Annals of Physics 485 (2026) 170326

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Normalizing Fock space states in static spacetimes

Jesse Huhtala *, Iiro Vilja

Department of Physics and Astronomy, University of Turku, 20014 Turku, Finland

ARTICLE INFO

Keywords:
Quantum field theory
Wave packet
Curved spacetime
Fock space

ABSTRACT

In quantum field theory, sharp momentum states have to be normalized to be in Fock space. We investigate two different normalization schemes: box normalization and wave packets. These methods are equivalent in flat spacetimes, but turn out to produce different results in curved spacetimes, specifically in those that break translation invariance. This means that scattering processes have to be defined in relation to the normalization scheme used, rather than being independent of it as is the case in flat spacetime. We prove this and provide an illustrative example.

1. Introduction

Curved spacetime quantum field theory is by now a mature field with multiple textbooks [1,2] and a research community of healthy size. Despite many decades of research, there are still foundational questions of interest relating to measurement [3] and the appropriate mathematical framework in which to place quantum field theory [4–7]. Concepts that are straightforward in flat spacetime are not always so in curved spacetimes, and it is one of these foundational concepts we wish to investigate in this article.

Our aim is to deal with the normalization of Fock space states in static spacetimes in the conventional QFT framework. As is well known, sharp momentum states, while heuristically convenient, are not actually in the Fock space. To obtain proper Fock space states, one should either use wave packets [8] or "put the system in a box" [9], typically with periodic boundary conditions. In flat spacetime, it does not matter which method one chooses; the end result is entirely equivalent after all the appropriate limits are taken. No wave packet or box dependence remains. We will show that this is not the case in curved spacetimes for scattering processes — rather, the end result depends on which method one chooses for normalization.

This matters because it has been shown that wave packet corrections may change the results of scattering calculations even in flat spacetimes. Ishikawa and collaborators showed that wave packet interference terms change decay rates in flat spacetimes for light particles [10–16]. In this work, we will show that there are always similar corrections in spacetimes which are not translationally invariant and in which momentum is thus not conserved.

Scattering calculations have been of recent interest in the literature [17–19], but many of these calculations have been done in translationally invariant spacetimes, where the issue investigated in this work does not appear — though expanding universes have other well-known issues of their own [2]. Since S-matrix elements are fundamental components of any quantum field theory, we believe the corrections in spatially inhomogeneous spacetimes are worth investigating.

It is well-known that one has to make various arbitrary choices when defining a quantum field theory on a curved spacetime, but much of the discussion has been about differing particle definitions and arbitrariness in the choice of modes. Fock space state normalization has received much less attention, each author normalizing their states in whichever way seems convenient. Thus, we wish to close this gap by providing a general and systematic investigation of the issues relating to Fock space state normalization.

We first examine the difference between the normalization methods in static spacetimes. We then provide an explicit example. Finally, we discuss implications and physical justifications for choosing one method over the other.

* Corresponding author.
E-mail addresses: jejohuh@utu.fi (J. Huhtala), vilja@utu.fi (I. Vilja).

https://doi.org/10.1016/j.aop.2025.170326
Received 23 September 2025; Accepted 15 December 2025
Available online 16 December 2025
0003-4916/

2. Packet and box normalizations

We will now derive the curved spacetime box and wave packet normalization formulae for a single massive scalar particle of mass M decaying to two scalars of some other – possibly zero – mass m in a static spacetime g in D+1 dimensions, $D \in \mathbb{N}$. We have chosen this particular process since it is the minimal example that illustrates our point, but of course similar reasoning applies to other processes in static spacetimes. We will perform the calculation at tree level, because the essential point does not change in higher orders and we wish to avoid dealing with renormalization. Since issues of normalizing the state are usually glossed over in flat spacetime quantum field theory, the reader may wish to consult Appendix A to see a simple 1+1 dimensional derivation of the usual decay rate formula in both the box and wave packet normalizations in full detail. Alternatively, those simple cases are found in less specific detail in Peskin&Schroder [8] for wave packets or Folland [9] for box normalization.

A few notes about decay rates are in order. In both the box and the wave packet normalization, we shall assume that the limit $T \to \infty$ makes sense. Unstable particles cannot really, of course, be created in the infinitely distant past. Moreover, the decay rate is usually defined in the reference frame where the momentum of the decaying particle is 0. Since we do not have a good definition of momentum in general, we cannot do this. The relation of $d\Gamma$ to flat spacetime decay rate is by analogy: we are really just dealing with an abstract S-matrix element. We will nevertheless use the term "decay rate" to maintain the analogy with flat spacetimes, and do not further concern ourselves with these issues, as they are equally present in both the box and wave packet normalizations. For our purposes, the limit $T \to \infty$ is formal.

The D+1 dimensional Klein–Gordon equation for metric g and mass m (correspondingly M) is given in the minimal coupling by

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\right] + m^2\right)\phi(x) = 0. \tag{1}$$

We could also use a more general equation, but this introduces nothing to our analysis. We will call the in-state field ϕ and the outgoing fields ψ with associated annihilation (creation) operators $b_k(b_k^\dagger)$, $a_k(a_k^\dagger)$. The mode functions – solutions of (1) – are denoted $f_k^m(\mathbf{x})e^{iE_k(t)}$ for the out-state and $f_k^M(\mathbf{x})e^{i\omega_k t}$ for the in-state. This decomposition can always be found in static spacetimes [2]. The interaction is given by

$$\mathcal{L}_{\text{int}} = -\lambda \int d^{D+1}x\sqrt{-g(x)}\phi(x)\psi(x)^2. \tag{2}$$

where $-g(x)$ is the determinant of the metric. We emphasize that in a general static spacetime, the index k of the mode functions is not a momentum — it is just a variable indexing the states of the field. The full interpretation as momentum is only available in spatially flat spacetimes, where it is a conserved quantity. We shall also assume that an appropriate set of mode functions solving the Klein–Gordon equation has been chosen; there is some arbitrariness in this choice, but it is not essential to our argument, since the general form of our results does not depend on the particular choices of mode functions. Our use of k, p, etc. for indexing is merely conventional [20]. Our convention for the commutator is $[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^D 2E_{\mathbf{k}}\delta^{(D)}(\mathbf{k}-\mathbf{k}')$.

2.1. Wave packet normalization

Since we do not know the distribution of out-states, we will use wave packets that simulate a detector of finite resolution. We do this by smoothing the states over a D-dimensional cube of side length ϵ. Hence our out-states are

$$|\Delta_k\rangle = \int_{\Delta_k} \widehat{d\mathbf{p}}\, a_{\mathbf{p}}^\dagger|0\rangle, \tag{3}$$

$$\widehat{d\mathbf{p}} = \frac{d^D\mathbf{p}}{\sqrt{(2\pi)^D 2E_{\mathbf{p}}V_\epsilon}}. \tag{4}$$

Here, Δ_k is the integral over a cube of volume V_ϵ centered on $\mathbf{k}$. This state is normalized:

$$\langle\Delta_k|\Delta_k\rangle = 1. \tag{5}$$

It also has the following useful Taylor expansion when $\epsilon \ll 1$:

$$\int_{\Delta_k} d\mathbf{p}h(\mathbf{p}) \approx V_\epsilon h(\mathbf{k}), \tag{6}$$

where h($\mathbf{k}$) is a function smooth almost everywhere.

For the in-state, we will use a more general wave packet such that

$$|g_{\mathbf{k}}^\sigma\rangle = \int \widetilde{d\mathbf{p}}\, g_{\mathbf{k}}^\sigma(\mathbf{p})a_{\mathbf{p}}^\dagger|0\rangle \tag{7}$$

$$\widetilde{d\mathbf{p}} = \frac{d^D\mathbf{p}}{\sqrt{(2\pi)^D 2E_{\mathbf{p}}}} \tag{8}$$

where we understand the integral to go over the whole k-spectrum. The central value of the wave packet is $\mathbf{k}$ and σ a parameter controlling its width, such as standard deviation. For normalization we require $\langle g^{\sigma}_{\mathbf{k}} | g^{\sigma}_{\mathbf{k}} \rangle = 1$, which leads to

$$\int d^D \mathbf{p} |g^{\sigma}_{\mathbf{k}}(\mathbf{p})|^2 = 1. \tag{9}$$

We demand that the wave packet is sharp. Practically, this means that

$$\int d\mathbf{p} g^{\sigma}_{\mathbf{k}}(\mathbf{p}) h(\mathbf{p}) \approx h(\mathbf{k}) \int d\mathbf{p} g^{\sigma}_{\mathbf{k}}(\mathbf{p}), \tag{10}$$

for a function h smooth almost everywhere. This is a standard assumption found in e.g. [8]. One wave packet satisfying the foregoing properties would be a Gaussian:

$$g^{\sigma}_{\mathbf{k}}(\mathbf{p}) = \frac{1}{(\pi\sigma^2)^{D/4}} \exp\left(\frac{(\mathbf{k}-\mathbf{p})^2}{2\sigma^2} \right). \tag{11}$$

With these preliminaries, we define the average decay rate as

$$\Delta\Gamma^{p_1}_{k_1 k_2} = \lim_{T\to\infty} \frac{1}{T} |\langle \text{out}, \Delta_{k_1} \Delta_{k_2} | g^{\sigma}_{\mathbf{p}}, \text{in} \rangle|^2 \tag{12}$$

where T is the total coordinate time of the interaction. We can obtain the tree-level result for the amplitude by using, for example, the LSZ theorem in curved spacetime [21]:

$$\begin{aligned} &\langle \text{out}, \Delta_{k_1} \Delta_{k_2} | g^{\sigma}_{\mathbf{p}}, \text{in} \rangle \\ &= \int_{\Delta_{k_1}} \widehat{d\mathbf{k}} \int_{\Delta k_1} \widehat{d\mathbf{k}'} \int \widetilde{d\mathbf{p}} \int d^{D+1}\mathbf{x}_1 d^{D+1}\mathbf{x}_2 d\mathbf{y} f^m_{\mathbf{k}}(\mathbf{x}_1) f^m_{\mathbf{k}'}(\mathbf{x}_2) f^M_{\mathbf{k}'}(\mathbf{y}) \mathcal{K}^m_{\mathbf{x}_1} \mathcal{K}^m_{\mathbf{x}_2} \mathcal{K}^M_{\mathbf{y}} \\ &\times \langle 0 | T\phi(\mathbf{y})\psi(\mathbf{x}_1)\psi(\mathbf{x}_2)|0\rangle \sqrt{-g(x_1)}\sqrt{-g(x_2)}\sqrt{-g(y)} \end{aligned} \tag{13}$$

with

$$\mathcal{K}^m_x = \frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g} g^{\mu\nu} \partial_\nu \right] + m^2, \tag{14}$$

$$\mathcal{K}^m_x G^m_F(x,y) = \frac{\delta(x-y)}{\sqrt{-g(x)}}, \tag{15}$$

$$G^m_F(x,y) = \langle 0 | T\psi_0(x)\psi_0(y)|0\rangle, \tag{16}$$

where in the last equation 0 indicates a free field and holds also for ϕ with the appropriate mass. The following analysis applies similarly to fermions and vectors, for which the formulae can be found in [22]. At tree level, we see after an application of Wick's theorem that

$$\begin{aligned} \Delta\Gamma^{p_1}_{k_1 k_2} &= \lim_{T\to\infty} \frac{\lambda^2}{T} \int_{\Delta_{k_1}} \widehat{d\mathbf{k}}\widehat{d\mathbf{k}'} \int_{\Delta_{k_2}} \widehat{d\mathbf{q}}\widehat{d\mathbf{q}'} \int \widetilde{d\mathbf{p}}\widetilde{d\mathbf{p}'} g^{\sigma}_{\mathbf{p_1}}(\mathbf{p}) g^{\sigma}_{\mathbf{p_1}}(\mathbf{p}') \int d^{D+1}\mathbf{x} d^{D+1}\mathbf{x}' \\ &\times \left[4\sqrt{-g(x)}\sqrt{-g(x')} f^m_{\mathbf{k}}(\mathbf{x}) f^m_{\mathbf{k}'}(\mathbf{x}) f^m_{\mathbf{q}}(\mathbf{x}) f^m_{\mathbf{q}'}(\mathbf{x}') f^M_{\mathbf{p}}(\mathbf{x}') f^M_{\mathbf{p}'}(\mathbf{x}') e^{i\Delta E t - i\Delta E' t'} \right]. \end{aligned} \tag{17}$$

with $\Delta E = E_k + E_{k'} - \omega_p$ and $\Delta E' = E_q + E_{q'} - \omega_{p'}$. We take $\Delta\Gamma^{p_1}_{k_1 k_2}$ to be the decay rate over the small $\mathbf{k}$ intervals; this will approach the differential decay rate as the intervals get smaller. We have chosen the spatial parts of the wave functions to be real with no loss of generality. We now use (6) and (10) to obtain

$$\begin{aligned} \Delta\Gamma^{p_1}_{k_1 k_2} =& \frac{4\lambda^2 V_{\epsilon_2} V_{\epsilon_1}}{(2\pi)^{3D} 8 E_{k_1} E_{k_2} \omega_p} \lim_{T\to\infty} \frac{1}{T} \left| \int_0^T dt e^{i(E_{k_1}+E_{k_2}-\omega_p)t} \right|^2 \\ &\times \left| \int d^D\mathbf{x} \sqrt{-g(x)} f^m_{\mathbf{k_1}}(\mathbf{x}) f^m_{\mathbf{k_2}}(\mathbf{x}) f^M_{\mathbf{p_1}}(\mathbf{x}) \right|^2 \int d\mathbf{p} d\mathbf{p}' g^{\sigma}_{\mathbf{p_1}}(\mathbf{p}) g^{\sigma}_{\mathbf{p_1}}(\mathbf{p}'). \end{aligned} \tag{18}$$

The limit $T \to \infty$ produces $2\pi\delta(E_{k_1} + E_{k_2} - \omega_{p_1})$, and thus we have

$$\begin{aligned} \Delta\Gamma^{p_1}_{k_1 k_2} =& \frac{4\lambda^2 V_{\epsilon_1} V_{\epsilon_2} \delta(E_{k_1} + E_{k_2} - \omega_{p_1})}{(2\pi)^{3D-1} 8 E_{k_1} E_{k_2} \omega_p} \left| \int d\mathbf{x} \sqrt{-g(x)} f^m_{\mathbf{k_1}}(\mathbf{x}) f^m_{\mathbf{k_2}}(\mathbf{x}) f^M_{\mathbf{p_1}}(\mathbf{x}) \right|^2 \\ &\times \int d\mathbf{p} d\mathbf{p}' g^{\sigma}_{\mathbf{p_1}}(\mathbf{p}) g^{\sigma}_{\mathbf{p_1}}(\mathbf{p}') \end{aligned} \tag{19}$$

As $V_{\epsilon_1}, V_{\epsilon_2}$ get smaller, we obtain the differential decay rate in terms of dk_1 and dk_2.

The last factor is a wave packet interference contribution which is not present in Minkowski spacetime. We are unable to rid ourselves of the wave packet dependence due to the lack of symmetries. In particular, it is the lack of translation invariance that leads to the extra factor. Quantum field theories, and thus Fock space states, are defined on time-slices with equal-time commutators; the definition of the states themselves loses symmetry when the translation invariance is not present. Physically, the lost symmetry is momentum conservation, because momentum conservation is generated by translation invariance.

We now turn to box normalization.

2.2. Box normalization

The idea of normalizing states with a box is to discretize the spectrum of the Klein–Gordon operator on the left hand side of (1). This is done by choosing some region of the manifold as the "box", limiting the theory to this box and then setting up boundary conditions that discretize the spectrum. We assume this is possible for static metrics of the kind we are interested in — though it is not entirely obvious that every operator of the type (1) can be discretized this way. In this section, we will understand the mode functions $f^m_{\mathbf{k}}(\mathbf{x})$ to satisfy the appropriate boundary conditions. The appropriate commutation relation is now $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = V(2\pi)^D 2E_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'}$ with V the coordinate volume of the box.

Thus a normalized state is

$$|k_{\text{box}}\rangle = V^{-1/2}((2\pi)^D 2E_k)^{-1/2} a^\dagger_k |0\rangle. \tag{20}$$

Then, with the integral measure for the states being $(2\pi)^{D/2} V d^D\mathbf{k}$ at the limit of large V, we have

$$\begin{aligned} d\Gamma^{p_1}_{k_1k_2} &= \lim_{T\to\infty} \frac{1}{T} \frac{4V^2 dk_1 dk_2}{(2\pi)^{2D} 8E_{k_1}E_{k_2}\omega_{p_1}V^3} \left| \int_0^T dt e^{i(E_{k_1}+E_{k_2}-\omega_{p_1})t} \right|^2 \\ &\times \left| \int_{\text{box}} d^D\mathbf{x}\sqrt{-g(x)} f^m_{\mathbf{k_1}}(\mathbf{x}) f^m_{\mathbf{k_2}}(\mathbf{x}) f^M_{\mathbf{p_1}}(\mathbf{x}) \right|^2 \\ &= \frac{dk_1 dk_2 \delta(E_{\mathbf{k_1}} + E_{\mathbf{k_2}} - \omega_{\mathbf{p_1}})}{(2\pi)^{2D-1} 2E_{k_1}E_{k_2}\omega_{p_1}V} \left| \int_{\text{box}} d^D\mathbf{x}\sqrt{-g(x)} f^m_{\mathbf{k_1}}(\mathbf{x}) f^m_{\mathbf{k_2}}(\mathbf{x}) f^M_{\mathbf{p_1}}(\mathbf{x}) \right|^2. \end{aligned} \tag{21}$$

In flat spacetimes, our choice of a box of constant V would make no difference due to the translation invariance of the theory. In curved spacetimes, though, different regions of the theory are obviously not equivalent. The limit $V \to \infty$ is trivially easy to take in flat spacetime (it is of the same kind as the limit $T \to \infty$), but hard or impossible to see in general static spacetimes.

In flat spacetimes, wave packet and box normalizations produce the same end result, since the details of both choices vanish at the appropriate limits [8,9]. Clearly, (19) and (21) have a different form. The integral of mode functions is calculated over a limited region of the spacetime, and each such region will in general have different solutions. The set of solutions for each region further depends on the imposed boundary conditions, adding a further ambiguity. The wave packet normalization, on the other hand, produces a wave-packet dependent interference factor, but the mode functions are defined in the entire spacetime. Both of these issues fundamentally result from the lack of momentum conservation in the theory.

To complete the argument that the box and wave packet normalizations really produce different results, we would need to show that the limits $V \to \infty$ (in the box) and $\sigma \to 0$ in the wave packets really produce different results. This is, of course, impossible in the general case, since we have a high degree of freedom in both the choice of box and the wave packets.

Momenta can be taken to be normal distributed in flat spacetime, so a Gaussian wave packet is reasonable. The limit $\sigma \to 0$ of the Gaussian also produces a delta function, which can be used to derive e.g. the decay rate (see Appendix A). In curved spacetimes where momentum is not conserved, neither of these requirements is reasonable: with no translation symmetry, there is no good definition of momentum, and no reason why it should be normal distributed around some value. It does not make sense to require the delta function limit, either, since that requirement also results from conservation laws.

Indeed, with a Gaussian wave packet, the limit $\sigma \to 0$ for the integral in (19) is zero, as will be seen in Section 3. An "infinitely sharp" limit such as we could reasonably use in flat spacetime therefore produces ill-defined results.

In box normalization, the end result of taking limit $V \to \infty$ is also sensitive to the global structure of the spacetime in the sense that to take the limit, the metric (and potential boundary conditions) have to be known in the entire spacetime. In the wave packet normalization, we define the mode functions in the entire spacetime, and the equivalent limit depends only on details of the wave packet, where we have considerable freedom. Either way, we are left with what seems to be an arbitrary choice, either about wave packets or the box. We will illustrate these points presently.

3. Example

In this section, we treat a simple model that illustrates the results of Section 2. We will use the 1+1 dimensional metric

$$g_{\mu\nu} = (A + B\tanh(x))\eta_{\mu\nu} \tag{22}$$

with $A, B \in \mathbb{R}, A, B > 0$ and $A + B\tanh(x) > 0\ \forall x \in \mathbb{R}$. This metric is asymptotically Minkowskian as $x \to \infty, x \to -\infty$.

3.1. Wave packet

First, we need to solve (1) for metric (22). This solution is already known when the metric depends on time instead of space, and exchanging time and space is trivial in 1+1 dimensional spacetime. Hence we only quote the result [2]:

$$\begin{aligned} &f^{m,M}_k(x)e^{i\omega_k t} \\ &\quad= \mathcal{N} e^{i\omega_k t - ig_+ x - ig_- \ln(2\cosh(x))} {}_2\mathrm{F}_1\left(1 + (ig_-), ig_-; 1 - ig_{\text{left}}; \frac{1}{2}(1 + \tanh(x))\right) \end{aligned} \tag{23}$$

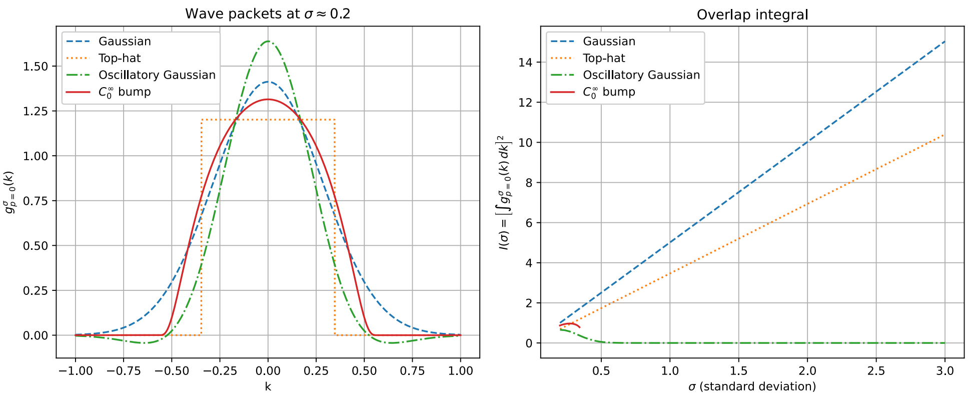

Fig. 1. Schematic comparison of different wave packets, normalized numerically and centered on $p = 0$. The Gaussian and top hat functions have their typical definitions; the oscillating Gaussian is $g^\sigma_{p=0}(k) \propto e^{-(k)^2/(2\sigma'^2)} \cos(\alpha k)$ with $\alpha = 3$ and σ' chosen to match the chosen standard deviation; the smooth function of compact support is of the form $g^\sigma_{p=0}(k) \propto (1+\beta k^2)\exp(-\frac{1}{1-k^2})$ in [−1,1] and 0 elsewhere. Note that the compactly supported function cannot have an arbitrarily large standard deviation.

with

$$g_{\text{left}} = (\omega_k^2 + m^2(B - A))^{1/2}, \tag{24}$$

$$g_{\text{right}} = (\omega_k^2 - m^2(A + B))^{1/2}, \tag{25}$$

$$g_\pm = \frac{1}{2}(g_{\text{left}} \pm g_{\text{right}}), \tag{26}$$

and correspondingly with the incoming mass M. $\mathcal{N}$ is chosen so as to fulfill the canonical commutation relations, and ${}_2\mathrm{F}_1$ is a hypergeometric function. The solution is chosen so that asymptotically as $x \to -\infty$, it reduces to the flat spacetime plane wave solution. There is another hypergeometric solution with similar parameters that reduces to the plane wave in $x \to \infty$, but we will not explicitly write it here. Note that the energy ω_k depends on k, the variable which indexes our states, in some complicated way that does not concern us here; in this case, the variable is continuous.

We are faced with the choice of wave packet. There is really not much reason to choose one over the other; if momentum were a good variable, we might think that a Gaussian, normal-distributed set of initial momenta is physically reasonable. Alas, the space has no translation invariance, and so this logic does not straightforwardly work.

We do, however, have an asymptotically flat spacetime, so we might guess that a Gaussian is a reasonable choice anyway, at least outside of the immediate vicinity of $x = 0$. In that case, we get from (19)

$$\begin{aligned}\Delta\Gamma^{p_1}_{k_1k_2} =& \frac{4\lambda^2 V_{\epsilon_1}V_{\epsilon_2}\delta(E_{k_1}+E_{k_2}-\omega_{p_1})}{(2\pi)^2 8E_{k_1}E_{k_2}\omega_p}\left|\int dx(A+B\tanh(x))f^m_{k_1}(x)f^m_{k_2}(x)f^M_{p_1}(x)\right|^2\\ &\times\int dp\,dp'\frac{1}{(\pi\sigma)^{1/2}}\exp\left(-\frac{(p-p_1)^2}{2\sigma}\right)\exp\left(\frac{-(p'-p_1)^2}{2\sigma}\right).\end{aligned} \tag{27}$$

We have not written out the hypergeometric functions explicitly, as there is no hope of analytically computing the integral. The overlap term is

$$\int dp\,dp'\frac{1}{(\pi\sigma)^{1/2}}\exp\left(-\frac{(p-p_1)^2}{2\sigma}\right)\exp\left(-\frac{(p'-p_1)^2}{2\sigma}\right) = \sqrt{\pi}\sigma. \tag{28}$$

Hence, we are left with an explicit wave packet dependence. In other words, we must now define our decay rate with respect to a particular choice of wave packet. We the shapes of various wavepackets in Fig. 1, along with their overlap integrals as a function of standard deviation.

3.2. Box normalization

We note immediately that the choice of box makes a crucial difference. Suppose we take the box $[x_0 - \frac{1}{2}L, x_0 + \frac{1}{2}L]$ with $x_0 \in \mathbb{R}, x_0 \gg 1,\ L \in \mathbb{R}, L > 0$. Then

$$g_{\mu\nu}(x) = (A + B\tanh(x))\eta_{\mu\nu} \approx (A + B)\eta_{\mu\nu}, \tag{29}$$

inside the box. Thus, in this box, we have in (21) with periodic boundary conditions

$$\frac{1}{V}\left|\int_{\text{box}} dx\sqrt{-g(x)}f^m_{k_1}(x)f^m_{k_2}(x)f^M_{p_1}(x)\right|^2 = \frac{1}{V}\left|\int_{\text{box}} dx(A+B)e^{i(k_1+k_2-ip_1)x}\right|^2 \tag{30}$$

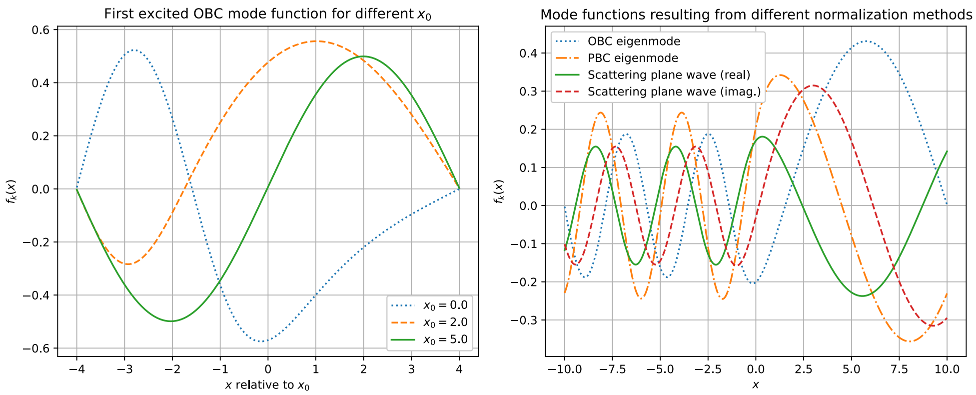

Fig. 2. In this figure, "OBC" means open boundary conditions (zero at the edges of the box), "PBC" means periodic boundary conditions. In the figure on the left, we have plotted the first excited-state mode functions for boxes centered on different x_0. On the right, we compare box solutions of different boundary conditions to a mode function that becomes a plane wave as $x \to -\infty$, i.e. an asymptotic scattering state. We can see that each of these choices results in vastly different solutions and accordingly different decay rates.

with k_i, p_i discretized with e.g. $k_1 = n\pi/L$. This is just the flat spacetime result with a scaling factor $A + B$. On the other hand, suppose that $x_0 \approx 0$. Then a reasonable approximation is

$$g_{\mu\nu}(x) \approx (A + Bx)\eta_{\mu\nu} \tag{31}$$

and the solution to (1) (with open boundary conditions where $f_i^m(-L/2) = f_i^m(L/2) = 0$) is

$$f_i^m(x) = \mathcal{N}\Big[\mathrm{Ai}(y(x,\omega_i))\mathrm{Bi}(y(-L,\omega_i)) - \mathrm{Bi}(y(,\omega_i))\mathrm{Ai}(y(-L,\omega_i))\Big], \tag{32}$$

$$y(x,\omega) = \frac{Am^2 - \omega_i^2 + Bm^2 x}{(bm^2)^{2/3}}. \tag{33}$$

Here $\mathcal{N}$ is a normalization factor chosen so that the commutation relations of our QFT are satisfied. Such a $\mathcal{N}$ can always be found since Eq. (1) with metric (29) is a regular Sturm–Liouville problem, and therefore the solutions form an orthogonal set. There would then be a corresponding solution for the mass M. Note that ω_i – the set of allowed energies – is now discrete and has to be chosen so as to satisfy the boundary conditions.

Thus, different choices of box and associated boundary conditions produce different results. This would not be a problem if we could just take the solution for the whole space, restrict it to a box and ultimately take the limit once the integrals have been computed. This procedure works in flat spacetime, but is unfortunately far from trivial in general static spacetime. How would one go about taking the limit $V \to \infty$ for complicated integrals over hypergeometric functions, for instance? In practical terms, the best one can do is to choose a large box and compute the integrals in it, but the choices of box then define differing decay rates.

In Fig. 2, we show how different the mode solutions of Eq. (1) can be with choices of different boundary conditions and box locations.

3.3. Preparation devices

One possible method to avoid the problems described in this article is to introduce a *preparation device* for particles, the states of which are themselves discrete and thus well-normalized. In this section, we will briefly sketch an example, but will not enter in to details.

To be concrete, let us take a monopole system with two states: ground state $|g\rangle$ and excited state $|e\rangle$. The monopole moment operator is

$$\hat{\mu}(t) = e^{i\Delta t} b^\dagger + e^{-i\Delta t} b \tag{34}$$

with $b, b^\dagger$ the monopole annihilation and creation operators and Δ the energy gap. Working in 1+1 dimensions, we couple the monopole to the incoming field linearly:

$$\hat{U}_{\text{prep}} = T\exp\left(-i\int dt' H(t')\right) \tag{35}$$

$$\hat{H}_{\text{prep}} = \epsilon\chi(t)\hat{\mu}(t)\int dx g(x)\phi(x,t), \tag{36}$$

where ϵ is the coupling strength, $g(x)$ is a spatial smearing function, $\chi(t)$ a time smearing function; we could reasonably choose $\chi(t)=1$, but let us keep the discussion general. We can write the first order expansion

$$\hat{U}_{\text{prep}} \approx 1 - i\epsilon \int dt dx \chi(t) g(x) \phi(x,t). \tag{37}$$

We now make the following adjustment: rather than starting from a wave packet state $|g_p^\sigma, \text{in}\rangle$, we instead start from the excited state of the detector. Further, supposing $\hat{U}_{\text{int}}$ is the particle interaction, we assume that the preparation and the particle interaction are separate processes, such that we can write the total interactions in the system as $\hat{U}_{\text{total}} \approx \hat{U}_{\text{int}}\hat{U}_{\text{prep}}$. The amplitude we are interested in is then

$$\langle \Delta_{k_1}\Delta_{k_2}|\hat{U}_{\text{int}}\hat{U}_{\text{prep}}|e\rangle \approx -i\epsilon \langle \Delta_{k_1}\Delta_{k_2}|\hat{U}_{\text{int}}\left[\int dt dx \chi(t) g(x)\phi(x,t)\right]|e\rangle \tag{38}$$

$$= -i\epsilon \int \widetilde{dp} \langle \Delta_{k_1}\Delta_{k_2}|\hat{U}_{\text{int}}|p\rangle\langle p|\left[\int dt dx \chi(t) g(x)\phi(x,t)\right]|e\rangle \tag{39}$$

$$= -i\epsilon \int \widetilde{dp} dt \chi(t) \langle \Delta_{k_1}\Delta_{k_2}|\hat{U}_{\text{int}}|p\rangle \int dx g(x) f_p^M(x) e^{i\omega_p t - i\Delta t}. \tag{40}$$

In (39), we used the resolution of the identity on the Fock space. The smearing function $g(x)$ – which might be determined by the geometry of the preparation device – now provides the wave packet. Note that if the mode function f_p^M were a plane wave, the integral over x would just be a Fourier transform, and we would directly get a momentum space wave packet $\tilde{g}(p)$. Even without this, though, we have in principle constructed a wave packet from the preparation device.

This is, of course, a different theory than directly using a particle in-state. However, if a reasonable model for the particle production may be constructed, it does provide a robust physical justification for the choice of wave packet.

4. Discussion

We have defined the decay rate of an unstable particle in two different ways found in the flat spacetime literature and found that these methods are not straightforwardly equivalent in curved spacetimes. We have also demonstrated this effect with a concrete example.

We emphasize that these results are really about the formal properties of scattering matrix elements in a curved spacetime QFT. Interpreting quantities like our $\Gamma_{k_1 k_2}^{p_1}$ as decay rates is tricky in curved spacetimes, because we cannot be sure that the limit $T \to \infty$ makes sense in an arbitrary spacetime. The states are moreover not labeled by momenta. All of these issues are caused by the lack of symmetry inherent in curved space QFTs, and each of them has to be investigated on a case-by-case basis once a spacetime has been chosen. For instance, the metric (29) provides at least asymptotic ($x \ll 1, x \gg 1$) definitions of momentum. Note that even though we did take the limit $T \to \infty$, this is not necessary for the ambiguity to appear, as the ambiguity we have discussed relates to the definition of Fock space states; if T is finite, not even energy would be conserved, but that would not remove the problems relating to momentum states.

There is one more notable way of normalizing the Fock space states, and that is discretizing the spacetime, i.e. putting it on a lattice. However, discretizing curved spacetimes inevitably ends up losing geometric information (see [23] and references therein); curvature cannot be perfectly represented on a grid, and thus the limit of zero lattice spacing does not necessarily recover the correct continuous manifold. We chose to limit ourselves to theories that can recover the correct metric, but these discrete theories are certainly interesting in their own right, and have their own ambiguities.

We computed the decay rate to the first order tree-level only. Of course, the same analysis would apply for higher orders with loops, except the procedure would now be technically complicated. It seems inevitable that in higher orders e.g. trying to take the limit $V \to \infty$ would be even more challenging than at tree-level, but this would not fundamentally change the analysis.

Another question we did not address is the validity of the sharp wave packet assumption, Eq. (10). In principle, we are free to choose any wave packet that satisfies the normalization conditions and possible boundary conditions, and thus one might imagine we can typically choose a wave packet that satisfies (10). However, in flat spacetime the wave packet would determine the initial distribution of momenta, so not every wave packet is equally as physically reasonable. A normal-distributed set of states seems plausible, but a Gaussian might not be a valid choice in every spacetime due to boundary conditions, nor is it a priori an appropriate initial distribution even in the ones where it is technically allowed. If we cannot assume (10), then the result is instead

$$\Gamma_{k_1 k_2}^{p_1} = \lim_{T\to\infty} \frac{4V_\epsilon^{k_1} V_\epsilon^{k_2} \lambda^2}{T} \int \widetilde{d\mathbf{p}}\widetilde{d\mathbf{p}'} g_{\mathbf{p_1}}^\sigma(\mathbf{p}) g_{\mathbf{p_1}}^\sigma(\mathbf{p}') \int d^{D+1}\mathbf{x} d^{D+1}\mathbf{x}' \sqrt{-g(x)}\sqrt{-g(x')}$$
$$\times \left[f_{\mathbf{k}}^m(\mathbf{x}) f_{\mathbf{k}'}^m(\mathbf{x}) f_{\mathbf{q}}^m(\mathbf{x}) f_{\mathbf{q}'}^m(\mathbf{x}') f_{\mathbf{p}}^M(\mathbf{x}') f_{\mathbf{p}'}^M(\mathbf{x}') e^{i\Delta E t - i\Delta E' t'} \right], \tag{41}$$

which is considerably more complicated. Note that even the limit $T \to \infty$ is now not easy to take, since the exponential time parts have different energy arguments.

We also note that when defining quantum fields formally as operator-valued distributions, one uses classes of test functions; in the general curved spacetime case, these must be functions of compact support [24,25], thus ruling out e.g. Gaussian packets. However, in the case of a continuous momentum spectrum and asymptotic scattering states, it is possible to use the more general Schwartz functions – which include Gaussians – for normalizing the Fock space states. Functions of compact support are only strictly required for defining local observables, and not asymptotic scattering states.

We wish to emphasize that while arbitrary choices always have to be made in curved spacetime QFTs, in the case of Fock space normalization, we really seem to have no unique method to choose one wave packet over the other or even to choose packets over box normalization. Even in the case of an asymptotically flat spacetime we have seen that the normalizations produce different results, even though we have an asymptotic definition of momentum.

In what cases might there be a physically well-motivated choice for normalization? First of all, it is possible that the box normalization for the fields produces spectra incompatible with energy conservation. Because the spectra of both fields of masses m and M are discrete, it is possible that there are no pairs of energies $\{E_i, E_j\}$ in the spectrum of the field of mass m such that $E_i + E_j = \omega_k$ where ω_k is in the spectrum of the field of mass M. In that case, we can rule out box normalization. On the other hand, if the equation of motion is only solvable in a compact region of the spacetime, then box normalization may be the natural choice.

A more intriguing possibility is knowledge of the *mechanism for particle production*. In recent literature, discrete quantum systems – especially Unruh–deWitt detectors – have been used to define a measurement scheme for quantum field theory [26]. As we showed in Section 3, Unruh–deWitt systems can equally well be used to emit particles, and then the coupling of the Unruh–deWitt monopole to the field determines the wave packet. If the preparation method for the particles is known, it provides a physically well-justified construction for a wave packet. It would also be conceptually harmonious with a detector-based measurement theory.

These results have implications for any scattering calculation done in spacetimes in which momentum is not conserved and the spectrum is uncountably infinite (labeled by a continuous variable). Clearly, the foregoing reasoning applies just as well to e.g. the scattering of two particles or other multi-particle processes. The result is, then, of general formal importance: if we wish to be careful in defining our Fock states, we must inevitably make a choice of how to normalize them, and these choices are not straightforwardly equivalent. Hence, scattering elements are only defined with respect to a particular normalization scheme. The work of Ishikawa et al. [10,11,15] and Edery [16] shows that wave packets cause extra terms to appear even in flat spacetimes under certain conditions; we have showed that extra factors of a similar type in fact appear always when momentum is not conserved. We have also provided a general form for the scattering elements of the differing normalization schemes and given a general recipe for investigating such differences.

CRediT authorship contribution statement

Jesse Huhtala: Writing – original draft, Investigation, Conceptualization. **Iiro Vilja:** Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare no competing interests.

Appendix A. Flat spacetime formulae

We present here 1+1 dimensional derivations for flat spacetime decay rate formulae. This is for the reader's convenience, as all the details are usually not spelled out in textbooks, since in flat spacetimes it does not matter which way you normalize your states.

A.1. Wave packets

Using the conventions of Section 2 explicitly with a Gaussian, we get

$$\Gamma^{p_1}_{k_1k_2} = \lim_{T\to\infty}\frac{\lambda^2}{T}\int_{\Delta_{k_1}}\widehat{dk}\widehat{dk'}\int_{\Delta_{k_2}}\widehat{dq}\widehat{dq'}\int\widetilde{dp}\widetilde{dp'}g^{\sigma}_{p_1}(p)g^{\sigma}_{p_1}(p')$$
$$\times\left[4\int d^2xd^2x'e^{ikx-iE_kt}e^{ik'x-iE_{k'}t}e^{iqx-iE_qt}e^{iq'x-iE_{q'}t}e^{-ipx+i\omega_pt}e^{-ip'x+i\omega_{p'}t}\right] \tag{42}$$
$$\approx\frac{4V^{k_1}_{\epsilon}V^{k_2}_{\epsilon}}{(2\pi)^2 4E_{k_1}E_{k_2}}\lim_{T\to\infty}\frac{\lambda^2}{T}\int\widetilde{dp}\widetilde{dp'}g^{\sigma}_{p_1}(p)g^{\sigma}_{p_1}(p')(2\pi)^2\delta(k_1+k_2-p)\delta(k_1+k_2-p')$$
$$\times\left[\int_0^T dtdt'e^{i(E_{k_1}+E_{k_2}-\omega_p)t-i(E_{k_1}+E_{k_2}-\omega_{p'})t}\right]. \tag{43}$$

In the second line we used (6) and integrated over the spatial coordinate. Assuming now (10) we find

$$\Gamma^{p_1}_{k_1k_2} = \frac{4V^{k_1}_{\epsilon}V^{k_2}_{\epsilon}}{(2\pi)8E_{k_1}E_{k_2}\omega_{p_1}}\lim_{T\to\infty}\frac{\lambda^2}{T}\int dpdp'g^{\sigma}_{p_1}(p)g^{\sigma}_{p_1}(p')\delta(k_1+k_2-p)\delta(k_1+k_2-p') \tag{44}$$
$$\times\left|\int_0^T dte^{i(E_{k_1}+E_{k_2}-\omega_{p_1})t}\right|^2. \tag{45}$$

Performing the integrals over p, p' and taking the limit, we find

$$\Gamma^{p_1}_{k_1k_2} = \frac{4V^{k_1}_{\epsilon}V^{k_2}_{\epsilon}\lambda^2}{8E_{k_1}E_{k_2}\omega_{p_1}}|g^{\sigma}_{p_1}(k_1+k_2)|^2\delta(E_{k_1}+E_{k_2}-\omega_{p_1}). \tag{46}$$

As we approach the limit of sharp in and out states with $\epsilon \to 0$ and $\sigma \to 0$, we find

$$\Gamma^{p_1}_{k_1k_2} = \frac{dk_1 dk_2 \lambda^2}{2E_{k_1}E_{k_2}\omega_{p_1}}\delta^{(2)}(p_1 - k_1 - k_2) \tag{47}$$

where we have used the fact that the limit of a Gaussian as its width approaches 0 is a Dirac delta function. This is the standard result to tree level in the interaction we used.

A.2. Box normalization

We use the same conventions as in Section 2.2. Specializing to 1+1, we have

$$d\Gamma^{p_1}_{k_1k_2} = \frac{dk_1 dk_2 \delta(E_{\mathbf{k_1}} + E_{\mathbf{k_2}} - \omega_{\mathbf{p_1}})}{(2\pi)2E_{k_1}E_{k_2}\omega_{p_1}V}\left|\int_0^V d\mathbf{x} f^m_{\mathbf{k_1}}(\mathbf{x}) f^m_{\mathbf{k_2}}(\mathbf{x}) f^M_{\mathbf{p_1}}(\mathbf{x})\right|^2 \tag{48}$$

We have chosen the box to go from 0 to V; this choice is arbitrary as the mode functions are translation invariant. Since we are now in flat spacetime, we know the spatial part of these functions explicitly; they are of course

$$f^{m,M}_k(x) = \exp(ikx) \quad \text{with} \quad k = \frac{n\pi}{L}, \quad n \in \mathbb{Z}. \tag{49}$$

The limit of sharp momentum states is equivalent to taking $V \to \infty$. Hence, we compute:

$$\lim_{V\to\infty}\frac{1}{V}\left|\int_0^V d\mathbf{x} f^m_{\mathbf{k_1}}(\mathbf{x}) f^m_{\mathbf{k_2}}(\mathbf{x}) f^M_{\mathbf{p_1}}(\mathbf{x})\right|^2 = 2\pi\delta(k_1 + k_2 - p_1) \tag{50}$$

Thus we end up with the result

$$\Gamma^{p_1}_{k_1k_2} = \frac{dk_1 dk_2 \lambda^2}{2E_{k_1}E_{k_2}\omega_{p_1}}\delta^{(2)}(p_1 - k_1 - k_2), \tag{51}$$

which is the also the wave packet result, as expected.

Data availability

No data was used for the research described in the article.

References

[1] V. Mukhanov, S. Winitzki, Introduction to Quantum Effects in Gravity, Cambridge University Press, 2007, http://dx.doi.org/10.1017/CBO9780511809149.
[2] N.D. Birrell, P.C.W. Davies, Quantum Fields in Curved Space, Cambridge University Press, 1982, http://dx.doi.org/10.1017/CBO9780511622632.
[3] C.J. Fewster, R. Verch, Quantum fields and local measurements, Comm. Math. Phys. 378 (2020) 851–889, http://dx.doi.org/10.1007/s00220-020-03800-6.
[4] C.J. Fewster, E. Hawkins, C. Minz, K. Rejzner, Local structure of sprinkled causal sets, Phys. Rev. D 103 (2021) http://dx.doi.org/10.1103/PhysRevD.103.086020.
[5] S. Hollands, R.M. Wald, Local wick polynomials and time ordered products of quantum fields in curved spacetime, Comm. Math. Phys. 223 (2001) 289–326.
[6] S. Hollands, R.M. Wald, Existence of local covariant time ordered products of quantum fields in curved spacetime, Comm. Math. Phys. 231 (2002) 309–345.
[7] S. Hollands, R.M. Wald, Axiomatic quantum field theory in curved spacetime, Comm. Math. Phys. 293 (2010) 85–125, http://dx.doi.org/10.1007/s00220-009-0880-7, http://arxiv.org/abs/0803.2003.
[8] E. Peskin, D. Schroeder, An Introduction to Quantum Field Theory, CRC Press, 1995, http://dx.doi.org/10.1201/9780429503559.
[9] G. Folland, Quantum Field Theory: A Tourist Guide for Mathematicians, American Mathematical Society, 2008.
[10] K. Ishikawa, T. Shimomura, Generalized S-matrix in mixed representation, Progr. Theoret. Phys. 114 (6) (2005) 1201–1234, http://dx.doi.org/10.1143/PTP.114.1201, URL https://academic.oup.com/ptp/article-lookup/doi/10.1143/PTP.114.1201.
[11] K. Ishikawa, Y. Tobita, Finite-size corrections to Fermi's golden rule: I. Decay rates, Prog. Theor. Exp. Phys. 2013 (7) (2013) 73B02–0, http://dx.doi.org/10.1093/ptep/ptt049, URL https://academic.oup.com/ptep/article-lookup/doi/10.1093/ptep/ptt049.
[12] K. Ishikawa, Y. Tobita, Matter-enhanced transition probabilities in quantum field theory, Ann. Physics 344 (2014) 118–178, http://dx.doi.org/10.1016/j.aop.2014.02.007, URL https://linkinghub.elsevier.com/retrieve/pii/S0003491614000323.
[13] K. Ishikawa, T. Tajima, Y. Tobita, Anomalous radiative transitions, Prog. Theor. Exp. Phys. 2015 (1) (2015) 13B02–0, http://dx.doi.org/10.1093/ptep/ptu168, URL https://academic.oup.com/ptep/article-lookup/doi/10.1093/ptep/ptu168.
[14] K. Ishikawa, Y. Tobita, Finite-size corrections to Fermi's Golden rule II: Quasi-stationary composite states, 2016, http://dx.doi.org/10.48550/ARXIV.1607.08522, Publisher: arXiv Version Number: 2, URL https://arxiv.org/abs/1607.08522.
[15] K. Ishikawa, K.-y. Oda, Particle decay in Gaussian wave-packet formalism revisited, Prog. Theor. Exp. Phys. 2018 (12) (2018) http://dx.doi.org/10.1093/ptep/pty127, URL https://academic.oup.com/ptep/article/doi/10.1093/ptep/pty127/5250519.
[16] A. Edery, Wave packets in QFT: Leading order width corrections to decay rates and clock behavior under Lorentz boosts, Phys. Rev. D 104 (12) (2021) 125015, http://dx.doi.org/10.1103/PhysRevD.104.125015, URL https://link.aps.org/doi/10.1103/PhysRevD.104.125015.
[17] J. Lankinen, I. Vilja, Decay of a massive particle in a stiff-matter-dominated universe, Phys. Rev. D 96 (2017) http://dx.doi.org/10.1103/PhysRevD.96.105026.
[18] J. Lankinen, I. Vilja, Decaying massive particles in the matter and radiation dominated eras, Phys. Rev. D 97 (2018) http://dx.doi.org/10.1103/PhysRevD.97.065004.
[19] J. Lankinen, I. Vilja, Particle decay in expanding Friedmann-Robertson-Walker universes, Phys. Rev. D 98 (2018) http://dx.doi.org/10.1103/PhysRevD.98.045010.
[20] W.G. Unruh, Notes on black-hole evaporation, Phys. Rev. D 14 (4) (1976) 870–892, http://dx.doi.org/10.1103/PhysRevD.14.870, URL https://link.aps.org/doi/10.1103/PhysRevD.14.870.

[21] N.D. Birrell, J.G. Taylor, Analysis of interacting quantum field theory in curved spacetime, J. Math. Phys. 21 (1980) 1740–1760, http://dx.doi.org/10.1063/1.524622.
[22] J. Huhtala, I. Vilja, Practical derivations of fermion and gauge boson reduction formulae in curved spacetimes, Eur. Phys. J. C 84 (6) (2024) 624, http://dx.doi.org/10.1140/epjc/s10052-024-12996-z, URL https://link.springer.com/10.1140/epjc/s10052-024-12996-z.
[23] K. Crane, M. Wardetzky, A glimpse into discrete differential geometry, Notices Amer. Math. Soc. 64 (10) (2017) 1153–1159, http://dx.doi.org/10.1090/noti1578, URL http://www.ams.org/notices/201710/rnoti-p1153.pdf.
[24] R.M. Wald, Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics, first ed., in: Chicago Lectures in Physics, 1994.
[25] S.A. Fulling, Aspects of quantum field theory in curved spacetime, in: London Mathematical Society Student Texts, Cambridge University Press, 1989.
[26] J. Polo-Gómez, L.J. Garay, E. Martín-Martínez, A detector-based measurement theory for quantum field theory, Phys. Rev. D 105 (6) (2022) 065003, http://dx.doi.org/10.1103/PhysRevD.105.065003, URL https://link.aps.org/doi/10.1103/PhysRevD.105.065003.