The Journal of Geometric Analysis (2024) 34:154
https://doi.org/10.1007/s12220-024-01599-5

Uniformly Perfect Sets, Hausdorff Dimension, and Conformal Capacity

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Received: 17 November 2023 / Accepted: 22 February 2024

Abstract
Using the definition of uniformly perfect sets in terms of convergent sequences, we apply lower bounds for the Hausdorff content of a uniformly perfect subset E of $\mathbb{R}^n$ to prove new explicit lower bounds for the Hausdorff dimension of E. These results also yield lower bounds for capacity test functions, which we introduce, and enable us to characterize domains of $\mathbb{R}^n$ with uniformly perfect boundaries. Moreover, we show that an alternative method to define capacity test functions can be based on the Whitney decomposition of the domain considered.

Keywords Condenser capacity · Invariant metrics · Modulus of a curve family · Uniformly perfect set · Whitney cubes

Mathematics Subject Classification Primary 30F45; Secondary 30C85

1 Introduction

Conformal invariants, in particular the modulus of a curve family and the conformal capacity of a condenser, are fundamental tools of geometric function theory and quasiconformal mappings [9–11, 13, 32]. For applications, many upper and lower bounds for conformal invariants have been derived in terms of various geometric functionals.

In memoriam: Pentti Järvi 1942–2021.

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Published online: 06 April 2024

All this research shows that the metric structure of the boundary has a strong influence on the intrinsic geometry of the domain of the mappings studied. Indeed, many results originally proven for functions defined in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$, $n \geq 2$, can be extended to the case of subdomains $G \subset \mathbb{R}^n$ if the boundary ∂G is "thick enough" in the sense of capacity, or, more precisely, if the boundary is uniformly perfect. The thickness of the boundary has a strong influence on the intrinsic geometry of the domain and thus it also gives a restriction on the oscillation of a function defined in G. We give several new characterizations of uniformly perfect sets.

A *condenser* is a pair (G, E) where $G \subset \mathbb{R}^n$, $n \geq 2$, is a domain and $E \subset G$ is a compact set [9, 11, 13]. A compact set $E \subset \mathbb{R}^n$ is of conformal capacity zero if, for some closed ball $B \subset \mathbb{R}^n \setminus E$, the condenser $(\mathbb{R}^n \setminus B, E)$ has capacity zero, written as $\text{cap}\,(E) = 0$ with notations of Definition 4.1. Sets of capacity zero are very thin, their Hausdorff dimensions are zero [27, p. 120, Cor. 2], [13, Lemma 9.11], and they often have the role of negligible exceptional sets in potential theory or geometric function theory. Note that, due to the Möbius invariance of the conformal capacity, the notions of zero and positive capacity immediately extend to compact subsets of the Möbius space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$.

Here our goal is to study those subsets of $\mathbb{R}^n$ that have a positive capacity instead. However, the structure of sets of positive capacity can sometimes be highly dichotomic, for instance, in the case of $E_1 \cup E_2$ where $E_1 \subset \mathbb{R}^2$ is a segment and E_2 is a point not contained in E_1. This kind of a dichotomy makes working with these sets difficult, but a subclass of sets with positive capacity, *uniformly perfect sets*, has certain natural properties useful for our purposes. During the past two decades, uniformly perfect sets have become ubiquitous for instance in geometric function theory [3, 10], analysis on metric spaces [14, 19], hyperbolic geometry [4, 17] and in the study of complex dynamics and Kleinian groups [7, 28].

We begin by giving a variant of the definition of uniformly perfect sets in terms of convergent sequences as follows. For $0 < c < 1$, let $\text{UP}_n(c)$ denote the collection of compact sets E in $\mathbb{R}^n$ with $\text{card}\,(E) \geq 2$ satisfying the condition

$$\{x \in E : cr < |x - a| < r\} \neq \emptyset \quad \text{for all } a \in E \quad \text{and} \quad 0 < r < d(E)/2.$$

We say that a set is uniformly perfect if it is in the class $\text{UP}_n(c)$ for some $c \in (0, 1)$.

The first lemma has an important role in the sequel.

Lemma 1.1 (T. Sugawa [30, Proposition 7.4]) *Let $E \in \text{UP}_n(c)$ for some $0 < c < 1$. Then for every $a \in E$, $a \neq \infty$, the Hausdorff content of E has the following lower bound*

$$\Lambda^\beta(E \cap \overline{B}^n(a, r)) \geq \frac{r^\beta}{2 \cdot 3^n}, \quad 0 < r < d(E)/2, \ \textit{where } \beta = \frac{\log 2}{\log(3/c)}.$$

Moreover, the Hausdorff dimension $\dim_H(E)$ of E is at least $(\log 2)/\log(3/c)$.

The explicit bounds we obtain in this paper depend on the above result of Sugawa, however, we prove it here with the above refined form of the constant β. Moreover, we also apply ideas from the work of Reshetnyak [26, 27] and Martio [20], see also

Remark 5.4, but now the constants are explicit which is crucial for what follows. In the study of uniformly perfect sets, similar methods were also applied by Järvi and Vuorinen [15, Thm 4.1, p. 522]. Our results here yield explicit constants for several characterizations of uniform perfectness such as the following main result.

Theorem 1.2 *Let $E \in \mathrm{UP}_n(c)$ for some $0 < c < 1$. Then for every $a \in E$, $a \neq \infty$, and all $r \in (0, d(E))$ the following lower bound for the conformal capacity* $\mathrm{cap}(a, E, r)$ *of the condenser $(B^n(a, 2r), E \cap \overline{B}^n(a, r))$ holds*

$$\mathrm{cap}(a, E, r) \geq \frac{1}{2 \cdot 3^n M_1(n, \beta)}, \quad \beta = \frac{\log 2}{\log(3/c)},$$

where $M_1(n, \beta)$, given in (5.10), is an explicit constant depending only on n and β.

There are many equivalent definitions of uniformly perfect sets. The definition given in [15] says that a set E is α-uniformly perfect, if the moduli of the ring domains separating the set have the upper bound α. We show in Sect. 6 that this definition is quantitatively equivalent with the definition of $\mathrm{UP}_n(c)$. Moreover, we prove this equivalence with explicite constants, a fact which would enable one to give explicit constants for instance in some earlier results such as in [15, Thm 4.1].

Suppose now that $G \subset \mathbb{R}^n$ is a domain, its boundary ∂G is of positive capacity, and define $u_\alpha : G \to (0, \infty)$ by

$$u_\alpha(z) = \mathrm{cap}\,(G, \overline{B}^n(z, \alpha d(z, \partial G))), \quad \alpha \in (0, 1),$$

for $z \in G$ where $d(z, \partial G)$ is the distance from z to ∂G. We call $u_\alpha(z)$ *the capacity test function* of G at the point $z \in G$. The numerical value of the capacity test function depends clearly on G, z, and α, but we omit G from the notation because it is usually understood from the context. Clearly, the capacity test function is invariant under similarity transformations. We will also show that it is continuous as a function of both $z \in G$ and $\alpha \in (0, 1)$. For the purpose of this paper, it is enough to choose e.g. $\alpha = 1/2$.

Analysing the capacity test function $u_\alpha(z)$ further, we show that, for a fixed α, it satisfies *the Harnack inequality* as a function of z, a property which has a number of consequences. First, for every $z_0 \in G$, we see that $u_\alpha(z_0) > 0$, because the boundary was assumed to be of positive capacity. Second, by fixing $z_0 \in G$, we see by a standard chaining argument [13, p. 96, Lemma 6.23 and p. 84] that $u_\alpha(z)/u_\alpha(z_0)$ has a positive explicit minorant for a large class of domains, so called φ-uniform domains. This minorant, depending on φ, $d(z, \partial G)$ and Harnack parameters, shows that $u_\alpha(z)$ cannot approach 0 arbitrarily fast when z moves far away from z_0 or when $z \to \partial G$. Under the stronger requirement that ∂G be uniformly perfect, it follows that $u_\alpha(z)$ is bounded from below by a constant $c > 0$. These observations lead to the following new characterization of uniformly perfect sets.

Theorem 1.3 *The boundary of a domain $G \subset \mathbb{R}^n$ is uniformly perfect if and only if there exists a constant $s > 0$ such that $u_\alpha(x) \geq s$ for all $x \in G$.*

Many characterizations are known for plane domains with uniformly perfect boundaries and often these characterizations are given in terms of hyperbolic geometry [10, pp. 342–344], [17], [4]. Because the hyperbolic geometry cannot be used in dimensions $n \geq 3$, we use here another tool, the *Whitney decomposition* of a domain $G \subset \mathbb{R}^n$, which has numerous applications to geometric function theory and harmonic analysis [6, 10, 29]. The Whitney decomposition represents G as a countable union of non-overlapping cubes with edge lengths equal to 2^{-k}, $k \in \mathbb{Z}$, where the edge length is proportional to the distance from a cube to the boundary of the domain [29]. Martio and Vuorinen [21] applied this decomposition to establish upper bounds for the metric size of the boundary ∂G in terms of N_k, the number of cubes with edge length equal to 2^{-k}. Their method was based on imposing growth bounds for N_k when $k \to \infty$ and depending on the growth rate, the conclusion was either an upper bound for the Minkowski dimension of the boundary or a sufficient condition for the boundary to be of capacity zero. In our next main result, Theorem 1.4, we use Whitney decomposition "in the opposite direction". Indeed, we employ Whitney cubes as test sets for the capacity structure of the boundary and obtain the following characterization of uniform perfectness in all dimensions $n \geq 2$. Whitney cubes also have applications to the study of surface area estimation of the level sets of the distance function [16].

Theorem 1.4 *The boundary of a domain $G \subset \mathbb{R}^n$ is uniformly perfect if and only if there exists a constant $s > 0$ such that, for every Whitney cube $Q \subset G$,*

$$\text{cap}\,(\mathbb{R}^n \setminus Q, \partial G) \geq s.$$

By definition, see the property 8.1(3) below, every Whitney cube $Q \subset G$ satisfies $d(Q)/d(Q, \partial G) > 1/4$. Thus we see that the next theorem generalizes Theorem 1.4.

Theorem 1.5 *Let $G \subset \mathbb{R}^n$ be a domain and $E \subset G$ a compact set. If E and ∂G are uniformly perfect, then*

$$\text{cap}\,(G, E) \geq s \log(1 + d(E)/d(E, \partial G)),$$

where $s > 0$ is a constant depending only on the dimension n and the uniform perfectness parameters of E and ∂G.

This theorem is well-known if both E and ∂G are continua, see [33, Lemma 7.38, Notes 7.60].

2 Preliminary Results

In this preliminary section we recall some basic facts about metrics and quasiconformal homeomorphisms. Moreover, we prove a few propositions which are results of technical character, essential for the proofs of the main theorems in subsequent sections.

The following notations will be used: The Euclidean diameter of the non-empty set J is $d(J) = \sup\{|x - y| \mid x, y \in J\}$. The Euclidean distance between two non-empty

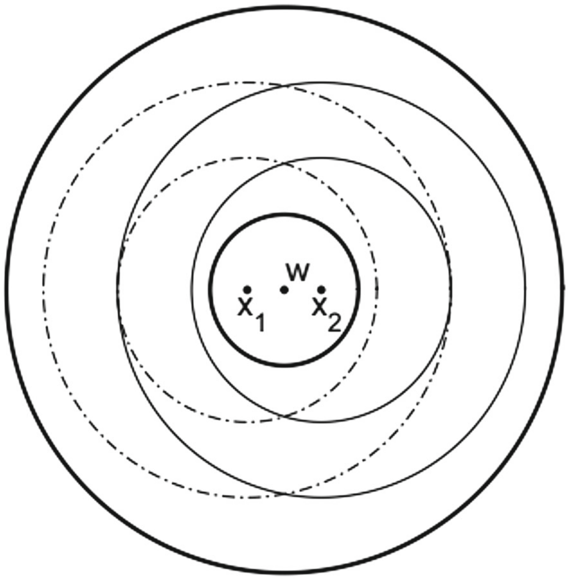


Fig. 1 For the proof of Proposition 2.2. An annulus centered at x_1 and its translation centered at x_2 (marked with solid and dash-dot markers, resp.) are both subsets of a larger annulus (marked with thick marker) centered at the midpoint $w = (x_1 + x_2)/2$. The larger annulus is a common superannulus of the two smaller annuli

sets $J, K \subset \mathbb{R}^n$ is $d(J, K) = \inf\{|x-y| \mid x \in J, y \in K\}$ and the distance from a point x to the set J is $d(x, J) = d(\{x\}, J)$. Thus, for all points x in a domain $G \subsetneq \mathbb{R}^n$, the Euclidean distance from x to the boundary ∂G is denoted by $d(x, \partial G)$, the Euclidean open ball with a center x and a radius r by $B^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, the corresponding closed ball by $\overline{B}^n(x, r)$ and their boundary by $S^{n-1}(x, r) = \partial B^n(x, r)$. If the center x or the radius r are not otherwise specified, assume that $x = 0$ and $r = 1$. The unit ball is denoted by $\mathbb{B}^n$. We denote by Ω_n the volume of the unit ball $\mathbb{B}^n$ and by ω_{n-1} the area of the unit sphere S^{n-1}. As well known, $\omega_{n-1} = n\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

For $x \in \mathbb{R}^n$, $0 < a < b$, we use the following notation for an annulus centered at x

$$R(x, b, a) = \overline{B}^n(x, b) \setminus B^n(x, a). \tag{2.1}$$

The first proposition shows that an annulus and its translation are both subsets of a larger annulus. This larger annulus is a superannulus for both of the two smaller annuli, i.e. the smaller annuli are separating the two boundary components of the larger annulus (Fig 1).

Proposition 2.2 *Let* $0 < a < b$ *and* $\tau \geq 2$. *If* $x_1, x_2 \in \mathbb{R}^n$, $|x_1 - x_2| < a$, *and* $w = (x_1 + x_2)/2$, *then for* $j = 1, 2$

$$R(x_j, b, a) \subset R(w, d, c), \quad \frac{d}{c} \leq \frac{b}{a}\frac{1 + |x_1 - x_2|/b}{1 - |x_1 - x_2|/a},$$

where $c = a - |x_1 - x_2|$, $d = b + |x_1 - x_2|$. *In particular, if* $|x_1 - x_2| < a/\tau^2$, *then for* $j = 1, 2$

$$R(x_j, \tau b, a/\tau) \subset R(w, \tau^2 b, a/\tau^2).$$

Moreover, $R(w, \tau^2 b, a/\tau^2) \subset B^n(x_j, 2\tau^2 b)$, *for* $j = 1, 2$.

Proof The claims follow from the triangle inequality and we prove here only the second one. Without loss of generality we may assume that $j = 1$. Fix $u \in R(x_1, \tau b, a/\tau)$. Then $\tau b > |u - x_1| > a/\tau$ and

$$|u - w| \geq |u - x_1 + x_1 - w| \geq |u - x_1| - |x_1 - w| \geq a/\tau - a/\tau^2 \geq a/\tau^2$$

where the last inequality holds because $\tau \geq 2$. Similarly,

$$|u - w| \leq |u - x_1 + x_1 - w| \leq |u - x_1| + |x_1 - w| \leq \tau b + a/\tau^2 \leq \tau^2 b$$

because $\tau \geq 2$. Therefore $a/\tau^2 \leq |u - w| \leq \tau^2 b$ and hence $u \in R(w, \tau^2 b, a/\tau^2)$. The last assertion also follows easily from the triangle inequality. □

Topology in $\mathbb{R}^n$ or in its one point compactification $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is the metric topology defined by the chordal metric. The *chordal (spherical) metric* is the function $q : \overline{\mathbb{R}}^n \times \overline{\mathbb{R}}^n \to [0, 1]$,

$$q(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \text{ if } x \neq \infty \neq y, \quad q(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}.$$

Thus, for instance, an unbounded domain $G \subset \mathbb{R}^n$ has ∞ as one of its boundary points.

For the sake of convenient reference we record the following simple inequality.

Proposition 2.3 *For* $a > 0$ *and* $p \geq 1$ *the inequality*

$$t - a \geq \lambda^p \sqrt{1 + t}, \quad \lambda > 1,$$

holds for all $t \geq \lambda^{2p} + \lambda^p \sqrt{1 + a} + a$.

Proof The inequality clearly holds for all large values of t, larger than $|x|$, where x is a root of $(x - a)^2 - \lambda^{2p}(x + 1) = 0$. By the quadratic formula we see that both roots have absolute value less than $\lambda^{2p} + \lambda^p \sqrt{1 + a} + a$. □

For two points in a domain, the next result gives an estimate for the number of annular domains separating these points. This result has an important role in the proof of one of our main results in Sect. 7.

Proposition 2.4 *Let* $G \subset \mathbb{R}^n$ *be a domain, let* E *be a compact subset of* G, *and choose* $x, y \in E$ *such that* $d(x, \partial G) = d(E, \partial G)$ *and* $|x - y| \geq d(E)/2$. *Fix* $z_0 \in \partial G$ *so that*

$|x - z_0| = d(x, \partial G)$ and let $w = (x + z_0)/2$, $\lambda > 1$. If $d(E)/d(E, \partial G) > \lambda^p + 2$ for some integer $p \geq 1$, then there are p disjoint annular domains

$$B^n(w, \lambda^m d(x, \partial G)/2) \setminus \overline{B}^n(w, \lambda^{m-1} d(x, \partial G)/2), \quad m = 1, ..., p$$

separating x and y. Moreover, for $d(E)/d(E, \partial G) > \max\{\lambda^p + 2, \lambda^2 + 2\lambda + 2\}$, we have

$$p > c \log(1 + d(E)/d(E, \partial G)), \quad c = 1/(2 \log \lambda).$$

Proof Because $\overline{B}^n(w, \lambda^p d(x, \partial G)/2)$ contains the largest annulus, it is enough to show that $|w - y| > \lambda^p d(x, \partial G)/2$. By the triangle inequality,

$$|y - w| \geq |x - y| - |x - w| \quad \Leftrightarrow \quad \frac{|y - w|}{|x - w|} \geq \frac{|x - y|}{|x - w|} - 1\,.$$

Further,

$$\begin{aligned} |y - w| \quad & \geq \left(\frac{|x - y|}{|x - w|} - 1 \right) |x - w| \geq \left(\frac{d(E)}{d(E, \partial G)} - 1\right) d(x, \partial G)/2 \\ & \geq (\lambda^p + 1) d(x, \partial G)/2 \end{aligned}$$

as desired. To prove the second claim, the lower bound for p, fix an integer $p \geq 1$ with

$$\lambda^{p+1} + 2 \geq u > \lambda^p + 2, \quad u = d(E)/d(E, \partial G).$$

Then

$$p + 1 \geq \frac{\log(u - 2)}{\log \lambda}$$

holds. To find a lower bound for p, we observe that the inequality

$$p + 1 \geq \frac{\log(u - 2)}{\log \lambda} > \frac{\log(1 + u)}{2 \log \lambda} + 1 \tag{2.5}$$

holds iff

$$\left(\frac{u - 2}{\lambda} \right)^2 > 1 + u \Leftrightarrow u^2 - (4 + \lambda^2) u + 4 - \lambda^2 > 0.$$

By Proposition 2.3, this holds for all $u > \lambda^2 + 2\lambda + 2$. By (2.5), we see that this yields also the desired lower bound for p. □

2.6 Quasiconformal Maps and Moduli of Curve Families

Quasiconformal homeomorphisms $f : G \to G' = f(G)$ between domains $G, G' \subset \mathbb{R}^n$ are commonly defined in terms of moduli of curve families. For the basic properties of the modulus $\mathsf{M}(\Gamma)$ of a curve family Γ, the reader is referred to [2, 11, 14], [32, 6.1, p. 16], [13]. According to Väisälä's book [32], K-quasiconformal maps are characterized by the inequality

$$\mathsf{M}(f\Gamma)/K \le \mathsf{M}(\Gamma) \le K\mathsf{M}(f\Gamma)$$

for every family of curves Γ in G where $f(\Gamma) = \{f(\gamma) : \gamma \in \Gamma\}$.

The following monotonicity of the moduli of curve families is quite useful in various estimates. Let Γ_1, Γ_2 be two curve families in $\overline{\mathbb{R}}^n$. We say that Γ_2 is minorized by Γ_1 and write $\Gamma_2 > \Gamma_1$ for it, if every curve $\gamma \in \Gamma_2$ has a subcurve belonging to Γ_1. For instance, $\Gamma_2 > \Gamma_1$ if $\Gamma_2 \subset \Gamma_1$.

Lemma 2.7 [32, §6]

(1) *If* $\Gamma_2 > \Gamma_1$, *then* $\mathsf{M}(\Gamma_2) \le \mathsf{M}(\Gamma_1)$.
(2) *For a collection of curve families* Γ_i $(i = 1, 2, \dots, N)$,

$$\mathsf{M}\left(\bigcup_j \Gamma_i\right) \le \sum_{i=1}^{N} \mathsf{M}(\Gamma_i).$$

Moreover, equality holds if the curve families are separate.

Here the families Γ_i are said to be *separate* if they are contained in pairwise disjoint Borel sets E_i (see also [11, §4.2.2]).

Let G be a domain in $\overline{\mathbb{R}}^n$ and $E, F \subset \overline{G}$. In what follows, $\Delta(E, F; G)$ will stand for the family of all the curves that are in G except for the endpoints, and that have one endpoint in the set E and another endpoint in F [32, pp. 11–25], [13]. When $G = \mathbb{R}^n$ or $G = \overline{\mathbb{R}}^n$, we often write $\Delta(E, F; G) = \Delta(E, F)$.

Lemma 2.8 [32, Thm 7.5, p. 22] *If* $0 < a < b$ *and* $D = \overline{B}^n(b) \backslash B^n(a)$,

$$\mathsf{M}(\Delta(S^{n-1}(a), S^{n-1}(b); D)) = \omega_{n-1}(\log(b/a))^{1-n},$$

where ω_{n-1} *is the* $(n-1)$*-dimensional surface area of the unit sphere* S^{n-1}.

Corollary 2.9 *Let* $0 < a < b$ *and* $E, F \subset D \equiv \overline{B}^n(b) \backslash B^n(a)$, *and let*

$$\mathsf{M}(\Delta(E, F; \mathbb{R}^n)) \ge c > 0.$$

Then there exists $\lambda = \lambda(n, c) > 1$ *such that for all* $t \ge \lambda$

$$\mathsf{M}(\widetilde{\Delta}) \ge c/2, \quad \widetilde{\Delta} = \Delta(E, F; B^n(0, tb) \setminus B^n(0, a/t)).$$

Proof By the subadditivity of the modulus, Lemma 2.7,

$$\mathsf{M}(\Delta(E, F; \mathbb{R}^n)) \le \mathsf{M}(\widetilde{\Delta}) + \mathsf{M}(\Delta(E, F; \mathbb{R}^n) \setminus \widetilde{\Delta})$$

whereas by Lemma 2.8

$$\mathsf{M}(\Delta(E, F; \mathbb{R}^n) \setminus \widetilde{\Delta}) \le 2\omega_{n-1}(\log t)^{1-n} \le c/2$$

for all $t \ge \lambda \equiv \exp((4\omega_{n-1}/c)^{1/(n-1)}) > 1$. □

The *Teichmüller ring* is a domain in $\mathbb{R}^n$ with the complementary components $[-e_1, 0]$ and $[se_1, \infty], s > 0$. The modulus of the family of all curves joining these boundary components, denoted by $\tau_n(s)$, is a decreasing homeomorphism $\tau_n : (0, \infty) \to (0, \infty)$ and admits the following lower bound

$$\tau_n(s) \ge c_n \log\left(1 + \frac{2(1 + \sqrt{1+s})}{s}\right) \ge 2c_n \log(1 + \frac{1}{\sqrt{s}}),$$
$$c_n = B(\frac{1}{2(n-1)}, \frac{1}{2})^{1-n}\omega_{n-2}, \tag{2.10}$$

where $B(\cdot, \cdot)$ is the beta function, $c_2 = 2/\pi$ [13, p. 114]. For $n = 2$, τ_2 can be expressed explicitly in terms of complete elliptic integrals [13, p. 123].

The function τ_n often occurs as a lower bound for moduli of curve families like in the following lemma, based on the spherical symmetrization of condensers. This lemma has found many applications because it provides, for a pair of non-degenerate continua E and F, an explicit connection between the geometric quantity $d(E, F)/\min\{d(E), d(F)\}$ and the modulus of the family of all curves joining the continua. Also a similar upper bound holds [33, 7.42], [13, Rmk 9.30], but the upper bound will not be needed here.

Lemma 2.11 *Let E and F be continua in $\mathbb{B}^n$ with $d(E), d(F) > 0$. Then*

(1) $\mathsf{M}(\Delta(E, F; \mathbb{R}^n)) \ge \tau_n(4m^2 + 4m)$,
(2) $\mathsf{M}(\Delta(E, F; \mathbb{B}^n)) \ge \frac{1}{2}\tau_n(4m^2 + 4m)$, *where* $m = d(E, F)/\min\{d(E), d(F)\}$.

Proof If $d(E; F) = 0$, then $\mathsf{M}(\Delta(E, F; \mathbb{B}^n)) = \mathsf{M}(\Delta(E, F; \mathbb{R}^n)) = \infty$ by [13, 7.22] (or [33, 5.33]). If $d(E; F) > 0$, the proof of(1) follows from [13, Lemma 9.26] (or [33, 7.38]) and the proof of (2) from (1) and [13, Lemma 7.14] (or [33, 5.22]). □

2.12 Quasiconformal Self-Homeomorphism of a Domain

For a proper subdomain G of $\mathbb{R}^n$ and for a fixed point $x \in G$, we define a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(z) = z$ if $z = x$ or $|x - z| \ge d(x, \partial G)$. Furthermore, for $0 < \alpha < \beta < 1$, the mapping f fulfills $f(S^{n-1}(x, \alpha d(x, \partial G))) = S^{n-1}(x, \beta\, d(x, \partial G))$.

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be the radial map $x \mapsto |x|^{a-1}x = h(x)$, $a \neq 0$, defined in [32, 16.2, p. 49]. Suppose $0 < \alpha, \beta < 1$ and we want to choose a so that $h(S^{n-1}(\alpha)) = S^{n-1}(\beta)$. Then

$$\alpha^a = \beta \quad \Leftrightarrow \quad a = \frac{\log(\beta)}{\log(\alpha)} \tag{2.13}$$

and, as shown in [32, 16.2], the maximal dilatation of this map is $K(h) = \max\{a^{n-1}, a^{1-n}\}$. By definition, $h(z) = z$ for $z \in \{0\} \cup S^{n-1}$. Define now a function g such that $g(z) = h(z)$ for all $z \in \overline{\mathbb{B}}^n$ but $g(z) = z$ whenever $|z| \geq 1$. By [32, Thm 35.1, p. 118], the dilatation of the mapping g is same as the one of h.

Fix now x in G and let $r = d(x, \partial G)$. Let f be a radial K-quasiconformal map defined by $f(z) = rg((z - x)/r)$, $z \in \mathbb{R}^n$. Then $f(x) = x$, $f(z) = z$ for all $z \in G \backslash B^n(x, r)$ and $f(B^n(x, \alpha r)) = B^n(x, \beta r)$, similarly as above.

We summarize the above arguments as a lemma.

Lemma 2.14 *For a proper subdomain G of $\mathbb{R}^n$ and for a fixed point $x \in G$, there exists a quasiconformal homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(G) = G$, and $f(z) = z$ if $z = x$ or $|x - z| \geq d(x, \partial G)$. Furthermore, for* $0 < \alpha < \beta < 1$,

$$f(S^{n-1}(x, \alpha d(x, \partial G))) = S^{n-1}(x, \beta\, d(x, \partial G)), \quad K(f) = \max\{a^{n-1}, a^{1-n}\},$$

where a is the number in (2.13).

3 Harnack Inequality and Capacity Test Functions

In the dimensions $n \geq 3$, quasihyperbolic distances defined below are widely used as substitutes of hyperbolic distances.

3.1 Quasihyperbolic Metric

For a domain $G \subsetneq \mathbb{R}^n$, the *quasihyperbolic metric* k_G is defined by [11, p. 39], [13, p. 68]

$$k_G(x, y) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{|dz|}{d(z, \partial G)}, \quad x, y \in G,$$

where Γ is the family of all rectifiable curves in G joining x and y. This infimum is attained when γ is the *quasihyperbolic geodesic segment* joining x and y. The hyperbolic metric of $\mathbb{B}^n$ can be also defined in terms of a similar length minimizing property, with the weight function $2/(1 - |x|^2)$. In many ways the quasihyperbolic metric is similar to the hyperbolic metric, see [13, Chap. 5], but unfortunately its values are known only in a few special cases. Fortunately, some lower bounds can be given in terms of the j_G metric and upper bounds can be given for a large class of domains as we will now show.

The *distance ratio metric* is defined in a domain $G \subsetneq \mathbb{R}^n$ as the function $j_G : G \times G \to [0, \infty)$,

$$j_G(x, y) = \log(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}}).$$

The lower bound

$$j_G(x, y) \le k_G(x, y)$$

holds for an arbitrary domain $G \subsetneq \mathbb{R}^n$ and all $x, y \in G$ [13, Cor. 5.6, p. 69].

For the upper bound we introduce a class of domains for which we have a simple upper bound of the quasihyperbolic distance. This upper bound combined with the above lower bound provide handy estimates for many applications.

3.2 φ-Uniform Domains

We say that a domain G is φ-*uniform* if for all $x, y \in G$

$$k_G(x, y) \le \varphi(|x - y|/\min\{d(x, \partial G), d(y, \partial G)\}).$$

The special case $\varphi(t) = c\log(1 + t), c > 1$, yields the so called *uniform domains* which are ubiquitous in geometric function theory [13, p. 84], [14]. For instance, balls and half-spaces and their images under quasiconformal mappings of $\mathbb{R}^n$ belong to this class of domains. It is easy to check that all convex domains are φ-uniform with $\varphi(t) \equiv t$. The strip domain $\{z \in \mathbb{C} : 0 < \mathrm{Im} z < 1\}$ is φ-uniform but not uniform.

3.3 Harnack Functions [13, p. 96]

Let $G \subset \mathbb{R}^n$ be a domain and let $u : G \to (0, \infty)$ be a continuous function. We say that u is a *Harnack function* with parameters $(s, C), 0 < s < 1 < C$, if for every $z \in G$ and all $x \in \overline{B}^n(z, sd(z, \partial G))$

$$u(z) \le Cu(x).$$

It follows easily from the definition of the quasihyperbolic metric, see [13, p. 69, Lemma 5.7], that the balls $\overline{B}^n(z, sd(z, \partial G))$ in the definition of a Harnack function have quasihyperbolic diameters majorized by a constant depending on s only. For given $x, y \in G$ one can now estimate for a Harnack function u the quotient $u(x)/u(y)$ using the quasihyperbolic distance $k_G(x, y)$ in a simple way as shown in [13, pp. 94–95]. In fact, we fix a quasihyperbolic geodesic in G joining x and y [13, p. 68, Lemma 5.1], and cover it optimally, using as few balls $\overline{B}^n(z, sd(z, \partial G))$ as possible. In this way, we obtain the next lemma.

Lemma 3.4 [13, Lemma 6.23, p. 96] *Let $u : G \to (0, \infty)$ be a Harnack function with parameters (s, C).*

(1) *Then*

$$u(x) \le C^{1+t}\, u(y), \quad t = k_G(x,y)/d_1(s),$$

for $x, y \in G$ *where* $d_1(s) = 2\log(1+s)$.

(2) *If a domain* $G \subset \mathbb{B}^n, 0 \in G$, *is* φ*-uniform, then for* $d(y, \partial G) < d(0, \partial G)$

$$u(y)/u(0) > C^{-1-\varphi(2/d(y,\partial G))/d_1(s)}.$$

We next start our study of the capacity test function

$$u_\alpha(z) = \mathrm{cap}\,(G, \overline{B}^n(z, \alpha d(z, \partial G))), \quad \alpha \in (0,1). \tag{3.5}$$

and examine its dependence on α and z when the other argument is fixed. It turns out that the dependence of the capacity test function on α is controlled by standard ring domain capacity estimates from 2.12 whereas, as a function of z, it is continuous and satisfies a Harnack condition.

Lemma 3.6 *Let* $G \subset \mathbb{R}^n$, $x \in G$, $0 < \alpha < \beta < 1$, *and* $a = \log(\beta)/\log(\alpha)$. *Then*

$$\mathsf{M}(\Gamma_\alpha) \le \mathsf{M}(\Gamma_\beta) \le a^{n-1}\, \mathsf{M}(\Gamma_\alpha),$$

where $\Gamma_s = \Delta(S^{n-1}(x, sd(x,\partial G)), \partial G; G)$ *for* $s = \alpha, \beta$. *In other words,*

$$u_\alpha(x) \le u_\beta(x) \le a^{n-1} u_\alpha(x).$$

Proof The first inequality follows from Lemma 2.7 and, by using the quasiconformal map f of 2.12, we see that the second inequality holds. □

The above result shows that for a fixed $x \in G$, $u_\alpha(x)$ is continuous with respect to the parameter α because $a \to 1$ when $\beta \to \alpha$. The next result shows, among other things, that for a fixed $\alpha \in (0,1)$, $u_\alpha(x)$ is continuous as a function of x. This continuity follows from the domain monotonicity of the capacity (4.4) and Lemma 3.6.

Theorem 3.7 *Let* G *be a domain in* $\mathbb{R}^n$ *with boundary of positive capacity. Let* $\alpha \in (0,1)$ *and choose* $s \in (0,1)$ *so small that* $(1+s)\alpha + s < 1$. *The capacity test function* u_α *of* G *is continuous on* G *and satisfies the Harnack inequality with parameters* (s, C), *where*

$$C = \left(\frac{\log((1+s)\alpha + s)}{\log \alpha}\right)^{n-1}.$$

Proof For $z \in G$, we take $x \in \overline{B}^n(z, s\, d(z, \partial G))$; namely,

$$|x - z| \le s\, d(z, \partial G). \tag{3.8}$$

It follows from the triangle inequality and the inequality (3.8) that

$$d(x,\partial G) \le |x-z| + d(z,\partial G) \le (1+s)d(z,\partial G)\,, \tag{3.9}$$

$$(1-s)d(z,\partial G) \le d(z,\partial G) - |x-z| \le d(x,\partial G)\,. \tag{3.10}$$

Note here that $\overline{B}^n(a,r_a) \subset \overline{B}^n(b,r_b)$ if and only if $|a-b|+r_a \le r_b$. By using the inequalities (3.8) and (3.9), we have

$$|x-z| + \alpha d(x,\partial G) \le s\, d(z,\partial G) + (1+s)\alpha\, d(z,\partial G) = ((1+s)\alpha + s)\, d(z,\partial G)\,.$$

Similarly, by the inequalities (3.8) and (3.10),

$$|x-z| + \alpha d(z,\partial G)) \le (s+\alpha)d(z,\partial G) \le \frac{s+\alpha}{1-s} d(x,\partial G)\,.$$

Therefore, we have

$$\overline{B}^n(x,\alpha\, d(x,\partial G)) \subset \overline{B}^n(z,((1+s)\alpha+s)d(z,\partial G)),$$
$$\overline{B}^n(z,\alpha\, d(z,\partial G)) \subset \overline{B}^n(x,\beta\, d(x,\partial G)),$$

where $\beta = (\alpha+s)/(1-s)$. In particular, with the help of Lemma 3.6, we obtain

$$u_\alpha(x) \le u_{(1+s)\alpha+s}(z) \le C u_\alpha(z)\,,$$
$$u_\alpha(z) \le u_\beta(x) \le D u_\alpha(x),$$

where $D = [\log\beta/\log\alpha]^{n-1}$. The first inequality is nothing but the required Harnack inequality. Since $C < D$, we obtain

$$|u_\alpha(z) - u_\alpha(x)| \le D = \left(\frac{\log[(\alpha+s)/(1-s)]}{\log\alpha}\right)^{n-1}.$$

Now the continuity of u_α follows because $D \to 1$ as $s \to 0$. □

Let $G \subset \mathbb{R}^n$ be a domain with $\mathrm{cap}\,(\partial G) > 0$, $z_1 \in G$ and fix $\alpha = 1/4$. Theorem 3.7 and Lemma 3.4 show that, perhaps surprisingly, the speed of decrease of the function $u_\alpha(z)/u_\alpha(z_1)$ to 0 when $k_G(z_1,z) \to \infty$ or $z \to \partial G$ is controlled from below by the Harnack parameters given by Theorem 3.7 and by $k_G(z_1,z)$.

4 Capacity Test Function

Various capacities are widely applied in geometric function theory to investigate the metric size of sets [9, 10]. We use here the conformal capacity of condensers and prove several lemmas involving this capacity. We begin by pointing out the connection between the condenser capacity and the modulus of a curve family. These lemmas, together with the superannulus Proposition 2.2, are applied to prove Lemma 4.8, which will be a key tool for the proof of a main result in Sect. 7.

A domain $D \subset \overline{\mathbb{R}}^n$ is called a *ring* if its complement $\overline{\mathbb{R}}^n \setminus D$ has exactly two components C_0 and C_1. Sometimes, we write $D = R(C_0, C_1)$. We say that a ring $D = R(C_0, C_1)$ *separates* a set E, if $E \subset C_0 \cup C_1$ and if E meets both of C_0 and C_1. As in [13, 7.16, p. 120], the (conformal) *modulus* of a ring $D = R(C_0, C_1)$ is defined by

$$\mathrm{mod}(D) = \left(\frac{\mathsf{M}(\Delta(C_0, C_1))}{\omega_{n-1}}\right)^{1/(1-n)}$$

and its capacity is $\mathrm{cap}\,(D) = \mathsf{M}(\Delta(C_0, C_1))$.

Definition 4.1 [13, Def. 9.2, p. 150] A pair $E = (A, C)$ where $A \subset \mathbb{R}^n$ is open and non-empty, and $C \subset A$ is compact and non-empty is called a *condenser*. The *capacity* of this condenser E is

$$\mathrm{cap}(E) = \inf_u \int_{\mathbb{R}^n} |\nabla u|^n dm,$$

where the infimum is taken over the family of all non-negative ACL^n functions u with compact support in A such that $u(x) \geq 1$ for $x \in C$. A compact set E is of *capacity zero*, denoted by $\mathrm{cap}\, E = 0$, if $\mathrm{cap}\,(A, C) = 0$ for some bounded domain A, $C \subset A$. Otherwise we denote $\mathrm{cap}\, E > 0$ and say that E is of positive capacity. Note that the definition of capacity zero does not depend on the open bounded set A [13, pp. 150–153].

For the definition of ACL and ACL^n mappings, see [32, Def. 26.2, p. 88; Def. 26.5 p. 89], [11, 6.4]. It is useful to recall the close connection between the modulus of a curve family and capacity, because many properties of curve families yield similar properties for the capacity.

Remark 4.2 [11, p. 164, Thm 5.2.3], [13, Thm 9.6, p. 152] The capacity of a condenser $E = (A, C)$ can also be expressed in terms of a modulus of a curve family as follows:

$$\mathrm{cap}(E) = \mathsf{M}(\Delta(C, \partial A; A))\,.$$

Lemma 4.3 *Let $E \subset \mathbb{R}^n$ be a compact set, $x \in \mathbb{R}^n$, and let $0 < r < s < t$. Then*

$$A \geq \mathrm{cap}\,(B^n(x,t), E \cap \overline{B}^n(x,r)) \geq \left(\frac{\log(s/r)}{\log(t/r)}\right)^{n-1} A,$$
$$A \equiv \mathrm{cap}\,(B^n(x,s), E \cap \overline{B}^n(x,r)).$$

Proof The proof follows immediately from Remark 4.2 and Lemma 2.14. □

Let $E = (A, C)$ be a condenser and A_1 a domain with $A \subset A_1$. It follows readily from the definition of the capacity (and also from Lemma 2.7) that the following *domain monotonicity property* holds

$$\mathrm{cap}\,(A_1, C) \le \mathrm{cap}\,(A, C). \tag{4.4}$$

For a compact set $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $r > 0$, we introduce the notation

$$\mathrm{cap}(x, E, r) = \mathsf{M}(\Delta(x, E, r)), \quad \Delta(x, E, r) = \Delta(S^{n-1}(x, 2r), E \cap \overline{B}^n(x, r); \mathbb{R}^n). \tag{4.5}$$

The condition $\mathrm{cap}(x, E, r) > 0$ gives information about the size of the set E in a neighborhood of the point x. The next lemma shows that there is a substantial part of the set E, in the sense of capacity, in an annulus centered at x.

Lemma 4.6 *Let $E \subset \mathbb{R}^n$ be a compact set, $x \in \mathbb{R}^n$, and $r > 0$ and suppose that*

$$\mathsf{M}(\Delta(x, E, r)) = c > 0.$$

If $\Delta_1 = \Delta(S^{n-1}(x, 2r), E \cap \overline{B}^n(x, r); B^n(x, 2r))$, then

$$\mathsf{M}(\Delta(x, E, r)) = \mathsf{M}(\Delta_1).$$

Fix $\lambda = \lambda(n, c) = \max\{t, 2\}$, where $t > 1$ satisfies

$$\omega_{n-1}(\log 2t)^{1-n} = c/2,$$

and, for $\sigma > 1$, let

$$\begin{aligned}&\widetilde{\Delta}(x, E, r, \sigma)\\ &\quad = \Delta(S^{n-1}(x, 2r), E \cap (\overline{B}^n(x, r) \setminus B^n(x, r/\sigma)); \overline{B}^n(x, 2r) \setminus B^n(x, r/\sigma)).\end{aligned}$$

Then for every $\sigma \ge \lambda$,

$$\mathsf{M}(\widetilde{\Delta}(x, E, r, \sigma)) \ge c/2.$$

Proof The equality $\mathsf{M}(\Delta(x, E, r)) = \mathsf{M}(\Delta_1)$ is a basic property of the modulus, see [32, p. 33, Thm 11.3]. By the subadditivity of the modulus in Lemma 2.7,

$$\mathsf{M}(\Delta(x, E, r)) = \mathsf{M}(\Delta_1) \le \mathsf{M}(\widetilde{\Delta}(x, E, r, \sigma)) + \mathsf{M}(\Delta_1 \setminus \widetilde{\Delta}(x, E, r, \sigma))$$

and, by the choice of λ and Lemmas 2.7 and 2.8,

$$\mathsf{M}(\Delta_1 \setminus \widetilde{\Delta}(x, E, r, \sigma)) \le c/2$$

for all $\sigma \geq \lambda$. Finally,

$$\mathsf{M}(\widetilde{\Delta}(x, E, r, \sigma)) \geq \mathsf{M}(\Delta_1) - \mathsf{M}(\Delta_1 \setminus \widetilde{\Delta}(x, E, r, \sigma)) \geq c - c/2 = c/2.$$

□

According to Lemma 4.6 we can find, under the above assumptions, a substantial portion of the set E in the annulus $\overline{B}^n(x, r) \setminus B^n(x, r/\lambda)$ for $x \in E$. We need to use this type of annuli for two disjoint sets E and F, which are close enough to each other and then to find a lower bound for the modulus of the curve family joining the respective substantial portions of each set. These annuli are translated versions of each other and we can use Proposition 2.2 to find a common superannulus for both annuli and consider the joining curves in this superannulus. To quantify this idea, we need a comparison principle of the modulus of a curve family from [33, p. 61, Lemma. 5.35], [11, p. 182, Thm 5.5.1].

Lemma 4.7 (1) [33, p. 61, Lemma 5.35] *Let G be a domain in $\overline{\mathbb{R}}^n$, let $F_j \subset G$, $j = 1, 2, 3, 4$, and let $\Gamma_{ij} = \Delta(F_i, F_j; G)$, $1 \leq i, j \leq 4$. Then*

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{\, \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{24}),\ \inf \mathsf{M}\big(\Delta(|\gamma_{13}|, |\gamma_{24}|; G)\big)\, \},$$

where the infimum is taken over all rectifiable curves $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$.

(2) [33, p. 63, 5.41 and 5.42] *If $F_j \subset B^n(z, s) \subset G$, $j = 1, 2, 3, 4$, and and there exists $t > 0$ such that for all $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$*

$$d(|\gamma_{13}|) \geq ts, \quad d(|\gamma_{24}|) \geq ts,$$

then there is a constant $v = v(n, b/a, t)$ such that

$$\mathsf{M}(\Gamma_{12}) \geq v \min\{\, \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{24})\, \}.$$

Proof The first claim (1) is proved in the cited reference. The second claim also follows easily from the cited reference but for clarity we include the details here. We apply the comparison principle of part (1) to get a lower bound for $\mathsf{M}(\Gamma_{12})$. Because $F_j \subset B^n(z, s) \subset G$, $j = 1, 2, 3, 4$, it follows from Lemma 2.10 that the infimum in the lower bound of (1) is at least $v_1(n, t) \equiv \frac{1}{2}\tau_n(4m^2 + 4m)$, $m = 2/t$, and thus

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\{\, \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}), v_1(n, t)\}.$$

By Lemma 2.8

$$\max\Big\{\, \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23})\, \Big\} \leq A \equiv \omega_{n-1}\Big(\log \frac{b}{a}\Big)^{1-n}$$

and further

$$\mathsf{M}(\Gamma_{12}) \geq 3^{-n} \min\Big\{\, \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}),\ \frac{1}{A}\Big(v_1(n, t)\Big) \min\Big\{\, \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23})\, \Big\}\Big\}$$

$$\geq v(n, b/a, t) \min\{ \mathsf{M}(\Gamma_{13}),\ \mathsf{M}(\Gamma_{23}) \}$$

where $v(n, b/a, t) = 3^{-n} \min\{ 1,\ \frac{1}{A} v_1(n, t) \}$. □

The next lemma will be a key tool in Sect. 7. It is based on three earlier results:

(a) the superannulus Proposition 2.2,
(b) the substantial subset selection Lemma 4.6,
(c) the comparison principle for the modulus of a curve family, Lemma 4.7.

Recall the notation $R(x, r', r)$ for $0 < r < r' < +\infty$ and $x \in \mathbb{R}^n$ from (2.1).

Lemma 4.8 *Let $n \geq 2$, $c > 0$ and $\lambda = \lambda(n, c) \geq 2$ be as in Lemma 4.6. Let $E, F \subset \mathbb{R}^n$ be compact sets, $x, y \in \mathbb{R}^n$, and $r > 0$ and suppose that*

$$\mathsf{M}(\Delta(x, E, r)) = c > 0, \quad \mathsf{M}(\Delta(y, F, r)) = c > 0.$$

If $|x - y| < r/\lambda^2$, then there exist constants $\tau = \tau(n, c) \geq 2$, $d = d(n, c) > 0$ such that with $w = (x + y)/2$

$$\mathsf{M}(\Delta(E, F; R(w, \tau^2 r, r/\tau^2))) \geq d.$$

Proof By Proposition 2.2, we see that

$$R(x, r, r/\lambda) \cup R(y, r, r/\lambda) \subset R(w, \lambda^2 r, r/\lambda^2)$$

and that $R(w, \lambda^2 r, r/\lambda^2)$ is a common superannulus for the two smaller annuli. We apply the comparison principle for the modulus, Lemma 4.7(2), with

$$F_1 = E \cap R(x, r, r/\lambda), \quad F_2 = F \cap R(y, r, r/\lambda),$$
$$F_3 = S^{n-1}(x, 2r), \quad F_4 = S^{n-1}(y, 2r).$$

Observe first that, in the notation of Lemma 4.7, for all $\gamma_{13} \in \Gamma_{13}$ and $\gamma_{24} \in \Gamma_{24}$

$$d(|\gamma_{13}|) \geq r, \quad d(|\gamma_{24}|) \geq r$$

and that, by Proposition 2.2, $F_j \subset R(w, 2\lambda^2 r, r/\lambda^2)$ for $j = 1, 2, 3, 4$. By Lemma 4.7(2), we have

$$\mathsf{M}(\Delta(F_1, F_2; \mathbb{R}^n)) \geq d_2(n, c) > 0.$$

Using Lemma 4.6 we obtain the numbers $d = d_2(n, c)/2$ and $\tau = \sqrt{2}\lambda$. □

5 Hausdorff Content and Lower Estimate of Capacity

In this section, we discuss lower bounds for the capacity in terms of the Hausdorff h-content. Our main references are O. Martio [20] and Yu.G. Reshetnyak [26], [27, pp. 110–120]. Reshetnyak also cites an earlier lemma of H. Cartan 1928 and gives its proof based on the work of L.V. Ahlfors [1] (cf. R. Nevanlinna [23, p. 141]). We give here a short review of the earlier relevant results and, for the reader's benefit, outline sketchy proofs.

We start with the next covering lemma [18, p. 197]. In the following, we denote by χ_E the characteristic function of a set $E \subset \mathbb{R}^n$; that is, $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise.

Lemma 5.1 *Let n be an integer with $n \geq 2$ and let A be a set in $\mathbb{R}^n$. Suppose that a radius $r(x) > 0$ is assigned for each point $x \in A$ in such a way that $\sup_{x \in A} r(x) < +\infty$. Then one can find a countable subset $\{x_k\}$ of A such that*

$$\chi_A(x) \leq \sum_k \chi_{B_k}(x) \leq N_n, \quad x \in \mathbb{R}^n, \tag{5.2}$$

where $B_k = B^n(x_k, r(x_k))$ and N_n is a constant depending only on n.

The above inequalities mean that A is covered by the family of balls $\{B_k\}$ and the number of overlapping of the covering is at most N_n. It is an interesting problem to find the best possible number N_n^* for the constant N_n in the above lemma. For $\varphi \in (0, \pi]$, we will say that a subset V of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is φ-separated if the angle subtended by the two line segments PO and QO is at least φ for distinct points P, Q in V. We denote by $\nu(n, \varphi)$ the maximal cardinal number of φ-separated subsets V of S^{n-1}. For instance, $\nu(n, \pi) = 2$. It is clear that $\nu(n, \varphi) \leq \nu(n, \varphi')$ for $0 < \varphi' < \varphi$. The proof of the above lemma in p. 199 of [18] tells us that

$$N_n^* \leq 1 + \inf_{\varphi < \pi/3} \nu(n, \varphi).$$

On the other hand, a standard compactness argument leads to the left continuity of $\nu(n, \varphi)$; that is, $\nu(n, \varphi') \to \nu(n, \varphi)$ as $\varphi' \to \varphi^-$. Hence, we have $N_n^* \leq 1 + \nu(n, \pi/3)$. Here we note that the number $\nu(n, \pi/3)$ is known as the *kissing number* $\kappa(n)$ in dimension n [8]. This number is closely related to other important issues such as sphere packing problems. For instance, it is known that $\kappa(2) = 6, \kappa(3) = 12, \kappa(4) = 24$. However, it is difficult to determine $\kappa(n)$ in general. The true value of $\kappa(5)$ is not determined up to the present. By using the special nature of the lattices E_8 and Λ_{24}, Viazovska determined $\kappa(8)$ and, later with her collaborators, $\kappa(24)$ and won a Fields medal in 2022 [24]. In summary, we can state the following.

Lemma 5.3 *The minimal number N_n^* of the bound N_n in Lemma 5.1 satisfies the inequality $N_n^* \leq \kappa(n) + 1$, where $\kappa(n)$ is the kissing number in dimension n.*

Let $h(r)$ be a measure function, that is, a monotone increasing continuous function on $0 < r < +\infty$ with $h(r) \to 0$ as $r \to 0$ and $h(r) \to +\infty$ as $r \to +\infty$. The

h-Hausdorff content of a set $E \subset \mathbb{R}^n$ is defined as

$$\Lambda_h(E) = \inf \left\{ \sum_k h(r_k) : E \subset \bigcup_k B^n(a_k, r_k) \right\}.$$

For a positive number $\beta > 0$, and for $h(r) = r^\beta$, the h-Hausdorff content is called the β-dimensional Hausdorff content and denoted by $\Lambda^\beta(E)$. Recall that the Hausdorff dimension dim E of E is characterized as the infimum of $\beta > 0$ with $\Lambda^\beta(E) = 0$.

The next lemma constitutes a key step in the proof of Martio's Theorem 5.7 below. In the proof below we give explicit estimates of the relevant constants.

Remark 5.4 The next lemma has a long history which goes back to H. Cartan and L.V. Ahlfors [1], [23, p. 141]. Yu.G. Reshetnyak [26], [27, Lemma 3.7, p.115], extended their two dimensional work to the case of $\mathbb{R}^n$ and applied the result to prove a lower bound for the capacity in terms of the Hausdorff content. O. Martio, in turn, made use of these results in his paper [20] which is one of our key references.

Lemma 5.5 (Lemma 2.6 in [20]) *Let σ be a positive finite measure on $\mathbb{R}^n$ and h be a measure function. We denote by T the set of those points $x \in \mathbb{R}^n$ for which the inequality $\sigma(B^n(x, r)) \le h(r)$ holds for all $r > 0$. Then $\Lambda_h(\mathbb{R}^n \backslash T) \le N_n^* \sigma(\mathbb{R}^n)$, where N_n^* is given in Lemma 5.3.*

Proof We choose $r_0 > 0$ so that $h(r_0) = \sigma(\mathbb{R}^n)$ and let $A = \mathbb{R}^n \setminus T$. For each $x \in A$, by definition, there is a positive $r(x)$ such that $\sigma(B^n(x, r(x))) > h(r(x))$. Note that $r(x) \le r_0$. We now apply Lemma 5.1 to extract a countable set $\{x_k\}$ from A so that $B_k = B^n(x_k, r(x_k))$ satisfy (5.2). Then

$$\begin{aligned}\Lambda_h(A) &\le \sum_k h(r(x_k)) < \sum_k \sigma(B_k) \\ &= \sum_k \int \chi_{B_k} d\sigma = \int \sum_k \chi_{B_k} d\sigma \le N_n^* \sigma(\mathbb{R}^n).\end{aligned}$$ □

By making use of the preceding lemma, Martio proved the following result. For the proof, see Lemma 2.8 in [20].

Lemma 5.6 *Let $1 < p \le n$ and $\alpha > 0$. For a function $u \ge 0$ in $L^p(\mathbb{R}^n)$ with support in $B^n(0, r_1)$, the set F of points $x \in \mathbb{R}^n$ satisfying the inequality*

$$\begin{aligned} v(x) \ := \int_{\mathbb{R}^n} u(y)|x-y|^{1-n} dm(y) > \Omega_n^{1-1/p} \left(\frac{n-1}{\alpha} \int_0^{r_1} h(t)^{1/p} t^{-n/p} dt \right. \\ \left. + r_1^{1-n/p} \|u\|_p \right) \end{aligned}$$

admits the estimate $\Lambda_h(F) \le N_n(\alpha \|u\|_p)^p$, where N_n is the number in Lemma 5.1.

Let

$$K_n = 2^{-n} \omega_{n-1}^n \Omega_n^{1-n} = \frac{1}{\Gamma(\frac{n}{2}+1)} \left(\frac{n\sqrt{\pi}}{2} \right)^n.$$

The following theorem is a special case of Theorem 3.1 in [20] when $p = n$. Since an explicit form of the constant M_1 is not given in [20], we give an outline of the proof with a concrete form of M_1.

Theorem 5.7 *Suppose that a measure function $h(r)$ satisfies the inequality*

$$I(r) = \int_0^{2r} h(t)^{1/n} t^{-1} dt \le A h(r)^{1/n}, \quad 0 < r \le r_0,$$

for some constants $A > 0$ and $r_0 > 0$. Let E be a closed set in $\mathbb{R}^n$. Then

$$\frac{\Lambda_h(E \cap \overline{B}^n(x, r))}{h(r)} \le M_1 \cdot \mathrm{cap}\,(x, E, r), \quad x \in \mathbb{R}^n,\ 0 < r \le r_0, \tag{5.8}$$

where M_1 is the positive constant given by

$$M_1 = \max \left\{ 2^n A^n \left(\frac{n-1}{n} \right)^n N_n,\ 1/K_n \right\}. \tag{5.9}$$

Proof We may assume $x = 0$ and write $B_r = B^n(0, r)$ for short. Because $E \cap B_r \subset B_{r'}, r < r'$, by the definition of the Hausdorff content it is clear that $\Lambda_h(E \cap B_r) \le h(r)$. Since $M_1 \ge 1/K_n$, the required inequality holds trivially when $\mathrm{cap}\,(0, E, r) \ge K_n$. Thus we may assume that $\mathrm{cap}\,(0, E, r) < K_n$. By the definition of capacity, for each $r > 0$ and a small enough $\varepsilon > 0$, we may choose a smooth function $w \ge 0$ on $\mathbb{R}^n$ with support in B_{2r} so that $w > 1$ on $E \cap B_r$ and so that

$$\|\nabla w\|_n^n = \int_{\mathbb{R}^n} |\nabla w|^n dm < \mathrm{cap}\,(B_{2r}, E \cap B_r) + \varepsilon = \mathrm{cap}\,(0, E, r) + \varepsilon < K_n,$$

where dm denotes the Lebesgue measure.

We apply Lemma 5.6 with $r_1 = 2r$ and $p = n$ to the function $u = |\nabla w|/\omega_{n-1}$ and by the above inequality we may choose α so that

$$\Omega_n^{1-1/n} \left(\frac{n-1}{\alpha} I(r) + \frac{\|\nabla w\|_n}{\omega_{n-1}} \right) = 1.$$

Since $\|\nabla w\|_n < K_n^{1/n}$, we have the inequality $\|\nabla w\|_n/\omega_{n-1} < \Omega_n^{1/n-1}/2$, which enables us to estimate α as

$$\alpha = \frac{(n-1)I(r)}{\Omega_n^{1/n-1} - \|u\|_n} < \frac{2(n-1)I(r)}{\Omega_n^{1/n-1}} \le 2A(n-1)\Omega_n^{1-1/n} h(r)^{1/n}$$

for $0 < r \le r_0$. By the representation formula (see [20, (2.2)])

$$w(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\nabla w(y) \cdot (x - y)}{|x - y|^n} dm(y),$$

we have the inequality

$$w(x) \le \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla w(y)||x-y|}{|x-y|^n} dm(y) = \int_{\mathbb{R}^n} u(y)|x-y|^{1-n} dm(y) = v(x).$$

Let F be as in Lemma 5.6. In particular, we have $E \cap B_r \subset F$ and thus by Lemma 5.6

$$\Lambda_h(E \cap B_r) \le \Lambda_h(F) \le N_n(\alpha \|u\|_n)^n \le N_n \, d_n \, h(r)(\mathrm{cap}\,(0, E, r) + \varepsilon),$$

where

$$d_n = \frac{2^n A^n (n-1)^n}{n\Omega_n} \left(\frac{\Omega_n}{\omega_{n-1}}\right)^{n-1} = 2^n A^n (\frac{n-1}{n})^n.$$

Letting $\varepsilon \to 0$, we obtain finally

$$\Lambda_h(E \cap B_r) \le 2^n A^n \left(\frac{n-1}{n}\right)^n N_n \, h(r) \, \mathrm{cap}\,(0, E, r).$$

Now, in conjunction with Lemma 5.3, the required inequality follows. □

When $h(r) = r^\beta$ for some $0 < \beta \le n$, we obtain

$$0 \int_0^{2r} h(t)^{1/n} t^{-1} dt = \int_0^{2r} t^{\beta/n - 1} dt = \frac{n}{\beta}(2r)^{\beta/n} = A\, h(r)^{1/n}, \quad A = \frac{n}{\beta} 2^{\beta/n}.$$

Hence, in this case, the constant M_1 in (5.9) may be expressed by

$$M_1(n, \beta) = \max\left\{2^{\beta+n}\left(\frac{n-1}{\beta}\right)^n N_n^*, \; 1/K_n\right\}. \tag{5.10}$$

For instance, when $n = 2$, we have $N_2^* = 7$ and thus $M_1(2, \beta) = 28(2^\beta/\beta^2)$.

6 Uniform Perfectness and Capacity

In this section, we study the connection between potential theoretic thickness of sets, as expressed in terms of capacity, and uniform perfectness. The notion of uniform perfectness was first used by Beardon and Pommerenke [5] in two dimensions. Later, Pommerenke [25] found a characterization of uniform perfectness in terms of the logarithmic capacity. One of the main results of Järvi and Vuorinen [15] was a characterization of uniform perfectness for the general dimension in terms of the quantity $\mathrm{cap}\,(x, E, r)$ defined in Sect. 5. The novel feature in our work is to give an explicit form for $\mathrm{cap}\,(x, E, r)$ in terms of the dimension n and the parameter c of $\mathrm{UP}_n(c)$ sets. Then we will prove Lemma 1.1 and Theorem 1.2 given in Introduction.

Definition 6.1 For $c \in (0,1)$, let $\mathrm{UP}_n(c)$ denote the collection of compact sets E in $\mathbb{R}^n$ with card $(E) \geq 2$ satisfying the condition

$$\{x \in E : cr < |x-a| < r\} \neq \emptyset \quad \text{for all } a \in E \quad \text{and} \quad 0 < r < d(E)/2.$$

A set is called *uniformly perfect* if it is of class $\mathrm{UP}_n(c)$ for some c.

Lemma 6.2 (Proposition 7.4 in [30]) *Let I be the index set $\{1, 2, \ldots, p\}$ and $0 < c < 1$ and $r > 0$. Suppose that a sequence of families of closed balls*

$$\overline{B}_{i_1 i_2 \ldots i_k} = \overline{B}^n(a_{i_1 i_2 \ldots i_k}, c^k r), \quad i_1, i_2, \ldots, i_k \in I,$$

for $k = 1, 2, 3, \ldots$, satisfies the following two conditions:

(i) $\overline{B}_{i_1 i_2 \ldots i_{k-1} i} \cap \overline{B}_{i_1 i_2 \ldots i_{k-1} j} = \emptyset$ *for $i, j \in I$ with $i \neq j$ and $i_1, \ldots, i_{k-1} \in I$.*
(ii) $\overline{B}_{i_1 i_2 \ldots i_k} \subset \overline{B}_{i_1 i_2 \ldots i_{k-1}}$ *for $i_1, \ldots, i_k \in I$.*

Then the set

$$K = \bigcap_{k=1}^{\infty} \bigcup_{(i_1, \ldots, i_k) \in I^k} \overline{B}_{i_1 i_2 \ldots i_k}$$

satisfies the inequality $\Lambda^\beta(K) \geq r^\beta / (p \cdot 3^n)$, where $\beta = -(\log p)/\log c$, and its Hausdorff dimension is β.

This lemma has the following important corollary.

Corollary 6.3 *Let $E \in \mathrm{UP}_n(c)$ for some $0 < c < 1$. Then the Hausdorff dimension $\dim_H(E)$ of E is at least $(\log 2)/\log(3/c)$, which is independent of the dimension n.*

Remark 6.4 Recall that the Cantor middle-third set K has Hausdorff dimension $(\log 2)/\log 3 \approx 0.63093$ [22, p. 60]. One can check that $K \in \mathrm{UP}_2(2/5)$ and the number $2/5$ cannot be increased. The above corollary thus implies that $\dim_H(K) \geq (\log 2)/\log(15/2) \approx 0.34401$.

Theorem 6.5 *Let $E \in \mathrm{UP}_n(c)$ for some $0 < c < 1$. Then for every $a \in E$, $a \neq \infty$,*

$$\Lambda^\beta(E \cap \overline{B}^n(a, r)) \geq \frac{r^\beta}{2 \cdot 3^n}, \quad 0 < r < d(E)/2, \ \text{where } \beta = \frac{\log 2}{\log(3/c)}. \tag{6.6}$$

$$\mathrm{cap}\,(x, E, r)) \geq \frac{1}{2 \cdot 3^n M_1(n, \beta)}, \quad 0 < r < d(E)/2, \ \text{where } \beta = \frac{\log 2}{\log(3/c)} \tag{6.7}$$

and where M_1 is as in (5.10).

Proof Set $\overline{B} = \overline{B}^n(a, r)$. Let $a_1 = a$ and take a point, say a_2, from the set $\{x \in E : 2cr/3 < |x - a| < 2r/3\}$, which is non-empty by assumption. Then we define $\overline{B}_i = \overline{B}^n(a_i, r_1)$ for $i = 1, 2$ and $r_1 = cr/3$. Since $|a_1 - a_2| > 2cr/3 = 2r_1$, we have $\overline{B}_1 \cap \overline{B}_2 = \emptyset$. Also, by $|a - a_2| + r_1 < 2r/3 + cr/3 < r$, we confirm that $\overline{B}_2 \subset \overline{B}$.

Next we let $a_{i1} = a_i$ for $i = 1, 2$ and choose a_{i2} from the set $\{x \in E : 2cr_1/3 < |x - a_i| < 2r_1/3\}$. Then we define $\overline{B}_{i_1 i_2} = \overline{B}^n(a_{i_1 i_2}, r_2)$ for $i_1, i_2 \in I = \{1, 2\}$ and $r_2 = cr_1/3 = (c/3)^2 r$. We can proceed inductively to define families of disjoint closed balls $\overline{B}_{i_1 i_2 \ldots i_k} = \overline{B}^n(a_{i_1 i_2 \ldots i_k}, r_k)$ $(i_1, i_2, \ldots, i_k \in I)$ for $k = 1, 2, 3, \ldots,$ and $r_k = (c/3)^k r$, in such a way that $\overline{B}_{i_1 i_2 \ldots i_k} \subset \overline{B}_{i_1 i_2 \ldots i_{k-1}}$. We finally set

$$K = \bigcap_{k=1}^{\infty} \bigcup_{(i_1, \ldots, i_k) \in I^k} \overline{B}_{i_1 i_2 \ldots i_k}.$$

Then K is a Cantor set and satisfies the inequality $\Lambda^\beta(K) \geq r^\beta/(2 \cdot 3^n)$ by Lemma 6.2. Since $K \subset E \cap \overline{B}^n(a, r)$ by construction, the proof of (6.6) is complete.

The proof of (6.7) follows from Theorem 5.8 and (6.6). □

6.8 Proofs of Lemma 1.1 and Theorem 1.2.

Theorem 6.5 also proves Lemma 1.1 and Theorem 1.2. □

Next we estimate the parameter c in terms of a cap (x, E, r) lower bound.

Lemma 6.9 *Suppose that a closed set E in $\overline{\mathbb{R}}^n$ satisfies* cap $(x, E, r) \geq \sigma$ *for $x \in E$ and $0 < r \leq d(E)$ for a constant $\sigma > 0$. Then $E \in \mathrm{UP}_n(c)$ for any $c > 0$ with $c \leq 2\exp\{-(\omega_{n-1}/\sigma)^{1/(n-1)}\}$.*

Proof Suppose that $E \cap R(a, \alpha r, r) = \emptyset$ for some $0 < r < d(E)/2$ and $a \in E$. Then $E \cap \overline{B}^n(a, r) \subset \overline{B}^n(a, \alpha r)$. By Lemma 2.8, we now obtain

$$\sigma \leq \mathrm{cap}\,(B^n(a, 2r), E \cap \overline{B}^n(a, r)) = \mathrm{cap}\,(B^n(a, 2r), \overline{B}^n(a, \alpha r)) = \omega_{n-1}\left(\log\frac{2}{\alpha}\right)^{1-n},$$

which implies $\alpha \geq 2\exp\{-(\omega_{n-1}/\sigma)^{1/(n-1)}\}$.

6.10 Equivalent Characterization of Uniform Perfectness

According to [15] a closed set $E \subset \mathbb{R}^n$ with card$E \geq 2$ is α- uniformly perfect, $\alpha > 0$, if there is no ring domain D separating E with mod$D > \alpha$. The following lemma shows that this notion is quantitatively equivalent to our notion of $\mathrm{UP}_n(c)$, with explicit constants.

Lemma 6.11 *Let E be a closed set in $\mathbb{R}^n$ containing at least two points.*

(1) *If E is α-uniformly perfect, then $E \in \mathrm{UP}_n(e^{-\alpha})$.*
(2) *If $E \in \mathrm{UP}_n(c)$, then E is α-uniformly perfect for $\alpha < A_n + \log(3/c)$, where A_n is a positive constant depending only on n.*

Proof This result is contained in the proof of Theorem 3.3 in [12]. □

According to [12], the constant A_n admits the following majorant for $n \geq 2$

$$A_n \leq 2\log\frac{(1+\sqrt{2})\lambda_n}{2}; \quad 4 \leq \lambda_n \leq 2^{n/(n-1)}e^{n(n-2)/(n-1)}.$$

The constant λ_n is the so called Grötzsch constant [11–13]. Observe that we take r in $0 < r < \mathrm{d}(E)/2$ in the above definition 6.1 of $\mathrm{UP}_n(c)$, whereas in [12] it is required that $0 < r < \mathrm{d}(E)$.

7 Proof of Theorems 1.5 and 1.3

In this section our goal is to prove one of the main results of this paper, Theorem 1.5, which gives a lower bound for

$$\mathrm{cap}(G, E)$$

when $G \subset \mathbb{R}^n$ is a domain and $E \subset G$ is a compact set and both E and ∂G are uniformly perfect. The proof is based on the results given in earlier sections and it is divided into three cases: (a) $d(E)/d(E, \partial G)$ is small (Lemma 7.3), (b) $d(E)/d(E, \partial G)$ is large (Lemma 7.4), (c) neither (a) nor (b) holds. These three cases form the logical structure of the proof of Theorem 7.5 which immediately yields the proof of Theorem 1.5. For the case (b) we apply Proposition 2.4 and Lemma 4.8 to construct a sequence of separate annuli with the parameter λ adjusted so that each annulus contains a substantial portion of both E and ∂G.

Then using Theorem 1.5 we also prove Theorem 1.3.

In Lemma 4.6 we proved that, for a compact set $E \subset \mathbb{R}^n$ of positive capacity, the condition $\mathrm{cap}(x, E, r) = c > 0$ implies the existence of $\lambda = \lambda(n, c)$ such that the set

$$E \cap (\overline{B}^n(x, r) \setminus B^n(x, r/\lambda))$$

is quite substantial. Now for a uniformly perfect set E and all $x \in E$, we see by Theorem 6.5 that for $0 < r < d(E)$ the sets

$$E \cap (\overline{B}^n(x, r_k) \setminus B^n(x, r_{k+1}))$$

are substantial for all $k = 1, 2, \ldots,$ where $r_1 = r, r_{k+1} = r_k/\lambda$. Observe that these sets are subsets of separate annuli centered at x.

Our first result in this section, Lemma 7.3, yields a lower bound for the modulus of the family of all curves joining for a pair of compact sets F_1 and F_2, in terms of the respective capacities, the dimension n and a set separation parameter $t \in (0, 1/2)$. This result is a counterpart of Lemma 2.11 which gives a similar lower bound for a pair of continua E and F. The parameter t now plays the role of $d(E, F)/\min\{d(E), d(F)\}$.

Proposition 7.1 *Let $K > 1, n \geq 2$, and*

$$g(x) = \log(1+x)\left(\log\frac{K}{x}\right)^{n-1}, \quad 0 < x \leq 1.$$

Then $g(x) \leq K((n-1)/e)^{n-1}$ for all $0 < x \leq 1$. In particular, for $K > 1 > x > 0$

$$\left(\log\frac{K}{x}\right)^{1-n} \geq \frac{1}{K}\left(\frac{e}{n-1}\right)^{n-1}\log(1+x). \tag{7.2}$$

Proof Let $h : [0,\infty) \to [0,\infty)$ be defined by $h(u) = e^{-u}u^{n-1}$. Then $h(0) = 0, h(\infty) = 0$, and h has its only maximal value at $u = n-1$, equal to $h(n-1) = ((n-1)/e)^{n-1}$. Setting $u = \log(K/x)$ and applying $\log(1+t) \leq t, t \geq 0$, yields

$$g(Ke^{-u}) = u^{n-1}\log(1+Ke^{-u}) \leq Kh(u) \leq Kh(n-1).$$

For $K > 1$ let

$$s(n,K) = \inf_{0<x<1}\frac{(\log(K/x))^{1-n}}{\log(1+x)}.$$

The above upper bound for the function h shows that for all $x \in (0,1]$

$$(\log(K/x))^{1-n} \geq s(n,K)\log(1+x) \geq \frac{1}{K}(e/(n-1))^{n-1}\log(1+x)$$

which completes the proof. □

Lemma 7.3 *Let $t \in (0,1/2]$ and let $F_j \subset \overline{B}^n((j-1)e_1, t)$ be compact sets with*

$$\delta_j = \mathrm{cap}((j-1)e_1, F_j, t) > 0, \quad j = 1,2.$$

Then there exists a constant $\mu_n > 0$ depending only on n, such that

$$\mathsf{M}(\Delta(F_1, F_2; \mathbb{R}^n)) \geq \mu_n \min\{\delta_1, \delta_2\}\log(1+t).$$

Proof Let $G = B^n(\frac{1}{2}e_1, \frac{3}{2})$, let $F_3 = \partial G$, and let $\Delta_{j3} = \Delta(F_j, F_3; G)$, $j = 1,2$. Then $G \subset B^n(e_1, 2)$, $G \subset B^n(0,2)$ and Lemma 4.3 with the triple of radii $\{t, 2t, 2\}$ yields

$$\mathsf{M}(\Delta_{j3}) = \mathrm{cap}(G, F_j) \geq u(t)\delta_j, \quad u(t) \equiv \left(\frac{\log 2}{\log(2/t)}\right)^{n-1}, j = 1,2.$$

By the choice of G for all $\gamma_j \in \Delta_{j3}$ we have

$$d(|\gamma_j|) \geq 1 - t \geq 1/2, \quad j = 1,2,$$

and by Lemma 2.10

$$\mathsf{M}(\Delta(|\gamma_1|, |\gamma_2|; G)) \geq \frac{1}{2}\tau_n(4m^2 + 4m)$$

where $m = 2/(1-t) \leq 4$ and $4m^2 + 4m \leq 80$. Because $\mathsf{M}(\Delta_{j3}) \leq A \equiv \omega_{n-1}(\log 2)^{1-n}$, we see that $\tau_n(80)/2 \geq (\tau_n(80)/(2A)) \min\{\delta_1, \delta_2\}$ and hence by Lemma 4.7, [33, 5.35]

$$\begin{aligned}\mathsf{M}(\Delta(F_1, F_2; \mathbb{R}^n)) &\geq 3^{-n} \min\{\mathsf{M}(\Delta_{13}), \mathsf{M}(\Delta_{23}), \tau_n(80)/2\} \\ &\geq 3^{-n} \min\left\{1, \frac{\tau_n(80)}{2A}\right\} u(t) \min\{\delta_1, \delta_2\}.\end{aligned}$$

Finally, we estimate $u(t)$ using (7.2)

$$u(t) \geq (\log 2)^{n-1} \frac{1}{2} \left(\frac{e}{n-1}\right)^{n-1} \log(1+t).$$

In conclusion, we can choose

$$\mu_n = 3^{-n} \min\left\{1, \frac{\tau_n(80)}{2A}\right\} (\log 2)^{n-1} \frac{1}{2} \left(\frac{e}{n-1}\right)^{n-1}.$$

□

Lemma 7.4 *Let $G \subset \mathbb{R}^n$ be a domain and let $E \subset G$ be a compact set, and suppose that for all $r \in (0, \min\{d(\partial G), d(E)\})$ and for all $z_1 \in E$, $z_2 \in \partial G$*

$$\mathrm{cap}\,(z_1, E, r) \geq \delta, \quad \mathrm{cap}\,(z_2, \partial G, r) \geq \delta > 0\,.$$

Fix $x, y \in E$, $z_0 \in \partial G$ such that $|x - z_0| = d(E, \partial G)$, $|x - y| \geq d(E)/2$, let $\lambda = \lambda(\delta, n) \geq 4$ be the number given by Lemma 4.8, and denoted there as τ^2. If $d(E)/d(E, \partial G) > \lambda^{2p} + 2$ for some integer $p \geq 1$, then

$$\mathsf{M}(\Delta(E, \partial G; G)) \geq d(n, 2)p\,\delta/4$$

where the constant d is as in Lemma 4.8.

Proof Let $w = (x + z_0)/2$ and consider the separate curve families in the annuli of Proposition 2.4 and apply Lemma 4.8. □

Theorem 7.5 *Let G, E and $\delta > 0$ be as in Lemma 7.4. Then there exists a constant $s > 0$ depending only on n and δ such that*

$$\mathrm{cap}\,(G, E) \geq s\, \log(1 + d(E)/d(E, \partial G)).$$

Proof Let $\lambda = \lambda(n, \delta) \geq 4$ be as in Lemma 4.8 denoted by τ^2.

Case A: In the case $d(E)/d(E, \partial G) \leq 1/2$, the proof follows from Lemma 7.3 with a constant $c_A = \mu_n \delta$. Indeed, fix $a \in E, b \in \partial G$ with $|a - b| = d(E, \partial G)$ and with $t = d(E)/d(E, \partial G)$ and $F_1 = E \cap B^n(a, d(E))$, $F_2 = \partial G \cap B^n(b, d(E))$ apply Lemma 7.3. Observe that $d(\partial G) > d(E)$ and the class of $\mathrm{UP}_n(c)$ is invariant under similarity transformations.

Case B: Consider next the case $t \equiv d(E)/d(E, \partial G) \geq t_0$ where $t_0 = t_0(n, \delta) > 1$ is defined as the number with $\log(1+t_0) = \log(\lambda^6+2)$. Then $\log(1+t) \geq \log(\lambda^{2p}+2)$ for some integer $p \geq 3$. Fix an integer $p \geq 3$ with

$$\log(\lambda^{2(p+1)} + 2) > \log(1 + t) \geq \log(\lambda^{2p} + 2).$$

Then by Proposition 2.3

$$2(p + 1) \log \lambda \geq \log(t - 1) \geq \frac{1}{2} \log(1 + t) + 2 \log \lambda,$$

for all $t \geq \lambda^4 + 2\lambda^2 + 1$. Because $\lambda > 3$ we see that $t_0 = \lambda^6 + 1 \geq 3\lambda^5 \geq \lambda^4 + 2\lambda^2 + 1$ and hence for all $t \geq t_0$

$$p \geq \frac{1}{4 \log \lambda} \log(1 + t)$$

and it follows from Lemma 7.4 that

$$\mathrm{cap}\,(G, E) \geq d(n, 2)p\delta/4 \geq (d(n, 2)\,\delta/4)\, \frac{1}{4 \log \lambda} \log(1 + t) = c_B \log(1 + t).$$

Case C: Finally, we consider the case $d(E)/d(E, \partial G) \in (1/2, t_0)$ where $t_0 = t_0(n, \delta)$ is as in Case B. Then by Case B, $t_0 \geq 3$ and

$$d(E, \partial G)/2 \leq d(E) \leq t_0 d(E, \partial G) \equiv T. \tag{7.6}$$

Fix $x_0 \in E$ and $z \in \partial G$ with $|x_0 - z| = d(E, \partial G)$ and observe that $E \cup B^n(z, 2d(E, \partial G)) \subset B^n(x_0, T)$. For the application of the comparison principle of Lemma 4.7 employ the following notation

$$F_1 = E, \quad F_2 = \overline{B}^n(z, d(E, \partial G)) \cap \partial G, \quad F_3 = S^{n-1}(x_0, 2T).$$

Let

$$\Gamma_{13} = \Delta(F_1, F_3; \mathbb{R}^n), \quad \Gamma_{23} = \Delta(F_2, F_3; \mathbb{R}^n).$$

Observe first that $F_1 \cup F_2 \subset \overline{B}^n(x_0, T)$ and $F_3 \cap B^n(x_0, 2T) = \emptyset$. Then by [13, Remark 7.8, Lemma 7.1 (2)] and Lemma 4.7(2)

$$\mathsf{M}(\Delta(E, \partial G; G)) = \mathsf{M}(\Delta(E, \partial G; \mathbb{R}^n)) \geq \mathsf{M}(\Delta(F_1, F_2; \mathbb{R}^n))$$

$$\geq d(n,2)\min\{\mathsf{M}(\Gamma_{13}),\mathsf{M}(\Gamma_{23})\}.$$

By Lemma 4.3 and (7.2)

$$\begin{aligned} M(\Gamma_{13}) &\geq \delta\left(\frac{\log 2}{\log(2T/d(E))}\right)^{n-1} \geq d_3\log(1+d(E)/(2T)) \\ &\geq \frac{d_3}{2t_0}\log(1+d(E)/d(E,\partial G)) \end{aligned}$$

where (7.6) and Bernoulli's inequality were applied. In the same way we have

$$\mathsf{M}(\Gamma_{23}) \geq \delta\left(\frac{\log 2}{\log(2T/d(E))}\right)^{n-1} \geq d_3\log(1+d(E)/(2T)).$$

Because $T/d(E,\partial G) = t_0$ by Lemma 4.3 and (7.2) we obtain

$$\begin{aligned} M(\Gamma_{23}) &\geq c((\log 2)/\log((2T+d(E,\partial G))/d(E,\partial G)))^{n-1} \\ &\geq d_4\log(1+\frac{d(E)/t_0}{(2t_0+1)d(E,\partial G)}) \geq c_C\log(1+d(E)/d(E,\partial G)). \end{aligned}$$

With the constant $\min\{c_A,c_B,c_C\}$ we have proved the theorem. □

7.7 Proof of Theorem 1.5.

The proof follows from Theorem 7.5. □

7.8 Proof of Theorem 1.3.

Suppose first that ∂G is uniformly perfect and let $z\in G$. Then

$$d(E)/d(E,\partial G) \geq 2\alpha/(1-\alpha)$$

for $E=\overline{B}^n(z,\alpha d(z,\partial G))$ and by Theorem 1.5

$$u_\alpha(z) = \mathrm{cap}(G,E) \geq s\log(1+2\alpha/(1-\alpha)).$$

For the proof of the converse implication we may assume by Theorem 3.7 without loss of generality that $\alpha=1/2$ and suppose that $u_{1/2}(z)\geq\gamma>0$, for all $z\in G$ and write $F=\overline{\mathbb{R}}^n\setminus G$ for short. Take $c\in(0,1)$ so that

$$-\log c = \left(\frac{2^n\omega_{n-1}}{\gamma}\right)^{1/(n-1)}.$$

We now show that the annulus $cr<|x-a|<r$ meets F for all $a\in F$ and $0<r<+\infty$. On the contrary, we suppose that there exist $a\in F\backslash\{\infty\}$ and $r>0$

such that the annulus $A = R(a, r, cr)$ separates F. It is easy to see that a is not an isolated point of F. Thus, by decreasing r if necessary, we may assume that there is a point $\xi \in \partial G$ with $|\xi - a| = cr$. Let $x_0 = a + s(\xi - a)$, where $s = 1/\sqrt{c}$. Then $|x_0 - a| = \sqrt{c}r$ and thus $x_0 \in A$. Since $\sqrt{c} - c \le 1 - \sqrt{c}$, we have

$$\delta := d(x_0, \partial G) = |x_0 - \xi| = d(x_0, \partial A) = |x_0 - a| - cr = (s-1)cr,$$

Set $C_0 = \{x : |x - a| \le cr\}$, $C_1 = \{x : |x - a| \ge r\}$ and $B = \overline{B}^n(x_0, \delta/2)$. Then

$$\Delta(B, \partial A; A) \subset \Delta(B, C_0; A) \cup \Delta(B, C_1; A)$$

and hence

$$\begin{aligned} \mathrm{cap}\,(A, B) &\le \mathsf{M}(\Delta(B, C_0; A)) + \mathsf{M}(\Delta(B, C_1; A)) \\ &\le \mathsf{M}(\Delta(B, C_0; \overline{\mathbb{R}}^n)) + \mathsf{M}(\Delta(B, C_1; \overline{\mathbb{R}}^n)). \end{aligned}$$

Let $R_j = \overline{\mathbb{R}}^n \setminus (B \cup C_j)$ for $j = 0, 1$. Since $A_0 = R(a, \sqrt{c}r, cr)$ is a subring of R_0, we have

$$\mathrm{mod}(R_0) \ge \mathrm{mod}(A_0) = \log s.$$

Similarly, since $A_1 = R(x_0, r - \sqrt{c}r - \delta/2, \delta/2)$ is a subring of R_1, we have $\mathrm{mod}(R_1) \ge \mathrm{mod}(A_1)$. We now observe that

$$\begin{aligned} \mathrm{mod}(A_1) &= \log \frac{r - \sqrt{c}r - \delta/2}{\delta/2} \\ &= \log \frac{s^2cr - scr - (s-1)cr/2}{(s-1)cr/2} = \log(2s - 1) > \log s. \end{aligned}$$

In view of the relation $\mathsf{M}(\Delta(B, C_j; \overline{\mathbb{R}}^n)) = \omega_{n-1}/(\mathrm{mod}(R_j))^{n-1}$ for $j = 0, 1$, we see that

$$\mathrm{cap}\,(A, B) \le \frac{\omega_{n-1}}{(\mathrm{mod}(R_0))^{n-1}} + \frac{\omega_{n-1}}{(\mathrm{mod}(R_1))^{n-1}} < \frac{2\omega_{n-1}}{(\log s)^{n-1}} = \frac{2^n\omega_{n-1}}{(-\log c)^{n-1}}$$

Since $\Delta(B, \partial G; G) > \Delta(B, \partial A; A)$, Lemma 2.7 yields

$$\mathrm{cap}\,(G, B) \le \mathrm{cap}\,(A, B) < \frac{2\omega_{n-1}}{(-2\log c)^{n-1}}.$$

By assumption, $\mathrm{cap}\,(G, B) = u_{1/2}(x_0) \ge \gamma$ so that $(-\log c)^{n-1} < 2^n\omega_{n-1}/\gamma$, which is impossible by the choice of c. Hence, we have shown the required assertion and conclude that $F \in \mathrm{UP}_n(c)$. □

8 Whitney Cubes and Uniform Perfectness

Next, we will study the condenser capacity by using Whitney cubes.

8.1 Whitney Decomposition

If $G \subset \mathbb{R}^2$ is a bounded domain, we can clearly represent it as a countable union of non-overlapping closed squares. By the Whitney decomposition theorem, we can choose these squares Q_j^k, $k \in \mathbb{Z}, 0 \leq j \leq N_k$, so that they have pairwise disjoint interiors and sides parallel to the coordinate axes and the following properties are fulfilled:

(1) every cube Q_j^k, $0 \leq j \leq N_k$, has sidelength 2^{-k},
(2) $G = \cup_{k,j} Q_j^k$,
(3) $d(Q_j^k) \leq d(Q_j^k, \partial G) < 4d(Q_j^k)$.

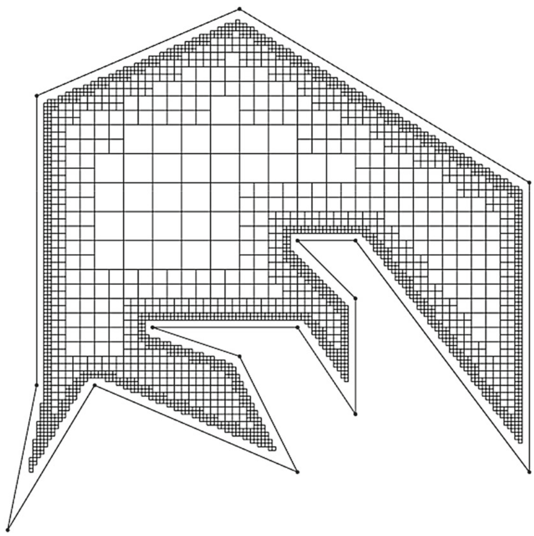
These kinds of squares Q_j^k are *Whitney squares* and this definition can be clearly extended to the general case $G \subset \mathbb{R}^n$, $n \geq 2$, so that we will have n-dimensional closed hypercubes called *Whitney cubes* instead of just squares [29, Thm 1, p. 167]. Note that the Whitney cubes of a domain G resemble in a way hyperbolic balls of $\mathbb{B}^n$ with a constant radius (Fig. 2).

8.2 Whitney Cubes and u_α

For a domain $G \subset \mathbb{R}^n$, $\alpha \in (0, 1)$ and $z \in G$, let

$$u_\alpha(z) = \mathrm{cap}\,(G, \overline{B}^n(z, \alpha\, d(z, \partial G)))$$

Fig. 2 Whitney decomposition of a polygon. The picture was generated with software by D. E. Marshall



and $G = \cup Q_j^k$ be a Whitney decomposition. Then Q_j^k has a side length 2^{-k} and

$$d(Q_j^k) \le d(Q_j^k, \partial G) < 4d(Q_j^k) = 4 \cdot 2^{-k}\sqrt{n}.$$

If m_j^k is the midpoint of Q_j^k, then clearly

$$B^n\left(m_j^k, \frac{d(Q_j^k)}{2\sqrt{n}}\right) \subset Q_j^k \subset \overline{B}^n\left(m_j^k, \frac{d(Q_j^k)}{2}\right), \tag{8.3}$$

$$\left(1 + \frac{1}{2\sqrt{n}}\right) d(Q_j^k) \le d(m_j^k, \partial G) \le \left(4 + \frac{1}{2}\right) d(Q_j^k). \tag{8.4}$$

From (8.3) and (8.4), we see that

$$u_\gamma(m_j^k) \le \operatorname{cap}(\mathbb{R}^n \setminus Q_j^k, \partial G), \quad \gamma = \frac{d(Q_j^k)/(2\sqrt{n})}{9d(Q_j^k)/2} = \frac{1}{9\sqrt{n}},$$

$$u_\eta(m_j^k) \ge \operatorname{cap}(\mathbb{R}^n \setminus Q_j^k, \partial G), \quad \eta = \frac{d(Q_j^k)/2}{(1 + 1/(2\sqrt{n}))d(Q_j^k)} = \frac{\sqrt{n}}{1 + 2\sqrt{n}}.$$

By Lemma 3.6, we can now observe that

$$u_\gamma(m_j^k) \le \operatorname{cap}(\mathbb{R}^n \setminus Q_j^k, \partial G) \le d_1 u_\gamma(m_j^k), \quad d_1 = d_1(n) = \left(\frac{9\sqrt{n}}{1 + 2\sqrt{n}}\right)^{n-1}.$$

The next lemma follows from the the above observations.

Lemma 8.5 *For a cube Q_j^k in the Whitney decomposition of the domain $G \subset \mathbb{R}^n$, let m_k^j be its midpoint. Then there exists a constant $d_1 > 0$ only depending on n such that*

$$u_\gamma(m_j^k) \le \operatorname{cap}(\mathbb{R}^n \setminus Q_j^k, \partial G) \le d_1 u_\gamma(m_j^k)$$

holds with $\gamma = 1/(9\sqrt{n})$.

Lemma 8.6 *Let Q_j^k be as in Lemma 8.5. Then there exists a constant $d_2 > 0$ only depending on n such that, for all $z \in Q_j^k$,*

$$u_\gamma(z)/d_2 \le \operatorname{cap}(\mathbb{R}^n \setminus Q_j^k, \partial G) \le d_2 u_\gamma(z)$$

holds with $\gamma = 1/(9\sqrt{n})$.

Proof By Theorem 3.7, $u_\gamma(z)$ satisfies the Harnack inequality in G. Therefore by [13, Lemma 6.23], there is a constant $d_3 > 0$ only depending on n such that

$$u_\gamma(z)/d_3 \le u_\gamma(m_j^k) \le d_3 u_\gamma(z).$$

The proof follows now from Lemma 8.5. □

8.7 Proof of Theorem 1.4

The proof follows from Lemma 8.6 and Theorem 1.3. □

Acknowledgements The authors are grateful to Prof. Akihiro Munemasa for information about the kissing numbers of combinatorial geometry. We are indebted to Prof. Don Marshall for a permission to use his software for plotting Whitney cubes. The second author would like to express his thanks to the Department of Mathematics and Statistics, University of Turku, for its hospitality and support during the visit to Turku, Finland in 2022.

Funding Open Access funding provided by University of Turku (including Turku University Central Hospital). The research of the first author was funded by Magnus Ehrnrooth Foundation. The research of the second and the third authors was supported in part by JSPS KAKENHI Grant Number JP17H02847.

Data availability Not applicable

References

1. Ahlfors, L.V.: Ein Satz von Henri Cartan und seine Anwendung auf die Theorie der meromorphen Funktionen. Soc. Scient. Fenn. Comm. Phys.-Math. **16**, 1–19 (1930)
2. Ahlfors, L.V.: Conformal Invariants: Topics in Geometric Function Theory. McGraw-Hill, New York (1973)
3. Avkhadiev, F.G., Wirths, K.-J.: Schwarz-Pick Type Inequalities. Frontiers in Mathematics. Birkhäuser, Basel (2009)
4. Beardon, A.F., Minda, D.: The hyperbolic metric and geometric function theory. In: Proceedings of International Workshop on Quasiconformal Mappings and their Applications (IWQCMA05) (2006), pp. 9–56
5. Beardon, A.F., Pommerenke, Ch.: The Poincaré metric of plane domains. J. Lond. Math. Soc. (2) **18**(3), 475–483 (1978)
6. Bishop, C.J.: Harmonic measure: algorithms and applications. In: Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited Lectures, pp. 1511–1537. World Science, Hackensack (2018)
7. Bishop, C.J., Jones, P.W.: Hausdorff dimension and Kleinian groups. Acta Math. **179**, 1–39 (1997)
8. Boyvalenkov, P., Dodunekov, S., Musin, O.: A survey on the kissing numbers. Serdica Math. J. **38**, 507–522 (2012)
9. Dubinin, V.N.: Condenser Capacities and Symmetrization in Geometric Function Theory. Birkhäuser, Boston (2014)
10. Garnett, J.B., Marshall, D.E.: Harmonic Measure. Cambridge University Press, Cambridge (2008)
11. Gehring, F.W., Martin, G.J., Palka, B.P.: An Introduction to the Theory of Higher Dimensional Quasiconformal Mappings. Mathematics. Surveys and Monographs, vol. 216. AMS, Providence (2017)
12. Golberg, A., Sugawa, T., Vuorinen, M.: Teichmüller's theorem in higher dimensions and its applications. Comput. Methods Funct. Theory **20**, 539–558 (2020)

13. Hariri, P., Klén, R., Vuorinen, M.: Conformally Invariant Metrics and Quasiconformal Mappings. Springer Monographs in Mathematics. Springer, Berlin (2020)
14. Heinonen, J.: Lectures on Analysis on Metric Spaces. Universitext, Springer, Berlin (2001)
15. Järvi, P., Vuorinen, M.: Uniformly perfect sets and quasiregular mappings. J. Lond. Math. Soc. **54**, 515–529 (1996)
16. Käenmäki, A., Lehrbäck, J., Vuorinen, M.: Dimensions, Whitney covers, and tubular neighborhoods. Indiana Univ. Math. J. **62**(6), 1861–1889 (2013)
17. Keen, L., Lakic, N.: Hyperbolic Geometry from a Local Viewpoint. London Mathematical Society Student Texts, vol. 68. Cambridge University Press, Cambridge (2007)
18. Landkof, N.S.: Foundations of Modern Potential Theory. Springer, Berlin (1972)
19. Lewis, J.L.: Uniformly fat sets. Trans. Am. Math. Soc. **308**(1), 177–196 (1988)
20. Martio, O.: Capacity and measure densities. Ann. Acad. Sci. Fenn. Ser. **4**(1), 109–118 (1979)
21. Martio, O., Vuorinen, M.: Whitney cubes, p-capacity, and Minkowski content. Exposition. Math. **5**(1), 17–40 (1987)
22. Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability. Cambridge Studies in Advanced Mathematics, vol. 44. Cambridge University Press, Cambridge (1995)
23. Nevanlinna, R.: Analytic Functions. Springer, Berlin (1970)
24. Okounkov, A.: The magic of 8 and 24. Proc. Int. Cong. Math. 1, 492–545 (2022). https://doi.org/10.4171/ICM2022/203
25. Pommerenke, Ch.: Uniformly perfect sets and the Poincaré metric. Arch. Math. **32**, 192–199 (1979)
26. Reshetnyak, Yu.G.: The concept of capacity in the theory of functions with generalized derivatives. (Russian) Sibirsk. Mat. Ž. **10**, 1109–1138 (1969)
27. Reshetnyak, Yu.G.: Space Mappings with Bounded Distortion. Translated from the Russian by H. H. McFaden. Translations of Mathematical Monographs, vol. 73. AMS, Providence (1989). xvi+362 pp
28. Shiga, H., Sugawa, T.: Kleinian groups and geometric function theory. In: Ohsika, K., Papadopoulos, A. (eds.) The Tradition of Thurston III, Geometry and Dynamics. Springer, Berlin (2024)
29. Stein, E.: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series, vol. 30. Princeton University Press, Princeton (1970)
30. Sugawa, T.: Various domain constants related to uniform perfectness. Complex Variables Theory Appl. **36**, 311–345 (1998)
31. Sugawa, T., Vuorinen, M., Zhang, T.: Conformally invariant complete metrics. Math. Proc. Camb. Philos. Soc. **174**(2), 273–300 (2023)
32. Väisälä, J.: Lectures on n-Dimensional Quasiconformal Mappings. Lecture Notes in Mathematics, vol. 229. Springer, Berlin (1971)
33. Vuorinen, M.: Conformal Geometry and Quasiregular Mappings. Lecture Notes in Mathematics, vol. 1319. Springer, Berlin (1988)