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Inequalities for the generalized point pair function*

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Abstract. We study a new generalized version of the point pair function defined with a constant $\alpha > 0$. We prove that this function is a quasi-metric for all values of $\alpha > 0$ and compare it to several hyperbolic-type metrics, such as the j^*-metric, the triangular ratio metric, and the hyperbolic metric. Most of the inequalities presented here have the best possible constants in terms of α. Furthermore, we research the distortion of the generalized point pair function under conformal and quasiregular mappings.

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1 Introduction

Several different metrics can be used to study conformal, quasiconformal, quasiregular, or other types of mappings, which are an important subject of study in the geometric function theory [5, 9, 10]. In the plane the hyperbolic metric is useful for this purpose because of its invariance properties, but, unfortunately, it can be defined only in special cases in dimensions $n \geqslant 3$. For this reason, researchers have introduced numerous new hyperbolic-type metrics [3, 8, 13], which are designed after the hyperbolic metric so that they can measure the distances between points by taking their location with respect to the domain boundary into account.

Let $G \subsetneq \mathbb{R}^n$ be a domain. For all points $x \in G$, denote the Euclidean distance to the boundary by $d_G(x) = \inf_{z\in\partial G}|x-z|$. For $\alpha > 0$, define then the function $p_G^\alpha : G \times G \to [0, 1)$ by [2, p. 1391, (5.1)]

$$p_G^\alpha(x,y) = \frac{|x-y|}{\sqrt{|x-y|^2 + \alpha d_G(x)d_G(y)}}. \tag{1.1}$$

This function is called the *generalized point pair function*. It is derived from the point pair function p_G, which is the particular case $\alpha = 4$ of definition (1.1). The point pair function was originally introduced in [1] and studied further in [2, 7, 11, 12, 14]. It was observed to be a useful tool for creating bounds for the hyperbolic metric and proved to be a quasi-metric for all domains $G \subsetneq \mathbb{R}^n$ with the constant less than or equal to $\sqrt{5}/2$ [2, p. 1388, Thm. 4.14]. In 2022, Dautova et al. [2] introduced the generalized point pair function by replacing the constant 4 in the definition of the point pair function by a general constant $\alpha > 0$.

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Considering the point pair function is very well justified if the domain G is the upper half-space $\mathbb{H}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n\colon\ x_n > 0\}$. This is because for all points $x, y \in \mathbb{H}^n$, the distance $p_{\mathbb{H}^n}(x, y)$ in the point pair function is equal to the distance $\mathrm{th}(\rho_{\mathbb{H}^n}(x, y)/2)$, where th is the hyperbolic tangent, and $\rho_{\mathbb{H}^n}(x, y)$ is the hyperbolic metric defined in $\mathbb{H}^n$. However, defining this function with another constant instead of 4 might be more reasonable in some domains; this is why studying the generalized point pair function for values of $\alpha > 0$ is useful. For instance, for $0 < \alpha \leqslant 12$, it is known that the generalized point pair function is a metric in the domains $\mathbb{R}^+$ [2, Thm. 5.2, p. 1391], $\mathbb{R}^n \setminus \{0\}$ [2, p. 1395, Thm. 5.11], and $\mathbb{H}^n$ [2, p. 1396, Thm. 5.13]. Consequently, our aim is to study this function further by comparing it to several hyperbolic-type metrics.

The structure of this paper is as follows. In Section 2, we give the necessary notations and definitions. In Section 3, we study the inequality between the generalized point pair function and the hyperbolic-type metric, known as the j^*-metric, and prove that the generalized point pair function is a quasi-metric in every domain $G \subsetneq \mathbb{R}^n$. In Section 4, we give similar inequalities for the triangular ratio metric and the t-metric. Finally, in Section 5, we present the inequalities between the generalized point pair function and the hyperbolic metric and use them to get some results for the distortion of the distances in the generalized point pair function under conformal and quasiregular mappings.

2 Preliminaries

Recall that a function $d : G \times G \to \mathbb{R}$ is a metric in a domain G if for all $x, y, z \in G$, (i) $d(x, y) \geqslant 0$ and $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$, and (iii) $d(x, y) \leqslant d(x, z) + d(z, y)$. Property (iii) is called the *triangle inequality*. If a function fulfills the two first properties and the relaxed version

$$d(x, y) \leqslant c\big(d(x, z) + d(z, y)\big)$$

of the triangle inequality with a constant c independent of the choice of the points x, y, z, then we call it a *quasi-metric*. Note that such a function is sometimes called a semimetric, a metametric, or an inframetric.

The Euclidean open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted $B^n(x, r)$, and its sphere is denoted $S^{n-1}(x, r)$. A Euclidean line segment with endpoints $x, y \in \mathbb{R}^n$ is denoted $[x, y]$. The argument of a complex number $x \in \mathbb{C} \setminus \{0\}$ is $\mathrm{Arg}(x)$. Denote also $d_G(x) = \inf_{z \in \partial G} |x - z|$ for $x \in \mathbb{R}^n$ as in Introduction.

Define the original *point pair function* $p_G : G \times G \to [0, 1)$ [1, p. 685], [7, p. 1124, 2.4] as

$$p_G(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4 d_G(x) d_G(y)}}.$$

The generalized point pair function is as in (1.1). To avoid possible confusion, note that p_G means that $\alpha = 4$, and if α is unspecified, then we mean the generalized version p_G^α.

Next, consider the following hyperbolic-type metrics. The *distance ratio metric* $j_G : G \times G \to [0, \infty)$ introduced by Gehring and Osgood [4] is defined as [1, p. 685]

$$j_G(x, y) = \log\left(1 + \frac{|x - y|}{\min\{d_G(x), d_G(y)\}}\right).$$

This expression can be modified as in [7, p. 1123, 2.2 and p. 1124, Lemma 2.1] to define the *j^*-metric* $j_G^* : G \times G \to [0, 1]$ as

$$j_G^*(x, y) = \mathrm{th}\,\frac{j_G(x, y)}{2} = \frac{|x - y|}{|x - y| + 2\min\{d_G(x), d_G(y)\}}.$$

The *triangular ratio metric* $s_G : G \times G \to [0, 1]$, originally introduced by Hästö [8] in 2002, is defined as [1, p. 683, (1.1)]

$$s_G(x, y) = \frac{|x - y|}{\inf_{z \in \partial G}(|x - z| + |z - y|)}.$$

Furthermore, the *t-metric* $t_G : G \times G \to [0,1)$, defined as

$$t_G(x,y) = \frac{|x-y|}{|x-y| + d_G(x) + d_G(y)},$$

was recently introduced in [13], but we must note that unlike the distance ratio metric or the triangular ratio metric, this metric is not necessarily of hyperbolic type because the closures of its balls are not always compact in the domain G.

Use notations sh, ch and th for the hyperbolic sine, cosine, and tangent. Denote by $\mathbb{H}^n$ the upper half-space $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n\colon\ x_n > 0\}$ and by $\mathbb{B}^n$ the Poincaré unit ball $\{x \in \mathbb{R}^n\colon\ |x| < 1\}$. In these two domains, the hyperbolic metric has the following formulas [6, p. 52, (4.8) and p. 55, (4.14)]:

$$\operatorname{ch} \rho_{\mathbb{H}^n}(x,y) = 1 + \frac{|x-y|^2}{2d_{\mathbb{H}^n}(x)d_{\mathbb{H}^n}(y)}, \quad x,y \in \mathbb{H}^n,$$

$$\operatorname{sh}^2 \frac{\rho_{\mathbb{B}^n}(x,y)}{2} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}, \quad x,y \in \mathbb{B}^n.$$

In the two-dimensional disk, we have

$$\operatorname{th} \frac{\rho_{\mathbb{B}^2}(x,y)}{2} = \left| \frac{x-y}{1-x\overline{y}} \right|,$$

where $\overline{y}$ is the complex conjugate of y.

3 Inequalities with the j^*-metric

In this section, we first find the inequalities between the generalized point pair function and the j^*-metric, and then we study the sharpness of the established inequalities and use them to prove that the generalized point pair function is a quasi-metric.

Theorem 1. *For all $x, y \in G \subsetneq \mathbb{R}^n$ and $\alpha > 0$, we have the inequality*

$$\min\left\{1, \frac{2}{\sqrt{\alpha}}\right\} j^*_G(x,y) \leqslant p^\alpha_G(x,y) \leqslant \sqrt{\frac{\alpha+4}{\alpha}}\, j^*_G(x,y).$$

For the domain $G = \mathbb{H}^n$, these constants are the best possible in terms of α. In fact, the first constant is the best possible in terms of α for every choice of the domain $G \subsetneq \mathbb{R}^n$.

Proof. By symmetry we can fix distinct points $x, y \in G$ such that $d_G(x) \leqslant d_G(y)$. Now

$$\frac{j^*_G(x,y)}{p^\alpha_G(x,y)} = \frac{\sqrt{|x-y|^2 + \alpha d_G(x) d_G(y)}}{|x-y| + 2d_G(x)}. \tag{3.1}$$

Clearly, for fixed choices of $d_G(x)$ and $|x-y|$, this quotient is increasing with respect to $d_G(y)$. Because of the triangle inequality, $d_G(y) \leqslant d_G(x) + |x-y|$, so the value of $d_G(y)$ is limited to the closed interval from $d_G(x)$ to $d_G(x) + |x-y|$. Consequently, quotient (3.1) is at minimum with respect to $d_G(y)$ when $d_G(y) = d_G(x)$ and at maximum when $d_G(y) = d_G(x) + |x-y|$.

Let us first find the minimum of quotient (3.1) in the case $d_G(y)=d_G(x)$. By differentiation we have

$$\frac{\partial}{\partial|x-y|}\left(\frac{\sqrt{|x-y|^2+\alpha d_G(x)^2}}{|x-y|+2d_G(x)}\right)=\frac{d_G(x)(2|x-y|-\alpha d_G(x))}{\sqrt{|x-y|^2+\alpha d_G(x)^2}(|x-y|+2d_G(x))^2}=0$$
$$\iff\quad |x-y|=\frac{\alpha}{2}d_G(x).$$

We see that the stationary point above is a minimum, and since this quotient is $\sqrt{\alpha/(\alpha+4)}$ at $|x-y|=\alpha d_G(x)/2$, this is the minimum of quotient (3.1).

As explained above, we need to fix $d_G(y)=d_G(x)+|x-y|$ to find the maximum value of quotient (3.1). By differentiation we have

$$\frac{\partial}{\partial|x-y|}\left(\frac{\sqrt{|x-y|^2+\alpha d_G(x)(d_G(x)+|x-y|)}}{|x-y|+2d_G(x)}\right)$$
$$=\frac{d_G(x)|x-y|(4-\alpha)}{2\sqrt{|x-y|^2+\alpha d_G(x)(d_G(x)+|x-y|)}(|x-y|+2d_G(x))^2}\geqslant 0$$
$$\iff\quad \alpha\leqslant 4.$$

Because this quotient is either decreasing or increasing with respect to $|x-y|$ depending on α, its maximum has one of the limit values

$$\lim_{|x-y|\to 0^+}\frac{\sqrt{|x-y|^2+\alpha d_G(x)(d_G(x)+|x-y|)}}{|x-y|+2d_G(x)}=\frac{\sqrt{\alpha}}{2},$$
$$\lim_{|x-y|\to\infty}\frac{\sqrt{|x-y|^2+\alpha d_G(x)(d_G(x)+|x-y|)}}{|x-y|+2d_G(x)}=1.$$

Consequently, the supremum of quotient (3.1) is $\max\{1,\sqrt{\alpha}/2\}$.

Consider now the domain $G=\mathbb{H}^n$. Quotient (3.1) attains its minimum $\sqrt{\alpha/(\alpha+4)}$ when $x=(0,\ldots,0,1)$ and $y=(\alpha/2,0,\ldots,0,1)$ as in Fig. 1(a). Similarly, it approaches its maximum value $\max\{1,\sqrt{\alpha}/2\}$ when $x=(0,\ldots,0,1)$ and $y=(0,\ldots,0,1+k)$ as $k\to 0^+$ if $\alpha<4$ or $k\to\infty$ if $\alpha\geqslant 4$. Consequently, $\mathbb{H}^n$ is an example of a domain in which the found extreme values are the best possible constants in terms of α. Also, $\max\{1,\sqrt{\alpha}/2\}$ is the best possible upper bound for quotient (3.1) in terms of α, regardless of how the domain G is chosen, because we can always fix points $y\in G$, $z\in S^{n-1}(y,d_G(y))\cap(\partial G)$, and $x=y+k(z-y)$ with either $k\to 0^+$ or $k\to 1^-$, depending on α, so that we attain the limit value of quotient (3.1).

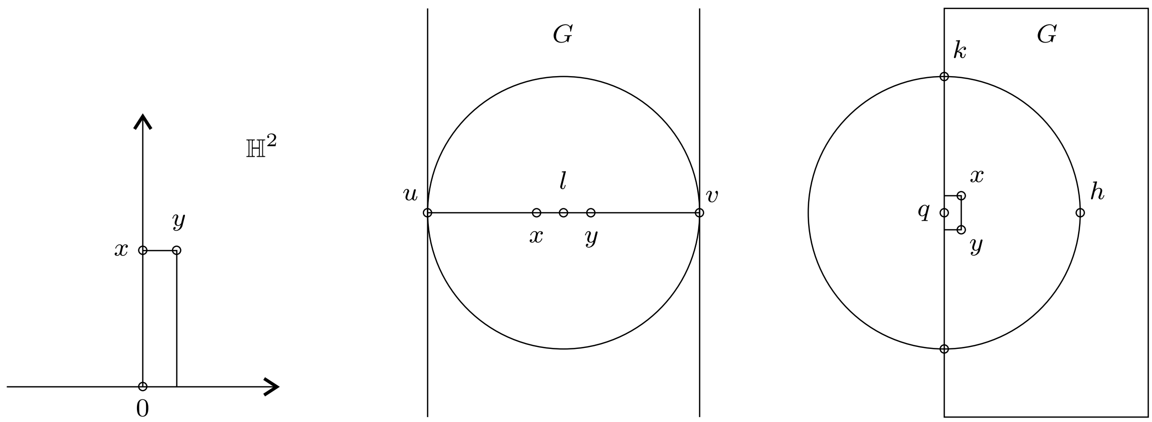
From this the theorem follows, but note that we consider the reciprocals of the found extreme values since the bounds are presented for the function $p^\alpha_G(x,y)$. $\square$

Although the latter constant in Theorem 1 is not sharp for some choices of $G\subsetneq\mathbb{R}^n$, it follows from the next result that this constant is the best possible in several common domains such as $\mathbb{B}^n$, $\mathbb{H}^n$, and $\mathbb{R}^n\setminus(\{0\}\cup\{1\})$.

Lemma 1. *For $G\subsetneq\mathbb{R}^n$ and $\alpha>0$, $\sqrt{(\alpha+4)/\alpha}$ is the best possible constant c in terms of α such that the inequality $p^\alpha_G(x,y)\leqslant cj^*_G(x,y)$ holds for all points $x,y\in G$ if*

(i) *G contains an open ball such that the end points of one of its diameters belong to the boundary ∂G, or*
(ii) *G contains an open half-ball, but one of its diameters is fully on the boundary ∂G.*

Proof. The inequality holds with $c=\sqrt{(\alpha+4)/\alpha}$ according to Theorem 1, and we can trivially verify that the equality holds here if $d_G(x)=d_G(y)$ and $|x-y|=\alpha d_G(x)/2$.



(a) $G = \mathbb{H}^2$, $\alpha = 0.5$ (b) G as in Lemma 1(i), $\alpha = 1$ (c) G as in Lemma 1(ii), $\alpha = 4$

Figure 1. Points $x, y \in G$ such that the equality $p_G^\alpha(x,y) = \sqrt{(\alpha+4)/\alpha}\, j_G^*(x,y)$ holds for different domains G and values of $\alpha > 0$.

Consider the first case where there are some points $u, v \in S^1(l, d_G(l)) \cap (\partial G)$ for $l = (u+v)/2$. Fix then $x = l + \alpha(u-l)/(\alpha+4)$ and $y = l + \alpha(v-l)/(\alpha+4)$; see Fig. 1(b). We will have $d_G(x) = 4d_G(l)/(\alpha+4) = d_G(y)$ and $|x-y| = \alpha d_G(x)/2$.

Suppose then that for $q \in \partial G$, $r > 0$, and $h \in S^{n-1}(q,r)$, the half-ball

$$\bigl\{z \in B^n(q,r)\colon\ |z-h| > \bigl|z - \bigl(h + 2r(q-h)\bigr)\bigr|\bigr\}$$

is included in G, but

$$\bigl[k,\ k + 2(q-k)\bigr] \subset \partial G \quad \text{for } k \in \bigl\{z \in S^{n-1}(q,r)\colon\ |z-h| = \bigl|z - \bigl(h + 2r(q-h)\bigr)\bigr|\bigr\}.$$

Fix

$$x = q + \frac{h-q}{\alpha+4} + \alpha\frac{k-q}{4(\alpha+4)}, \qquad y = q + \frac{h-q}{\alpha+4} - \alpha\frac{k-q}{4(\alpha+4)}$$

as in Fig. 1(c). Now $d_G(x) = d_G(y) = r/(\alpha+4)$ and $|x-y| = \alpha r/(2(\alpha+4)) = \alpha d_G(x)/2$, so the result follows. □

Lemma 2. *For all $x, y \in \mathbb{R}^n \setminus \{0\}$ and $\alpha > 0$, we have the inequalities*

$$\frac{2}{\sqrt{\alpha}} j^*_{\mathbb{R}^n\setminus\{0\}}(x,y) \leqslant p^\alpha_{\mathbb{R}^n\setminus\{0\}}(x,y) \leqslant \sqrt{1 + \frac{4}{\alpha}}\, j^*_{\mathbb{R}^n\setminus\{0\}}(x,y) \quad \textit{if } \alpha \leqslant 4$$

and

$$j^*_{\mathbb{R}^n\setminus\{0\}}(x,y) \leqslant p^\alpha_{\mathbb{R}^n\setminus\{0\}}(x,y) \leqslant \max\biggl\{1, \frac{4}{\sqrt{\alpha+4}}\biggr\} j^*_{\mathbb{R}^n\setminus\{0\}}(x,y) \quad \textit{if } \alpha > 4$$

with the best possible constants in terms of α.

Proof. The left sides of both inequalities follow from Theorem 1, according to which they also have the best possible constants. By symmetry assume that $|x| \leqslant |y|$ for the points $x, y \in G = \mathbb{R}^n \setminus \{0\}$. Let k be the angle between the vectors from the origin to x and to y. By writing the distance $|x-y|$ with law of cosines we will have

$$\frac{p^{\alpha}_{\mathbb{R}^n\setminus\{0\}}(x,y)}{j^*_{\mathbb{R}^n\setminus\{0\}}(x,y)}=\frac{\sqrt{|x|^2+|y|^2-2|x||y|\cos k}+2|x|}{\sqrt{|x|^2+|y|^2-2|x||y|\cos k+\alpha|x||y|}}. \tag{3.2}$$

To prove the right side of the inequalities in the lemma, we need to find the maximum value of this quotient.

By differentiation we have

$$\begin{aligned}&\frac{\partial}{\partial\cos k}\left(\frac{\sqrt{|x|^2+|y|^2-2|x||y|\cos k}+2|x|}{\sqrt{|x|^2+|y|^2+(\alpha-2\cos k)|x||y|}}\right)\\&=\frac{|x|^2|y|(2\sqrt{|x|^2+|y|^2-2|x||y|\cos k}-\alpha|y|)}{\sqrt{|x|^2+|y|^2-2|x||y|\cos k}(|x|^2+|y|^2+(\alpha-2\cos k)|x||y|)^{3/2}}=0\\&\iff\quad\cos k=\frac{4|x|^2+4|y|^2-\alpha^2|y|^2}{8|x||y|}.\end{aligned}$$

This stationary point is a maximum. It fulfills $-1\leqslant\cos k\leqslant 1$ if and only if

$$-4\big(|y|-|x|\big)^2\leqslant\alpha^2|y|^2\leqslant 4\big(|x|+|y|\big)^2\quad\iff\quad(\alpha-2)|y|\leqslant 2|x|,$$

which is only possible for $\alpha\leqslant 4$ given the limitation $|x|\leqslant|y|$. If $\cos k$ is equivalent to the stationary point above, then quotient (3.2) becomes $\sqrt{1+4|x|/(\alpha|y|)}$, which is decreasing with respect to $|y|$ and attains its maximum value $\sqrt{1+4/\alpha}$ at $|y|=|x|$. Consequently, this is the maximum value of quotient (3.2) if $\alpha\leqslant 4$.

Suppose then that $\alpha>4$. Now we must choose either $\cos k=-1$ or $\cos k=1$ to find the maximum value of quotient (3.2). If $\cos k=-1$, then this quotient becomes

$$\frac{|y|+3|x|}{\sqrt{|x|^2+|y|^2+(\alpha+2)|x||y|}}. \tag{3.3}$$

By differentiation we have

$$\frac{\partial}{\partial|y|}\left(\frac{|y|+3|x|}{\sqrt{|x|^2+|y|^2+(\alpha+2)|x||y|}}\right)=\frac{-|x|((2+\alpha)|x|+|y|/2)}{(|x|^2+|y|^2+(\alpha+2)|x||y|)^{3/2}}<0,$$

so quotient (3.3) is decreasing with respect to $|y|$ and has the maximum value $4/\sqrt{\alpha+4}$ at $|y|=|x|$.

If $\cos k=1$ instead, then quotient (3.2) becomes

$$\frac{|x|+|y|}{\sqrt{|x|^2+|y|^2+(\alpha-2)|x||y|}}. \tag{3.4}$$

By differentiation we have

$$\frac{\partial}{\partial|y|}\left(\frac{|x|+|y|}{\sqrt{|x|^2+|y|^2+(\alpha-2)|x||y|}}\right)=\frac{|x|(|y|-|x|)(\alpha/2-2)}{(|x|^2+|y|^2+(\alpha-2)|x||y|)^{3/2}}\geqslant 0$$

if $\alpha>4$ and $|x|\leqslant|y|$. Consequently, quotient (3.4) is increasing with respect to $|y|$ and its maximum has the limit value 1 obtained as $|y|\to\infty$. Thus the result follows. □

Corollary 1. *For all $G\subsetneq\mathbb{R}^n$ and $\alpha>0$, the function $p^{\alpha}_G(x,y)$ is a quasi-metric with a constant $\sqrt{\alpha+4}/2$ if $\alpha\leqslant 4$ and $\sqrt{(\alpha+4)/\alpha}$ if $\alpha>4$.*

Proof. From Theorem 1 and the fact that $j^*_G(x,y)$ is a metric it follows that

$$\begin{aligned}p^\alpha_G(x,y) &\leqslant \sqrt{\frac{\alpha+4}{\alpha}}j^*_G(x,y) \leqslant \sqrt{\frac{\alpha+4}{\alpha}}\big(j^*_G(x,z)+j^*_G(z,y)\big)\\ &\leqslant \sqrt{\frac{\alpha+4}{\alpha}}\max\left\{1,\frac{\sqrt{\alpha}}{2}\right\}\big(p^\alpha_G(x,z)+p^\alpha_G(z,y)\big). \qquad \square\end{aligned}$$

Note that for the domain $G=\mathbb{R}^n\setminus\{0\}$, the inequalities of Lemma 2 would give better constants for Corollary 1, but it is already proven that the generalized point pair function is a metric in $\mathbb{R}^n\setminus\{0\}$ for all $0<\alpha\leqslant 12$ [2, p. 1395, Thm. 5.11].

4 Inequalities with the triangular ratio metric and t-metric

Let us now find the inequalities between the generalized point pair function and the triangular ratio metric and t-metric.

Lemma 3. *For all $x,y\in G\subsetneq\mathbb{R}^n$ and $\beta>\alpha>0$, we have the inequality*

$$p^\beta_G(x,y)\leqslant p^\alpha_G(x,y)\leqslant\sqrt{\frac{\beta}{\alpha}}p^\beta_G(x,y)$$

with the best possible constants in terms of α.

Proof. Clearly, the quotient

$$\frac{p^\alpha_G(x,y)}{p^\beta_G(x,y)}=\sqrt{\frac{|x-y|^2+\beta d_G(x)d_G(y)}{|x-y|^2+\alpha d_G(x)d_G(y)}}$$

attains its minimum value 1 when either x or y approaches boundary so that $d_G(x)\to 0^+$ or $d_G(y)\to 0^+$, and its maximum value $\sqrt{\beta/\alpha}$ when the points x and y approach each other so that $|x-y|\to 0^+$. $\square$

Lemma 4. *For all $x,y\in G\subsetneq\mathbb{R}^n$ and $\alpha>0$,*

$$\max\left\{\frac{1}{2},\frac{\sqrt{\alpha}}{2\sqrt{2}}\right\}s_G(x,y)\leqslant p^\alpha_G(x,y)\leqslant\sqrt{\frac{\alpha+4}{\alpha}}s_G(x,y)\quad \textit{if } \alpha\leqslant 4,$$
$$\frac{1}{\sqrt{2}}s_G(x,y)\leqslant p^\alpha_G(x,y)\leqslant\sqrt{\frac{\alpha+4}{\alpha}}s_G(x,y)\quad \textit{if } \alpha>4,$$

and if G is convex, then the left sides of these inequalities can be improved by replacing the constants $\max\{1/2,\sqrt{\alpha}/(2\sqrt{2})\}$ and $1/\sqrt{2}$ by $\max\{1/\sqrt{2},\sqrt{\alpha}/2\}$ and 1,respectively.

Proof. By [7, p. 1124, Lemma 2.1 and p. 1125, Lemma 2.2] the inequality $j^*_G(x,y)\leqslant s_G(x,y)\leqslant 2j^*_G(x,y)$ holds for all $x,y\in G\subsetneq\mathbb{R}^n$, and by combining this with Theorem 1 we have

$$\min\left\{\frac{1}{2},\frac{1}{\sqrt{\alpha}}\right\}s_G(x,y)\leqslant p^\alpha_G(x,y)\leqslant\sqrt{\frac{\alpha+4}{\alpha}}s_G(x,y). \tag{4.1}$$

From Lemma 3 it also follows that

$$\min\left\{1,\frac{\sqrt{\alpha}}{2}\right\}p_G(x,y)\leqslant p^\alpha_G(x,y)\leqslant\max\left\{1,\frac{\sqrt{\alpha}}{2}\right\}p_G(x,y) \tag{4.2}$$

and, by [14, Thm. 3.6], $1/\sqrt{2}p_G(x,y)\leqslant s_G(x,y)\leqslant\sqrt{2}p_G(x,y)$. Consequently,

$$\frac{1}{\sqrt{2}}\min\left\{1,\frac{\sqrt{\alpha}}{2}\right\}s_G(x,y)\leqslant p_G^\alpha(x,y)\leqslant\max\left\{\sqrt{2},\sqrt{\frac{\alpha}{2}}\right\}s_G(x,y). \tag{4.3}$$

Let us now combine inequalities (4.1) and (4.3). Note that

$$\max\left\{\min\left\{\frac{1}{2},\frac{1}{\sqrt{\alpha}}\right\},\ \min\left\{\frac{1}{\sqrt{2}},\frac{\sqrt{\alpha}}{2\sqrt{2}}\right\}\right\}=\begin{cases}1/2 & \text{if } \alpha\leqslant 2,\\ \sqrt{\alpha}/(2\sqrt{2}) & \text{if } 2<\alpha\leqslant 4\ ,\\ 1/\sqrt{2} & \text{if } \alpha>4\end{cases}$$

and

$$\sqrt{\frac{\alpha+4}{\alpha}}\leqslant\max\left\{\sqrt{2},\sqrt{\frac{\alpha}{2}}\right\},$$

from which the first part of the lemma follows.

Suppose then that G is convex. By [7, p. 1129, Thm. 2.9(i)], in this case, $s_G(x,y)\leqslant\sqrt{2}j_G^*(x,y)$, so it follows from Theorem 1 that

$$\min\left\{\frac{1}{\sqrt{2}},\sqrt{\frac{2}{\alpha}}\right\}s_G(x,y)\leqslant p_G^\alpha(x,y).$$

Furthermore, $s_G(x,y)\leqslant p_G(x,y)$ in a convex domain G by [6, p. 197, Lemma 11.6(1)], so it follows from inequality (4.2) that

$$\min\left\{1,\frac{\sqrt{\alpha}}{2}\right\}s_G(x,y)\leqslant p_G^\alpha(x,y).$$

The rest of the lemma follows from the two inequalities above as

$$\max\left\{\min\left\{\frac{1}{\sqrt{2}},\sqrt{\frac{2}{\alpha}}\right\},\ \min\left\{1,\frac{\sqrt{\alpha}}{2}\right\}\right\}=\begin{cases}1/\sqrt{2} & \text{if } \alpha\leqslant 2,\\ \sqrt{\alpha}/2 & \text{if } 2<\alpha\leqslant 4,\\ 1 & \text{if } \alpha>4.\end{cases}\qquad\square$$

Lemma 5. *For all $x,y\in G\subsetneq\mathbb{R}^n$ and $\alpha>0$, we have the inequalities*

$$t_G(x,y)\leqslant p_G^\alpha(x,y)\leqslant\frac{4}{\sqrt{\alpha(4-\alpha)}}t_G(x,y)\quad \textit{if } \alpha<2$$

and

$$\min\left\{1,\frac{2}{\sqrt{\alpha}}\right\}t_G(x,y)\leqslant p_G^\alpha(x,y)\leqslant 2t_G(x,y)\quad \textit{if } \alpha\geqslant 2$$

with the best possible constants in terms of α.

Proof. Consider the quotient

$$\frac{p_G^\alpha(x,y)}{t_G(x,y)}=\frac{|x-y|+d_G(x)+d_G(y)}{\sqrt{|x-y|^2+\alpha d_G(x)d_G(y)}}. \tag{4.4}$$

By differentiation we have

$$\frac{\partial}{\partial d_G(y)}\left(\frac{|x-y|+d_G(x)+d_G(y)}{\sqrt{|x-y|^2+\alpha d_G(x)d_G(y)}}\right)=\frac{|x-y|^2+\frac{\alpha}{2}d_G(x)(d_G(y)-|x-y|-d_G(x))}{2(|x-y|^2+\alpha d_G(x)d_G(y))^{3/2}}=0$$
$$\iff\quad d_G(y)=|x-y|+d_G(x)-\frac{2|x-y|^2}{\alpha d_G(x)}.$$

The stationary point above is a minimum. By symmetry let us assume that $d_G(x)\leqslant d_G(y)$. It follows from the triangle inequality that $d_G(y)\leqslant |x-y|+d_G(x)$. Consequently, we can choose $d_G(y)=|x-y|+d_G(x)-2|x-y|^2/(\alpha d_G(x))$ if and only if

$$d_G(x)\leqslant |x-y|+d_G(x)-\frac{2|x-y|^2}{\alpha d_G(x)}\leqslant |x-y|+d_G(x)\quad\iff\quad 0\leqslant |x-y|\leqslant\frac{\alpha d_G(x)}{2}.$$

Suppose first that $|x-y|\leqslant \alpha d_G(x)/2$. If $d_G(y)=|x-y|+d_G(x)-2|x-y|^2/(\alpha d_G(x))$, then quotient (4.4) becomes

$$\frac{2}{\alpha d_G(x)}\sqrt{\alpha d_G(x)\big(|x-y|+d_G(x)\big)-|x-y|^2}.$$

By differentiation we have

$$\frac{\partial}{\partial|x-y|}\big(\alpha d_G(x)\big(|x-y|+d_G(x)\big)-|x-y|^2\big)=\alpha d_G(x)-2|x-y|,$$

so the expression is increasing with respect to $|x-y|$ when $|x-y|\leqslant(\alpha d_G(x))/2$. It has the limit value

$$\lim_{|x-y|\to 0^+}\left(\frac{2}{\alpha d_G(x)}\sqrt{\alpha d_G(x)\big(|x-y|+d_G(x)\big)-|x-y|^2}\right)=\frac{2}{\sqrt{\alpha}}.$$

Consider then the case $|x-y|>\alpha d_G(x)/2$. Now quotient (4.4) is increasing with respect to $d_G(y)$. If $d_G(y)=d_G(x)$, then quotient (4.4) becomes

$$\frac{|x-y|+2d_G(x)}{\sqrt{|x-y|^2+\alpha d_G(x)^2}}. \tag{4.5}$$

By differentiation we have

$$\frac{\partial}{\partial|x-y|}\left(\frac{|x-y|+2d_G(x)}{\sqrt{|x-y|^2+\alpha d_G(x)^2}}\right)=\frac{d_G(x)(\alpha d_G(x)-2)}{(|x-y|^2+\alpha d_G(x)^2)^{3/2}}\geqslant 0\quad\iff\quad d_G(x)\geqslant\frac{2}{\alpha}.$$

Consequently, quotient (4.4) is monotonic with respect to $|x-y|$. Quotient (4.4) has the limit value $\sqrt{1+4/\alpha}$ as $|x-y|\to\alpha d_G(x)/2$ and the limit value 1 as $|x-y|\to\infty$, out of which 1 is smaller. Thus it follows that the infimum of the quotient (4.4) is $\min\{1,2/\sqrt{\alpha}\}$.

It follows from the earlier differentiation of quotient (4.4) that it is at maximum with respect to $d_G(y)$ in one of the end points of the interval $[d_G(x),d_G(x)+|x-y|]$. If $d_G(y)=d_G(x)$, then quotient (4.4) is quotient (4.5), which was noted to be monotonic with respect to $|x-y|$. The maximum value of quotient (4.5) has either the limit value $2/\sqrt{\alpha}$ obtained as $|x-y|\to 0^+$ or 1 obtained as $|x-y|\to\infty$, depending on if $\alpha\leqslant 4$ or not.

If $d_G(y)=d_G(x)+|x-y|$, then quotient (4.4) becomes

$$\frac{2(|x-y|+d_G(x))}{\sqrt{|x-y|^2+\alpha d_G(x)(d_G(x)+|x-y|)}}. \tag{4.6}$$

By differentiation we have

$$\frac{\partial}{\partial|x-y|}\left(\frac{2(|x-y|+d_G(x))}{\sqrt{|x-y|^2+\alpha d_G(x)(d_G(x)+|x-y|)}}\right)=\frac{d_G(x)((\alpha-2)|x-y|+\alpha d_G(x))}{(|x-y|^2+\alpha d_G(x)(d_G(x)+|x-y|))^{3/2}}.$$

The derivative above is positive if either $\alpha\geqslant 2$ or $\alpha<2$ and $|x-y|<\alpha d_G(x)/(2-\alpha)$. Consequently, if $\alpha\geqslant 2$, then quotient (4.6) is increasing with respect to $|x-y|$, and its maximum has the limit value 2 obtained as $|x-y|\to\infty$. If $\alpha<2$ instead, then the maximum of quotient (4.6) is $4/\sqrt{\alpha(4-\alpha)}$ at $|x-y|=\alpha d_G(x)/(2-\alpha)$. Because these values are greater than the limit values of the maximum values of quotient (4.5), it follows that the supremum of quotient (4.4) is either $4/\sqrt{\alpha(4-\alpha)}$ if $\alpha<2$ and 2 if $\alpha\geqslant 2$. The inequalities of the lemma now follow.

The limit value $2/\sqrt{\alpha}$ of quotient (4.4) can be obtained in any domain $G\subsetneq\mathbb{R}^n$ by choosing $y\in B^n(x,d_G(x))$ for any fixed point $x\in G$ so that $d_G(y)\to d_G(x)$ and $|x-y|\to 0^+$. Similarly, the limit value 1 can be found by choosing $x,y\in G$ so that $d_G(x),d_G(y)\to 0^+$. If $\alpha<2$, then the value $4/\sqrt{\alpha(4-\alpha)}$ of quotient (4.4) is possible to find by fixing $x\in G$, $z\in S^{n-1}(x,d_G(x))\cap(\partial G)$, and $y=x+\alpha(z-x)/2$. Furthermore, the limit value 2 of quotient (4.4) can be obtained if we fix $x\in G$, $z\in S^{n-1}(x,d_G(x))\cap(\partial G)$ and $y=z+k(x-z)$ with $k\to 0^+$, because then $d_G(y)\to 0^+$ but $|x-y|\to d_G(x)>0$. It follows from this that we have the best constants in terms of α regardless of the choice of the domain G. □

5 Inequalities with the hyperbolic metric

In this section, we first study the inequalities between the generalized point pair function and the hyperbolic metric in the upper half-space and the unit ball and then study the distortion of the generalized point pair function under Möbius and quasiregular mappings.

Corollary 2. *For all $x,y\in\mathbb{H}^n$ and $\alpha>0$, we have the inequality*

$$\min\left\{1,\frac{\sqrt{\alpha}}{2}\right\}\operatorname{th}\frac{\rho_{\mathbb{H}^n}(x,y)}{2}\leqslant p^\alpha_{\mathbb{H}^n}(x,y)\leqslant\max\left\{1,\frac{2}{\sqrt{\alpha}}\right\}\operatorname{th}\frac{\rho_{\mathbb{H}^n}(x,y)}{2}$$

with the best possible constants in terms of α.

Proof. The results follows from Lemma 3 and the fact that $\operatorname{th}(\rho_{\mathbb{H}^n}(x,y)/2)=p_{\mathbb{H}^n}(x,y)$ by [6, p. 460]. □

Theorem 2. *For all $x,y\in\mathbb{B}^n$ and $\alpha>0$, we have the inequality*

$$\min\left\{1,\frac{1}{\sqrt{\alpha}}\right\}\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\leqslant p^\alpha_{\mathbb{B}^n}(x,y)\leqslant\max\left\{1,\frac{2}{\sqrt{\alpha}}\right\}\operatorname{th}\frac{\rho_{\mathbb{B}^n}(x,y)}{2}$$

with the best possible constants in terms of α.

Proof. The values of $p^\alpha_{\mathbb{B}^n}(x,y)$ and $\rho_{\mathbb{B}^n}(x,y)$ only depend on how the points x,y are fixed on the intersection of the unit ball and the two-dimensional plane containing x, y, and the origin, so it suffices to prove this inequality in the case $n=2$ by studying the quotient

$$\frac{p^\alpha_{\mathbb{B}^2}(x,y)}{\operatorname{th}(\rho_{\mathbb{B}^2}(x,y)/2)}=\frac{|1-x\overline{y}|}{\sqrt{|x-y|^2+\alpha(1-|x|)(1-|y|)}}. \tag{5.1}$$

If $y = 0$, then quotient (5.1) becomes

$$\frac{1}{\sqrt{|x|^2 + \alpha(1 - |x|)}}, \tag{5.2}$$

which approaches $1/\sqrt{\alpha}$ as $|x| \to 0^+$ and 1 as $|x| \to 1^-$. By differentiation we have

$$\frac{\partial}{\partial |x|}\bigl(|x|^2 + \alpha\bigl(1 - |x|\bigr)\bigr) = 2|x| - \alpha = 0 \quad\Longleftrightarrow\quad |x| = \frac{\alpha}{2}.$$

It follows that if $\alpha < 2$, then the maximum of quotient (5.2) is $2/\sqrt{\alpha(4-\alpha)}$. Otherwise, the maximum is 1. The minimum of quotient (5.2) is also 1 or $1/\sqrt{\alpha}$, depending on if $\alpha < 1$ or not. By symmetry these are the extreme values of quotient (5.1) also in the case $x = 0$.

Suppose then that $x \neq 0 \neq y$. Let $k \in [0, \pi]$ be the angle between the vectors from the origin to x and y, or, equivalently, $k = \operatorname{Arg}(x/y)$. By the law of cosines,

$$|1 - x\overline{y}| = \bigl|1 - |x||y|\mathrm{e}^{\operatorname{Arg}(x/y)i}\bigr| = \sqrt{1 + |x|^2|y|^2 - 2|x||y|\cos k},$$
$$|x - y| = \sqrt{|x|^2 + |y|^2 - 2|x||y|\cos k}.$$

Consequently, quotient (5.1) can be written as

$$\sqrt{\frac{1 + |x|^2|y|^2 - 2|x||y|\cos k}{|x|^2 + |y|^2 - 2|x||y|\cos k + \alpha(1 - |x|)(1 - |y|)}}. \tag{5.3}$$

By differentiation we have

$$\frac{\partial}{\partial \cos k}\left(\frac{1 + |x|^2|y|^2 - 2|x||y|\cos k}{|x|^2 + |y|^2 - 2|x||y|\cos k + \alpha(1 - |x|)(1 - |y|)}\right)$$
$$= \frac{2|x||y|(1 - |x|)(1 - |y|)((1 + |x|)(1 + |y|) - \alpha)}{(|x|^2 + |y|^2 - 2|x||y|\cos k + \alpha(1 - |x|)(1 - |y|))^2}.$$

Thus quotient (5.3) is monotonic with respect to $\cos k$ and is at minimum when $\cos k = -1$ and at maximum when $\cos k = 1$ or vice versa, depending on if $\alpha < (1 + |x|)(1 + |y|)$ or not.

Let us first consider the case $\cos k = -1$. Now quotient (5.3) becomes

$$\sqrt{\frac{(1 + |x||y|)^2}{(|x| + |y|)^2 + \alpha(1 - |x|)(1 - |y|)}}. \tag{5.4}$$

By differentiation we have

$$\frac{\partial}{\partial |y|}\left(\frac{(1 + |x||y|)^2}{(|x| + |y|)^2 + \alpha(1 - |x|)(1 - |y|)}\right)$$
$$= \frac{(1 - |x|)(1 + |x||y|)(\alpha(1 - |x||y| + 2|x|) - 2(1 + |x|)(|x| + |y|))}{(|x| + |y|)^2 + \alpha(1 - |x|)(1 - |y|)^2}.$$

We see that the only stationary point of quotient (5.4) with respect to $|y|$ is a maximum. However, the maximum of quotient (5.4) is the maximum of quotient (5.3) if and only if $\alpha \geqslant (1 + |x|)(1 + |y|)$. Because the stationary point fulfills

$$\alpha=\frac{2(1+|x|)(|x|+|y|)}{(1-|x||y|+2|x|)}<\big(1+|x|\big)\big(1+|y|\big)\quad\iff\quad 0<\big(1-|y|\big)\big(1+|x||y|\big),$$

it cannot be the maximum of quotient (5.3). Thus quotient (5.4) can offer extreme values of quotient (5.3) only as $|y|\to0^+$ or $|y|\to1^-$. The case $y=0$ was already considered earlier, and if $|y|\to1^-$, then quotient (5.4) approaches 1.

Let us next consider the case $\cos k=1$, where quotient (5.3) is

$$\sqrt{\frac{(1-|x||y|)^2}{(|x|-|y|)^2+\alpha(1-|x|)(1-|y|)}}. \tag{5.5}$$

By differentiation we have

$$\begin{aligned}&\frac{\partial}{\partial|y|}\left(\frac{(1-|x||y|)^2}{(|x|-|y|)^2+\alpha(1-|x|)(1-|y|)}\right)\\&\quad=\frac{(1-|x|)(1-|x||y|)(\alpha(1+|x||y|-2|x|)+2(1+|x|)(|x|-|y|))}{(|x|-|y|)^2+\alpha(1-|x|)(1-|y|)^2}=0\\&\iff\quad |y|=\frac{\alpha(1-2|x|)+2|x|(1+|x|)}{2(1+|x|)-\alpha|x|}.\end{aligned}\tag{5.6}$$

If $|y|$ is in (5.6), then quotient (5.5) is

$$\sqrt{\frac{4(|x|^2+(2-\alpha)|x|+1)^2}{\alpha((4-\alpha)|x|^2+(\alpha^2-6\alpha+8)|x|+4-\alpha)}}. \tag{5.7}$$

Again, by differentiation we have

$$\begin{aligned}&\frac{\partial}{\partial|x|}\left(\frac{(|x|^2+(2-\alpha)|x|+1)^2}{(4-\alpha)|x|^2+(\alpha^2-6\alpha+8)|x|+4-\alpha}\right)=\frac{\alpha-2(1+|x|)}{\alpha-4}=0\\&\iff\quad |x|=\frac{\alpha-2}{2}.\end{aligned}$$

Quotient (5.7) is 1 at $x=(\alpha-2)/2$. If $|x|\to1^-$, then $|y|\to1^-$ if $|y|$ is as in (5.6), and quotient (5.7) approaches $2/\sqrt{\alpha}$. Quotient (5.5) approaches 1 as $|y|\to1^-$.

Thus all the potential extreme values of quotient (5.1) and their limit values are 1, $1/\sqrt{\alpha}$, $2/\sqrt{\alpha}$, and, if $\alpha<2$, $2/\sqrt{\alpha(4-\alpha)}$. Note that $2/\sqrt{\alpha(4-\alpha)}$ is never an extreme value of this quotient because it is obtained only if $0<\alpha<2$ and the inequality $1/\sqrt{\alpha}<2/\sqrt{\alpha(4-\alpha)}<2/\sqrt{\alpha}$ holds for all $0<\alpha<3$. The inequality of the theorem follows, and since the values are either extreme values of quotient (5.1) or their limit values, there are no better constants in terms of α. $\square$

Corollary 3. *For all $x,y\in\mathbb{B}^n$ and $\alpha>0$ and any conformal mapping $f:\mathbb{B}^n\to\mathbb{B}^n=f(\mathbb{B}^n)$,*

$$\min\left\{\frac{\sqrt{\alpha}}{2},\frac{1}{2},\frac{1}{\sqrt{\alpha}}\right\}p^{\alpha}_{\mathbb{B}^n}(x,y)\leqslant p^{\alpha}_{\mathbb{B}^n}\big(f(x),f(y)\big)\leqslant\max\left\{\frac{2}{\sqrt{\alpha}},2,\sqrt{\alpha}\right\}p^{\alpha}_{\mathbb{B}^n}(x,y).$$

Proof. By Theorem 2 and the conformal invariance of the hyperbolic metric,

$$\begin{aligned} p^{\alpha}_{\mathbb{B}^n}(f(x), f(y)) &\leqslant \max\left\{1, \frac{2}{\sqrt{\alpha}}\right\} \operatorname{th} \frac{\rho_{\mathbb{B}^n}(f(x), f(y))}{2} = \max\left\{1, \frac{2}{\sqrt{\alpha}}\right\} \operatorname{th} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \\ &\leqslant \max\left\{1, \frac{2}{\sqrt{\alpha}}\right\} \frac{1}{\min\{1, 1/\sqrt{\alpha}\}} p^{\alpha}_{\mathbb{B}^n}(x, y) = \max\left\{\frac{2}{\sqrt{\alpha}}, 2, \sqrt{\alpha}\right\} p^{\alpha}_{\mathbb{B}^n}(x, y), \end{aligned}$$

and since the inverse mapping f^{-1} of any conformal mapping is another conformal mapping, the first part of inequality follows directly from this. □

Consider the Möbius transformation $T_a : \mathbb{B}^2 \to \mathbb{B}^2$ defined as $T_a(z) = (z - a)/(1 - \overline{a}z)$. It has been observed that the Lipschitz constant of this mapping seems to be $1 + |a|$ for several intrinsic metrics and quasi-metrics defined in the unit disk, including the triangular ratio metric [1, p. 684, Conj. 1.6], the j^*-metric [12], the t-metric [13, Conj. 4.4], the point pair function [12], and the Barrlund metric [3, p. 25, Conj. 4.3]. Computer tests suggest that this also holds for the generalized point pair function regardless of the value of $\alpha > 0$.

Conjecture 1. For all $x, y, a \in \mathbb{B}^2$ and $\alpha > 0$,

$$\frac{1}{1 + |a|} p^{\alpha}_{\mathbb{B}^2}(x, y) \leqslant p^{\alpha}_{\mathbb{B}^2}\big(T_a(x), T_a(y)\big) \leqslant \big(1 + |a|\big) p^{\alpha}_{\mathbb{B}^2}(x, y).$$

DEFINITION 1. (See [6, pp. 288–289].) Let $G \subset \mathbb{R}^n$ be a domain. See [6, p. 149, Def. 9.1] for a definition of a function that is absolute continuous on lines, abbreviated as ACL. Denote the derivative of f at x by $f'(x)$ and the Jacobian determinant of f at x by $J_f(x)$. A mapping $f : G \to \mathbb{R}^n$ is quasiregular if it is ACL^n and there is constant $K \geqslant 1$ such that

$$\big|f'(x)\big|^n \leqslant K J_f(x), \qquad \big|f'(x)\big| = \max_{|h|=1} \big|f'(x)h\big|, \tag{5.8}$$

holds almost everywhere (a.e.) in G. If f is quasiregular, then the smallest $K \geqslant 1$ with which (5.8) holds is the outer dilatation of f, denoted by $K_O(f)$, and the smallest $K \geqslant 1$ such that the inequality

$$J_f(x) \leqslant K l\big(f'(x)\big)^n, \quad l\big(f'(x)\big) = \min_{|h|=1} \big|f'(x)h\big|,$$

holds a.e. in G is the inner dilatation of f, denoted by $K_I(f)$. A quasiregular mapping f is K-quasiregular if

$$\max\big\{K_O(f), K_I(f)\big\} \leqslant K.$$

By [6, p. 157, (9.5) and p. 158, (9.6)] define the constant

$$\log \lambda_n = \lim_{t \to \infty} \left(\left(\frac{\gamma_n(t)}{\omega_{n-1}} \right)^{n-1} - \log t \right),$$

where γ_n is the Grötzsch capacity defined as in [6, p. 121, (7.17)]. By [6, p. 122, (7.18)], in the two-dimensional case, $\gamma_2(t) = 4\mathcal{K}(1/t)/\mathcal{K}(\sqrt{1 - 1/t^2})$, where $\mathcal{K}$ is a complete elliptic integral of the first kind. This integral is defined as

$$\mathcal{K}(r) = \int_0^1 \frac{1}{\sqrt{(1 - x^2)(1 - r^2x^2)}} \, \mathrm{d}x, \quad 0 < r < 1,$$

and can be computed with ready functions in many programming languages.

Theorem 3. (*See* [6, *p.* 300, *Thm.* 16.2(1)].) *If $G, G' \in \{\mathbb{H}^n, \mathbb{B}^n\}$ and $f : G \to G'$ is a nonconstant K-quasiregular mapping with $f(G) \subset G'$, then*

$$\text{th}\,\frac{\rho_{G'}(f(x), f(y))}{2} \leqslant \lambda_n^{1-c}\left(\text{th}\,\frac{\rho_G(x,y)}{2}\right)^c,$$

where $c = K_I(f)^{1/(1-n)}$.

Corollary 4. *If $f : \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$ is a nonconstant K-quasiregular mapping, then for all $x, y \in \mathbb{B}^n$ and $\alpha > 0$,*

$$p^\alpha_{\mathbb{B}^n}\big(f(x), f(y)\big) \leqslant \lambda_n^{1-c} \max\left\{1, \frac{2}{\sqrt{\alpha}}, (\sqrt{\alpha})^c, 2(\sqrt{\alpha})^{c-1}\right\} p^\alpha_{\mathbb{B}^n}(x,y)^c,$$

where $c = K_I(f)^{1/(1-n)}$.

Proof. By Theorems 2 and 3,

$$\begin{aligned} p^\alpha_{\mathbb{B}^n}\big(f(x), f(y)\big) &\leqslant \max\left\{1, \frac{2}{\sqrt{\alpha}}\right\} \text{th}\,\frac{\rho_{\mathbb{B}^n}(f(x), f(y))}{2} \leqslant \lambda_n^{1-c}\max\left\{1, \frac{2}{\sqrt{\alpha}}\right\}\left(\text{th}\,\frac{\rho_{\mathbb{B}^n}(x,y)}{2}\right)^c \\ &\leqslant \lambda_n^{1-c}\max\left\{1, \frac{2}{\sqrt{\alpha}}\right\}\left(\frac{1}{\min\{1, 1/\sqrt{\alpha}\}} p^\alpha_{\mathbb{B}^n}(x,y)\right)^c \\ &= \lambda_n^{1-c}\max\left\{1, \frac{2}{\sqrt{\alpha}}, (\sqrt{\alpha})^c, 2(\sqrt{\alpha})^{c-1}\right\}\big(p^\alpha_{\mathbb{B}^n}(x,y)\big)^c. \qquad \square \end{aligned}$$

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