Locating-dominating codes in paths

Geoffrey Exoo
Department of Mathematics and Computer Science
Indiana State University
Terre Haute, IN 47809, USA
gexoo@indstate.edu

Ville Junnila*
Turku Centre for Computer Science TUCS and
Department of Mathematics
University of Turku, FI-20014 Turku, Finland
viljun@utu.fi

Tero Laihonen
Department of Mathematics
University of Turku, FI-20014 Turku, Finland
terolai@utu.fi

Corresponding author:
Ville Junnila
Department of Mathematics
University of Turku, FI-20014 Turku, Finland
E-mail: viljun@utu.fi
Telephone: +358 2 333 6675
Fax: +358 2 333 6595

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Abstract

Bertrand, Charon, Hudry and Lobstein studied, in their paper in 2004, r-locating-dominating codes in paths $\mathcal{P}_n$. They conjectured that if $r \geq 2$ is a fixed integer, then the smallest cardinality of an r-locating-dominating code in $\mathcal{P}_n$, denoted by $M_r^{LD}(\mathcal{P}_n)$, satisfies $M_r^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$ for infinitely many values of n. We prove that this conjecture holds. In fact, we show a stronger result saying that for any $r \geq 3$ we have $M_r^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$ for all $n \geq n_r$ when n_r is large enough. In addition, we solve a conjecture on location-domination with segments of even length in the infinite path.

Keywords: Locating-dominating code; optimal code; domination; graph; path
Running head: Location-domination in paths

1 Introduction

Let $G = (V, E)$ be a simple connected and undirected graph with V as the set of vertices and E as the set of edges. Let u and v be vertices in V. If u and v are adjacent to each other, then the edge between u and v is denoted by uv. The *distance* $d(u, v)$ is the number of edges in any shortest path between u and v. Let r be a positive integer. We say that u r*-covers* v if the distance $d(u, v)$ is at most r. The *ball of radius* r *centered at* u is defined as

$$B_r(u) = \{x \in V \mid d(u, x) \leq r\}.$$

A nonempty subset of V is called a *code*, and its elements are called *codewords*. Let $C \subseteq V$ be a code and u be a vertex in V. An *I-set* (or an *identifying set*) of the vertex u with respect to the code C is defined as

$$I_r(C; u) = I_r(u) = B_r(u) \cap C.$$

Definition 1.1. Let r be a positive integer. A code $C \subseteq V$ is said to be r-*locating-dominating* in G if for all $u, v \in V \setminus C$ the set $I_r(C; u)$ is nonempty and

$$I_r(C; u) \neq I_r(C; v).$$

Let X and Y be subsets of V. The *symmetric difference* of X and Y is defined as $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. We say that the vertices u and v are r*-separated* by a code $C \subseteq V$ (or by a codeword of C) if the symmetric difference $I_r(C; u) \triangle I_r(C; v)$ is nonempty. The definition of r-locating dominating codes can now be reformulated as follows: $C \subseteq V$ is an r-locating-dominating code in G if and only if for all $u, v \in V \setminus C$ the vertex u is r-covered by a codeword of C and

$$I_r(C; u) \triangle I_r(C; v) \neq \emptyset.$$

The smallest cardinality of an r-locating-dominating code in a finite graph G is denoted by $M_r^{LD}(G)$. Notice that there always exists an r-locating-dominating code in G. An r-locating-dominating code attaining the smallest cardinality is called *optimal*. In [4], it is shown that the problem of determining $M_r^{LD}(G)$ is NP-hard.

Locating-dominating codes are also known as locating-dominating *sets* in the literature. The concept of locating-dominating codes was first introduced by

Slater in [12, 14, 15] and later generalized by Carson in [3]. Locating-dominating codes have been since studied in various papers such as [2], [5], [6], [8], [9], [13], [16], [17] and [18]. For other papers on the subject, we refer to the Web site [11]. Moreover, location-domination in paths has been examined in [1] and [7] (for cycles see [?]).

Let n be a positive integer. A path $\mathcal{P}_n = (V_n, E_n)$ is a graph such that the set of vertices is defined as $V_n = \{v_i \mid i = 0, 1, \ldots, n-1\}$ and the set of edges is defined as

$$E_n = \{v_i v_{i+1} \mid i = 0, 1, \ldots, n-2\}.$$

In [14], Slater showed that $M_1^{LD}(\mathcal{P}_n) = \lceil 2n/5 \rceil$. Bertrand *et al.* [1] provide the following lower bound for $r \geq 2$.

Theorem 1.2. *Let n and r be integers such that $n \geq 1$ and $r \geq 2$. Then we have*

$$M_r^{LD}(\mathcal{P}_n) \geq \left\lceil \frac{n+1}{3} \right\rceil. \tag{1}$$

Moreover, in [1], it is conjectured that for any fixed $r \geq 2$, there exist infinitely many values of n such that $M_r^{LD}(\mathcal{P}_n)$ attains the previous lower bound. In [7], it is shown that $M_2^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$ for any n. Hence, the conjecture holds when $r = 2$. In Section 4 and Section 5, we prove that the conjecture also holds when $r \geq 3$. Moreover, we show that for any $r \geq 3$ we have $M_r^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$ for all $n \geq n_r$ when n_r is large enough ($n_r = \mathcal{O}(r^3)$).

In Section 2, we begin by introducing some basic results concerning r-locating-dominating codes in paths. In Section 3, we continue by considering r-locating-dominating codes in paths $\mathcal{P}_n$ with small n (compared to r). Then, in Section 5, we present optimal 3- and 4-locating-dominating codes in $\mathcal{P}_n$ for all n. Finally, in Section 6, we solve the conjecture stated in [1, Conjecture 2], which considers location-domination with segments of even lengths in the infinite path.

2 Basics

Let C be a nonempty subset of V_n. We first present a useful characterization of r-locating-dominating codes in paths. For this, we need the concept of C-consecutive vertices introduced in [1]. Let i and j be positive integers such that $0 \leq i < j \leq n-1$. We say that (v_i, v_j) is a pair of *C-consecutive vertices* in $\mathcal{P}_n$ if $v_i, v_j \in V_n \setminus C$ and $v_k \in C$ for $0 \leq i < k < j \leq n-1$. Now we are ready to present the following characterization, which is introduced in [1, Remark 3].

Lemma 2.1 ([1]). *Let r be a positive integer. A code $C \subseteq V_n$ is r-locating-dominating in $\mathcal{P}_n$ if and only if each vertex $u \in V_n \setminus C$ is r-covered by a codeword of C and for each pair (u, v) of C-consecutive vertices in $\mathcal{P}_n$ the vertices u and v are r-separated by a codeword of C.*

The following theorem provides a handy property on the size of the optimal r-locating-dominating codes in $\mathcal{P}_n$.

Theorem 2.2. *Let n and r be positive integers. Then we have*

$$M_r^{LD}(\mathcal{P}_n) \leq M_r^{LD}(\mathcal{P}_{n+1}) \leq M_r^{LD}(\mathcal{P}_n) + 1.$$

Proof. Consider first the inequality $M_r^{LD}(\mathcal{P}_n) \leq M_r^{LD}(\mathcal{P}_{n+1})$. Let $C \subseteq V_{n+1} = \{v_0, v_1, \ldots, v_n\}$ be an r-locating-dominating code in $\mathcal{P}_{n+1}$. Assume first that the vertex $v_n \notin C$. Now it is obvious that C is also an r-locating-dominating code in $\mathcal{P}_n$.

Assume then that $v_n \in C$. Denote by X the set of pairs of C-consecutive vertices in $\mathcal{P}_n$. There exists at most one pair $(u, v) \in X$ such that the codeword v_n belongs to the symmetric difference of $I_r(u)$ and $I_r(v)$. If there is no such pair of C-consecutive vertices, then it is clear that $(C \setminus \{v_n\}) \cup \{v_{n-1}\}$ is an r-locating-dominating code in $\mathcal{P}_n$. Assume then that (v_i, v_j) with $i < j$ is the unique pair of C-consecutive vertices such that $v_n \in I_r(v_i) \triangle I_r(v_j)$. Now define $C' = (C \setminus \{v_n\}) \cup \{v_j\}$. Since all the pairs of C-consecutive vertices belonging to $X \setminus \{(v_i, v_j)\}$ are r-separated by a codeword of C', then it is easy to conclude that all the pairs of C'-consecutive vertices are r-separated by a codeword of C' in $\mathcal{P}_n$. Notice that if a vertex is r-covered by v_n, then it is also r-covered by v_j. Therefore, each vertex in V_n is r-covered by a codeword of C'. Thus, by Lemma 2.1, C' is an r-locating-dominating code in $\mathcal{P}_n$. In conclusion, we have $M_r^{LD}(\mathcal{P}_n) \leq M_r^{LD}(\mathcal{P}_{n+1})$.

Let then $C \subseteq V_n$ be an r-locating-dominating code in $\mathcal{P}_n$. Since $C \cup \{v_n\}$ is an r-locating-dominating code in $\mathcal{P}_{n+1}$, we immediately have $M_r^{LD}(\mathcal{P}_{n+1}) \leq M_r^{LD}(\mathcal{P}_n) + 1$. □

In what follows, we present a couple of lemmas that are useful in determining the smallest cardinalities of r-locating-dominating codes in paths with a small number of vertices in Section 3. The first lemma says that an r-locating-dominating code in $\mathcal{P}_n$ is such that at least r of both the first and the last $2r+1$ vertices of the path are codewords.

Lemma 2.3. *Let C be an r-locating-dominating code in $\mathcal{P}_n$ and n be an integer such that $n \geq 2r + 1$.*

(i) The intersection $C \cap \{v_0, v_1, \ldots, v_{2r}\}$ contains at least r vertices.

(ii) The intersection $C \cap \{v_{n-2r-1}, v_{n-2r}, \ldots, v_{n-1}\}$ contains at least r vertices.

Proof. Let C be an r-locating-dominating code in $\mathcal{P}_n$. Denote $\{v_0, v_1, \ldots, v_r\}$ by Q_r. Assume that there are k codewords in $C \cap Q_r$ with $0 \leq k \leq r - 1$. (Notice that if $k \geq r$, then the case (i) immediately follows.) Now there are $r - k$ pairs (u, v) of C-consecutive vertices such that $u \in Q_r$ and $v \in Q_r$. Notice that if (u, v) and (u', v') are such pairs of C-consecutive vertices, then the symmetric differences $I_r(u) \triangle I_r(v)$ and $I_r(u') \triangle I_r(v')$ are subsets of $\{v_{r+1}, v_{r+2}, \ldots, v_{2r}\}$ and the intersection of the symmetric differences $I_r(u) \triangle I_r(v)$ and $I_r(u') \triangle I_r(v')$ is empty. Hence, there are at least $r - k$ codewords in $\{v_{r+1}, v_{r+2}, \ldots, v_{2r}\}$. Thus, the claim (i) follows.

The case (ii) follows by symmetry. □

The second lemma says that an r-locating-dominating code in $\mathcal{P}_n$ is such that any set of $3r + 1$ consecutive vertices in the path contains at least r codewords.

Lemma 2.4. *Let C be an r-locating-dominating code in $\mathcal{P}_n$ and n be an integer such that $n \geq 3r + 1$. For $i = 0, 1, \ldots, n - 3r - 1$, the set*

$$\{v_i, v_{i+1}, \ldots, v_{i+3r}\} \subseteq V_n$$

contains at least r codewords of C.

Proof. Let C be an r-locating-dominating code in $\mathcal{P}_n$ and i be an integer such that $0 \le i \le n - 3r - 1$. Denote $\{v_{i+r}, v_{i+r+1}, \dots, v_{i+2r}\}$ by Q_r. Assume that there are k codewords in $C \cap Q_r$ with $0 \le k \le r-1$. Now there are $r-k$ pairs (u, v) of C-consecutive vertices such that $u \in Q_r$ and $v \in Q_r$. Notice that if (u, v) and (u', v') are such pairs of C-consecutive vertices, then it is easy to see that the symmetric differences $I_r(u) \bigtriangleup I_r(v)$ and $I_r(u') \bigtriangleup I_r(v')$ are subsets of $\{v_i, v_{i+1}, \dots, v_{i+r-1}\} \cup \{v_{i+2r+1}, v_{i+2r+2}, \dots, v_{i+3r}\}$ and the intersection of the symmetric differences $I_r(u) \bigtriangleup I_r(v)$ and $I_r(u') \bigtriangleup I_r(v')$ is empty. Hence, there are at least $r-k$ codewords in $\{v_i, v_{i+1}, \dots, v_{i+r-1}\} \cup \{v_{i+2r+1}, v_{i+2r+2}, \dots, v_{i+3r}\}$. Thus, the claim follows. □

3 Paths with a small number of vertices

Let r be a positive integer. In this section, we determine the exact values of $M_r^{LD}(\mathcal{P}_n)$ when $1 \le n \le 7r+3$. We also present a new lower bound on $M_r^{LD}(\mathcal{P}_n)$ (improving the previous lower bound of Theorem 1.2) for some specific lengths n of the paths.

Consider then the exact values of $M_r^{LD}(\mathcal{P}_n)$ when $1 \le n \le 7r+3$. Clearly, we have $M_r^{LD}(\mathcal{P}_1) = 1$. The exact values of $M_r^{LD}(\mathcal{P}_n)$, when $2 \le n \le 7r+3$, are given in the following theorem. Previously, in [1], it has been shown that $M_r^{LD}(\mathcal{P}_{3r+1}) = M_r^{LD}(\mathcal{P}_{3r+2}) = r+1$ and $M_r^{LD}(\mathcal{P}_{3r+3}) = r+2$.

Theorem 3.1. *Let r be an integer such that $r \ge 2$. Then we have the following results for $2 \le n \le 7r+3$:*

1) If $2 \le n \le r+1$, then $M_r^{LD}(\mathcal{P}_n) = n-1$.

2) If $r+2 \le n \le 2r+1$, then $M_r^{LD}(\mathcal{P}_n) = r$.

3) If $2r+2 \le n \le 3r+2$, then $M_r^{LD}(\mathcal{P}_n) = r+1$.

4) If $n = 3r+3$, then $M_r^{LD}(\mathcal{P}_n) = r+2$.

5) If $3r+4 \le n \le 4r+2$, then $M_r^{LD}(\mathcal{P}_n) = n-2(r+1)$.

6) If $4r+3 \le n \le 5r+2$, then $M_r^{LD}(\mathcal{P}_n) = 2r$.

7) If $5r+3 \le n \le 6r+2$, then $M_r^{LD}(\mathcal{P}_n) = 2r+1$.

8) If $6r+3 \le n \le 6r+5$, then $M_r^{LD}(\mathcal{P}_n) = 2r+2$.

9) If $6r+6 \le n \le 7r+3$, then $M_r^{LD}(\mathcal{P}_n) = n-4r-3$.

Proof. Let C be an r-locating-dominating code in $\mathcal{P}_n$.

1) Assume that $2 \le n \le r+1$. Now it is obvious that $B_r(u) = V_n$ for all $u \in V_n$. Hence, it is immediate that $M_r(\mathcal{P}_n) = n-1$.

2) Assume that $r+2 \le n \le 2r+1$. Now, by Theorem 2.2, we have $M_r^{LD}(\mathcal{P}_n) \ge M_r^{LD}(\mathcal{P}_{r+1}) = r$. On the other hand, using Lemma 2.1, it is easy to verify that $D_2 = \{v_0, v_1, \dots, v_{r-2}\} \cup \{v_{2r}\}$ is an r-locating-dominating code in $\mathcal{P}_{2r+1}$ with r codewords. Therefore, by Theorem 2.2, $M_r^{LD}(\mathcal{P}_n) = r$ when $r+2 \le n \le 2r+1$.

3) Assume that $2r+2 \le n \le 3r+2$. Consider first the path $\mathcal{P}_{2r+2}$. It is easy to conclude that each codeword can r-separate at most one pair of C-consecutive vertices in $\mathcal{P}_{2r+2}$. The number of pairs of C-consecutive vertices in $\mathcal{P}_{2r+2}$ is equal to $2r+2-|C|-1$. Therefore, we have the following inequality:

$$|C| \ge 2r+1-|C| \iff |C| \ge \frac{2r+1}{2}.$$

Thus, by the previous inequality and Theorem 2.2, $M_r^{LD}(\mathcal{P}_n) \ge M_r^{LD}(\mathcal{P}_{2r+2}) \ge r+1$. The code $D_3 = \{v_r, v_{r+1}, \ldots, v_{2r-1}\} \cup \{v_{3r}\}$ introduced in [1] is r-locating-dominating in $\mathcal{P}_{3r+2}$. Therefore, $M_r^{LD}(\mathcal{P}_n) = r+1$ when $2r+2 \le n \le 3r+2$.

4) In [1], it is shown that $D_4 = \{v_0\} \cup \{v_{r+1}, v_{r+2}, \ldots, v_{2r}\} \cup \{v_{3r+2}\}$ is an r-locating-dominating code in $\mathcal{P}_{3r+3}$. Hence, by Theorem 1.2, we have $M_r^{LD}(\mathcal{P}_{3r+3}) = r+2$.

5) Assume that $3r+4 \le n \le 4r+2$. Now we can denote $n = 3r+3+p$, where $1 \le p \le r-1$. By Lemma 2.3, subsets $\{v_0, v_1, \ldots, v_{2r}\}$ and $\{v_{r+p+2}, v_{r+p+3}, \ldots, v_{3r+p+2}\}$ both contain at least r codewords of C. The number of vertices in the intersection of these subsets is equal to $r-p-1$. Therefore, we have

$$|C| \ge r-p-1+2(r-(r-p-1)) = r+p+1.$$

On the other hand, using Lemma 2.1, it is straightforward to verify that $D_5 = \{v_1\} \cup \{v_{r+2}, v_{r+3}, \ldots, v_{2r+p}\} \cup \{v_{3r+p+1}\}$ is an r-locating-dominating code in $\mathcal{P}_n$. Thus, $M_r^{LD}(\mathcal{P}_{3r+3+p}) = r+p+1 = n-2(r+1)$ when $3r+4 \le n \le 4r+2$.

6) Assume that $4r+3 \le n \le 5r+2$. By Theorem 2.2, we have $M_r^{LD}(\mathcal{P}_n) \ge M_r^{LD}(\mathcal{P}_{4r+2}) = 2r$. Then define

$$D_6 = \{v_0\} \cup \{v_{r+2}, v_{r+3}, \ldots, v_{2r}\} \cup \{v_{3r+1}, v_{3r+2}, \ldots, v_{4r-1}\} \cup \{v_{5r+1}\}.$$

The number of vertices in D_6 is equal to $2r$ and, by Lemma 2.1, it can be easily verified that D_6 is an r-locating-dominating code in $\mathcal{P}_{5r+2}$. Therefore, by Theorem 2.2, $M_r^{LD}(\mathcal{P}_n) = 2r$ when $4r+3 \le n \le 5r+2$.

7) Assume that $5r+3 \le n \le 6r+2$. Let us first show that $M_r^{LD}(\mathcal{P}_{5r+3}) \ge 2r+1$. Assume to the contrary that C is an r-locating-dominating code in $\mathcal{P}_{5r+3}$ with at most $2r$ codewords. By Lemma 2.3, we know that both $\{v_0, v_1, \ldots, v_{2r}\}$ and $\{v_{3r+2}, v_{3r+3}, \ldots, v_{5r+2}\}$ contain at least r codewords of C. Hence, there are no codewords of C in $\{v_{2r+1}, v_{2r+2}, \ldots, v_{3r+1}\}$. Therefore, since all the pairs (u,v) of C-consecutive vertices in $\mathcal{P}_{5r+3}$ such that $u, v \in \{v_0, v_1, \ldots, v_{2r+1}\}$ are r-separated by a codeword of C, then the codewords of C belonging to $\{v_0, v_1, \ldots, v_{2r+1}\}$ form an r-locating-dominating code in $\mathcal{P}_{2r+2}$ with r codewords. This is a contradiction with the case 3). Thus, by Theorem 2.2, $M_r^{LD}(\mathcal{P}_n) \ge M_r^{LD}(\mathcal{P}_{5r+3}) \ge 2r+1$. Define then

$$D_7 = \{v_r, v_{r+1}, \ldots, v_{2r-1}\} \cup \{v_{3r}\} \cup \{v_{4r+2}, v_{4r+3}, \ldots, v_{5r}\} \cup \{v_{5r+2}\}.$$

Using Lemma 2.1, it is straightforward to verify that D_7 is an r-locating-dominating code in $\mathcal{P}_{6r+2}$ with $2r+1$ codewords. Thus, $M_r^{LD}(\mathcal{P}_n) = 2r+1$ when $5r+3 \le n \le 6r+2$.

8) Assume that $6r+3 \le n \le 6r+5$. By Theorem 1.2, we have $M_r^{LD}(\mathcal{P}_n) \ge 2r+2$. Define then

$$D_8 = \{v_1, v_{r+1}\} \cup \{v_{r+3}, v_{r+4}, \ldots, v_{2r}\} \cup \{v_{3r+1}, v_{3r+3}\} \cup \{v_{4r+4}, v_{4r+5}, \ldots, v_{5r+1}\} \cup \{v_{5r+3}, v_{6r+3}\}.$$

By Lemma 2.1, D_8 is an r-locating-dominating code in $\mathcal{P}_{6r+5}$ with $2r+2$ vertices. Thus, $M_r^{LD}(\mathcal{P}_n) = 2r+2$ when $6r+3 \leq n \leq 6r+5$.

9) Assume that $6r+6 \leq n \leq 7r+3$. Now we can denote $n = 6r+5+p$, where $1 \leq p \leq r-2$. Consider first the path $\mathcal{P}_{7r+3}$. By Lemma 2.3, the subsets $\{v_0, v_1, \ldots, v_{2r}\}$ and $\{v_{5r+2}, v_{5r+3}, \ldots, v_{7r+2}\}$ of V_{7r+3} both contain at least r codewords of C. By Lemma 2.4, the same also holds for the subset $\{v_{2r+1}, v_{2r+2}, \ldots, v_{5r+1}\}$. Therefore, $M_r^{LD}(\mathcal{P}_{7r+3}) \geq 3r$. Thus, by Theorem 2.2 and the fact that $M_r^{LD}(\mathcal{P}_{6r+5}) = 2r+2$, we have $M_r^{LD}(\mathcal{P}_{6r+5+p}) = 2r+2+p$ when $1 \leq p \leq r-2$. In other words, $M_r^{LD}(\mathcal{P}_n) = n-4r-3$ when $6r+6 \leq n \leq 7r+3$. □

By generalizing the lower bound in the case 9) of the previous proof, the following theorem is immediately obtained.

Theorem 3.2. *Let r be a positive integer and $n = 2(2r+1)+p(3r+1)$ where $p \geq 0$ is an integer. Then we have*

$$M_r^{LD}(\mathcal{P}_n) \geq (p+2)r.$$

Using the notations of the previous theorem, the lower bound of Theorem 1.2 implies that

$$M_r^{LD}(\mathcal{P}_n) \geq \left\lceil \frac{n+1}{3} \right\rceil = (p+1)r+1+\left\lceil \frac{r+p}{3} \right\rceil.$$

By straightforward calculations, it can be shown that $(p+2)r > (p+1)r+1+\lceil (r+p)/3 \rceil$ if and only if $0 \leq p \leq 2r-6$. Thus, the previous theorem gives improvements on the previously known lower bound when $n = 2(2r+1)+p(3r+1)$ and $0 \leq p \leq 2r-6$.

By applying Theorem 2.2 to the previous lower bound, we also obtain new lower bounds for some other values of n. For example, by Theorem 1.2, we have $M_5^{LD}(\mathcal{P}_{56}) \geq 19$. However, by Theorem 3.2, we have $M_5^{LD}(\mathcal{P}_{54}) \geq 20$ and, therefore, $M_5^{LD}(\mathcal{P}_{56}) \geq M_5^{LD}(\mathcal{P}_{54}) \geq 20$.

The values given by the lower bound of Theorem 3.2 are sometimes optimal. For example, when $r=5$ and $p=4$, we have $M_5^{LD}(\mathcal{P}_{86}) \geq 30$. On the other hand,

$$D_{86} = \{v_2, v_6, v_8, v_9, v_{10}, v_{12}, v_{17}, v_{21}, v_{24}, v_{25}, v_{27}, v_{29}, v_{33}, v_{37}, v_{41}, v_{43}, v_{45}, v_{46}, v_{53}, v_{55}, v_{59}, v_{61}, v_{62}, v_{63}, v_{71}, v_{75}, v_{76}, v_{78}, v_{79}, v_{83}\}$$

is a 5-locating-dominating code in $\mathcal{P}_{86}$. Therefore, $M_5^{LD}(\mathcal{P}_{86}) = 30$.

4 Paths with a large number of vertices

Let n be a positive integer and r be an integer such that $r \geq 5$. In this section, we show that the size of an optimal r-locating-dominating code in $\mathcal{P}_n$ is equal to $\lceil (n+1)/3 \rceil$ for all $n \geq n_r$ when n_r is large enough. The proof of this is based on the result of Theorem 4.3 saying that if $n = 3r+2+p((r-3)(6r+3)+3r+3)+q(6r+3)$, where p and q are non-negative integers, then we have $M_r^{LD}(\mathcal{P}_n) \leq \lceil (n+1)/3 \rceil$. The proof of Theorem 4.3 is illustrated in the following example when $r=5$.

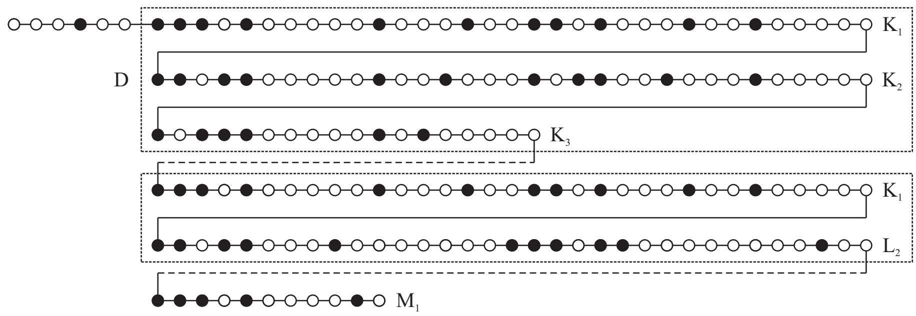

Figure 1: The r-locating-dominating code C_1 illustrated when $r = 5$.

Example 4.1. Assume that $r = 5$. Let p and q be non-negative integers. In what follows, we show that if $n = 3r + 2 + p((r-3)(6r+3) + 3r + 3) + q(6r + 3) = 17 + 84p + 33q$, then $M_5^{LD}(\mathcal{P}_n) \leq \lceil (n+1)/3 \rceil$. In Figures 1 and 2, first consider the pattern D (the upper dashed box in the figures), which is formed by concatenating the patterns K_1, K_2 and K_3, which are of lengths $6r+3$, $6r+3$ and $3r+3$, respectively. The pattern D is of length $(r-3)(6r+3)+3r+3 = 84$ and contains $((r-3)(6r+3)+3r+3)/3 = 28$ codewords, i.e. $1/3$ of the vertices of D are codewords. Moreover, it is easy to verify that D is a 5-locating-dominating code in a cycle of length 84 (compare this with Lemma 4.2). Similarly, the pattern (the lower dashed box in the figures) formed by K_1 and L_2, which is of length $2(6r + 3) = 66$ and contains $(2(6r + 3))/3 = 22$ codewords, is a 5-locating-dominating code in a cycle of length 66.

The actual 5-locating-dominating code in $\mathcal{P}_n$ depends on the parity of q. Assume first that q is even, i.e. $q = 2q'$ for some integer q'. The code C_1 is now defined as in Figure 1, where the pattern D is repeated p times and the pattern formed by K_1 and L_2 is repeated q' times. Since the patterns D and the one formed by K_1 and L_2 are 5-locating-dominating codes, respectively, in cycles of lengths 84 and 66, it is straightforward to verify that C_1 is a 5-locating-dominating code in $\mathcal{P}_n$ (by Lemma 2.1). Similarly, it can be shown that the code C_2 defined in Figure 2 is 5-locating-dominating in $\mathcal{P}_n$ when q is odd, i.e. $q = 2q' + 1$ for some integer q'. Therefore, if $n = 17 + 84p + 33q$, we have $M_r^{LD}(\mathcal{P}_n) \leq 6 + 28p + 11q = \lceil (n+1)/3 \rceil$.

For the formal proof of Theorem 4.3, we first need to introduce some preliminary definitions and results. Let i and s be non-negative integers. First, for $1 \leq i \leq r-2$, define

$$M_i(s) = \left(\bigcup_{\substack{j=0 \\ j \neq r-i-1}}^{r-1} \{v_{s+j}\} \right) \cup \{v_{s+2r-i}\}$$

and $M_i'(s) = M_i(s) \setminus \{v_{s+2r-i}\}$. Notice that $|M_i(s)| = r$. Furthermore, for

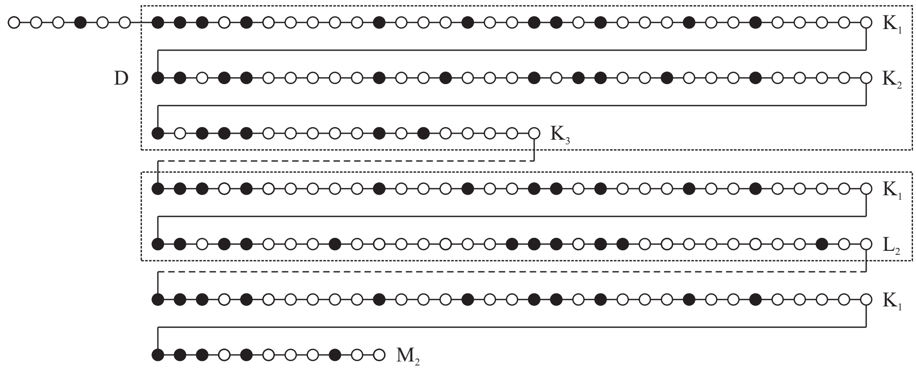

Figure 2: The r-locating-dominating code C_2 illustrated when $r = 5$.

$1 \leq i \leq r-3$, define

$$K_i(s) = M'_i(s) \cup \{v_{s+2r}, v_{s+3r-i}\} \cup \left(\bigcup_{\substack{j=3r+2 \\ j \neq 4r-i}}^{4r} \{v_{s+j}\} \right) \cup \{v_{s+5r-i}, v_{s+5r+2}\},$$

and $K_{r-2}(s) = M'_{r-2}(s) \cup \{v_{s+2r}, v_{s+2r+2}\}$. Notice that for $i = 1, 2, \ldots, r-3$, we have $|K_i(s)| = 2r + 1$ and $|K_{r-2}(s)| = r + 1$. Finally, define

$$\begin{aligned} L_1(s) = M_1(s) &\cup \left(\bigcup_{j=3r+1}^{4r-1} \{v_{s+j}\} \right) \cup \{v_{s+4r+1}, v_{s+6r+1}\} \\ &\cup \left(\bigcup_{j=6r+3}^{7r+1} \{v_{s+j}\} \right) \cup \{v_{s+8r+3}\} \end{aligned}$$

and, for $2 \leq i \leq r-2$, define

$$L_i(s) = M_i(s) \cup \left(\bigcup_{\substack{j=3r+1 \\ j \neq 4r-i+1}}^{4r+1} \{v_{s+j}\} \right) \cup \{v_{s+6r-i+2}\}.$$

Notice that $|L_1(s)| = 3r + 1$ and $|L_i(s)| = 2r + 1$ when $2 \leq i \leq r - 2$.

As in Example 4.1, denote by K_i, L_i and M_i the patterns $\{v_s, v_{s+1}, \ldots, v_{s+\ell-1}\}$ where the codewords are determined by $K_i(s)$, $L_i(s)$ and $M_i(s)$, respectively. The length ℓ of each pattern K_i and L_i is equal to three times the number of codewords in the pattern. For example, the length of the pattern L_1 is equal to $9r + 3$ (see the case (iv) below). The length of the pattern M_i is equal to $2r+1$. The following lemma says for general $r \geq 5$ that the patterns K_i, L_i and M_i can be concatenated to form r-locating dominating codes as in Example 4.1 (because the beginning of each of them contains $M'_i(s)$).

Lemma 4.2. *Let n and s be positive integers, and let r be an integer such that $r \geq 5$. Let C be a code in $\mathcal{P}_n$.*

(i) *Let i be an integer such that $1 \leq i \leq r-3$. If $K_i(s) \cup M'_{i+1}(s+6r+3) \subseteq C$, then each pair (v_{j_1}, v_{j_2}) of C-consecutive vertices in $\mathcal{P}_n$ such that $s \leq j_1 \leq s+7r+2$ and $s \leq j_2 \leq s+7r+2$ is r-separated by a codeword of C.*

(ii) *If $K_{r-2}(s) \cup M'_1(s+3r+3) \subseteq C$, then each pair (v_{j_1}, v_{j_2}) of C-consecutive vertices in $\mathcal{P}_n$ such that $s \leq j_1 \leq s+4r+2$ and $s \leq j_2 \leq s+4r+2$ is r-separated by a codeword of C.*

(iii) *Let i be an integer such that $2 \leq i \leq r-2$. If $L_i(s) \cup M'_{i-1}(s+6r+3) \subseteq C$, then each pair (v_{j_1}, v_{j_2}) of C-consecutive vertices in $\mathcal{P}_n$ such that $s \leq j_1 \leq s+7r+2$ and $s \leq j_2 \leq s+7r+2$ is r-separated by a codeword of C.*

(iv) *If $L_1(s) \cup M'_{r-2}(s+9r+3) \subseteq C$, then each pair (v_{j_1}, v_{j_2}) of C-consecutive vertices in $\mathcal{P}_n$ such that $s \leq j_1 \leq s+10r+2$ and $s \leq j_2 \leq s+10r+2$ is r-separated by a codeword of C.*

Proof. (i) Let i be an integer with $1 \leq i \leq r-3$ and $C \subseteq V_n$ a code such that $K_i(s) \cup M'_{i+1}(s+6r+3) \subseteq C$. Consider then the symmetric differences $B_r(v_{j_1}) \triangle B_r(v_{j_2})$, where (v_{j_1}, v_{j_2}) are pairs of C-consecutive vertices such that $s \leq j_1 \leq s+7r+2$ and $s \leq j_2 \leq s+7r+2$. For the following considerations, notice that

$$M'_{i+1}(s+6r+3) = \bigcup_{\substack{j=6r+3 \\ j \neq 7r-i+1}}^{7r+2} \{v_{s+j}\}.$$

Let k be a positive integer. If $s+r \leq k \leq s+2r-i-2$, $s+2r-i \leq k \leq s+2r-2$, $s+4r+2 \leq k \leq s+5r-i-2$ or $s+5r-i+1 \leq k \leq s+5r$, then it is straightforward to verify that the vertex v_{k-r} belongs to the symmetric difference $I_r(v_k) \triangle I_r(v_{k+1})$. If $s+2r+1 \leq k \leq s+3r-i-2$, $s+3r-i+1 \leq k \leq s+3r-1$, $s+5r+3 \leq k \leq s+6r-i-1$ or $s+6r-i+1 \leq k \leq s+6r+1$, then it can be seen that the vertex v_{k+r+1} belongs to the symmetric difference $I_r(v_k) \triangle I_r(v_{k+1})$. Moreover, we have that

$$\begin{aligned}
v_{s+2r} &\in I_r(v_{s+r-i-1}) \triangle I_r(v_{s+r}), \\
v_{s+3r-i} &\in I_r(v_{s+2r-i-1}) \triangle I_r(v_{s+2r-i}), \\
v_{s+r-1} &\in I_r(v_{s+2r-1}) \triangle I_r(v_{s+2r+1}), \\
v_{s+4r-i+1} &\in I_r(v_{s+3r-i-1}) \triangle I_r(v_{s+3r-i+1}), \\
v_{s+2r} &\in I_r(v_{s+3r}) \triangle I_r(v_{s+3r+1}), \\
v_{s+5r-i} &\in I_r(v_{s+3r+1}) \triangle I_r(v_{s+4r-i}), \\
v_{s+3r-i} &\in I_r(v_{s+4r-i}) \triangle I_r(v_{s+4r+1}), \\
v_{s+5r+2} &\in I_r(v_{s+4r+1}) \triangle I_r(v_{s+4r+2}), \\
v_{s+4r-i-1} &\in I_r(v_{s+5r-i-1}) \triangle I_r(v_{s+5r-i+1}), \\
v_{s+6r+3} &\in I_r(v_{s+5r+1}) \triangle I_r(v_{s+5r+3}), \\
v_{s+5r-i} &\in I_r(v_{s+6r-i}) \triangle I_r(v_{s+6r-i+1}) \text{ and} \\
v_{s+5r+2} &\in I_r(v_{s+6r+2}) \triangle I_r(v_{s+7r-i+1}).
\end{aligned}$$

In conclusion, all the pairs (v_{j_1}, v_{j_2}) of C-consecutive vertices in $\mathcal{P}_n$ such that $s \leq j_1 \leq s+7r+2$ and $s \leq j_2 \leq s+7r+2$ are r-separated by a codeword of C.

The proofs of the cases (ii), (iii) and (iv) are analogous to the first one. □

The following theorem now proves the conjecture stated in [1, Conjecture 1] when $r \geq 5$.

Theorem 4.3. *Let $r \geq 5$ be an integer and $n = 3r + 2 + p((r - 3)(6r + 3) + 3r + 3) + q(6r + 3)$, where p and q are non-negative integers. Then we have*

$$M_r^{LD}(\mathcal{P}_n) \leq \left\lceil \frac{n+1}{3} \right\rceil .$$

Proof. Let $r \geq 5$ be an integer and $n = 3r+2+p((r-3)(6r+3)+3r+3)+q(6r+3)$, where p and q are non-negative integers. Let s be a non-negative integer and define

$$D(s) = \bigcup_{i=0}^{r-3} K_{i+1}(s + i(6r + 3)).$$

Assume that q is even, i.e. $q = 2q'$ for some integer q'. Define then

$$\begin{aligned} C_1 = \{v_{r-2}\} &\cup \bigcup_{j=0}^{p-1} D(r + 1 + j((r - 3)(6r + 3) + 3r + 3)) \\ &\cup \bigcup_{j=0}^{q'-1} K_1(r + 1 + p((r - 3)(6r + 3) + 3r + 3) + 2j(6r + 3)) \\ &\cup \bigcup_{j=0}^{q'-1} L_2(r + 1 + p((r - 3)(6r + 3) + 3r + 3) + (2j + 1)(6r + 3)) \\ &\cup M_1(r + 1 + p((r - 3)(6r + 3) + 3r + 3) + q(6r + 3)). \end{aligned}$$

Notice that if $r = 5$, this definition of C_1 coincides with the one of Example 4.1. (Recall also the length of the patterns K_i, L_i and M_i as described earlier.) As in the previous example, C_1 is formed by concatenating the patterns K_i, L_i and M_i. Since $M_i'(s) \subseteq K_i(s)$ and $M_i'(s) \subseteq L_i(s)$, Lemma 4.2 applies to each occurrence of $K_i(s)$ and $L_i(s)$ in C_1. Therefore, each pair (v_j, v_k) of C_1-consecutive vertices in $\mathcal{P}_n$ such that $r+1 \leq j \leq n-r-2$ and $r+1 \leq k \leq n-r-2$ is r-separated by a codeword of C_1. Hence, it is easy to see that each pair of C_1-consecutive vertices in $\mathcal{P}_n$ is r-separated by C_1. Since there are no $2r + 1$ consecutive vertices belonging to $V_n \setminus C_1$ in $\mathcal{P}_n$, all the vertices in $\mathcal{P}_n$ are r-covered by a codeword of C_1. Thus, by Lemma 2.1, it is easy to conclude that C_1 is an r-locating-dominating code in $\mathcal{P}_n$ with $\lceil (n + 1)/3 \rceil$ vertices.

Assume then that q is odd, i.e. $q = 2q' + 1$ for some integer q'. Define then

$$\begin{aligned} C_2 = \{v_{r-2}\} &\cup \bigcup_{j=0}^{p-1} D(r + 1 + j((r - 3)(6r + 3) + 3r + 3)) \\ &\cup \bigcup_{j=0}^{q'} K_1(r + 1 + p((r - 3)(6r + 3) + 3r + 3) + 2j(6r + 3)) \\ &\cup \bigcup_{j=0}^{q'-1} L_2(r + 1 + p((r - 3)(6r + 3) + 3r + 3) + (2j + 1)(6r + 3)) \\ &\cup M_2(r + 1 + p((r - 3)(6r + 3) + 3r + 3) + q(6r + 3)). \end{aligned}$$

Similarly, as in the previous case, it can be shown that C_2 is an r-locating-dominating code in $\mathcal{P}_n$ with $\lceil (n+1)/3 \rceil$ vertices. □

In [10, Theorem 8.3], the following theorem is presented. This theorem turns out useful in future considerations.

Theorem 4.4 ([10]). *Let a and b be positive integers such that the greatest common divisor of a and b is equal to 1. Then, for any integer $n > ab - a - b$, there exist such non-negative integers p and q that $n = pa + qb$.*

The length of the path in Theorem 4.3 can be written as follows:

$$\begin{aligned} n &= 3r + 2 + p((r-3)(6r+3) + 3r + 3) + q(6r+3) \\ &= 3r + 2 + 3(p((r-3)(2r+1) + r + 1) + q(2r+1)). \end{aligned}$$

The greatest common divisor of $(r-3)(2r+1)+r+1$ and $2r+1$ is equal to 1. Thus, by Theorem 4.4, if n' is an integer such that $n' \geq 2r((r-3)(2r+1)+r)$, then there exist non-negative integers p and q such that $n' = p((r-3)(2r+1)+r+1)+q(2r+1)$. Therefore, if n is an integer such that $n \geq 3r+2+3\cdot 2r((r-3)(2r+1)+r)$ and $n \equiv 2 \pmod 3$, then there exist integers $p \geq 0$ and $q \geq 0$ such that $n = 3r+2+p((r-3)(6r+3)+3r+3)+q(6r+3)$.

Assume that $n \geq 3r+2+6r((r-3)(2r+1)+r)$ and $n = 3k+2$, where k is an integer. Combining the lower bound of Theorem 1.2, Theorem 2.2 and Theorem 4.3, we obtain

$$k+1 \leq M_r^{LD}(\mathcal{P}_{3k}) \leq M_r^{LD}(\mathcal{P}_{3k+1}) \leq M_r^{LD}(\mathcal{P}_{3k+2}) \leq k+1.$$

Therefore, $M_r^{LD}(\mathcal{P}_{3k}) = M_r^{LD}(\mathcal{P}_{3k+1}) = M_r^{LD}(\mathcal{P}_{3k+2}) = k+1$. Thus, the following theorem immediately follows.

Theorem 4.5. *Let r be a positive integer such that $r \geq 5$. If $n \geq 3r+2+6r((r-3)(2r+1)+r)$, then*

$$M_r^{LD}(\mathcal{P}_n) = \left\lceil \frac{n+1}{3} \right\rceil.$$

Theorem 4.3 provides one approach to form r-locating-dominating codes in paths using Lemma 4.2. However, this lemma can also be applied in other ways. For example, when k is an integer such that $0 \leq k \leq r-3$,

$$\begin{aligned} C(k) = \{v_{r-2}\} \cup L_1(r+1) \cup \left(\bigcup_{j=0}^{k-1} L_{r-2-j}(10r+4+j(6r+3)) \right) \\ \cup M_{r-2-k}(10r+4+k(6r+3)) \end{aligned}$$

is an optimal r-locating-dominating code in $\mathcal{P}_n$ with $n = 12r+5+k(6r+3)$. Notice that the optimal r-locating-dominating codes in paths of these lengths cannot be obtained using Theorem 4.3.

5 The exact values of $M_3^{LD}(\mathcal{P}_n)$ and $M_4^{LD}(\mathcal{P}_n)$

Let n be a positive integer. In this section, we solve the exact values of $M_3^{LD}(\mathcal{P}_n)$ and $M_4^{LD}(\mathcal{P}_n)$ for all n. In order to do this, we first need to present some preliminary definitions and results.

Define an infinite path $\mathcal{P}_\infty = (V_\infty, E_\infty)$, where $V_\infty = \{v_i \mid i \in \mathbb{Z}\}$ and $E_\infty = \{v_i v_{i+1} \mid i \in \mathbb{Z}\}$. Define then

$$C = \{v_i \in V_\infty \mid i \equiv 0, 2 \bmod 6\}.$$

In [7], it is stated that if r is an integer such that $r \geq 2$ and $r \equiv 1, 2, 3$ or $4 \pmod 6$, then C is an r-locating-dominating code in $\mathcal{P}_\infty$. This result is rephrased in the following lemma when $r = 3$ or $r = 4$.

Lemma 5.1. *Let n and k be integers such that*

$$D = \{v_k, v_{k+2}, v_{k+6}, v_{k+8}, v_{k+12}, v_{k+14}\} \subseteq V_n.$$

If a pair (v_i, v_j) of D-consecutive vertices in $\mathcal{P}_n$ is such that $k+5 \leq i \leq k+13$ and $k+5 \leq j \leq k+13$, then v_i and v_j are 3- and 4-separated by a codeword of D. Moreover, each vertex $v_i \in V_n \setminus D$ such that $k+6 \leq i \leq k+11$ is 3- and 4-covered by a codeword of D.

Consider then r-locating-dominating codes in $\mathcal{P}_n$ when $r = 3$. By Theorem 3.1, the exact values of $M_3^{LD}(\mathcal{P}_n)$ are known when $1 \leq n \leq 24$. Let p be an integer such that $p \geq 1$. Define

$$D_1(p) = \{v_1\} \cup \left(\bigcup_{i=0}^{p} \{v_{4+6i}, v_{6+6i}\} \right) \cup \{v_{9+6p}, v_{14+6p}, v_{15+6p}, v_{17+6p}\}$$

and

$$D_2(p) = \{v_1\} \cup \left(\bigcup_{i=0}^{p} \{v_{4+6i}, v_{6+6i}\} \right) \cup \{v_{10+6p}, v_{12+6p}, v_{16+6p}, v_{18+6p}, v_{21+6p}\}.$$

It is straightforward to verify that $D_1(1)$ and $D_2(1)$ are 3-locating-dominating codes in $\mathcal{P}_{26}$ and $\mathcal{P}_{29}$, respectively. Therefore, using Lemma 5.1, it is easy to conclude that $D_1(p)$ and $D_2(p)$ are 3-locating-dominating codes in $\mathcal{P}_{20+6p}$ and $\mathcal{P}_{23+6p}$, respectively, when $p \geq 2$. Moreover, by Theorem 1.2 and Theorem 2.2, we have

$$|D_1(p)| \geq M_3^{LD}(\mathcal{P}_{20+6p}) \geq M_3^{LD}(\mathcal{P}_{19+6p}) \geq M_3^{LD}(\mathcal{P}_{18+6p}) \geq 7 + 2p$$

and

$$|D_2(p)| \geq M_3^{LD}(\mathcal{P}_{23+6p}) \geq M_3^{LD}(\mathcal{P}_{22+6p}) \geq M_3^{LD}(\mathcal{P}_{21+6p}) \geq 8 + 2p.$$

Since $|D_1(p)| = 7+2p$ and $|D_2(p)| = 8+2p$, we have that $M_3^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$ for any $n \geq 24$. In conclusion, all the values of $M_3^{LD}(\mathcal{P}_n)$ are determined.

Consider then r-locating-dominating codes in $\mathcal{P}_n$ when $r = 4$. By Theorem 3.1, the exact values of $M_4^{LD}(\mathcal{P}_n)$ are known when $1 \leq n \leq 31$. Assume now that $p \geq 0$. Define

$$D_3(p) = \{v_1, v_5, v_7, v_8\} \cup \left(\bigcup_{i=0}^{p} \{v_{13+6i}, v_{15+6i}\} \right) \cup \{v_{20+6p}, v_{21+6p}, v_{23+6p}, v_{27+6p}\}$$

and

$$D_4(p) = \{v_1, v_5, v_7, v_8\} \cup \left(\bigcup_{i=0}^{p} \{v_{13+6i}, v_{15+6i}\} \right) \cup \{v_{20+6p}, v_{21+6p}, v_{23+6p}, v_{28+6p}\}$$
$$\cup \{v_{31+6p}, v_{34+6p}, v_{36+6p}, v_{39+6p}, v_{42+6p}, v_{47+6p}, v_{49+6p}, v_{50+6p}, v_{53+6p}\}.$$

It is straightforward to verify that $D_3(0)$, $D_3(1)$, $D_4(0)$ and $D_4(1)$ are 4-locating-dominating codes in $\mathcal{P}_{29}$, $\mathcal{P}_{35}$, $\mathcal{P}_{56}$ and $\mathcal{P}_{62}$, respectively. Therefore, using Lemma 5.1, it is easy to conclude that $D_1(p)$ and $D_2(p)$ are 4-locating-dominating codes in $\mathcal{P}_{29+6p}$ and $\mathcal{P}_{56+6p}$, respectively, when $p \geq 2$. Moreover, by Theorem 1.2 and Theorem 2.2, we have

$$|D_3(p)| \geq M_4^{LD}(\mathcal{P}_{29+6p}) \geq M_4^{LD}(\mathcal{P}_{28+6p}) \geq M_4^{LD}(\mathcal{P}_{27+6p}) \geq 10 + 2p$$

and

$$|D_4(p)| \geq M_4^{LD}(\mathcal{P}_{56+6p}) \geq M_4^{LD}(\mathcal{P}_{55+6p}) \geq M_4^{LD}(\mathcal{P}_{54+6p}) \geq 19 + 2p.$$

Since $|D_3(p)| = 10 + 2p$ and $|D_4(p)| = 19 + 2p$, we have that $M_4^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$ when $27 + 6p \leq n \leq 29 + 6p$ and $54 + 6p \leq n \leq 56 + 6p$ ($p \geq 0$). In conclusion, the values of $M_4^{LD}(\mathcal{P}_n)$ are determined except when $n = 32$, $36 \leq n \leq 38$, $42 \leq n \leq 44$ or $48 \leq n \leq 50$.

By Theorem 3.1, we have $M_4^{LD}(\mathcal{P}_{31}) = 12$. Therefore, by Theorem 2.2, since $M_4^{LD}(\mathcal{P}_{35}) = 12$, we also have that $M_4^{LD}(\mathcal{P}_{32}) = 12$. Define then

$$D_{37} = \{v_2, v_3, v_5, v_6, v_{13}, v_{16}, v_{17}, v_{19}, v_{23}, v_{29}, v_{30}, v_{31}, v_{33}\},$$

$$D_{43} = \{v_2, v_3, v_5, v_8, v_{10}, v_{16}, v_{18}, v_{21}, v_{23}, v_{24}, v_{31}, v_{34}, v_{35}, v_{37}, v_{41}\}$$

and

$$D_{49} = \{v_2, v_5, v_6, v_8, v_{13}, v_{16}, v_{19}, v_{20}, v_{26}, v_{27}, v_{30}, v_{33}, v_{38}, v_{40}, v_{41}, v_{42}, v_{48}\}.$$

It is easy to verify that D_{37}, D_{43} and D_{49} are 4-locating-dominating codes in $\mathcal{P}_{37}$, $\mathcal{P}_{43}$ and $\mathcal{P}_{49}$ attaining the lower bound of Theorem 1.2, respectively. Therefore, by Theorem 2.2, we also have the optimal 4-locating-dominating codes for the paths $\mathcal{P}_{36}$, $\mathcal{P}_{42}$ and $\mathcal{P}_{48}$. By Theorem 3.2, we have $M_4^{LD}(\mathcal{P}_{44}) \geq 16$. On the other hand, we have $M_r^{LD}(\mathcal{P}_{44}) \leq M_r^{LD}(\mathcal{P}_{45}) = 16$. Hence, $M_4^{LD}(\mathcal{P}_{44}) = 16$.

Now the only open values are $M_4^{LD}(\mathcal{P}_{38})$ and $M_4^{LD}(\mathcal{P}_{50})$. By the previous constructions, we know that $M_4^{LD}(\mathcal{P}_{38}) \leq M_4^{LD}(\mathcal{P}_{39}) = 14$ and $M_4^{LD}(\mathcal{P}_{50}) \leq M_4^{LD}(\mathcal{P}_{51}) = 18$. By an exhaustive computer search, we have been able to prove that there are no 4-locating-dominating codes in $\mathcal{P}_{38}$ and $\mathcal{P}_{50}$ with 13 and 17 codewords, respectively. Hence, $M_4^{LD}(\mathcal{P}_{38}) = 14$ and $M_4^{LD}(\mathcal{P}_{50}) = 18$. In conclusion, all the values of $M_4^{LD}(\mathcal{P}_n)$ are determined.

6 On the conjecture of even segment lengths

In this section, the focus is on the infinite path $\mathcal{P}_\infty$. Previously, we have considered the balls $B_r(v_i) = \{v_j \in V_\infty \mid i - r \leq j \leq i + r\}$, $i \in \mathbb{Z}$, of size (or *length*) $2r + 1$, which is necessarily odd. In [1], also the case where a ball or rather a *segment* can have an even length is considered in P_∞. Clearly, the 'center' of

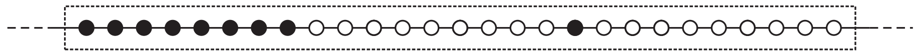

Figure 3: The code C of Theorem 6.2 illustrated when $k = 3$. The code is formed by repeating the pattern in the dashed box infinitely many times to the left and to the right.

the segment of even size is not a vertex of V_∞, so we also need to choose how to associate a segment with a codeword. Notice that this prevents the usual symmetry

$$v_j \in B_r(v_i) \Leftrightarrow v_i \in B_r(v_j)$$

which we earlier often used. In what follows, we always associate a segment in the same way with every codeword.

The problem is stated analogously after selecting the association of a segment with a codeword: how to place the codewords (segments) in P_∞ in such a way that every vertex of V_∞, which is not in the code, belongs to at least one segment and no two non-codewords belong to the same set of segments. Again, we would like to have as small density of a code as possible. The *density* of a code C is defined as usually

$$D(C) = \limsup_{n\to\infty} \frac{|Q_n \cap C|}{|Q_n|}$$

where $Q_n = \{v_i \in V_\infty \mid -n \le i \le n\}$.

In [1], it is pointed out that the choice how to associate a segment with a codeword affects on the minimum density of a locating-dominating code in P_∞. However, it is shown in Theorem 16 of [1] that no matter how one chooses the association with a codeword, the smallest density is at least $1/3$.

Related to this lower bound, the following conjecture is given in [1].

Conjecture 6.1. *Let s be a positive integer divisible by 6. Then we can achieve the density $1/3$ for a locating-dominating code using segments of length s in P_∞.*

In the next theorem we shall confirm this conjecture.

Theorem 6.2. *Let s be a positive integer divisible by 6. There exists a code $C \subseteq V_\infty$ and an assignment of a segment of length s with a codeword such that C is locating-dominating in P_∞ with density $1/3$.*

Proof. Let s be a positive integer with $s = 6k$ and $k \ge 1$. Denote $S = \{0, 1, 2, \ldots, 3k-2, 6k-1\}$. Take

$$C = \{v_i \in V_\infty \mid i \equiv x \bmod 9k \text{ for some } x \in S\}.$$

In Figure 3, the code C is illustrated when $k = 3$. Let us associate, for all the codewords $v_c \in C$, the segment as follows: $\widetilde{B}_s(v_c) = \widetilde{B}_{6k}(v_c) = \{v_{c-3k+1}, \ldots, v_c, \ldots, v_{c+3k}\}$. Clearly, the density of the code is $1/3$. Next we show that C is locating-dominating in P_∞ by determining any vertex $v_i \in V_\infty \setminus C$ with the aid of the segments of codewords it belongs to.

First of all, every non-codeword v_i belongs to some segment, namely to a segment associated with $v_{c_1} \in C$ for some $c_1 \equiv 3k-2 \pmod{9k}$ or with $v_{c_2} \in C$ for some $c_2 \equiv 6k-1 \pmod{9k}$.

Suppose first that there exists a codeword $v_c \in C$ such that $c \equiv 6k-1 \pmod{9k}$ with $v_i \in \widetilde{B}_s(v_c)$. If there is no other codeword to whose segment v_i belongs, then $v_i = v_{c+1}$. Assume then that we have at least one codeword $v_{c'}$ for which $c' > c$ and to whose segment v_i belongs. Let $c_1 = \max\{a \in \mathbb{Z} \mid v_i \in \widetilde{B}_s(v_a), v_a \in C\}$. Consequently, $v_i = v_{c_1-3k+1}$. Suppose now that we do not have codewords with larger index c' than c for which $v_i \in \widetilde{B}_s(v_{c'})$. Let $c_2 = \min\{a \in \mathbb{Z} \mid v_i \in \widetilde{B}_s(v_a), v_a \in C\}$. Then $v_i = v_{c_2+3k}$.

Suppose finally that none of the codewords v_c such that $v_i \in \widetilde{B}_s(v_c)$ satisfies $c \equiv 6k-1 \pmod{9k}$. Now $v_i = v_{c_2+3k-1}$ where again $c_2 = \min\{a \in \mathbb{Z} \mid v_i \in \widetilde{B}_s(v_a), v_a \in C\}$. This completes the proof. □

Locating-dominating codes achieving the density 1/3 for the even segment lengths satisfying $s \not\equiv 0 \pmod 6$, can be found in [1].

7 Conclusions

Previously, the exact values of $M_1^{LD}(\mathcal{P}_n)$ and $M_2^{LD}(\mathcal{P}_n)$ are known due to [14] and [7], respectively. In Section 5, we computed the exact values of $M_3^{LD}(\mathcal{P}_n)$ and $M_4^{LD}(\mathcal{P}_n)$. In Section 3, the exact values of $M_r^{LD}(\mathcal{P}_n)$ have been determined when $1 \leq n \leq 7r+3$. Furthermore, by Theorem 4.5, we have that $M_r^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$ when $n \geq 3r+2+3(2r+1)((r-3)(2r+1)+r)$. In conclusion, although some of the exact values of $M_r^{LD}(\mathcal{P}_n)$ are known when $7r+3 < n < 3r+2+3(2r+1)((r-3)(2r+1)+r)$, the question remains open in general.

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