K. Matomäki and J. Merikoski (2023) "Siegel Zeros, Twin Primes, Goldbach's Conjecture, and Primes in Short Intervals,"
International Mathematics Research Notices, Vol. 00, No. 00, pp. 1–48
https://doi.org/10.1093/imrn/rnad069

Siegel Zeros, Twin Primes, Goldbach's Conjecture, and Primes in Short Intervals

Kaisa Matomäki[1] and Jori Merikoski[2,*]

[1]Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland and [2]Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter (550), Woodstock Road, Oxford, OX2 6GG, UK

We study the distribution of prime numbers under the unlikely assumption that Siegel zeros exist. In particular, we prove for

$$\sum_{n\le X} \Lambda(n)\Lambda(\pm n+h)$$

an asymptotic formula that holds uniformly for $h = O(X)$. Such an asymptotic formula has been previously obtained only for fixed h in which case our result quantitatively improves those of Heath-Brown (1983) and Tao and Teräväinen (2021). Since our main theorems work also for large h, we can derive new results concerning connections between Siegel zeros and the Goldbach conjecture and between Siegel zeros and primes in almost all very short intervals.

1 Introduction

While the proof of the twin prime conjecture is a distant goal, Heath-Brown [9] proved in 1983 that if there are infinitely many Siegel zeros, then there are infinitely many twin primes. More precisely, Heath-Brown showed that if, for a Dirichlet character χ (mod q), the Dirichlet L-function $L(s,\chi)$ has a real zero at $s = \beta_0$ with

$$\beta_0 = 1 - \frac{1}{\eta \log q} \tag{1}$$

Received February 5, 2022; Revised November 2, 2022; Accepted February 27, 2023
Communicated by Prof. Sound Soundarajan

for some $\eta \geq 10$, then, for any $h \geq 1$ and $X \in [q^{250}, q^{500}]$, one has

$$\sum_{n \leq X} \Lambda(n)\Lambda(n+h) = X\mathfrak{S}_h + O_h\left(\frac{X}{\log\log \eta}\right),$$

where

$$\mathfrak{S}_h := \mathbf{1}_{2|h} \cdot 2 \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|h \\ p>2}} \left(1 + \frac{1}{p-2}\right) \asymp \mathbf{1}_{2|h} \frac{h}{\varphi(h)}.$$

Actually, Heath-Brown proved a more general result for sums of the form

$$\sum_{n \leq X} \Lambda(\alpha_1 n + \beta_1)\Lambda(\alpha_2 n + \beta_2).$$

Very recently, Heath-Brown's result was quantitatively improved by Tao and Teräväinen [20] who showed that, for any $h \geq 1$ and $X \in [q^{41/2+\varepsilon}, q^{\eta^{1/2}}]$, one has

$$\sum_{n \leq X} \Lambda(n)\Lambda(n+h) = X\mathfrak{S}_h + O_h\left(\frac{X}{\log^{1/20} \eta}\right). \tag{2}$$

This is a special case of their more general theorem on "Hardy–Littlewood–Chowla" type correlations (with $k \leq 2$)

$$\sum_{n \leq X} \Lambda(n+h_1) \cdots \Lambda(n+h_k) \lambda_{\text{Liouville}}(n+h_1') \cdots \lambda_{\text{Liouville}}(n+h_\ell'), \tag{3}$$

where $\lambda_{\text{Liouville}}$ denotes the usual Liouville function $\lambda_{\text{Liouville}}(n) = (-1)^{\Omega(n)}$ (we reserve the symbol λ for $1 * \chi$ to match our notations with previous articles).

In this paper, we will prove (2) with better error term and with uniformity in the shift h. Furthermore, despite leading to stronger results, our proof is less involved. Before turning to the strongest formulations of our theorems, let us state two corollaries. The first corollary improves on (2).

Corollary 1.1. Let $h \in \mathbb{N}$. Let χ be a primitive quadratic character modulo $q \geq 2$ and assume that $L(s, \chi)$ has a real zero β_0 such that

$$\beta_0 = 1 - \frac{1}{\eta \log q}$$

for some $\eta \geq 10$.

(i) Let $C \geq 1$. For $X \in [q^{10}, q^{10 \log \eta}]$, we have

$$\sum_{n \leq X} \Lambda(n)\Lambda(n+h) = X\mathfrak{S}_h + O_{h,C}\left(X \exp(-C\sqrt{\log \eta})\right).$$

(ii) Let $\varepsilon > 0$. For $X = q^V$ with $V \in [10 \log \eta, \eta^{1-\varepsilon}]$, we have

$$\sum_{n \leq X} \Lambda(n)\Lambda(n+h) = X\mathfrak{S}_h + O_{h,\varepsilon}\left(X\frac{\log^6 \eta}{\eta/V}\right).$$

Note that for $X \in [q^{10 \log \eta}, q^{10 \log^4 \eta}]$, the error term is $O(\eta^{-1}X \log^{10} \eta)$, where the dependency on η is best that can be hoped for apart from the power of $\log \eta$. This is because in the presence of exceptional zeros we expect a secondary main term of size $\asymp_h X/\eta$ in this range. In principle, it might be possible to evaluate this secondary main term precisely and thus get a better error term. However, this appears to be a difficult problem as it is closely related to evaluating correlations of higher order divisor functions τ_k.

Our main results are uniform with respect to h, which allows us to attack also the Goldbach conjecture. There has been recent activity on the relation between Siegel zeros and the Goldbach conjecture, see for example, [3, 6]. The asymptotic form of Goldbach's conjecture claims that, for all $h \geq 4$,

$$\sum_{\substack{n_1, n_2 \leq h \\ n_1 + n_2 = h}} \Lambda(n_1)\Lambda(n_2) = (1 + o(1))\mathfrak{S}_h \cdot h. \tag{4}$$

We show that if a weak form of (4) holds, then there cannot be Siegel zeros.

Corollary 1.2. Let $\delta > 0$. There exists $\eta = \eta(\delta) \geq 100$ such that the following holds. Let $q \geq 2$ be such that there exists a primitive quadratic character $\chi \pmod{q}$. Assume that there exists an even $h \in [q^{10}, q^{\eta^{99/100}}]$ such that $q \mid h$ and

$$\delta\mathfrak{S}_h \cdot h \leq \sum_{\substack{n_1, n_2 \leq h \\ n_1 + n_2 = h}} \Lambda(n_1)\Lambda(n_2) \leq (2 - \delta)\mathfrak{S}_h \cdot h. \tag{5}$$

Then the Dirichlet L-function $L(s,\chi)$ does not have a real zero β_0 with

$$\beta_0 \geq 1 - \frac{1}{\eta \log q}.$$

This improves on a recent result of Friedlander, Goldston, Iwaniec, and Suriajaya [3] who got a similar conclusion assuming that (5) holds for several $h \equiv 0 \pmod q$. In fact, our result is even stronger and we only need the lower bound in (5) if $\chi(-1) = -1$ and similarly only the upper bound in (5) if $\chi(-1) = 1$.

We now turn to the precise statements of our main results. When we work uniformly with respect to h, we get an additional main term when (h,q) has size close to q. To explain why, let us consider the most transparent case $h = q$. It is well known that if there is a Siegel zero, then the primes fail to be equidistributed $\pmod q$. More precisely, we have, for $(a,q) = 1$, when q, χ, and β_0 are as in Corollary 1.1 with η large (see e.g., [12, Theorem 5.27]),

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) = \frac{x}{\varphi(q)}\left(1 - \frac{\chi(a)}{\beta_0 x^{1-\beta_0}}\right) + O\left(x \exp\left(\frac{-c\log x}{\sqrt{\log x} + \log q}\right)\log^4 q\right).$$

Hence, when η is large, the residue classes $a \pmod q$ with $\chi(a) = -1$ contain about twice as many primes as one would expect, whereas the residue classes $a \pmod q$ with $\chi(a) = 1$ contain very few primes. Consequently, one would expect that, for even q, we have

$$\sum_{n \leq X} \Lambda(n)\Lambda(n+q) \approx 2\mathfrak{S}_q X,$$

which is twice of the expected main term. The following general theorem confirms this intuition. Here we extend $\Lambda(n)$ to negative integers by defining $\Lambda(-n) = \Lambda(n)$ and similarly below for other arithmetic functions.

Theorem 1.3. Let $C \geq 1$ and $\varepsilon > 0$. Let χ be a primitive quadratic character modulo $q \geq 2$. Write $q = 2^r q'$ with $r \geq 0$ and $2 \nmid q'$. Assume that $L(s,\chi)$ has a real zero β_0 such that

$$\beta_0 = 1 - \frac{1}{\eta \log q}$$

for some $\eta \geq 10$. Let $X = q^V$ with $V \geq 10$ and $0 \neq h = O(X)$. We have

$$\sum_{n \leq X} \Lambda(n)\Lambda(n+h) = X\mathfrak{S}_h \left(1 + \mathbf{1}_{\varphi(2^r)|h}(-1)^{\frac{h}{\varphi(2^r)}} \prod_{\substack{p|q' \\ p\nmid h}} \frac{-1}{p-2}\right)$$
$$+ O_{C,\varepsilon}\left(\frac{|h|}{\varphi(h)}X\left(\exp(-C\sqrt{V\log\eta}) + \exp(-C(\log X)^{3/5-\varepsilon}) + \frac{V(\log\eta)^6}{\eta}\right)\right).$$

Note that the main term vanishes for some even h, for instance when $3 \mid q$ and $h = 2q/3$ (note that q' is necessarily square-free (see e.g., [12, Section 3.3]) and so in this case $3 \nmid h$).

We get a similar result concerning Goldbach's conjecture.

Theorem 1.4. Let $C \geq 1$ and $\varepsilon > 0$. Let χ be a primitive quadratic character modulo $q \geq 2$. Write $q = 2^r q'$ with $r \geq 0$ and $2 \nmid q'$. Assume that $L(s,\chi)$ has a real zero β_0 such that

$$\beta_0 = 1 - \frac{1}{\eta \log q}$$

for some $\eta \geq 10$. Let $h \geq q^{10}$ be an integer. Writing $V := \frac{\log h}{\log q} \geq 10$, we have

$$\sum_{\substack{n_1,n_2 \leq h \\ n_1+n_2=h}} \Lambda(n_1)\Lambda(n_2) = h\mathfrak{S}_h \left(1 + \chi(-1)\mathbf{1}_{\varphi(2^r)|h}(-1)^{\frac{h}{\varphi(2^r)}} \prod_{\substack{p|q' \\ p\nmid h}} \frac{-1}{p-2}\right)$$
$$+ O_{C,\varepsilon}\left(\frac{h}{\varphi(h)}h\left(\exp(-C\sqrt{V\log\eta}) + \exp(-C(\log h)^{3/5-\varepsilon}) + \frac{V(\log\eta)^6}{\eta}\right)\right).$$

Corollaries 1.1(i) and 1.2 immediately follow from Theorems 1.3 and 1.4 since by Siegel's theorem (see e.g., [18, Theorem 11.14 combined with (11.10)])

$$\eta \ll_\varepsilon q^\varepsilon. \tag{6}$$

For $X \geq q^{10\log\eta}$, the quantity

$$\frac{1}{\eta} = \exp(-\log\eta) \gg \exp(-\sqrt{(\log q)(\log\eta)}) \gg \exp(-\sqrt{\log X}),$$

dominates $\exp(-C(\log X)^{3/5-\varepsilon})$ and $\exp(-CV\sqrt{\log \eta})$, so also Corollary 1.1(ii) follows from Theorem 1.3.

We will also prove a corollary concerning the distribution of primes in almost all short intervals. A probabilistic model predicts that

$$\sum_{y<p\leq y+H} 1 = (1+o(1))\frac{H}{\log X} \tag{7}$$

holds for almost all $y \in [X, 2X]$ as soon as $H/\log X \to \infty$. Selberg [19] has shown this for $H/\log^2 X \to \infty$ assuming the Riemann hypothesis. Assuming also the Pair correlation conjecture [17], Heath-Brown [8] proved the result predicted by the probabilistic model. Unconditionally, the best current results are an asymptotic formula (7) for almost all $y \in [X, 2X]$ when $H \geq X^{1/6+o(1)}$ (see e.g., [7, Theorem 9.1]), and a lower bound when $H \geq X^{1/20+\varepsilon}$ (see Jia [14]).

Here we prove the asymptotic formula (7) for almost all y with H in the range predicted by the probabilistic model under the unlikely assumption of existence of Siegel zeros.

Corollary 1.5. Let $C \geq 2$ and $\varepsilon > 0$. Let χ be a primitive quadratic character modulo $q \geq 2$ and assume that $L(s,\chi)$ has a real zero β_0 such that

$$\beta_0 = 1 - \frac{1}{\eta \log q}$$

for some $\eta \geq 10$.

Let $X \geq q^{10}$, write $V := \frac{\log X}{\log q} \geq 10$, and let $2 \leq H \leq X^{1/3}$. Then

$$\int_X^{2X} \left(\sum_{y<n\leq y+H} \Lambda(n) - H \right)^2 \mathrm{d}y$$
$$\ll_{C,\varepsilon} HX\log X + H^2 X \left(\exp(-C\sqrt{V\log\eta}) + \exp(-C(\log X)^{3/5-\varepsilon}) + \frac{V\log^6 \eta}{\eta} \right).$$

This implies that as soon as

$$\eta \to \infty, \quad \frac{H}{\log X} \to \infty, \quad \text{and} \quad q^{10} \leq X \leq q^{\eta^{1-\delta}},$$

we get the asymptotic formula

$$\sum_{y<p\leq y+H} 1 = (1+o(1))\frac{H}{\log y}$$

for almost all $y \in [X, 2X]$.

Remark 1.6. Under the assumption of the existence of Siegel zeros, one can show that the Pair correlation conjecture [17] fails drastically, that is, almost all zeros of $\zeta(s)$ in some range depending on q lie on a lattice with normalized distances in $(\frac{1}{2}+o(1))\mathbb{Z}$ (the so-called alternative hypothesis, this follows for example, from [1, Proposition 9.2] assuming $\eta = \log^{100} q$ for zeros of height $T = \exp(\log^{10} q)$). Note however that in [8], instead of the full Pair correlation conjecture, Heath-Brown only requires a weaker upper bound for Montgomery's [17] function F, which is consistent with the alternative hypothesis.

Remark 1.7. All our results hold also if instead of existence of Siegel zeros we assume that $L(s,\chi)$ takes a small value at $s=1$, that is, assuming

$$L(1,\chi) \leq \frac{1}{\eta \log q}.$$

Indeed, by [18, Theorem 11.4] (using [18, (11.7)] in the contrapositive direction) this implies that $L(s,\chi)$ has a real zero β_0 with

$$1-\beta_0 \ll L(1,\chi) \ll \frac{1}{\eta \log q}.$$

In the opposite direction, the best current result loses essentially two factors of $\log q$, that is, we have $L(1,\chi) \ll (1-\beta_0)(\log^2 q)/\log\log q$ by work of Friedlander and Iwaniec [5]. For this reason, we state our results in terms of Siegel zeros.

Note that in many articles such as [4, 16], one has to assume that χ is exceptional in a very strong sense, that is, $L(1,\chi) \ll \log^{-C} q$ for some large constant C. Combining the ideas from this paper with the argument in [16], it is possible to replace this strong assumption by (1) with a similar dependency on η as in Corollary 1.1.

Let us here briefly discuss why having a Siegel zero is a helpful assumption; we will discuss the proof strategy rigorously and in more detail in Section 2.

If $L(s,\chi)$ has a zero at $s=\beta_0$ with β_0 close to 1, then

$$L(1,\chi)^{-1}=\sum_n \frac{\mu(n)\chi(n)}{n}=\prod_p\left(1+\frac{\mu(p)\chi(p)}{p}\right)$$

is large, so that $\chi(p)=\mu(p)$ for most primes (in a wide range depending on q) and heuristically we have

$$\Lambda=\mu*\log\approx\chi*\log, \tag{8}$$

so that we can hope to replace Λ by $\chi*\log$. But the function $\chi*\log$ is of similar complexity as the divisor function $\tau=1*1$ when the modulus q is small compared to x.

Hence, in order to prove our theorems, we need to show that the contribution of the error in the approximation (8) is small as well as study

$$\sum_{n\le X}(\chi*\log)(n)(\chi*\log)(\pm n+h),$$

which has similar complexity as divisor correlations. Friedlander and Iwaniec [4] have shown that the error in (8) can be controlled if we can solve the corresponding ternary divisor problem. In our case, we cannot, but we can still deal with the error, once we restrict both sides to numbers without large prime factors. In general, for sequences with relatively large (logarithmic) density, one can exploit crude bounds to get a result without knowledge of the corresponding ternary divisor problem.

Notation

We write $\mathbf{1}_P$ for the indicator function of the claim P. We write $\Lambda(n)$, $\mu(n)$, $\varphi(n)$, $\tau(n)$ for the von Mangoldt function, Möbius function, Euler φ-function, and the divisor function. These functions are understood to equal 0 for non-positive integers. For arithmetic functions $f,g\colon\mathbb{N}\to\mathbb{C}$, we define the Dirichlet convolution

$$(f*g)(n):=\sum_{n=km}f(k)g(m).$$

For $f\colon\mathbb{R}\to\mathbb{C}$ and $g\colon\mathbb{R}\to\mathbb{R}^+$, we write $f(x)=O(g(x))$ or $f(x)\ll g(x)$ if there exists a constant $C>0$ such that $|f(x)|\le Cg(x)$ for every x. Furthermore, for positive valued f and g, we write $f(x)\asymp g(x)$ when $g(x)\ll f(x)\ll g(x)$. If there is a subscript (e.g., $O_k(g(x))$), then the implied constant is allowed to depend on the parameter(s) in the subscript.

We say that a function $g\colon \mathbb{R} \to \mathbb{R}$ is smooth if it has derivatives of all orders.

For $u \in \mathbb{C}$, we write $e(u) := e(2\pi i u)$ and for $q \in \mathbb{N}$ we write $e_q(u) = e(u/q)$. For any function

$$g \in L^1(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C}\colon \int_{-\infty}^{\infty} |f(x)| \mathrm{d}x < \infty \right\},$$

we denote by $\widehat{g}$ the Fourier transform

$$\widehat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e(-\xi x) \mathrm{d}x.$$

For $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, we write $\overline{a}$ for the inverse of $a \pmod q$ (the modulus will be clear from the context, e.g., in $e(\frac{c\overline{u}}{v})$ the inverse is $\pmod v$).

2 Initial Steps

We start by replacing the sums in Theorems 1.3 and 1.4 by smoothed variants. Let $\delta = X^{-\varepsilon}$ for some small $\varepsilon > 0$, and let $g : \mathbb{R} \to [0, 1]$ be a smooth function that is supported on $[1, 2]$ and equals 1 on $[1 + \delta, 2 - \delta]$. Assume further that the derivatives of g satisfy $g^{(j)}(x) \ll \delta^{-j}$ for every x.

In case of Theorem 1.3, we first decompose the summation condition $n \leq X$ dyadically into conditions $n \in (x, 2x]$ with $x \leq X/2$. We estimate the contribution of $x \leq X^{1-\varepsilon/4}$ trivially, and for the remaining x, we replace the condition $\mathbf{1}_{n\in(x,2x]}$ by $g(n/x)$ with an error term $O(\delta x \log^2 X) = O(X^{1-\varepsilon/2})$. Thus, it suffices to show that

$$\sum_n g\left(\frac{n}{X}\right) \Lambda(n)\Lambda(n+h) = \int g\left(\frac{y}{X}\right) \mathrm{d}y \cdot \mathfrak{S}_h \left(1 + \mathbf{1}_{\varphi(2^r)|h} (-1)^{\frac{h}{\varphi(2^r)}} \prod_{\substack{p|q' \\ p\nmid h}} \frac{-1}{p-2} \right)$$
$$+ O_{C,\varepsilon}\left(\frac{h}{\varphi(h)} X \left(\exp(-C\sqrt{V \log \eta}) + \exp(-C(\log X)^{3/5-\varepsilon}) + \frac{V(\log \eta)^6}{\eta} \right)\right),$$

whenever $0 < h \leq X^{1+\varepsilon/2}$ (we can assume that h is positive by symmetry).

In case of Theorem 1.4, note that by symmetry, we can concentrate on the case $n_1 < h/2 < n_2$ and that when $n_1 \leq h^{1-\varepsilon/4}$, we can use a trivial estimate. Arguing as above with similar g, we see that it suffices to show that

$$\sum_n g\left(\frac{n}{X}\right)\Lambda(n)\Lambda(h-n) = \int g\left(\frac{y}{X}\right)\mathrm{d}y \cdot \mathfrak{S}_h\left(1+\chi(-1)\mathbf{1}_{\varphi(2^r)\mid h}(-1)^{\frac{h}{\varphi(2^r)}}\prod_{\substack{p\mid q'\\ p\nmid h}}\frac{-1}{p-2}\right)$$
$$+\,O_C\left(\frac{h}{\varphi(h)}X\left(\exp(-CV\sqrt{\log\eta})+\exp(-C(\log h)^{3/5-\varepsilon})+\frac{V(\log\eta)^6}{\eta}\right)\right).$$

for every $X \in [h^{1-\varepsilon/3}, h/4]$.

We shall deal with Theorems 1.3 and 1.4 simultaneously, and thus consider

$$\sum_n g\left(\frac{n}{X}\right)\Lambda(n)\Lambda(\pm n+h).$$

Note that when n is in the support of $g(n/X)$ with X as above, we have $\pm n + h \geq \max\{X, h/2\}$.

Let χ be a quadratic character modulo q. Following [4, 16], we write

$$\lambda := 1 * \chi \quad \text{and} \quad \lambda' := \chi * \log, \tag{9}$$

so that

$$\lambda * \Lambda = (1*\chi)*(\mu*\log) = (1*\mu)*(\chi*\log) = \lambda'.$$

Note also that $\lambda(n) \geq 0$ and $\lambda'(n) \geq \Lambda(n) \geq 0$ (since $\lambda' = \lambda * \Lambda$). By above

$$\lambda'(n) = (\lambda * \Lambda)(n) = \Lambda(n) + \sum_{\substack{n=km\\ m>1}}\Lambda(k)\lambda(m), \tag{10}$$

so we have obtained a formula for the error term in the approximation (8). Similarly, as in [16], we now restrict this approximation to rough numbers to ensure that m is large. For large m, we expect by the assumption of existence of Siegel zeros that the function $\lambda(m)$ is lacunary, which will make the sum in (10) small.

For $w \geq 2$ and $k \in \mathbb{N}$, write

$$P(w) = \prod_{p<w} p \quad \text{and} \quad P_k(w) = \prod_{\substack{p<w\\ p\nmid k}} p.$$

Let u be a large parameter to be chosen later (see (41)) and write $z := X^{1/u}$. Adding the condition $(n, qP(z)) = 1$ to both sides of (10), we get

$$\Lambda(n) = \lambda'(n)\mathbf{1}_{(n,qP(z))=1} - \sum_{\substack{n=km \\ m\geq z \\ (km,qP(z))=1}} \Lambda(k)\lambda(m) + O\Bigg(\log n \cdot \mathbf{1}_{\substack{n=p^{\nu} \\ p\mid qP(z)}}\Bigg). \tag{11}$$

We define

$$c_n := \sum_{\substack{n=km \\ m\geq z \\ (km,P(z))=1}} \Lambda(k)\lambda(m). \tag{12}$$

Note that $0 \leq \lambda'(n) \leq \tau(n)\log n$ and $0 \leq c_n \leq \mathbf{1}_{(n,P(z))=1}\tau(n)^2 \log n$. When $g\colon \mathbb{R} \to [0,1]$ is a smooth function supported on $[1,2]$, we obtain

$$\sum_n g\left(\frac{n}{X}\right)\Lambda(n)\Lambda(\pm n+h) = \sum_{\substack{n \\ (n(\pm n+h),qP(z))=1}} g\left(\frac{n}{X}\right)\lambda'(n)\lambda'(\pm n+h) \tag{13}$$

$$+ O\left((z+\omega(q))\log^2 X + \log X \sum_{\substack{n\leq 2X \\ (n(\pm n+h),qP(z))=1}} (c_n\tau(\pm n+h)^2 + \tau(n)c_{\pm n+h})\right).$$

We deal with the error term using the following lemma, which will quickly follow from Henriot's bound on correlations of multiplicative functions (see Section 3.1 for the proof).

Lemma 2.1. Let c_n be as in (12), let $X \geq 3, u \geq 2$, and $z = X^{1/u}$. Then, for any $0 < |h| \leq X^{10}$, we have

$$\sum_{\substack{n\leq 4X \\ (n(\pm n+h),P(z))=1}} c_n\tau(\pm n+h)^2 \ll \frac{h}{\varphi(h)}\frac{X}{\log X}\left(u^4 \sum_{\substack{z\leq m\leq 4X/z \\ (m,P(z))=1}} \frac{\lambda(m)}{m} + \frac{u^{10}}{z}\right).$$

Here the sum over m can be estimated in terms of η. The following lemma will be proved in Section 4.

Lemma 2.2. Let χ be a primitive quadratic character modulo $q \geq 2$. Assume that $L(s, \chi)$ has a real zero β_0 such that

$$\beta_0 = 1 - \frac{1}{\eta \log q}$$

for some $\eta \geq 10$. Let $z = q^v$ for some $v \in \mathbb{R}_+$. Then for any$Y > z$

$$\sum_{\substack{z \leq m \leq Y \\ (m, P(z))=1}} \frac{\lambda(m)}{m} \ll \left(\frac{1}{v^2 \eta^{v/2}} + \frac{v}{\eta} \cdot \frac{\log Y}{\log z} + \frac{1}{z} \right) \left(\frac{\log Y}{\log z} \right)^2 .$$

To deal with the main term in (13), we use the following proposition, which we will prove in Section 5. Note that this proposition is unconditional and gives an asymptotic formula for generalized divisor function correlations over rough numbers. The parameter β refers to the β-sieve described in Lemma 3.2.

Proposition 2.3. Let $\delta > 0$. Let $X \geq 2$ and $M_1, M_2, N_1, N_2 \geq 1$ be such that

$$M_1 N_1 \asymp X, \quad X \ll M_2 N_2 \ll \delta^{-1} X, \quad \text{and} \quad M_j \ll N_j.$$

Let $h \in \mathbb{Z}_+$. Let $q \geq 1$, and let $\chi_1, \chi_2, \psi_1, \psi_2$ be real characters $\pmod q$. Let $f \colon \mathbb{R}_+^4 \to [0, 1]$ be smooth and compactly supported and suppose that for all $j \geq 0$ and $v \in \{1, 2\}$,

$$\frac{\partial^j}{\partial x_v^j} f(x_1, x_2, y_1, y_2), \quad \frac{\partial^j}{\partial y_v^j} f(x_1, x_2, y_1, y_2) \ll_j \delta^{-j}.$$

Let $A \in \mathbb{N}$. Assume that β is sufficiently large in terms of A and $u \geq 1000\beta$. Write $z = X^{1/u}$ and, for $r \geq 0$,

$$z_r := z^{((\beta-1)/\beta)^r}. \tag{14}$$

Then

$$
\begin{aligned}
&\sum_{\substack{m_1,m_2,n_1,n_2\\ \pm m_1n_1+h=m_2n_2\\ (m_1m_2n_1n_2,P(z))=1}} \chi_1(m_1)\chi_2(m_2)\psi_1(n_1)\psi_2(n_2)f\left(\frac{m_1}{M_1},\frac{m_2}{M_2},\frac{n_1}{N_1},\frac{n_2}{N_2}\right)\\
&= \prod_{\substack{p<z\\ p\mid h, p\nmid q}}\left(1-\frac{1}{p}\right)\prod_{\substack{p<z\\ p\nmid hq}}\left(1-\frac{2}{p}\right)\cdot\frac{1}{q}\sum_{\gamma(q)}\psi_1(\gamma)\psi_2(\pm\gamma+h)\\
&\cdot \sum_{\substack{m_1,m_2\\ (m_1m_2,P(z))=1}} \frac{\chi_1(m_1)\chi_2(m_2)\psi_1(m_1)\psi_2(m_2)}{m_1m_2}\int f\left(\frac{m_1}{M_1},\frac{m_2}{M_2},\frac{y}{m_1N_1},\frac{\pm y+h}{m_2N_2}\right)\mathrm{d}y\\
&+ O\Bigg(\sum_{\substack{r_1,r_2,r_3\geq 0\\ \max r_j\geq \frac{u}{1000}-\beta}} 2^{-A(r_1+r_2+r_3)} \sum_{\substack{m_1,m_2,n_1,n_2\\ \pm m_1n_1+h=m_2n_2\\ (n_1,P(z_{r_1}))=(n_2,P_h(z_{r_2}))=1\\ (m_1,P(z_{r_3}))=(m_2,hP(z))=1}} (\tau(m_1)\tau(n_1)\tau(n_2))^{A+1}\\
&\quad \cdot f\left(\frac{m_1}{M_1},\frac{m_2}{M_2},\frac{n_1}{N_1},\frac{n_2}{N_2}\right)\Bigg) + O_A\left(\delta^{-3}X^{7/9}q^2+\frac{h}{\varphi(h)}\frac{X}{\log^2 X}\left(\frac{u^6}{z}+e^{-Au/3000}\right)\right).
\end{aligned}
$$

As an application of Henriot's bound (see Lemma 3.1 below), we will prove in Section 3.1 the following lemma concerning the error term.

Lemma 2.4. Let $X \geq 2$, and let $h \in \mathbb{Z}_+$ be such that $h = X^{O(1)}$. Let $A \in \mathbb{N}$. Assume that β is sufficiently large in terms of A and $u \geq 1000\beta$. Write $z = X^{1/u}$ and, for $r \geq 0$, let z_r be as in (14). Then

$$
\begin{aligned}
&\sum_{\substack{r_1,r_2,r_3\geq 0\\ \max r_j\geq \frac{u}{1000}-\beta}} 2^{-A(r_1+r_2+r_3)} \sum_{\substack{m_1,m_2,n_1,n_2\\ m_1n_1\leq 10X\\ \pm m_1n_1+h=m_2n_2\\ (n_1,P(z_{r_1}))=(n_2,P_h(z_{r_2}))=1\\ (m_1,P(z_{r_3}))=(m_2,hP(z))=1}} (\tau(m_1)\tau(n_1)\tau(n_2))^{A+1} \\
&\ll_A \frac{h}{\varphi(h)}\frac{X}{\log^2 X}e^{-Au/2000}.
\end{aligned} \tag{15}
$$

In Section 6, we will prove the following lemma, which helps us in evaluating the main term we obtain from Proposition 2.3.

Lemma 2.5. Let χ be a primitive quadratic character modulo $q \geq 2$. Assume that $L(s,\chi)$ has a real zero β_0 such that

$$\beta_0 = 1 - \frac{1}{\eta \log q}$$

for some $\eta \geq 10$. Let $X \geq 3, u \geq 2$, and $z = X^{1/u} = q^v$. Assume that $X^{1/10} \leq N \leq X^2$ and $1 \leq y \leq X^2$. Then, for any $\varepsilon > 0$ and $A, C \geq 2$,

$$\sum_{\substack{n \leq N \\ (n,P(z))=1}} \frac{\chi(n)\log(y/n)}{n} = (1 + O_{A,C,\varepsilon}(\mathcal{E})) \prod_{p<z} \left(1 - \frac{1}{p}\right)^{-1}$$

with

$$\mathcal{E} = \frac{u^4}{v^2 \eta^{v/2}} + \frac{vu^5}{\eta} + \frac{u^4}{z} + e^{-Au/3000} + \exp(-C \log^{3/5-\varepsilon} X).$$

The complete character sums resulting from Proposition 2.3 will be evaluated using the following elementary lemma, which will be proved in Section 3.4. In the case of a prime modulus, these sums are a special case of Jacobsthal sums and their exact evaluation goes back to [13].

Lemma 2.6. Let $q \geq 2$, and let χ be a primitive quadratic character of modulus $q = 2^r q'$ with $r \geq 0$ and $2 \nmid q'$. Let χ_0 be the principal character $\pmod{q}$, and let h be an even integer. Then

$$\frac{1}{q} \sum_{m(q)} \chi_0(m)\chi_0(\pm m + h) = \prod_{p \mid (q,h)} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \mid q \\ p \nmid h}} \left(1 - \frac{2}{p}\right), \tag{16}$$

$$\frac{1}{q} \sum_{m(q)} \chi(m)\chi_0(\pm m + h) = \frac{-\chi(\mp h)}{q}, \tag{17}$$

$$\frac{1}{q} \sum_{m(q)} \chi(m)\chi(\pm m + h) = \mathbf{1}_{\varphi(2^r) \mid h} (-1)^{\frac{h}{\varphi(2^r)}} \chi(\pm 1) \prod_{p \mid (q,h)} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \mid q \\ p \nmid h}} \frac{-1}{p}. \tag{18}$$

Precise evaluation of the character sum (18) is, in addition to keeping track of some dependencies, the reason we can deal with the case (h, q) close to q in Theorems 1.3 and 1.4. As Tao and Teräväinen [20] consider the more general problem (3), they face more complicated character sums and need to apply Weil's bound, and thus they do not obtain similar uniformity in h.

In Section 7, we shall use the results stated here to prove Theorems 1.3 and 1.4. Then, in Section 8, we deduce Corollary 1.5 from Theorem 1.3.

Remark 2.7. Before moving on, let us briefly discuss how our approach differs from that of Tao and Teräväinen [20]. We establish Proposition 2.3 using the β-sieve whereas Tao and Teräväinen [20] use Selberg's sieve in their arguments. Using the β-sieve makes replacing the condition $\mathbf{1}_{(n,P(z))=1}$ by a type I sum more transparent since the remainder is easier to analyse, see Lemma 3.2(i) for a convenient bound. This difference of the two sieves is discussed also in [2, Section 10.2]. Thanks to using the β-sieve our arguments are considerably simpler than those in [20], and we get improved error terms automatically.

More precisely, the starting point of [20] is to replace $\Lambda(n)$ by $(\chi * \log)(n)\nu(n)$, where $\nu(n)$ are Selberg sieve weights. Replacing $\Lambda(n)$ by $\Lambda(n)\nu(n)$ is almost immediate thanks to the support of $\Lambda(n)$. However, replacing $\Lambda(n)\nu(n)$ by $(\chi * \log)(n)\nu(n)$ is somewhat more complicated and this leads to weaker error terms. Also later the Selberg sieve coefficients complicate matters in [20]. Despite this, it might be possible to argue more carefully with the Selberg sieve and obtain error terms comparable to ours.

3 Lemmas

3.1 Multiplicative functions

We will use some standard estimates for multiplicative functions. Note first that, when $f\colon \mathbb{N} \to \mathbb{C}$ is a divisor-bounded (i.e., $f(n) \le \tau(n)^A$ for some $A \ge 1$) multiplicative function, we have

$$\sum_{n\le X} \frac{|f(n)|}{n} \ll \prod_{p\le X}\left(1+\frac{|f(p)|}{p}\right). \tag{19}$$

Furthermore, by Mertens' theorem,

$$\prod_{w<p\le z}\left(1+\frac{k}{p}\right) \asymp \left(\frac{\log z}{\log w}\right)^{k}. \tag{20}$$

We shall also need the following consequences of Henriot's bound [10, 11].

Lemma 3.1. Let $X \geq 2$ and $2 \leq w_1, w_2 \leq X$. Let also $1 \leq |h| \leq X^{10}$.

(i) Let $m_1, m_2 \geq 0$. Then

$$\sum_{n \leq X} \tau(n)^{m_1} \mathbf{1}_{(n,P(w_1))=1} \tau(\pm n + h)^{m_2} \mathbf{1}_{(\pm n+h, P_h(w_2))=1} \ll_{m_1,m_2} \frac{h}{\varphi(h)} \cdot \frac{X}{\log^2 X} \left(\frac{\log X}{\log w_1}\right)^{2^{m_1}+1} \left(\frac{\log X}{\log w_2}\right)^{2^{m_2}}.$$

(ii) Let $m \in \mathbb{Z}$ be such that $(m, hP(w_2)) = 1$ and $m = X^{O(1)}$. Then

$$\sum_{n \leq X} \mathbf{1}_{(n,P(w_1))=1} \mathbf{1}_{(\pm mn+h, P(w_2))=1} \tau(\pm mn + h)^2 \ll \frac{h}{\varphi(h)} \frac{X}{\log^2 X} \frac{\log X}{\log w_1} \left(\frac{\log X}{\log w_2}\right)^4.$$

Proof. **Proof of (i).** We apply Henriot's bound [10, Theorem 3] with

$$\begin{aligned} &Q_1(n) = n,\ Q_2(n) = \pm n + h,\ D = h^2, \\ &F(n_1, n_2) = \mathbf{1}_{(n_1,P(w_1))=1} \mathbf{1}_{(n_2,P_h(w_2))=1} \tau(n_1)^{m_1} \tau(n_2)^{m_2}, \\ &\rho_{Q_1}(n) = |\{v \pmod n : v \equiv 0(n)\}| = 1, \\ &\rho_{Q_2}(n) = |\{v \pmod n : \pm v + h \equiv 0(n)\}| = 1 \\ &\rho(p) = |\{v \pmod p : v(\pm v + h) \equiv 0(p)\}| = \begin{cases} 2 & \text{if } p \nmid h \\ 1 & \text{otherwise,} \end{cases} \\ &\Delta_D = \prod_{p \mid h} \Bigg(1 + F(p,1) \frac{|\{n(p^2) : p \| n, p \nmid \pm n + h\}|}{p^2} + F(1,p) \frac{|\{n(p^2) : p \nmid n, p \| \pm n + h\}|}{p^2} \\ &\qquad + F(p,p) \frac{|\{n(p^2) : p \| n, p \| \pm n + h\}|}{p^2}\Bigg) \ll \prod_{\substack{p \mid h \\ p \geq w_1}} \left(1 + \frac{2^{m_1+m_2}}{p}\right) \end{aligned}$$

(note that the terms in Δ_D involving $F(p,1)$ and $F(1,p)$ vanish since $p \mid h$). Applying [10, Theorem 3] and then (19), we obtain

$$\begin{aligned} &\sum_{n \leq X} \tau(n)^{m_1} \mathbf{1}_{(n,P(w_1))=1} \tau(\pm n + h)^{m_2} \mathbf{1}_{(\pm n+h, P_h(w_2))=1} \\ &\ll \Delta_D X \prod_{p \leq X} \left(1 - \frac{\rho(p)}{p}\right) \sum_{\substack{n_1 n_2 \leq X \\ (n_1 n_2, D)=1}} F(n_1, n_2) \frac{\rho_{Q_1}(n) \rho_{Q_2}(n)}{n_1 n_2} \\ &\ll X \prod_{\substack{p \mid h \\ p \geq w_1}} \left(1 + \frac{2^{m_1+m_2}}{p}\right) \cdot \prod_{p \leq X} \left(1 - \frac{2}{p}\right) \prod_{p \mid h} \left(1 + \frac{1}{p}\right) \\ &\quad \cdot \prod_{\substack{w_1 \leq p \leq X \\ p \nmid h}} \left(1 + \frac{2^{m_1}}{p}\right) \prod_{\substack{w_2 \leq p \leq X \\ p \nmid h}} \left(1 + \frac{2^{m_2}}{p}\right). \end{aligned}$$

Here

$$\prod_{\substack{p\mid h\\ p\geq w_1}}\left(1+\frac{2^{m_1+m_2}}{p}\right)\ll\exp\left(2^{m_1+m_2}\sum_{\substack{p\mid h\\ p\geq w_1}}\frac{1}{p}\right)\ll\exp\left(2^{m_1+m_2}\sum_{p\leq 10\frac{\log X}{\log w_1}}\frac{1}{p}\right)$$
$$\ll_{m_1,m_2}\left(\log\frac{\log X}{\log w_1}\right)^{2^{m_1+m_2}}\ll_{m_1,m_2}\frac{\log X}{\log w_1},$$

and (i) follows from (20).

Proof of (ii). We use a similar application of Henriot's bound [10, Theorem 3]—this time

$$Q_1(n)=n,\ Q_2(n)=\pm mn+h,\ D=h^2,\ F(n_1,n_2)=\mathbf{1}_{(n_1,P(w_1))=1}\mathbf{1}_{(n_2,P(w_2))=1}\tau(n_2)^2,$$
$$\rho_{Q_1}(n)=|\{v(n)\colon v\equiv 0(n)\}|=1$$
$$\rho_{Q_2}(n)=|\{v\pmod n\colon \pm vm+h\equiv 0(n)\}|=\begin{cases}1 & \text{if } (n,m)=1\\ 0 & \text{otherwise.}\end{cases}$$
$$\rho(p)=|\{v\pmod p\colon v(\pm vm+h)\equiv 0(p)\}|=\begin{cases}2 & \text{if } p\nmid hm\\ 1 & \text{otherwise,}\end{cases}$$

and

$$\Delta_D=\prod_{p\mid h}\Bigg(1+F(p,1)\frac{|\{n(p^2)\colon p\| n,p\nmid\pm mn+h\}|}{p^2}+F(1,p)\frac{|\{n(p^2)\colon p\nmid n,p\|\pm mn+h\}|}{p^2}$$
$$+F(p,p)\frac{|\{n(p^2)\colon p\| n,p\|\pm mn+h\}|}{p^2}\Bigg)\ll\prod_{\substack{p\mid h\\ p\geq w_2}}\left(1+\frac{4}{p}\right)$$

(where again the terms in Δ_D involving $F(1,p)$ and $F(p,1)$ vanish since $p\mid h$). Applying [10, Theorem 3] and then (19), we obtain

$$\sum_{n\leq X}\mathbf{1}_{(n,P(w_1))=1}\mathbf{1}_{(\pm mn+h,P(w_2))=1}\tau(\pm mn+h)^2$$
$$\ll\Delta_D X\prod_{p\leq X}\left(1-\frac{\rho(p)}{p}\right)\sum_{\substack{n_1n_2\leq X\\ (n_1n_2,D)=1}}F(n_1,n_2)\frac{\rho_{Q_1}(n)\rho_{Q_2}(n)}{n_1n_2}$$
$$\ll X\prod_{\substack{p\mid h\\ p\geq w_2}}\left(1+\frac{4}{p}\right)\cdot\prod_{p\leq X}\left(1-\frac{2}{p}\right)\prod_{\substack{p\leq X\\ p\mid hm}}\left(1+\frac{1}{p}\right)\cdot\prod_{\substack{w_1\leq p\leq X\\ p\nmid h}}\left(1+\frac{1}{p}\right)\prod_{\substack{w_2\leq p\leq X\\ p\nmid hm}}\left(1+\frac{2^2}{p}\right).$$

Since $(m, hP(w_2)) = 1$, this is

$$\ll \frac{h}{\varphi(h)} \frac{X}{\log^2 X} \frac{\log X}{\log w_1} \left(\frac{\log X}{\log w_2}\right)^4 \prod_{\substack{w_2 \le p \le X \\ p \mid m}} \left(1 + \frac{1}{p} - \frac{4}{p}\right)$$

and the claim (ii) follows. ■

Now Lemmas 2.1 and 2.4 follow quickly:

Proof of Lemma 2.1. Consider

$$\sum_{n \le 4X} c_n \tau(\pm n + h)^2 \mathbf{1}_{(\pm n + h, P(z)) = 1}. \tag{21}$$

Using $c_n \le \mathbf{1}_{(n,P(z))=1} \tau(n)^2 \log n$, we see that those n with $(n, h) > 1$ contribute

$$\ll \log X \sum_{\substack{n \le 4X \\ (n,h) > 1}} \tau(n)^2 \mathbf{1}_{(n,P(z))=1} \tau(\pm n + h)^2 \mathbf{1}_{(\pm n + h, P(z)) = 1}$$

$$\ll \log X \sum_{\substack{p \mid h \\ z \le p \le 4X}} \sum_{n \le 4X/p} \tau(n)^2 \mathbf{1}_{(n,P(z))=1} \tau((\pm n + h)/p)^2 \mathbf{1}_{((\pm n + h)/p, P(z)) = 1}.$$

By Lemma 3.1(i), this is

$$\ll \frac{h}{\varphi(h)} X \log X \sum_{\substack{p \mid h \\ z \le p \le 4X}} \frac{u^9}{p \log^2(8X/p)} \ll \frac{h}{\varphi(h)} \frac{X}{\log X} \frac{u^{10}}{z}$$

since $z = X^{1/u} \le X^{1/2}$. On the other hand, those n with $(n, h) = 1$ contribute to (21) by (12)

$$\ll \sum_{\substack{z \le m \le 4X/z \\ (m, hP(z)) = 1}} \lambda(m) \sum_{\substack{z \le \ell \le 4X/m \\ (\pm \ell m + h, P(z)) = 1}} \Lambda(\ell) \tau(\pm \ell m + h)^2$$

$$\ll \log X \sum_{\substack{z \le m \le 4X/z \\ (m, hP(z)) = 1}} \lambda(m) \sum_{\substack{z \le \ell \le 4X/m \\ (\pm \ell m + h, P(z)) = 1}} \mathbf{1}_{(\ell, P(X^{1/4})) = 1} \tau(\pm \ell m + h)^2.$$

The claim follows from applying Lemma 3.1(ii) to the inner sum. ■

Proof of Lemma 2.4. We have, for any $r_1, r_2, r_3 \geq 0$,

$$\sum_{\substack{m_1,m_2,n_1,n_2\\ m_1n_1\leq 10X\\ \pm m_1n_1+h=m_2n_2\\ (n_1,P(z_{r_1}))=(n_2,P_h(z_{r_2}))=1\\ (m_1,P(z_{r_3}))=(m_2,hP(z))=1}} (\tau(m_1)\tau(n_1)\tau(n_2))^{A+1}$$

$$\ll \sum_{n\leq 10X} \tau(n)^{2A+3}\tau(\pm n+h)^{A+2}\mathbf{1}_{(n,P(\min\{z_{r_1},z_{r_3}\}))=1}\mathbf{1}_{(\pm n+h,P_h(z_{r_2}))=1}.$$

By Lemma 3.1(i) and recalling the definition of z_r from (14), the left-hand side of (15) is

$$\ll_A \frac{h}{\varphi(h)}\frac{X}{\log^2 X}\sum_{\substack{r_1,r_2,r_3\geq 0\\ \max r_j\geq \frac{u}{1000}-\beta}} 2^{-A(r_1+r_2+r_3)}\left(\frac{\log X}{\log\min\{z_{r_1},z_{r_3}\}}\right)^{2^{2A+3}+1}\left(\frac{\log X}{\log z_{r_2}}\right)^{2^{A+2}}$$

$$\ll \frac{h}{\varphi(h)}\frac{X}{\log^2 X}u^{2^{2A+5}}\sum_{\substack{r_1,r_2,r_3\geq 0\\ \max r_j\geq \frac{u}{1000}-\beta}} 2^{-A(r_1+r_2+r_3)}\left(\frac{\beta}{\beta-1}\right)^{2^{2A+4}(r_1+r_2+r_3)}.$$

Once β is large enough in terms of A, this is

$$\ll_A \frac{h}{\varphi(h)}\frac{X}{\log^2 X}u^{2^{2A+5}}2^{-0.9Au/1000} \ll_A \frac{h}{\varphi(h)}\frac{X}{\log^2 X}e^{-Au/2000}.$$

■

3.2 Sieves

The following is a technical version of the fundamental lemma of the sieve. For the standard version, see for example, [2, Lemma 6.11].

Lemma 3.2. Let $\theta \in (0,1/3)$ and $A \in \mathbb{N}$. There exists $\beta_1 = \beta_1(A) \geq 2$ such that the following holds for any $\beta \geq \beta_1$ and $u \geq \beta/\theta$.

Let $X \geq 2$, write $D = X^{\theta}$ and $z = X^{1/u}$, and define

$$z_r := z^{((\beta-1)/\beta)^r}.$$

Then there exist coefficients λ_d with $|\lambda_d| \leq 1$ supported on $d \leq D$ such that the following hold.

(i) For every $n \in \mathbb{N}$ and any $v \in \mathbb{N}$,

$$\mathbf{1}_{(n,P_v(z))=1} = \sum_{d|(n,P_v(z))} \lambda_d + O\left(\sum_{r \geq u\theta - \beta} 2^{-Ar} \mathbf{1}_{(n,P_v(z_r))=1} \tau(n)^{A+1} \right).$$

(ii) If $g\colon \mathbb{N} \to [-1,1]$ is a multiplicative function for which $|g(p)| \leq 2/p$ for every prime p, then

$$\sum_{d|P(z)} \lambda_d g(d) = (1 + O_{\beta,A}(e^{-A\theta u/2})) \prod_{p<z} (1 - g(p)).$$

Proof. Define

$$\mathcal{D} := \{d = p_1 \cdots p_r \mid P(z) \colon p_1 > p_2 > \ldots > p_r, p_1 \cdots p_m p_m^{\beta} < D \text{ for all odd } m\}$$

and define the upper bound β-sieve weights $\lambda_d = \mu(d) \mathbf{1}_{d \in \mathcal{D}}$.

Proof of (i). Consider first the case $v = 1$. By the definition of λ_d, we have (see e.g., [2, (6.29) with $\mathcal{A} = \{n\}$]),

$$\mathbf{1}_{(n,P(z))=1} = \sum_{d|(n,P(z))} \lambda_d - \sum_{r \text{ odd}} S_r(n),$$

where

$$S_r(n) := \sum_{\substack{n = p_1 \cdots p_r k \\ p_r < p_{r-1} < \ldots < p_1 < z \\ p_1 p_2 \cdots p_r p_r^{\beta} \geq D \\ p_1 \cdots p_h p_h^{\beta} < D \text{ for all odd } h < r}} \mathbf{1}_{(k,P(p_r))=1}.$$

Since $p_j < z$ and $p_1 p_2 \cdots p_r p_r^{\beta} \geq D$, the sum in $S_r(n)$ is non-empty only if $(r + \beta)/u \geq \theta \iff r \geq u\theta - \beta$.

Furthermore, since $\frac{\log D}{\log z} = u\theta \geq \beta$, one can easily show that, for every r, in the sum defining $S_r(n)$ one has $p_r \geq z_r$ (see e.g., [12, Section 6.3]). We write $m = p_1 \cdots p_r$ so that obviously $2^{A(\omega(m)-r)} \geq 1$. Hence

$$S_r(n) \leq \sum_{\substack{n = mk \\ p|mk \implies p \geq z_r}} 2^{A\omega(m) - Ar} \leq 2^{-Ar} \tau(n)^{A+1} \mathbf{1}_{(n,P(z_r))=1},$$

and (i) follows in case $v = 1$.

When $v > 1$, we write $n = n'v'$ with $(n', v) = 1$ and $p \mid v' \implies p \mid v$. Then $(n, P_v(w)) = (n', P(w))$ for every $w \geq 2$. Hence, by the case $v = 1$, we obtain

$$\begin{aligned}\mathbf{1}_{(n,P_v(z))=1} = \mathbf{1}_{(n',P(z))=1} &= \sum_{d\mid(n',P(z))} \lambda_d + O\left(\sum_{r\geq u\theta-\beta} 2^{-Ar}\mathbf{1}_{(n',P(z_r))=1}\tau(n')^{A+1}\right)\\ &= \sum_{d\mid(n,P_v(z))} \lambda_d + O\left(\sum_{r\geq u\theta-\beta} 2^{-Ar}\mathbf{1}_{(n,P_v(z_r))=1}\tau(n)^{A+1}\right)\end{aligned}$$

as claimed.

Proof of (ii). By the definition of λ_d, we have (see e.g., [2, (6.31)])

$$\sum_{d\mid P(z)} \lambda_d g(d) = \prod_{p<z}(1-g(p)) - \sum_{r \text{ odd}} V_r(z), \tag{22}$$

where

$$V_r(z) := \sum_{\substack{p_r<p_{r-1}<\ldots<p_1<z\\ p_1p_2\cdots p_rp_r^{\beta}\geq D\\ p_1\cdots p_hp_h^{\beta}<D \text{ for all odd } h<r}} g(p_1\cdots p_r)\prod_{p<p_r}(1-g(p)).$$

As in (i), only $r \geq u\theta - \beta$ contribute and $p_r \geq z_r$. Hence, writing $m = p_1 \cdots p_r$ and using again $1 \leq 2^{A(\omega(m)-r)}$, we obtain

$$V_r(z) \leq \prod_{p<z}(1-g(p))\prod_{z_r\leq p\leq z}(1+|g(p)|)\sum_{\substack{m\\ p\mid m \implies z_r\leq p<z}} 2^{A(\omega(m)-r)}|\mu(m)|g(m).$$

Since $|g(p)| \leq 2/p$, we get, using (19) and (20),

$$\begin{aligned}V_r(z) &\leq 2^{-Ar}\prod_{p<z}(1-g(p))\prod_{z_r\leq p\leq z}\left(1+\frac{2}{p}\right)\left(1+\frac{2^{A+1}}{p}\right)\\ &\ll 2^{-Ar}\left(\frac{\log z}{\log z_r}\right)^{2^{A+1}+2}\prod_{p<z}(1-g(p))\\ &\ll 2^{-Ar}\left(\frac{\beta}{\beta-1}\right)^{(2^{A+1}+2)r}\prod_{p<z}(1-g(p)).\end{aligned}$$

Now, once β is large enough in terms of A,

$$\sum_{r\geq u\theta-\beta} 2^{-Ar}\left(\frac{\beta}{\beta-1}\right)^{(2^{A+1}+2)r} \ll_{\beta,A} 2^{-0.9Au\theta} \leq e^{-Au\theta/2}$$

and the claim follows. ■

Lemma 3.3. Let the assumptions and λ_d be as in Lemma 3.2 and let $v, q \in \mathbb{N}$. Then

$$\sum_{\substack{d_1,d_2|P(z)\\(d_2,d_1v)=1\\(d_1d_2,q)=1}} \frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2} = \left(1+O_{\beta,A}\left(e^{-Au\theta/2}\right)\right)\prod_{\substack{p<z\\p\nmid q}}\left(1-\frac{1}{p}\right)\prod_{\substack{p<z\\p\nmid vq}}\left(1-\frac{1}{\varphi(p)}\right). \tag{23}$$

Proof. We have to be somewhat careful here due to the condition $(d_1, d_2) = 1$ (see also [2, Section 5.9]).

By (22) with $g(d) = \frac{1_{(d,d_2q)=1}}{d}$, we can write

$$\sum_{\substack{d_1|P(z)\\(d_1,d_2q)=1}} \frac{\lambda_{d_1}}{d_1} = \prod_{\substack{p<z\\p\nmid d_2q}}\left(1-\frac{1}{p}\right) - \sum_{\substack{r\geq u\theta-\beta\\r\text{ odd}}} \sum_{\substack{p_r<p_{r-1}<\ldots<p_1<z\\p_1p_2\cdots p_rp_r^{\beta}\geq D\\p_1\cdots p_hp_h^{\beta}<D\text{ for all odd }h<r\\p_j\nmid d_2q}} \frac{1}{p_1\cdots p_r}\prod_{\substack{p<p_r\\p\nmid d_2q}}\left(1-\frac{1}{p}\right). \tag{24}$$

Using Lemma 3.2(ii) we see that the first term on the right-hand side contributes to the left-hand side of (23)

$$\begin{aligned}\sum_{\substack{d_2|P(z)\\(d_2,vq)=1}} \frac{\lambda_{d_2}}{d_2}\prod_{\substack{p<z\\p\nmid d_2q}}\left(1-\frac{1}{p}\right) &= \prod_{\substack{p<z\\p\nmid q}}\left(1-\frac{1}{p}\right)\sum_{\substack{d_2|P(z)\\(d_2,vq)=1}}\frac{\lambda_{d_2}}{\varphi(d_2)}\\ &= \prod_{\substack{p<z\\p\nmid q}}\left(1-\frac{1}{p}\right)(1+O_{\beta,A}(e^{-Au\theta/2}))\prod_{\substack{p<z\\p\nmid vq}}\left(1-\frac{1}{\varphi(p)}\right).\end{aligned}$$

Let us now consider the contribution of the r-sum in (24) to the left-hand side of (23). Writing $m = p_1\cdots p_{r-1}$ so that $2^{A(\omega(m)-r)} \geq 1/2^A$ and recalling from the proof of

Lemma 3.2 that $p_r \ge z_r$, this contribution is

$$\ll_A \sum_{r\ge u\theta-\beta} 2^{-Ar} \sum_{\substack{z_r\le p_r<z\\ p_r\nmid q}} \frac{1}{p_r} \sum_{\substack{m\le D/p_r\\ (m,q)=1\\ p|m \implies p_r<p<z}} \frac{2^{A\omega(m)}}{m} \prod_{\substack{p<p_r\\ p\nmid q}}\left(1-\frac{1}{p}\right)$$
$$\cdot \left| \sum_{\substack{d_2|P(z)\\ (d_2,vqmp_r)=1}} \frac{\lambda_{d_2}}{d_2} \prod_{\substack{p<p_r\\ p|d_2}}\left(1-\frac{1}{p}\right)^{-1}\right|.$$

Applying Lemma 3.2(ii) and recombining the variables m and $p_r \ge z_r$, and applying then (19) and (20), this is

$$\ll \sum_{r\ge u\theta-\beta} 2^{-Ar} \sum_{\substack{m\le D\\ (m,q)=1\\ p|m \implies z_r\le p<z}} \frac{2^{A\omega(m)}}{\varphi(m)} \prod_{\substack{p<z_r\\ p\nmid q}}\left(1-\frac{1}{p}\right) \prod_{\substack{p<z\\ p\nmid vq}}\left(1-\frac{1}{\varphi(p)}\right)$$
$$\ll \prod_{\substack{p<z\\ p\nmid vq}}\left(1-\frac{1}{\varphi(p)}\right) \prod_{\substack{p<z\\ p\nmid q}}\left(1-\frac{1}{p}\right) \sum_{r\ge u\theta-\beta} 2^{-Ar} \prod_{\substack{z_r\le p<z\\ p\nmid q}}\left(1+\frac{2^A+1}{p}\right)$$
$$\ll \prod_{\substack{p<z\\ p\nmid vq}}\left(1-\frac{1}{\varphi(p)}\right) \prod_{\substack{p<z\\ p\nmid q}}\left(1-\frac{1}{p}\right) \sum_{r\ge u\theta-\beta} 2^{-Ar} \left(\frac{\beta}{\beta-1}\right)^{(2^A+1)r}$$
$$\ll_{\beta,A} e^{-Au\theta/2} \prod_{\substack{p<z\\ p\nmid vq}}\left(1-\frac{1}{\varphi(p)}\right) \prod_{\substack{p<z\\ p\nmid q}}\left(1-\frac{1}{p}\right)$$

once β is large enough. ■

3.3 Poisson summation

Let us state the version of the Poisson summation formula we shall use.

Lemma 3.4. Let $b \in \mathbb{Z}$, $d, q \in \mathbb{N}$ with $(d, bq) = 1$, let $f\colon \mathbb{R} \to \mathbb{C}$ be such that $f, \widehat{f} \in L^1(\mathbb{R})$ and have bounded variation. Let $g\colon \mathbb{R} \to \mathbb{C}$ be q-periodic. Then

$$\sum_{m\equiv b(d)} f(m)g(m) = \frac{1}{dq}\sum_{h} \widehat{f}\left(\frac{h}{dq}\right) \sum_{\substack{m(dq)\\ m\equiv b(d)}} g(m)e\left(\frac{hm}{dq}\right).$$

Proof. Write

$$\sum_{m\equiv b(d)} f(m)g(m) = \sum_{a(q)} g(a) \sum_{\substack{m\equiv b(d)\\ m\equiv a(q)}} f(m) = \sum_{a(q)} g(a) \sum_{k} f(c_a + dqk),$$

where c_a is such that $c_a \equiv b(d)$ and $c_a \equiv a(q)$. By Poisson summation (see e.g., [12, formula (4.24)], the above equals

$$\sum_{a(q)} g(a) \frac{1}{dq} \sum_{h} \widehat{f}\left(\frac{h}{dq}\right) e\left(\frac{hc_a}{dq}\right) = \frac{1}{dq} \sum_{h} \widehat{f}\left(\frac{h}{dq}\right) \sum_{a(q)} g(a) e\left(\frac{hc_a}{dq}\right)$$
$$= \frac{1}{dq} \sum_{h} \widehat{f}\left(\frac{h}{dq}\right) \sum_{\substack{c(dq)\\ c\equiv b(d)}} g(c) e\left(\frac{hc}{dq}\right).$$

■

3.4 Character sums

Proof of Lemma 2.6. Since χ is a primitive quadratic character of modulus $q = 2^r q'$ with $2 \nmid q'$, we have that $r \in \{0, 2, 3\}$ and q' is square-free (see e.g., [12, Section 3.3]). Hence, by the Chinese reminder theorem, it suffices to consider the prime case $q = p > 2$ and the case $q = 2^r$ with $r \in \{2, 3\}$.

Proof of (16). For $p > 2$,

$$\sum_{m(p)} \chi_0(m)\chi_0(\pm m + h) = \begin{cases} p - 1 = \varphi(p) & \text{if } p \mid h; \\ p - 2 & \text{if } p \nmid h. \end{cases}$$

and, for $r \in \{2, 3\}$ and even h,

$$\sum_{m(2^r)} \chi_0(m)\chi_0(\pm m + h) = \sum_{\substack{m \pmod{2^r}\\ (m,2)=1}} 1 = 2^{r-1} = \varphi(2^r).$$

Combining these, (16) follows.

Proof of (17). For $p > 2$,

$$\sum_{m(p)} \chi(m)\chi_0(\pm m + h) = \sum_{m(p)} \chi(m) - \chi(\mp h) = -\chi(\mp h),$$

and, for $r \in \{2, 3\}$ and even h,

$$\sum_{m(2^r)} \chi(m)\chi_0(\pm m + h) = \sum_{m(2^r)} \chi(m) = 0 = -\chi(\mp h).$$

Combining these, (17) follows.

Proof of (18). For $p > 2$,

$$\sum_{m(p)} \chi(m(\pm m + h)) = \sum_{\substack{m(p)\\(m,p)=1}} \chi(m(\pm m + h)) = \chi(\pm 1) \sum_{\substack{m(p)\\(m,p)=1}} \chi(m)^2 \chi(1 \pm h\overline{m})$$

$$= \chi(\pm 1) \sum_{\substack{m(p)\\(m,p)=1}} \chi(1 \pm hm) = \chi(\pm 1) \left(\sum_{m(p)} \chi(1 \pm hm) - \chi(1) \right) = \chi(\pm 1) \begin{cases} p - 1 & \text{if } p \mid h; \\ -1 & \text{if } p \nmid h. \end{cases}$$

Similarly, for $r = 2, 3$ and even h,

$$\sum_{m(2^r)} \chi(m(\pm m + h)) = \sum_{\substack{m(2^r)\\(m,2)=1}} \chi(m(\pm m + h)) = \chi(\pm 1) \sum_{\substack{m(2^r)\\(m,2)=1}} \chi(m)^2 \chi(1 \pm h\overline{m})$$

$$= \chi(\pm 1) \sum_{\substack{m(2^r)\\(m,2)=1}} \chi(1 \pm hm) = \chi(\pm 1) \left(\sum_{m(2^r)} \chi(1 \pm hm) - \frac{1}{2} \sum_{m(2^r)} \chi(1 \pm 2hm) \right).$$

It is easy to see that, for $r = 2, 3$ and even h, the expression in the parentheses equals

$$2^r \mathbf{1}_{2^r \mid h} - 2^{r-1} \mathbf{1}_{2^{r-1} \mid h} = 2^{r-1} \mathbf{1}_{2^{r-1} \mid h} (-1)^{\frac{h}{2^{r-1}}} = \varphi(2^r) \mathbf{1}_{\varphi(2^r) \mid h} (-1)^{\frac{h}{\varphi(2^r)}},$$

where the last formulation is such that it is 1 for $r = 0$ when h is even. Combining these, (17) follows. ■

3.5 Kloosterman sums

For integers a, b, c with $c \geq 1$, we denote the Kloosterman sum by

$$S(a, b; c) := \sum_{\substack{n\,(c)\\(n,c)=1}} e_c(an + b\bar{n}).$$

We shall use the Weil bound (see e.g., [12, Corollary 11.12]).

Lemma 3.5. Let $a, b \in \mathbb{Z}$ and $c \in \mathbb{N}$. Then, for any $\varepsilon > 0$,

$$S(a, b; c) \ll_\varepsilon c^{1/2+\varepsilon}(a, b, c)^{1/2}.$$

Remark 3.6. We could alternatively use a more elementary bound $S(a, b; c) \ll_\varepsilon c^{3/4+\varepsilon}(a, b, c)^{1/4}$ due to Kloosterman (see [15, Lemma 4] with $\Lambda = 1$), and this would only somewhat increase the exponent of q in the conditions of the type $X \geq q^{10}$ and $h \geq q^{10}$ in our theorems and corollaries.

We also need the following rough bounds.

Lemma 3.7. Let $L \geq 1$. For any integer $q \neq 0$, we have

$$\sum_{1 \leq \ell \leq L} (\ell, q) \leq \tau(q) L$$

and

$$\sum_{1 \leq \ell \leq L} (\ell, q) \frac{\ell}{\varphi(\ell)} \ll \tau(q)^2 L.$$

Proof. Writing $d = (\ell, q)$ and $\ell = kd$, we have

$$\sum_{1 \leq \ell \leq L} (\ell, q) \leq \sum_{d | q} d \sum_{1 \leq k \leq \frac{L}{d}} 1 \leq L \sum_{d | q} 1 = \tau(q) L.$$

Similarly, using also $\varphi(dk) \geq \varphi(d)\varphi(k)$ and $\sum_{k \leq K} \frac{k}{\varphi(k)} \ll K$, we get

$$\sum_{1 \leq \ell \leq L} (\ell, q) \frac{\ell}{\varphi(\ell)} \leq \sum_{d | q} \frac{d^2}{\varphi(d)} \sum_{1 \leq k \leq L/d} \frac{k}{\varphi(k)} \ll \sum_{d | q} \frac{Ld}{\varphi(d)} = L \prod_{p | q} \left(1 + \frac{p}{\varphi(p)}\right) \leq \tau(q)^2 L.$$

■

From Lemma 3.5, we obtain the following bound for incomplete Kloosterman sums via Poisson summation.

Lemma 3.8. Let $\delta, \varepsilon > 0$, and $N \geq 1$. Let $F : \mathbb{R} \to \mathbb{C}$ be a bounded smooth compactly supported function and suppose that for some $\delta \in (0, 1)$ we have $|F''| \ll \delta^{-2}$. Then, for any integers α, d, e, q, k with $d, q \geq 1$ and $(eq, d) = 1$, we have

$$\sum_{\substack{n\equiv\alpha\ (q)\\(n,d)=1}} F\left(\frac{n}{N}\right) e_d(k\overline{en}) \ll_\varepsilon \delta^{-1} d^{1/2+\varepsilon} + \frac{N(d,k)}{dq}.$$

Proof. Applying Poisson summation (Lemma 3.4 with d-periodic $g(n) = \mathbf{1}_{(n,d)=1} e_d(k\overline{en})$), we obtain

$$\sum_{\substack{n\equiv\alpha\ (q)\\(n,d)=1}} F\left(\frac{n}{N}\right) e_d(k\overline{en}) = \frac{N}{dq}\sum_h \widehat{F}\left(\frac{hN}{dq}\right) \sum_{\substack{n\ (dq)\\ n\equiv\alpha\ (q)\\(n,d)=1}} e_d(k\overline{en}) e_{dq}(hn).$$

Since $(d,q) = 1$, by the Chinese remainder theorem, we can write $n = \alpha d\bar{d} + \beta q\bar{q}$, where $\bar{d}$ is inverse $\pmod q$ and $\bar{q}$ is inverse $\pmod d$ to get

$$\begin{aligned}\sum_{\substack{n\equiv\alpha\ (q)\\(n,d)=1}} F\left(\frac{n}{N}\right) e_d(k\overline{en}) &= \frac{N}{dq}\sum_h \widehat{F}\left(\frac{hN}{dq}\right) e_q(h\alpha\bar{d}) \sum_{\substack{\beta\ (d)\\(\beta,d)=1}} e_d(h\bar{q}\beta + k\overline{e\beta}) \\ &= \frac{N}{dq}\sum_h \widehat{F}\left(\frac{hN}{dq}\right) e_q(h\alpha\bar{d}) S(h\bar{q}, k\bar{e}; d).\end{aligned} \tag{25}$$

For $h = 0$, we have a Ramanujan sum (see e.g., [12, (3.1) and (3.5)])

$$S(0, k\bar{e}; d) = \sum_{\substack{n\ (d)\\(n,d)=1}} e_d(k\overline{en}) \ll (d,k).$$

The contribution from this to (25) is

$$\ll \frac{N(d,k)}{dq}|\widehat{F}(0)| \ll \frac{N(d,k)}{dq}.$$

For $h \neq 0$, we get by integration by parts

$$\widehat{F}\left(\frac{hN}{dq}\right) \ll \min\left\{1, \left|\delta\frac{hN}{dq}\right|^{-2}\right\}.$$

Hence, by Lemmas 3.5 and 3.7, we get

$$\frac{N}{dq}\sum_{h\neq 0}\widehat{F}\left(\frac{hN}{dq}\right)e_q(h\alpha\bar{d})S(h\bar{q},k\bar{e};d) \ll_\varepsilon d^{1/2+\varepsilon/2}\frac{N}{dq}\sum_{h\neq 0}(h,d)\min\left\{1,\left|\delta\frac{hN}{dq}\right|^{-2}\right\}$$

$$\ll_\varepsilon \delta^{-1}d^{1/2+\varepsilon}.$$

■

4 Proof of Lemma 2.2

The following lemma is the same as [20, Proposition 3.5], but we provide the proof also here for completeness.

Lemma 4.1. Let χ be a primitive quadratic character modulo $q \geq 2$. Assume that $L(s,\chi)$ has a real zero β_0 such that

$$\beta_0 = 1 - \frac{1}{\eta \log q}$$

for some $\eta \geq 10$. Let $\delta > 0$. Then, for any $Y > q^{1/2+\delta}$, one has

$$\sum_{q^{1/2+\delta} < p \leq Y} \frac{\lambda(p)}{p} \ll_\delta \frac{\log Y}{\eta \log q},$$

and, for any $k \geq 2$, one has

$$\sum_{q^{(1/2+\delta)/k} < p \leq q^{(1/2+\delta)/(k-1)}} \frac{\lambda(p)}{p} \ll_\delta \frac{k}{\eta^{1/k}}.$$

Proof. We may assume that η is large since otherwise the claims are trivial. By [18, Exercise 3(g) in Section 11.2.1], we have for any $y \geq q^{1/2+\delta}$

$$\sum_{m\leq y}\frac{\lambda(m)}{m} = L(1,\chi)(\log y + \gamma) + L'(1,\chi) + O_\delta(q^{-\delta/3}),$$

where γ is the Euler–Mascheroni constant (see [18, formula (1.27)]). Using Siegel's bound $L(1,\chi) \gg_\delta q^{-\delta/4}$ (see e.g., [18, Theorem 11.14]), we get

$$\sum_{m\leq y}\frac{\lambda(m)}{m} = L(1,\chi)\left(\log y + \frac{L'(1,\chi)}{L(1,\chi)} + O_\delta(1)\right). \tag{26}$$

By [18, Theorem 11.4], we have

$$\frac{L'(1,\chi)}{L(1,\chi)} = \eta \log q + O(\log q) \asymp \eta \log q$$

as we assumed that η is large. Plugging this into (26) with $y = q^{1/2+\delta}$, we obtain

$$\sum_{m \le q^{1/2+\delta}} \frac{\lambda(m)}{m} \gg_\delta \eta L(1,\chi) \log q. \tag{27}$$

Also, subtracting (26) for $y = q^{1/2+\delta}$ and $y = Yq^{1/2+\delta}$ gives

$$\sum_{q^{1/2+\delta} < m \le Yq^{1/2+\delta}} \frac{\lambda(m)}{m} = L(1,\chi)(\log Y + O_\delta(1)). \tag{28}$$

On the other hand, by non-negativity and multiplicativity of $\lambda(n)$, we get

$$\sum_{q^{1/2+\delta} < m \le Yq^{1/2+\delta}} \frac{\lambda(m)}{m} \ge \sum_{\substack{q^{1/2+\delta} < m \le Yq^{1/2+\delta} \\ m = pn,\, n \le q^{1/2+\delta} < p}} \frac{\lambda(m)}{m} \ge \sum_{n \le q^{1/2+\delta}} \frac{\lambda(n)}{n} \sum_{q^{1/2+\delta} < p \le Y} \frac{\lambda(p)}{p}.$$

The first bound in the lemma now follows by (28) and (27).

Similarly, for the second bound, we write $m = np_1 \cdots p_k$ to get

$$\sum_{q^{1/2+\delta} < m \le q^3} \frac{\lambda(m)}{m} \ge \sum_{\substack{q^{1/2+\delta} < m \le q^3 \\ m = p_1 \cdots p_k n,\, n \le q^{1/2+\delta} \\ q^{(1/2+\delta)/k} < p_1, \ldots, p_k \le q^{(1/2+\delta)/(k-1)} \\ p_1, \ldots, p_k \nmid n,\, p_j \text{ distinct}}} \frac{\lambda(m)}{m}. \tag{29}$$

Now, for $m \le q^3$, we have

$$\sum_{\substack{p_1 \cdots p_k \mid m \\ q^{(1/2+\delta)/k} < p_1, \ldots, p_k \le q^{(1/2+\delta)/(k-1)}}} 1 \le k! \binom{6k}{k} \le k!\,(1+1)^{6k} = k!\,2^{6k}.$$

Using this and (29), we see that

$$\sum_{q^{1/2+\delta} < m \le q^3} \frac{\lambda(m)}{m} \gg \frac{1}{k!\,2^{6k}} \sum_{n \le q^{1/2+\delta}} \frac{\lambda(n)}{n} \sum_{\substack{q^{(1/2+\delta)/k} < p_1, \ldots, p_k \le q^{(1/2+\delta)/(k-1)} \\ p_1, \ldots, p_k \nmid n \\ p_j \text{ distinct}}} \frac{\lambda(p_1) \cdots \lambda(p_k)}{p_1 \cdots p_k}.$$

Once we have fixed $p_1, \dots, p_{j-1}$, to make sure that p_j is distinct from $p_1, \dots, p_{j-1}$ we have to exclude $j-1 \le k$ primes. To ensure that also $p_j \nmid n$, we have to further remove at most k primes. Let $\mathcal{P}$ be the set of $2k$ smallest primes $> q^{(1/2+\delta)/k}$ such that $\lambda(p) = 2$. Then by positivity we get

$$\sum_{q^{1/2+\delta} < m \le q^3} \frac{\lambda(m)}{m} \gg \frac{1}{k!\, 2^{6k}} \sum_{n \le q^{1/2+\delta}} \frac{\lambda(n)}{n} \Bigg(\sum_{\substack{q^{(1/2+\delta)/k} < p \le q^{(1/2+\delta)/(k-1)} \\ p \notin \mathcal{P}}} \frac{\lambda(p)}{p} \Bigg)^k.$$

Using (28) and (27), we get

$$\Bigg(\sum_{\substack{q^{(1/2+\delta)/k} < p \le q^{(1/2+\delta)/(k-1)} \\ p \notin \mathcal{P}}} \frac{\lambda(p)}{p} \Bigg)^k \ll_\delta \frac{2^{6k} k!}{\eta},$$

which gives us

$$\sum_{\substack{q^{(1/2+\delta)/k} < p \le q^{(1/2+\delta)/(k-1)} \\ p \notin \mathcal{P}}} \frac{\lambda(p)}{p} \ll \frac{k}{\eta^{1/k}}.$$

The primes $p \in \mathcal{P}$ can be inserted back in with an error term $O(4kq^{-(1/2+\delta)/k})$. By Siegel's bound (6), this error is $O(k\eta^{-1/k})$, and the second claim follows. ■

Proof of Lemma 2.2. First, note that we can assume that

$$\frac{1}{v^2 \eta^{v/2}} + \frac{v}{\eta} \frac{\log Y}{\log z} \le 1 \tag{30}$$

since otherwise the claim follows trivially from $\lambda(m) \le \tau(m)$, (19), and (20).

We write

$$\sum_{\substack{z \le m \le Y \\ (m, P(z))=1}} \frac{\lambda(m)}{m} = \sum_{\substack{z \le m \le Y \\ (m, P(z))=1}} \frac{|\mu(m)|\lambda(m)}{m} + \sum_{\substack{z \le m \le Y \\ (m, P(z))=1}} \frac{(1-|\mu(m)|)\lambda(m)}{m}. \tag{31}$$

For the second sum, we get by (19) and (20)

$$\sum_{\substack{z \le m \le Y \\ (m, P(z))=1}} \frac{(1-|\mu(m)|)\lambda(m)}{m} \ll \sum_{p \ge z} \frac{1}{p^2} \sum_{\substack{m \le Y/p^2 \\ (m, P(z))=1}} \frac{\tau(m)}{m} \ll \frac{1}{z} \left(\frac{\log Y}{\log z} \right)^2.$$

In the first sum on the right-hand side of (31), we have

$$\sum_{\substack{z\le m\le Y\\(m,P(z))=1}}\frac{|\mu(m)|\lambda(m)}{m}\le\prod_{z\le p\le Y}\left(1+\frac{\lambda(p)}{p}\right)-1\le\prod_{z\le p\le Y}\exp\left(\frac{\lambda(p)}{p}\right)-1$$
$$=\exp\left(\sum_{z\le p\le Y}\frac{\lambda(p)}{p}\right)-1. \tag{32}$$

Let $\delta>0$ be small and let

$$K:=\left\lceil\frac{1/2+\delta}{v}\right\rceil\ge\max\left\{1,\frac{1/2+\delta}{v}\right\},$$

so that $z=q^v\ge q^{\frac{1/2+\delta}{K}}$. It is easy to see that either $K\le 2/v$ or $K=1$. Then by Lemma 4.1

$$\sum_{z\le p\le Y}\frac{\lambda(p)}{p}\le\sum_{2\le k\le K}\sum_{q^{(1/2+\delta)/k}<p\le q^{(1/2+\delta)/(k-1)}}\frac{\lambda(p)}{p}+\sum_{q^{1/2+\delta}<p\le Y}\frac{\lambda(p)}{p}$$
$$\ll_\delta K^2\eta^{-1/K}+\frac{\log Y}{\eta\log q}\ll\frac{1}{v^2\eta^{v/2}}+\frac{v}{\eta}\frac{\log Y}{\log z}.$$

Plugging this in (32) and using (30), we obtain

$$\sum_{\substack{z\le m\le Y\\(m,P(z))=1}}\frac{|\mu(m)|\lambda(m)}{m}\le\exp\left(\sum_{z\le p\le Y}\frac{\lambda(p)}{p}\right)-1\ll\frac{1}{v^2\eta^{v/2}}+\frac{v}{\eta}\frac{\log Y}{\log z}$$

and the claim follows. ■

5 Proof of Proposition 2.3

Let us study

$$S^{\pm}:=\sum_{\substack{m_1,m_2,n_1,n_2\\ \pm m_1n_1+h=m_2n_2\\(m_1m_2n_1n_2,P(z))=1}}\chi_1(m_1)\chi_2(m_2)\psi_1(n_1)\psi_2(n_2)f\left(\frac{m_1}{M_1},\frac{m_2}{M_2},\frac{n_1}{N_1},\frac{n_2}{N_2}\right).$$

First we wish to make the variable n_2 implicit, so we write the above as

$$\sum_{\substack{m_1,m_2,n_1,n_2\\ \pm m_1n_1+h=m_2n_2\\(m_1m_2n_1n_2,P(z))=1}}\chi_1(m_1)\chi_2(m_2)\psi_1(n_1)\psi_2\left(\frac{\pm m_1n_1+h}{m_2}\right)f\left(\frac{m_1}{M_1},\frac{m_2}{M_2},\frac{n_1}{N_1},\frac{\pm m_1n_1+h}{m_2N_2}\right).$$

Let us now show that we can replace the conditions $(m_2, P(z)) = 1$ and $(n_2, P(z)) = 1$ by the conditions $(m_2, hP(z)) = 1$ and $(n_2, P_h(z)) = 1$. These two sets of conditions are equivalent unless $(m_2 n_2, h) > 1$. But $(m_2 n_2, h) \mid m_1 n_1$, so in this case there must be a prime $p \geq z$ such that $p \mid m_2 n_2$, $p \mid h$ and $p \mid m_1 n_1$. Using Lemma 3.1(i), we see that the error introduced from these changes of summation conditions is

$$\begin{aligned} &\ll \sum_{\substack{p \mid h \\ z \leq p \ll X}} \sum_{\substack{m \ll X/p \\ (\pm m + h/p, P_h(z))=1 \\ (m, P(z))=1}} \tau(m)\tau(\pm m + h/p) \ll \frac{h}{\varphi(h)} X u^5 \sum_{\substack{p \mid h \\ z \leq p \ll X}} \frac{1}{p \log^2(X/p)} \\ &\ll \frac{h}{\varphi(h)} \frac{X}{\log^2 X} \frac{u^6}{z}. \end{aligned}$$

Hence we can replace the condition $(m_2 n_2, P(z)) = 1$ by conditions $(m_2, hP(z)) = 1$ and $(n_2, P_h(z)) = 1$.

Next we shall replace the conditions $(m_1 n_1, P(z)) = 1$ and $(n_2, P_h(z)) = 1$ by sieve weights. By Lemma 3.2(i) with $\theta = 1/1000$, we can write

$$\mathbf{1}_{(n_2, P_h(z))=1} = \sum_{\substack{d_2 \mid (n_2, P(z)) \\ (d_2, h)=1}} \lambda_{d_2} + O\left(\sum_{r_2 \geq u/1000 - \beta} 2^{-Ar_2} \mathbf{1}_{(n_2, P_h(z_{r_2}))=1} \tau(n_2)^{A+1} \right)$$

$$\mathbf{1}_{(n_1, P(z))=1} = \sum_{d_1 \mid (n_1, P(z))} \lambda_{d_1} + O\left(\sum_{r_1 \geq u/1000 - \beta} 2^{-Ar_1} \mathbf{1}_{(n_1, P(z_{r_1}))=1} \tau(n_1)^{A+1} \right)$$

$$\mathbf{1}_{(m_1, P(z))=1} = \sum_{e \mid (m_1, P(z))} \lambda_e + O\left(\sum_{r_3 \geq u/1000 - \beta} 2^{-Ar_3} \mathbf{1}_{(m_1, P(z_{r_3}))=1} \tau(m_1)^{A+1} \right).$$

Using this, and noticing that for example, $\mathbf{1}_{(m_1, P(z))=1} \leq 2^{-A \cdot 0} \mathbf{1}_{(m_1, P(z_0))=1} \tau(m_1)^{A+1}$, we obtain

$$\begin{aligned} S^{\pm} = &\sum_{\substack{d_1 d_2 \mid P(z) \\ (d_2, h)=1}} \lambda_{d_1} \lambda_{d_2} \sum_{e \mid P(z)} \lambda_e \sum_{\substack{m_1, m_2 \\ (m_2, hP(z))=1}} \chi_1(em_1)\chi_2(m_2)\psi_2(m_2) W(d_1, d_2, em_1, m_2) \\ &+ O\left(\frac{h}{\varphi(h)} \frac{X}{\log^2 X} \frac{u^6}{z} \right) + O\Bigg(\sum_{\substack{r_1, r_2, r_3 \geq 0 \\ \max r_j \geq \frac{u}{1000} - \beta}} 2^{-A(r_1 + r_2 + r_3)} \\ &\cdot \sum_{\substack{m_1, m_2, n_1, n_2 \\ \pm m_1 n_1 + h = m_2 n_2 \\ (n_1, P(z_{r_1})) = (n_2, P_h(z_{r_2}))=1 \\ (m_1, P(z_{r_3})) = (m_2, hP(z))=1}} (\tau(m_1)\tau(n_1)\tau(n_2))^{A+1} f\left(\frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{n_1}{N_1}, \frac{\pm m_1 n_1 + h}{m_2 N_2} \right) \Bigg) \end{aligned}$$

with

$$W(d_1,d_2,em_1,m_2) := \sum_{\substack{n_1 \\ \pm em_1d_1n_1 \equiv -h (d_2m_2)}} \psi_1(d_1n_1)\psi_2(\pm em_1d_1n_1+h) \cdot f\left(\frac{em_1}{M_1}, \frac{m_2}{M_2}, \frac{d_1n_1}{N_1}, \frac{\pm em_1d_1n_1+h}{m_2N_2}\right).$$

Thanks to Lemma 2.4, we can concentrate on the main term. Note that since $(d_2m_2, h) = 1$, we can add the condition $(em_1d_1, d_2m_2) = 1$. Due to the support of χ and ψ (noting that $d_2m_2 \mid \pm em_1d_1n_1 + h$), we can also add conditions $(d_1d_2em_1m_2, q) = 1$. Hence, we need to study

$$\sum_{\substack{d_1d_2|P(z) \\ (d_2,d_1h)=1 \\ (d_1d_2,q)=1}} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{e|P(z) \\ (e,d_2q)=1}} \lambda_e \sum_{\substack{m_1,m_2 \\ (m_2,hP(z))=1 \\ (em_1d_1,d_2m_2)=1 \\ (m_1m_2,q)=1}} \chi_1(em_1)\chi_2(m_2)\psi_2(m_2)W(d_1,d_2,em_1,m_2). \tag{33}$$

Now

$$\int f\left(\frac{em_1}{M_1}, \frac{m_2}{M_2}, \frac{d_1y}{N_1}, \frac{\pm em_1d_1y+h}{m_2N_2}\right) e(ky)\mathrm{d}y$$
$$= \frac{1}{em_1d_1} \int f\left(\frac{em_1}{M_1}, \frac{m_2}{M_2}, \frac{y}{em_1N_1}, \frac{\pm y+h}{m_2N_2}\right) e\left(k\frac{y}{em_1d_1}\right) \mathrm{d}y,$$

so by Poisson summation (Lemma 3.4), we get

$$W(d_1,d_2,em_1,m_2) = \frac{1}{d_1d_2em_1m_2q} \sum_{k\in\mathbb{Z}} \int f\left(\frac{em_1}{M_1}, \frac{m_2}{M_2}, \frac{y}{em_1N_1}, \frac{\pm y+h}{m_2N_2}\right) \cdot e\left(k\frac{y}{d_1d_2em_1m_2q}\right) dy \sum_{\substack{n\, (qd_2m_2) \\ \pm n \equiv -h\overline{d_1em_1} (d_2m_2)}} \psi_1(d_1n)\psi_2(\pm em_1d_1n+h)e_{qd_2m_2}(kn). \tag{34}$$

By the Chinese remainder theorem, we can write

$$n = \mp q\bar{q}h\overline{d_1em_1} + d_2m_2\gamma,$$

so that

$$\begin{aligned}\sum_{\substack{n\ (qd_2m_2)\\ n\equiv -h\overline{d_1em_1}\,(d_2m_2)}} \quad & \psi_1(d_1n)\psi_2(\pm em_1d_1n+h)e_{qd_2m_2}(kn)\\ &= e_{d_2m_2}(\mp\bar{q}hk\overline{d_1em_1})\sum_{\gamma\ (q)}\psi_1(d_1d_2m_2\gamma)\psi_2(\pm\gamma d_1d_2em_1m_2+h)e_q(k\gamma).\end{aligned} \tag{35}$$

5.1 Contribution from $k = 0$

Since in (33) we have $(em_1d_1d_2m_2, q) = 1$, writing $\gamma' = \gamma em_1d_1d_2m_2$ we get

$$\begin{aligned}\sum_{\gamma\ (q)}\psi_1(d_1d_2m_2\gamma)\psi_2(\pm\gamma d_1d_2em_1m_2+h) &= \psi_1(em_1)\sum_{\gamma'\ (q)}\psi_1(\gamma')\psi_2(\pm\gamma'+h)\\ &=: \psi_1(em_1)U^{\pm}(h;q),\end{aligned}$$

say. Recombining the variables e, m_1, we see that the contribution from the part of (34) with $k = 0$ to (33) is

$$\begin{aligned}\frac{U^{\pm}(h;q)}{q}\sum_{\substack{m_1,m_2\\ (m_2,hP(z))=1\\ (m_1,m_2)=1}}\frac{\chi_1(m_1)\chi_2(m_2)\psi_1(m_1)\psi_2(m_2)}{m_1m_2}\Bigg(\sum_{\substack{d_1,d_2\mid P(z)\\ (d_2,d_1hm_1)=1\\ (d_1d_2,q)=1}}\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\Bigg)\\ \cdot\int f\left(\frac{m_1}{M_1},\frac{m_2}{M_2},\frac{y}{m_1N_1},\frac{\pm y+h}{m_2N_2}\right)\mathrm{d}y\sum_{e\mid (m_1,P(z))}\lambda_e\end{aligned} \tag{36}$$

(here we were able to drop the condition $(d_1, m_2) = 1$ since $d_1 \mid P(z)$ and $(m_2, P(z)) = 1$). Notice that by Lemma 3.3 with $\theta = 1/1000$ and (16) we have, for $m_1 \in \mathbb{N}$,

$$\begin{aligned}&\frac{U^{\pm}(h;q)}{q}\sum_{\substack{d_1,d_2\mid P(z)\\ (d_2,d_1hm_1)=1\\ (d_1d_2,q)=1}}\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\\ &= \frac{U^{\pm}(h;q)}{q}\left(1+O_A\left(e^{-Au/2000}\right)\right)\prod_{\substack{p<z\\ p\nmid q}}\left(1-\frac{1}{p}\right)\prod_{\substack{p<z\\ p\nmid hqm_1}}\left(1-\frac{1}{\varphi(p)}\right)\\ &\ll \frac{u^2}{\log^2 X}\frac{h}{\varphi(h)}\frac{m_1}{\varphi(m_1)}.\end{aligned} \tag{37}$$

Using Lemma 3.2(i) and (37), we see that replacing the sum over e in (36) by $\mathbf{1}_{(m_1,P(z))=1}$ affects (36) by

$$\ll \frac{h}{\varphi(h)}\frac{X}{\log^2 X}u^2 \sum_{r\geq u/1000-\beta} 2^{-Ar} \sum_{\substack{m_1\ll M_1, m_2\ll M_2\\ (m_1,P(z_r))=(m_2,P(z))=1}} \frac{\tau(m_1)^{A+1}}{\varphi(m_1)m_2}.$$

Using (19) and (20), and taking β sufficiently large in terms of A, this is, as in our earlier arguments,

$$\ll_A \frac{h}{\varphi(h)}\frac{X}{\log^2 X}e^{-Au/3000}.$$

Using also the equality in (37), handling the error term with (19) and (20), we obtain that (36) equals

$$\frac{U^{\pm}(h;q)}{q} \sum_{\substack{m_1,m_2\\ (m_1m_2,P(z))=1\\ (m_1,m_2)=1}} \frac{\chi_1(m_1)\chi_2(m_2)\psi_1(m_1)\psi_2(m_2)}{m_1m_2} \prod_{\substack{p<z\\ p\nmid q}}\left(1-\frac{1}{p}\right)\prod_{\substack{p<z\\ p\nmid hq}}\left(1-\frac{1}{\varphi(p)}\right)$$
$$\cdot \int f\left(\frac{m_1}{M_1},\frac{m_2}{M_2},\frac{y}{m_1N_1},\frac{\pm y+h}{m_2N_2}\right)\mathrm{d}y + O\left(\frac{h}{\varphi(h)}\frac{X}{\log^2 X}e^{-Au/3000}\right)$$

Finally, we need to remove the condition $(m_1,m_2)=1$. Since $(m_1m_2,P(z))=1$, using (37) and then (19) and (20), this introduces an error

$$\ll \frac{h}{\varphi(h)}\frac{X}{\log^2 X}u^2 \sum_{z\leq p\leq X}\frac{1}{p(p-1)} \sum_{\substack{m_1,m_2\leq X\\ (m_1m_2,P(z))=1}} \frac{1}{\varphi(m_1)m_2} \ll \frac{h}{\varphi(h)}\frac{X}{\log^2 X}\frac{u^4}{z}.$$

5.2 Contribution from $k\neq 0$

By partial integration we have, for $k\neq 0$,

$$\int f\left(\frac{em_1}{M_1},\frac{m_2}{M_2},\frac{y}{em_1N_1},\frac{\pm y+h}{m_2N_2}\right)e\left(k\frac{y}{d_1d_2em_1m_2q}\right)\mathrm{d}y$$
$$\ll_j \left(\frac{k}{d_1d_2M_1M_2q}\right)^{-j}\left(\frac{\delta^{-1}}{M_1N_1}+\frac{\delta^{-1}}{M_2N_2}\right)^j \ll_j \left(\frac{\delta^{-1}d_1d_2M_1M_2q}{kX}\right)^j.$$

Hence, $|k| > X^{\varepsilon}\frac{qd_1d_2M_1M_2}{\delta X}$ contribute $O(X^{-100})$ to (34). The remaining part contributes to (33) by (34) and (35)

$$\sum_{\substack{d_1d_2|P(z)\\(d_2,d_1h)=1\\(d_1d_2,q)=1}} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{e|P(z)\\(e,d_2q)=1}} \lambda_e \sum_{\substack{m_1,m_2\\(m_2,hP(z))=1\\(em_1d_1,d_2m_2)=1\\(m_1m_2,q)=1}} \chi_1(em_1)\chi_2(m_2)\psi_2(m_2) \tag{38}$$

$$\cdot\frac{1}{qd_1d_2em_1m_2}\sum_{0<|k|\le\frac{qd_1d_2M_1M_2}{\delta X^{1-\varepsilon}}}\int f\left(\frac{em_1}{M_1},\frac{m_2}{M_2},\frac{y}{em_1N_1},\frac{\pm y+h}{m_2N_2}\right)e\left(k\frac{y}{d_1d_2em_1m_2q}\right)\mathrm{d}y$$

$$\cdot\, e_{d_2m_2}(\mp\bar{q}hk\overline{d_1em_1})\sum_{\gamma\,(q)}\psi_1(d_1d_2m_2\gamma)\psi_2(\pm\gamma d_1d_2em_1m_2+h)e_q(k\gamma).$$

We write

$$\int f\left(\frac{em_1}{M_1},\frac{m_2}{M_2},\frac{y}{em_1N_1},\frac{\pm y+h}{m_2N_2}\right)e\left(k\frac{y}{d_1d_2em_1m_2q}\right)\mathrm{d}y$$
$$= m_1\int f\left(\frac{em_1}{M_1},\frac{m_2}{M_2},\frac{y}{eN_1},\frac{\pm ym_1+h}{m_2N_2}\right)e\left(k\frac{y}{d_1d_2em_2q}\right)\mathrm{d}y$$

and rearrange (38) so that the sum over m_1 is innermost. Splitting also the sum over m_1 into congruence classes modulo q, (38) is bounded by

$$\ll_{\varepsilon} q^2X^{\varepsilon}\sum_{\substack{e|P(z)\\e\le D}}\sum_{\substack{(m_2,P(z))=1\\m_2\ll M_2}}\sum_{\substack{d_1d_2|P(z)\\d_1,d_2\le D\\(d_2,h)=1}}\mathbf{1}_{(d_1eq,d_2m_2)=1} \tag{39}$$
$$\cdot\frac{X}{qd_1d_2M_1M_2}\sum_{0<|k|\le\frac{qd_1d_2M_1M_2}{\delta X^{1-\varepsilon}}}\left|\Upsilon(h,k,e,q,d_1;d_2m_2)\right|,$$

where, for some reduced residue $\alpha\ (q)$ and $y\asymp eN_1$, we have

$$\Upsilon(h,k,e,q,d_1;d_2m_2) := \sum_{\substack{m_1\equiv\alpha\,(q)\\(m_1,d_2m_2)=1}} f\left(\frac{em_1}{M_1},\frac{m_2}{M_2},\frac{y}{eN_1},\frac{\pm ym_1+h}{m_2N_2}\right)e_{d_2m_2}(\mp\bar{q}hk\overline{d_1em_1}),$$

which is an incomplete Kloosterman sum modulo d_2m_2. Applying Lemma 3.8, we get

$$\Upsilon(h,k,e,q,d_1;d_2m_2)\ll_{\varepsilon}\delta^{-1}(d_2m_2)^{1/2+\varepsilon}+\frac{M_1(hk,d_2m_2)}{ed_2m_2q}.$$

Hence (39) is bounded by (using Lemma 3.7 and that $D = X^{1/1000}$ and $M_2 \ll (M_2N_2)^{1/2} \ll (\delta^{-1}X)^{1/2}$)

$$\begin{aligned}&\ll_\varepsilon q^2X^{2\varepsilon}\Bigg(\delta^{-2}D^{3+1/2}\sum_{m_2\ll M_2} m_2^{1/2}\\ &\qquad\qquad + \sum_{e\le D}\sum_{\substack{d_1,d_2\le D\\ m_2\ll M_2}} \frac{X}{qd_1d_2M_1M_2}\sum_{0<|k|\le \frac{qd_1d_2M_1M_2}{\delta X^{1-\varepsilon}}} \frac{M_1(hk,d_2m_2)}{ed_2m_2q}\Bigg)\\ &\ll_\varepsilon q^2X^{2\varepsilon}\delta^{-2}D^{7/2}(\delta^{-1}X)^{(1/2)\cdot(3/2)} + qX^{4\varepsilon}\delta^{-1}DM_1 \ll \delta^{-3}X^{7/9}q^2,\end{aligned}$$

and the claim follows.

6 Proof of Lemma 2.5

Recalling that $z = X^{1/u}$, by (20), it suffices to show that

$$\begin{aligned}\sum_{\substack{n\le N\\ (n,P(z))=1}} \frac{\chi(n)\log(y/n)}{n} &= \prod_{p<z}\left(1-\frac{1}{p}\right)^{-1} + O\left(\left(\frac{u^3}{v^2\eta^{v/2}} + \frac{u^4v}{\eta}\right)\log X\right)\\ &\quad + O_A\left(\left(\frac{u^3}{z} + e^{-Au/3000}\right)\log X\right) + O_{\varepsilon,C}\left(\exp(-C\log^{3/5-\varepsilon} X)\right).\end{aligned}$$

We first replace $\chi(n)$ by $\mu(n)$ in the sum. We have

$$\chi(n) = (\lambda * \mu)(n) = \mu(n) + \sum_{\substack{n=km\\ m>1}} \mu(k)\lambda(m),$$

so that (analogously to (11))

$$\sum_{\substack{n\le N\\ (n,P(z))=1}} \frac{\chi(n)\log(y/n)}{n} = \sum_{\substack{n\le N\\ (n,P(z))=1}} \frac{\mu(n)\log(y/n)}{n} + \sum_{\substack{km\le N\\ (km,P(z))=1\\ m\ge z}} \frac{\mu(k)\lambda(m)\log\frac{y}{km}}{km}. \tag{40}$$

Using Lemma 2.2, the second term gives (by (19) and (20)) an admissible contribution

$$\ll \log X \sum_{\substack{k\le N\\ (k,P(z))=1}} \frac{1}{k} \sum_{\substack{z\le m\le N\\ (m,P(z))=1}} \frac{\lambda(m)}{m} \ll u\log X\left(\frac{1}{v^2\eta^{v/2}} + \frac{uv}{\eta} + \frac{1}{z}\right)u^2.$$

The first sum on the right-hand side of (40) is by (19) and (20)

$$\sum_{\substack{n\leq N\\(n,P(z))=1}}\frac{\lambda_{\text{Liouville}}(n)\log y/n}{n}+O\left(\log X\sum_{p\geq z}\frac{1}{p^2}\sum_{\substack{n\leq N/p^2\\(n,P(z))=1}}\frac{1}{n}\right)$$
$$=\sum_{\substack{n\leq N\\(n,P(z))=1}}\frac{\lambda_{\text{Liouville}}(n)\log y/n}{n}+O\left(\frac{u\log X}{z}\right),$$

where $\lambda_{\text{Liouville}}(n)$ denotes the Liouville function.

We deal with the condition $(n,P(z))=1$ using Lemma 3.2(i) with $\theta=1/1000$. This gives

$$\sum_{\substack{n\leq N\\(n,P(z))=1}}\frac{\lambda_{\text{Liouville}}(n)\log y/n}{n}=\sum_{d|P(z)}\frac{\lambda_d\lambda_{\text{Liouville}}(d)}{d}\sum_{n\leq N/d}\frac{\lambda_{\text{Liouville}}(n)\log\frac{y}{dn}}{n}$$
$$+O\left(\log X\sum_{r\geq u/1000-\beta}2^{-Ar}\sum_{n\leq N}1_{(n,P(z_r))=1}\frac{\tau(n)^{A+1}}{n}\right).$$

By (19) and (20), the error term is

$$\ll_A\log X\sum_{r\geq u/1000-\beta}2^{-Ar}\left(\frac{\log X}{\log z_r}\right)^{2^{A+1}}\ll\log X\sum_{r\geq u/1000-\beta}2^{-Ar}u^{2^{A+1}}\left(\frac{\beta}{\beta-1}\right)^{2^{A+1}r}$$
$$\ll_A e^{-Au/3000}\log X$$

when β is sufficiently large in terms of A.

For $\Re s>1$, let

$$F(s):=\sum_{n\in\mathbb{N}}\frac{\lambda_{\text{Liouville}}(n)}{n^s}=\prod_{p\in\mathbb{P}}\left(1+\frac{1}{p^s}\right)^{-1}=\frac{1}{\zeta(s)}\prod_{p\in\mathbb{P}}\frac{1}{1-\frac{1}{p^{2s}}},$$

so that

$$\sum_n\frac{\lambda_{\text{Liouville}}(n)\log\frac{y}{dn}}{n^s}=F(s)\log\frac{y}{d}+F'(s).$$

Note that $F(s)$ has a simple zero at $s=1$, whereas $F'(s)$ is holomorphic but non-zero at $s=1$.

By Perron's formula (see e.g., [18, Corollary 5.3]), we have, for $T := \exp(2C\log^{3/5-\varepsilon} X)$,

$$\sum_{d|P(z)} \frac{\lambda_d \lambda_{\text{Liouville}}(d)}{d} \sum_{n\le N/d} \frac{\lambda_{\text{Liouville}}(n)\log\frac{y}{dn}}{n}$$
$$= \sum_{d|P(z)} \frac{\lambda_d \lambda_{\text{Liouville}}(d)}{d} \frac{1}{2\pi i}\int_{1/\log N-iT}^{1/\log N+iT} \left(F(s+1)\log\frac{y}{d} + F'(s+1)\right)\frac{(N/d)^s}{s}\mathrm{d}s + O\left(\frac{\log^5 X}{T}\right).$$

We move the integration line to $\Re s = -1/\log^{2/3+\varepsilon} T$, staying in the zero-free region for $\zeta(s)$ (see e.g., [12, Theorem 8.29]). From the residue at $s = 0$, we get a main term

$$\sum_{d|P(z)} \frac{\lambda_d \lambda_{\text{Liouville}}(d)}{d} F'(1) = \sum_{d|P(z)} \frac{\lambda_d \lambda_{\text{Liouville}}(d)}{d} \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p^2}}.$$

By Lemma 3.2(ii), this equals

$$(1+O_A(e^{-Au/2000}))\prod_{p<z}\left(1+\frac{1}{p}\right)\prod_{p\in\mathbb{P}}\frac{1}{\left(1+\frac{1}{p}\right)\left(1-\frac{1}{p}\right)}$$
$$= \left(1+O_A\left(e^{-Au/2000}+\frac{1}{z}\right)\right)\prod_{p<z}\left(1-\frac{1}{p}\right)^{-1}.$$

Let us now consider the remaining integral. For $s = -1/\log^{2/3+\varepsilon} T + it$ with $|t| \le T$, one has (see e.g., [12, Theorem 8.29])

$$\frac{1}{\zeta(s+1)} \ll \log^{2/3+\varepsilon} T \quad \text{and} \quad \frac{\zeta'(s+1)}{\zeta(s+1)} \ll \log^{2/3+\varepsilon} T.$$

Hence, the remaining integral contributes

$$\ll \sum_{d\le N^{1/1000}} \frac{1}{d}\left|\int_{-1/\log^{2/3+\varepsilon} T-iT}^{-1/\log^{2/3+\varepsilon} T+iT}\left(F(s+1)\log\frac{y}{d} - F'(s+1)\right)\frac{(N/d)^s}{s}\mathrm{d}s\right|$$
$$\ll \log^2 X\cdot\log^3 T\cdot N^{-1/(2\log^{2/3+\varepsilon} T)} \ll_{C,\varepsilon} \exp(-C\log^{3/5-\varepsilon} X),$$

and the claim follows.

7 Proof of Theorems 1.3 and 1.4

Theorems 1.3 and 1.4 are trivial unless h is even. Furthermore, they follow from Lemma 3.1(i) with $w_1 = w_2 = X^{1/4}$ and $m_1 = m_2 = 0$ unless η is large. We will assume these as well as $\eta \ll q^{\varepsilon}$ from (6).

As explained in beginning of Section 2, it suffices to study, with g as there,

$$\sum_n g\left(\frac{n}{X}\right) \Lambda(n)\Lambda(\pm n + h),$$

where in case of $+$-sign we have $0 \neq |h| \leq X^{1+\varepsilon/2}$ and in case of $-$-sign we have $X \in [h^{1-\varepsilon/3}, h/4]$, for some small $\varepsilon > 0$. Here we have done a dyadic splitting of n, discarding $n \leq X^{1-\varepsilon/2}$. Hence now $q = X^{1/V'}$ for some $V' \in [(1-\varepsilon/2)V, V]$.

Define $z := X^{1/u}$ for some u to be chosen shortly (in (41)). Then $z = q^{V'/u}$. By (13) and Lemmas 2.1 and 2.2, we have

$$\sum_n g\left(\frac{n}{X}\right) \Lambda(n)\Lambda(\pm n + h) = \sum_{\substack{n \\ (n(\pm n+h), qP(z))=1}} g\left(\frac{n}{X}\right) \lambda'(n)\lambda'(\pm n + h)$$
$$+ O\left(\frac{h}{\varphi(h)} X \left(\frac{u^8}{V^2 \eta^{V/(3u)}} + \frac{u^6 V}{\eta} + \frac{u^{10}}{q^{V/(2u)}}\right) + (z + \omega(q)) \log^2 X\right).$$

To balance the error terms with errors of the type $e^{-Au/3000}$ that we will encounter later, we choose

$$u = \min\left\{\frac{\sqrt{V \log \eta}}{10C}, \log \eta\right\}. \tag{41}$$

With this choice, the error terms are acceptable and we can concentrate on the main term.

By the definition of λ' (see (9)), here

$$\sum_{\substack{n \\ (n(\pm n+h), qP(z))=1}} g\left(\frac{n}{X}\right) \lambda'(n)\lambda'(\pm n + h)$$
$$= \sum_{\substack{m_1, m_2, n_1, n_2 \\ \pm m_1 n_1 + h = m_2 n_2 \\ (m_1 n_1 m_2 n_2, P(z))=1}} g\left(\frac{m_1 n_1}{X}\right) \chi(m_1)\chi(m_2)\chi_0(n_1)\chi_0(n_2) \log n_1 \log n_2. \tag{42}$$

Let $X_\pm := \pm X + h \asymp \max\{X, h\} \ll X^{1+\varepsilon/2}$, so that $m_2 n_2 \asymp X_\pm$. We make a smooth partition of the variables m_j. We let $\Psi\colon \mathbb{R} \to [0,1]$ be a smooth function for which $\Psi(x) = 0$ for $x \le 1$ and $\Psi(x) = 1$ for $x \ge 2$. Define then $F\colon \mathbb{R}_+ \to [0,1]$ by

$$F(x) = \begin{cases} \Psi(x) & \text{if } 0 < x \le 2 \\ 1 - \Psi(x/2) & \text{if } x > 2. \end{cases}$$

The function $F(x)$ is supported on $[1,4]$ and gives a smooth partition of unity

$$\sum_{j \in \mathbb{Z}} F\left(\frac{x}{2^j}\right) = 1 \quad \text{for all } x > 0.$$

We write $N_1 = X/M_1$ and $N_2 = X_\pm/M_2 \asymp \max\{X, h\}/M_2$. Then when $m_j \in [M_j, 4M_j]$, in (42), then $n_j \in [N_j/20, 20N_j]$. Let $h\colon \mathbb{R}_{\ge 0} \to [0,1]$ be a smooth function supported on $[1/40, 40]$ such that $h(x) = 1$ for $x \in [1/20, 20]$, and write

$$f_{M_1,M_2}(x_1, x_2, y_1, y_2) = g(x_1 y_1) F(x_1) F(x_2) h(y_1) h(y_2) \frac{\log(y_1 N_1) \log(y_2 N_2)}{\log^2 X}.$$

Then

$$\begin{aligned} &\sum_{\substack{n \\ (n(\pm n+h), qP(z))=1}} g\left(\frac{n}{X}\right) \lambda'(n) \lambda'(\pm n + h) \\ &= \log^2 X \sum_{\substack{M_1 = 2^{i_1}, M_2 = 2^{i_2} \\ 1/4 \le M_1 \le 4X \\ 1/4 \le M_2 \le 4X_\pm}} \sum_{\substack{m_1, m_2, n_1, n_2 \\ \pm m_1 n_1 + h = m_2 n_2 \\ (m_1 n_1 m_2 n_2, P(z))=1}} f_{M_1,M_2}\left(\frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{n_1}{N_1}, \frac{n_2}{N_2}\right) \\ &\qquad \cdot \chi(m_1)\chi(m_2)\chi_0(n_1)\chi_0(n_2) \\ &=: \Sigma^\pm_{S,S} + \Sigma^\pm_{S,L} + \Sigma^\pm_{L,S} + \Sigma^\pm_{L,L}, \end{aligned}$$

say, where $\Sigma^\pm_{S,S}$ corresponds to $M_1 \le X^{1/2}$ and $M_2 \le X_\pm^{1/2}$, $\Sigma^\pm_{S,L}$ corresponds to $M_1 \le X^{1/2}$ and $M_2 > X_\pm^{1/2}$, $\Sigma^\pm_{L,S}$ corresponds to $M_1 > X^{1/2}$ and $M_2 \le X_\pm^{1/2}$, and $\Sigma^\pm_{L,L}$ corresponds to $M_1 \ge X^{1/2}$ and $M_2 \ge X_\pm^{1/2}$.

Let us start with $\Sigma^{\pm}_{S,S}$. Applying Proposition 2.3 with $\psi_1 = \psi_2 = \chi_0, \chi_1 = \chi_2 = \chi$, and $\delta = X^{-1/1000}$, handling the error term with Lemma 2.4, we see that, for any $A \geq 1$,

$$\Sigma^{\pm}_{S,S} = \log^2 X \prod_{\substack{p<z \\ p\mid h, p\nmid q}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid hq}} \left(1-\frac{2}{p}\right) \cdot \frac{1}{q} \sum_{\gamma(q)} \chi_0(\gamma)\chi_0(\pm\gamma+h)$$

$$\cdot \sum_{\substack{M_1=2^{i_1}, M_2=2^{i_2} \\ 1/4\leq M_1\leq 4X \\ 1/4\leq M_2\leq 4X_{\pm}}} \sum_{\substack{m_1,m_2 \\ (m_1m_2, P(z))=1}} \frac{\chi(m_1)\chi(m_2)}{m_1m_2} \int f_{M_1,M_2}\left(\frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{y}{m_1N_1}, \frac{\pm y+h}{m_2N_2}\right) \mathrm{d}y$$

$$+ O_A\left(\frac{h}{\varphi(h)} X\left(e^{-Au/3000} + \frac{u^6}{z}\right) + X^{7/9+3/1000} q^2\right).$$

Taking $A = 10^6 C^2$, the error terms are acceptable by our choice of u in (41). Let us denote the main term by $\widetilde{\Sigma}^{\pm}_{S,S}$. There

$$\sum_{\substack{M_1=2^{i_1}, M_2=2^{i_2} \\ 1/4\leq M_1\leq X^{1/2} \\ 1/4\leq M_2\leq X_{\pm}^{1/2}}} \sum_{\substack{m_1,m_2 \\ (m_1m_2, P(z))=1}} \frac{\chi(m_1)\chi(m_2)}{m_1m_2} \int f_{M_1,M_2}\left(\frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{y}{m_1N_1}, \frac{\pm y+h}{m_2N_2}\right) \mathrm{d}y$$

$$= \int g\left(\frac{y}{X}\right) \sum_{\substack{m_1,m_2 \\ (m_1m_2, P(z))=1}} \frac{\chi(m_1)\chi(m_2) \log\frac{y}{m_1} \log\frac{\pm y+h}{m_2}}{m_1m_2 \cdot \log^2 X} \sum_{\substack{M_1=2^{i_1}, M_2=2^{i_2} \\ 1/4\leq M_1\leq X^{1/2} \\ 1/4\leq M_2\leq X_{\pm}^{1/2}}} F\left(\frac{m_1}{M_1}\right) F\left(\frac{m_2}{M_2}\right) \mathrm{d}y.$$

Note that there is no dependency between m_1 and m_2,

$$\sum_{\substack{M_1=2^{i_1} \\ 1/4\leq M_1\leq X^{1/2}}} F\left(\frac{m_1}{M_1}\right) = \begin{cases} 1 & \text{if } m_1 \leq X^{1/2}/4; \\ 0 & \text{if } m_1 \geq 4X^{1/2}, \end{cases}$$

and

$$\sum_{\substack{M_2=2^{i_2} \\ 1/4\leq M_2\leq X_{\pm}^{1/2}}} F\left(\frac{m_2}{M_2}\right) = \begin{cases} 1 & \text{if } m_2 \leq X_{\pm}^{1/2}/4; \\ 0 & \text{if } m_2 \geq 4X_{\pm}^{1/2}. \end{cases}$$

Hence, by partial summation and Lemmas 2.5 (with $v = V'/u$) and 2.6, we obtain

$$\widetilde{\Sigma}^{\pm}_{S,S} = \prod_{\substack{p<z \\ p\mid h, p\nmid q}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid hq}} \left(1-\frac{2}{p}\right) \prod_{p\mid (q,h)} \left(1-\frac{1}{p}\right) \prod_{\substack{p\mid q \\ p\nmid h}} \left(1-\frac{2}{p}\right) \prod_{p<z} \left(1-\frac{1}{p}\right)^{-2} \int g\left(\frac{y}{X}\right) \mathrm{d}y$$

$$\cdot \left(1 + O_A\left(\frac{u^6}{V^2 \eta^{V/(3u)}} + \frac{Vu^4}{\eta} + \frac{u^4}{z} + e^{-Au/3000}\right) + O_{C,\varepsilon}\left(\exp(-C\log^{3/5-\varepsilon} X)\right)\right)^2.$$

Here

$$\begin{aligned}
&\prod_{\substack{p<z \\ p\mid h, p\nmid q}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z \\ p\nmid hq}} \left(1-\frac{2}{p}\right) \prod_{p\mid (q,h)} \left(1-\frac{1}{p}\right) \prod_{\substack{p\mid q \\ p\nmid h}} \left(1-\frac{2}{p}\right) \prod_{p<z} \left(1-\frac{1}{p}\right)^{-2} \\
&= 2 \prod_{2<p<z} \frac{1-\frac{2}{p}}{\left(1-\frac{1}{p}\right)^2} \cdot \left(1 + O\left(\frac{u}{z}\right)\right) \prod_{\substack{p\mid (h,q) \\ p>2}} \frac{1-\frac{1}{p}}{1-\frac{2}{p}} \prod_{\substack{p\mid h \\ p\nmid q \\ p>2}} \frac{1-\frac{1}{p}}{1-\frac{2}{p}} \qquad (43) \\
&= 2\left(1 + O\left(\frac{u}{z}\right)\right) \prod_{2<p<z} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p\mid h \\ p>2}} \left(1 + \frac{1}{p-2}\right).
\end{aligned}$$

Hence

$$\widetilde{\Sigma}^{\pm}_{S,S} = \mathfrak{S}_h \int g\left(\frac{y}{X}\right) \mathrm{d}y$$

$$+ O_{A,C,\varepsilon}\left(\frac{h}{\varphi(h)} X \left(\frac{u^6}{V^2\eta^{V/(3u)}} + \frac{Vu^4}{\eta} + \frac{u^4}{q^{V/(2u)}} + e^{-Au/3000} + \exp(-C\log^{3/5-\varepsilon} X)\right)\right).$$

The error terms are acceptable when $A = 10^6 C^2$ by our choice of u in (41).

For $\Sigma^{\pm}_{L,L}$, we use Proposition 2.3 with $\chi_1 = \chi_2 = \chi_0$, $\psi_1 = \psi_2 = \chi$, $\delta = X^{-1/1000}$ and the roles of M_j and N_j interchanged. Handling the error term with Lemma 2.4, we see that, for any $A \geq 1$,

$$\begin{aligned}\Sigma^{\pm}_{L,L} = &\log^2 X \prod_{\substack{p<z \\ p \mid h, p \nmid q}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p<z \\ p \nmid hq}} \left(1 - \frac{2}{p}\right) \cdot \frac{1}{q} \sum_{\gamma(q)} \chi(\gamma)\chi(\pm\gamma + h) \\ &\cdot \sum_{\substack{M_1=2^{i_1}, M_2=2^{i_2} \\ X^{1/2}<M_1\leq 4X \\ X_{\pm}^{1/2}<M_2\leq 4X_{\pm}}} \sum_{\substack{n_1,n_2 \\ (n_1n_2,P(z))=1}} \frac{\chi(n_1)\chi(n_2)}{n_1n_2} \int f_{M_1,M_2}\left(\frac{y}{M_1n_1}, \frac{\pm y+h}{M_2n_2}, \frac{n_1}{N_1}, \frac{n_2}{N_2}\right) \mathrm{d}y \\ &+ O_A\left(\frac{h}{\varphi(h)} X \left(e^{-Au/3000} + \frac{u^6}{z}\right) + X^{7/9+3/1000} q^2\right).\end{aligned}$$

Taking $A = 10^6 C^2$, the error term is again sufficiently small by our choice of u in (41). We denote the main term by $\widetilde{\Sigma}^{\pm}_{L,L}$. Recall that $2 \mid h$ so that by Lemma 2.6

$$\sum_{m(q)} \chi(m(\pm m + h)) = \chi(\pm 1) \sum_{m(q)} \chi_0(m)\chi_0(\pm m + h) \cdot \mathbf{1}_{\varphi(2^r)\mid h} (-1)^{\frac{h}{\varphi(2^r)}} \prod_{\substack{p \mid q' \\ p \nmid h}} \frac{-1}{p-2}.$$

Furthermore,

$$\begin{aligned}&\sum_{\substack{M_1=2^{i_1}, M_2=2^{i_2} \\ X^{1/2}<M_1\leq 4X \\ X_{\pm}^{1/2}<M_2\leq 4X_{\pm}}} \sum_{\substack{n_1,n_2 \\ (n_1n_2,P(z))=1}} \frac{\chi(n_1)\chi(n_2)}{n_1n_2} \int f_{M_1,M_2}\left(\frac{y}{M_1n_1}, \frac{\pm y+h}{M_2n_2}, \frac{n_1}{N_1}, \frac{n_2}{N_2}\right) \mathrm{d}y \\ &= \int g\left(\frac{y}{X}\right) \sum_{\substack{n_1,n_2 \\ (n_1n_2,P(z))=1}} \frac{\chi(n_1)\chi(n_2)\log(n_1)\log(n_2)}{n_1n_2} \\ &\quad \cdot \sum_{\substack{M_1=2^{i_1}, M_2=2^{i_2} \\ X^{1/2}<M_1\leq 4X \\ X_{\pm}^{1/2}<M_2\leq 4X_{\pm}}} F\left(\frac{y}{M_1n_1}\right) F\left(\frac{\pm y+h}{M_2n_2}\right) \mathrm{d}y.\end{aligned}$$

Similarly to the case of $\Sigma^{\pm}_{S,S}$, we can use partial summation, Lemma 2.5 (with $y = 1$), and (43) to obtain

$$\begin{aligned}\widetilde{\Sigma}^{\pm}_{L,L} = &\mathfrak{S}_h \chi(\pm 1) \int g\left(\frac{y}{X}\right) \mathrm{d}y \cdot \mathbf{1}_{\varphi(2^r)\mid h} (-1)^{\frac{h}{\varphi(2^r)}} \prod_{\substack{p \mid q' \\ p \nmid h}} \frac{-1}{p-2} \\ &+ O\left(\frac{h}{\varphi(h)} X \left(\frac{u^6}{V^2 \eta^{V/(3u)}} + \frac{Vu^4}{\eta} + \frac{u^4}{q^{V/(2u)}} + u e^{-Au/3000} + \exp(-C \log^{3/5-\varepsilon} X)\right)\right).\end{aligned}$$

The error term is again acceptable by (41).

We handle $\Sigma^{\pm}_{S,L}$ and $\Sigma^{\pm}_{L,S}$ similarly. The error terms are the same as before, whereas the main term from Proposition 2.3 is by Lemma 2.6, (19), (20), and (6)

$$\begin{aligned}
&\ll X\log^2 X \prod_{\substack{p<z\\ p\mid h, p\nmid q}} \left(1-\frac{1}{p}\right) \prod_{\substack{p<z\\ p\nmid hq}} \left(1-\frac{2}{p}\right) \cdot \frac{\mathbf{1}_{(h,q)=1}}{q} \sum_{\substack{m_1 \ll X^{1/2}, m_2 \ll X_{\pm}^{1/2}\\ (m_1 m_2, P(z))=1}} \frac{1}{m_1 m_2} \\
&\ll_{\varepsilon} \frac{h}{\varphi(h)} X \frac{u^4}{q^{1-\varepsilon}} \ll \frac{h}{\varphi(h)} X \frac{1}{\eta}.
\end{aligned}$$

Collecting everything together the claims follow.

8 Proof of Corollary 1.5

Squaring out and applying the prime number theorem, we see that

$$\begin{aligned}
&\int_X^{2X} \left(\sum_{y<n\le y+H} \Lambda(n) - H \right)^2 \mathrm{d}y \\
&\quad = \int_X^{2X} \left(\sum_{y<n\le y+H} \Lambda(n) \right)^2 - 2H \sum_{y<n\le y+H} \Lambda(n) + H^2 \mathrm{d}y \\
&\quad = \sum_{|h|\le H} \sum_{\substack{n_1,n_2\\ n_1=n_2+h}} \Lambda(n_1)\Lambda(n_2) \int_X^{2X} \mathbf{1}_{n_1,n_1-h\in(y,y+H]} \mathrm{d}y - H^2 X \\
&\qquad + O_{C,\varepsilon}\left(H^3 \log X + \frac{H^2 X}{\exp(C \log^{3/5-\varepsilon} X)} \right).
\end{aligned}$$

The first term on the right-hand side equals

$$\begin{aligned}
&\sum_{|h|\le H} (H-|h|) \sum_{X<n_1\le 2X} \Lambda(n_1)\Lambda(n_1+h) + O\left(H^3 \log^2 X\right) \\
&= H \sum_{X<n\le 2X} \Lambda(n)^2 + \sum_{0<|h|\le H} (H-|h|) \sum_{X<n\le 2X} \Lambda(n)\Lambda(n+h) + O\left(H^3\log^2 X\right).
\end{aligned}$$

The first term on the right-hand side is by the prime number theorem $\ll HX\log X$. Applying Theorem 1.3 to the second term, noting that $\sum_{0<|h|\le H} \frac{h}{\varphi(h)} \ll H$, we see that it suffices to show that

$$\sum_{\substack{0<|h|\le H\\ h \text{ even}}} (H-|h|)\mathfrak{S}_h \left(1+\mathbf{1}_{\varphi(2^r)|h}(-1)^{\frac{h}{\varphi(2^r)}}\prod_{\substack{p|q'\\ p\nmid h}}\frac{-1}{p-2}\right) = H^2 + O\left(H\log X + \eta^{-1}H^2\right).$$

Recall that q' is necessarily square-free (see e.g., [12, Section 3.3]). Using the bound $\mathfrak{S}_h \ll h/\varphi(h)$ and Lemma 3.7, we see that the term depending on q contributes

$$\ll H \sum_{0<|h|\le H}\frac{h}{\varphi(h)}\prod_{\substack{p|q'\\ p\nmid h}}\frac{1}{p-2} \ll H\prod_{p|q'}\frac{1}{p-2}\sum_{0<|h|\le H}\frac{h}{\varphi(h)}(h,q') \ll \frac{H^2}{q^{1/2}},$$

which is admissible by (6). Hence, it remains to show that

$$2\sum_{\substack{0<|h|\le H\\ h \text{ even}}}(H-|h|)\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p|h\\ p>2}}\left(1+\frac{1}{p-2}\right) = H^2 + O\left(H\log X\right). \tag{44}$$

Now

$$\begin{aligned}\sum_{\substack{0<|h|\le H\\ h \text{ even}}}(H-|h|)\prod_{\substack{p|h\\ p>2}}\left(1+\frac{1}{p-2}\right) &= \sum_{j=2}^{H}\sum_{\substack{0<|h|<j\\ h \text{ even}}}\prod_{\substack{p|h\\ p>2}}\left(1+\frac{1}{p-2}\right)\\ &= \sum_{j=2}^{H}\sum_{0<|h|<j/2}\prod_{\substack{p|h\\ p>2}}\left(1+\frac{1}{p-2}\right).\end{aligned}$$

Define a multiplicative function f such that

$$f(p^\nu) = \begin{cases} 0 & \text{if } p=2 \text{ or } \nu \ge 2;\\ \frac{1}{p-2} & \text{otherwise.}\end{cases}$$

Then

$$\sum_{0<|h|<j/2} \prod_{\substack{p|h \\ p>2}} \left(1+\frac{1}{p-2}\right) = \sum_{0<|h|<j/2} \sum_{r|h} f(r) = \sum_{r<j/2} f(r) \sum_{0<|h|<j/2r} 1$$

$$= j \sum_{r<j/2} \frac{f(r)}{r} + O\left(\sum_{r<j/2} f(r)\right) = j \prod_{p>2} \left(1+\frac{1}{p(p-2)}\right) + O(\log j)$$

$$= j \prod_{p>2} \left(1-\frac{1}{(p-1)^2}\right)^{-1} + O(\log j).$$

Thus, the left-hand side of (44) equals

$$2\sum_{j=2}^{H} j + O\left(H \log H\right) = H^2 + O\left(H \log X\right)$$

as claimed.

Funding

This work was supported by the Academy of Finland [285894 to K.M. and 333707 to J.M.]; and the European Research Council under the European Union's Horizon 2020 research and innovation programme [851318 to J.M.].

Acknowledgments

We are grateful to Terence Tao and Joni Teräväinen for helpful discussions and to Andrew Granville for providing us material concerning the relationship between Siegel zeros and $L(1,\chi)$. We are also grateful to the anonymous referees for helpful comments.

References

[1] Iwaniec, H. and B. Conrey. "Spacing of zeros of hecke l-functions and the class number problem." *Acta Arith.* 103, no. 3 (2002): 259–312.

[2] Friedlander, J. and H. Iwaniec. *Opera de Cribro*. American Mathematical Society Colloquium Publications, vol. 57. Providence, RI: American Mathematical Society, 2010.

[3] Friedlander, J. B., D. A. Goldston, H. Iwaniec, and A. I. Suriajaya. "Exceptional zeros and the Goldbach problem." *J. Number Theory* 233 (2022): 78–86.

[4] Friedlander, J. B. and H. Iwaniec. "The illusory sieve." *Int. J. Number Theory* 1, no. 4 (2005): 459–94.

[5] Friedlander, J. B. and H. Iwaniec. "A note on Dirichlet L-functions." *Expo. Math.* 36, no. 3–4 (2018): 343–50.

[6] Goldston, D. A. and A. I. Suriajaya. "Note on the Goldbach conjecture and Landau–Siegel zeros." Preprint, arXiv:2104.09407v1, 2021.

[7] Harman, G. *Prime-Detecting Sieves*. London Mathematical Society Monographs Series, vol. 33. Princeton, NJ: Princeton University Press, 2007.

[8] Heath-Brown, D. R. "Gaps between primes, and the pair correlation of zeros of the zeta function." *Acta Arith.* 41, no. 1 (1982): 85–99.

[9] Heath-Brown, D. R. "Prime twins and Siegel zeros." *Proc. London Math. Soc. (3)* 47, no. 2 (1983): 193–224.

[10] Henriot, K. "Nair–Tenenbaum bounds uniform with respect to the discriminant." *Math. Proc. Cambridge Philos. Soc.* 152, no. 3 (2012): 405–24.

[11] Henriot, K. "Nair–Tenenbaum uniform with respect to the discriminant—erratum [mr2911138]." *Math. Proc. Cambridge Philos. Soc.* 157, no. 2 (2014): 375–7.

[12] Iwaniec, H. and E. Kowalski. *Analytic Number Theory*. American Mathematical Society Colloquium Publications, vol. 53. Providence, RI: American Mathematical Society, 2004.

[13] Jacobsthal, E. "Über die darstellung der primzahlen der form 4n + 1 als summe zweier quadrate." *J. Reine Angew. Math.* 132 (1907): 238–45.

[14] Jia, C. "Almost all short intervals containing prime numbers." *Acta Arith.* 76, no. 1 (1996): 21–84.

[15] Kloosterman, H. D. "On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$." *Acta Math.* 49, no. 3–4 (1927): 407–64.

[16] Merikoski, J. "Exceptional characters and prime numbers in sparse sets." Pre-print, 2021.

[17] Montgomery, H. L. "The pair correlation of zeros of the zeta function." *Analytic Number Theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972)*. 181–93, 1973.

[18] Montgomery, H. L. and R. C. Vaughan. *Multiplicative Number Theory. I. Classical Theory*. Cambridge Studies in Advanced Mathematics, vol. 97. Cambridge: Cambridge University Press, 2007.

[19] Selberg, A. "On the normal density of primes in small intervals, and the difference between consecutive primes." *Arch. Math. Naturvid.* 47, no. 6 (1943): 87–105.

[20] Tao, T. and J. Teräväinen. "The Hardy–Littlewood–Chowla conjecture in the presence of a Siegel zero." *J. London Math. Soc.* 106, no. 4 (2022): 3317–78.