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Hedging temperature risk with CDD and HDD temperature futures

Fred Espen Benth[1] | **Jukka Lempa**[2]

[1]Department of Mathematics, University of Oslo, Oslo, Norway

[2]Department of Mathematics and Statistics, University of Turku, Turku, Finland

Correspondence
Jukka Lempa, Department of Mathematics and Statistics, University of Turku, FIN–20014, Turku, Finland.
Email: jumile@utu.fi

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Abstract

This paper is concerned with managing risk exposure to temperature using weather derivatives. We consider hedging temperature risk using so-called HDD- and CDD-index futures, which are instruments written on temperatures in specific locations over specific time periods. The temperatures are modelled as continuous-time autoregressive (CARMA) processes and pricing of the hedging instrument is done under an equivalent pricing measure. We develop hedging strategies for locations, cutoff temperatures, and time periods different to the ones in the traded contracts, allowing for more flexibility in the hedging application. The dynamic hedging strategies are expressed explicitly by the term structure of the volatility. We also provide numerical case studies with temperatures following a CAR(3)-process to illustrate the temporal behaviour of the hedge under different scenarios.

KEYWORDS

continuous-time autoregressive processes, dynamic hedging, futures contracts, temperature risk

1 | INTRODUCTION

The Chicago Mercentile Exchange (CME)* offers a variety of temperature futures to hedge temperature risk. These futures contracts are settled on temperature indices measured in nine cities scattered around in the US, in addition to London, Amsterdam and Tokyo. The temperature indices, being the so-called Heating-Degree-Day and Cooling-Degree-Day (usually referred to as HDD and CDD), are measured over two pre-defined time strips in the winter and summer season with a cutoff temperature of 65°F (or, equivalently, 18°C), respectively. For London, Amsterdam and Tokyo, futures on the cumulative average temperature (CAT) index is also traded.

These futures contracts are tailored to swap a floating index into a fixed one, and thereby one can offset a risk exposure to temperature. This is relevant for electricity producers who may want to hedge their volume risk exposure to demand, induced by heat or cold spells. Tourism and leisure industry are other branches where one may be interested to insure income against undesirable temperature events. Finally, in agriculture one is depending on favourable temperatures in growth and harvesting seasons, say.

The risk exposure to temperature of a business is depending on a spatial location or area, a given period in time as well as a potential cutoff temperature. For example, electricity demand in Finland, Norway and Sweden is proportional to the HDD-index of a country-wide average temperature with a cutoff of 15.5°C (see Grochowicz et al.,[1] where the CDD-index

*https://cmegroup.com.

with same cutoff is also argued for in other areas of Europe where cooling drives the demand). In the US, one has a similar dependence of a CDD-index in power demand in summer periods due to air-condition cooling. A ski-resort, located in a mountain area near New York say, may wish to hedge their temperature risk in the winter period. Their risk is the local temperature, whereas the natural CME hedging instrument is for New York city. We mention Stefani et al.[2] for applications of weather derivatives for insurance purposes (see the extensive list of references therein for other sources and application areas of weather derivatives).

In this paper, we develop explicit hedges in traded CDD and HDD futures, when the exposure is in different time periods and spatial locations than those traded at the CME, as well as for possibly different cutoff temperatures in the indices in question. These hedges are represented as portfolios of the traded HDD and CDD futures at the CME, such that the desired location, time period or cutoff temperature is optimally reached. For example, we derive portfolios of HDD and CDD futures reaching a different cutoff temperature than the traded ones with 18°C. In effect, we are constructing new synthetic futures contracts which replicate as good as possible tailormade cutoff temperatures in desired locations and time periods.

Modelling the temperature dynamics as a seasonal continuous-time autoregressive moving average (CARMA) process, we apply the Clark–Ocone formula (see Nualart [Reference 3, prop. 1.3.5]) along with Malliavin calculus (see e.g., Nualart[3] for an introduction) to derive explicit expressions for the hedges. Apart from a mathematical treatment, we also provide some numerical case studies to gain further insight. In fact, risk exposure to CDD and HDD temperature risk over different time periods and cutoff temperatures than the traded ones can be partly replicated by a dynamic trading strategy in CDD and HDD futures offered at CME. The position in the traded temperature futures should follow a ratio defined by the term structure of volatility. On the other hand, a risk exposure in a different location can only be partly offset, where the hedging efficiency is proportional to the correlation between temperatures in the two locations in question. A spatial hedging problem for weather derivatives was analysed in Barth, Benth and Potthoff,[4] where the optimal hedge was derived by minimizing the quadratic error based on a spatio-temporal random field model for temperature variations. We also would like to mention the paper by Benth and Detering,[5] where the hedging of Asian-style energy market options are analysed. The futures that we focus on are indeed Asian-style, paying a sum of “call/put-options” on temperatures. In Benth and Detering,[5] partial hedging is considered when you are not allowed to update your hedge during the delivery period due to liquidity constraints. They also focus on hedging of calls and puts on futures, which is a different problem than what we focus on here.

In Benth and Šaltytė Benth,[6] CARMA-processes are proposed as models for the stochastic temperature variations on a daily scale in a given location. This class of models is empirically validated for several locations in Europe in a number of studies (see Reference 6 and references therein, as well as Härdle and Lopez-Cabrera[7] for Asian temperatures and Swishchuk and Cui[8] for Canadian temperatures.). Eggen et al.[9] have used CARMA-processes to model the time-dynamics of stratospheric temperatures. Several extensions have been proposed, for example regime-switching models (Türkvatan, Hayfavi and Omay[10]) and the inclusion of stochastic volatility (Alfonsi and Vadillo[11]). In Benth and Šaltytė Benth[6] one finds a detailed analysis of temperature futures pricing based on the arbitrage pricing theory in mathematical finance (see e.g., Björk[12]). Other popular approaches for pricing of weather derivatives, like burn analysis and indifference pricing, are discussed, as well as derivatives on wind and precipitation. Extensive treatments of weather derivatives and modeling are found in Jewson and Brix,[13] Alexandridis and Zapranis,[14] and Benth and Šaltytė Benth.[6]

Our results are presented as follows. In the next section, we define the CARMA-process which will form the basis for our temperature dynamics. Next, in Section 3, we introduce CDD and HDD futures, and provide an expression for the futures price. This will form the basis of our main results in Section 4 on hedges when measurement periods, locations or cutoff temperatures differ from the traded ones. Some numerical case studies are presented in Section 5, while we conclude in the last section.

2 | TEMPERATURE MODEL

We model the temperature dynamics as a seasonal continuous-time autoregressive moving average (CARMA) process

$$T(t) = S(t) + Y(t), \tag{1}$$

for $t \geq 0$ and S being a deterministic seasonality function assumed to be continuous. To specify the random fluctuations Y around the seasonal mean S, consider for $p \in \mathbb{N}$ the $\mathbb{R}^p$-valued process X given as the solution of the Itô-equation

$$dX(t) = AX(t)dt + \sigma(t)\mathbf{e}_p dW(t),$$

where W is a one-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$, $\mathbf{e}_p$ is pth basis vector of $\mathbb{R}^p$, and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \cdots & -\alpha_1 \end{pmatrix}.$$

We suppose that α_k are positive constants. Finally, the volatility σ is assumed to be a continuous deterministic function. By Itô's formula it is readily seen that

$$X(s) = \exp(A(s-t))X(t) + \int_t^s \sigma(u)\exp(A(s-u))\mathbf{e}_p dW(u), \tag{2}$$

for $0 \leq t \leq s$. Finally, Y is given by

$$Y(t) = \mathbf{b}'X(t), \tag{3}$$

where $\mathbf{b} \in \mathbb{R}^p$ is the vector $\mathbf{b} = (b_0, b_1, \ldots, b_q = 1, 0, \ldots, 0)'$ for $q \in \mathbb{N}, q < p$. The process Y is called a CARMA(p, q)-process. Several empirical studies have shown that seasonal CARMA(3,0)-processes (also frequently referred to as CAR(3)-processes, where $\mathbf{b} = \mathbf{e}_1$) explains the temperature dynamics very well (see e.g., Härdle and Lopez-Cabrera,[7] Benth and Šaltytė Benth,[6] Eggen et al.,[9] and Swichchuk and Cui[8]). Moreover, de-seasonalized temperatures are naturally stationary, something which is accounted for by the model whenever A has eigenvalues with negative real part (see Benth and Šaltytė Benth[6] for more on this).

3 | CDD AND HDD FUTURES

The cooling-degree days (CDD) index over the measurement period $[\tau_1, \tau_2]$ is defined as

$$\int_{\tau_1}^{\tau_2} \max(T(u) - C, 0)du,$$

where C is the cutoff temperature. Similarly, the heating-degree days (HDD) index is defined as

$$\int_{\tau_1}^{\tau_2} \max(C - T(u), 0)du.$$

In the marketplace, $C = 65°$F, equivalently, $C = 18°$C. Also, we remark in passing that we have chosen the continuous-time definition of the CDD and HDD indices, being approximations of the true discrete-time aggregation of the daily average temperature (see Jewson and Brix[13] and Benth and Šaltytė Benth[6])

We investigate futures contracts written on the CDD index over the specified period $[\tau_1, \tau_2]$ (the case of futures on the HDD-index is completely analogous). We suppose that the CDD-futures contract delivers the money-equivalent of the CDD-index at the end of the measurement period, with a money-conversion factor 1. The futures price at time $t \leq \tau_2$

written on the CDD index is therefore the $\mathbb{F}$-adapted stochastic process defined as

$$F_{CDD}(t;\tau_1,\tau_2) = \mathbb{E}^{\mathbb{Q}}\left[\int_{\tau_1\vee t}^{\tau_2} \max(T(u)-C,0)du \middle| \mathcal{F}_t\right].$$

Here, $\mathbb{Q}$ is a pricing measure. Following the argument in Benth, Šaltytė Benth and Koekebakker,[15] the pricing measure can be any measure $\mathbb{Q} \sim \mathbb{P}$ for which the conditional expectation exists (i.e., the CDD-index is integrable with respect to $\mathbb{Q}$). This is due to the fact that the only tradeable instrument is the treasury bill, the discounted price of which is a martingale under every equivalent measure. Remark that we use the convention that the delivery of the contract is the remaining CDD-index when we have entered the measurement period, that is, the CDD-index measured over $[t,\tau_2]$ when $t\in[\tau_1,\tau_2]$.

To scale down the possible choices of $\mathbb{Q}$, we consider the parameterized class of probability measures $\{\mathbb{Q}^\theta\}$, such that the process

$$W^\theta(t) := W(t) - \int_0^t \theta(u)du$$

is a Wiener process under the measure $\mathbb{Q}^\theta$. The parameterizing functions θ are real, bounded and piecewise continuous; in this case, we can apply the Girsanov theorem to conclude that such measures exist. Moreover, by the preserved Gaussianity of the process X under $\mathbb{Q}^\theta$ and the linear growth of the CDD- and HDD-indices, we are ensured that the conditional expectations are well-defined. We denote the expectation with respect to $\mathbb{Q}^\theta$ as $\mathbb{E}^\theta$.

Proposition 1. *The CDD-futures price is given by*

$$F_{CDD}(t;\tau_1,\tau_2) = \int_{\tau_1\vee t}^{\tau_2} v(t,u)\Psi\left(\frac{m(t,u,\mathbf{b}'e^{A(u-t)}X(t),C)}{v(t,u)}\right)du,$$

where $\Psi(x) = x\Phi(x)+\Phi'(x)$, Φ is the standard normal cumulative probability distribution function and

$$m(t,u,x,C) = S(t) + x + \int_t^u \sigma(s)\theta(s)\mathbf{b}'e^{A(u-s)}\mathbf{e}_p ds - C,$$
$$v^2(t,u) = \int_t^u \sigma^2(s)\left(\mathbf{b}'e^{A(u-s)}\mathbf{e}_p\right)^2 ds.$$

Moreover, the CDD-futures price dynamics can be written as

$$dF_{CDD}(t;\tau_1,\tau_2) = \sigma(t)\int_{\tau_1\vee t}^{\tau_2} \mathbf{b}'e^{A(u-t)}\mathbf{e}_p\Phi\left(\frac{m(t,u,\mathbf{b}'e^{A(u-t)}X(t),C)}{v(t,u)}\right)dudW^\theta(t)$$

Proof. See Benth, Šaltytė Benth and Koekebakker,[16] Propositions 5 and 6, where in particular the representation in (2) of X is applied. ■

The term structure of the volatility of the CDD-futures contract reads as

$$\Sigma_{CDD}(t;\tau_1,\tau_2,C) = \sigma(t)\int_{\tau_1\vee t}^{\tau_2} \mathbf{b}'e^{A(u-t)}\mathbf{e}_p\Phi\left(\frac{m(t,u,\mathbf{b}'e^{A(u-t)}X(t),C)}{v(t,u)}\right)du, \quad (4)$$

for $t<\tau_2$. We have emphasised the dependency of the time interval τ_1 and τ_2 as well as the cutoff C in the notation for Σ_{CDD}. We then have

$$dF_{CDD}(t;\tau_1,\tau_2) = \Sigma_{CDD}(t;\tau_1,\tau_2,C)dW^\theta(t)$$

for the CDD-futures price dynamics. This volatility term structure becomes useful in expressing hedges, as we will see below.

According to Benth and Šaltytė Benth [Reference 6, Prop. 5.6], the HDD-futures price (with measurement period $[\tau_1, \tau_2]$ and cutoff temperature C), is

$$F_{HDD}(t; \tau_1, \tau_2) = \int_{\tau_1 \vee t}^{\tau_2} v(t,u)\Psi\left(-\frac{m(t,u,\mathbf{b}' e^{A(u-t)}X(t), C)}{v(t,u)}\right)du. \tag{5}$$

Notice that the only difference to the $F_{CDD}(t; \tau_1, \tau_2)$ is the sign change inside the function Ψ. Moreover, again from Benth and Šaltytė Benth [Reference 6, Prop. 5.6], the dynamics becomes

$$dF_{HDD}(t; \tau_1, \tau_2) = -\sigma(t)\Sigma_{HDD}(t; \tau_1, \tau_2, C)dW^{\theta}(t), \tag{6}$$

where

$$\Sigma_{HDD}(t; \tau_1, \tau_2, C) = \sigma(t)\int_{\tau_1 \vee t}^{\tau_2} \mathbf{b}' e^{A(u-t)}\mathbf{e}_p \Phi\left(-\frac{m(t,u,\mathbf{b}' e^{A(u-t)}X(t), C)}{v(t,u)}\right)du, \tag{7}$$

for $t < \tau_2$. Notice again a minus sign appearing compared with $\Sigma_{CDD}(t; \tau_1, \tau_2, C)$, this time in the argument of Φ. The other minus sign in front of $\sigma(t)$ may seem odd at first sight. But this is only indicating that whenever the noise moves upwards (i.e., informally $dW^{\theta} > 0$), means that the CDD-futures price goes up while the HDD-futures price goes down (i.e., $dF_{CDD} > 0$ while $dF_{HDD} < 0$). Naturally, the two prices move in opposite directions with the random movements of the noise.

4 | HEDGING TEMPERATURE RISK

In this section, we provide the theoretical expressions for hedging various temperature risk exposures with given contracts in the market. First we consider the problem of hedging a CDD index, measured over a time interval which does not coincide with that traded in the market, and with a different cutoff temperature. Next we consider a geographical hedging problem, where your temperature risk is at a different location than those cities traded at CME. We provide hedges in both cases, for which we can offset parts of the risk by strategies in tradeable contracts.

4.1 | Temporal and cutoff hedge

Assume that there is a CDD-futures contract available for trade with delivery period $[\tau_1, \tau_2]$ and cutoff temperature C. Recall from Proposition 1 its price dynamics with volatility term structure $\Sigma_{CDD}(t; \tau_1, \tau_2, C)$ in (4). We consider hedging of the payoff

$$\int_{\eta_1}^{\eta_2} \max(T(u) - \tilde{C}, 0)du. \tag{8}$$

where the cutoff temperature $\tilde{C}$ and the delivery period $[\eta_1, \eta_2]$ are not necessarily equal to their counterparts in the traded contract. For future easy reference, denote the instantaneous payoff by

$$P(u) = \max(T(u) - \tilde{C}, 0). \tag{9}$$

Without loss of generality, we may assume that $\tau_2 > \eta_1$; if $\tau_2 < \eta_1$, the futures contract has expired before η_1 and hedging is not possible. We split the payoff (8) as

$$\int_{\eta_1}^{\eta_2 \wedge \tau_2} P(u)du + \int_{\eta_2 \wedge \tau_2}^{\eta_2} P(u)du.$$

The second integral corresponds to a period that cannot be hedged due to the expiry of the hedging instrument. Thus the relevant payoff to consider is

$$\int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du. \tag{10}$$

We provide a hedge in CDD-futures at the market for this risk in the next Proposition:

Proposition 2. *Assume that, for any $t < \tau_2$, $\Sigma_{CDD}(t;\tau_1,\tau_2,C) > 0$. We can hedge the temperature risk in (10) by the continuous investment strategy given by the position $H_{CDD}(t) := H_{\mathrm{CDD}}(t;\tau_1,\tau_2,\eta_1,C,\eta_2,\widetilde{C})$ for $t < \eta_2$ in the CDD-contract traded in the market, with,*

$$H_{\mathrm{CDD}}(t) = \frac{\Sigma_{CDD}(t,\eta_1,\eta_2\wedge\tau_2,\widetilde{C})}{\Sigma_{CDD}(t,\tau_1,\tau_2,C)}$$

Here, Σ_{CDD} is defined in (4).

Proof. By the Clark–Ocone formula (see Nualart [Reference 3, Prop. 1.3.5]) and Proposition 1, we obtain the representation

$$\begin{aligned} P(u) &= \mathbb{E}^\theta[P(u)] + \int_0^u \mathbb{E}^\theta[D_tP(u)|\mathcal{F}_t]dW^\theta(t) \\ &= \mathbb{E}^\theta[P(u)] + \int_0^u \frac{\mathbb{E}^\theta[D_tP(u)|\mathcal{F}_t]}{\Sigma_{CDD}(t;\tau_1,\tau_2,C)}dF_{CDD}(t;\tau_1,\tau_2). \end{aligned}$$

Here, D_t is the Malliavin derivative. Hence we can replicate the instantaneous payoff $P(u)$ by choosing the contract with price dynamics $F_{CDD}(\cdot;\tau_1,\tau_2)$ and maintaining the position $\frac{\mathbb{E}^\theta[D_tP(u)|\mathcal{F}_t]}{\Sigma_{CDD}(t;\tau_1,\tau_2,C)}$ by a continuous rebalancing. By changing the order of integration, we find that

$$\begin{aligned} \int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du &= \mathbb{E}^\theta\left[\int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du\right] \\ &\quad + \int_{\eta_1}^{\eta_2\wedge\tau_2}\left(\int_0^u \frac{\mathbb{E}^\theta[D_tP(u)|\mathcal{F}_t]}{\Sigma_{CDD}(t;\tau_1,\tau_2,C)}dF_{CDD}(t;\tau_1,\tau_2)\right)du \\ &= \mathbb{E}^\theta\left[\int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du\right] \\ &\quad + \int_0^{\eta_2\wedge\tau_2}\left(\frac{\int_{\eta_1\vee t}^{\eta_2\wedge\tau_2}\mathbb{E}^\theta[D_tP(u)|\mathcal{F}_t]du}{\Sigma_{CDD}(t;\tau_1,\tau_2,C)}\right)dF_{CDD}(t;\tau_1,\tau_2). \end{aligned}$$

We compute the Malliavin derivative $D_tP(u)$ for $u \geq t$. By the chain rule of the Malliavin derivative, we find that (strictly speaking, the max-function is non-differentiable at the kink $\widetilde{C}$, but we can avoid this obstacle by smoothing the max-function or arguing that it is non-differentiable in just one point of measure zero)

$$\begin{aligned} D_t\max(T(u)-\widetilde{C},0) &= \mathbf{1}(T(u)>\widetilde{C})D_tT(u) \\ &= \mathbf{1}(T(u)>\widetilde{C})\sigma(t)\mathbf{b}'e^{A(u-t)}\mathbf{e}_p. \end{aligned}$$

This yields the expression

$$\begin{aligned} &\int_{\eta_1\vee t}^{\eta_2\wedge\tau_2}\mathbb{E}^\theta[D_tP(u)|\mathcal{F}_t]du \\ &\quad = \sigma(t)\int_{\eta_1\vee t}^{\eta_2\wedge\tau_2}\mathbf{b}'e^{A(u-t)}\mathbf{e}_p\mathbb{Q}^\theta[T(u)>\tilde{C}|\mathcal{F}_t]du \end{aligned}$$

$$= \sigma(t)\int_{\eta_1\vee t}^{\eta_2\wedge\tau_2} \mathbf{b}' e^{A(u-t)}\mathbf{e}_p \Phi\left(\frac{m(t,u,\mathbf{b}' e^{A(u-t)}X(t),\widetilde{C})}{v(t,u)}\right)du,$$

where m and v are given in Proposition 1. Thus, the hedging portfolio reads as

$$H_{CDD}(t) = \frac{\sigma(t)\int_{\eta_1\vee t}^{\eta_2\wedge\tau_2} \mathbf{b}' e^{A(u-t)}\mathbf{e}_p \Phi\left(\frac{m(t,u,\mathbf{b}' e^{A(u-t)}X(t),\tilde{C})}{v(t,u)}\right)du}{\Sigma_{CDD}(t;\tau_1,\tau_2,C)}$$

for $t < \eta_2$. Appealing to the definition of Σ_{CDD} in (4) yields the result. ■

The condition $\Sigma_{CDD}(t;\tau_1,\tau_2,C)$ being positive might seem unnecessary at first sight since it is the volatility of the CDD-futures price dynamics. Recalling (4), Φ is the cumulative probability distribution which is always positive, as is also naturally the volatility σ of the temperature dynamics. However, the term $\mathbf{b}'\exp(A(t-u))\mathbf{e}_p$ might be negative, and the integral could therefore very well become non-positive. Usually, $\mathbf{b} = \mathbf{e}_1$, and with $p = 1$ positivity is ensured, so this is something that may happen for CAR-models of the temperature of higher-order.

Notice from the definition in (4) that Σ_{CDD} depends explicitly on the stochastic process $X(t)$ and thus the hedge ratio becomes stochastic. Moreover, $X(t)$ defines the temperature dynamics, so the position in the CDD-futures is, roughly speaking, being updated according to the temperature at time t. If we model the temperature as a CAR(3)-dynamics, $X(t) = (X_1(t), X_2(t), X_3(t))$, and by definition $X_1(t) = T(t) - S(t)$. Thus, the hedge depends on the current (time-t) de-seasonalized temperature. By definition of the CAR(3)-dynamics, $X_2(t) = X_1'(t)$ and $X_3(t) = X_1''(t)$ which means that the hedge ratio also depends on the current de-seasonalized temperature gradient as well as its curvature (the gradient of the temperature gradient). We can numerically assess the values of these by observing temperature changes locally prior but close to t.

From the expression of the hedge ratio in Proposition 2, we readily find

Corollary 1. *Under the condition of Proposition 2, it holds that*

$$\lim_{\substack{\eta_1\to\tau_1\\ \eta_2\to\tau_2\\ \tilde{C}\to C}} H_{CDD}(t;\tau_1,\tau_2,C,\eta_1,\eta_2,\tilde{C}) = 1.$$

Thus, if our temporal and cutoff temperature risk exposure is close to the payoff of the tradeable CDD-futures, the hedge ratio is close to one. Let us now fix $\eta_1 = \tau_1$ and $\eta_2 = \tau_2$. If $\widetilde{C} < C$, we find from the definition of m in Proposition 1 that

$$m(t,u,x,\widetilde{C}) > m(t,u,x,C)$$

As Φ is an increasing function, the numerator of the hedge ratio in Proposition 2 will be larger than the denominator. Thus, in that case $H_{CDD}(t) > 1$. The opposite, that is, $H_{CDD}(t) < 1$, holds when $\widetilde{C} > C$. We also observe that we can exactly replicate the temperature risk when $\eta_1 = \tau_1$ and $\eta_2 = \tau_2$.

By repeating the above arguments for a temperature risk exposure in the HDD-index, and using an HDD-futures contract as hedging instrument, it is readily seen that we obtain a hedge

$$H_{HDD}(t) = \frac{\Sigma_{HDD}(t,\eta_1,\eta_2\wedge\tau_2,\widetilde{C})}{\Sigma_{HDD}(t,\tau_1,\tau_2,C)}$$

where we recall the HDD volatility term structure in (7). We could even hedge a CDD-type of risk by an HDD-future, or vice versa, and the result would be hedges that can be expressed in terms of the ratio of the HDD and the CDD volatility term structures.

It is finally worth remarking that the hedge constructed in Proposition 2 replicates

$$\int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du - \mathbb{E}^{\theta}\left[\int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du\right],$$

as we can see from the Clark–Ocone representation in the proof. The second term is the futures price if one could trade in a contract paying $\int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du$. Indeed, the hedging strategy is giving the same payoff as entering a position where one is short $\int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du$ in return for a fixed price $\mathbb{E}^\theta\left[\int_{\eta_1}^{\eta_2\wedge\tau_2} P(u)du\right]$. Thus, we have constructed a synthetic futures contract with given time period $[\eta_1, \eta_2 \wedge \tau_2]$ and cutoff $\widetilde{C}$ temperature from the available market traded futures.

4.2 | Geographical hedge

Consider a hedging problem involving two locations, being first a city where CME offers temperature futures and a second location where temperature risk is not traded. The hedging question is how one can exploit the tradeable futures to mitigate temperature risk exposure in the second location. We label for convenience the two locations of interest by 1 and 2, and assume that a CDD-futures contract with measurement period $[\tau_1, \tau_2]$ and cutoff temperature C is traded in location 1.

As indicated, we study the hedging of CDD-temperature risk in location 2 using the contract traded in location 1. To this end, assume that the temperatures in locations 1 and 2 are modelled as in Section 2, following, respectively, the dynamics

$$T_1(t) = S_1(t) + \mathbf{b}_1' X_1(t),$$
$$T_2(t) = S_2(t) + \mathbf{b}_2' X_2(t),$$

where

$$dX_1(t) = A_1 X_1(t)dt + \sigma_1(t)\mathbf{e}_p dW_1(t),$$
$$dX_2(t) = A_2 X_2(t)dt + \sigma_2(t)\mathbf{e}_p dW_2(t).$$

Here, $(W_1, W_2)^T$ is a correlated two dimensional Brownian motion. More specifically, consider

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} B \\ \rho B + \sqrt{1-\rho^2} Z \end{pmatrix},$$

where $\rho \in [-1, 1]$, and B and Z are independent one-dimensional Brownian motions. By the Girsanov theorem, we obtain an equivalent measure $\mathbb{Q}^\theta$, such that

$$W_1^\theta(t) = W_1(t) - \int_0^t \theta(u)du,$$
$$W_2^\theta(t) = W_2(t) - \rho\int_0^t \theta(u)du,$$

are $\mathbb{Q}^\theta$-Brownian motions. Moreover, we note that

$$W_2^\theta(t) = \rho W_1^\theta(t) + \sqrt{1-\rho^2} Z(t). \tag{11}$$

Assume we want to hedge the temperature risk with payoff as in (8) in location 2, that is,

$$\int_{\eta_1}^{\eta_2} \max(T_2(u) - \widetilde{C}, 0)du, \tag{12}$$

where as before the cutoff temperature $\widetilde{C}$ and the delivery period $[\eta_1, \eta_2]$ may be different to the specifications of the tradeable CDD-futures contract in location 1. However, as in the previous Subsection, we assume that $\tau_2 > \eta_1$.

We further introduce the notation $\Sigma_{CDD,i}(t;\tau_1,\tau_2,C)$ for the term structure given in (4) but labelled by the location, $i=1,2$,

Proposition 3. *Assume that, for any* $t<\tau_2$, $\Sigma_{CDD,1}(t;\tau_1,\tau_2,C)>0$. *We can hedge the temperature risk in* (12) *by the position denoted by* $H^1_{CDD}(t):=H^1_{CDD}(t;\tau_1,\tau_2,C,\eta_1,\eta_2,\widetilde{C})$ *for* $t<\eta_2$ *in the CDD contract traded in location 1, with,*

$$H^1_{CDD}(t)=\rho\frac{\Sigma_{CDD,2}(t,\eta_1,\eta_2\wedge\tau_2,\widetilde{C})}{\Sigma_{CDD,1}(t,\tau_1,\tau_2,C)}.$$

Proof. Consider the instantaneous payoff

$$P_2(u)=\max(T_2(u)-\tilde{C},0)$$

in location 2. By the Clark–Ocone formula and Equation (11), we find that

$$\begin{aligned}
P_2(u)&=\mathbb{E}^\theta\left[P_2(u)du\right]+\int_0^u\mathbb{E}^\theta[D_tP_2(u)|\mathcal{F}_t]dW_2^\theta(t)\\
&=\mathbb{E}^\theta[P_2(u)]+\rho\int_0^u\mathbb{E}^\theta[D_tP_2(u)|\mathcal{F}_t]dW_1^\theta(t)\\
&\quad+\sqrt{1-\rho^2}\int_0^u\mathbb{E}^\theta[D_tP_2(u)|\mathcal{F}_t]dZ(t)\\
&=\mathbb{E}^\theta[P_2(u)]+\rho\int_0^u\frac{\mathbb{E}^\theta[D_tP_2(u)|\mathcal{F}_t]}{\Sigma_{CDD,1}(t;\tau_1,\tau_2,C)}dF^1_{CDD}(t;\tau_1,\tau_2)\\
&\quad+\sqrt{1-\rho^2}\int_0^u\mathbb{E}^\theta[D_tP_2(u)|\mathcal{F}_t]dZ(t).
\end{aligned}$$

Here $\Sigma_{CDD,1}$ is the term structure for volatility of the futures contract in location 1 and dF^1_{CDD} is the $\mathbb{Q}^\theta$-dynamics for the futures price in location 1. We can interpret the dF^1_{CDD}-integral as the hedgeable part of the payoff and the dZ-integral as the nonhedgeable (idiosyncratic) part of the payoff.

By arguing similarly as in the proof of Proposition 2, we obtain the hedging portfolio

$$H^1_{CDD}(t)=\rho\frac{\sigma_2(t)\int_{\eta_1\vee t}^{\eta_2\wedge\tau_2}\mathbf{b}_2'e^{A_2(u-t)}\mathbf{e}_p\Phi\left(\frac{m_2(t,u,\mathbf{b}_2'e^{A_2(u-t)}X_2(t),\tilde{C})}{v_2(t,u)}\right)du}{\sigma_1(t)\int_{\tau_1\vee t}^{\tau_2}\mathbf{b}_1'e^{A_1(u-t)}\mathbf{e}_p\Phi\left(\frac{m_1(t,u,\mathbf{b}_1'e^{A_1(u-t)}X_1(t),C)}{v_1(t,u)}\right)du},$$

where the functions m_i and v_i are defined as in Proposition 1 and referring to the locations via $i=1,2$. This concludes the proof. ∎

We observe that again the optimal hedge in the tradeable contract is to hold a ratio defined by the term structure of volatilities, but in this spatial context, the ratio involves the parameters in the respective temperature models as well. Additionally, the ratio is scaled by the correlation coefficient ρ. Naturally, if ρ is close to 1, it is to be expected that the coefficients of the temperature model in location 2 are similar to those of location 1 as well, and the ratio becomes similar to the expression derived in Proposition 2. With ρ being negative, we simply reverse the position compared with the same hedge ratio when ρ is positive (i.e., short instead of long, or vice versa).

Inspecting the proof of Proposition 3, we have an expression for the nonhedgeable part of the temperature risk. Indeed, one sees after some calculations as in the proof of Proposition 2 that this term becomes

$$\sqrt{1-\rho^2}\int_0^{\eta_2\wedge\tau_1}\Sigma_{CDD,2}(t;\eta_1,\eta_2\wedge\tau_2,\widetilde{C})dZ(t). \tag{13}$$

The stronger the correlation between the two locations is, the more of the idiosyncratic risk can be hedged by the CDD-futures. If the correlation between the two locations is zero, then evidently we cannot hedge any of the risk in location 2 by the tradeable CDD-futures in location 1. The constructed synthetic futures will in this case only partly replicate the desired futures contract in the give location.

We remark that in Barth, Benth and Potthoff,[4] a similar spatial hedging problem is considered. They consider a spatio-temporal random field model for the temperature dynamics, and ask for the minimal-variance hedge for the risk exposure in one location using tradeable futures in a finite number of other locations. We extend their work by deriving an explicit dynamic portfolio based on the martingale representation of Clark–Ocone's theorem. We obtain, moreover, an explicit formula for the non-hedgeable part, without resorting to numerical minimization of the variance. We also would like to mention the recent paper by D'Aversa et al.[17] dealing with geographical basis risk. In their study, an efficient cutoff temperature is derived by minimizing the static quadratic deviation of a collection of CDD or HDD temperature futures payoff functions in different geographical locations against a reference payoff function.

5 NUMERICAL CASE STUDIES

We perform some numerical case studies to illustrate the theoretical hedges that we have derived in Propositions 2 and 3 of the previous Section. In the case studies, we focus on CDD-futures and temperature risk exposure.

We fix a CAR(3)-model for the temperature dynamics inspired by the estimated model for the city of Vilnius, Lithuania, found in Benth and Šaltytė Benth.[6] Thus, $p = 3$ and $T(t)$ follows the dynamics in (1) with $Y(t) = \mathbf{e}_1'X(t)$ and X given in (2). The A-matrix has coefficients $\alpha_1 = 2.034$, $\alpha_2 = 1.311$ and $\alpha_3 = 0.187$ (see [Reference 6, p. 99]), which yields that A has eigenvalues with negative real parts and thus defines a stationary dynamics X for de-seasonalized temperatures. To simplify matters, we have assumed a constant seasonality function $S(t) \equiv 18$ for Vilnius as well as a constant volatility σ. We let $\sigma = \sqrt{5.76}$, which is the level of estimated seasonality for Vilnius (see [Reference 6, tab. 3.2, p. 46]). We notice that both the actual seasonality function and the volatility for Vilnius display significant variations over the year, but in our case studies here we focus on a fixed level which reflects to some degree the summer period of the temperature dynamics in Vilnius.

In the calculation of the hedges in the different cases, we see from the formulas in Propositions 2 and 3 that we need to compute ratios of the term structure of volatility given in (4). This term structure depends on the current state of $X(t)$. Letting time be measured on a *daily* scale, we have from (2)

$$X(t+1) = \exp(A)X(t) + \sigma \int_t^{t+1} \exp(A(t+1-s))\mathbf{e}_3 dB(s).$$

We use a simple time-discretization to simulate approximative paths of $X(t)$ which is given by

$$x(t+1) = \exp(A)x(t) + \sigma \exp(A)\mathbf{e}_3 \epsilon(t) \tag{14}$$

where $(\epsilon(t))_{t=0,1,2,\dots}$ are IID standard normal random variables. We note that the noise part in the above three-variate time series $(x(t))_{t=0,1,2,\dots}$ is an approximation of the term $\sigma\int_t^{t+1} \exp(A(t+1-s)\mathbf{e}_3 dB(s)$. Moreover, we recall that $B(t+1) - B(t)$ are independent and standard normally distributed random variables when $t = 0, 1, 2, \dots$. In the numerics, we simulate first 100 days of X in a burn-in period to have a reasonable starting point for the temperature dynamics, which is not polluted by an artificially set initial state.

In the calculation of the function m appearing in the hedges through the term structure of volatility (4) (see Proposition 1 for the definition of this function), we choose a risk premium $\theta = 0$ for simplicity. The risk premium must be calibrated from traded CDD-futures prices. Rather than speculating on the level and sign of the risk premium, we suppose that the market prices by the expectation hypothesis. We refer the reader to Härdle and Lopez-Cabrera[7] for a theoretical and empirical analysis of the risk premium of temperature derivatives.

The function $v(t, u)$ defined in Proposition 1 as well as the CDD-volatility term structure (4) are defined as integrals. In the case studies, a simple Riemann sum approximation is used to calculate these expressions.

In the case studies, we let the tradeable CDD-futures have a measurement period covering "next month", that is, setting $\tau_1 = 30$ and $\tau_2 = 60$. The range of times are daily $t = 0, 1, \dots, 60$. Furthermore, we fix $C = 18$ throughout. The further numerical analysis was performed using Matlab.

In the first case study, we analyse numerically the impact of hedging a CDD temperature risk exposure for cutoff temperatures different than $C = 18$. That is, let us fix $\eta_1 = \tau_1$ and $\eta_2 = \tau_2$, and consider $\widetilde{C} = 20$. In Figure 1 we have plotted the hedge together with the temperature dynamics.

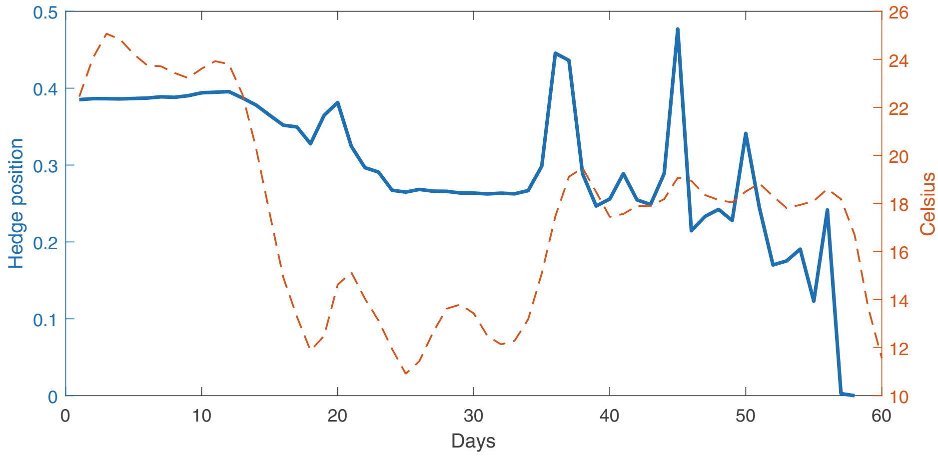

FIGURE 1 Hedge in CDD-futures (blue) when cutoff is $\widetilde{C} = 20$, and temperature dynamics (dashed red).

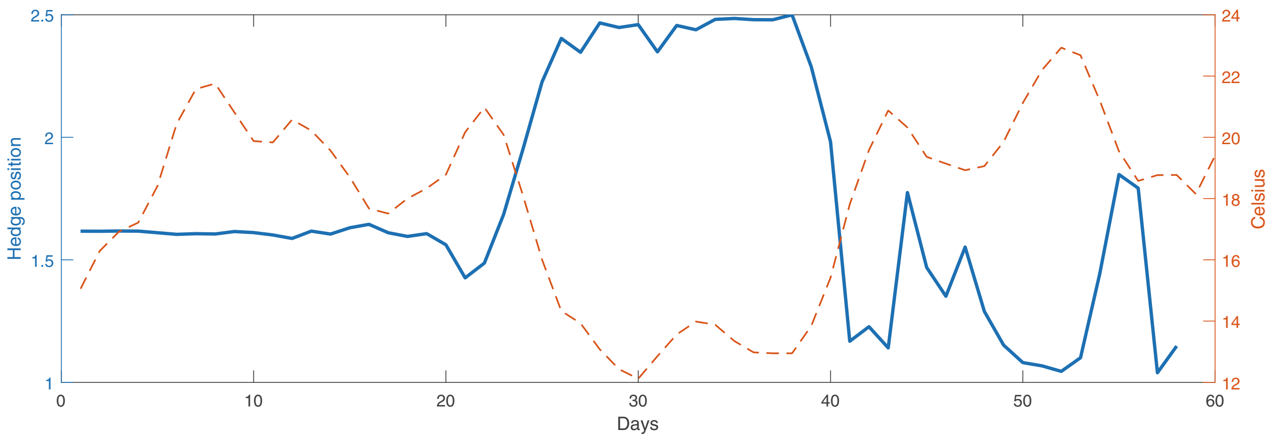

FIGURE 2 Hedge in CDD-futures (blue) when cutoff is $\widetilde{C} = 16$, and temperature dynamics (dashed red).

Obviously, a temperature exposure with cutoff $\widetilde{C} = 20$ is less risky (i.e., less likely to have non-zero value) than $C = 18$, which is reflected in the hedge being less than 1. In fact, we shall have a position less than 0.5 in the CDD-futures up to end of measurement at day 60. We also observe a downward trending position in the CDD-futures with an overall downward trending temperature towards the start of the measurement period on day 30. This is in line with the risk perception, as temperatures down to 12–14°C makes the exposure very low and we need little coverage in the CDD-futures. However, at around day 35 the temperature increases rather abruptly to around 19°C, which is followed by a strong increase in the hedge position from around 0.3 to approximately 0.45. As we move towards the end of the measurement period we clearly see that the hedge position becomes increasingly more sensitive to the temperature variations, which is natural as we have a contract with gradually shorter measurement period left and the impact of temperature is less smoothed out. Towards to end the hedge drops to zero, reflecting a temperature decreasing downwards and making it highly unlikely that there will be any risk that it may bounce back to a level above 20°C.

In Figure 2 we depict the hedge position when the cutoff in the risk exposure is $\widetilde{C} = 16$°C, which is a more risky position than the payoff from the CDD-futures. We have simulated a new temperature path. Now we see that we should be long more than one CDD-contract. Interestingly, we scale up to to about 2.5 CDD-futures slightly before the start of the measurement period when the temperature goes down. This reflects that although the probability for positive payoff in both the CDD-futures is reduced, it is more so for the risk exposure, and thus we must increase our position to compensate. From about day 40 the temperature goes up, and we can reduce our position in the hedge.

We have also explored the case when $\eta_1 = 45$, that is, the temperature risk exposure is only covering the second half of the CDD-futures measurement period. In Figure 3 we have plotted the hedging position when $\widetilde{C} = 16$°C. Since the period is shorter in the risk exposure, the hedge is stable and close to zero until around day 30 when it increases

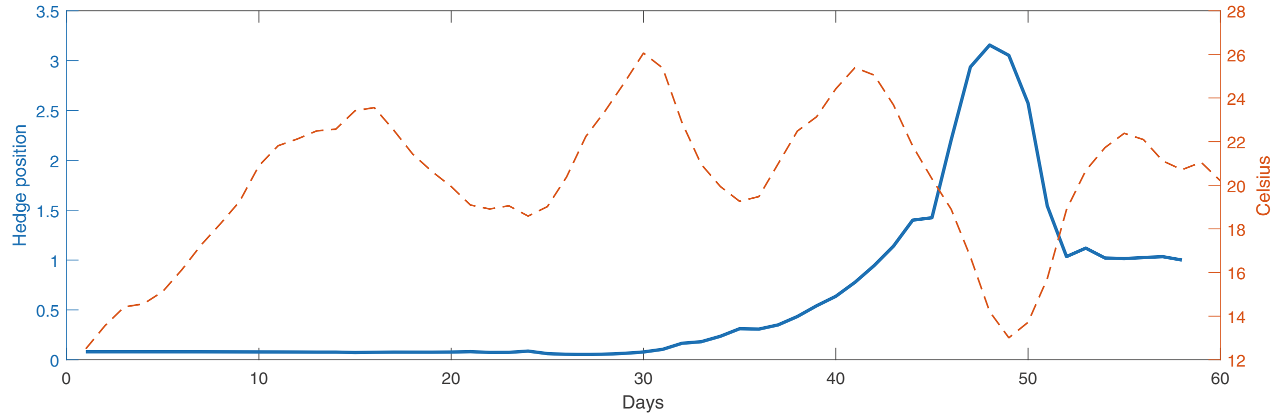

FIGURE 3 Hedge in CDD-futures (blue) when cutoff is $\widetilde{C} = 16$ and $\eta_1 = 45$, and temperature dynamics (dashed red).

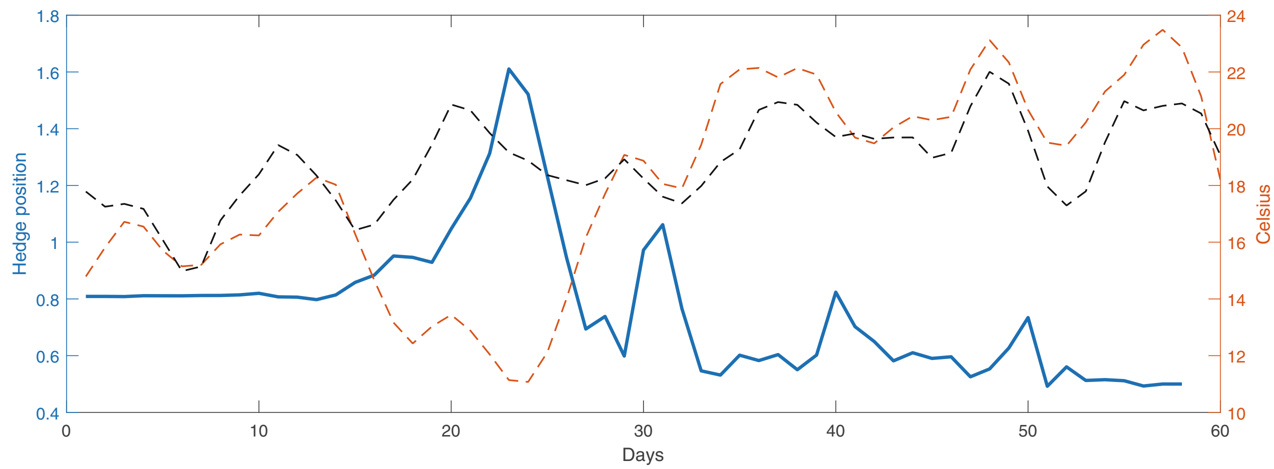

FIGURE 4 Hedge in CDD-futures (blue) when cutoff is $\widetilde{C} = 16$, and correlated temperature dynamics (dashed red is temperature in location 1, dashed black in location 2).

and crosses one at around day 42. The temperatures are well above 16°C in this period (except in the very beginning). From day 45 the temperature drops significantly, which forces the hedge to take an even higher long position of above three. We are now in the measurement period of the temperature exposure. The temperature increases and stabilises above 20 in the last week of the considered time span, where the hedge reduces its long position down towards one.

We end our numerical case studies with an investigation of the spatial hedge, where we have exposure in one location and can trade in another. We assume that the CAR(3)-characteristics are equal in the temperature dynamics of the two locations, except that the noises are correlated by $\rho = 0.5$. In Figure 4 we have depicted the hedge position in the tradeable contract where we suppose that $\widetilde{C} = 16$ in the temperature risk exposure. As is evident from the graphs, the two temperatures are following each other rather closely, except from a period around 15 to 30 days where they move in opposite directions. The temperature in location 1, the location where there is a CDD-futures to hedge in, drops whereas the temperature where we are exposed to risk tends upwards. The hedge makes then a large upward adjustment, where it nearly doubles from its level 0.8. When entering the measurement period, the temperatures in the two locations starts to move in close resemblance, and the hedge is stabilising to a low level tending towards 0.5.

6 | CONCLUDING REMARKS

We have derived analytical expressions for (partly) replicating temperature risk exposure of CDD and HDD-type. In particular, we have shown how one can use traded CDD (or HDD)-futures to hedge different cutoff temperatures and measurement periods. The hedges prescribe positions in the tradeable futures being a ratio of volatility term structures, which again depends on the current path characteristics of the temperature. We have also seen how one can hedge temperature risk exposures in different locations than the traded futures are settled, where we find that the hedges are proportional to the correlation coefficient of the temperature dynamics in the two locations.

A temperature risk exposure may be more complex than a simple CDD or HDD-like index. One could imagine linear combinations of such, or the desire to enter specific combinations to offset just parts of the risk exposure. This could be accommodated by our theoretical results by looking at linear combinations of hedges, where each part refers to a specific CDD or HDD-index.

The marketplace offers several futures contracts for trade, and a relevant extension of our analysis is to see how one could use many futures in combination to hedge. This is particularly relevant when the measurement periods are not coinciding with the traded ones, as for example we saw in Subsection 4.1 where the payoff is split into two. In a spatial hedging context, one could potentially bring in more locations where traded futures are available, and play on the different correlations with the location where there is temperature risk exposure.

Liquidity might be low for some traded futures, especially when the measurement period is far into the future. Thus, a continuously updated hedge must be approximated by a piecewise one, where updates may happen at a set of discrete times (possibly random). To find the best such hedge is a problem which is interesting, and left for the future. A simple crude approximation could be to use discrete updating based on the value of the continuous hedge at the times in question.

Other weather-dependent futures could also be analysed if one finds tradeable instruments in the market. One such is for example wind futures in Germany, traded at the EEX (see Benth and Pircalabu[18] and Benth, Christensen and Rohde[19] for pricing and hedging spatial wind risk with such wind futures). Furthermore, our approach could also have applications in constructing synthetic option contracts. Since the cutoff temperature in CDD and HDD futures is analogous to the strike price in call and put options, we could perform a similar analysis as provided in this paper to construct synthetic call and put options having prescribed strikes, in terms of portfolios of traded call and put options. This could be interesting in markets where certain options (i.e., strikes) have very low liquidity, or the desired strike is not available for trade. We note that such a study would require different stochastic models for the underlying, as well as a simplification as no time period is involved except for Asian-type options.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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