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INTRINSIC GEOMETRY AND BOUNDARY STRUCTURE OF PLANE DOMAINS

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Abstract. For a non-empty compact set E in a proper subdomain Ω of the complex plane, we denote the diameter of E and the distance from E to the boundary of Ω by $d(E)$ and $d(E,\partial\Omega)$, respectively. The quantity $d(E)/d(E,\partial\Omega)$ is invariant under similarities and plays an important role in Geometric Function Theory. In the present paper, when Ω has the hyperbolic distance $h_\Omega(z,w)$, we consider the infimum $\kappa(\Omega)$ of the quantity $h_\Omega(E)/\log(1+d(E)/d(E,\partial\Omega))$ over compact subsets E of Ω with at least two points, where $h_\Omega(E)$ stands for the hyperbolic diameter of the set E. We denote the upper half-plane by $\mathbb{H}$. Our main results claim that $\kappa(\Omega)$ is positive if and only if the boundary of Ω is uniformly perfect and that the inequality $\kappa(\Omega)\le\kappa(\mathbb{H})$ holds for all Ω, where equality holds precisely when Ω is convex.

1. Introduction

Let Ω be a domain in the complex plane $\mathbb{C}$ with the hyperbolic metric $\rho_\Omega(z)|dz|$ of Gaussian curvature -1 [1]. The celebrated Uniformization Theorem [2, p. 81] guarantees the existence of ρ_Ω for a domain Ω when its boundary $\partial\Omega$ contains at least three points. Such a domain is called hyperbolic. Here and in what follows, the boundary of a domain is taken with respect to the Riemann sphere $\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$.

The function $\rho_\Omega(z)$ is sometimes called the *hyperbolic density* of Ω. For instance, for the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and the upper half-plane $\mathbb{H}=\{z\in\mathbb{C}:\operatorname{Im} z>0\}$, the hyperbolic densities are given by $\rho_{\mathbb{D}}(z)=2/(1-|z|^2)$ and $\rho_{\mathbb{H}}(z)=1/\operatorname{Im} z$, respectively. Let $h_\Omega(z_1,z_2)$ denote the *hyperbolic distance* induced by $\rho_\Omega(z)|dz|$ and $d(z,\partial\Omega)$ the Euclidean distance from a point $z\in\Omega$ to the boundary $\partial\Omega$. Then we have the inequality $\rho_\Omega(z)\le 2/d(z,\partial\Omega)$ for each $z\in\Omega$ as a simple consequence of Schwarz' Lemma [3, (2.1)]. On the other hand, the inequality $\rho_\Omega(z)\ge 1/(2d(z,\partial\Omega))$ holds for a simply connected domain Ω [3, (2.2)], [1, p. 35 Thm 8.6], [4, p.34, (3.2.1)].

The distance on Ω induced by the continuous Riemannian metric $|dz|/d(z,\partial\Omega)$ is called the *quasihyperbolic distance* and denoted by $k_\Omega(z_1,z_2)$ [5]. We now have the inequality $h_\Omega(z_1,z_2)\le 2k_\Omega(z_1,z_2)$ for a general domain Ω and $h_\Omega(z_1,z_2)\ge k_\Omega(z_1,z_2)/2$ for a simply connected domain Ω. These two inequalities are very handy, because there are many

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estimates for quasihyperbolic distances whereas hyperbolic distances are not easy to estimate because the density function $\rho_\Omega(z)$ depends on the local boundary structure in the vicinity of z in a subtle manner [3], [2, p.241, Thm 14.5.2], [6]. It should be noticed that the second estimate does not apply to general domains, because the hyperbolic distance is not bounded from below by a constant multiple of the quasihyperbolic distance, for instance, if the domain has isolated boundary points. To measure the similarity between h_Ω and k_Ω, the domain functional [7]

$$c(\Omega) = \inf_{z\in\Omega} \rho_\Omega(z)d(z,\partial\Omega) = \inf_{z_1,z_2\in\Omega,\ z_1\neq z_2} \frac{h_\Omega(z_1,z_2)}{k_\Omega(z_1,z_2)} \tag{1.1}$$

is useful, where the second equality will be proven in the next section. By the above observations, we have $c(\Omega) \le 2$ for a general domain Ω and $c(\Omega) \ge 1/2$ for a simply connected domain Ω. But more is known about this domain constant.

Theorem A. *Let Ω be a hyperbolic domain in $\mathbb{C}$. Then $c(\Omega) \le 1$ with equality if and only if Ω is convex. Furthermore, $c(\Omega) > 0$ if and only if $\partial\Omega$ is uniformly perfect.*

The general inequality $c(\Omega) \le 1$ is due to Harmelin and Minda [7] and the equality condition is due to Mejía and Minda [8]. The last assertion is due to Beardon and Pommerenke [3]. Here, a closed set E in $\widehat{\mathbb{C}}$ with $\operatorname{card}(E) \ge 2$ is said to be *uniformly perfect* if there is a constant $0 < \alpha < 1$ such that the closed annulus $\alpha r \le |z-a| \le r$ meets E whenever $a \in E$ and $0 < r < d(E)$. Here and hereafter, $\operatorname{card}(E)$ denotes the cardinality of the set E and $d(E)$ is the Euclidean diameter of E. In other words, $d(E) = \sup_{z,w\in E}|z-w|$. We set $d(E) = +\infty$ when $\infty \in E$. For uniformly perfect sets, we refer to [9], [10, pp. 343-345], [11], [12], [13] and [14]. Uniform perfectness has many applications in potential theory, metric spaces, Kleinian groups and complex dynamics as well as geometric function theory; see, in addition to the above references, for instance [9], [15], [16] and [17].

In their work about the quasihyperbolic metric, Gehring and Palka [5] also introduced the *distance-ratio metric*

$$j_\Omega(z_1,z_2) = \log\left(1 + \frac{|z_1-z_2|}{\min\{d(z_1,\partial\Omega), d(z_2,\partial\Omega)\}}\right)$$

for $z_1, z_2 \in \Omega$, see also [18, p.61]. They proved that $j_\Omega(z_1,z_2) \le k_\Omega(z_1,z_2)$ holds always. It is also known that j_Ω satisfies the triangle inequality on Ω [18, p.59, Lemma 4.6]. The opposite inequality characterises so called uniform domains: a domain Ω is *uniform* if and only if there exists a constant $b > 0$ such that the inequality

$$k_\Omega(z_1,z_2) \le b j_\Omega(z_1,z_2)$$

holds, see Gehring and Osgood [19] and [18, p.84]. These domains are ubiquitous in geometric function theory [4].

It is a natural and interesting question to ask what can be said if we replace k_Ω by h_Ω. Our answer is the following result.

Theorem 1.2. *Let Ω be a hyperbolic domain in $\mathbb{C}$. There is a constant $c > 0$ such that $cj_\Omega(z_1, z_2) \le h_\Omega(z_1, z_2)$ for all $z_1, z_2 \in \Omega$ if and only if the boundary of Ω in $\widehat{\mathbb{C}}$ is uniformly perfect.*

In conjunction with the Gehring-Osgood theorem [19, pp.59-60], we have the following result.

Corollary 1.3. *Let Ω be a hyperbolic domain in $\mathbb{C}$. Then the hyperbolic metric h_Ω is comparable with the distance-ratio metric j_Ω if and only if Ω is uniform and has uniformly perfect boundary.*

Indeed, if for some constants $0 < c_1 \le c_2$,

$$c_1 j_\Omega(z_1, z_2) \le h_\Omega(z_1, z_2) \le c_2 j_\Omega(z_1, z_2), \quad \text{for } z_1, z_2 \in \Omega,$$

we first see that $\partial\Omega$ is uniformly perfect. Then h_Ω is comparable with k_Ω by Theorem A. We now conclude that Ω is uniform by the Gehring-Osgood theorem. The converse follows readily from Theorem 1.2 and the Gehring-Osgood theorem.

For a subset E of Ω with $\operatorname{card}(E) \ge 2$, we define the set functionals

$$h_\Omega(E) = \sup_{z_1, z_2 \in E} h_\Omega(z_1, z_2) \quad \text{and} \quad J_\Omega(E) = \log\left(1 + \frac{d(E)}{d(E, \partial\Omega)}\right).$$

Here and hereafter, $d(E, F)$ denotes the Euclidean distance between the sets E and F. For a singleton $E = \{z\}$, we write $d(\{z\}, F) = d(z, F) = d(F, z)$. We will use the following monotonicity property frequently in the sequel: $h_\Omega(E) \le h_\Omega(E')$ and $J_\Omega(E) \le J_\Omega(E')$ for $E \subset E' \subset \Omega$. We note that $h_\Omega(E)$ is the hyperbolic diameter of E in Ω and that $J_\Omega(E)$ is important in connection with capacity estimates of E (see, for instance, [20]). We now consider the domain constant

$$\kappa(\Omega) = \inf_E \frac{h_\Omega(E)}{J_\Omega(E)},$$

where E runs over all compact subsets of Ω with $\operatorname{card}(E) \ge 2$. As the following result tells, the two domain constants $c(\Omega)$ and $\kappa(\Omega)$ are comparable.

Theorem 1.4. *Let Ω be a hyperbolic domain in $\mathbb{C}$. Then the double inequality*

$$\frac{c(\Omega)}{2} \le \kappa(\Omega) \le c(\Omega)$$

holds. In particular, $\kappa(\Omega) > 0$ if and only if $\partial\Omega$ is uniformly perfect.

It is a little surprising that the quantity $\kappa(\Omega)$ behaves like $c(\Omega)$ in the following sense (compare with Theorem A).

Theorem 1.5. *Let Ω be a hyperbolic domain in $\mathbb{C}$. Then, the inequality $\kappa(\Omega) \le \kappa(\mathbb{H})$ holds, where equality holds if and only if Ω is convex.*

In view of the above theorem, we are curious about the value of $\kappa(\mathbb{H})$. However, it seems difficult to evaluate it in a simple form. Since $c(\mathbb{H}) = 1$, the first part of Theorem 1.4 implies $1/2 \le \kappa(\mathbb{H}) \le 1$. We will prove later that the inequality $\kappa(\mathbb{H}) < 1$ holds and give a numerical approximation of the value of $\kappa(\mathbb{H})$ in Theorem 4.18, thus answering a problem formulated in [18, p.455, item (12)].

The existence of an extremal configuration of the set E for the functional $h_\Omega(E)/J_\Omega(E)$ is more subtle. We will prove the following result in the final section. We note that a convex domain in $\mathbb{C}$ carries the hyperbolic metric unless it is $\mathbb{C}$ itself.

Theorem 1.6. *Let Ω be a convex proper subdomain of $\mathbb{C}$. There exists a compact subset E in Ω satisfying $\kappa(\Omega) = h_\Omega(E)/J_\Omega(E)$ if and only if Ω is a half-plane.*

When Ω is the upper half-plane $\mathbb{H}$, there exists a three-point set E^* of the form $\{i, z_1, z_2\}$ constituting a hyperbolic equilateral triangle with $\kappa(\mathbb{H}) = h_\mathbb{H}(E^*)/J_\mathbb{H}(E^*)$, $1 < \operatorname{Im} z_j$ $(j = 1, 2)$ and $z_1 = -\overline{z_2}$. Moreover, such an extremal three-point set is unique up to similarities keeping $\mathbb{H}$ invariant.

In view of the application given in the final section, it is important to have a lower bound of $\kappa(\Omega)$ when Ω is simply connected. We consider the number

$$\kappa_0 = \inf_{\Omega} \kappa(\Omega), \tag{1.7}$$

where Ω runs over all simply connected proper subdomains of $\mathbb{C}$. By Theorem 1.4 and the well-known estimate $c(\Omega) \ge 1/2$, we obtain $\kappa_0 \ge 1/4$. On the other hand, when Ω is the slit domain $\Omega_0 = \mathbb{C} \setminus (-\infty, 0]$, numerically we have $\kappa(\Omega_0) \le h_{\Omega_0}(E)/J_{\Omega_0}(E) = 0.4251604\ldots$ for $E = \{w_0, w_1, w_2\}$, $w_0 = 1$, $w_1 = 2.121820474 + 1.198476681i$, $w_2 = \bar{w}_1$. Note that $h_{\Omega_0}(w_0, w_1) = h_{\Omega_0}(w_0, w_2) \approx h_{\Omega_0}(w_1, w_2)$. Thus, we have the following corollary.

Corollary 1.8. $1/4 \le \kappa_0 < 0.4251605$.

It is an open problem to determine the value κ_0.

The organization of this paper is as follows. In Section 2, preliminary results cocerning the domain constant $\kappa(\Omega)$ are given and Theorems 1.2 and 1.4 are proved. Section 3 is devoted to the proof of Theorem 1.5. We determine extremal configurations of three-point sets E with respect to the set functional $h_\mathbb{H}(E)/J_\mathbb{H}(E)$ and prove Theorem 1.6 in Section 4. We also give numerical observations on the quantity $\kappa(\mathbb{H})$. We will apply our results to lower estimation of the capacity of a condenser in the final section.

2. Preliminaries

In this section, we prove several simple preliminary results. We begin with the proof of the second equality in (1.1). To distinguish the both sides of (1.1), for a while, we write

$$c(\Omega) = \inf_{z \in \Omega} \rho_\Omega(z) d(z, \partial\Omega) \quad \text{and} \quad c'(\Omega) = \inf_{z, w \in \Omega} \frac{h_\Omega(z, w)}{k_\Omega(z, w)}.$$

We will prove that $c(\Omega)=c'(\Omega)$. Since $\rho_\Omega(z)\ge c(\Omega)/d(z,\partial\Omega)$, we easily obtain $h_\Omega(z_1,z_2)\ge c(\Omega)k_\Omega(z_1,z_2)$. Hence, $c'(\Omega)\ge c(\Omega)$. On the other hand, by the formula

$$\lim_{w\to z}\frac{h_\Omega(z,w)}{k_\Omega(z,w)}=\lim_{w\to z}\frac{h_\Omega(z,w)}{|z-w|}\cdot\frac{|z-w|}{k_\Omega(z,w)}=\frac{\rho_\Omega(z)}{1/d(z,\partial\Omega)}=\rho_\Omega(z)d(z,\partial\Omega),$$

we have $c(\Omega)\ge c'(\Omega)$. Thus, we are done.

For the analysis of domain constants, we introduce some variants of the domain constant $\kappa(\Omega)$. First, we replace h_Ω with k_Ω and define the domain constant

$$\hat\kappa(\Omega)=\inf_E\frac{k_\Omega(E)}{\log(1+d(E)/d(E,\partial\Omega))}\,,$$

where the infimum is taken over all compact subsets E of Ω with $\operatorname{card}(E)\ge 2$. Here, $k_\Omega(E)$ denotes the quasihyperbolic diameter of E. We also define the following auxiliary domain constants for integers $n\ge 2$:

$$\kappa_n(\Omega)=\inf_{E\subset\Omega,\operatorname{card}(E)=n}\frac{h_\Omega(E)}{\log(1+d(E)/d(E,\partial\Omega))}$$

and

$$\hat\kappa_n(\Omega)=\inf_{E\subset\Omega,\operatorname{card}(E)=n}\frac{k_\Omega(E)}{\log(1+d(E)/d(E,\partial\Omega))}\,.$$

For $E=\{z_1,\dots,z_n\}$, letting $z_n\to z_{n-1}$, we observe that

$$\kappa_2(\Omega)\ge\kappa_3(\Omega)\ge\cdots\ge\kappa(\Omega)$$

and

$$\hat\kappa_2(\Omega)\ge\hat\kappa_3(\Omega)\ge\cdots\ge\hat\kappa(\Omega)\,.$$

For these domain constants, we have the following results. In particular, we see that $\kappa_n(\Omega)=\kappa(\Omega)$ and $\hat\kappa_n(\Omega)=\hat\kappa(\Omega)$ for every $n\ge 3$.

Lemma 2.1. (i) $\hat\kappa_2(\Omega)\ge 1$.

(ii) $\kappa_3(\Omega)=\kappa(\Omega)$ *and* $\hat\kappa_3(\Omega)=\hat\kappa(\Omega)$.

(iii) $\kappa_2(\Omega)\le 2\kappa_3(\Omega)$ *and* $\hat\kappa_2(\Omega)\le 2\hat\kappa_3(\Omega)$.

Proof. Part (i) is clear from the Gehring-Palka inequality $k_\Omega(z_1,z_2)\ge j_\Omega(z_1,z_2)$. Let E be an arbitrary compact set in Ω with $\operatorname{card}(E)\ge 2$. Take $z_0,z_1,z_2\in E$ so that $d(E)=|z_1-z_2|$ and $d(E,\partial\Omega)=d(z_0,\partial\Omega)$ and let $E_0=\{z_0,z_1,z_2\}$. (Note that one of the points z_1,z_2 may be the same as z_0.) Then

$$\begin{aligned}h_\Omega(E)\ge h_\Omega(E_0)&\ge\kappa_3(\Omega)\log(1+d(E_0)/d(E_0,\partial\Omega))\\&=\kappa_3(\Omega)\log(1+|z_1-z_2|/d(z_0,\partial\Omega))\\&=\kappa_3(\Omega)\log(1+d(E)/d(E,\partial\Omega)).\end{aligned}\tag{2.2}$$

Taking the infimum over compact subsets E of Ω, we obtain the inequality $\kappa(\Omega) \ge \kappa_3(\Omega)$. Since $\kappa(\Omega) \le \kappa_3(\Omega)$ as we noted above, we conclude $\kappa(\Omega) = \kappa_3(\Omega)$. In the same way, we can verify $\hat{\kappa}(\Omega) = \hat{\kappa}_3(\Omega)$.

Finally, we prove part (iii). Let $E \subset \Omega$ with $\operatorname{card}(E) = 3$ and choose $z_0 \in E$ so that $d(E, \partial\Omega) = d(z_0, \partial\Omega)$. Also choose $z_1, z_2 \in E$ so that $d(E) = |z_1 - z_2|$. Then

$$\begin{aligned}
&\log(1 + d(E)/d(E, \partial\Omega)) \\
&= \log(1 + |z_1 - z_2|/d(z_0, \partial\Omega)) \\
&\le \log(1 + (|z_1 - z_0| + |z_2 - z_0|)/d(z_0, \partial\Omega)) \\
&\le \log(1 + |z_1 - z_0|/d(z_0, \partial\Omega)) + \log(1 + |z_2 - z_0|/d(z_0, \partial\Omega)) \\
&\le \kappa_2(\Omega)^{-1}(h_\Omega(z_1, z_0) + h_\Omega(z_2, z_0)) \\
&\le 2h_\Omega(E)/\kappa_2(\Omega),
\end{aligned}$$

which implies $\kappa_2(\Omega) \le 2\kappa_3(\Omega)$. In the same way, we can prove the other inequality. □

We need also the following simple lemma.

Lemma 2.3. *For a hyperbolic domain Ω in $\mathbb{C}$, the inequality $\kappa_2(\Omega) \le c(\Omega)$ holds.*

Proof. Noting the formula

$$\lim_{w \to z} \frac{h_\Omega(z, w)}{j_\Omega(z, w)} = \rho_\Omega(z) d(z, \partial\Omega),$$

we have

$$\kappa_2(\Omega) = \inf_{z \ne w} \frac{h_\Omega(z, w)}{j_\Omega(z, w)} \le \inf_{z \ne w} \rho_\Omega(z) d(z, \partial\Omega) = c(\Omega).$$

□

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. By the above lemma and the inequality $h_\Omega(x, y) \ge c(\Omega)k_\Omega(x, y)$, for an arbitrary compact set E in Ω, we have

$$\begin{aligned}
\frac{h_\Omega(E)}{\log(1 + d(E)/d(E, \partial\Omega))} &\ge \frac{c(\Omega)k_\Omega(E)}{\log(1 + d(E)/d(E, \partial\Omega))} \\
&\ge c(\Omega)\hat{\kappa}(\Omega) \ge \frac{c(\Omega)}{2}\hat{\kappa}_2(\Omega) \ge \frac{c(\Omega)}{2}.
\end{aligned}$$

Hence we have $\kappa(\Omega) \ge c(\Omega)/2$. The other inequality follows from Lemma 2.3:

$$\kappa(\Omega) \le \kappa_2(\Omega) \le c(\Omega).$$

□

We now prove Theorem 1.2.

Proof of Theorem 1.2. Assume that $c\, j_\Omega(z_1,z_2) \le h_\Omega(z_1,z_2)$ for $z_1,z_2\in\Omega$. Then $\kappa_2(\Omega)\ge c$. By Lemma 2.1 and Theorem 1.4, we obtain

$$c(\Omega)\ge\kappa(\Omega)\ge\frac{1}{2}\kappa_2(\Omega)\ge\frac{c}{2}>0.$$

Thus, $\partial\Omega$ is uniformly perfect. Conversely, if $\partial\Omega$ is uniformly perfect, similarly we obtain $\kappa_2(\Omega)\ge\kappa(\Omega)\ge c(\Omega)/2>0$. Thus, $c\, j_\Omega(z_1,z_2)\le h_\Omega(z_1,z_2)$ holds with $c=\kappa_2(\Omega)>0$. □

3. Proof of Theorem 1.5

In this section, we will prove Theorem 1.5 step by step. We begin with the following result.

Lemma 3.1. *For any hyperbolic domain Ω in $\mathbb{C}$, the inequality $\kappa(\Omega)\le\kappa(\mathbb{D})$ holds.*

Proof. By definition, for a given $\varepsilon>0$, there is a compact subset E of $\mathbb{D}$ such that

$$\frac{h_{\mathbb{D}}(E)}{J_{\mathbb{D}}(E)}<\kappa(\mathbb{D})+\varepsilon.$$

Moreover, by rotating E if necessary, we may further assume that the nearest point of the boundary $\partial\mathbb{D}$ to E is 1. Namely, $d(E,\partial\mathbb{D})=d(E,1)$.

Let Ω be an arbitrary hyperbolic domain in $\mathbb{C}$. For an arbitrarily fixed point $z_0\in\Omega$, choose $\zeta_0\in\partial\Omega$ so that $d(z_0,\partial\Omega)=|z_0-\zeta_0|$. Since $\kappa(\Omega)$ is invariant under similarities, we may assume that $z_0=0$ and $\zeta_0=1$. Then $\mathbb{D}\subset\Omega$. By the domain monotonicity of the hyperbolic metric, we have $h_\Omega(E)\le h_{\mathbb{D}}(E)$. On the other hand, we have $d(E,\partial\Omega)=d(E,1)=d(E,\partial\mathbb{D})$ so that $J_\Omega(E)=J_{\mathbb{D}}(E)$. Hence,

$$\kappa(\mathbb{D})+\varepsilon>\frac{h_{\mathbb{D}}(E)}{J_{\mathbb{D}}(E)}\ge\frac{h_\Omega(E)}{J_\Omega(E)}\ge\kappa(\Omega).$$

Since $\varepsilon>0$ is arbitrary, we obtain the required inequality $\kappa(\mathbb{D})\ge\kappa(\Omega)$. □

Remark 3.2. Note that the set functional $J_D(E)$ in the above proof is not the same thing as the diameter of E in the j_D metric

$$j_D(E)=\sup\{j_D(x,y):x,y\in E\}\,.$$

It is easy to see that the inequality

$$J_D(E)/2\le j_D(E)\le J_D(E)$$

holds for all $E\subset D$, with equality in the second inequality if E is a disk, card $(E)=2$, or card $(E)=3$ and the triangle with vertices E is either equilateral or a so-called Reuleaux triangle.

Moreover, for a half-plane, we have the following result.

Lemma 3.3. *Let H be an open half-plane in $\mathbb{C}$. Then $\kappa(\mathbb{D})=\kappa(H)$.*

Proof. By Lemma 3.1, it is enough to prove the inequality $\kappa(H) \ge \kappa(\mathbb{D})$. We choose the right half-plane $\{z : \operatorname{Re} z > 0\}$ as H. For every $\varepsilon > 0$, we can find a compact subset E of H such that

$$\frac{h_H(E)}{J_H(E)} < \kappa(H) + \varepsilon.$$

Let ζ_0 be the nearest boundary point to E. For simplicity, we assume that $\zeta_0 = 0$. For $R > 0$, we denote the disk $\{z : |z - R| < R\}$ by Δ_R. For a large enough R, $E \subset \Delta_R$ and $d(E, \partial\Delta_R) = d(E, 0) = d(E, \partial H)$ so that $J_H(E) = J_{\Delta_R}(E)$. On the other hand, since

$$\rho_{\Delta_R}(z) = \frac{2R}{R^2 - |z-R|^2} = \frac{1}{\operatorname{Re} z - |z|^2/(2R)} \to \frac{1}{\operatorname{Re} z} = \rho_H(z)$$

locally uniformly on $\mathbb{H}$, we obtain $h_{\Delta_R}(E) \to h_H(E)$ as $R \to +\infty$. Noting the inequality

$$h_{\Delta_R}(E)/J_{\Delta_R}(E) \ge \kappa(\Delta_R) = \kappa(\mathbb{D}),$$

we have

$$\frac{h_H(E)}{J_H(E)} = \lim_{R\to+\infty} \frac{h_{\Delta_R}(E)}{J_{\Delta_R}(E)} \ge \kappa(\mathbb{D}).$$

Hence, $\kappa(H) + \varepsilon > \kappa(\mathbb{D})$. Since $\varepsilon > 0$ was arbitrary, we obtain the inequality $\kappa(H) \ge \kappa(\mathbb{D})$ as required. □

We next prove the following lemma.

Lemma 3.4. *Let Ω be a convex domain in $\mathbb{C}$ with $\Omega \ne \mathbb{C}$. Then $\kappa(\Omega) = \kappa(\mathbb{D})$.*

Proof. Let E be any compact subset of Ω. Take $\zeta_0 \in \partial\Omega$ so that $d(E, \partial\Omega) = d(E, \zeta_0)$. Since Ω is convex, there is a supporting line, say, L at the point ζ_0. Let H be the connected component of $\mathbb{C} \setminus L$ containing Ω. Then $\Omega \subset H$ and $\zeta_0 \in \partial H = L$. Since $d(E, \partial H) = d(E, \zeta_0) = d(E, \partial\Omega)$, we obtain

$$\frac{h_\Omega(E)}{J_\Omega(E)} \ge \frac{h_H(E)}{J_H(E)} \ge \kappa(H) = \kappa(\mathbb{D}).$$

Here, we used Lemma 3.3. Taking the infimum over E, we obtain the inequality $\kappa(\Omega) \ge \kappa(\mathbb{D})$. Recalling Lemma 3.1, we have the desired relation. □

To deduce the equality condition is the most subtle part in the proof of Theorem 1.5. A key ingredient is Keogh's lemma about non-convex domains. See Figure 1.

Lemma 3.5 (Keogh [21])**.** *Suppose that a domain Ω in $\mathbb{C}$ is not convex. Then there are two open disks Δ_1 and Δ_2 whose boundaries intersect perpendicularly such that $G = \Delta_1 \setminus \overline{\Delta_2}$ is contained in Ω and the midpoint ζ_0 of the concave boundary arc $\Delta_1 \cap \partial\Delta_2$ of G lies on the boundary $\partial\Omega$ of Ω.*

We are now ready to prove the following result, which is the last piece of the proof of Theorem 1.5.

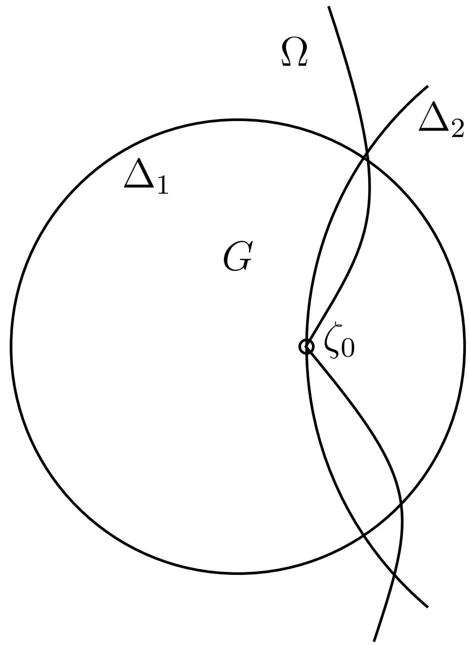

FIGURE 1. The domain $G = \Delta_1 \backslash \overline{\Delta_2}$ in Ω

Lemma 3.6. *Let Ω be a non-convex domain in $\mathbb{C}$. Then $\kappa(\Omega) < \kappa(\mathbb{D})$.*

Proof. We find open disks Δ_1, Δ_2 as in Keogh's lemma so that $G = \Delta_1 \setminus \overline{\Delta_2} \subset \Omega$ and the midpoint ζ_0 of the concave boundary arc of G is contained in $\partial\Omega$. We may assume that $\Delta_1 = \mathbb{D}$ and $\zeta_0 = a \in (0, 1)$ so that the center of Δ_2 lies on the real axis. Then the second disk Δ_2 is the image of the right half-plane H under the Möbius transformation

$$T(z) = \frac{z + a}{1 + az}.$$

Thus, $G = T(\mathbb{D}_-)$, where $\mathbb{D}_-$ is the left half $\{z \in \mathbb{D} : \operatorname{Re} z < 0\}$ of the unit disk. We now construct a conformal map f of the upper half-plane $\mathbb{H}$ onto G as follows. We denote the analytic automorphism $(1 + z/2)/(1 - z/2)$ of $\mathbb{H}$ by M. Note that M maps the positive imaginary axis $i\,\mathbb{R}_+ = \{iy : 0 < y < +\infty\}$ onto the upper half of the unit circle $|\zeta| = 1$. The function $S(\zeta) = \sqrt{\zeta}$ maps $\mathbb{H}$ onto the first quadrant $D = \{w : \operatorname{Re} w > 0, \operatorname{Im} w > 0\}$. Then the Möbius transformation $L(w) = i(w - 1)/(w + 1)$ maps D onto the left half $\mathbb{D}_-$ of $\mathbb{D}$. Hence, the function $f = T \circ L \circ S \circ M$ maps $\mathbb{H}$ onto G in such a way that $f(i\,\mathbb{R}_+) = (-1, a)$. More concretely, f is expressed by

$$f(z) = T\left(i \frac{\sqrt{1 + z/2} - \sqrt{1 - z/2}}{\sqrt{1 + z/2} + \sqrt{1 - z/2}} \right).$$

In view of this form, we see that $f(z)$ is analytic on $|z| < 1$. (This follows also from the Schwarz reflection principle.) Therefore, we can expand $f(z)$ about $z = 0$ as follows:

$$f(z) = a + a_1 z + a_2 z^2 + \cdots \quad (|z| < 1).$$

By a straightforward computation, we have here

$$a_1 = \frac{i}{4}(1 - a^2), \quad a_2 = \frac{a}{16}(1 - a^2)$$

and therefore

$$A := \frac{a_2}{a_1} = \frac{a}{4i}. \tag{3.7}$$

Let $E_x := xE^* = \{xz_j : j = 0, 1, 2\}$ for $0 < x < 1$, where $E^* = \{z_0, z_1, z_2\} \subset \mathbb{H}$ with $z_0 = i$ is the set in Theorem 1.6 and thus $\kappa(\mathbb{H}) = h_{\mathbb{H}}(E^*)/J_{\mathbb{H}}(E^*)$. Let $w_j = f(xz_j)$ and set $E'_x = f(E_x) = \{w_j : j = 0, 1, 2\}$. Since $f(xz) = a + a_1 xz + O(x^2)$ as $x \to 0$ locally uniformly in z, $d(E'_x) = |w_1 - w_2|$ and $d(E'_x, \partial G) = d(w_0, \partial G) = d(w_0, \Delta_1 \cap \partial\Delta_2)$ for a small enough $x > 0$. Note here that $w_0 = f(xz_0) = f(ix) \in (0, a)$ because $f(i\,\mathbb{R}_+) = (-1, a)$. Hence, $d(E'_x, \partial G) = d(w_0, \Delta_1 \cap \partial\Delta_2) = d(w_0, a)$. We now look at the quantity

$$F(x) = \frac{d(E'_x)}{d(E'_x, \partial G)} = \frac{|w_1 - w_2|}{|w_0 - a|} = \left|\frac{w_1 - w_2}{w_0 - a}\right|.$$

We observe that

$$W = \frac{w_1 - w_2}{w_0 - a} = \frac{f(xz_1) - f(xz_2)}{f(xz_0) - f(0)}$$

is even analytic in $x \in \mathbb{D}$ and we compute

$$\begin{aligned} W &= \frac{a_1 x(z_1 - z_2) + a_2 x^2 (z_1^2 - z_2^2) + O(x^3)}{a_1 x z_0 + a_2 x^2 z_0^2 + O(x^3)} \\ &= \frac{z_1 - z_2}{z_0} \cdot \frac{1 + Ax(z_1 + z_2) + O(x^2)}{1 + Axz_0 + O(x^2)} \\ &= \frac{z_1 - z_2}{z_0} \cdot \big[1 + Ax(z_1 + z_2 - z_0) + O(x^2)\big], \end{aligned}$$

where $A = a_2/a_1 = a/(4i)$ by (3.7). Hence $F(x) = |W|$ is real analytic in $-1 < x < 1$ and

$$\begin{aligned} F(x) &= \frac{|z_1 - z_2|}{|z_0|}\Big\{1 + \operatorname{Re}\big[Ax(z_1 + z_2 - z_0)\big] + O(x^2)\Big\} \\ &= \frac{|z_1 - z_2|}{|z_0|}\left\{1 + \frac{ax}{4}\operatorname{Im}(z_1 + z_2 - z_0) + O(x^2)\right\} \end{aligned}$$

as $x \to 0$. Since $\operatorname{Im} z_j = d(z_j, \partial\mathbb{H}) > d(z_0, \partial\mathbb{H})$ for $j = 1, 2$, we have

$$F(0) = \frac{|z_1 - z_2|}{|z_0|} = \frac{d(E^*)}{d(E^*, \partial\mathbb{H})} \quad \text{and} \quad F'(0) = \frac{a|z_1 - z_2|}{4|z_0|}\operatorname{Im}(z_1 + z_2 - z_0) > 0.$$

In particular, $F(x)$ is strictly increasing at $x = 0$ and thus $F(x) > F(0)$ for small enough $x > 0$. Since $G \subset \Omega$, we have the inequality $h_\Omega(E'_x) \le h_G(E'_x)$. We also note that

$$d(E'_x, \partial\Omega) \ge d(E'_x, \partial G) = d(w_0, a) \ge d(E'_x, \partial\Omega),$$

because $a \in \partial\Omega$, and therefore $d(E'_x, \partial\Omega) = d(E'_x, \partial G)$ so that $J_\Omega(E'_x) = J_G(E'_x)$. Moreover, since the hyperbolic distance is conformally invariant, $h_G(E'_x) = h_G(f(E_x)) = h_{\mathbb{H}}(E_x) = h_{\mathbb{H}}(E^*)$. Hence, for a small enough $x > 0$,

$$\begin{aligned}\kappa(\Omega) \le \frac{h_\Omega(E'_x)}{J_\Omega(E'_x)} \le \frac{h_G(E'_x)}{J_G(E'_x)} &= \frac{h_{\mathbb{H}}(E^*)}{\log(1 + F(x))} \\ &< \frac{h_{\mathbb{H}}(E^*)}{\log(1 + F(0))} = \frac{h_{\mathbb{H}}(E^*)}{J_{\mathbb{H}}(E^*)} = \kappa(\mathbb{H}).\end{aligned}$$

The proof is finished. □

Now Theorem 1.5 follows from Lemmas 3.1, 3.4 and 3.6.

4. Extremal configuration of three points in $\mathbb{H}$

In this section, we work to find extremal configurations of three-point sets E in the upper half-plane for the functional $h_{\mathbb{H}}(E)/J_{\mathbb{H}}(E)$. Since the both quantities $h_{\mathbb{H}}(E)$ and $J_{\mathbb{H}}(E)$ are invariant under the affine mappings of the form $z \mapsto az + b$ with $a > 0$ and $b \in \mathbb{R}$, we may restrict our attention to the family $\mathcal{E}$ of three-point subsets E of $\mathbb{H}$ containing $i = \sqrt{-1}$ with $d(E, \partial\mathbb{H}) = d(i, \partial\mathbb{H}) = 1$. Namely, the infimum in the definition of $\kappa_3(\mathbb{H})$ may be limited to $\mathcal{E}$:

$$\kappa_3(\mathbb{H}) = \inf_{E \in \mathcal{E}} \frac{h_{\mathbb{H}}(E)}{J_{\mathbb{H}}(E)} = \inf_{E \in \mathcal{E}} \frac{h_{\mathbb{H}}(E)}{\log(1 + d(E))}.$$

Our goal in this section is to determine the extremal sets E for which the above infimum is attained, and to compute (at least numerically) the value of $\kappa_3(\mathbb{H})$. First, we note the following fact for the upper half-plane $\mathbb{H}$. Though the result is essentially known (e.g., [18, Lemma 4.9 (2)]), we give a short proof for convenience of the reader.

Lemma 4.1.

$$\kappa_2(\mathbb{H}) = \inf_{z_1, z_2 \in \mathbb{H}} \frac{h_{\mathbb{H}}(z_1, z_2)}{j_{\mathbb{H}}(z_1, z_2)} = 1.$$

Proof. Note that $\rho_{\mathbb{H}}(z) = 1/\operatorname{Re} z = 1/d(z, \partial\mathbb{H})$. Hence, we have $h_{\mathbb{H}}(z, w) = k_{\mathbb{H}}(z, w)$ for $z, w \in \mathbb{H}$. Thus, the inequality $j_{\mathbb{H}}(z, w) \le h_{\mathbb{H}}(z, w)$ is nothing but the Gehring-Palka inequality [5]. Hence, we have $\kappa_2(\mathbb{H}) \ge 1$. On the other hand, by Lemma 2.3, we have $\kappa_2(\mathbb{H}) \le c(\mathbb{H}) \le 1$, where the last inequality follows from Theorem A. □

We will write

$$\Delta(z_0, r) = \{z \in \mathbb{H} : h_{\mathbb{H}}(z, z_0) < r\} = \{z : |z - z_0| < \rho |z - \bar{z}_0|\}$$

for the open hyperbolic disk in $\mathbb{H}$ centered at $z_0 \in \mathbb{H}$ with hyperbolic radius $r > 0$, where $\rho = \tanh(r/2) = (e^r - 1)/(e^r + 1) \in (0, 1)$ and denote its closure by $\overline{\Delta}(z_0, r)$. We need the following elementary fact for the proof of Lemma 4.15, which will be a key result below.

Lemma 4.2. *Let C be the boundary circle of the hyperbolic disk $\Delta(z_0, r)$ in $\mathbb{H}$.*

(i) *The Euclidean distance $|z - z_0|$ between $z \in C$ and z_0 takes its maximum at the top of C and its minimum at the bottom of C.*

(ii) *The Euclidean diameter of the circle C is $2(\operatorname{Im} z_0)\sinh r$.*

(iii) *The hyperbolic distance of the endpoints of an arbitrary diameter of the circle C is at least equal to $\varphi(r)$ given in* (4.3).

Proof. We write $z_0 = x_0 + iy_0$. It is well known (see, e.g., [18, (4.11)]) that the boundary of $\Delta(z_0, r)$ is the Euclidean circle $|z - c| = R$, where

$$c = x_0 + iy_0 \cosh r \quad \text{and} \quad R = y_0 \sinh r.$$

Since $\operatorname{Re} z_0 = \operatorname{Re} c$ and $\operatorname{Im} z_0 < \operatorname{Im} c$, it is evident that $|z - z_0|$ is maximized at $z = c + iR$ and minimized at $z = c - iR$ on C. The proof of the first assertion is now complete. The second assertion is clear because the Euclidean diameter of C is $2R$. It is clear that the diameter of the circle C with the minimal hyperbolic diameter is $[c - R, c + R]$. We now compute the hyperbolic distance

$$\begin{aligned} h_{\mathbb{H}}(c + R, c - R) &= h_{\mathbb{H}}(i \cosh r + \sinh r, i \cosh r - \sinh r) \\ &= 2 \operatorname{artanh} \frac{\sinh r}{\sqrt{\cosh 2r}} \\ &= \log \frac{\sqrt{\cosh 2r} + \sinh r}{\sqrt{\cosh 2r} - \sinh r} \\ &= 2 \log \frac{\sqrt{\cosh 2r} + \sinh r}{\cosh r} =: \varphi(r). \end{aligned} \tag{4.3}$$

Then the third assertion follows. □

Remark 4.4. By geometry, we see that $|c + i\,Re^{\pm i\theta} - z_0|$ is strictly decreasing in $0 < \theta < \pi$, which will be needed in the proof of Lemma 4.15.

We remark also that the sharp upper bound of the hyperbolic distance of the endpoints of a diameter of C is $h_{\mathbb{H}}(c + iR, c - iR) = 2r$. By the form of $\varphi(r)$, we also see that $\varphi(r) \to \log \frac{\sqrt{2}+1}{\sqrt{2}-1} = 2\log(\sqrt{2} + 1) = 1.7627\ldots$ as $r \to +\infty$.

In order to find the extremal configuration, we divide the family $\mathcal{E}$ into one-parameter subfamilies. More concretely, for $u > 0$, let $\mathcal{E}(u)$ be the subfamily of $\mathcal{E}$ consisting of sets E with $h_{\mathbb{H}}(E) = 2u$. Then

$$\kappa_3(\mathbb{H}) = \inf_{0<u<+\infty} \inf_{E \in \mathcal{E}(u)} \frac{2u}{J_{\mathbb{H}}(E)} = \inf_{0<u<+\infty} \frac{2u}{\log(1 + M(u))}, \tag{4.5}$$

where

$$M(u) = \sup_{E\in\mathcal{E}(u)} d(E) \tag{4.6}$$

Our task is to find the extremal configuration of $E \in \mathcal{E}(u)$ for the functional $d(E)$. We first define a candidate of the extremal set. For a given number $u > 0$, we choose $t > 0$ and $\theta \in (0, \pi/2)$ such that

$$h_{\mathbb{H}}(ie^{t+i\theta}, ie^{t-i\theta}) = h_{\mathbb{H}}(ie^{t+i\theta}, i) = 2u.$$

In other words, we choose t and θ so that the set $E^*(u) = \{i, ie^{t+i\theta}, ie^{t-i\theta}\}$ forms the vertices of a hyperbolic equilateral triangle with sidelength $2u$. We now give formulae describing θ and t in terms of u. Since $h_{\mathbb{H}}(ie^{t+i\theta}, ie^t) = u$, we obtain $u = 2\,\mathrm{artanh}\,(\tan(\theta/2))$ and thus

$$\theta = 2\arctan(\tanh(u/2)). \tag{4.7}$$

Moreover, by the hyperbolic cosine formula for a hyperbolic right triangle [22, Thm 7.11.1, p. 146], we have

$$\cosh t = \cosh h_{\mathbb{H}}(ie^t, i) = \frac{\cosh h_{\mathbb{H}}(ie^{t+i\theta}, i)}{\cosh h_{\mathbb{H}}(ie^{t+i\theta}, ie^t)} = \frac{\cosh 2u}{\cosh u}.$$

Hence,

$$t = \mathrm{arcosh}\,((\cosh 2u)/\cosh u). \tag{4.8}$$

We now compute

$$|ie^{t+i\theta} - ie^{t-i\theta}| = 2e^t \sin\theta = \chi(u),$$

where

$$\begin{aligned} \chi(u) &= 2e^{\mathrm{arcosh}\,((\cosh 2u)/\cosh u)} \sin\left[2\arctan\tanh(u/2)\right] \\ &= 2\frac{\cosh 2u + \sqrt{(\cosh^2 2u) - (\cosh^2 u)}}{\cosh u} \cdot \tanh u \\ &= \frac{2\sinh u}{1+\sinh^2 u}\left[1 + 2\sinh^2 u + \sinh u\sqrt{3 + 4\sinh^2 u}\right]. \end{aligned} \tag{4.9}$$

Note that $\chi(u) \le d(E^*(u))$. In the same way, we compute

$$\begin{aligned} \mathrm{Im}\left(ie^{t+i\theta}\right) &= e^t\cos\theta = e^{\mathrm{arcosh}\,((\cosh 2u)/\cosh u)} \cos\left[2\arctan\tanh(u/2)\right] \\ &= \frac{\cosh 2u + \sqrt{\cosh^2 2u - \cosh^2 u}}{\cosh u} \cdot \frac{1}{\cosh u} > \frac{\cosh 2u}{\cosh^2 u} > 1\,. \end{aligned}$$

Therefore, we obtain $d(E^*(u), \partial\mathbb{H}) = 1$ for every $u > 0$. We summarize the above observations in the following lemma.

Lemma 4.10. *The set $E^*(u)$ of the vertices of the hyperbolic equilateral triangle in $\mathbb{H}$ with sidelength $2u$ constructed above belongs to $\mathcal{E}(u)$ for every $u > 0$.*

We make further preparatory observations.

Lemma 4.11. *If* $0 < u \le \log(11/4) \approx 1.0116$*, then* $d(E^*(u)) = \chi(u)$ *and*

$$\frac{2u}{\log(1+M(u))} < 1.$$

Proof. We will prove the inequality

$$\frac{2u}{\log(1+\chi(u))} < 1 \tag{4.12}$$

for $0 < u \le \log(11/4)$. Since $E^*(u) \in \mathcal{E}(u)$ by Lemma 4.10, we have $M(u) \ge d(E^*(u)) \ge \chi(u)$. Thus, the second assertion will follow from (4.12).

By using the elementary inequality $\sqrt{3+4\sinh^2 u} > \sqrt{3+3\sinh^2 u} = \sqrt{3}\cosh u$ for $u > 0$, we obtain the estimate

$$\chi(u) > \frac{2\sinh u}{1+\sinh^2 u}\big[1 + 2\sinh^2 u + \sqrt{3}\sinh u\cosh u\big].$$

Thus, we have

$$\begin{aligned}\chi(u) + 1 - e^{2u} &\ge \frac{2\sinh u}{1+\sinh^2 u}\big[1 + 2\sinh^2 u + \sqrt{3}\sinh u\cosh u\big] + 1 - e^{2u} \\ &= \frac{(\sqrt{3}-1)(e^u+1)(e^u-1)^2 P(e^u-1)}{e^u(e^{2u}+1)^2},\end{aligned}$$

where $P(T)$ is the polynomial given by

$$P(T) = 4 + (7+\sqrt{3})T + 4T^2 - \sqrt{3}T^3 - \frac{1+\sqrt{3}}{2}T^4.$$

We now estimate $P(T)$ for $T \ge 0$ from below:

$$P(T) \ge 4 + 8T + 4T^2 - 2T^3 - 2T^4 = 2(1+T)(2+2T-T^3).$$

Since $Q(T) = 2 + 2T - T^3$ is concave on $[0, +\infty)$, we have

$$Q(T) \ge \min\{Q(0), Q(7/4)\} = 9/64 > 0 \quad \text{for } 0 \le T \le 7/4.$$

Hence, we have proved that $e^{2u} < 1 + \chi(u)$ and thus (4.12) holds for $0 < u \le \log(11/4)$.

Finally, we prove that $d(E^*(u)) = \chi(u)$ for such u. Indeed, the inequality

$$|ie^{t+i\theta} - ie^{t-i\theta}| < |ie^{t+i\theta} - i|$$

would hold otherwise. Then the two-point subset $E = \{i, ie^{t+i\theta}\}$ of $E^*(u)$ satisfies $h_{\mathbb{H}}(E) = 2u$, $d(E) = d(E^*(u))$ and $d(E, \partial\mathbb{H}) = d(E^*(u), \partial\mathbb{H}) = 1$. Thus, we would have

$$\frac{2u}{\log(1+\chi(u))} > \frac{2u}{J_{\mathbb{H}}(E^*(u))} = \frac{2u}{J_{\mathbb{H}}(E)} \ge \kappa_2(\mathbb{H}) = 1$$

by Lemma 4.1. This contradicts (4.12). In this way, we have proved that $d(E^*(u)) = \chi(u)$. □

Lemma 4.13. *Let $0 < u < +\infty$. The condition $\varphi(2u) \geq 2u$ holds if and only if $u \leq u_0$, where φ is given in* (4.3) *and $u_0 \approx 0.831443$ is the positive solution to the equation $4\cosh^4 u = \cosh 4u$.*

Proof. We observe that for $u > 0$,

$$\begin{aligned} & \varphi(2u) = 2\,\mathrm{artanh}\,\big[(\sinh 2u)/\sqrt{\cosh 4u}\big] < 2u \\ \Leftrightarrow \quad & \frac{\sinh 2u}{\sqrt{\cosh 4u}} = \frac{2\sinh u\cosh u}{\sqrt{\cosh 4u}} < \tanh u = \frac{\sinh u}{\cosh u} \\ \Leftrightarrow \quad & 4 < \frac{\cosh 4u}{\cosh^4 u}. \end{aligned}$$

Since $(\cosh 4u)/\cosh^4 u$ increases from 1 to 8 when u moves from 0 to $+\infty$, there exists a unique number $u_0 > 0$ satisfying the relation $4 = (\cosh 4u_0)/\cosh^4 u_0$. We now see that $\varphi(2u) < 2u$ if and only if $u > u_0$. □

The following elementary result is also needed later.

Lemma 4.14. *The function $f(x) = x/\log(1 + 2\sinh x)$ strictly increases from $1/2$ to 1 as x moves from 0 to $+\infty$.*

Proof. Because $f(x) = x/\log(e^x - e^{-x} + 1)$, differentiation yields

$$f'(x) = h(x)/\big[\log(e^x - e^{-x} + 1)\big]^2, \quad \text{where } h(x) = \log(e^x - e^{-x} + 1) - \frac{x(e^x + e^{-x})}{e^x - e^{-x} + 1}.$$

Further, we have

$$h'(x) = -\frac{x(e^x - e^{-x})}{e^x - e^{-x} + 1} + \frac{x(e^x + e^{-x})^2}{(e^x - e^{-x} + 1)^2} = \frac{x(e^{-x} - e^x + 4)}{(e^x - e^{-x} + 1)^2} = \frac{2x(2 - \sinh x)}{(1 + 2\sinh x)^2}.$$

We now see that $h'(x) > 0$ for $0 < x <\ \mathrm{arsinh}\, 2$ and $h'(x) < 0$ for $\mathrm{arsinh}\, 2 < x$. Since $h(0) = 0$ and

$$h(x) = x + \log(1 + e^{-x} - e^{-2x}) - x\frac{1 + e^{-2x}}{1 + e^{-x} - e^{-2x}} = O(xe^{-x}) = o(1)$$

as $x \to +\infty$, the function $h(x)$ is positive for all $x > 0$. Hence, $f'(x) > 0$ for all $x > 0$, which implies that $f(x)$ is strictly increasing in $x > 0$. It is easy to see that $f(x) \to 1/2$ as $x \to 0$ and that $f(x) \to 1$ as $x \to +\infty$. □

We are ready to prove our result.

Lemma 4.15. *Let $u > 0$. Then the quantity $M(u)$ defined in* (4.6) *is evaluated as*

$$M(u) = \begin{cases} \chi(u) & \text{if } 0 < u < u_0, \\ 2\sinh 2u & \text{if } u_0 \leq u, \end{cases}$$

where $\chi(u)$ is given in (4.9) *and $u_0 \approx 0.831443$ is the positive solution to the equation $4\cosh^4 u = \cosh 4u$. Moreover, when $0 < u < u_0$, a set $E \in \mathcal{E}(u)$ satisfies $d(E) = M(u)$ if and only if $E = E^*(u)$.*

Proof. We denote the circle $\partial\Delta(i, 2u)$ by C in the following. Since every $E \in \mathcal{E}(u)$ is contained in the closed disk $\overline{\Delta}(i, 2u)$, the diameter $d(E)$ is at most $2\sinh 2u$ by Lemma 4.2(ii). Hence, we observe that

$$M(u) \le 2\sinh 2u, \quad u > 0.$$

First, we assume that $u \ge u_0$; equivalently by Lemma 4.13, $\varphi(2u) \le 2u$. Let z_1, z_2 be the endpoints of the horizontal diameter of the boundary circle $C = \partial\Delta(i, 2u)$. Note that $\operatorname{Im} z_j = \cosh 2u > 1$. Then, by Lemma 4.2(iii), $h_{\mathbb{H}}(z_1, z_2) = \varphi(2u) \le 2u$. Thus, $E = \{i, z_1, z_2\} \in \mathcal{E}(u)$ which implies $d(E) = 2\sinh 2u \le M(u)$. Therefore, we have proved that $M(u) = 2\sinh 2u$. Note that the extremal set E is not necessarily unique when $\varphi(2u) < 2u$ (for instance, we can rotate the diameter a little about the Euclidean center of C).

Next, we assume that $u < u_0$; namely, $\varphi(2u) > 2u$. We prove that there exists a set $E_0 \in \mathcal{E}(u)$ attaining the supremum in (4.6); namely, $M(u) = d(E_0)$. Indeed, by definition, we can find a sequence of sets E_k in $\mathcal{E}(u)$ such that $d(E_k) \to M(u)$ as $k \to \infty$. Since each $E \in \mathcal{E}(u)$ is contained in the closed hyperbolic disk $\overline{\Delta}(i, 2u)$, by passing to a subsequence if necessary, we may assume that $E_k = \{i, z_k, w_k\}$ and $z_k \to z_\infty$ and $w_k \to w_\infty$ as $k \to \infty$ for some $z_\infty, w_\infty \in \overline{\Delta}(i, 2u)$. By continuity, we have $d(E_\infty) = M(u)$ for $E_\infty = \{i, z_\infty, w_\infty\}$. We have to check that E_∞ belongs to $\mathcal{E}(u)$. If E_∞ consists only of two points, by Lemma 4.1,

$$\log(1 + M(u)) \le J_{\mathbb{H}}(E_\infty) \le h_{\mathbb{H}}(E_\infty) = 2u,$$

which contradicts Lemma 4.11 because $u \le u_0 < \log(11/4)$. We have proved the claim.

Now assume that $E_0 = \{i, z_0, w_0\} \in \mathcal{E}(u)$ satisfies $d(E_0) = M(u)$. By assumption, we have $z_0 \in \overline{\Delta}(i, 2u) \cap \overline{\Delta}(w_0, 2u)$. Observe that $z_0 \in \partial\Delta(i, 2u) = C$ in the present situation. In fact, let $r = h_{\mathbb{H}}(z_0, w_0)$ and suppose $h_{\mathbb{H}}(z_0, i) < 2u$. Then z_0 can be moved along the circle $\partial\Delta(w_0, r)$ upwards a bit to get a new point z_0' in such a way that

$$\operatorname{Im} z_0 < \operatorname{Im} z_0', \;\; h_{\mathbb{H}}(z_0', i) < 2u, \;\; h_{\mathbb{H}}(z_0', w_0) = r \quad \text{and } |z_0 - w_0| < |z_0' - w_0|$$

by Lemma 4.2 and Remark 4.4. Hence we would have $h_{\mathbb{H}}(E_0') = h_{\mathbb{H}}(E_0)$ and $d(E_0) < d(E_0')$ for $E_0' = \{i, z_0', w_0\}$. This, however, violates the initial assumption that $d(E_0) = M(u)$. Therefore, we conclude that $h_{\mathbb{H}}(z_0, i) = 2u$. In the same way, we obtain $h_{\mathbb{H}}(w_0, i) = 2u$. We can further prove, as before (cf. the proof of Lemma 4.11), that $|z_0 - w_0| = d(E_0)$.

The remaining task is now to determine the configuration of the points z_0, w_0 on the circle C maximizing the quantity $|z_0 - w_0|$ under the constraints $h_{\mathbb{H}}(z_0, w_0) \le 2u$ and $\min\{\operatorname{Im} z_0, \operatorname{Im} w_0\} \ge 1$. We recall that the hyperbolic distance of the endpoints of an arbitrary Euclidean diameter of C is at least $\varphi(2u)$ by Lemma 4.2(iii). We first suppose that $\varphi(2u) < 2u$. Let C_0 be the shorter component of $C \setminus \{z_0, w_0\}$. It is evident that the chord $|z_0 - w_0|$ is shortest when (and only when) z_0 and w_0 are situated symmetrically with respect to the imaginary axis. Therefore, we have

$$E_0 = E^*(u) \quad \text{and } M(u) = 2u/\log(1 + d(E^*(u))) = \xi(u)\,.$$

By the above proof, uniqueness of the extremal set for $0 < u \le u_0$ is clear. Thus, the proof is now complete. □

Remark 4.16. In view of Lemmas 4.11 and 4.14, as a corollary of the last lemma, we have the inequality

$$\inf_{E\in\mathcal{E}(u)} \frac{2u}{J_{\mathbb{H}}(E)} = \frac{2u}{\log(1 + M(u))} < 1$$

for every $u > 0$.

We are now in a position to prove the following theorem.

Theorem 4.17. *There is a zero $u = u^*$ of the derivative $\xi'(u)$ of the function*

$$\xi(u) = \frac{2u}{\log(1 + \chi(u))}$$

in the interval $0 < u < u_0 \approx 0.83$ such that

$$\kappa(\mathbb{H}) = \frac{h_{\mathbb{H}}(z^*, w^*)}{\log(1 + |z^* - w^*|)} = \frac{h_{\mathbb{H}}(E^*)}{\log(1 + d(E^*)/d(E^*, \partial\mathbb{H}))},$$

where u_0 is given in Lemma 4.13, $E^* = E^*(u^*) = \{i, z^*, w^*\}$, $z^* = ie^{t^*+i\theta^*}$, $w^* = ie^{t^*-i\theta^*}$ and t^*, θ^* are given in (4.8) and (4.7), respectively, for $u = u^*$. Moreover, if $\kappa(\mathbb{H}) = h_{\mathbb{H}}(E)/\log(1 + d(E)/d(E, \partial\mathbb{H}))$ for a three-point set E in $\mathbb{H}$, then there are real numbers a, b with $a > 0$ such that $E = aE^* + b$.

Proof. Lemma 4.15 implies that for $u \ge u_0 = 0.831\ldots$,

$$\frac{2u}{\log(1 + M(u))} = \frac{2u}{\log(1 + 2\sinh 2u)}.$$

Since the function $x/\log(1 + 2\sinh x)$ is increasing in $0 < x < +\infty$ by Lemma 4.14, we can restrict the range of the infimum in (4.5) to $(0, u_0]$:

$$\kappa(\mathbb{H}) = \kappa_3(\mathbb{H}) = \inf_{0<u\le u_0} \frac{2u}{\log(1 + M(u))} = \inf_{0<u\le u_0} \frac{2u}{\log(1 + \chi(u))} = \inf_{0<u\le u_0} \xi(u),$$

where $\chi(u)$ is given in (4.9). By the form of $\chi(u)$ in (4.9), we observe that $\chi(u) = 2u + 2\sqrt{3}u^2 + O(u^3)$ as $u \to 0^+$. Thus, we obtain $\xi(u) \ge 2u/\log(1+\chi(u)) = 1-(\sqrt{3}-1)u+O(u^2)$ as $u \to 0^+$. In particular, $\xi(0^+) = 1$ and $\xi'(0^+) = 1-\sqrt{3} < 0$. Since $\xi'(u_0) = 0.1917\cdots > 0$, the above infimum of $\xi(u)$ is attained at its critical point in $(0, u_0)$.

The last assertion easily follows from the uniqueness of the extremal set in Lemma 4.15. The proof is now complete. □

See Figure 2 for the graph of the function $2u/\log(1+M(u))$. By numerical computations, we obtain $u^* \approx 0.432335123777, t^* \approx 0.727535978839, \theta^* \approx 0.419463976058$, and $\kappa(\mathbb{H}) = \xi(u^*) \approx 0.8750987500145$. Note that by Theorem 1.5 $\kappa(\mathbb{H}) = \kappa(\Omega)$ for a convex hyperbolic domain Ω. In conclusion, we have the following theorem:

Theorem 4.18. *For any convex hyperbolic domain* Ω*,* $\kappa(\Omega) \approx 0.875098750014$*.*

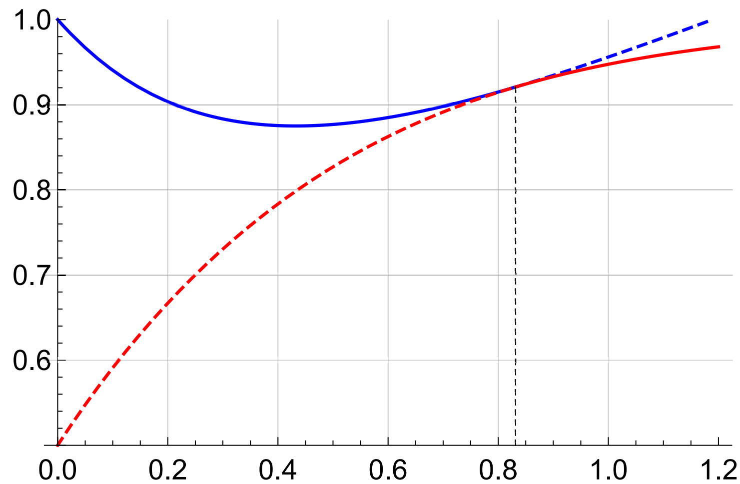

FIGURE 2. The graph of $2u/\log(1+M(u))$ (the thick line); the blue curve indicates the graph of $\xi(u)$ and the red one does the graph of $2u/\log(1+2\sinh 2u)$

Finally, we prove Theorem 1.6.

Proof of Theorem 1.6. It remains to prove the first assertion. Let $\Omega \subsetneq \mathbb{C}$ be a convex domain and suppose that $\kappa(\Omega) = h_\Omega(E)/J_\Omega(E)$ for a compact subset E of Ω. As in the proof of Lemma 2.1, we take points $z_0, z_1, z_2 \in E$ so that $d(E) = |z_1 - z_2|$ and $d(E, \partial\Omega) = d(z_0, \partial\Omega)$ and let $E_0 = \{z_0, z_1, z_2\}$. (Since $\kappa(\Omega) = \kappa(\mathbb{H}) < 1$, the set E_0 contains exactly three points.) By Lemma 2.1, we have $\kappa_3(\Omega) = \kappa(\Omega)$. Thus, in the chain of inequalities (2.2), the last term is the same as the initial term. Thus, we have $h_\Omega(E) = h_\Omega(E_0)$. Hence $\kappa(\Omega) = h_\Omega(E_0)/J_\Omega(E_0)$.

Let $\zeta_0 \in \partial\Omega$ be such that $d(E_0, \partial\Omega) = d(z_0, \partial\Omega) = |z_0 - \zeta_0|$. Take a half-plane H as in the proof of Lemma 3.4 such that $\Omega \subset H$ and $z_0 \in \partial H$. Then $J_\Omega(E_0) = J_H(E_0)$ and $h_\Omega(E_0) \geq h_H(E_0)$. If Ω is a proper subdomain of H, then we would have $h_H(E_0) < h_\Omega(E_0)$. Thus,

$$\kappa(H) \leq \frac{h_H(E_0)}{J_H(E_0)} < \frac{h_\Omega(E_0)}{J_\Omega(E_0)} = \kappa(\Omega).$$

On the other hand, Theorem 1.5 yields $\kappa(H) = \kappa(\Omega)$, which is a contradiction. Thus, Ω equals H, a half-plane. $\square$

5. Application to Capacity Estimation

Finally, we apply the results above to capacity estimation. First, we recall some basic notions.

Definition 5.1. [18, Def. 9.2, p. 150] A pair (Ω, E) of a domain Ω in $\mathbb{C}$ and a non-empty compact subset E of Ω is called a *condenser*. The *capacity* of this condenser is defined to be

$$\operatorname{cap}(\Omega, E) = \inf_u \iint_{\mathbb{C}} |\nabla u(z)|^2 dxdy \quad (z = x + iy),$$

where the infimum is taken over the family of all non-negative functions u in the Sobolev class $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ with compact support in Ω such that $u(z) \geq 1$ for $z \in E$.

If Ω is a simply connected proper subdomain of $\mathbb{C}$ and E is a (non-degenerate) continuum in Ω such that the set $R = \Omega \setminus E$ is a doubly connected domain (a ring), then its modulus is known to be $2\pi/\operatorname{cap}(\Omega, E)$.

We define the homeomorphism $\mu : (0, 1) \to \mathbb{R}^+$ by the formula (see, e.g., [18, 7.4.1, p. 122])

$$\mu(r) = \frac{\pi}{2} \cdot \frac{\mathcal{K}(\sqrt{1 - r^2})}{\mathcal{K}(r)},$$

where $\mathcal{K}(r)$ is Legendre's complete elliptic integral of the first kind defined by

$$\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - r^2x^2)}}.$$

It is known that $\mu(r)$ represents the modulus of the Grötzsch ring $\mathbb{D} \setminus [0, r]$. In particular, $\mu(r)$ decreases from $+\infty$ to 0 as r moves from 0 to 1. We note that $2\pi/\mu(r)$ is the capacity of $\mathbb{D} \setminus [0, r]$. For later convenience, we put

$$\Phi(x) = \frac{2\pi}{\mu(\tanh(x/2))}, \quad 0 < x < \infty.$$

Note that $\Phi(x)$ increases from 0 to $+\infty$ as x moves from 0 to $+\infty$. We are ready to give the main result in this section. Recall that $J_\Omega(E) = \log(1 + d(E)/d(E, \partial\Omega))$.

Theorem 5.2. *Let E be a continuum in a simply connected domain $\Omega \subsetneq \mathbb{C}$. Then the following are valid.*

(i) *The inequality*

$$\operatorname{cap}(\Omega, E) \geq \Phi(\kappa(\Omega)J_\Omega(E)) \geq \Phi(\kappa_0 J_\Omega(E))$$

holds, where κ_0 is given in (1.7)*.*

(ii) *If Ω is convex,*

$$\operatorname{cap}(\Omega, E) \geq \Phi(\kappa_1 J_\Omega(E))),$$

where $\kappa_1 = \kappa(\mathbb{D}) > 0.87509875$.

Proof. Let $f : \Omega \to \mathbb{D}$ be a conformal homeomorphism and set $E' = f(E)$. Since the capacity and the hyperbolic distance are conformally invariant, we obtain

$$\mathrm{cap}\,(\Omega, E) = \mathrm{cap}\,(\mathbb{D}, E') \geq \Phi(h_{\mathbb{D}}(E')) = \Phi(h_\Omega(E)),$$

where we used a consequence of the circular symmetrization (see [18, Lemma 9.20, p. 163]). Other parts follow from Corollary 1.8 and Theorem 4.18. □

Example 5.3. Consider next an example where $\Omega = \{z : -1 < \mathrm{Im}\, z < 1\}$ and $E = [1, 2]$. Because Ω is convex, it follows from Theorem 5.2 that

$$\mathrm{cap}\,(\Omega, E) \geq \Phi(\kappa_1 J_\Omega(E)) \approx \frac{2\pi}{\mu(0.43754937 \log 2))} > 2.4288.$$

By applying the circular (spherical) symmetrization (see [18, 9.1, pp. 155-157]) with the origin as a center and x-axis as the symmetrization axis. Observe first that the negative x-axis is contained in the complement of the symmetrized condenser whereas $[1, 2]$ remains invariant and hence

$$\mathrm{cap}\,(\Omega, E) \geq \tau_2(1) = 2,$$

where $\tau_2(t)$ denotes the capacity of the Teichmüller ring $\mathbb{C} \setminus ([-1, 0] \cup [t, +\infty))$ for $t > 0$ (see [18, 7.3, pp. 120]), which is a weaker lower bound for the capacity than what we proved above. On the other hand, if we take into account that the whole left half-plane is contained in the complement of the symmetrized condenser, we obtain

$$\mathrm{cap}\,(\Omega, E) \geq \Phi(\log 2) = \frac{2\pi}{\mu(\tanh(\log \sqrt{2}))} \approx 2.55852.$$

Hence the value of our bound given in Theorem 5.2 lies between these two bounds obtained by symmetrization. Finally, let us find the exact value of $\mathrm{cap}\,(\Omega, E)$. Obviously, $\mathrm{cap}\,(\Omega, E) = \mathrm{cap}\,(\Omega, E_0)$, where $E_0 = [0, 1]$. Note that the function $f(z) = \dfrac{2}{\pi} \log \dfrac{1+z}{1-z}$ maps the unit disk $\mathbb{D}$ onto Ω and that $f^{-1}(E_0) = [0, \tanh(\pi/4)]$. Thus

$$\mathrm{cap}\,(\Omega, E) = \mathrm{cap}\,(\Omega, E_0) = \mathrm{cap}\,(\mathbb{D}, [0, \tanh(\pi/4)]) = \frac{2\pi}{\mu(\tanh(\pi/4))} = \Phi(\frac{\pi}{2}) \approx 3.75108.$$

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