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Systematic Review of Research on Pedagogical Content Knowledge in Mathematics: Insights from a Topic-Specific Approach

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Abstract
One of the most important characteristics of mathematics teachers is their knowledge enabling them to provide learning experiences that promote students' understanding, defined as pedagogical content knowledge by Shulman in 1986. In recent decades, the number of studies on mathematics teachers' pedagogical content knowledge has steadily increased. The current review captures the results of 237 studies published in the last ten years (2013-2022), thematically addressing two components included in all definitions of pedagogical content knowledge - knowledge of instructional strategies and representations and knowledge of student learning difficulties. We approach this review from a topic-specific perspective presenting results of PCK research on nine mathematical topics, the most research relating to rational numbers, geometry and algebra. Altogether 83 unique topic-specific learning difficulties were mentioned in PCK measures including procedural and conceptual difficulties or combinations of them. Teaching strategies were reported less frequently and some strategies such as the use of manipulatives and visualizations were mentioned in relation to many mathematical topics.

Keywords Systematic literature review · Mathematics education · Pedagogical content knowledge · Mathematical knowledge for teaching

1 Introduction

To improve mathematics teacher education, it is crucial to recognize that school mathematics is unique and distinct from, but connected to, the broader scientific understanding of mathematics (Watson, 2008). It is widely agreed that mathematics teachers need to have a deep understanding of curriculum content. However, they also need to know their students (Hill et al., 2008) and be able to explain mathematical concepts, provide age-appropriate feedback on students' thinking and learning (NCTM, 2000).

Although scientific knowledge of mathematics is important (Hill et al., 2008; Lee, 2007), it is not enough to ensure high-quality mathematics teaching (Blömeke et al., 2022; Stigler & Miller, 2018). Mathematics teachers are not always aware of the challenges students face in understanding the subject and they often lack knowledge on how to explain difficult mathematical concepts in an insightful way (Depaepe et al., 2015).

2 Theoretical Framework

2.1 Conceptualisation of Pedagogical Content Knowledge

Shulman (1986) emphasized the importance of teachers' specialised knowledge required for teaching content, which he called pedagogical content knowledge (PCK). PCK consists of a synthesis of pedagogical knowledge (general knowledge of teaching methods) and content knowledge (Depaepe et al., 2013) that allows teachers to organise

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their knowledge from a teaching perspective and use it as a foundation to help students understand specific concepts (Cochran, 1997).

Over the past three decades, the idea of separating mathematics teachers' general content knowledge (CK) from PCK has become increasingly popular, and numerous researchers have related the PCK concept to teachers' competencies either related to their behaviour (e.g., quality of instruction) and/or beliefs and knowledge (Depaepe et al., 2013). However, many researchers raise questions about the nature of this particular type of knowledge (Mason, 2008; van Driel & Berry, 2012) or suggest different names and definitions for it (Ball et al., 2008). For instance, Hill et al. (2008) questioned the distinction between CK and PCK and proposed the more integrated concepts of mathematical knowledge for teaching (MKT) and content knowledge for teaching mathematics (CKTM) (Hill et al., 2008). Subsequent researchers have analysed the dimensionality of mathematics teachers' knowledge, with mixed results (summary by Copur-Gencturk & Tolar, 2022).

Researchers have also raised questions concerning the appropriate level of granularity for PCK (Chan & Hume, 2019), asking whether it mainly relates to general disciplinary principles and practices or whether its value lies in its topic specificity (van Driel et al., 1998).

2.2 Approaches to the Study of Mathematics Teachers' PCK

Most studies on PCK have been based on questionnaires, but some have employed more dynamic measures, such as video-based assessments in view of simulating more realistic classroom situations (Stigler & Miller, 2018). Few questionnaire studies have described in detail how the items used to analyse PCK were constructed. In a large-scale study, Hill et al. (2008) explained their development of questionnaire items to measure teachers' content knowledge and students' (KCS), which is one of the components of MKT. They mainly considered empirical studies on the development and learning of mathematics, but drew also on their own experiences as mathematics teachers.

In line with the methodological approach followed by Hill et al. (2008), empirical research has provided a basis for creating test items to analyse teachers' knowledge of students' mathematical thinking and learning. Several studies have focused on specific mathematical topics that learners at different levels of education find challenging, such as algebra (Wang, 2015) and rational numbers (Lortie-Forgues et al., 2015). Many studies have uncovered typical errors made by students regarding certain mathematical concepts and procedures. However, errors can be random slips, but from a pedagogical point of view, frequent, systematic errors, such as regarding 'buggy procedures' in written arithmetic (Brown & VanLehn, 1980), are more important.

Examining student errors merely in procedural terms is too simplistic, because, in many cases, procedural errors may have a deeper conceptual basis (De Corte et al., 1991). Thus, when analysing PCK, researchers such as Depaepe et al. (2015) and Hill et al. (2008) have understood students' errors as being caused by deeper misconceptions, such as a lack of understanding of the decimal system (Fuson, 1992) or an inability to recognise the difference between integers, natural and rational numbers (Van Hoof et al., 2017). The term 'misconception' is often used to refer to erroneous mathematical thoughts and beliefs (overview – Jamaludin & Maat, 2020). According to a narrower definition, the term refers to errors that are hard to correct because they are based on deeply entrenched concepts and beliefs that can only be overcome by means of conceptual change (Vosniadou & Verschaffel, 2004).

The other aspect of PCK refers to teachers' knowledge about teaching methods relevant for various mathematical topics. PCK measure are not always evidence-based but relay also on general theoretical ideas or researchers' own experiences (Hill et al., 2008).

2.3 Reviews and Large-Scale Studies

Depaepe et al. (2013) examined the conceptualisation of PCK in empirical mathematics education research. The review included 60 research articles, and has considered conceptualisations of PCK in the studies, the research methods used, and the mathematical subfields studied. Fractions, algebra, and functions were the most frequently mentioned mathematical subfields in terms of measuring PCK. Furthermore, they distinguished between cognitive and situated approaches to assess PCK. However, although the study was informative regarding the aspects examined, it did not provide detailed insights into the specific learning difficulties about these mathematical subtopics and/or the strategies for teaching these mathematical subtopics to students reported in the reviewed articles.

In another review study, Şimşek and Boz (2016) analysed 56 studies published between 2004 and 2015 on Turkish mathematics teachers' PCK. They reported that algebra was the topic most frequently studied by mathematics researchers, followed by geometry and calculus (referred as 'fundamental mathematics').

Although these reviews concerning mathematics teachers' PCK have summarised important general aspects of the relevant theories, methods, and content areas, they have not provided detailed information on the specific mathematical topics addressed in the PCK studies and the related learning difficulties and teaching strategies—information that is important for supporting preservice and in-service teacher education (Kolitsoe Moru & Qhobela, 2013).

Smaller-scale, more specific reviews and meta-analyses have also been conducted, such as reviews comparing the effectiveness of mathematics teaching methods in general (with PCK elements) (Pellegrini et al., 2021) and in different areas of mathematics, including fractions (Misquitta, 2011), geometry (Uwurukundo et al., 2020) and algebra (Haas, 2005).

One of the large-scale studies focusing on PCK is the TEDS-M [Teacher Education and Development Study in Mathematics] (Tatto et al., 2008). The focus of this international study was to understand and improve mathematics teacher education. Findings showed that PCK is a critical component of teacher effectiveness and quality of teaching and that it is related to student achievement and classroom performance. In the TEDS-M study (e.g., Blömeke et al., 2012; Kaiser & Blömeke, 2013), which included pre-service teachers in their final year of teacher education across different countries, different competence levels were observed for CK as well as for PCK.

Another example of large-scale studies is the COACTIV [Cognitively Activating Instruction], which build a theoretical framework of teachers' professional competences on Shulman's approach (Baumert & Kunter, 2013). The key finding of this project was that CK is a necessary but not sufficient basis for teachers' PCK and that PCK has a stronger influence on fostering instructional quality and student learning outcomes compared to CK (Kunter et al., 2013).

2.4 The Current Study

Given the value of a topic-specific approach nature PCK (van Driel et al., 1998) the current review, different from other reviews in the field, explicitly takes a topic-specific approach. For each mathematical topic, content-specific learning difficulties and instructional strategies that are analysed and/or mentioned throughout the studies of mathematics teachers' PCK are described. This approach sheds a light on the extent to which various mathematics topics are applied in PCK measures and as such provides a deeper understanding on what is meant by PCK in mathematics education.

Because the role of PCK in mathematics teaching and learning has expanded greatly after the review of Depaepe et al. (2013), an updated review study is needed. As noted by Depaepe et al. (2013), all studies on mathematics teachers' PCK have focused on two components originally mentioned by Shulman (1987): (1) knowledge of students' learning difficulties and (2) knowledge of instructional strategies and representations. Therefore, in this review, we considered both components. In this review study, we looked at student errors at procedural and conceptual levels. Teaching strategies were analysed in relation to the mathematical topics referred in the studies.

Our research question: *which common learning difficulties and which accompanying instructional strategies have researchers addressed, investigated, and/or identified in studies on mathematics teachers' PCK?*

3 Data Collection

3.1 Databases and Search Terms

Following the PRISMA guidelines (Moher et al., 2010), we collected the studies for this research from the EBSCO and Web of Science (WoS) databases available in Lithuania (Appendix 1).

The search terms were 'pedagogical content knowledge OR PCK OR mathematics knowledge for teaching OR MKT OR didactics knowledge OR pedagogical content knowing OR PCKG OR CKTM OR content knowledge for teaching mathematics AND (mathematics or math) NOT (STEAM OR STEM) (abbr. 'science, technology, engineering, (art) and mathematics') OR physics OR biology OR chemistry OR TPACK OR TPCK (both are abbr. for 'technological pedagogical content knowledge') OR PTICK (abbr. 'pedagogical technology integration content knowledge').

We only considered articles published in peer-reviewed journals, written in English, and published between 2013 and 2023 (between 1 January 2013 and 31 December 2022 for EBSCO; January 2013 and December 2022 for WoS).

3.2 Search Results

The EBSCO search returned 988 articles, and the WoS search yielded 1,073 articles. We obtained a total of 2,061 hits at the first screening. Before the first screening, we removed records duplicated in EBSCO and WoS ($n = 266$) and two other records that did not qualify as research articles. We screened the articles to determine their suitability for the study, and the reasons for rejection are presented in Table 1.

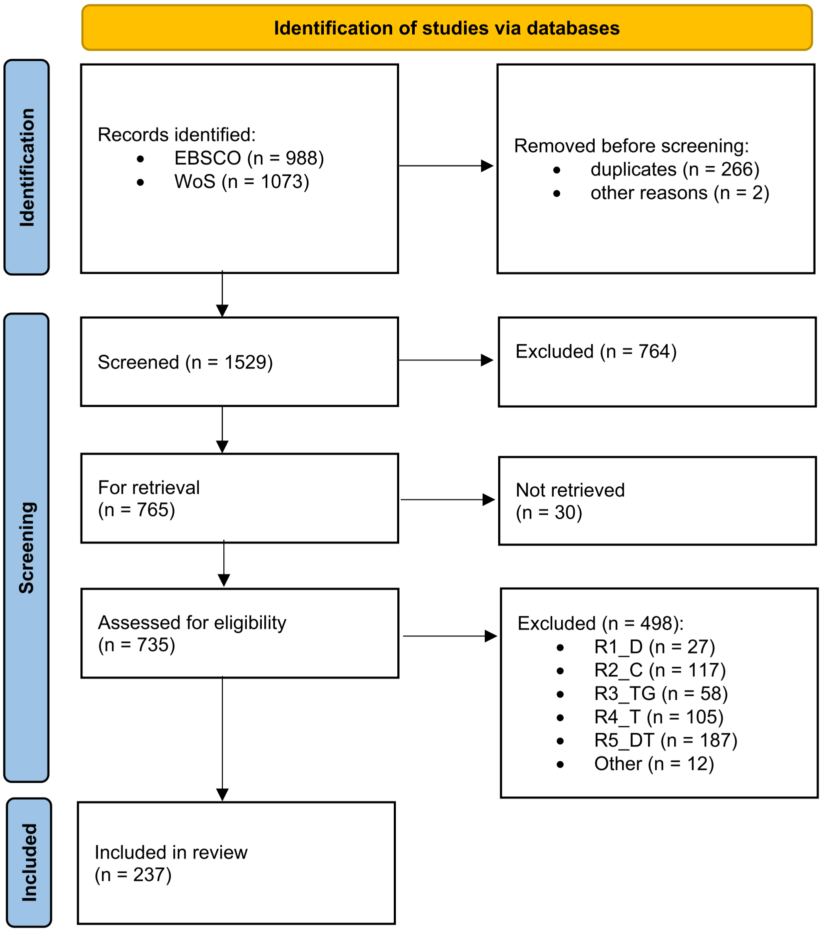
Reason R5 was only applicable for full text screening, meaning that the article was otherwise suitable, but the details did not meet the purpose of the current study. The required details were:

- Reporting of students' mathematical errors or misconceptions based on a test or questionnaire, a measure of teachers' PCK (or any other related concept), or teachers' responses during interviews. Teachers' mathematical errors or misconceptions were not considered appropriate for the current study because they were related to teachers' CK rather than their PCK about student errors or misconceptions.
- Reporting of teaching strategies aimed at preventing or correcting students' mathematical errors or misconceptions, commonly identified through interviews with teachers, lesson observations, and sometimes tests or questionnaires for measuring teachers' PCK.

Table 1 Rejection reasons

R1_D: domain-related reasons	R2_C: concept-related reasons	R3_TG: target group reasons	R4_T: type of study reasons	R5: no details mentioned in the article
STEM/STEAM (R1_D1): financial/business (R1_D2): music, language, or literacy (R1_D3).	TPACK/TPCK (R2_C1): beliefs, perceptions, or engagement (R2_C2).	Teacher-educators or pupils (R3_TG1); special education (R3_TG2).	Reviews, meta-analyses (R4_T1); textbook or course material analyses (R4_T2).	

Fig. 1 PRISMA chart



3.3 Screening Process

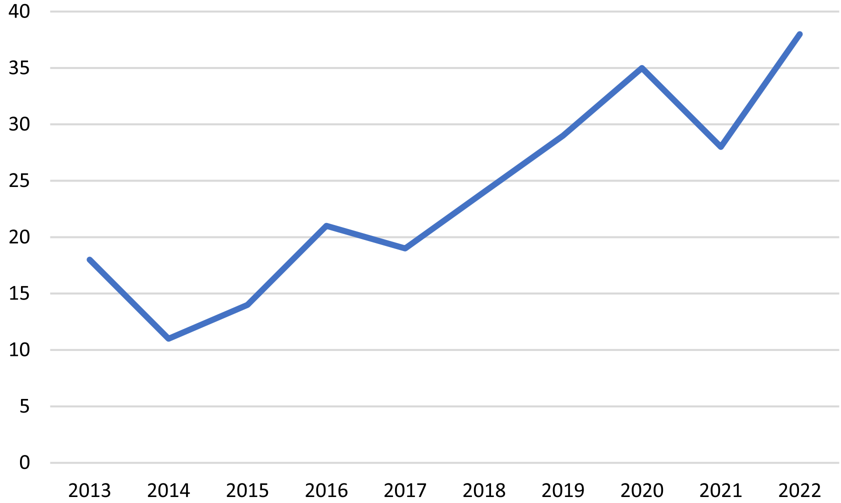
The screening process was conducted using a PRISMA flowchart (Moher et al., 2010), as shown in Fig. 1.

During the initial title and abstract screening, two researchers double-screened 10% of the articles for suitability, and the interrater reliability score was .76 (Cohen's kappa). Disagreements were discussed until a consensus was reached.

The title and abstract screening (1st stage) resulted in 764 studies being rejected for further screening. Reasons for rejection were R1_D (168 hits; R1_D1, 110; R1_D2, 11; R1_D3, 47), R2_C (145 hits; R2_C1, 74; R2_C2, 71), R3_TG (334; R3_TG1, 306; R2_TG2, 28), and R4_T (131 hits; R4_T1, 59; R4_T2, 72). In addition, 18 duplicates and 2 retracted studies were rejected for further screening (an article could be rejected for more than one reason).

The full-text screening (2nd stage) included 765 articles, 30 of which could not be retrieved within the available timeline. Thus, the second screening comprised of 735 articles. After the 2nd stage, 498 articles were rejected, resulting in 237 articles. The reasons for rejection were R1_D (27 hits; R1_D1, 11; R1_D2, 3; R1_D3, 13), R2_C (117 hits; R2_C1, 40; R2_C2, 77), R3_TG (58 hits; R3_TG1, 55; R3_TG2, 3), R4_T (105 hits; R4_T1, 73; R4_T2, 32), and R5_DT (187 hits; duplicates, 10; full text not in English, 2). As at the 1st screening stage, there could be more than one reason for rejection. Of all articles, 10% were double-screened, and the two scorers agreed perfectly about which studies should be included in the research.

Fig. 2 Number of studies published each year



The references of the articles included in the review are given in supplementary information material.

3.4 Analysis

The review of the selected articles was conducted in two stages. First, a within-case analysis (Miles & Huberman, 1994) was conducted for the selected 237 articles. In this phase, the areas of interest (AOIs) in each article (e.g. paragraphs describing teaching strategies, pictures presenting errors, questionnaire items) were first marked and assigned to one of the nine mathematical topics. Specific aspects were then extracted from the AOIs. If an article contained an example item from a PCK questionnaire and this item contained student errors or faulty reasoning (e.g. pupils' answers regarding many numbers between $\frac{3}{8}$ and $\frac{6}{8}$), a misconception was identified from the presented item and then assigned to a specific topic. At the end of this phase, detailed information about students' learning difficulties and related teaching strategies was extracted from all the articles.

Based on the results of the within-case analysis, a cross-case analysis was conducted in which the unit of analysis was changed from articles to mathematical topics. Aspects of learning difficulties and teaching strategies were grouped within each topic. In some cases, clear subtopics emerged (e.g. decimals under the rational number topic). The cross-case analysis allowed us to identify commonalities among findings. ~30% of the articles were checked by the second researcher and there were no major disagreements.

Based on the basic classification of learning difficulties, we conducted a second coding of articles into three categories characterizing the way how the learning difficulties are presented:

- *Procedural* [Pro] if the description of the topic was predominately procedural without explicit references to underlying conceptions;
- *Procedural/conceptual* [Pro/Con] when the description was essentially both procedural and conceptual;
- *Conceptual* [Con] if the main description of the topic was conceptual and possible procedural elements were not explicitly mentioned.

The coding was made by the first and second author so that both independently coded a subset of 30% of items. Cohen's Kappa was .64, which is considered as substantial agreement (McHugh, 2012).

4 Results

4.1 Descriptives

From the 237 studies, 118 covered preservice teachers, 107 in-service teachers, and 12 both preservice and in-service teachers. Regarding school level, 83 studies covered primary, 131 lower secondary, and 62 upper-secondary school. Some studies covered several school levels.

The number of studies published in each year reflects a trend of increasing interest in mathematics teachers' PCK in educational research (Fig. 2).

The region of each study was determined by the country in which the participating teachers were working or studying. Some studies included more than one country, in which case, all were treated as contributors to the total. The number of studies conducted in each continent was as follows: Africa 24, Australia 18, Europe 58, North America 57, and South America 16. For two studies, the participating countries could not be identified from the study reports and were categorised as unspecified. Following the trends reported by Depaepe et al. (2013), USA was still the leading country for studies ($n = 51$); Turkey was the second with 40 articles published between 2013 and 2022 (Fig. 3)

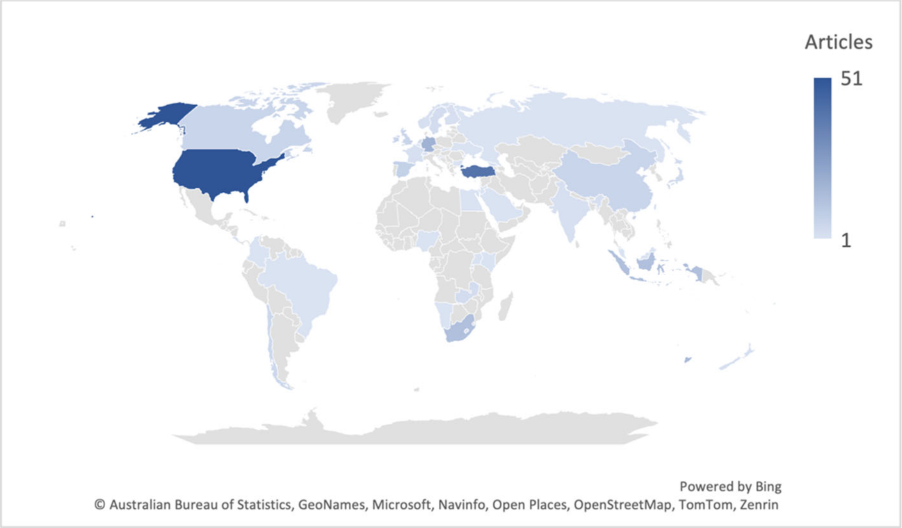


Fig. 3 Publications of mathematics teachers' PCK around the world

In terms of the research instruments used in the studies, questionnaires (also referred as 'tests') were by far the most popular research instrument (n = 133), followed by interviews (incl. stimulated recall, n = 83), classroom observations (incl. video/audio analysis, n = 72), document analyses (n = 30), and focus group interviews (n = 15). A combination of two or more research instruments was used in over half of the studies (n = 170).

4.2 Specific Mathematical Topics and Teaching Strategies Mentioned in PCK Studies

The findings are organized into nine mathematics topics (whole numbers, rational numbers, geometry, algebra, functions, calculus, statistics and probability, problem solving and measurement) typical for school mathematics (Mullis et al., 2016).

4.2.1 Basic Arithmetic and Whole Numbers

The topics of whole numbers and basic arithmetic were mentioned in 55 articles. The studies included the following subtopics: arithmetic operations with natural numbers, challenges related to word problems and number properties (Table 2).

Most PCK measures on the topic involved procedural challenges in applying the algorithms for the four basic arithmetic operations. In some studies, these were presented without any reference to conceptual problems. In typical tasks, teachers were asked what the procedural error is (e.g., $502 - 6 = 406$) when applying the subtraction algorithm (Charalambous, 2016, #43, Pro). Ozdemir et al. (2017, #162, Pro) presented a series of place value mistakes and asked prospective teachers to help student to apply the correct procedure.

Some studies such as Norton (2019, #155, Pro/Con) analysed preservice teachers' abilities to notice procedural errors in written subtraction algorithms and the conceptual problems causing them.

Purely conceptual difficulties were mentioned in PCK measures mainly in relation to the number concept. Most of them referred to misconceptions in learning negative numbers (e.g., Kaur, 2017, #106, Con). These findings are in line with studies on student learning, which show that understanding negative numbers requires conceptual change in terms of restructuring natural number based prior conceptions (e.g. Vlassis, 2004).

Most of the instructional strategies (Table 3) mentioned in the PCK studies related to the use of manipulatives or other external representations when teaching arithmetic operations and word problems, such as "bears and shells" (Fisher, 2018, #66). Conceptually deeper instructional strategies related to overcoming the limited way to teach multiplication only as repeated addition. Several strategies such as number line, area calculation, or Cartesian product methods were mentioned (Barnby, 2013, #20).

4.2.2 Rational Numbers

The most frequently occurring topic was teachers' PCK of rational numbers that was addressed in 74 articles (Table 4). This is consistent with the findings of research on the teaching and learning of mathematics. Large-scale studies have confirmed the importance of learning fractions for later mathematical success (e.g. Siegler et al., 2012).

As can be seen in Table 4, many studies focused on students' conceptual understanding or both procedural and conceptual aspects. The most frequently cited conceptual challenge was related to the conceptual complexity of rational

Table 2 Learning difficulties related to whole number arithmetic

Learning difficulty	Type	N of studies
Arithmetic operations		
Place value errors - erroneous grouping of numbers when adding or subtracting	Pro (162, 189)	2
	Pro/Con (14, 156, 183, 190)	4
Erroneous borrowing directly from hundreds to ones if there were zero tens	Pro (43, 81)	2
	Pro/Con (156)	1
Errors in applying the subtraction algorithm, such as erroneously subtracting the smaller number from the larger number	Pro (158)	1
	Pro/Con (14)	1
Including place value errors - incorrect multiplication	Pro (48, 81, 91, 123, 155, 158, 168)	7
Errors related to long division (incl. remainder)	Pro (94, 123, 158)	3
	Con (46, 162)	2
The 'loss of zero' problem in long division algorithm	Pro (50)	1
	Pro/Con (75)	1
Inability to recognize divisibility rules	Pro (5, 23)	2
	Pro/Con (43, 189, 226)	3
Errors related to the application of divisibility rules	Con (23, 68, 189)	3
Word problems		
Superficial problem-solving caused by focusing on key words that implied a particular operation	Pro (69, 85, 147)	3
Formulating mathematical problem from word problem	Con (25, 36, 136, 147, 158)	5
Number properties		
Problems in recognising and/or explaining prime	Con (77, 106, 123)	3
Problems in recognising and/or explaining odd and even numbers	Con (152, 161, 186)	3
Question of whether zero is an even number	Con (80, 161, 189)	3
Challenges in understanding negative numbers, particularly double negation	Con (86, 106, 111, 112, 117, 119, 125, 162)	8

Table 3 Teaching strategies mentioned in the PCK studies

Teaching strategies	Studies
Basic arithmetic	
Use of manipulatives	66, 85, 133, 183, 127, 133, 156
Using different multiplication representations and strategies	19, 20, 47, 51, 68, 176
Word problem solving	
The requirement for visual representations as an instructional strategy for word problem-solving	25, 147, 156, 176
Number properties	
Anecdotal instructional justifications for double negation rule (e.g. 'the enemy of my enemy is my friend')	125, 161

numbers. Many of the PCK studies mentioned the challenge of understanding the relationship between representations of rational numbers, e.g. fraction's relation to decimals and percents (e.g., Güler, 2019, #75, Con). Others mentioned the complexity of the fraction concept as part-whole, ratio, operator, quotient and measure, which is challenging for learners. Barnby (2013, #20, Con) emphasized that if only the part-whole representation with pizza models is used in class, students rely too much on this model and struggle to understand, e.g., fractions on a number line.

Table 4 Learning difficulties related to rational numbers

Learning difficulty	Type	N of studies
Errors related to relationship between fraction, decimal number and percent	Pro (58, 97, 113, 151)	4
	Con (5, 75, 80, 86, 87, 92, 100, 110, 113, 122, 146, 158, 173, 186, 189, 195, 205, 206, 219)	19
Errors related to relationships between fractional interpretations (measure, quotient, operator and ratio)	Con (20, 59, 61, 69, 78, 87, 96, 100, 106, 137, 154, 159, 162, 188, 196, 225, 233)	17
Mistakes concerning the 'whole' in fractions	Pro/Con (122, 151, 161, 195)	4
	Con (13, 20, 58, 61, 68, 86, 88, 100, 116, 173, 216, 219)	12
Errors related to concept of proper fractions	Con (72, 125)	2
Natural number bias - incorrect approaches to the addition or subtraction of fractions	Pro/Con (75, 100, 126, 158, 162, 164, 175, 188)	8
	Con (43, 58, 181, 183)	4
Place value errors - multiplying decimal numbers	Pro/Con (71, 122, 123, 155, 175, 182, 206)	7
Natural number bias - (mis)understanding the decimal number size and calculation	Con (92, 97, 116, 123)	4
Errors related to density of rational numbers	Con (20, 23, 43, 87, 97, 118, 158, 181, 216, 219)	10
Incorrect approach of multiplication of fractions	Pro (81, 92, 158, 181, 188)	5
	Pro/Con (116, 122, 125, 175, 195)	5
Inability to represent multiplication and division operations	Pro/Con (12, 49, 49, 58, 89, 101, 113, 122, 159, 175, 211)	11

Students' difficulties to understand the whole in arithmetic operations with fractions were reported in many studies. E.g., Ozdemir et al. (2017, #161, Pro/Con) presented preservice and in-service teachers with a student's solutions such as 11/15 + 20/25 = 31/40 and asked them how they would resolve this situation for the student. They concluded that none of the teachers gave an explanation for the student's deeper failure to understand the whole and the different parts of it, but only suggested the correct algorithm for the students to learn.

Similar examples of student solutions that incorrectly add numerators and denominators when adding fractions have been presented in many other studies (e.g., Lannin, 2013, #126, Pro/Con). Many of the examples of students' errors in arithmetic operations, magnitude, or density of rational numbers presented in the studies in this review are related to misconceptions described as natural number bias in recent research on mathematics learning (Van Hoof et al., 2017). However, the term was not used in most of these PCK studies.

Most of the teaching strategies related to rational numbers mentioned in the review articles dealt with the use of external representations and manipulatives in teaching fractions and decimals, and a few studies suggested the use of everyday situations where rational numbers are needed (Table 5). Number line representations were almost never mentioned in the teaching strategies of the PCK studies, although number line-based training of rational numbers is strongly emphasised in recent research on learning and teaching rational numbers (e.g. McMullen et al., 2023).

4.2.3 Geometry

Sixty-nine studies focused on geometry, including several subtopics related to learning difficulties about various geometric concepts (Table 6). Most of the conceptual challenges were related to limited understanding of geometric concepts such as difficulties in identifying and classifying geometric objects. According to findings from studies on students' difficulties in learning geometry, errors in classifying geometric figures may be due to a lack of relevant knowledge (e.g. Özerem, 2012) or, in a few cases, deeper misconceptions (Kabaca et al., 2011). In most PCK studies no specific misconceptions on these topics were mentioned, but rather a lack of prior conceptions. E.g., Alkhateeb (2018, #8, Pro/Con) states that congruence of triangles is a difficult topic for students that requires a lot of support for successful learning.

However, there were also some examples of deeper misconceptions where prior knowledge and beliefs are systematically misleading and make it difficult to learn correct geometric knowledge without restructuring prior understanding.

Table 5 Teaching strategies related to rational numbers

Teaching strategies	Studies
Presenting fraction multiplication as repeated addition	12, 71, 188
Presenting fraction division as repeated subtraction	71
Using visual strategies (drawing, showing picture)	14, 20, 23, 58, 61, 72, 80, 87, 88, 100, 113, 116, 122, 161, 175, 183, 188, 196, 205, 211, 233
Using manipulatives (e.g. cut-outs of cake, pizza, chocolate, wooden blocks, beads)	14, 23, 86, 58
Including IT	55, 216
Contextualised instruction for percentages (e.g., tipping, taxes, discounts)	23, 57, 106, 172

Table 6 Learning difficulties related to geometry learning

Learning difficulty	Type	N of studies
Errors related to congruence and similarity	Pro/Con (8, 189)	2
	Con (104, 120, 149, 214)	4
Errors related to concept of angle	Pro/Con (13)	1
	Con (9, 45, 98)	3
Errors related to concept of symmetry	Pro/Con (60, 80, 126, 189)	4
	Con (69, 152)	2
Errors related to concept of trigonometric ratios	Pro/Con (138, 203)	2
	Con (134)	1
Faulty identification or classification of two-dimensional geometric figures (exc. triangles, quadrilaterals)	Pro (11, 126)	2
	Pro/Con (60, 179)	2
	Con (130)	1
Faulty identification or classification of quadrilaterals	Pro (25, 80)	2
	Pro/Con (37, 60, 225)	3
	Con (1, 2, 120, 217, 229, 230, 232)	7
Faulty identification or classification of triangles	Pro/Con (70, 178, 225)	3
	Con (8, 40, 104, 120, 149, 214, 229, 230)	8
Faulty identification or classification of angles	Pro/Con (13)	1
	Con (64)	1
Faulty identification or classification of three-dimensional geometric figures or their parts	Pro/Con (178, 189, 196)	3
	Con (11, 23, 114, 130, 132, 170, 194)	7
Errors related to the height of the parallelogram outside the figure	Pro (111, 112, 117)	3
	Pro/Con (14)	1
Errors related to the calculation (or evaluation) of the size of the area of a plane	Pro (111, 112, 117, 142, 153)	5
	Pro/Con (70, 108, 142)	3
	Con (60, 121, 149, 174, 232)	5
Errors related to the calculation (or evaluation) of the size of the area of a circle	Pro/Con (142)	1
	Con (106, 194)	2
Faulty evaluation of consistency of a volume	Pro (102)	1
	Pro/Con (14, 60, 178, 189, 196)	5
	Con (194)	1

Some of the PCK studies indicated the possibility that prior teaching may lead to common errors. Kleickmann (2013, 2015, #111, #112, Pro) and Krauss (2020, #117, Pro) present in their PCK measure inability to calculate the area of a parallelogram because the height line was not “inside” the figure without referring to conceptual knowledge. Baki (2017, #14, Pro/Con) presented the same procedural mistake but referred to lacking conceptual knowledge. According to

Table 7 Teaching strategies in geometry

Teaching strategies	Studies
Use of visualisations to teach the Pythagorean theorem	8, 40, 134, 201
Visualizing transformations to teach quadrilaterals	1, 2, 232
Incorporating manipulatives into teaching processes for two-dimensional figures	11, 14, 23, 62, 130, 152, 154
Incorporating manipulatives into teaching processes for three-dimensional figures	23, 62, 64, 106, 114, 130, 156, 174, 194, 196
Drawing activities for two-dimensional figures	13, 62, 98, 104
Drawing activities for three-dimensional figures	40, 130, 114
Drawing activities for pattern teaching	60, 152, 189
Use of interactive software	8, 70, 104, 130, 149, 233

Fukaya (2022, #69, Con), unclear summaries when teaching about axes of symmetry can lead to the misconception that there is only one axis of symmetry.

Only a few of the geometric topics addressed in the PCK studies were related to procedural errors. The area and volume calculations presented in the studies involve challenges that are mostly of a procedural nature.

Most of the teaching strategies proposed in the PCK studies involved the use of visualisations and manipulatives as well as the students' own drawing of geometric objects. Another teaching strategy was the use of interactive software, especially Geogebra (e.g., Alkhateeb, 2018, #10) (Table 7).

4.2.4 Algebra

The findings on the topic of algebra are based on 64 studies (Table 8) with algebraic expressions, equations and number properties as subtopics. Some of the difficulties were presented only as procedural mistakes with algebraic expressions without explicit references to conceptual background (e.g. Spangerber, 2020, #199, Pro).

However, most of the learning challenges mentioned in the PCK studies were related to misconceptions, which appeared as procedural but were also connected to conceptual understanding. The most frequent example of this type of learning difficulty concerned an inadequate understanding of equivalence in solving procedures (e.g. Daniel, 2021, #52, Pro/Con) and building up a conceptual interpretation of tasks where operations and variables appeared on both sides of the equal sign (Murtafiah, 2019, #153, Con). Misconceptions of equality is a topic that has been extensively explored in research on conceptual change in mathematics (e.g., Byrd et al., 2015).

Only very few studies mentioned teaching strategies related to algebra education and most of them proposed some kind of modelling (#47, 54, 67, 76, 106, 129, 157, 185, 220, 224, 233). E.g., Demonty (2018, #54) presents two problems where modelling, if done correctly, would lead to better understanding of the pattern presented and builds on algebraic generalisation.

4.2.5 Functions

Functions were mentioned in 31 studies. Two subtopics emerged: the concept of a function (incl. slope) and the function graphical representation (Table 9).

The concept of function is found to be very important for developing deeper understanding of mathematics (Chang et al., 2016), but there are many misconceptions related to the interpretation of functions (review by Leinhardt et al., 1990).

In line with the findings of research on students' learning challenges, PCK studies include assumption of false linearity. For example, Hatisaru (2020, #83, Con) and Marban (2020, #140, Con) described pupils' thinking of a function only as a straight line (only linear). Campbell (2014, #37, Pro/Con) uses questions about function graphs and the framing of the possible answers suggest possibility that the mistake in identifying the correct answer could be more procedural, while understanding how function acts in different circumstances is conceptual. Applying knowledge of functions can also be procedural as in reading the function graph and finding intersection points to realize what solution there is to equation plotted as a function (Karagöz-Akar, 2016, #104, Pro).

In contrast to the topic of algebra, teaching strategies for functions were much better documented in the PCK studies (Table 10).

To introduce concept of functions, Taşdan (2017, #208), Sintema (2020, #193) and other researchers present teaching through an example of a machine that has input and output.

To develop deeper understanding, researchers propose posing thought-provoking questions such as "What is a function?" as a start for a discussion (Busch, 2015, #34) to develop pupils' understanding.

Borke (2021, #27) presents mathematical modelling activities, for example, examination of gas price tendencies or magazine sells. The use of interactive drawing programs (e.g., Geogebra) was extensively covered in the reviewed studies (e.g., Emre-Akdogan, 2018, #62).

Table 8 Learning difficulties related to algebra

Learning difficulty	Type	N of studies
Algebraic expressions		
Errors related to algebraic expressions in simplification or grouping tasks	Pro (199)	1
	Con (64, 84, 102, 201)	4
Treating unequal terms as if they were equal	Pro/Con (168, 234)	2
	Con (67)	1
Errors related to concept of variables	Pro (33, 234)	2
	Pro/Con (52)	1
	Con (143, 168)	2
Equations		
Difficulties in solving atypical equations (e.g. variables on both sides, variable on the 'wrong' side, etc	Pro (74, 84, 108, 141)	4
	Pro/Con (80, 157, 220)	3
	Con (67, 153, 201)	3
Mistakes in solving quadratic equations	Pro/Con (93, 157, 161, 164, 207)	5
	Con (67)	1
Mistakes in applying formulae	Pro/Con (93, 157, 164)	3
	Con (17)	1
Errors related to 'sign changes' and/or 'cancellations'	Pro (141)	1
	Pro/Con (84, 102, 201)	3
	Con (67, 153)	2
Inability to interpret equation solutions geometrically	Pro (207)	1
	Con (67, 157, 233)	3
Number properties		
Lack of understanding of the commutative property or distributive property	Pro (141)	1
	Pro/Con (52, 80, 108)	3
	Con (67, 143)	2
Lack of understanding of the zero-product property	Pro/Con (93, 124)	2
Errors related to equality sign and modelling equality	Pro (141)	1
	Pro/Con (52, 84, 182)	3
	Con (143)	1
Wrongly changing the direction of the inequality sign	Pro (74, 76)	2
Errors related to the concept of quadratic inequality	Pro/Con (17, 218)	2
Difficulties in solving pattern problems	Con (54, 196, 223, 224, 231)	5
Errors related to proportional reasoning	Pro (37)	1
	Con (33, 74, 75, 106, 108, 126, 168, 196, 197, 222)	10
Errors related to rationality and/or roots	Pro/Con (5, 124)	2
	Con (75, 131, 157)	3
Errors related to the power of zero	Con (75, 161)	2
Errors related to absolute values	Con (41)	1
General		
Confusion between the terms 'equation' and 'expression'	Pro (125)	1
The difficulty of formulating either an algebraic expression or an equation from a word problem (or creating a problem, or modelling a problem)	Pro/Con (80, 99, 125, 185, 207)	5
	Con (54, 67, 84, 106, 121, 129, 132, 182, 189, 197, 217, 220, 223, 224, 226)	15

Table 9 Learning difficulties related to functions

Learning difficulty	Type	No of studies
Concept of the functions		
Inability to determine dependent and independent variables	Con (3, 27, 28, 208)	4
Inability to determine range, and domain	Pro/Con (37, 210)	2
	Con (27, 83, 104, 105, 148, 166, 193, 209, 217)	9
Difficulties in applying the univalence property	Pro/Con (123)	1
	Con (27, 83, 166, 178, 208, 209, 215, 237)	8
Faulty linking of a function to its various forms	Pro (31, 74, 83)	3
	Con (3, 7, 27, 34, 90, 103, 105, 128, 208, 209, 213, 237)	12
Inability to create a model from a problem	Con (28, 74, 90, 165)	4
Graphical representations		
Not being able to read or interpret graphs	Pro (17, 31, 148)	3
	Con (3, 27, 28, 90, 140)	5
Difficulties in plotting	Pro (17, 104, 105, 107, 148, 215)	6
The inability to determine the range and domain of data	Pro (148)	1
	Con (3, 83, 166)	3
Falsely assuming linearity	Con (3, 27, 83, 140 193)	5

Table 10 Teaching strategies in function topic

Teaching strategies	Studies
Introducing the concept of a function using a machine	3,7, 193, 208, 209
Using interactive drawing programmes to support visualisation	62, 104, 166
Use of real-world problems	3,7, 28, 62, 74, 103, 148, 165, 208, 213, 215

4.2.6 Calculus

Only ten PCK studies mentioned calculus, referring to conceptual difficulties and deeper misconceptions (Table 11). Studies on students' learning have revealed profound misconceptions and conceptual changes in the extension of number concepts when learning calculus (Lehtinen et al., 1997). In the PCK studies the conceptual challenges are related to general problems in understanding calculus concepts, in particular the infinitesimal concept. For example, the researchers present a misconception of the uniqueness of the derivative function (Pino-Fan, 2018, #171, Con), which means that students are unable to account for the constant. This is a well-known misconception in studies of student learning (e.g., Kiat, 2005). All the learning difficulties presented in this topic were conceptual.

Only two proposals for teaching strategies of calculus were mentioned in the PCK studies. To promote pupils' understanding of infinity and develop infinitesimal approach, discussion of infinity paradoxes can be used (Savuran, 2022, #184) like Zeno's paradox. One study (Ünver, 2015, #215) proposed the use of mathematics programs, such as Geogebra.

4.2.7 Statistics and Probability

The findings of statistics and probability come from 21 studies.

Misconceptions that are mentioned the most are the ones dealing with statistical concepts (Table 12). Studies on students' learning of statistics have shown that it is the abstract nature of statistics, which causes most of the difficulties in understanding (Sotos et al., 2007). For example, students are unable to interpret problems about averages because of lack of conceptual understanding (Callingham, 2016, #35, Con; Campbell, 2014, #37, Con; Wasserman, 2017, #86, Con). Similarly, misinterpretation of what standard deviation is (large means greater variability, small suggests points are close to mean) (Wasserman, 2017, #221, Con).

For probabilities, Matitaputty (2022, #144, Con) notes students' inability to distinguish which combinatorics formula is appropriate for a given situation. Maher (2022,

Table 11 Learning difficulties related to calculus

Learning difficulty	Type	N of studies
Errors related to concept of limit	Con (17, 24, 105, 139, 215)	5
Connecting limits with functions wrongly	Con (105, 171, 207)	3
Understanding the meaning of the infinitesimal	Con (184, 207, 215)	3
Incorrect treatment of infinity as a number	Con (184, 228)	2

Table 12 Learning difficulties related to statistics and probabilities

Learning difficulty	Type	N of studies
Statistics		
Confusion between the centre, spread, and range of distributions	Pro/Con (132)	1
Not being able to explain the meaning of standard deviation	Con (221)	1
Errors related to calculation and interpretation of arithmetic averages	Con (14)	1
	Pro/Con (204)	1
Errors related to the average, variability, and centrality of data	Pro (47)	1
	Pro/Con (132)	1
	Con (14, 35, 37, 86, 160, 191, 204, 221)	8
The misuse of visualisations, the faulty creation of misleading graphs and charts	Con (39, 160)	2
Misinterpretation of various graphs (inc. scatter plots, box-and-whiskers diagrams)	Pro/Con (132)	1
	Con (42, 160, 191)	3
Probabilities		
Understanding permutation principle, independence (of probabilistic events), and probability	Con (139, 144)	2
Misusing the formula	Pro/Con (144, 177)	2

#139, Con) presents pupils' inability to understand distribution. Misusing the formula could be procedural, but also could show lack of understanding of the concept (Matitaputty, 2022, #144, Pro/Con, Retnawati, 2018, #177, Pro/Con).

For statistics, most of the teaching strategies share the same core idea - extensive talking about statistical concepts (Table 13), e.g., Callingham (2016, #35) introduces use of faulty charts that could be discussed with students. For teaching data representations, Harr (2015, #82) invites to discuss a variety of data representations for a learning unit on patterns in data. For probabilities, mostly discussion-based strategies included discussing concepts or more specific aspects of the topics.

4.2.8 Measurement

Ten PCK studies were addressing measurement and only three mentioned learning difficulties (Table 14).

A typical misconception is the inability to use measurement parameters appropriately that can be a procedural mistake that is based on a misconception (Besser, 2020, #22, Pro/Con; Lim, 2013, #132; Pro/Con).

Teaching strategies (Table 15) for measurement include using manipulatives to learn, e.g., length - rope or ruler (Johar, 2021, #97), weight - scales (Johar, 2021, #97; Jong, 2021, #99) and using real-world examples e.g., temperature (Kang, 2016, #103), speed and area (Polman, 2021, #172), volume (Polman, 2021, #172, Slavit, 2018, #196).

4.2.9 Problem Solving, Reasoning, and Proof

Twenty-five PCK studies considered problem solving, with PCK tests highlighting frequently appearing problems in students' learning (Table 16). All of the learning difficulties in this topic were presented as conceptual.

Most of the findings on problem solving can be also linked to a specific mathematical topic (e.g. algebra and algebra-related problem-solving issues (Akyüz, 2020, #6), Con) (Table 16). However, for reasoning and proof, findings are more concept-related. Buchbinder (2022, #30, Con), asks what proving the theorem means and how to inverse

Table 13 Teaching strategies in teaching statistics and probabilities

Teaching strategies	Studies
Statistics	
Clarifying the underlying arithmetic mean, distribution, standard deviation, sample, population, and other concepts and clearly explaining the relationships between	37, 39, 42, 191, 221, 227, 233
Teacher- or peer-led discussion to help students better process new information and understand the topic	37, 191
Emphasising discussion-based analysis of given data representation examples (including incorrect ones)	35, 37, 39, 41, 82, 132, 233
Use of various data representations during the instructional process	37, 39, 82, 160, 233
Probabilities	
Explicit discussing the concepts	82, 187
Using contrasting cases (e.g. analysing fairness), analysing examples	144, 177, 178, 233
Conducting simple experiments	236
Presenting alternative solutions	187

Table 14 Misconceptions related to measurement

Learning difficulty	Type	N of studies
Inability to understand the data	Pro/Con (22, 132)	2
Not knowing the relationship between units of measurement	Pro (189)	1

Table 15 Teaching strategies in teaching measurement

Teaching strategies	Studies
Estimating with everyday objects before introducing standard units	141, 202
Using manipulatives or their visual representations	97, 99
Employing real-life examples	103, 172, 196

Table 16 Learning difficulties related to problem solving, reasoning, and proof

Learning difficulty	Type	N of studies
Being unable to apply mathematical knowledge to presented situations	Con (5, 23, 29, 57, 65, 67 68, 73, 150, 174, 177, 189, 196, 202)	14
Difficulties in applying more than one problem-solving strategy	Con (6, 26, 65, 145, 169)	5
Making unjustified assumptions and basing a solution on them	Con (22, 30, 45, 53, 169, 174, 177)	7
Inability to determine the validity of an argument	Con (16, 45, 177, 217)	4
Inability to present a valid argument	Con (16, 29, 30, 169, 217)	5

it - whether it is true and does not need proof or any other combination.

For the teaching strategies, findings were scarce. Two studies (#65, #44) mentioned group work and one study (#6) using diverse problem-solving strategies and possibilities to teach the topic. Buchbinder (2020, #29) mentions type of task "Is this a coincidence" that is used to teach how to avoid unjustified assumptions.

4.2.10 Summary of the Findings

The most frequently presented topics in the studies were rational numbers, geometry, algebra, and whole number arithmetic. The least popular topics were calculus and measurement.

The number of different topic-specific learning difficulties mentioned in studies was 84, most of which were related to algebra (20), whole numbers (14), geometry (13) and ra-

Table 17 Summary of the results

Topic	Different learning difficulties N	Procedural difficulties N (%)	Procedural/ conceptual difficulties N (%)	Conceptual difficulties N (%)	Teaching strategies N	Number of articles referring to topic[1]
Whole numbers	14	21 (36)	10 (17)	27 (47)	4	55
Rational numbers	10	9 (8)	34 (31)	68 (61)	6	74
Geometry	13	14 (16)	31 (36)	42 (48)	8	69
Algebra	20	13 (13)	34 (33)	55 (54)	1	64
Functions	9	13 (20)	3 (5)	49 (75)	3	31
Calculus	4	0 (0)	0 (0)	13 (100)	2	10
Statistics and probability	8	1 (8)	5 (42)	6 (50)	8	21
Measurement	2	2 (67)	1 (33)	0 (0)	3	10
Problem solving, reasoning, and proof	4	0 (0)	0 (0)	35 (100)	2	25
Total	84	74 (15)	116 (24)	295 (61)	37	237

[1] The number of articles is different than the sum of classified difficulties because one article can include different difficulties or only teaching strategies.

tional numbers (10). The total number of teaching strategies mentioned was 40, but some methods, such as using manipulatives, visualizations and computer applications, were mentioned in relation to several mathematical topics.

The distribution of procedural, procedural/conceptual and conceptual learning difficulties in different mathematical topics is presented in Table 17.

Most of the learning difficulties were presented as conceptual. The highest number of procedural difficulties was related to arithmetic. In most of the cases where articles presented procedural difficulties, there were some remarks of underlying conceptual difficulties.

5 Discussion

This review study examined learning difficulties and teaching strategies mentioned in 237 studies on pre-service and in-service mathematics teachers' PCK. The descriptive data showed the increasing popularity of the topic, a tendency to use more than one research instrument (the most popular instrument was a questionnaire), and internationally uneven contributions to the topic, with the US and Turkey making the greatest contributions.

The findings on the most frequently mentioned topics are consistent with the extent to which these topics have been studied in academic research on learning difficulties and misconceptions in mathematics (e.g. Neidorf et al., 2020).

Most of the learning difficulties were presented as conceptual. The nature of learning difficulties mentioned in PCK measurements varied between different topics. In whole number arithmetic many of the learning difficulties mentioned in the studies were procedural in nature, whereas in other topics such as geometry, algebra and rational numbers the difficulties were mainly described as conceptual or conceptual/procedural with explicit references to underlying misconceptions or lacking conceptual knowledge.

This review also revealed that teaching strategies tended to be presented in more general terms than learning difficulties. The same teaching strategies were mentioned in relation to several topics (e.g. the use of manipulatives to enrich the teaching of whole numbers and geometry), while learning difficulties were more specific and in most cases could only be linked to one topic, thus they were more frequently presented in research instruments than teaching strategies. One reason for this dominance of learning difficulties in PCK measures is that these difficulties can be evaluated somewhat independently of the cultural and educational context, whereas instructional strategies are typically more closely related to different cultural and pedagogical traditions or lack of experimental research on topic-specific teaching methods. The learning difficulties and teaching strategies were found to be only partly based on research evidence from research on students' mathematics learning and teaching interventions (see Hill et al., 2008). Instead of presenting explicit research evidence for the measurements, PCK studies often just copied items from previous PCK studies or developed items based on common beliefs within the mathematics education community. E.g., many PCK measures used similar items which have been used in recent research on natural number bias in rational number learning without any references to these studies.

There are many possible reasons for the vague use of research evidence in PCK measures. Whereas certain topics, such as basic arithmetic, rational numbers and algebra, have been investigated intensively, on others, such as calculus or measurements, the research evidence on learners' difficulties and teaching strategies for preventing or remedying these difficulties is very limited. Even in the extensively studied topics there is lack of systematic reviews and meta-analyses summarizing research evidence, which could be used in developing evidence-based PCK measures.

The results of the review may be biased due to the selection criteria, since books, book chapters, non-peer-reviewed articles, and dissertations were excluded. Second, only studies published in English were considered. Moreover, when conducting this review, several shortcomings that prevented a more detailed analysis have been identified. Although many studies were based on questionnaires, only a small number of them included the questionnaires in the articles, with many presenting only a few examples thus preventing more in-depth analysis. The same problem holds for the reviewed studies based on interviews, where short excerpts from the interviews limited the analysis.

To develop the scientific knowledge necessary in enhancing evidence based PCK measures in future, more basic research on students learning is needed in many topics of mathematics curricula which are currently underrepresented in research literature.

Future research on PCK could take a meta-analysis approach to identifying the possible effects of the level of teachers' PCK, including its components such as recognizing learning difficulties on teaching qualities and student learning. Further systematic reviews could focus on regions that were not included in the sample for this review due to the search criteria excluding studies not published in English.

With the increasing interest in the PCK of mathematics teachers, scholarly understanding of the concept has significantly advanced. However, as the present research indicates, there remains much to be explored.

6 Conclusion

The article presents a synthesis of more than 230 PCK studies in mathematics and shows that the studies focus predominantly on mathematical topics where students have difficulties or misconceptions. More than 80 different mathematical topics that are known to be difficult for students are described in the PCK measures. Many, but not all, of these are based on findings from on students' learning. This rich database can be used in teacher training and professional development, in the preparation of mathematics textbooks, and in the planning of new tools to analyse teachers' PCK. Much less was reported on topic-specific teaching strategies, which is a weakness of PCK measures.

Appendix 1

List of databases included in the EBSCO search (Lithuania access)

Education source	(n = 611)
ERIC	(n = 571)
Academic Search Ultimate	(n = 295)
Teacher Reference Center	(n = 97)
Applied Science & Technology Ulimate	(n = 35)
MasterFILE Premier	(n = 24)
MEDLINE	(n = 13)
Humanities Source Ultimate	(n = 10)
Business Source Ultimate	(n = 8)
APA PsycArticles	(n = 4)
Health Source: Nursing/Academic Edition	(n = 3)
Library, Information Science & Technology Abstracts	(n = 2)

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Declarations

Competing Interests The authors declare no competing interests.

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