\}$ of distinct elements $u, v \in V$. For a graph $G = (V, E)$, the sets $V = V(G)$ and $E = E(G)$ are called the *vertex set* and the *edge set*, respectively. If $\{x, y\} \in E(G)$, then the vertices x and y are called *adjacent*. A graph is called *finite* if $V(G)$ is finite.

Definition 2.1. Let G_1 and G_2 be graphs. A bijection $\Phi\colon V(G_1) \to V(G_2)$ is called an *isomorphism* of G_1 and G_2 if the equivalence

$$\{u, v\} \in E(G_1) \iff \{\Phi(u), \Phi(v)\} \in E(G_2)$$

holds for all $u, v \in V(G_1)$. The graphs are *isomorphic* if there is an isomorphism of these graphs.

Let G be a graph. A graph G_1 is a *subgraph* of G if

$$V(G_1) \subseteq V(G) \qquad \text{and} \qquad E(G_1) \subseteq E(G).$$

In this case we will write $G_1 \subseteq G$. If V_1 is nonempty subset of $V(G)$, $G_1 \subseteq G$, $V_1 = V(G_1)$ and $\{u, v\} \in E(G)$ implies $\{u, v\} \in E(G_1)$ for all $u, v \in V_1$, then we say that G_1 is an *induced subgraph* of G and that G_1 is induced by V_1.

A *path* is a finite graph P whose vertices can be numbered without repetitions so that

$$V(P) = \{x_1, \dots, x_k\} \quad \text{and} \quad E(P) = \{\{x_1, x_2\}, \dots, \{x_{k-1}, x_k\}\} \tag{2.1}$$

with $k \geqslant 2$. We will write $P = (x_1, \dots, x_k)$ or $P = P_{x_1,x_k}$ if P is a path satisfying (2.1) and said that P is a *path joining* x_1 *and* x_k.

A graph G is *connected* if for every two distinct vertices of G there is a path $P \subseteq G$ joining these vertices.

An infinite graph G of the form

$$V(G) = \{v_1, v_2, \dots, v_n, v_{n+1}, \dots\},$$
$$E(G) = \{\{v_1, v_2\}, \dots, \{v_n, v_{n+1}\}, \dots\},$$

where $v_i \neq v_j$ for $i \neq j$, is called a *ray*. We say that a graph is *rayless* if it contains no rays.

A finite graph C is a *cycle* if $|V(C)| \geqslant 3$ and there exists an enumeration of its vertices without repetition such that

$$\begin{aligned} V(C) &= \{x_1, \dots, x_n\}, \\ E(C) &= \{\{x_1, x_2\}, \dots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}. \end{aligned} \tag{2.2}$$

A cycle C satisfying (2.2) is called *n-cycle*. In what follows the n-cycles will be denoted by C_n.

Definition 2.2. Let G be a graph and let k be a cardinal number. The graph G is *k-partite* if the vertex set $V(G)$ can be partitioned into k non-void disjoint subsets, or parts, in such a way that no edge has both ends in the same part. A k-partite graph is *complete* if any two vertices in different parts are adjacent.

The complete k-partite finite graph with parts $V_1, \dots, V_k$ will be denoted by $K_{n_1,\dots,n_k}$ if we have $n_1 \leqslant n_2 \leqslant \dots \leqslant n_k$ and $|V_i| = n_i$, $\forall i \in \{1, \dots, k\}$.

Definition 2.3. Let G be a graph. A graph $\overline{G}$ is the *complement* of the graph G if the equality $V(\overline{G}) = V(G)$ holds and the equivalence

$$\{u, v\} \in E(\overline{G}) \Leftrightarrow \{u, v\} \notin E(G)$$

is valid for all distinct $u, v \in V(G)$.

The following proposition seems to be well known, but the authors cannot give an exact reference here.

Proposition 2.4. *Let $K_{n_1,\dots,n_k}$ and $K_{m_1,\dots,m_t}$ be complete multipartite finite graphs. Then the following statements are equivalent:*

(i) *The equality $k = t$ holds and, in addition, we have $n_i = m_i$ for every $i \in \{1, \dots, k\}$.*
(ii) *$K_{n_1,\dots,n_k}$ and $K_{m_1,\dots,m_t}$ are isomorphic.*

Proof. It follows directly from Definition 2.2 that $K_{n_1,\dots,n_k}$ and $K_{m_1,\dots,m_t}$ are isomorphic if statement (i) holds.

Suppose (ii) holds. Let $\overline{K}_{n_1,\dots,n_k}$ and $\overline{K}_{m_1,\dots,m_t}$ be the complements of $K_{n_1,\dots,n_k}$ and $K_{m_1,\dots,m_t}$ respectively. The Statement (ii) and Definitions 2.2 and 2.3 imply that $\overline{K}_{n_1,\dots,n_k}$ and $\overline{K}_{m_1,\dots,m_t}$ are isomorphic graphs.

Since $K_{n_1,\dots,n_k}$ and $K_{m_1,\dots,m_t}$ are complete multipartite graphs, the complements $\overline{K}_{n_1,\dots,n_k}$ and $\overline{K}_{m_1,\dots,m_t}$ are disjoint unions of complete graphs. Two complete graphs G_1 and G_2 are isomorphic if and only if they have the same number of vertices, $|V(G_1)| = |V(G_2)|$. Applying this statement to subgraphs of the graphs $\overline{K}_{n_1,\dots,n_k}$ and $\overline{K}_{m_1,\dots,m_t}$ induced by parts of $K_{n_1,\dots,n_k}$ and, respectively, by parts of $K_{m_1,\dots,m_t}$, one can easily prove the validity of statement (i). □

The following simple corollary of Proposition 2.4 describes the complete multipartite graph which are isomorphic to the cycle C_4.

Corollary 2.5. *Let $K = K_{n_1,\dots,n_k}$ be a complete multipartite finite graph. Then the following statements are equivalent:*

(i) *The equality $k = 2$ holds and, in addition, we have $n_1 = n_2 = 2$.*

(ii) *$K_{n_1,\dots,n_k}$ is isomorphic to the cycle C_4.*

Proof. The cycle C_4 admits an enumeration of its vertices such that

$$V(C_4) = \{x_1, x_2, x_3, x_4\}, \tag{2.3}$$

and

$$E(C_4) = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_1\}\}. \tag{2.4}$$

Using (2.3) and (2.4) it is easy to prove that C_4 is a complete bipartite graph with parts $\{x_1, x_3\}$ and $\{x_2, x_4\}$. Hence the implication (i)⇒(ii) is valid. The validity of (ii)⇒(i) follows from Proposition 2.4. □

As was shown in [7] the concepts of complete multipartite graphs and ultrametric spaces are closely related. In order to describe this relationship, we recall the definition of diametrical graphs.

The following definition is a modification of [34, Definition 2.1].

Definition 2.6. Let (X, d) be a metric space. Denote by $G_{X,d}$ a graph such that $V(G_{X,d}) = X$ and, for u, $v \in V(G_{X,d})$,

$$(\{u, v\} \in E(G_{X,d})) \iff (d(u, v) = \operatorname{diam} X \text{ and } u \neq v).$$

We call $G_{X,d}$ the *diametrical graph* of (X, d).

Example 2.7. The diametrical graphs of ultrametric spaces depicted in Figure 2.1 are the C_4 cycles depicted in Figure 2.2.

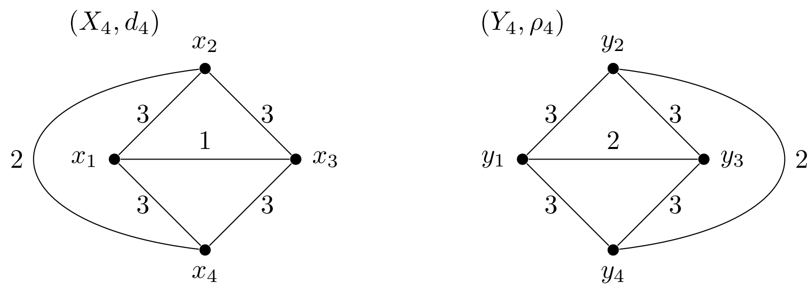


FIGURE 2.1. The four-point ultrametric spaces (X_4, d_4) and (Y_4, ρ_4).

The following theorem was proved in [7].

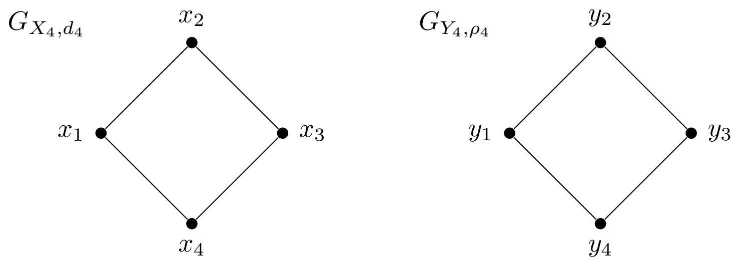


FIGURE 2.2. The diametrical graphs of (X_4, d_4) and (Y_4, ρ_4).

Theorem 2.8. *Let (X, ρ) be an ultrametric space with $2 \leqslant |X| < \infty$ and let $G_{X,\rho}$ be the diametrical graph of (X, ρ). Then $G_{X,\rho}$ is a complete k-partite finite graph with $k \geqslant 2$.*

Conversely, if G is a complete k-partite finite graph with $k \geqslant 2$, then there is an ultrametric $d\colon V(G) \times V(G) \to \mathbb{R}^+$ such that $G = G_{V,d}$.

Remark 2.9. Papers [3, 4] describe some other interconnections between complete multipartite graphs and ultrametric spaces.

Proposition 2.10. *Let (X, d) and (Y, ρ) be finite and weakly similar metric spaces with $|X| = |Y| \geqslant 2$. Then the diametrical graphs $G_{X,d}$ and $G_{Y,\rho}$ are isomorphic.*

Proof. Let $\Phi\colon X \to Y$ be a weak similarity between (X, d) and (Y, ρ), and let $x, y \in X$. Then, by Definition 1.11, the equality

$$d(x, y) = \operatorname{diam} X$$

holds if and only if

$$\rho\big(\Phi(x), \Phi(y)\big) = \operatorname{diam} Y.$$

Hence, the mapping $\Phi\colon X \to Y$ is an isomorphism of $G_{X,d}$ and $G_{Y,\rho}$ by Definition 2.1. $\square$

3. Preliminaries. Star graphs and **US**-spaces

A connected graph without cycles is called a *tree*.

A *labeled tree* $T(l)$ is a pair (T, l), where T is a tree and l is a function

$$l\colon V(T) \to \mathbb{R}^+.$$

Let $T = T(l)$ be a labeled tree. Following [9], we consider the mapping

$$d_l\colon V(T) \times V(T) \to \mathbb{R}^+,$$

$$d_l(u,v) := \begin{cases} 0, & \text{if } u = v, \\ \max\limits_{w \in V(P)} l(w), & \text{otherwise,} \end{cases} \tag{3.1}$$

where P denotes the unique path connecting u and v in T.

Theorem 3.1 ([9, Proposition 3.2])**.** *Let $T = T(l)$ be a labeled tree and d_l be defined by* (3.1)*. Then d_l is an ultrametric on $V(T)$ if and only if*

$$\max\{l(u), l(v)\} > 0 \tag{3.2}$$

holds for every edge $\{u,v\} \in E(T)$.

A labeling $l\colon V(T) \to \mathbb{R}^+$ is said to be *non-degenerate* if inequality (3.2) is satisfied for every edge $\{u,v\}$ of T.

The following theorem was proved in [16].

Theorem 3.2. *Let T be a tree. Then the following statements are equivalent:*

(i) *The ultrametric space $(V(T), d_l)$ is complete for every non-degenerate labeling $l\colon V(T) \to \mathbb{R}^+$.*
(ii) *T is rayless.*

Let us recall now the concept of star graphs.

Definition 3.3. A tree S is called a *star graph* if there exists a vertex $c \in V(S)$, referred to as the *center* of S, such that c is adjacent to every vertex of the set $V(S)\backslash\{c\}$, and no other edges exist, i.e.,

$$\{u,w\} \notin E(S) \quad \text{whenever} \quad u \neq c \neq w.$$

Now we can define the class **US** of ultrametric spaces as follows: An ultrametric space (X,d) belongs to **US** if there exists a labeled star graph $S(l)$ such that

$$X = V(S) \quad \text{and} \quad d = d_l,$$

where d_l is defined by (3.1) with $T = S$. In this case, we say that (X,d) is an **US**-space generated by $S(l)$.

The following two results were obtained in [23] and [13] respectively.

Theorem 3.4. *Let (X,d) be an ultrametric space. Then the following statements are equivalent:*

(i) $(X,d) \in \mathbf{US}$.
(ii) *There is $x_0 \in X$ such that the inequality*

$$d(x_0, x) \leqslant d(y, x) \tag{3.3}$$

holds whenever

$$x_0 \neq x \neq y. \tag{3.4}$$

Proposition 3.5. *Let (X, d) be an ultrametric space and let $x_0 \in X$. If, for $x, y \in X$, inequality* (3.3) *holds whenever we have* (3.4)*, then (X, d) is generated by labeled star graph $S(l)$ with the center x_0 and the labeling $l\colon X \to \mathbb{R}^+$ defined by*

$$l(x) := d(x, x_0)$$

for each $x \in X$.

The next two theorems were proved in [13].

Theorem 3.6. *Let (X, d) be an ultrametric space. Suppose that either X is finite, or X has a limit point. Then (X, d) is an* **US**-*space if and only if (X, d) contains no four-point subspace which is weakly similar to (X_4, d_4) or to (Y_4, ρ_4).*

Theorem 3.7. *Let (X,d) be an* **US**-*space. Then $(Y, d|_{Y\times Y})$ is also an* **US**-*space for every finite nonempty $Y \subseteq X$.*

The original purpose of this paper was to prove the following conjecture, formulated in [13].

Conjecture 3.8. *The following statements are equivalent for every infinite ultrametric space (X, d):*

(i) $(X, d) \notin$ **US**.
(ii) *(X, d) contains a four-point subspace which is weakly similar either to (X_4, d_4) or to (Y_4, ρ_4).*

4. Lemmas and Main result

Let us start from the following proposition.

Proposition 4.1. *Let (X, d) be an* **US**-*space. Then (X, d) is complete.*

Proof. Since $(X, d) \in$ **US** holds, there is a labeled star graph $S(l)$ such that $(X, d) = (V(S), d_l)$. As d_l is ultrametric, Theorem 3.1 shows that the labeling $l\colon V(S) \to \mathbb{R}^+$ is non-degenerate.

Definition 3.3 implies that S is rayless. Hence (X, d) is complete by Theorem 3.2. □

The next lemma rephrases Statement (ii) of Conjecture 3.8.

Lemma 4.2. *Let (X, ρ) be a four-point ultrametric space. Then the following statements are equivalent:*

(i) (X, ρ) *is weakly similar either to* (X_4, d_4) *or to* (Y_4, ρ_4).
(ii) *The diametrical graph* $G_{X,\rho}$ *is isomorphic to* C_4.

Proof. (i)⇒(ii). Suppose (X, ρ) is weakly similar either to (X_4, d_4) or to (Y_4, ρ_4). It was noted in Example 2.7 that the diametrical graphs G_{X_4,d_4} and G_{Y_4,ρ_4} are isomorphic to the cycle C_4. Consequently, the diametrical graphs $G_{X,\rho}$ and C_4 are also isomorphic by Proposition 2.10.

(ii)⇒(i). Suppose $G_{X,\rho}$ and C_4 are isomorphic. Then there is an enumeration of the points of X such that $G_{X,\rho}$ can be depicted by Figure 4.1

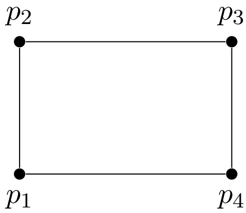


FIGURE 4.1. The diametrical graph $G_{X,\rho}$.

In this case, Definition 2.6 implies the strict inequalities

$$\rho(p_1, p_3) < \operatorname{diam} X \tag{4.1}$$

and

$$\rho(p_2, p_4) < \operatorname{diam} X. \tag{4.2}$$

If the equality

$$\rho(p_1, p_3) = \rho(p_2, p_4) \tag{4.3}$$

holds, then the mapping $\Phi_1 \colon X \to Y_4$ defined by $\Phi_1(p_i) = y_i$, $i = 1, 2, 3, 4$, is a weak similarity of the ultrametric spaces (X, ρ) and (Y_4, ρ_4).

Indeed, since $G_{X,\rho}$ and C_4 are isomorphic, inequalities (4.1), (4.2) and the definition of (Y_4, ρ_4) imply that

$$|D(X)| = |D(Y_4)| = 3,$$

i.e., the distance sets of (X, ρ) and (Y_4, ρ_4) are three-point subsets of $\mathbb{R}^+$. Hence there exists a unique strictly increasing bijective function

$$f_1 \colon D(Y_4) \to D(X).$$

Using Figures 2.1,2.2,4.1 we can see that Equality (1.9) holds for all $x, y \in X$ with $(X, d) = (X, \rho)$, $(Y, \delta) = (Y_4, \rho_4)$, and $f = f_1$.

Thus, by Definition 1.11, the mapping $\Phi_1 \colon X \to Y_4$ is a weak similarity of (X, ρ) and (Y_4, ρ_4).

If equality (4.3) is false, then we have either

$$\rho(p_1, p_3) < \rho(p_2, p_4), \tag{4.4}$$

or

$$\rho(p_2, p_4) < \rho(p_1, p_3). \tag{4.5}$$

Suppose that (4.4) holds. Then arguing as in the case of equality (4.3), we can show that the mapping $\Phi_2 \colon X \to X_4$, defined by $\Phi_2(p_i) = x_i$, $i = 1, 2, 3, 4$, is a weak similarity of (X, ρ) and (X_4, d_4).

To complete the proof we only note that the mapping $\Phi_3 \colon X \to X_4$ defined by equalities

$$\Phi_3(p_1) = x_2, \qquad \Phi_3(p_2) = x_1, \qquad \Phi_3(p_3) = x_4, \qquad \Phi_3(p_4) = x_3,$$

is the desired weak similarity of (X, ρ) and (X_4, d_4) for the case when inequality (4.5) holds. □

Lemma 4.3. *Let (X, d) be an infinite ultrametric space, x_0 and y_0 be two distinct fixed points of X. Suppose also that the following conditions hold:*

(i) *The inequality*

$$d(x_0, y_0) \leqslant d(p_1, p_2)$$

holds for all distinct $p_1, p_2 \in X$.

(ii) *(X, d) contains no four-point subspace whose diametrical graph is isomorphic to the cycle C_4.*

Then (X, d) is an **US***-space.*

Proof. Suppose that (i) and (ii) are satisfied. It is necessary to prove that

$$(X, d) \in \mathbf{US}. \tag{4.6}$$

Theorem 3.4 implies that (4.6) is valid if the inequality

$$d(x_0, v) \leqslant d(u, v) \tag{4.7}$$

holds for all distinct $u, v \in X$. If $u = x_0$, then (4.7) evidently holds for each $v \in X$. Therefore, it suffices to consider the case when $u \in X \backslash \{x_0\}$. Proposition 1.2 and condition (i) imply the equality

$$d(x_0, v) = d(y_0, v)$$

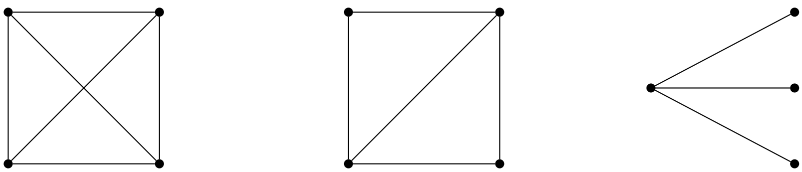
for every $v \in X \backslash \{x_0, y_0\}$. Consequently it suffices to prove (4.7) for the case when u and v are distinct points of $X \backslash \{x_0, y_0\}$.

Let x_0, y_0, u, and v be pairwise distinct, and let (A, d_A) be the ultrametric space with

$$A := \{x_0, y_0, u, v\}, \quad d_A := d|_{A \times A}.$$

In order to prove (4.7) we will describe the structure of the diametrical graph G_{A,d_A}. It is clear that G_{A,d_A} has four vertices. By Proposition 2.4, every four-vertex complete multipartite graph is isomorphic to exactly one of the graphs $K_{1,1,1,1}$, $K_{1,1,2}$, $K_{1,3}$ or $K_{2,2}$.

Theorem 2.8 implies that the diametrical graph G_{A,d_A} is a complete multipartite finite graph. By condition (ii), the diametrical graph G_{A,d_A} and the cycle C_4 are not isomorphic. The graph $K_{2,2}$ and the cycle C_4 are isomorphic by Corollary 3.1. Consequently, G_{A,d_A} is isomorphic to one of the graphs $K_{1,1,1,1}$, $K_{1,1,2}$ or $K_{1,3}$, see Figure 4.2.



$K_{1,1,1,1}$ $K_{1,1,2}$ $K_{1,3}$

FIGURE 4.2. G_{A,d_A} is isomorphic to one of the graphs $K_{1,1,1,1}$, $K_{1,1,2}$ or $K_{1,3}$.

Now we are ready to prove inequality (4.7).

If G_{A,d_A} and $K_{1,1,1,1}$ are isomorphic, then the distance between any two distinct points of (A, d_A) is equal to $\operatorname{diam} A$. Thus we have

$$d(x_0, v) = \operatorname{diam} A = d(u, v),$$

that implies (4.7).

If G_{A,d_A} is isomorphic to $K_{1,1,2}$, then there is the unique pair of distinct non-adjacent vertices of G_{A,d_A}. It follows from condition (i) that this pair coincides with $\{x_0, y_0\}$. Consequently the equality

$$d(u, v) = \operatorname{diam} A$$

holds, that again implies (4.7).

Suppose now that G_{A,d_A} is isomorphic to $K_{1,3}$. Then the ultrametric space (A, d_A) contains a unique point p such that the equality

$$d(p, a) = \operatorname{diam} A \tag{4.8}$$

holds for each $a \in A \backslash \{p\}$. Let us assume that $p \in \{x_0, y_0\}$. Then (i) implies the inequality

$$d(x_0, y_0) \leqslant d(a_1, a_2)$$

for all distinct $a_1, a_2 \in A$. Hence we get from (4.8) and Proposition 1.2 with $X = A$ and $\{x_1, x_2\} = \{x_0, y_0\}$ that

$$d(x_0, a) = \operatorname{diam} A$$

for each $a \in A \backslash \{x_0\}$, and

$$d(y_0, a) = \operatorname{diam} A$$

for each $a \in A\backslash\{y_0\}$. This contradicts to the uniqueness of the point $p \in A$ satisfying (4.8) for each $a \in A\backslash\{p\}$. Hence

$$p \in \{u, v\}.$$

Now using (4.8) with $p = u$, $a = v$ and with $p = v$, $a = u$, we obtain the equality

$$d(u, v) = \operatorname{diam} A,$$

which implies (4.7).

Thus, inequality (4.7) holds in all the cases. □

The following lemma can be considered as an extension of Theorem 3.6.

Lemma 4.4. *Let (X, d) be an infinite discrete ultrametric space satisfying the following conditions:*

(i) *The completion (Y, ρ) of (X, d) has a limit point c.*

(ii) *(X, d) contains no four-point subspace which is weakly similar to (X_4, d_4) or to (Y_4, ρ_4).*

Then (Y, ρ) is an **US***-space.*

Proof. Let $\Phi\colon X \to Y$ be an isometric embedding of (X, d) into (Y, ρ) such that $\Phi(X)$ is a dense subset of (Y, ρ). (This isometric embedding exists according to Definition 1.9). Let Y^* denote the union of the set $\Phi(X)$ and the singleton $\{c\}$:

$$Y^* = \Phi(X) \cup \{c\},$$

and ρ^* be the restriction of the ultrametric $\rho\colon Y \times Y \to \mathbb{R}^+$ on the set $Y^* \times Y^*$:

$$\rho^* = \rho|_{Y^* \times Y^*}.$$

It should be noted that $c \notin \Phi(X)$ since (X, d) is discrete and, consequently, $\Phi(X)$ is a discrete subset of (Y, ρ). Moreover, c is a limit point of the set $\Phi(X)$ because this set is dense in (Y, ρ) and c is a limit point of (Y, ρ).

We claim that (Y^*, ρ^*) *is an* **US***-space.*

Indeed, since (Y^*, ρ^*) contains a limit point, Theorem 3.6 implies that the ultrametric space (Y^*, ρ^*) is an **US**-space if (Y^*, ρ^*) has no four-point subspace which is weakly similar to (X_4, d_4) or (Y_4, ρ_4).

Let A be an four-point subset of the set Y^* and let $\rho^*_A = \rho^*|_{A \times A}$. Consider two cases.

a) If $c \notin A$, then condition (ii) implies that (A, ρ^*_A) is not weakly similar to (X_4, d_4) or to (Y_4, ρ_4).

b) Consider the case when $c \in A$.

Since c is a limit point of the set $\Phi(X)$ in the space (Y^*, ρ^*), there exists a point $p \in Y^*$ such that $p \notin A$ and

$$\rho^*(c,p) < \rho^*(a_1,a_2)$$

holds for all distinct $a_1, a_2 \in A$. Denote by A_p the set $A \cup \{p\}$ and consider the ultrametric space $(A_p, \rho^*_{A_p})$ in which the ultrametric $\rho^*_{A_p}$ is defined by

$$\rho^*_{A_p} = \rho^*|_{A_p \times A_p}.$$

Let $r \in (0, \infty)$ satisfy the double inequality

$$\rho^*(c,p) < r < \min\{\rho^*(a_1,a_2) : a_1 \neq a_2,\ a_1, a_2 \in A\}. \tag{4.9}$$

Then, using Lemma 1.5 and (4.9) we can prove that the inequality

$$\rho^*(c,p) \leqslant \rho^*(x_1,x_2) \tag{4.10}$$

holds for all distinct $x_1, x_2 \in A_p$. Denote by A_p^0 the set $(A\backslash\{c\}) \cup \{p\}$ and consider the ultrametric space $(A_p^0, \rho^0_{A_p})$ with

$$\rho^*_{A_p^0} = \rho^*|_{A_p^0 \times A_p^0}.$$

Since (4.10) holds for all different $x_1, x_2 \in A_p$, the conditions of Corollary 1.3 are satisfied for

$$(X,d) = (A_p, \rho^*_{A_p}), \qquad x_1 = p, \qquad x_2 = c.$$

Consequently the spaces (A, ρ^*_A) and $(A_p^0, \rho^*_{A_p^0})$ are isometric.

Thus, by condition (ii), the space (Y^*, ρ^*) contains no four-point subspaces which are weakly similar to (X_4, d_4) or to (Y_4, ρ_4). Since c is a limit point of the space (Y^*, ρ^*), it is an **US**-space by Theorem 3.6.

Now, Proposition 4.1 implies that (Y^*, ρ^*) is complete. Hence (Y^*, ρ^*) also is a completion of the ultrametric space (X, d). By Proposition 1.10 the spaces (Y^*, ρ^*) and (Y, ρ) are isometric. Thus (Y, ρ) is an **US**-space as required. □

Lemma 4.5. *Suppose that an infinite discrete ultrametric space (X,d) satisfies the following conditions:*

- (i) (X,d) *is not metrically discrete.*
- (ii) (X,d) *contains no four-point subspace whose diametrical graph is isomorphic to the cycle C_4.*

Then (X,d) contains a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of X.

Proof. Since (X, d) is not metrically discrete, Proposition 1.7 implies that there exist a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of X and a strictly decreasing sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$\lim_{n\to\infty} r_n = 0 \tag{4.11}$$

and

$$|B_{r_n}(x_n)| = 1, \quad |B_{kr_n}(x_n)| \geqslant 1 \tag{4.12}$$

for each $n \in \mathbb{N}$ and every $k \in (1, \infty)$.

We claim that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). By Proposition 1.8 the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if the following limit relation

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

holds. Note that it is satisfied if and only if

$$\limsup_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.13}$$

Suppose that (4.13) is false. Then there exist $c_1 > 0$ and a subsequence $(x_{n_m})_{m\in\mathbb{N}}$ of the sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$d(x_{n_m}, x_{n_m+1}) > c_1 \tag{4.14}$$

for every $m \in \mathbb{N}$. Using (4.11) and (4.14) we can find $m_0 \in \mathbb{N}$ such that

$$r_{n_{m_0}+1} < r_{n_{m_0}} < c_1. \tag{4.15}$$

Therefore, there exists $k \in (1, \infty)$ for which the double inequality

$$kr_{n_{m_0}+1} < kr_{n_{m_0}} < c_1 \tag{4.16}$$

is satisfied.

For the convenience, we introduce the following notation:

$$x^0 := x_{n_{m_0}}, \quad x^1 := x_{n_{m_0}+1}, \quad r^0 := kr_{n_{m_0}}, \quad r^1 := kr_{n_{m_0}+1}. \tag{4.17}$$

Consider the open balls $B_{r^0}(x^0)$ and $B_{r^1}(x^1)$. Then inequality (4.14) with $m = m_0$, double inequality (4.16) together with $x_m = x_{n_m} = x^0$ and Lemma 1.5 with $B_1 = B_{r^0}(x^0)$ and $B_2 = B_{r^1}(x^1)$ give us the equality

$$B_{r^0}(x^0) \cap B_{r^1}(x^1) = \varnothing. \tag{4.18}$$

The second inequality in (4.12) implies the existence of points $y^0 \in B_{r^0}(x^0)$ and $y^1 \in B_{r^1}(x^1)$ satisfying conditions

$$y_0 \neq x^0, \qquad y^1 \neq x^1. \tag{4.19}$$

Now using (4.18) and (4.19) we see that the point x^0, y^0, x^1 and y^1 are pairwise distinct. Let us denote by A the set $\{x^0, y^0, x^1, y^1\}$ and consider the ultrametric space (A, d_A) with

$$d_A := d_{A\times A}.$$

By Theorem 2.8 the diametrical graph G_{A,d_A} is a complete multipartite graph. We claim that *G_{A,d_A} and $K_{2,2}$ are isomorphic.* Indeed, note that the equalities

$$\operatorname{diam} A = d(x^0, x^1) = d(x^0, y^1) = d(y^0, x^1) = d(y^0, y^1) \tag{4.20}$$

hold by Lemma 1.5. Moreover, inequality (4.14) with $m = m_0$, double inequality (4.15), and equalities (4.17) give us the inequalities

$$d(y^0, x^0) < \operatorname{diam} A$$

and

$$d(y^1, x^1) < \operatorname{diam} A. \tag{4.21}$$

Now using (4.20) and (4.21), we see that G_{A,d_A} and $K_{2,2}$ are isomorphic.

Hence G_{A,d_A} and C_4 also are isomorphic by Corollary 2.5. The last statement contradicts condition (ii).

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d). □

The following theorem shows that Conjecture 3.8 is true.

Theorem 4.6. *Let (X, d) be an infinite ultrametric space. Then the following statements are equivalent:*

- (i) *There exists an ultrametric space $(Y, \rho) \in$ **US** such that (X, d) is isometric to a subspace of (Y, ρ).*
- (ii) *(X, d) contains no four-point subspace whose diametrical graph is isomorphic to the cycle C_4.*
- (iii) *(X, d) contains no four-point subspace which is weakly similar to (X_4, d_4) or to (Y_4, ρ_4).*

Proof. The equivalence (ii)⇔(iii) is valid by Lemma 4.1. Theorems 3.6 and 3.7 imply the validity of implication (i)⇒(ii). Therefore to prove the theorem under consideration we must show that statement (i) is true if the statements (ii) and (iii) hold.

Suppose that statements (ii) and (iii) are valid. Denote by Δ the infimum of the set

$$D_0(X) := \{d(x, y) : x, y \in X,\ x \neq y\}, \tag{4.22}$$

$$\Delta := \inf D_0(X). \tag{4.23}$$

The following three cases are posible:

(i_1) $\Delta = 0$,

(i_2) $\Delta > 0$ and $\Delta \in D_0(X)$,

(i_3) $\Delta > 0$ and $\Delta \notin D_0(X)$.

Below we will prove the implication ((ii) $\wedge$ (iii)) $\Rightarrow$ (i) in each of the cases (i_1)–(i_3) separately.

Suppose (i_1) holds. If (X, d) has a limit point, then (X, d) is an **US**-space by Theorem 3.6. Hence statement (i) holds with $(Y, \rho) = (X, d)$. If (X, d) contains no limit points, then, using Definition 1.6 and Proposition 1.7, we can prove that (X, d) is discrete but not metrically discrete. Consequently (X, d) contains a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points of X by Lemma 4.5.

Let (Y, ρ) be the completion of (X, d) and let $\Phi\colon X \to Y$ be an isometric embedding of (X, d) in (Y, ρ). Then the sequence $(\Phi(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y, ρ). Since (Y, ρ) is complete, the sequence $(\Phi(x_n))_{n\in\mathbb{N}}$ converges to a point $c \in Y$. Note that all points of the sequence $(\Phi(x_n))_{n\in\mathbb{N}}$ are pairwise distinct because the points of $(x_n)_{n\in\mathbb{N}}$ are pairwise distinct. Therefore, the point c is a limit point of Y. Now using Lemma 4.4 we obtain $(Y, \rho) \in$ **US**. Thus $\Phi : X \to Y$ is the desired isometric embedding of (X, d) into some **US**-space.

Assume that (i_2) holds. Then (X, d) contains two distinct points x_0 and y_0 such that

$$d(x_0, y_0) \leqslant d(y_1, y_2)$$

for all distinct y_1, $y_2 \in X$. Now, using Lemma 4.3, we see that (X, d) is an **US**-space. Thus statement (i) hold with $(Y, \rho) = (X, d)$.

Funally suppose that (i_3) holds. Then the mapping $d^0\colon X \times X \to \mathbb{R}^+$ defined by

$$d^0(x, y) = \begin{cases} d(x, y) - \Delta, & \text{if} \quad x \neq y, \\ 0, & \text{otherwise} \end{cases} \tag{4.24}$$

is an ultrametric on X.

Indeed, using (i_3) and (4.24) it is easy to see that

$$d^0(x, y) > 0 \quad \text{and} \quad d^0(x, y) = d^0(y, x)$$

hold for all distinct $x, y \in X$. The strong triangle inequality

$$d^0(x, y) \leqslant \max\{d^0(x, z), d^0(z, y)\} \tag{4.25}$$

directly follows from (4.24) if $x = y$.

Suppose x and y are distinct. Since d is an ultrametric on X, we have the strong triangle inequality

$$d(x, y) \leqslant \max\{d(x, z),\, d(z, y)\} \tag{4.26}$$

for all $x, y, z \in X$. The last inequality is equivalent to

$$d(x, y) - \Delta \leqslant \max\{d(x, z),\, d(z, y)\} - \Delta. \tag{4.27}$$

The relations

$$\begin{aligned}\max\{d(x,z),\, d(z,y)\} - \Delta &= \max\{d(x,z) - \Delta,\, d(z,y) - \Delta\}\\ &\leqslant \max\{d^0(x,z),\, d^0(z,y)\}\end{aligned}$$

and (4.26), (4.27) give us (4.25). Thus (X, d^0) is an ultrametric space as required.

Note that it directly follows from (4.22), (4.23), and (4.24) that

$$\inf\{d^0(x,y) : x, y \in X,\, x \neq y\} = 0. \tag{4.28}$$

Definition 1.6 and (i_3) imply that (X, d) is a discrete ultrametric space. Since (X, d) is a discrete ultrametric space, (i_3) and (4.24) imply that the ultrametric (X, d^0) also is discrete. Now using (4.28) and Proposition 1.7 we see that the discrete space (X, d^0) is not metrically discrete.

Let A be an arbitrary subset of X with $|A| = 4$, and let G_{A,d_A} and G_{A,d^0_A} be the diametrical graphs of the ultrametric spaces $(A, d|_{A\times A})$ and $(A, d^0|_{A\times A})$, respectively. Using (4.24) it is easy to see that the identity mapping

$$\mathrm{Id}_A \colon A \to A, \quad \mathrm{Id}_A(a) = a,$$

for each $a \in A$, is an isomorphism of the diametrical graphs G_{A,d_A} and G_{A,d^0_A}. Hence, by statement (ii), (X, d^0) contains no four-point subspace whose diametrical graph is isomorphic to the cycle C_4. The last statement, (iii) and Lemma 4.2 imply that (X, d^0) contains no four-point subspace which is weakly similar to (X_4, d_4) or to (Y_4, ρ_4). Now, arguing as in case (i_1), we can prove the existence of an **US**-space (Y, ρ^0) such that (X, d^0) is isometrically embeddable in (Y, ρ^0).

Let $\Phi^0 \colon X \to Y$ be an isometric embedding of (X, d^0) in (Y, ρ^0) and let the mapping $\rho \colon Y \times Y \to \mathbb{R}$ be defined by

$$\rho(x,y) := \begin{cases} \rho^0(x,y) + \Delta, & \text{if} \quad x \neq y,\\ 0, & \text{otherwise.}\end{cases} \tag{4.29}$$

As in the case of transition $d \to d^0$, it is easy to verify that $\rho \colon Y \times Y \to \mathbb{R}^+$ is an ultrametric on Y. Since (Y, ρ^0) is an **US**-space, Theorem 3.4 and (4.29) imply the membership relation $(Y, \rho) \in \mathbf{US}$. Moreover, the restriction of the ultrametric ρ on the set $\Phi^0(X) \times \Phi^0(X)$, where

$$\Phi^0(X) := \{\Phi^0(x) : x \in X\},$$

gives us an ultrametric space that is isometric to the space (X, d). Thus the mapping

$$(X, d) \xrightarrow{\mathrm{Id}_X} (X, d^0) \xrightarrow{\Phi^0} (Y, \rho^0) \xrightarrow{\mathrm{Id}_Y} (Y, \rho),$$

where Id_X and Id_Y are, respectively, the identical mappings on the sets X and Y, is the desired isometric embedding of (X, d) into **US**-space (Y, ρ).

Thus statement (i) holds in all cases. The proof is completed. □

Theorem 3.6, Theorem 3.7 and Theorem 4.6 give us the next result.

Corollary 4.7. *Let (X, d) be an infinite ultrametric space. Then (X, d) can be isometrically embedded in an* **US***-space if and only if every four-point subspace of (X, d) can be isometrically embedded in an* **US***-space.*

5. Conclusion. Expected results

Theorem 2.8 and Theorem 4.6 imply that an infinite ultrametric space (X, d) admits an isometric embedding in an **US**-space if and only if for each four-point $A \subseteq X$, the diametrical graph G_{A,d_A} is isomorphic to one of the graphs

$$K_{1,1,1,1}, \qquad K_{1,1,2}, \qquad K_{1,3}.$$

If G_{A,d_A} and $K_{1,1,1,1}$ are isomorphic for every four-point $A \subseteq X$, then it is easy to prove that (X, d) is equidistant, i.e. there is $t > 0$ such that

$$d(x, y) = t$$

for all distinct $x, y \in X$.

Thus we obtain the next simple proposition.

Proposition 5.1. *Let (X, d) be an infinite ultrametric space. Then the following statements are equivalent:*

(i) *(X, d) is equidistant.*

(ii) *The diametrical graph of every four-point subspace of (X, d) is isomorphic to $K_{1,1,1,1}$.*

Let us now turn to the graph $K_{1,1,2}$.

We hope that the following conjecture is true.

Conjecture 5.2. *Let (X, d) be an ultrametric space with $|X| \geqslant 4$. Then the following statements are equivalent:*

(i) *The diametrical graph of each four-point subspace of (X, d) is isomorphic to $K_{1,1,2}$.*

(ii) *Each four-point subspace of (X, d) is weakly isometric to (W_4, δ_4).*

(iii) *The space (X, d) is weakly isometric to (W_4, δ_4).*

Consider now two examples of **US**-spaces whose four-point subspaces have diametrical graph isomorphic to $K_{1,3}$.

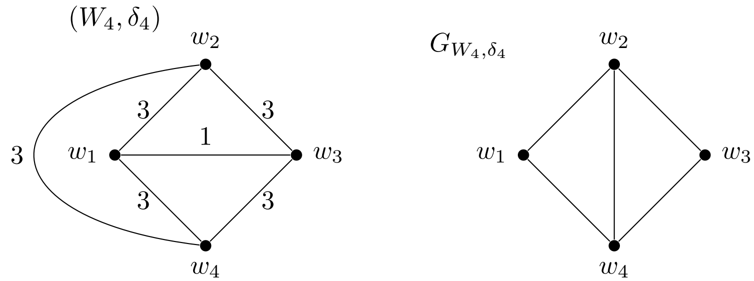


FIGURE 5.1. The four-point ultrametric spaces (W_4, δ_4) and its diametrical graph G_{W_4,δ_4}.

Example 5.3. Define an ultrametric $d^+ \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by the following formula:

$$d^+(p,q) = \begin{cases} 0, & \text{if} \quad p = q, \\ \max\{p,q\}, & \text{if} \quad p \neq q. \end{cases} \tag{5.1}$$

It was noted in [23] that $(\mathbb{R}^+, d^+) \in \mathbf{US}$. Moreover using (5.1) it is easy to see that the diametrical graphs of all four-point subspaces of $(\mathbb{R}^+, d^+)$ are isomorphic to $K_{1,3}$.

Example 5.4. See Figure 5.2 below.

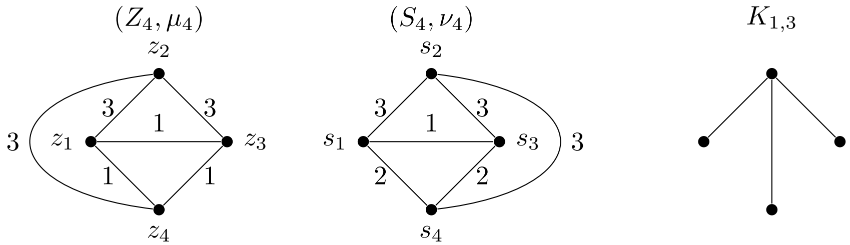


FIGURE 5.2. The diametrical graphs of the four-point ultrametric spaces (S_4, ν_4) and (Z_4, μ_4) are isomorphic to $K_{1,3}$.

The following hypothesis is very plausible.

Conjecture 5.5. *Let (X, d) be an ultrametric space with $|X| \geqslant 4$. Then the following statements are equivalent:*

(i) *The diametrical graph of each four-point subspace of (X, d) is isomorphic to $K_{1,3}$, but (X, d) contains no four-point subspaces which are weakly similar to (Z_4, μ_4).*

(ii) *All four-point subspace of* (X, d) *are weakly similar to* (S_4, ν_4).
(iii) *The space* (X, d) *admits an isometric embedding into* $(\mathbb{R}^+, d^+)$.

The ultrametric d^+ on $\mathbb{R}^+$ was introduced by Delhommé, Laflamme, Pouzet, and Sauer in [5]. Some results related to the ultrametric space $(\mathbb{R}^+, d^+)$ can be found in [10, 11, 15, 27–29].

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