

CHARACTERIZATION AND CLASSIFICATION OF SIGMOID FUNCTION AND GENERALIZED DISCRETE PROBABILITY DISTRIBUTIONS SERIES DEFINED BY AWOLERE–OLADIPO DERIVATIVE OPERATOR IN THE ANALYTIC UNIVALENT FUNCTION SPACE

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The sigmoid function is a special function emulating biological neurons in terms of signal processing or sending alerts to various departments of the brain for responses. It is also good for machine learning, while the generalized discrete probability distribution series is useful for modeling infectious diseases, genetic modeling, clinical trials, risk analysis, optimal pricing, ecological and climatic modeling, and conservation biology. We explore the convex combination and several other types of behaviors of the sigmoid function associated with the generalized discrete probability distribution series in the space of univalent function theory. This is done by introducing and studying a new derivative operator. Several coefficient inequalities and consequences of various choices of the parameters involved are discussed. Graphically, by using the Python software, various convex combinations of K_ϕ and g_ϕ are presented in terms of disease and intervention, policy and policy intervention, investment return, disease spread, infection and intervention, free market and intervention, and cost and efficiency relations.

1. Introduction

Let $f \in A$ represent the class of analytic functions of the form

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t,$$

normalized such that $f(0) = 0$ and $f'(0) - 1 = 0$, defined in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the class of analytic univalent functions. Let T denote the subclass of A consisting of functions whose none zero coefficients from second on, given by (see [19])

$$f(z) = z - \sum_{t=2}^{\infty} a_t z^t, \quad a_t \geq 0.$$

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Furthermore, let $g \in A$ be of the form

$$g(z) = z + \sum_{t=2}^{\infty} b_t z^t,$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z + \sum_{t=2}^{\infty} a_t b_t z^t = (g * f)(z).$$

Geometric Function Theory (GFT), a prominent branch of complex analysis, is dedicated to examining the geometric characteristics of analytic functions and their numerous applications across various mathematical domains [7, 8, 13], as well as related fields such as special functions [10], probability distributions [2, 16], physics, computer science, and engineering.

The Sigmoid function, defined as

$$g(z) = \frac{1}{1 + e^{-s}},$$

is particularly useful due to its differentiability, which is essential for weight-learning algorithms. It expands the hypothesis space that a neural network can represent, enabling the network to perform more complex learning tasks.

The Sigmoid function possesses several important properties:

1. It outputs real numbers between 0 and 1.
2. It maps a wide input domain to a narrow range of outputs.
3. It is information-preserving, as it is a one-to-one function.
4. It is monotonically increasing.

These characteristics make the Sigmoid function suitable for applications involving univalent functions [9].

2. New Derivative Operator (Awolere–Oladipo Derivative Operator)

Let $s \geq 0$, and define $\Delta(s)$ as

$$\Delta(s) = \frac{2}{1 + e^{-s}},$$

then

$$\Delta(s) = 1 + \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right]^m \right),$$

where $\Delta(s)$ is a modified Sigmoid function that has been investigated by several authors for various applications [4, 5, 9, 11].

Now, let Ω denote the sum of the convergent series of the form

$$\Omega = \sum_{n=0}^{\infty} a_n,$$

where $a_n \geq 0$ for all $n \in \mathbb{N}$. Also let $p(j)$ denote the probability mass function given as

$$p(j) = \frac{a_j}{\Omega}, \quad j \geq 0.$$

It is important to note that $p(j)$ is a valid probability mass function since $p(j) \geq 0$ and $\sum_j p(j) = 1$. Let

$$\Psi(x) = a_0 + a_1x + a_2x^2 + \dots,$$

then, from the fact that

$$\Omega = \sum_{n=0}^{\infty} a_n$$

the series μ converges for both $|x| < 1$ and $x = 1$.

Next, if X is a discrete random variable that takes values x_1, x_2, \dots with associated probabilities p_1, p_2, \dots , then the expected value of X , denoted by $E(X)$ is defined as

$$E(X) = \sum_{j=1}^{\infty} p_j x_j.$$

The r^{th} moment of a discrete probability distribution about $x = 0$ is defined by

$$\mu'_r = E(X^r),$$

where μ'_r represents the mean of the distribution, and the variance is given by

$$\mu'_2 - (\mu'_1)^2.$$

The moment about the origin is given as

$$\text{Mean} = \mu'_1 = \frac{\Psi'(1)}{\Omega},$$

$$\text{Variance} = \mu'_2 - (\mu'_1)^2$$

$$= \frac{1}{\Omega} \left[\Psi''(1) + \Psi'(1) - \frac{(\Psi'(1))^2}{\Omega} \right].$$

The moment generating function of a random variable X is denoted by $M_x(t)$, and is defined as

$$M_x(t) = E(e^{Xt})$$

and the moment generating function of a generalized discrete probability is given by

$$M_x(t) = \frac{\Psi(e^t)}{\Omega}.$$

By choosing specific values of a_n , various well known discrete probability distributions such as the Yule–Simon distribution, Logarithmic distribution, Poisson distribution, Binomial distribution, Zeta distribution, Geometric distribution, and Bernoulli distribution can be obtained [12, 13].

Differential operators for characterization and applications in a variety of domains have recently been introduced by numerous authors. These operators have been refined and extended for related purposes by other researchers. For a detailed discussion on different types of derivative operators and their applications, see [1, 3, 6], and [15].

For example, in 1975, Ruscheweyh [17] developed and studied the following derivative operator:

Let $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$, and defined n th order Ruscheweyh derivative $R^n f(z)$ as

$$R^n f(z) = z + \sum_{t=2}^{\infty} \epsilon_t(n) a_t z^t,$$

where

$$\epsilon_t = \epsilon(n, t) = \frac{(n+1)(n+2)\dots(n+t-1)}{(t-1)!},$$

and

$$\epsilon(0, t) = \frac{(n+1)_{t-1}}{(1)_{t-1}} = 1,$$

which implies that $R^0 f(z)$ has the form

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t.$$

Later, in 1983, Salagean [18] introduced and studied the derivative operator

$$D^n f(z) = z + \sum_{t=2}^{\infty} t^n a_t z^t.$$

Building on previous research, many scholars have introduced and analyzed various derivative operators. Of particular relevance to this study is the work presented in [14], where a novel generalized multiplier transformation was proposed to explore the function classes

$$TSQ_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\varpi, n) \quad \text{and} \quad TSQ_{\sigma, \xi, \omega, \varepsilon}^{\mu, \lambda, \eta}(\varpi, n; M_i),$$

leading to the derivation of several geometric properties. Additionally, Amourah and Yousef [3] introduced the derivative operator

$$D_{\mu, \lambda, \sigma}^m(\alpha, \beta),$$

which was employed to define new subclasses $\Omega_m^*(\delta, \lambda, \alpha, \beta, b)$ of analytic functions, thereby facilitating the investigation of their geometric characteristics.

In this study, we aim to introduce a novel generalized differential operator for power series, where the coefficients represent probabilities from a generalized distribution series. This operator is defined as follows:

$$K_\phi(z) = z + \sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega} z^t. \quad (1)$$

Furthermore, we define the following function:

$$g_\phi(z) = z - \sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega} z^t. \quad (2)$$

These formulations serve as the foundation for our investigation into their analytical properties and applications. For more information on (1) and (2) above, interested readers are referred to [9, 12], and the references therein.

Definition 2.1. *Let K_ϕ be of the form (1). We define a new generalized differential operator equipped with a generalized distribution series as follows:*

$$\begin{aligned} D_{\mu, \varphi, \sigma}^0(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \Omega)K_\phi(z) &= K_\phi(z), \\ D_{\mu, \varphi, \sigma}^1(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z) \\ &= \left(\Delta(s) - \frac{m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma)}{\mu + \varphi} \right) K_\phi(z) \\ &\quad + \frac{m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma)}{\mu + \varphi} {}_z K'_\phi(z) \\ &\quad + \frac{\tau + \Gamma}{\mu + \varphi} z^2 K''_\phi(z), \end{aligned}$$

and

$$\begin{aligned} D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z) \\ = z + \sum_{t=2}^{\infty} \left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \frac{a_{t-1}}{\Omega} z^t, \end{aligned} \quad (3)$$

$$n \in \mathbb{N}_0.$$

Here, the parameters satisfy

$$\alpha, \sigma \geq 0, \quad \beta, \varphi, \mu > 0, \quad \varphi \neq \alpha, \quad m, \Phi, \gamma, \tau, \Gamma, s \geq 0.$$

In the same vein, let $g_\phi(z)$ be as defined in equation (2), then

$$D_{\mu,\varphi,\sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)g_\phi(z) \\ = z - \sum_{t=2}^{\infty} \left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \frac{a_{t-1}}{\Omega} z^t.$$

Remark 2.1. By selecting appropriate parameter values, one can recover various well-established operators introduced by different authors and derive new derivative operators. This is further explained in the following remarks:

- $D_{0,1,1}^n(0, 1, 0, 0, 1, 0, 0, 0, \Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} t^n \frac{a_{t-1}}{\Omega} z^t.$
- $D_{\mu,\varphi,\sigma}^n(0, m, \Phi, 0, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z) \\ = z + \sum_{t=2}^{\infty} \left\{ 1 + \frac{(t-1) [m(\varphi - \Phi)^\sigma + \chi_1 + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \frac{a_{t-1}}{\Omega} z^t,$
where $\chi_1 = (\varphi - \alpha)(\beta - \sigma)$, and the resulting expression defines a new generalized derivative operator.
- $D_{\mu,\varphi,\sigma}^n(s, m, \Phi, 0, \beta, \tau, \Gamma, 0, \Omega)K_\phi(z) \\ = z + \sum_{t=2}^{\infty} \left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^\sigma + \chi_2 + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \frac{a_{t-1}}{\Omega} z^t,$
where $\chi_2 = \varphi(\beta - \sigma)$, and the resulting expression defines a new generalized derivative operator.
- $D_{\mu,\varphi,\sigma}^n(s, 0, \Phi, \gamma, \beta, 0, 0, \alpha, \Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} \left\{ \Delta(s) + \frac{(t-1)(\varphi - \alpha)(\beta - \sigma)}{\mu + \varphi} \right\}^n \frac{a_{t-1}}{\Omega} z^t,$ which results in a new derivative operator.
- $D_{\mu,\varphi,\sigma}^n(0, 0, 0, 0, \beta, 0, 0, \alpha, \Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} \left\{ 1 + \frac{(t-1) [(\varphi - \alpha)(\beta - \sigma)]}{\mu + \varphi} \right\}^n \frac{a_{t-1}}{\Omega} z^t.$ This expression results in a new generalized derivative operator.
- $D_{1-\varphi,\varphi,\sigma}^n(s, 0, 0, 0, \beta, 0, 0, \alpha, \Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} \{ \Delta(s) + (t-1)(\varphi - \alpha)(\beta - \sigma) \}^n \frac{a_{t-1}}{\Omega} z^t$ is a new derivative operator.
- $D_{1-\varphi,\varphi,\sigma}^n(0, 0, 0, 0, \beta, 0, 0, \alpha, \Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} \{ 1 + (t-1)(\varphi - \alpha)(\beta - \sigma) \}^n \frac{a_{t-1}}{\Omega} z^t,$ which results in a new derivative operator.
- $D_{1-\varphi,\varphi,0}^n(0, 0, 0, 0, \beta, 0, 0, \alpha, \Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} \{ 1 + (t-1)(\varphi - \alpha)\beta \}^n \frac{a_{t-1}}{\Omega} z^t,$ which results in a new derivative operator.
- $D_{1-\varphi,\varphi,0}^n(s, 0, 0, 0, 1, 0, 0, 0, \Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} \{ \Delta(s) + (t-1)\varphi \}^n \frac{a_{t-1}}{\Omega} z^t,$ which is a new derivative operator.

- $D_{1-\varphi,\varphi,0}^n(0,0,0,0,1,0,0,0,\Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} \{1 + (t-1)\varphi\}^n \frac{a_{t-1}}{\Omega} z^t$ which results in a new derivative operator.
- $D_{\mu,\varphi,0}^n(0,0,0,0,1,0,0,0,\Omega)K_\phi(z) = z + \sum_{t=2}^{\infty} \left\{ \frac{\mu + \varphi t}{\mu + \varphi} \right\}^n \frac{a_{t-1}}{\Omega} z^t$, which results in a new derivative operator.

Inspired primarily by recent studies and leveraging a newly developed differential operator, we introduce the following novel classes of analytic univalent functions.

Definition 2.2. Let $\kappa \geq 0$, $-\pi < \theta \leq \pi$, the function K_ϕ defined in (1) belongs to the class

$$TG_{\mu,\varphi,\sigma}^{s,m,\Phi,\gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega)$$

if it satisfies the following condition:

$$\Re \left\{ \frac{D_{\mu,\varphi,\sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z)}{z} \right\} \geq \frac{\kappa(1 + e^{i\theta})}{2} \left| \left[D_{\mu,\varphi,\sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z) \right]' - \frac{D_{\mu,\varphi,\sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z)}{z} \right|.$$

By selecting different parameter values, numerous new and existing classes of analytic functions can be obtained.

3. Main Results

Theorem 3.1. A function g_ϕ defined by (2) is said to belong to the class $TG_{\mu,\varphi,\sigma}^{s,m,\Phi,\gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega)$ if and only if

$$\sum_{t=2}^{\infty} \left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \times \left\{ \frac{2 + \kappa(1 + \cos \theta)(t-1)}{2} \right\} \left| \frac{a_{t-1}}{\Omega} \right| \leq 1.$$

Proof. Assume that K_ϕ , defined by (1), satisfies condition (3). Then, we consider the following expression:

$$\Re \left\{ \frac{D_{\mu,\varphi,\sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z)}{z} \right\} - \frac{\kappa(1 + e^{i\theta})}{2} \left| \left[D_{\mu,\varphi,\sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z) \right]' - \frac{D_{\mu,\varphi,\sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega)K_\phi(z)}{z} \right|.$$

Equivalently, we write

$$= 1 - \left| \left(1 - \sum_{t=2}^{\infty} \Theta(n, t) \frac{a_{t-1}}{\Omega} z^{t-1} \right) - 1 \right| \\ - \frac{\kappa(1 + e^{i\theta})}{2} \left| \left(1 - \sum_{t=2}^{\infty} t\Theta(n, t) \frac{a_{t-1}}{\Omega} z^{t-1} \right) - \left(1 - \sum_{t=2}^{\infty} \Theta(n, t) \frac{a_{t-1}}{\Omega} z^{t-1} \right) \right|.$$

Simplifying, we obtain

$$= 1 - \left| \sum_{t=2}^{\infty} \Theta(n, t) \frac{a_{t-1}}{\Omega} z^{t-1} \right| - \frac{\kappa(1 + e^{i\theta})}{2} \left| \sum_{t=2}^{\infty} (t-1) \Theta(n, t) \frac{a_{t-1}}{\Omega} z^{t-1} \right|, \quad |z| = r < 1, \\ \leq 1 - \sum_{t=2}^{\infty} |\Theta(n, t)| \left| \frac{a_{t-1}}{\Omega} \right| - \frac{\kappa(1 + e^{i\theta})}{2} \sum_{t=2}^{\infty} (t-1) |\Theta(n, t)| \left| \frac{a_{t-1}}{\Omega} \right| \\ = 1 - \sum_{t=2}^{\infty} |\Theta(n, t)| \left[\frac{2 + \kappa(1 + \cos \theta)}{2} \right] \left| \frac{a_{t-1}}{\Omega} \right| \geq 0, \quad (4)$$

where

$$\Theta(n, t) = \left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n.$$

Conversely, suppose that the following inequality holds:

$$\Re \left\{ \frac{D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_{\phi}(z)}{z} \right\} \\ - \frac{\kappa(1 + e^{i\theta})}{2} \left| \left[D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_{\phi}(z) \right]' - \frac{D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_{\phi}(z)}{z} \right| > 0$$

which means that

$$\Re \left\{ 1 - \sum_{t=2}^{\infty} |\Theta(n, t)| \left| \frac{a_{t-1}}{\Omega} \right| \right\} - \frac{\kappa(1 + \cos \theta)}{2} \sum_{t=2}^{\infty} (t-1) |\Theta(n, t)| \left| \frac{a_{t-1}}{\Omega} \right| > 0.$$

Letting z approach real values and taking the limit as $z \rightarrow 1$, we have

$$1 - \sum_{t=2}^{\infty} |\Theta(n, t)| \left| \frac{a_{t-1}}{\Omega} \right| - \frac{\kappa(1 + \cos \theta)}{2} \sum_{t=2}^{\infty} (t-1) |\Theta(n, t)| \left| \frac{a_{t-1}}{\Omega} \right| \geq 0$$

which implies that

$$\sum_{t=2}^{\infty} |\Theta(n, t)| \left[\frac{2 + \kappa(1 + \cos \theta)}{2} \right] \left| \frac{a_{t-1}}{\Omega} \right| \leq 1$$

by condition (2).

Remark 3.1. For any function $K_\phi \in TG_{\mu, \varphi, \sigma}^{s, m, \Phi, \gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega)$, we obtain

$$\frac{a_{t-1}}{\Omega} \leq \frac{1}{\left(\Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right)^n \cdot \left(\frac{2 + \kappa(1 + \cos \theta)(t-1)}{2} \right)},$$

and equality holds for the function

$$K_\phi(z) = z - \sum_{t=2}^{\infty} \frac{1}{\left(\Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right)^n \cdot \left(\frac{2 + \kappa(1 + \cos \theta)(t-1)}{2} \right)} z^t.$$

Theorem 3.2. A function K_ϕ , defined by (1), is said to belong to the class $TG_{\mu, \varphi, \sigma}^{s, m, \Phi, \gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega; A_i)$ if

$$\sum_{t=k+1}^{\infty} \left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \times \left\{ \frac{2 + \kappa(1 + \cos \theta)(t-1)}{2} \right\} \frac{a_{t-1}}{\Omega} \leq 1 - \sum_{i=2}^k A_i,$$

where $0 \leq A_i \leq 1$ and $0 \leq \sum_{i=2}^k A_i \leq 1$.

Proof. Suppose that K_ϕ belongs to the class $TG_{\mu, \varphi, \sigma}^{s, m, \Phi, \gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega; A_i)$. Then, by Definition 2.2, we have the following inequality for each $i = 2, 3, 4, \dots, k$

$$\frac{a_{i-1}}{\Omega} \leq \frac{A_i}{\left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \left\{ \frac{2 + \kappa(1 + \cos \theta)(t-1)}{2} \right\}}.$$

This implies that the sum of all such terms up to $i = k$, together with the tail sum from $t = k + 1$ to infinity, satisfies

$$\sum_{i=2}^k A_i \sum_{t=k+1}^{\infty} \left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n$$

$$\times \left\{ \frac{2 + \kappa(1 + \cos \theta)(t - 1)}{2} \right\} \frac{a_{t-1}}{\Omega} \leq 1.$$

Conversely, consider the quantity

$$\begin{aligned} & \Re \left\{ \frac{D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_\phi(z)}{z} \right\} \\ & \quad - \frac{\kappa(1 + e^{i\theta})}{2} \left| [D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_\phi(z)]' \right. \\ & \quad \quad \left. - \frac{D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_\phi(z)}{z} \right| \\ & \geq 1 - \left| \frac{D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_\phi(z)}{z} - 1 \right| \\ & \quad - \frac{\kappa(1 + e^{i\theta})}{2} \left| [D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_\phi(z)]' \right. \\ & \quad \quad \left. - \frac{D_{\mu, \varphi, \sigma}^n(s, m, \Phi, \gamma, \beta, \tau, \Gamma, \alpha, \Omega) K_\phi(z)}{z} \right| \\ & = 1 - \sum_{t=2}^{\infty} |\Theta(n, t)| \left| \frac{a_{t-1}}{\Omega} \right| - \frac{\kappa(1 + e^{i\theta})}{2} \sum_{t=2}^{\infty} (t - 1) |\Theta(n, t)| \left| \frac{a_{t-1}}{\Omega} \right| \\ & = 1 - \sum_{i=2}^k \Theta(n, t) \frac{\kappa(1 + \cos \theta)}{2} \left| \frac{a_{t-1}}{\Omega} \right| - \sum_{t=i+1}^k \Theta(n, t) \frac{\kappa(1 + \cos \theta)}{2} \left| \frac{a_{t-1}}{\Omega} \right| \\ & = 1 - \sum_{i=2}^k A_i - \sum_{t=i+1}^k \Theta(n, t) \frac{\kappa(1 + \cos \theta)(t - 1)}{2} \left| \frac{a_{t-1}}{\Omega} \right| \geq 0. \end{aligned}$$

Hence, it follows from Theorem 3.1 that K_ϕ belongs to the specified class.

Moreover, the result is sharp for the function defined by

$$\begin{aligned} K_\phi(z) = z - \sum_{i=2}^k \frac{A_i}{\left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \left(\frac{\kappa(1 + \cos \theta)(t-1)}{2} \right)} z^j \\ - \frac{1 - \sum_{i=2}^k A_i}{\left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \left(\frac{\kappa(1 + \cos \theta)(t-1)}{2} \right)} z^t. \end{aligned}$$

Corollary 3.1. Suppose that K_ϕ belongs to the class $TG_{\mu,\varphi,\sigma}^{s,m,\Phi,\gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega; A_i)$. Then

$$a_{t-1} \leq \frac{1 - \sum_{i=2}^k A_i}{\left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \left(\frac{\kappa(1 + \cos \theta)(t-1)}{2} \right)}, \quad (5)$$

$$t \geq k + 1.$$

The result is sharp for K_ϕ given by (1).

Theorem 3.3. The class $TG_{\mu,\varphi,\sigma}^{s,m,\Phi,\gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega; A_i)$ is convex.

Proof. Assume there exist functions $K_\phi, g_\phi \in TG_{\mu,\varphi,\sigma}^{s,m,\Phi,\gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega; A_i)$ such that

$$K_\phi(z) = z - \sum_{i=2}^k \frac{A_i}{\left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \left(\frac{\kappa(1 + \cos \theta)(t-1)}{2} \right)} z^j - \sum_{t=k+1}^{\infty} \frac{a_{t-1}}{\Omega}, \quad (6)$$

and

$$g_\phi(z) = z - \sum_{i=2}^k \frac{A_i}{\left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \left(\frac{\kappa(1 + \cos \theta)(t-1)}{2} \right)} z^j - \sum_{t=k+1}^{\infty} \frac{b_{t-1}}{\Omega}. \quad (7)$$

Let

$$0 \leq A_i \leq 1 \quad \text{and} \quad 0 \leq \sum_{i=2}^k A_i \leq 1.$$

Now, let $0 \leq \delta \leq 1$, and define the convex combination

$$E(z) = \delta K_\phi(z) + (1 - \delta)g_\phi(z), \quad (8)$$

where δ is a scalar weight used to control the influence and contribution of K_ϕ and g_ϕ . It is a hyperparameter designed to tune, shape, and validate the behavior of the convex combination, to optimize the bias-variance trade-off, and to align the model with empirical data.

By applying (6) and (7) in (8), we arrive

$$E(z) = z - \sum_{i=2}^k \frac{A_i}{\left\{ \Delta(s) + \frac{(t-1) [m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + t(\tau + \Gamma)]}{\mu + \varphi} \right\}^n \left(\frac{\kappa(1 + \cos \theta)(t-1)}{2} \right)} - \sum_{t=k+1}^{\infty} \left(\delta \frac{a_{t-1}}{\Omega} + (1 - \delta) \frac{b_{t-1}}{\Omega} \right) z^t.$$

Furthermore, observe that

$$\begin{aligned} & \sum_{t=k+1}^{\infty} \Theta(n, t) \left\{ \frac{2 + \kappa(1 + \cos \theta)(t-1)}{2} \right\} \left[\delta \frac{a_{t-1}}{\Omega} + (1 - \delta) \frac{b_{t-1}}{\Omega} \right] \\ &= \delta \sum_{t=k+1}^{\infty} \Theta(n, t) \left\{ \frac{2 + \kappa(1 + \cos \theta)(t-1)}{2} \right\} \frac{a_{t-1}}{\Omega} \\ & \quad + (1 - \delta) \sum_{t=k+1}^{\infty} \Theta(n, t) \left\{ \frac{2 + \kappa(1 + \cos \theta)(t-1)}{2} \right\} \frac{b_{t-1}}{\Omega} \\ & \leq \delta \left(1 - \sum_{i=2}^k A_i \right) + (1 - \delta) \left(1 - \sum_{i=2}^k A_i \right) \\ & = 1 - \sum_{i=2}^k A_i. \end{aligned}$$

Hence, $E(z) \in TG_{\mu, \varphi, \sigma}^{s, m, \Phi, \gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega; A_i)$.

Theorem 3.4. *Let the function g_ϕ , defined by (2), belong to the class $TG_{\mu, \varphi, \sigma}^{s, m, \Phi, \gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega)$. Then the following inequalities hold:*

$$\begin{aligned} & r - \frac{2\mu\varphi}{\left\{ \Delta(s)(\mu + \varphi) + m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma) \right\} \{2 + \kappa(1 + \cos \theta)\}} r^2 \\ & \leq |K_\phi(z)| \\ & \leq r + \frac{2\mu\varphi}{\left\{ \Delta(s)(\mu + \varphi) + m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma) \right\} \{2 + \kappa(1 + \cos \theta)\}} r^2 \end{aligned}$$

and

$$\leq 1 + \frac{4\mu\varphi}{\left\{ \Delta(s)(\mu + \varphi) + m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma) \right\} \{2 + \kappa(1 + \cos \theta)\}} r$$

$$\begin{aligned} &\leq |K'_\phi(z)| \\ &\leq 1 + \frac{4\mu\varphi}{\{\Delta(s)(\mu + \varphi) + m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma)\} \{2 + \kappa(1 + \cos \theta)\}} r. \end{aligned}$$

Proof. Since $K_\phi \in TG_{\mu,\varphi,\sigma}^{s,m,\Phi,\gamma}(\theta, \kappa, \beta, \tau, \Gamma, \alpha, \Omega)$, Theorem 3.1 readily yields the inequality

$$\sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega} \leq \frac{2\mu\varphi}{\{\Delta(s)(\mu + \varphi) + m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma)\} \{2 + \kappa(1 + \cos \theta)\}}.$$

Thus, for $|z| = r < 1$, we obtain

$$|K_\phi(z)| \leq |z| + \sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega} |z^t| \leq r + r^2 \sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega},$$

which implies

$$|K_\phi(z)| \leq r + \frac{2\mu\varphi}{\{\Delta(s)(\mu + \varphi) + m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma)\} \{2 + \kappa(1 + \cos \theta)\}} r^2.$$

Similarly,

$$|K_\phi(z)| \geq |z| - \sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega} |z^t| \leq r - r^2 \sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega},$$

and thus

$$|K_\phi(z)| \geq r - \frac{2\mu\varphi}{\{\Delta(s)(\mu + \varphi) + m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma)\} \{2 + \kappa(1 + \cos \theta)\}} r^2.$$

Also,

$$|K'_\phi(z)| \leq 1 + \sum_{t=2}^{\infty} t \frac{a_{t-1}}{\Omega} |z^t| \leq 1 + 2r \sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega},$$

and therefore,

$$|K'_\phi(z)| \leq 1 + \frac{4\mu\varphi}{\{\Delta(s)(\mu + \varphi) + m(\varphi - \Phi)^{\sigma(1-\gamma)} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma)\} \{2 + \kappa(1 + \cos \theta)\}} r.$$

Similarly,

$$|K'_\phi(z)| \geq 1 - 2r \sum_{t=2}^{\infty} \frac{a_{t-1}}{\Omega},$$

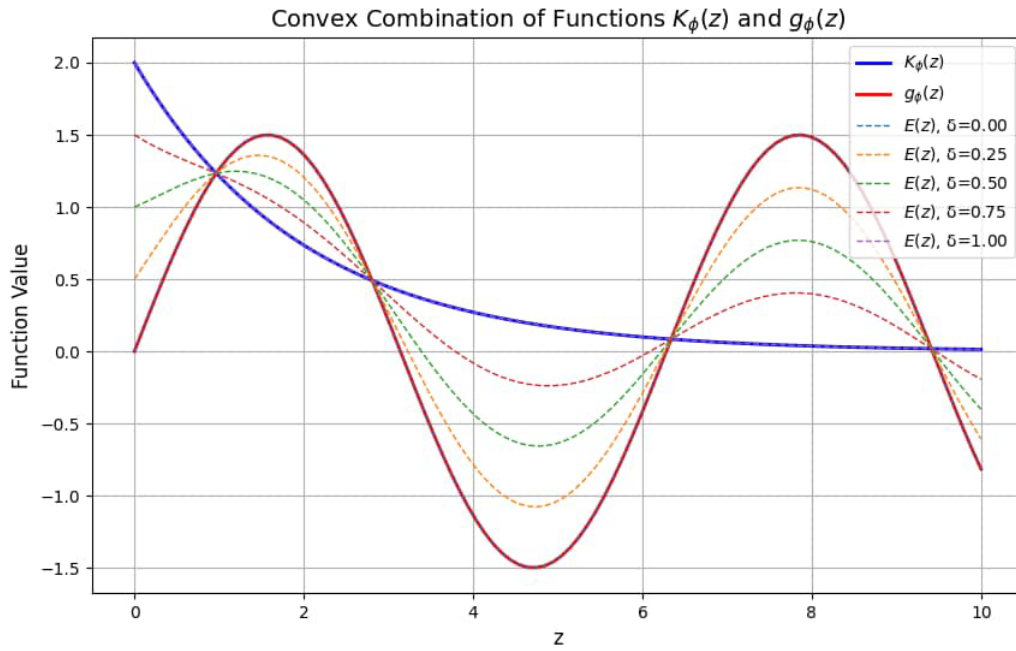


Fig. 1

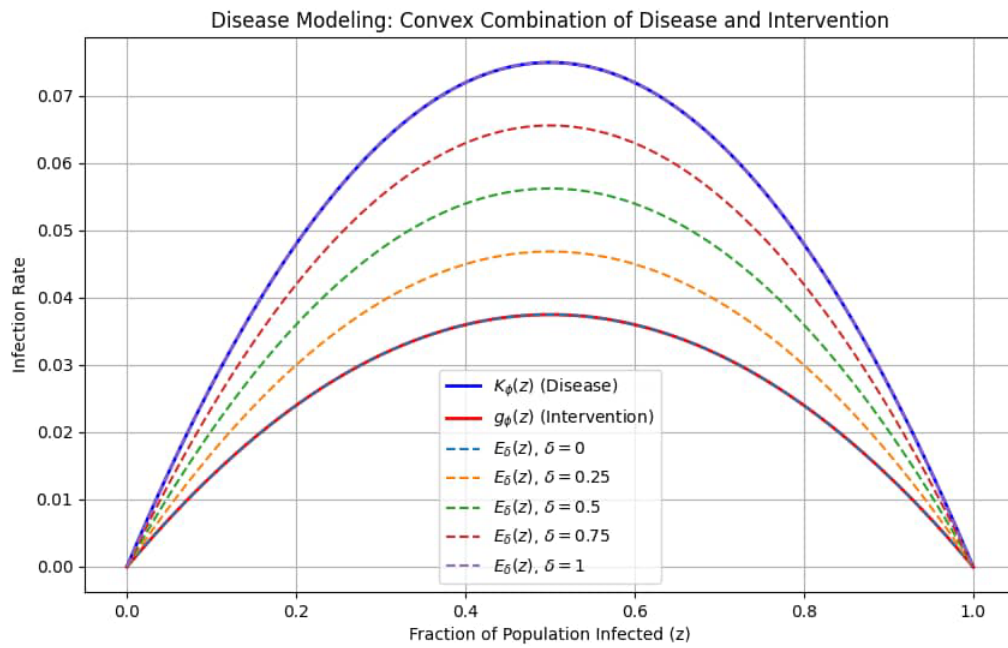


Fig. 2

which gives

$$|K'_\phi(z)| \geq 1 - \frac{2\mu\varphi}{\{\Delta(s)(\mu + \varphi) + m(\varphi - \Phi)\sigma^{1-\gamma} + (\varphi - \alpha)(\beta - \sigma) + 2(\tau + \Gamma)\} \{2 + \kappa(1 + \cos \theta)\}^r}.$$

This completes the proof.

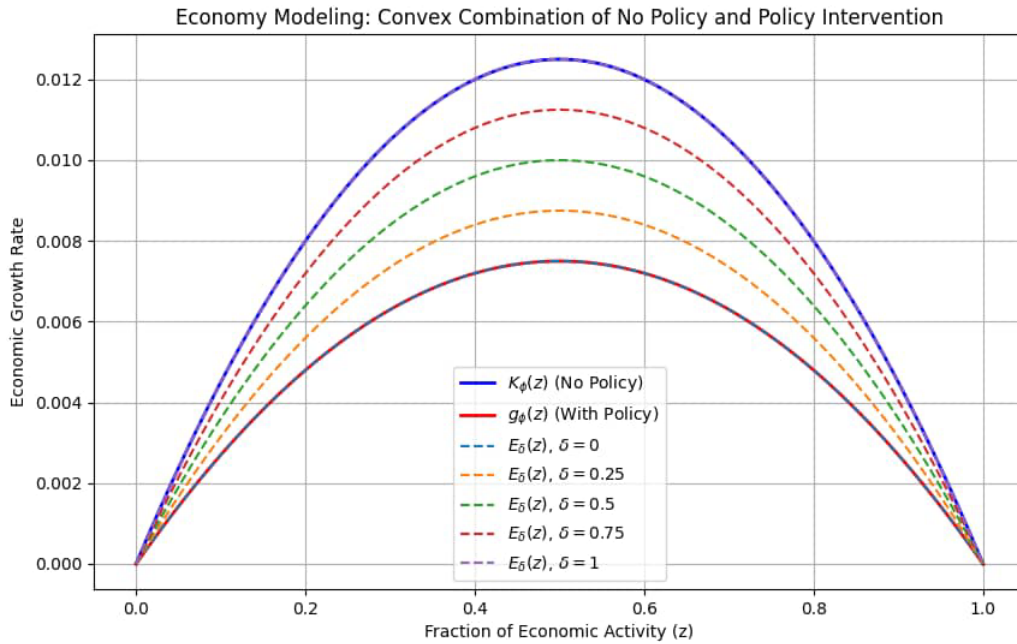


Fig. 3

Investment Return Visualization

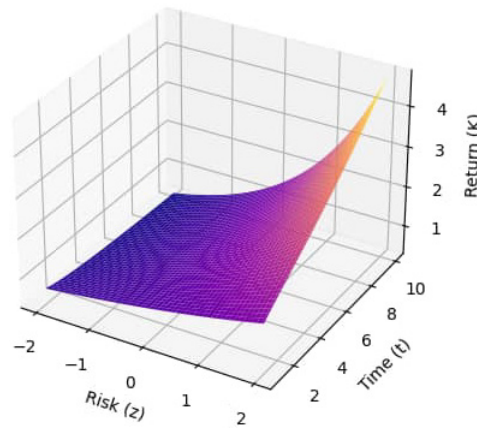


Fig. 4

4. Graphical Characterization of Some Obtained Results

In summary, the graphical presentations in this section aim to explore various aspects of our results, particularly Theorem 3.3, which concerns the convex combination of the functions K_ϕ and g_ϕ . These graphical results are driven by the hyperparameter δ , where $0 \leq \delta \leq 1$. This scalar weight controls the influence and contribution of K_ϕ and g_ϕ . It serves as a hyperparameter to tune, shape, and validate the behavior of the convex combination, to optimize the bias-variance trade-off, and to align the model with empirical data.

Using Python, we graphically illustrate the convex combination of our results across various real-life applications related to mathematics, such as in biomathematics (e.g., disease intervention, disease spread, and infection dynamics). For instance, in Fig. 2, K_ϕ may represent a strict lockdown function, while g_ϕ may correspond to a vaccine rollout model. In this context, $E(z)$ in equation (8) captures the blended intervention policy.

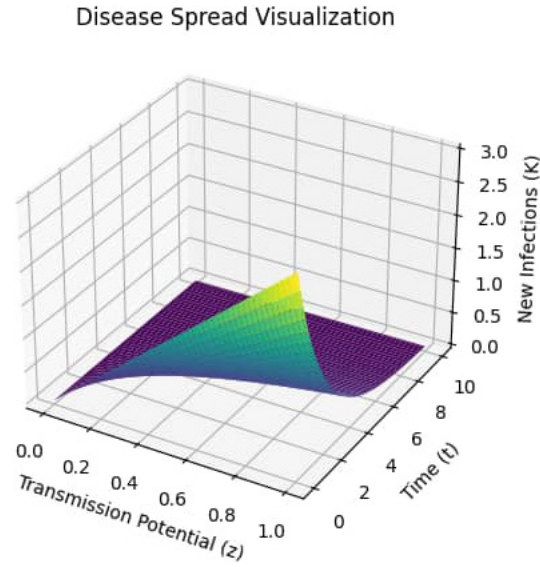


Fig. 5

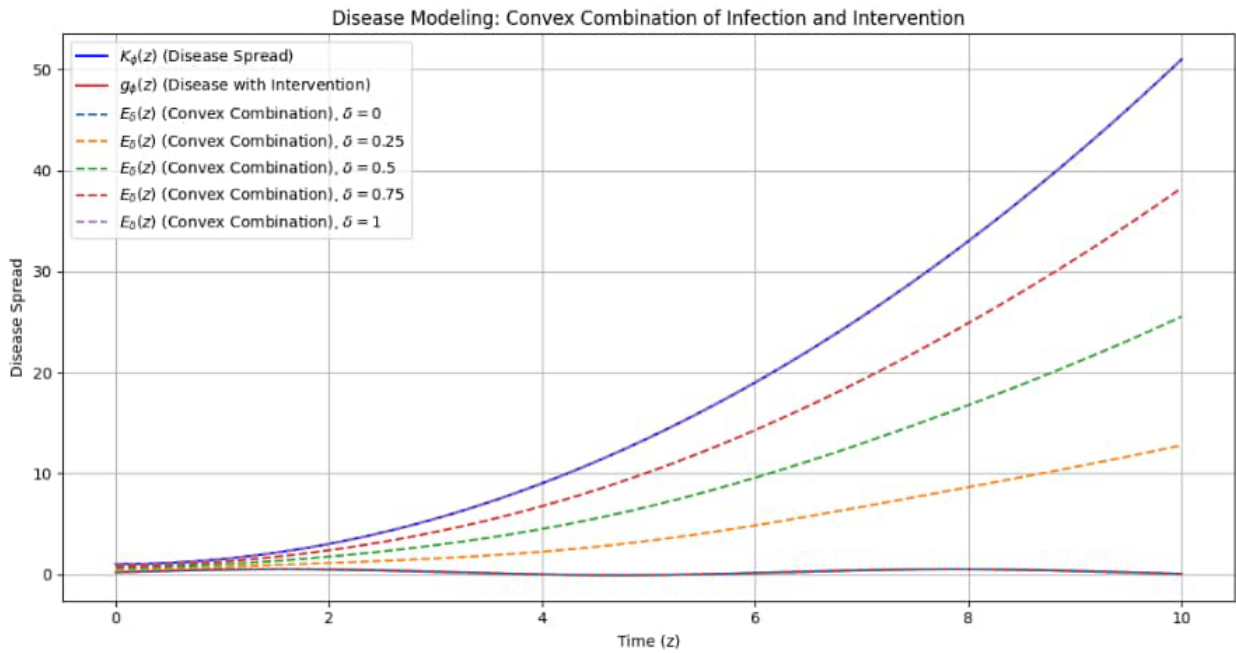


Fig. 6

In the context of management science (e.g., policy versus no-policy scenarios, investment returns, free-market versus intervention strategies, and cost-efficiency analysis), Figures 3 and 4 illustrate another application. Here, K_ϕ is interpreted as the return of a particular stock A, and g_ϕ as the return of stock B. If δ represents the proportion of wealth invested in stock A and $1 - \delta$ in stock B, then $E(z)$ in equation (8) denotes the expected return of the investment portfolio.

This type of convex combination is widely used in optimization, probability, machine learning, and analysis to interpolate or balance between two models, strategies, or estimates. These outcomes are achieved by adjusting the key parameters — particularly δ and the variable z . The resulting visualizations are presented below.

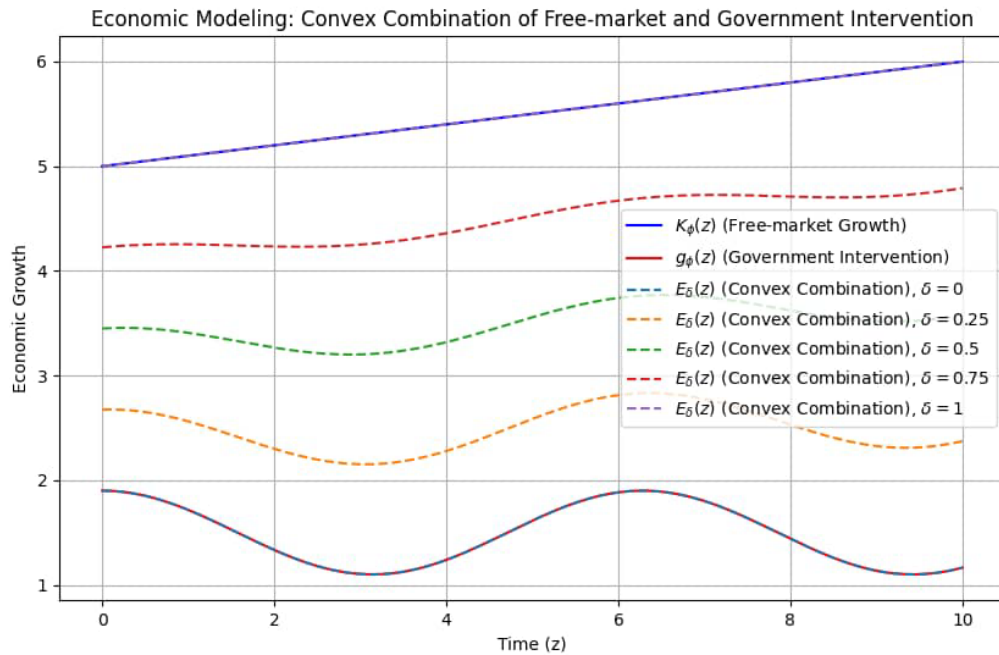


Fig. 7

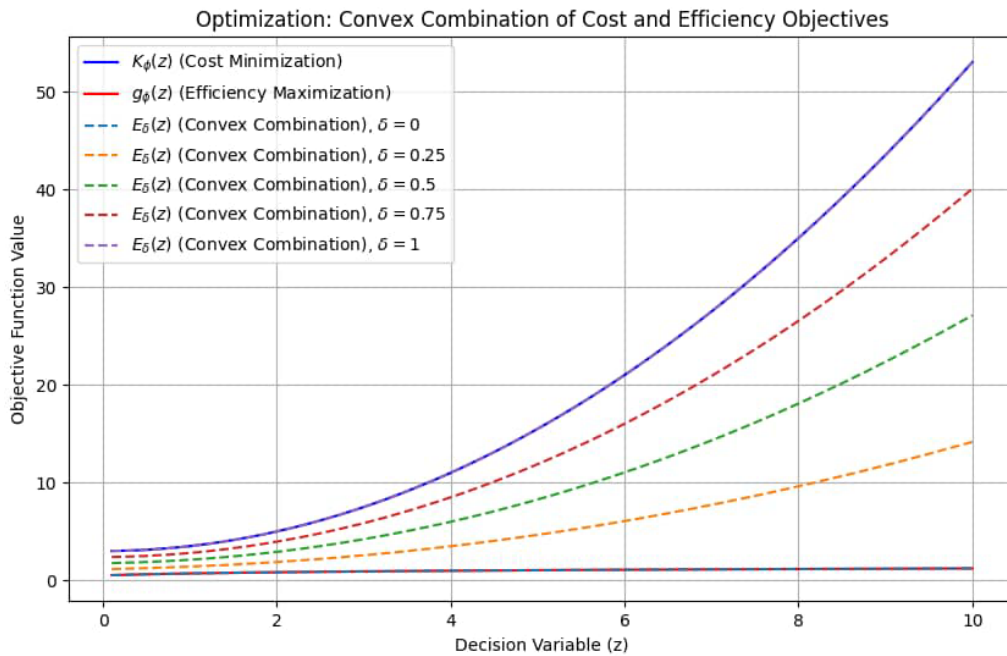


Fig. 8

5. Conclusion

We conducted a comprehensive analysis of the generalized discrete probability distribution incorporating the Sigmoid function. Various properties were established and characterized by exploring different parameter choices. Additionally, graphical representations were generated using Python to illustrate these findings.

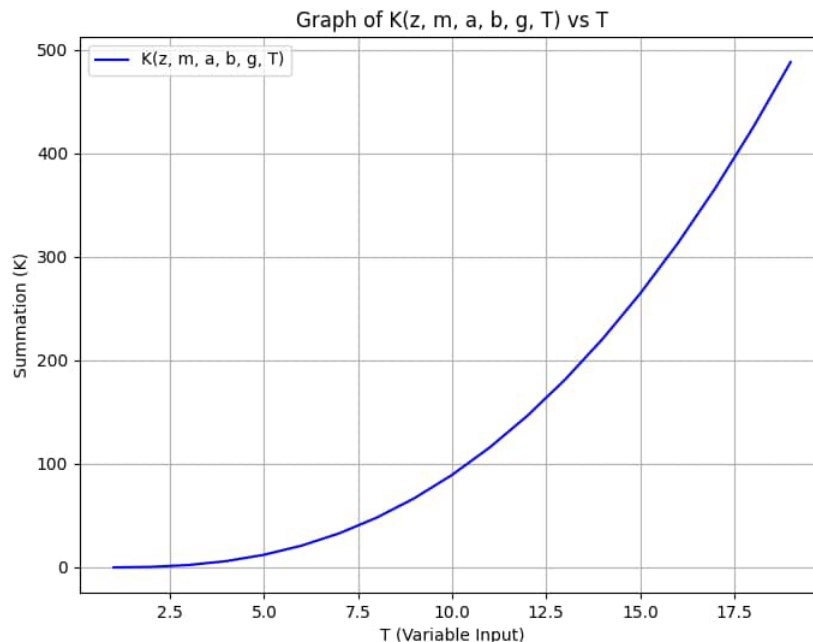


Fig. 9

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