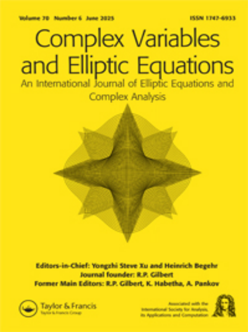

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On distortion of quasiregular mappings of the upper half plane

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ABSTRACT
We prove a sharp result for the distortion of a hyperbolic-type metric under K-quasiregular mappings of the upper half plane. The proof makes use of a new kind of Bernoulli inequality and the Schwarz lemma for quasiregular mappings.

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1. Introduction

Alongside the hyperbolic metric, several other metrics similar to it have become standard tools of geometric function theory [1–6]. In Ref. [7, p.42–48], the author lists twelve metrics recurrent in complex analysis. These metrics, sometimes called metrics of hyperbolic type, are typically not Möbius invariant, but they are often quasi-invariant and within a constant factor from the hyperbolic metric. Here we study one of these metrics, defined on a proper subdomain D of a metric space (X, d) as follows

$$h_{D,c}(x,y) = \log\left(1 + c\frac{d(x,y)}{\sqrt{d_D(x)d_D(y)}}\right), \quad c > 0, \tag{1}$$

where $d_D(x) = \mathrm{dist}(x, \partial D)$.

This metric has been studied in several very recent papers. In Ref. [8], the metric (1) was called the *geometric mean distance metric*. In Ref. [8] also comparison inequalities between the metric (1) and some other metrics were given and the geometry defined by the metric studied. For further work, see also Refs [9, 10].

Theorem 1.1 ([1, Thm 1.1]): *The function* (1) *is a metric for every* $c \geq 2$. *The constant* 2 *is best possible here.*

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Here the constant c is sharp in the case if the domain D is the unit disk $\mathbb{B}^2$. It was shown in [1, Rem. 3.29] that $h_{\mathbb{B}^2,c}$ is not a metric if $0 < c < 2$. For the half plane $\mathbb{H}^2$ case we prove the following result.

Theorem 1.2: *The function $h_{\mathbb{H}^2,c}$ is a metric for every $c \geqslant 1$.*

The behaviour of this metric under quasiconformal mappings was also studied in [1, Thm 4.9]. Our goal here is to prove the following new sharp result for quasiregular mappings of the upper half plane $\mathbb{H}^2$.

Theorem 1.3: *For $K \geq 1$ there exists a constant $\lambda(K) \in [1, \exp(\pi(K - 1/K)))$ such that for a K-quasiregular mapping $f : \mathbb{H}^2 \to \mathbb{H}^2 = f(\mathbb{H}^2)$ and for all $x, y \in \mathbb{H}^2, c \geq 1$,*

$$h_{\mathbb{H}^2,c}(f(x),f(y)) \leq \lambda(K)^{1/2}K^{1+c}\max\{h_{\mathbb{H}^2,c}(x,y)^{1/K}, h_{\mathbb{H}^2,c}(x,y)\}.$$

Theorem 1.3 is perhaps new also in the case when $K = 1, c > 1$, in which case the constant $\lambda(K)^{1/2}K^{1+c} = 1$, and the function f is analytic.
The constant $\lambda(K)$ is explicitly given in (13) with concrete estimates. The proof of Theorem 1.3 is based on two main components. The first component is the well-known Schwarz lemma for quasiregular mappings and associated estimates for special functions expressed in terms of complete elliptic integrals. The second component is a Bernoulli-type inequality, formulated as Theorem 3.1 that enables us to simplify the inequalities given by the Schwarz lemma.

2. Preliminary results

2.1. Hyperbolic geometry

We recall some basic formulas and notation for hyperbolic geometry from Ref. [11]. The hyperbolic metrics of the unit disk $\mathbb{B}^2$ and the upper half plane $\mathbb{H}^2$ are defined, resp., by

$$\operatorname{sh}\frac{\rho_{\mathbb{B}^2}(a,b)}{2} = \frac{|a-b|}{\sqrt{(1-|a|^2)(1-|b|^2)}}, \quad a, b \in \mathbb{B}^2, \tag{2}$$

and

$$\operatorname{ch}\rho_{\mathbb{H}^2}(x,y) = 1 + \frac{|x-y|^2}{2\operatorname{Im}(x)\operatorname{Im}(y)}, \quad x, y \in \mathbb{H}^2. \tag{3}$$

Above the symbols sh and ch stand for the hyperbolic sine and cosine functions. Their inverses are arsh and arch. Recalling that $2(\operatorname{ch} t - 1) = (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^2$ for $t \geq 0$, we obtain by (3) and the definition (1)

$$2\operatorname{sh}\frac{\rho_{\mathbb{H}^2}(x,y)}{2} = \sqrt{2(\operatorname{ch}\rho_{\mathbb{H}^2}(x,y) - 1)} = \frac{|x-y|}{\sqrt{\operatorname{Im}(x)\operatorname{Im}(y)}} = \frac{1}{c}\left(e^{h_{\mathbb{H}^2,c}(x,y)} - 1\right). \tag{4}$$

These two metrics are Möbius invariant: if $G, D \in \{\mathbb{B}^2, \mathbb{H}^2\}$ and $f : G \to D = f(G)$ is a Möbius transformation, then $\rho_G(x,y) = \rho_D(f(x),f(y))$ for all $x, y \in G$.

By the Bernoulli inequality [4, (5.6)] we have for $c_1 \geq c_2 \geq 1$ and all $t > 0$

$$\log(1 + c_1 t) \leq \frac{c_1}{c_2} \log(1 + c_2 t),$$

and hence for all $x, y \in D$

$$h_{D,c_1}(x,y) \leq \frac{c_1}{c_2} h_{D,c_2}(x,y). \tag{5}$$

Proposition 2.1 ([1, Prop. 2.5]): *If* $c > 0$ *we have*

(1)

$$\sqrt{2(\,\mathrm{ch}\rho_{\mathbb{H}^n}(x,y) - 1)} = \frac{\mathrm{e}^{h_{\mathbb{H}^n,c}(x,y)} - 1}{c}, \quad \textit{for all } x, y \in \mathbb{H}^n,$$

(2)

$$\mathrm{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2} \leq \frac{\mathrm{e}^{h_{\mathbb{B}^n,c}(x,y)} - 1}{c} \leq 2\,\mathrm{sh}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}, \quad \textit{for all } x, y \in \mathbb{B}^n.$$

We record the next three auxiliary results for the proof of Theorem 1.2 and give the proof based on these results.

Lemma 2.1: *For* $c \geq 1$ *and* $x \geq 1$, *the following inequality holds*

$$x \leq c\left(x - \frac{1}{x}\right) + 1.$$

Proof: The claim is equivalent to $x - 1 \leq c(x-1)(x+1)/x$ which clearly holds for $x \geq 1, c \geq 1$. ■

Lemma 2.2: *For* $c \geq 1$, *the following function is decreasing for* $x > 1$,

$$f(x) = \frac{\log\left(1 + c\left(x - \frac{1}{x}\right)\right)}{\log x}.$$

Proof: Consider the derivative of f,

$$\begin{aligned} f'(x) &= \frac{\frac{c\left(1+\frac{1}{x^2}\right)}{1+c\left(x-\frac{1}{x}\right)}\log x - \frac{1}{x}\log\left(1 + c\left(x - \frac{1}{x}\right)\right)}{(\log x)^2} \\ &= \frac{c(x^2+1)\log x - (x + c(x^2-1))\log\left(1 + c\left(x - \frac{1}{x}\right)\right)}{x^2\left(1 + c\left(x - \frac{1}{x}\right)\right)(\log x)^2}. \end{aligned} \tag{6}$$

Let $f_1(x)$ be the numerator of (6). Since $x^2\left(1 + c\left(x - \frac{1}{x}\right)\right)(\log x)^2 > 0$, we have to check whether $f_1(x) < 0$ for $x > 1$.

Now we have $f_1(1) = 0 - 0 = 0$ and

$$f_1'(x) = 2cx\log x - (2cx+1)\log\left(c\left(x - \frac{1}{x}\right) + 1\right). \tag{7}$$

As $1 \le x \le c\left(x - \frac{1}{x}\right) + 1$ holds by Lemma 2.1, we have $f_1'(x) < 0$.

Therefore $f_1(x) < 0$ holds and the assertion is obtained. ■

Proposition 2.2: *For a constant c with $c \ge 1$ and $t \ge 0$, let*

$$F(t) = \log\left(1 + c\sqrt{2(\cosh t - 1)}\right).$$

Then $F(0) = 0$ and $F : [0,\infty) \to [0,\infty)$ is increasing whereas $F(t)/t$ is decreasing for $t > 0$.

Proof: Because $2(\mathrm{ch}t - 1) = (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^2$ for $t \ge 0$, the function $F(t)/t$ can be expressed as

$$\frac{F(t)}{t} = \frac{\log\left(1 + c\sqrt{2(\cosh t - 1)}\right)}{t} = \frac{\log\left(1 + c\left(\mathrm{e}^{\frac{t}{2}} - \mathrm{e}^{-\frac{t}{2}}\right)\right)}{t}.$$

So, setting $\mathrm{e}^{\frac{t}{2}} = x$, we have

$$\frac{F(t)}{t} = \frac{F(2\log x)}{2\log x} = \frac{\log\left(1 + c\left(x - \frac{1}{x}\right)\right)}{2\log x}.$$

From the monotonicity of the exponential function and Lemma 2.2, the assertion is obtained. ■

Proof of Theorem 1.2.: Suppose that $g : [0,\infty) \to [0,\infty)$ is an increasing function with $g(0) = 0$ such that $g(t)/t$ is decreasing on $(0,\infty)$. Then by [4, p.80. Ex 5.24(1)] the subadditivity inequality $g(s+t) \le g(s) + g(t)$ holds for $s, t \ge 0$. We now apply this observation with the function F in place of g. By Proposition 2.2 we see that the function F satisfies, indeed, the above requirements for the function g. By (4) and the definition (1) we can write

$$h_{\mathbb{H}^2,c}(x,y) = F(\rho_{\mathbb{H}^2}(x,y)).$$

The triangle inequality follows now from the above subadditivity property of the function g. ■

In the next theorem we compare the metric $h_{\mathbb{H}^2,c}$ to the hyperbolic metric. To this end, the following inequality is needed [4, (4.12)]

$$2\log\left(1 + \sqrt{\frac{1}{2}(x-1)}\right) \le \operatorname{arch} x \le 2\log\left(1 + \sqrt{2(x-1)}\right), x \ge 1. \tag{8}$$

Theorem 2.1: *For all $x, y \in D \in \{\mathbb{B}^n, \mathbb{H}^n\}$ and all $c \ge 1$ we have*

$$h_{D,c}(x,y)/c \le \rho_D(x,y) \le 2h_{D,c}(x,y).$$

Proof: For convenience of notation write $D = \mathbb{H}^2$. Fix $x, y \in D$ and $c \geq 1$. Applying (4), (8), and the Bernoulli inequality we now have

$$\begin{aligned}\rho_D(x,y) &= \operatorname{arch}\left(1 + \frac{1}{2c^2}(e^{h_{D,c}(x,y)} - 1)^2\right) \geq 2\log\left(1 + \frac{1}{2c}(e^{h_{D,c}(x,y)} - 1)\right)\\ &\geq \frac{1}{c}\log(1 + (e^{h_{D,c}(x,y)} - 1)) = h_{D,c}(x,y)/c\end{aligned}$$

which proves the first inequality.

To prove the second inequality, it is enough to prove the case $c = 1$ because $h_{D,1}(x,y) \geq h_{D,c}(x,y)$ for $c \geq 1$. Again by (8)

$$\rho_D(x,y) = \operatorname{arch}\left(1 + \frac{1}{2}(e^{h_{D,1}(x,y)} - 1)^2\right) \leq 2\log\left(1 + (e^{h_{D,1}(x,y)} - 1)\right) = 2h_{D,1}(x,y)$$

completing the proof. ■

3. A Bernoulli-type inequality

The following theorem yields a Bernoulli-type inequality which is one of the main steps in the proof of Theorem 1.3. The proof of the Bernoulli-type inequality is based on three lemmas, Lemmas 3.1 A, 3.2 B, and 3.3 C, formulated below.

Theorem 3.1: *The following inequality holds*

$$\log\left(1 + 2c\max\left\{t^K, t^{1/K}\right\}\right) \leq K^{1+c}\max\left\{\log(1+2ct), \left(\log(1+2ct)\right)^{1/K}\right\}$$

for $c \geq 1,\ K \geq 1,\ t > 0.$

It will be shown in Remark 3.1 that Theorem 3.1 is sharp in the sense that the constant K^{1+c} cannot be replaced with K^2.

Lemma 3.1: ***A.*** *For* $c \geq 1,\ K \geq 1,$

$$\left(K^{1+c}\right)^{\frac{K}{K-1}} - 2c > 0.$$

Proof: Let $A(K) = (K^{1+c})^{\frac{K}{K-1}} - 2c$. Differentiation yields

$$A'(K) = \left(K^{1+c}\right)^{\frac{K}{K-1}} \frac{1}{(K-1)^2}(1+c)\left((K-1) - \log K\right).$$

Because $K \geq 1$, the inequality $(K-1) - \log K > 0$ holds, and we have $A'(K) > 0$.

Next, we will check $A(1) > 0$. Set $y = (K^{1+c})^{\frac{K}{K-1}}$ and consider $\log y = \frac{K}{K-1}(1+c)\log K$. Then,

$$\lim_{K\to 1} \log y = \lim_{K\to 1} (1+c)\frac{K\log K}{K-1} = \lim_{K\to 1}(1+c)\frac{(K\log K)'}{(K-1)'} = \lim_{K\to 1}(1+c)\frac{\log K + 1}{1}$$
$$= 1 + c.$$

So we have

$$\lim_{K\to 1} y = \lim_{K\to 1} \left(K^{1+c}\right)^{\frac{K}{K-1}} = \mathrm{e}^{1+c}.$$

Here, for $c > 1$, the inequality $\mathrm{e}^{1+c} - 2c > 0$ holds. Therefore,

$$A(1) = \lim_{K\to 1} \left(K^{1+c}\right)^{\frac{K}{K-1}} - 2c = \mathrm{e}^{1+c} - 2c > 0.$$

As we have $A(1) > 0$ and $A'(K) > 0$ for $K > 1$, the assertion $A(K) > 0$ is obtained. ∎

Lemma 3.2: ***B.*** (1) *For $K \geq 1$ and $t > 0$,*

$$B_1(t) = t^{1/K}\log t$$

attains its minimum $-\frac{K}{e}$ at $t = \mathrm{e}^{-K}$.

(2) *For $K \geq 1, c \geq 1$ and $\frac{e-1}{2c} \leq t < 1$, the following holds*

$$K^{1+c}\log(1+2ct) - \log(1+2ct^{1/K}) > 0.$$

Proof: (1) Consider the equation

$$B_1'(t) = \frac{1}{K}t^{\frac{1}{K}-1}\log t + t^{1/K}\frac{1}{t} = t^{\frac{1}{K}-1}\left(\frac{1}{K}\log t + 1\right) = 0.$$

Because $t > 0$, we have $\log t = -K$ that is $t = \mathrm{e}^{-K}$. So $B_1(t)$ attains the minimum at $t = \mathrm{e}^{-K}$. Moreover, the minimum value is given by

$$B_1(\mathrm{e}^{-K}) = (\mathrm{e}^{-K})^{1/K}\log\, \mathrm{e}^{-K} = -\frac{K}{e}.$$

(2) Let $B_2(K) = K^{1+c}\log(1+2ct) - \log(1+2ct^{1/K}) > 0$. For each fixed c, t with $c \geq 1$, $\frac{e-1}{2c} \leq t < 1$, we will show $B_2(K) > 0$. Observe that $\log(1+2ct) > 1$ holds in this case.

Differentiation and part (1) yield

$$\begin{aligned}\frac{\partial}{\partial K}B_2(K) &= \frac{2ct^{1/K}\log t}{K^2(1+2ct^{1/K})} + (1+c)K^c\log(1+2ct)\\ &\geq \frac{2c\left(-\frac{K}{e}\right)}{K^2(1+2ct^{1/K})} + (1+c)K^c\log(1+2ct) \quad \text{(from part (1))}\\ &\geq -\frac{2c}{Ke(1+2ct^{1/K})} + (1+c)K^c \quad (\text{from } \log(1+2ct) > 1)\\ &\geq -\frac{2c}{e^2} + (1+c) \quad (\text{from } K^c > K \geq 1,\ 1+2ct^{1/K} \geq 1+2ct \geq e)\\ &\geq -c+1+c > 0.\end{aligned}$$

Moreover,

$$B_2(1) = 1\cdot\log(1+2ct) - \log(1+2ct^1) = 0.$$

As we have $B_2(1) = 0$ and $\frac{\partial}{\partial K}B_2(K) > 0$ for $K \geq 1$, the assertion $B_2(K) > 0$ is obtained. ■

Lemma 3.3: ***C.*** *For $K \geq 1, c \geq 1,\ t \geq 1$, the function*

$$C(K) = (1+2ct)^K - (1+2ct^K)$$

is increasing.

Proof: The derivative is

$$\frac{\partial}{\partial K}C(K) = (1+2ct)^K\log(1+2ct) - 2ct^K\log t.$$

Observing that for $K \geq 1, t \geq 1, c \geq 1$, the following inequalities hold

$$(1+2ct)^K > 1+(2ct)^K > 2ct^K, \quad \text{and} \quad 1+2ct > t$$

we have

$$\frac{\partial}{\partial K}C(K) > (1+2ct)^K\log(1+2ct) - (1+2ct)^K\log t > 0$$

and hence $C(K)$ is increasing for $K \geq 1$. ■

Proof of Theorem 3.1.: For $K \geq 1, c \geq 1$ and $t > 0$, the following relations hold

$$\max\{t^K, t^{1/K}\} = \begin{cases} t^K & (t > 1)\\ t^{1/K} & (0 < t \leq 1)\end{cases} \tag{9}$$

and

$$\max\left\{\log(1+2ct), \left(\log(1+2ct)\right)^{1/K}\right\} = \begin{cases}\log(1+2ct) & \left(t > \dfrac{e-1}{2c}\right)\\ \left(\log(1+2ct)\right)^{1/K} & \left(0 < t \leq \dfrac{e-1}{2c}\right).\end{cases} \tag{10}$$

Because $c \geq 1$, we have $0 < \frac{e-1}{2c} < 1$. Therefore, to prove Theorem 3.1, it is sufficient to show the following inequalities A,B and C.

(A) For $c \geq 1,\ 0 < t \leq \frac{e-1}{2c},\ K \geq 1$, the following inequality holds

$$\log(1 + 2ct^{1/K}) \leq K^{1+c} \left(\log(1 + 2ct)\right)^{1/K}.$$

(B) For $c \geq 1,\ \frac{e-1}{2c} < t < 1,\ K \geq 1$, the following inequality holds

$$\log(1 + 2ct^{1/K}) \leq K^{1+c} \log(1 + 2ct).$$

(C) For $c \geq 1, t \geq 1, K \geq 1$, the following inequality holds

$$\log(1 + 2ct^{K}) \leq K^{1+c} \log(1 + 2ct).$$

Accordingly, we consider these three cases A,B,C.

(A) Fix c, K with $c \geq 1, K \geq 1$. In this case, we remark that $0 < \log(1 + 2ct) \leq 1$ as $0 < t \leq \frac{e-1}{2c}$. Let

$$a(t) := K^{1+c} \left(\log(1 + 2ct)\right)^{1/K} - \log(1 + 2ct^{1/K}). \tag{11}$$

Here, we will show that $a(t) \geq 0$ holds. First, we remark that $a(0) = K^{1+c}\log(1) - \log(1) = 0$. Next, we obtain

$$a'(t) = K^{1+c} \cdot \frac{1}{K} \cdot \frac{2c\left(\log(1 + 2ct)\right)^{\frac{1}{K}-1}}{1 + 2ct} - \frac{1}{K}\frac{2ct^{\frac{1}{K}-1}}{1 + 2ct^{\frac{1}{K}}}. \tag{12}$$

So, if we can show $a'(t) \geq 0$ for $0 < t \leq \frac{e-1}{2c}$, we obtain the assertion. Dividing the right side of (12) by $\frac{1}{K}(2c) > 0$, we have

$$A_0 := \frac{K}{2c}a'(t) = K^{1+c}\frac{\left(\log(1 + 2ct)\right)^{\frac{1}{K}-1}}{1 + 2ct} - \frac{t^{\frac{1}{K}-1}}{1 + 2ct^{\frac{1}{K}}}.$$

Multiplying the above constant A_0 by $(1 + 2ct)\left(1 + 2ct^{\frac{1}{K}}\right) > 0$ we have

$$A_1 := (1 + 2ct)\left(1 + 2ct^{\frac{1}{K}}\right)A_0 = K^{1+c}\left(1 + 2ct^{\frac{1}{K}}\right)\left(\log(1 + 2ct)\right)^{\frac{1}{K}-1} - t^{\frac{1}{K}-1}(1 + 2ct).$$

Here, remark that $1 + 2ct^{\frac{1}{K}} \geq 1 + 2ct$ holds because $0 < t < \frac{e-1}{2c} < 1,\ K > 1$. Therefore,

$$A_1 > (1 + 2ct)\left(K^{1+c}\left(\log(1 + 2ct)\right)^{\frac{1}{K}-1} - t^{\frac{1}{K}-1}\right).$$

Multiplying the above constant A_1 by $\frac{1}{1+2ct}(t\log(1 + 2ct))^{\frac{K-1}{K}} > 0$,

$$A_2 := K^{1+c}t^{\frac{K-1}{K}} - \left(\log(1 + 2ct)\right)^{\frac{K-1}{K}}.$$

To show $A_2 > 0$, it is sufficient to check $(K^{1+c})^{\frac{K}{K-1}} t > \log(1+2ct)$. Because $X - \log(X+1) > 0$ holds for $X > 0$, we have by Lemma 3.1 A

$$(K^{1+c})^{\frac{K}{K-1}} t - \log(1+2ct) \geq (K^{1+c})^{\frac{K}{K-1}} t - 2ct = t\left((K^{1+c})^{\frac{K}{K-1}} - 2c\right) > 0.$$

Therefore, for $0 < t < \frac{e-1}{2c}$, $A_2 > 0$ and $a'(t) > 0$ hold. Hence,

$$\log\left(1 + 2ct^{1/K}\right) \leq K^{1+c} \log(1+2ct)^{1/K}$$

is obtained in this case.

(B) Fix K, c and t with $K \geq 1, c \geq 1$ and $\frac{e-1}{2c} < t < 1$. In this case, we remark that $\log(1+2ct) > 1$. By Lemma 3.2 B (2) we see that the desired inequality holds.

(C) In this case, we remark that $\log(1+2ct) > 1$. By Lemma 3.3 C, the following inequality holds

$$(1+2ct)^K - (1+2ct^K) \geq 0$$

which implies

$$\log\left(1 + 2ct^K\right) \leq \log(1+2ct)^K.$$

This inequality yields the claim C with the better constant K in place of K^{1+c}.

In conclusion, the proof of Theorem 3.1 is complete. ■

Remark 3.1: *For,* $K = 1.2, c = 5$ *and* $t = 0.001$, *the following inequality does not hold*

$$\log(1 + 2c \max\{t^K, t^{1/K}\}) \leq K^2 \max\{(\log(1+2ct)), (\log(1+2ct))^{1/K}\}.$$

In conclusion, Theorem 3.1 *is sharp in the sense that the constant* K^{1+c} *can not be replaced with* K^2.

4. Proof of Theorem 1.3

We first recapitulate some fundamental facts about K-quasiregular mappings needed for the sequel. The definition of these mappings is given in Refs [4, p.288–289], [12, p.10–11]. For the proof of the main result, Theorem 1.3, we need some properties of an increasing homeomorphism $\varphi_{K,2} : [0,1] \to [0,1]$, [4, (9.13), p.167]

$$\varphi_{K,2}(r) = \frac{1}{\gamma_2^{-1}(K\gamma_2(1/r))} = \mu^{-1}(\mu(r)/K), \quad 0 < r < 1,\ K > 0.$$

This function is the special function of the Schwarz lemma in [4, Thm 16.2], and its properties and estimates are crucial for the proof. The Grötzsch capacity, a decreasing homeomorphism $\gamma_2 : (1,\infty) \to (0,\infty)$, is defined as Ref. [4, (7.18), p.122]

$$\gamma_2(1/r) = \frac{2\pi}{\mu(r)}, \quad \mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-r^2x^2)}}$$

with $0 < r < 1$. Denote then [13, (10.4), p.203]

$$\lambda(K) = \left(\frac{\varphi_{K,2}(1/\sqrt{2})}{\varphi_{1/K,2}(1/\sqrt{2})}\right)^2, \quad K > 1. \tag{13}$$

Trivially, $\lambda(1) = 1$ and, by Anderson et al. [13, Thm 10.35, p.219], $\lambda(K) < \mathrm{e}^{\pi(K-1/K)}$ for $K > 1$. The function $\varphi_{K,2}$ satisfies many identities and inequalities as shown in [13, Section 10]. In particular, for $t > 0, K \geq 1$, we have by [13, (10.3), 10.24]

$$\eta_K(t) \equiv \frac{\varphi_{K,2}(\sqrt{t/(1+t)})^2}{1-\varphi_{K,2}(\sqrt{t/(1+t)})^2} \leq \lambda(K)\max\{t^{1/K}, t^K\}. \tag{14}$$

Setting $\sqrt{t/(1+t)} = \operatorname{th}(u/2)$ we have $t = \operatorname{sh}^2(u/2)$ and the above inequality (14) attains the following form

$$\eta_K(\operatorname{sh}^2(u/2)) \equiv \frac{\varphi_{K,2}(\operatorname{th}(u/2))^2}{1-\varphi_{K,2}(\operatorname{th}(u/2))^2} \leq \lambda(K)\max\{(\operatorname{sh}^2(u/2))^{1/K}, (\operatorname{sh}^2(u/2))^K\}. \tag{15}$$

Proof of Theorem 1.3.: For short we write $\mathbb{H} = \mathbb{H}^2$ and fix $x, y \in \mathbb{H} \equiv \mathbb{H}^2$ and abbreviate $\rho = \rho_{\mathbb{H}}(x,y)$, $\rho' = \rho_{\mathbb{H}}(f(x),f(y))$. Now for $c > 0$

$$h_{\mathbb{H},c}(f(x),f(y)) = \log(1 + 2c\operatorname{sh}\frac{\rho'}{2}) = \log\left(1 + 2c\frac{\operatorname{th}\frac{\rho'}{2}}{\sqrt{1-\operatorname{th}^2\frac{\rho'}{2}}}\right)$$

and by the Schwarz lemma for quasiregular mappings [4, 16.2]

$$h_{\mathbb{H},c}(f(x),f(y)) \leq \log\left(1 + 2c\frac{\varphi_{K,2}(\operatorname{th}\frac{\rho}{2})}{\sqrt{1-\varphi_{K,2}(\operatorname{th}\frac{\rho}{2})^2}}\right).$$

By (15) we have

$$\log\left(1 + 2c\frac{\varphi_{K,2}(\operatorname{th}\frac{\rho}{2})}{\sqrt{1-\varphi_{K,2}(\operatorname{th}\frac{\rho}{2})^2}}\right) \leq \log\left(1 + 2c\lambda(K)^{1/2}\max\left\{(\operatorname{sh}\frac{\rho}{2})^{1/K}, \left(\operatorname{sh}\frac{\rho}{2}\right)^K\right\}\right)$$

and further by the Bernoulli inequality

$$\leq \lambda(K)^{1/2}\log\left(1 + 2c\max\left\{\left(\operatorname{sh}\frac{\rho}{2}\right)^{1/K}, (\operatorname{sh}\frac{\rho}{2})^K\right\}\right).$$

Finally, by Theorem 3.1 for $c \geq 1$

$$\leq K^{1+c}\lambda(K)^{1/2}\max\left\{\left(\log\left(1 + 2c\operatorname{sh}\frac{\rho}{2}\right)\right)^{1/K}, \log\left(1 + 2c\operatorname{sh}\frac{\rho}{2}\right)\right\}$$

$$= K^{1+c}\,\lambda(K)^{1/2}\max\{h_{\mathbb{H},c}(x,y)^{1/K}, h_{\mathbb{H},c}(x,y)\}.$$

■

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