UNIVERSITY OF TURKU

ABSTRACT

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Title	An application of the Black–Litterman model using exponential GARCH-in-mean model		
Supervisors	Prof. Mika Vaihekoski, M.Sc. Valtteri Peltonen		

Abstract

The Black–Litterman model is an alternative asset allocation method that was developed to challenge the modern portfolio theory in the international asset allocation decision-making context. The modern portfolio theory takes only the mean and the variance of the asset returns into account and therefore it is called "mean-variance optimization". There are several issues related to this model and the Black–Litterman model was developed to overcome these issues. The main contribution of the model is that it incorporates investors' subjective views to the model which the modern portfolio does not take into consideration.

This study applies exponential GARCH-in-mean model (EGARCH-M) to estimate proxies for the investors' views in the Black–Litterman model. EGARCH-M model was chosen as it takes many statistical properties of asset returns into consideration. The study is conducted for a theoretical portfolio consisting of twelve MSCI country indices from different economic backgrounds. The theoretical investors' views are built based on the results of EGARCH-M and the new revised vector of expected returns is computed using these inputs and the prior distribution of implied equilibrium returns and historical variance-covariance matrix. Finally the new weights are calculated and compared to the benchmark portfolio based on naïve allocation.

Key words	asset allocation, portfolio theory, CAPM, the Black–Litterman model, GARCH
Further information	

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Tiivistelmä

Black–Litterman -malli on sijoitusten allokointimalli, joka kehitettiin haastamaan moderni portfolioteoria globaaleissa sijoitusten allokointipäätöksissä. Moderni portfolioteoria keskittyy ainoastaan sijoituskohteiden keskiarvoon (mean) ja varianssiin (variance), joten sitä kutsutaan myös ns.mean-variance -optimoinniksi. Kyseiseen malliin liittyy useita ongelmia, jotka Black–Litterman malli pyrkii huomioimaan. Black–Litterman -mallin tärkein hyöty on se, että se ottaa huomioon sijoittajien subjektiiviset näkemykset.

Tässä tutkielmassa sovelletaan eksponentiaalista GARCH-in-mean -mallia (EGARCH-M) ja estimoidaan proxy -muuttujat sijoittajien näkemyksille Black–Litterman malliin. EGARCH-M -malli soveltuu tähän, sillä se ottaa useita tuottojen tilastollisia ominaisuuksia huomioon. Tässä tutkielmassa sovelletaan mallia indeksituottojen aikasarjadatalle käyttäen MSCI maaindeksejä erilaisista taloudellisista lähtökohdista. EGARCH – mallin tuloksista muodostetaan teoreettiset sijoittajien näkemykset ja Black–Litterman -mallin mukaiset tuotot lasketaan käyttäen näitä näkemyksiä sekä CAPM -malliin perustuvia tuottoja ja varianssi-kovarianssi matriisia. Lopuksi lasketaan uudet Black–Litterman -mallin mukaiset painot sijoituskohteille ja verrataan valittuun benchmarkiin, joka tässä tapauksessa on ns. naïve allocation eli sijoituskohteiden painot jaetaan tasaisesti portfoliossa.

Asiasanat	sijoitusten allokointi, portfolioteoria, CAPM, Black–Litterman -malli, GARCH
Muita tietoja	

UNIVERSITY OF TURKU
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AN APPLICATION OF THE BLACK-LITTERMAN MODEL USING EXPONENTIAL GARCH-IN-MEAN MODEL

Master´s Thesis
in Accounting and Finance

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1 INTRODUCTION

1.1 Background

Investors tend to prefer domestic assets in their portfolio selection even though the diversification benefits are widely known. This phenomenon is called home bias and it occurs when the capabilities of processing information are limited. When the home bias holds investors invest exclusively on domestic assets in lack of relevant information of global diversification benefits. (Mukherjee, Paul and Shankar 2017) Successful investors should understand the importance of diversification not only globally but also accross industries and asset classes (Markowitz 1952; Mukherjee et al. 2017).

Investors that are aware of the modern portfolio theory developed by Markowitz (1952) should not exhibit home bias or preferences for investing in a specific asset class or industry. The modern portfolio theory is one of the main finance theories and the theory of Markowitz (1991) together with the capital asset pricing model developed by Sharpe (1964) and Lintner (1965) have established the theory of microeconomics of the capital markets.

The basic idea of the modern portfolio theory is that rational investors seek to maximize the expected return under uncertainty. Based on the theory investors would choose a diversified portfolio to reduce risk. Without the assumption of uncertainty investors would select portfolios only based on the expected return. (Markowitz 1991) The asset allocation decisions reflect investors' individual risk preferences by means of risk aversion (Riley & Chow 1992). As investors make decisions only based on the expected return (mean) and the risk (variance) of the portfolio the model is also called "mean-variance model" (Fama & French 2004). There are several extensions to the traditional mean-variance model. For example Kraus and Litzenberger (1976) have included the effect of skewness to the model. However the traditional mean-variance remains the most useful version of the model and any additional moments added to the model do not increase attractiveness of the model significantly (Elton & Gruber 1998). One of the main findings of the modern portfolio theory is that investors should not focus only on the return and risk characteristics of the individual assets but instead understand the co-movement, namely covariance, of the assets. Taking this into account leads to a portfolio with less risk than in a situation where only the individual risk is considered. (Elton & Gruber 1998)

The return-risk trade-off is presented with the concept of efficient frontier which represents all the optimal portfolios that offer either the maximum return or minimum risk. The efficient frontier has different shapes depending on the correlation of the assets. When considering the riskless borrowing and lending, represented by capital market line, investors can combine the riskless asset with the risky asset in their portfolio selection. (Markowitz 1952) This is called two-fund separation theorem and it involves finding the greatest slope that combines the efficient frontier and the capital market line. The greatest slope is referred as Sharpe ratio or alternatively tangency portfolio. More generally the two-fund separation theorem refers to holding any two assets. (Elton 2011)

According to Markowitz (1991) the modern portfolio theory was developed to describe how the investors should behave when optimizing their portfolio while the capital asset pricing model (CAPM) developed by Sharpe (1964) and Litner (1965) is concerned of finding the equilibrium when the investors behave as the modern portfolio theory suggests. CAPM model describes the relationship between the expected return and the risk of the asset. It is a widely used tool to calculate the firm's cost of the capital and to measure portfolio performance. (Fama & French 2004) In this research CAPM model is used to calculate inputs for the Black–Litterman model. Sharpe (1964) and Litner (1965) extent the model of Markowitz (1952) so that they include two additional assumptions to the model. Firstly it is assumed that investors have an identical return distribution over a one-period horizon. Secondly it holds that investors can borrow and lend at riskless rate.

The Black–Litterman model was introduced by Fischer Black and Robert Litterman originally in 1990 to overcome several issues related to the modern portfolio theory (He & Litterman 1999). The main issues of the modern portfolio theory are large quantity of required input data, extreme portfolio weights, sensitivity of weights and inability to distinguish the strong views from the weak ones. (Drobetz 2001) The model combines the equilibrium market returns with the subjective investors' views on asset performance compared to a benchmark and uses Bayesian inference to estimate both the revised vector of expected returns and the optimal weights. (Black & Litterman 1990; Beach & Orlov 2007). In practice the Black–Litterman model combines the traditional mean-variance optimization model with the CAPM model (Black & Litterman 1992).

The Black–Litterman model is widely applied and estimated with different approahces. The perspectives vary based on the method used to incorporate investors' views. One could use analysts' recommendations as a reference and estimate the confidence on these views so that more weight is given to a view received from a skillful analyst (Idzorek 2005). Beach and Orlov (2007) argue that application of GARCH model for the forecast estimates might be preferable compared to the use of analyst views as GARCH models can take many empirical properties of asset returns into account.

Beach and Orlov (2007) and Duqi, Franci and Torluccio (2014) have estimated the Black–Litterman model by using an extension of General Autoregressive Conditional Heteroscedasticity model (GARCH) in forecasting the investors' views on volatility. Their argument for the selection of this model is that with the use of GARCH models, one can get better understanding of the regularities in the stock returns. For example it is known that asset returns tend to exhibit strong leptokurtosis and volatility clustering and GARCH models can be used to capture these kind of properties of the asset returns. (Duqi et al. 2014; Fama 1965) As an alternative to the volatility prediction models Bessler and Wolff (2017) build a portfolio optimization model that uses macroeconomic factors to predict industry returns. Their major finding is that when utilizing the industry-based returns it results to a higher portfolio performance compared to a situation where the estimation is done with the historical return estimates. Examples of more complex methods are described in the articles of Palomba (2008), which used Flexible Dynamic Conditional Correlations developed by Billio, Caporin and Gobbo (2006), and Lejeune (2009), which used VaR (Value at Risk) model to construct a fund-of-funds with an absolute return in the Black–Litterman context.

1.2 Aim of the study

This study is inspired by the articles of Beach and Orlov (2007) and Duqi et al. (2014) which appliy an exponential GARCH-in-mean model (EGARCH-M) to generate inputs to be used as views in the Black–Litterman model. EGARCH-M model generates estimates for the asset returns as well as variances and the views are formulated based on them. Moreover the residuals of the EGARCH-M model are used as an input for the uncertainty matrix of the investors' views. In the articles of Beach and Orlov (2007) and Duqi et al. (2014) the global portfolio is built from various asset classes but in this study we focus on equity indices. The data covers twelve Morgan Stanley Capital International (MSCI) country equity índices.

According to Beach and Orlov (2007) and Duqi et al. (2014) the purpose of their study is to select an econometric model which reflects the properties of the excess returns and volatility in the global portfolio, to utilize this model to generate volatility and return forecasts and to use the Black–Litterman model to generate returns and weights for the optimal portfolio. Both articles show that the risk-adjusted returns based on the Black–Litterman model exceed the CAPM returns. As a result they concluded that the Black–Litterman model can be understood as a useful tool to build global investment strategies. The actual research questions of this study are:

- What are the benefits for an investor when applying a strategy based on the Black–Litterman model?
- Can we consider EGARCH-M model a good method to estimate investors' views for the Black–Litterman model?

It is known that asset returns exhibit for example leptokurtosis, volatility clustering, asymmetry effect, mean reversion, leverage effect and heteroscedasticity. Beach and Orlov (2007) detect strong volatility clustering in their research and therefore modeling of conditional volatility is reasonable. GARCH models are used to capture the tendency of heteroscedasticity and leptokurtosis. Moreover exponential GARCH incorporates asymmetry effect and leverage effect in the estimation model. As EGARCH model takes most of the statistical properties of asset returns into account it is reasonable to use it in this research as well.

In the research of Beach and Orlov (2007) and Duqi et al. (2014) exponential GARCH model is extended with the arch-in-mean effect which adds the impact of variance to the mean equation. When including arch-in-mean effect it adds variance to the mean equation and the investors' views can be formulated from the expected returns estimated with EGARCH-M model. In the articles of Beach and Orlov (2007) and Duqi et al. (2014) the portfolio based on the Black–Litterman model beats the portfolio based on the traditional mean-variance optimization.

One of the aims of GARCH models is to estimate the future variance with the use of past variance and the forecast of variance in past periods. The choice of concentrating on predicting volatility is reasonable. It is important to understand not only the returns of assets but also the risk. Investors may focus too much on the asset returns without understanding the high risk. Volatility prediction is important in investment and risk management and it captures efficiently the international equity market risk. Moreover the volatility forecasts are able to take the time-varying property of asset return confidence intervals into account. (Beach & Orlov 2007)

The data used in this study consists of twelve MSCI total return equity indices covering the time horizon of 10 years (14.3.2008 – 14.3.2018). Due to the fact of using indices the effect of different biases, including home bias, is reduced. The benchmark used in this study is naïve allocation which means that each asset is given an equal weight (1/N) at the beginning of the estimation. In this case there are twelve assets so each asset gets a weight of (1/12). Based on the estimated implied equilibrium returns the assets are sorted to three different risk portfolios according to their performance at *t*. When sorting the assets to these three portfolios we can formulate three views. We aim to find assets from lower categories outperforming assets in the higher category. EGARCH-M model is used to generate excess returns which form the base for the investors' subjective views. Moreover the diagonal matrix for the uncertainty of the views is calculated from the variance-covariance matrix of the residuals of the EGARCH-M derived excess returns. Finally these inputs are used to estimate the revised vector of expected returns and the asset weights for the Black–Litterman model. ′For simplicity, the forecast horizon used in this study covers only two consecutive periods. In practice this means that the study is conducted twice for the same dataset but with different time horizons to get the out-of-sample forecasts for two subsequent periods. The results show how the forecasted Black–Litterman based returns and weights vary from one period to another.

The theory of this study is restricted to modern portfolio theory, capital asset pricing model and the Black–Litterman model. In addition the basic theory behind the GARCH models is introduced briefly. The background of the portfolio theory including the utility theory is not covered. Furthermore it is assumed that the reader has a basic knowledge of finance and is familiar with different models and parameters. The most complex mathematical formulas are not derived but their objective and parameters are explained.

1.3 Structure of the thesis

The study consists of theoretical background, empirical research and analysis of the results. The theoretical framework is divided into two chapters. The chapter 2 covers the modern portfolio theory, capital asset pricing model and the Black–Litterman model. With regard to the modern portfolio theory, the concept of risk and return, the efficient frontier and the principles of mean–variance optimization are explained. We discuss on the issues related to the modern portfolio theory and introduce the Black–Litterman model as a response to overcome these problems. The main concepts of the Black–Litterman model, the equilibrium market portfolio and the investors' views on the asset returns, are discussed and the model construction is explained. The subsequent chapter provides a short presentation to the GARCH models that are used in the empirical research.

The theoretical part is followed by the empirical section that aims to build a theoretical model to estimate the expected returns and weights in the framework of the Black–Litterman model. This is done by calculating the equilibrium implied returns and the historical variance-covariance matrix and incorporating the investors' views predicted with the EGARCH-M model as a proxy for the views. In the last part the main results of the empirical research are discussed.

2 THEORETICAL BACKGROUND

2.1 Capital asset pricing model and modern portfolio theory

2.1.1 Main concepts of portfolio management

Return and risk of the portfolio are the main concepts related to portfolio management. Campbell, Lo and MacKinlay (1997) argue that there are two main reasons to use returns instead of prices. Firstly returns are considered as a complete summary of the investment opportunity since the amount that an average investor invests does not affect the price changes in perfectly competitive markets. Secondly returns have more desirable statistical properties than prices.

The most common forms of returns are one-period simple return and continuosly compounded returns. One-period simple return can be calculated as follows:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1, \tag{1}$$

where R_t is return at time *t*, P_t is the asset price at time *t* and P_{t-1} is the asset price at time *t-1*. When dividends are taken into account the one-period simple return is computed in the following way:

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1, \tag{2}$$

where D_t denotes the dividend paid at time *t*. Continuosly compounded returns are computed as follows:

$$r_t = \ln(1 + R_t) = \ln\left(\frac{P_t + D_t}{P_{t-1}}\right), \tag{3}$$

where r_t is the continuosly compounded return at time *t*. These logarithmic returns are considered superior to simple returns for two main reasons. Continuosly compounded returns have more attractive statistical properties than simple returns. In addition the calculation of multi-period sum of returns is easier for continuosly compounded returns. The multi-period returns for simple returns and continuosly compounded returns are computed in the following way respectively:

$$R_{i,t} = \prod_{t=1}^{T}(R_{it} + 1) - 1$$

$$r_{i,t} = \sum_{t=1}^{T} r_{it}. \quad (4)$$

As shown in Equation (4) the multi-period simple return is a product of simple one-period gross returns while the equivalent multi-period return for logarithmic returns can be computed simply as a sum of one-period logarithmic returns. (Tsay 2005)

It is reasonable to use variance or it's square root, standard deviation, as a measure for the deviation from the average return. This is because the deviations from the average return can be either positive or negative and variance takes this into account by using squared values instead of original ones. Therefore all the deviations can be considered as positive values. (Elton 2011, 44-45)

When computing the variance of a portfolio, the concept of covariance has to be taken into consideration. Covariance is the measure of how the returns of two asset returns behave together. It gets positive values when the two asset returns go to the same direction simultaneously, either positive or negative. On the other hand it gets negative values when the returns move to the opposite directions. Zero covariance means that the asset returns are not linearly dependent on each other. Covariance can be computed in the following way:

$$\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j, \quad (5)$$

where σ_{ij} is covariance between the returns of asset *i* and *j*, ρ_{ij} is their correlation and σ_i and σ_j refer to standard deviations of assets *i* and *j*, respectively. This equation can be rewritten as follows:

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i\sigma_j}. \quad (6)$$

In Equation (6) the correlation between two assets is explained as a quotient of covariance and the product of individual standard deviations. The variable ρ_{ij} is called correlation coefficient. In fact, the only difference between the covariance and the correlation coefficient is that the latter one scales the values between -1 and 1. (Elton 2011, 51-52)

2.1.2 Capital asset pricing model

Capital asset pricing model (CAPM) was first introduced by Sharpe (1964). According to the theory there are four basic assumptions that should hold so that the model is valid. The first one states that all the investors aim to maximize the wealth only based on mean and variance of the portfolios. Secondly the effect of taxes or transaction costs should be eliminated. Thirdly it is assumed that the investors have homogenous expectations regarding the asset returns. This means that all the investors have the same expectations and an identical set of choices. Fourthly there are no limitations for investors to borrow and lend at the riskless rate of return. (Black, Jensen & Scholes 1972; Walters 2007) Lending at riskless rate is consideded as a purchase of a riskless asset, e.g. a short-term Treasury bill that has a certain return. Borrowing can be understood as selling this asset short. (Elton 2011, 83)

The main idea of CAPM model is the assumption of a linear correlation between the risk and the return of the asset. It can be written as follows:

$$E(R_i) = r_f + \beta_i(r_m - r_f), \tag{7}$$

where $E(R_i)$ is the expected return of the asset *i*, r_f is a risk-free rate of return, β_i is beta coefficient representing the risk of the asset *i* and r_m is the return of the market portfolio. Beta coefficient represents the idiosyncratic risk, i.e. the asset-specific risk and can be rewritten as follows:

$$\beta_i = \rho \frac{\sigma_i}{\sigma_m} = \frac{Cov(r_i, r_m)}{Var(r_m)}, \tag{8}$$

where ρ is the correlation coefficient, σ_i is the variance of the returns of asset *i* and σ_m is the variance of the market portfolio. (Walters 2011)

The total risk of an individual asset *i* can be expressed with the following equation:

$$\sigma_i = \beta_i \sigma_m + \sigma_{ei}, \tag{9}$$

where σ_i is the variance of the security i, β_i is the beta coefficient, σ_m is the market risk and σ_{ei} is the residual risk. The last term approaches zero when the number of the securities held in the portfolio gets large enough. Thus the first term on the right side of Equation (9) defines the risk of the security. This is market volatility multiplied by beta coefficient which cannot be diversified away. Therefore the term $\beta_i \sigma_m$ is called non-diversifiable systematic risk and the term σ_{ei} diversifiable risk. (Elton 2011, 134-135)

2.1.3 Background of the modern portfolio theory

According to Markowitz (1952) portfolio selection consists of two stages. The first one involves observation and experience followed by forecasts on future performance of the assets. The second stage starts with these forecasts and ends up with the selection of the optimal portfolio. The modern portfolio theory focuses on the second stage of this portfolio selection process.

The modern portfolio theory starts by rejecting the assumption that an optimal portfolio is the one that maximizes the expected return. This would imply that the investors would invest only on the asset that gives the highest return. However this approach does not take the risk into account. The portfolio that gives the highest return does not automatically guarantee the minimum risk. It suggests that any non-diversified portfolio is more desirable thant a diversified one. However investors cannot know the future returns with certainty. Therefore diversification should be taken into account to reduce uncertainty. When assuming that investors both maximize the expected return and diversify the portfolio, we can find a portfolio that gives the highest expected return and minimum risk. (Markowitz 1952)

Markowitz introduced the concept of efficient frontier that represents the efficient portfolios that maximize the expected return while minimizes the variance. In this framework investors can benefit from a higher return by taking additional risk or they can reduce risk by accepting a lower expected return. According to Markowitz the expected return and the variance of the portfolio can be constructed as follows:

$$E = \sum_{i=1}^{N} X_i \mu_i \qquad (10)$$

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} X_i X_j, \qquad (11)$$

where X_i and X_j are the weights invested in the assets i and j respectively, μ_i is the mean return of the asset i and σ_{ij} is the covariance between the two assets. Based on Equation (10) the expected return is calculated as a sum of the weighted mean returns. Equation (11) represents the risk of all the assets together. It is denoted as a sum of the product of the asset weights and their covariance. (Markowitz 1952)

When short-sales are not allowed it is assumed that all the wealth is invested, i.e. the sum of the proportions invested equals to one and that all the weights are zero or positive:

$$\sum_{i=1}^{N} X_i = 1$$

$$X_i \geq 0. \tag{12}$$

The optimal diversification is done by combining assets from different industries as their covariances are lower than within the same industry. This is simply because the performance of the assets in the same industry tend to move together. (Markowitz 1952)

2.1.4 Efficient frontier

Efficient frontier represents all the efficient portfolios that are consistent with the assumption of risk averse investors that prefer more to less. The efficient portfolios are the combinations of risky assets that either offer the highest return for the minimum level of risk or the lowest risk for the desired expected return. Graphically, the efficient frontier is the curve that determines the efficient portfolios in the return-standard deviation space. All the efficient portfolios are between the global minimum variance portfolio and the maximum return portfolio. Figure 1 represents the efficient frontier that starts from the point B and ends at the point C:

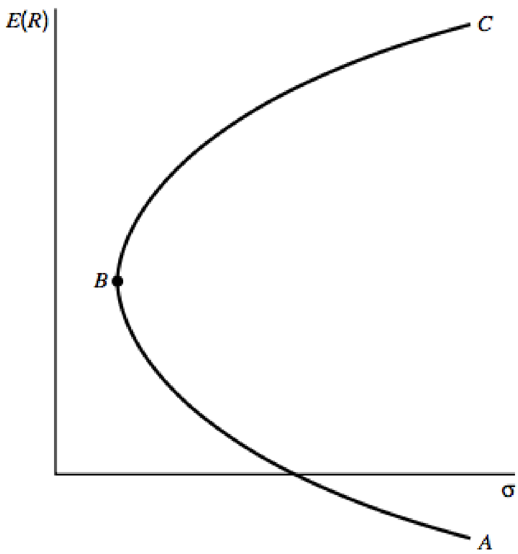

Figure 1: Efficient frontier

(Elton, 2011, 292)

In Figure 1 the point B represents the global mínimum variance portfolio and the point C is the maximum return portfolio. When short sales are allowed, the upper bound is infinite. (Elton 2011, 292)

It is assumed that investors can lend and borrow at a riskless rate which gives them a certain return with zero risk. Lending at riskless rate is considede d as a purchase of a riskless asset, e.g. a short-term Treasury bill that has a certain return. Borrowing can be understood as selling this asset short. Figure 2 represents the line that represents all the combinations of the risky portfolios with riskless lending and borrowing. The line starts from the point where standard deviation equals zero since the asset has zero risk. Moving to the right or left from the point A means that the risky portfolio is combined with riskless lending or borrowing respectively. The line starting from the point R_f is called capital market line and all the efficient portfolios lie on this line. (Elton 2011, 82-83; Walters 2011)

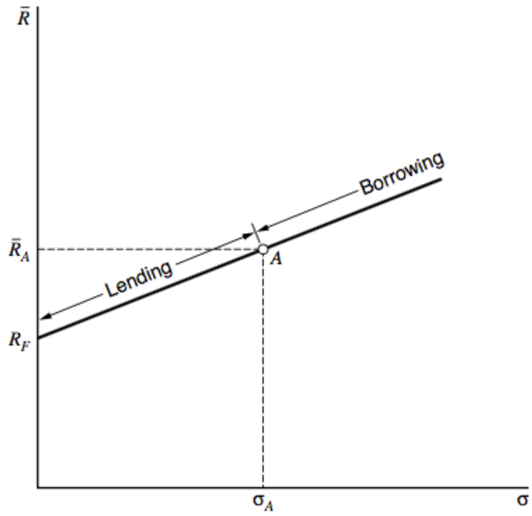

Figure 2: Capital market line with riskless lending and borrowing

(Elton, 2011, 83)

The portfolio risk consists of the individual risk of each asset and the covariance between all the individual asset pairs. It can be shown that when including a riskless asset the risk of the portfolio actually diminishes so that it only involves the risk of the risky asset. Equation (13) demonstrates how the average return of the portfolio is divided between a risky asset and a riskless asset:

$$\bar{R}_P = (1 - X)R_F + X\bar{R}_A, \qquad (13)$$

where $\bar{R}_P$ is the average return of the portfolio, X is the weight of the asset in the portfolio, R_F is the return of the risk-free asset and $\bar{R}_A$ is the average return of the risky asset A. The risk of the portfolio consisting of a risky asset and a riskless asset is demonstrated in Equation (14):

$$\sigma_P = \sqrt{(1-X)^2\sigma_F^2 + X^2\sigma_A^2 + 2X(1-X)\sigma_F\sigma_A\rho_{FA}}, \qquad (14)$$

where σ_F is the standard deviation of the riskless asset, σ_A is the standard deviation of the risky asset A and ρ_{FA} is the correlation between these two assets. After eliminating the terms that equal zero the risk of the portfolio becomes as follows:

$$\sigma_P = X\sigma_{A.} \qquad (15)$$

Equation (15) implies that the risk of the portfolio equals the risk of the risky asset A. When solving this equation with respect to the parameter X, we see that the weight of the risky asset X can be expressed as a quotient of the standard deviations of the portfolio and the risky asset A respectively:

$$X = \frac{\sigma_P}{\sigma_{A.}} \qquad (16)$$

By replacing this in Equation (13) we get the following equation for the return of the portfolio:

$$\bar{R}_P = R_F + \frac{(\bar{R}_A - R_F)}{\sigma_A}\sigma_{P.} \qquad (17)$$

(Elton 2011, 82-83)

Figure 3 shows how the capital market line and efficient frontier determine the optimal allocation of riskless and risky asset:

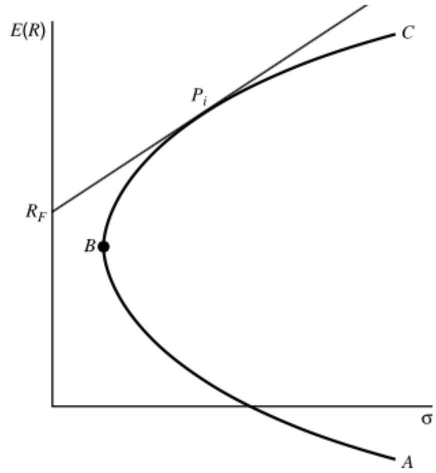

Figure 3: Efficient frontier with riskless lending and borrowing

(Elton, 2011, 293)

In Figure 3 the point P represents the market portfolio, i.e. the combination of all the risky assets. This usually refers to CAPM portfolio. It is assumed that all the investors have homogenous expectations and therefore their efficient frontier is identical. Furthermore all the investors have the same rate for riskless borrowing and lending. Investors may choose the optimal combination of portfolios from the capital market line depending on their risk preferences. They can choose to hold a combination of the market portfolio and riskless borrowing or lending. Alternatively they can invest only on the market portfolio or they may choose a combination of risky assets. (Elton 2011, 292-293)

When an investor holds a riskless asset and a risky market portfolio we can refer to the two-fund separation theorem. The optimal combination of these two assets can be found by identifying the greatest slope that connects capital market line and efficient frontier by starting from the riskless asset R_f. This is computed by maximizing the quotient of the excess return on top of the riskless return divided by the standard return of the portfolio:

$$\theta = \frac{\bar{R}_P - R_F}{\sigma_P}, \qquad (18)$$

where $\bar{R}_P$ is average return of the market portfolio, R_F is riskless return and σ_P is standard deviation of the market portfolio. In fact, this equation is the same as the latter term in Equation (17). This formula is called Sharpe ratio or alternatively tangency portfolio. When maximizing this equation all the wealth has to be invested so that Equation (19) holds:

$$\sum_{i=1}^{N} X_i = 1. \tag{19}$$

(Elton, 2011). Fabozzi, Focardi and Kolm (2010) introduce two main stages for the two-fund separation theorem: asset allocation and risky portfolio construction. The first one involves the decision of how to divide the investor's wealth between the riskless asset and the risky assets. The second one refers to the decision of how to allocate the wealth among risky assets.

The correlation between two assets determines the shape of the efficient frontier. When the correlation between two assets is 1, this means that the two assets are perfectly correlated and the efficient frontier is a straight line as depicted in Figure 4:

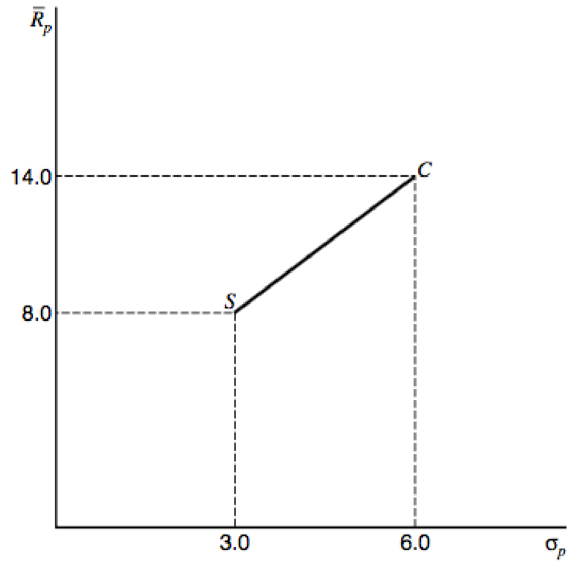

Figure 4: Efficient frontier when the correlation is $\rho = 1$

(Elton, 2011, 68)

In Figure 4, the points S and C are examples of portfolios that lie on the efficient frontier. By contrast, when the correlation between the securities is -1 this implies perfectly negative correlation. When this occurs, investors can combine two efficient portfolios so that the total risk is zero. Figure 5 shows how this is depicted in the return-standard deviation space:

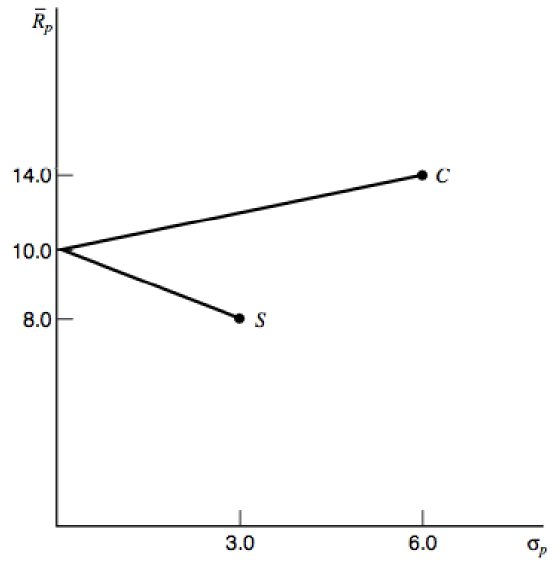

Figure 5: Efficient frontier when the correlation is $\rho = -1$

(Elton, 2011, 71)

In case when the correlation between two assets is zero and the assets are not correlated, this is represented as a curve as in Figure 6:

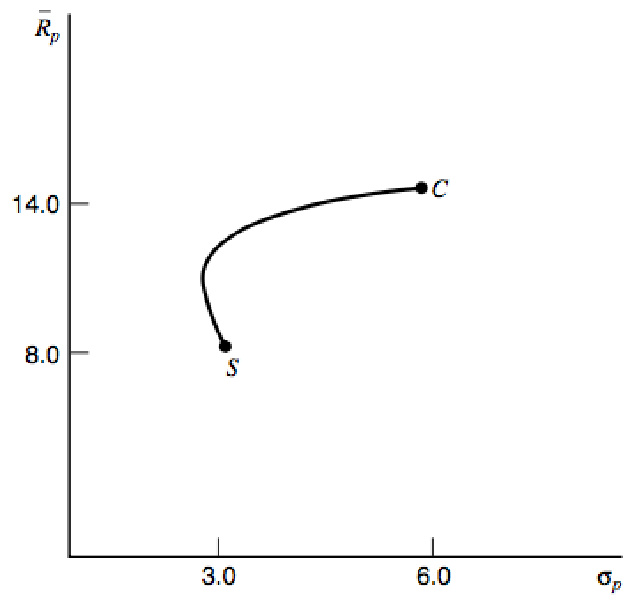

Figure 6: Efficient frontier when the correlation is $\rho = 0$

(Elton, 2011, 72)

2.1.5 Mean-variance optimization

Mean-variance optimization is performed by combining the estimated returns, volatilities and correlations of the selected assets and taking into consideration the investment choices of the investor. This results to the mean-variance efficient frontier that gives the best portfolios in terms of return-risk space. (Fabozzi, Gupta & Markowitz 2002) Figure 7 illustrates this process:

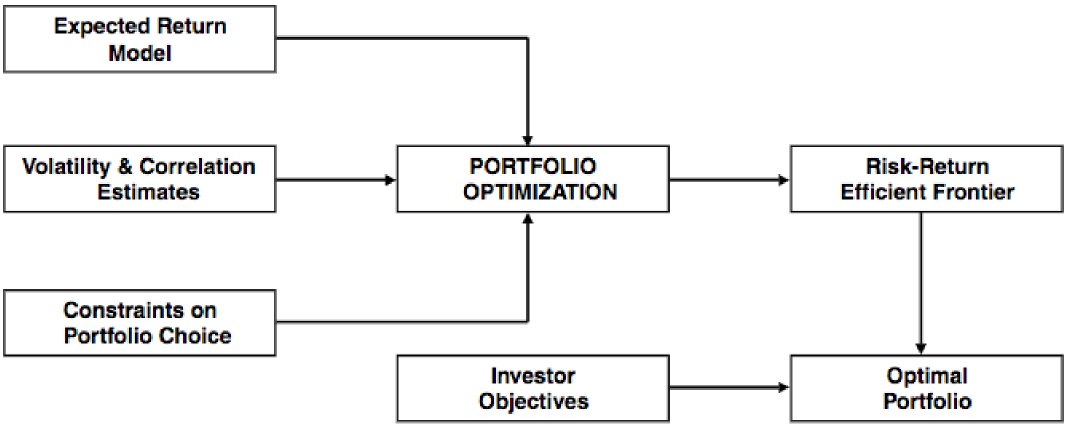

Figure 7: Mean-variance optimization process

(Fabozzi, Gupta & Markowitz, 2002, 8)

When assuming that there are N risky assets in the portfolio the proportion of each asset held in the portfolio is expressed with Nx1 vector $\mathbf{w} = (w_1, w_2, \dots, w_N)$ where each parameter represents the weight of an individual asset *i* in the portfolio. As for the risk of the assets we need to specify the variance-covariance matrix which is constructed by calculating the variance of each asset and the covariance of each asset pair in the portfolio. The variance-covariance matrix is constructed as follows:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_{NN} \end{pmatrix}, \qquad (20)$$

where σ_{11} is the variance of the asset 1, σ_{NN} the variance of the asset N and σ_{1N} and σ_{N1} are the covariances between these two assets. In fact, the covariance of a pair of assets is represented twice since σ_{1N} equals σ_{N1}. (Fabozzi et al. 2002, 317)

When using the weight vector and the covariance matrix, we can form equations for the expected return and the variance of the portfolio. Equation (21) represents the vector of expected returns:

$$\boldsymbol{\mu}_p = \boldsymbol{w}'\boldsymbol{\mu}, \tag{21}$$

where $\boldsymbol{w}$ is the weight vector and $\boldsymbol{\mu}$ is the mean return vector. The superscript refers to the transpose of the vector. The calculation for the variance of the portfolio is demonstrated in Equation (22):

$$\sigma_p^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}, \tag{22}$$

where $\boldsymbol{\Sigma}$ represents the variance-covariance matrix. The purpose of the mean-variance optimization is to minimize the variance of the portfolio. The function is minimized with respect to some constraints. One of them is the budget constraint which implies that the weights of the assets in the portfolio sum up to 1. Another constraint shows the average expected return μ_0 that an optimal portfolio should reach. The optimization problem and its constraints are constructed in the following way:

$$\min_{w} \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\,\boldsymbol{w}$$

$$s.t.\boldsymbol{w}'\boldsymbol{\mu} = \mu_0$$

$$w'1 = 1. \tag{23}$$

This is called risk minimization problem and can be solved by using Lagrangian multipliers. Equation (24) shows the Lagrangian function:

$$L = \frac{1}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + \lambda(1 - \boldsymbol{w}'1) + \gamma(\boldsymbol{\mu}_0 - \boldsymbol{w}'\boldsymbol{\mu}), \tag{24}$$

where λ and γ are Lagrangian multipliers to be used in estimation. (Fabozzi 2002, 317-318)

Lagrangian function is solved by using first order conditions which are partial derivatives with respect to each parameter. This starts by taking partial derivatives with respect to each parameter in Equation (24). Different equations are formed with respect to parameters **w**, λ and γ:

$$\frac{\partial L}{\partial w} = \boldsymbol{\Sigma}\boldsymbol{w} - \lambda 1 - \gamma\boldsymbol{\mu}_0 = 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - \boldsymbol{w}'\boldsymbol{1}$$

$$\frac{\partial L}{\partial \gamma} = \boldsymbol{\mu}_0 - \boldsymbol{w}'\boldsymbol{\mu}. \qquad (25)$$

When solving the first-order condition equation with respect to the vector **w** it gets the following form:

$$\boldsymbol{w} = \lambda\boldsymbol{\Sigma}^{-1}1 + \gamma\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \qquad \textbf{(26)}$$

Moreover the other two first-order condition equations are solved as follows:

$$1 = \boldsymbol{w}'\boldsymbol{1}$$

$$\boldsymbol{\mu}_0 = \boldsymbol{w}'\boldsymbol{\mu}. \qquad (27)$$

Equation (26) can be solved by using Equation (27):

$$\boldsymbol{w}'\boldsymbol{1} = \lambda\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1} + \gamma\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = 1$$

$$\boldsymbol{w}'\boldsymbol{\mu} = \lambda\boldsymbol{\mu}\boldsymbol{\Sigma}^{-1}\boldsymbol{1} + \gamma\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}_{0.} \qquad (27)$$

These equations can be solved by using the following matrix form:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\gamma} \end{pmatrix} = \begin{pmatrix} \boldsymbol{1} \\ \boldsymbol{\mu_0} \end{pmatrix}, \qquad (28)$$

where A = $\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1}$, B = $\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and C = $\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$. When combining Equation (22) and Equation (29) we can get the following equation for the variance equation:

$$\boldsymbol{\sigma}_0^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} = \frac{A\boldsymbol{\mu}_0^2 - 2B\boldsymbol{\mu}_0 + C}{\Delta}, \qquad (30)$$

where $\Delta = AC - B^2$. One can find each portfolio on the efficient frontier by calculating the value for the variance when the value for the mean return is known. Similarly we can derive an equation for the global mínimum variance portfolio:

$$\frac{d\sigma_0^2}{d\mu_0} = \frac{2A\boldsymbol{\mu}_0 - 2B}{\Delta} = 0. \qquad (31)$$

As shown in Equation (31) the global mínimum variance portfolio is actually the derivative with respect to the parameter $\boldsymbol{\mu}_0$. The optimal portfolio weights can expressed as follows:

$$\mathrm{w}=\frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \qquad (32)$$

(Fabozzi et al. 2010, 317-319) Equation (29), Equation (30) and Equation (31) are not derived here.

The portfolio optimization requires the definition of expected returns and variance-covariance matrix used. The most common way is to use historical estimates. However using historical information as a source of estimating future parameters is questionable. When focusing on information from a certain historical period, one can make false expectations about the future. (Fabozzi et al. 2002) There are also other methods, for example James-Stein estimators, Bayes-Stein estimators and CAPM based estimators. There is no common way to determine the method used but the choice can be evaluated in each context. For example, Grauer and Hakansson (1995) studied the use of all the above mentioned estimators in two settings: industry rotation and global asset allocation. They found out that James-Stein and Bayes-Stein estimators performed better than historical returns in industry rotation context but in the global asset allocation context it was the opposite. By contrast the CAPM estimators performed worse than historical estimators in both settings.

Another question to solve is the use of variance-covariance matrix. According to Ledoit and Wolf (2003) one should not use historical variance-covariance matrix but instead a shrinkage variance-covariance matrix. Historical variance-covariance matrix produces large estimation errors and this can be improved by the shrinkage method that pushes the extreme values closer to the average values to systematically reduce risk.

2.2 The Black-Litterman model

2.2.1 Theoretical background

In the framework of global portfolio management investors need to make important decisions on how to allocate the wealth accross different asset classes as well as the degree of hedging the currencies. Mostly the decision-making is done with a simple rule: maximize return and minimize risk. (Black & Litterman 1992) One major problem related to the traditional mean-variance optimization is that it often provides weights that imply large short position in some asset classes. When restricting the model only to long positions, this often results in zero weights as well as large weights in assets of small-cap markets. (Black & Litterman 1992) Moreover the estimation of the expected returns is complex and requires investors to have wide expertise in all the asset classes. Investors tend to rely on the historical average returns which are poor estimates for the future returns. The traditional models follow strictly the return distribution and the distinction between strong and weak views is not clear. (Black & Litterman 1992) Moreover, the use of traditional mean-variance optimization often leads to weights that do not make sense. The Black–Litterman model was developed to overcome most of these issues. (He & Litterman 1999)

The usefulness of the traditional models can be improved by incorporating the investors' views that move the neutral weights given by the market equilibrium portfolio to the direction of the investors' perceptions. This is the main contribution of the Black–Litterman model. (Black & Litterman 1992) The model was introduced by Fischer Black and Robert Litterman in 1990 to provide a new improved perspective to asset allocation. According to the model there exists an equilibrium market portfolio that should be used as a reference point to estimate the expected returns. Another contribution of the model is that it incorporates the investors' views on the asset returns. (Walters 2011) According to Cheung (2010) the Black–Litterman model is a portfolio construction tool that converts the investors' views into expected returns. The Black–Litterman model is based on Bayesian estimation method which relates the implied equilibrium market returns to the investor views to produce a revised vector of expected returns. (He & Litterman 1999)

The equilibrium market portfolio is the global equilibrium that is based on CAPM returns and it implies neutral weights. In the context of Black–Litterman model the investors can have either relative or absolute views on asset returns and adjust these neutral equilibrium values according to those views. Moreover they can evaluate their confidence on the views. If the investor has no views on assets, the weights are equal to the market equilibrium weights. (Black & Litterman 1992) In fact, the Black–Litterman portfolio is a combination of market equilibrium portfolio and the weighted sum of portfolios that incorporate the investors' views. A positive view represents bullish view i.e. the investor believes that the stock price will increase compared to the equilibrium and other stocks. (He & Litterman 1999) The weights can reflect the market equilibrium weights or they can be extreme compared to them depending on the investors' confidence on the views. Naturally the risk-taking tendency of the investor affects the weights. (Beach & Orlov 2007). The Black–Litterman model generates the optimal weights by estimating the certainty of the view and by computing the covariance both between the view and the equilibrium and among the views. (He & Litterman 1999)

The main benefit of the model is that it helps to avoid extreme values, i.e. it eliminates extreme corner solutions with extremely high expected return or a low estimate for volatility. The confidence of the views defines how much the estimates differ from the global equilibrium. (He & Litterman 1999) In practice the Black–Litterman model has been utilized by Goldman Sachs investment bank and it has been available for institutional investors and financial advisors worldwide. However the model has not reached too much attention in the literature mainly because of the complexity of the formulation of the view vector for the model. (Beach & Orlov 2007)

2.2.2 The parameters of the Black–Litterman model

According to the the Black–Litterman model, the expected returns are a combination of the equilibrium risk premiums and the expected returns based on the investors' subjective views. The construction of the model is depicted in Figure 8:

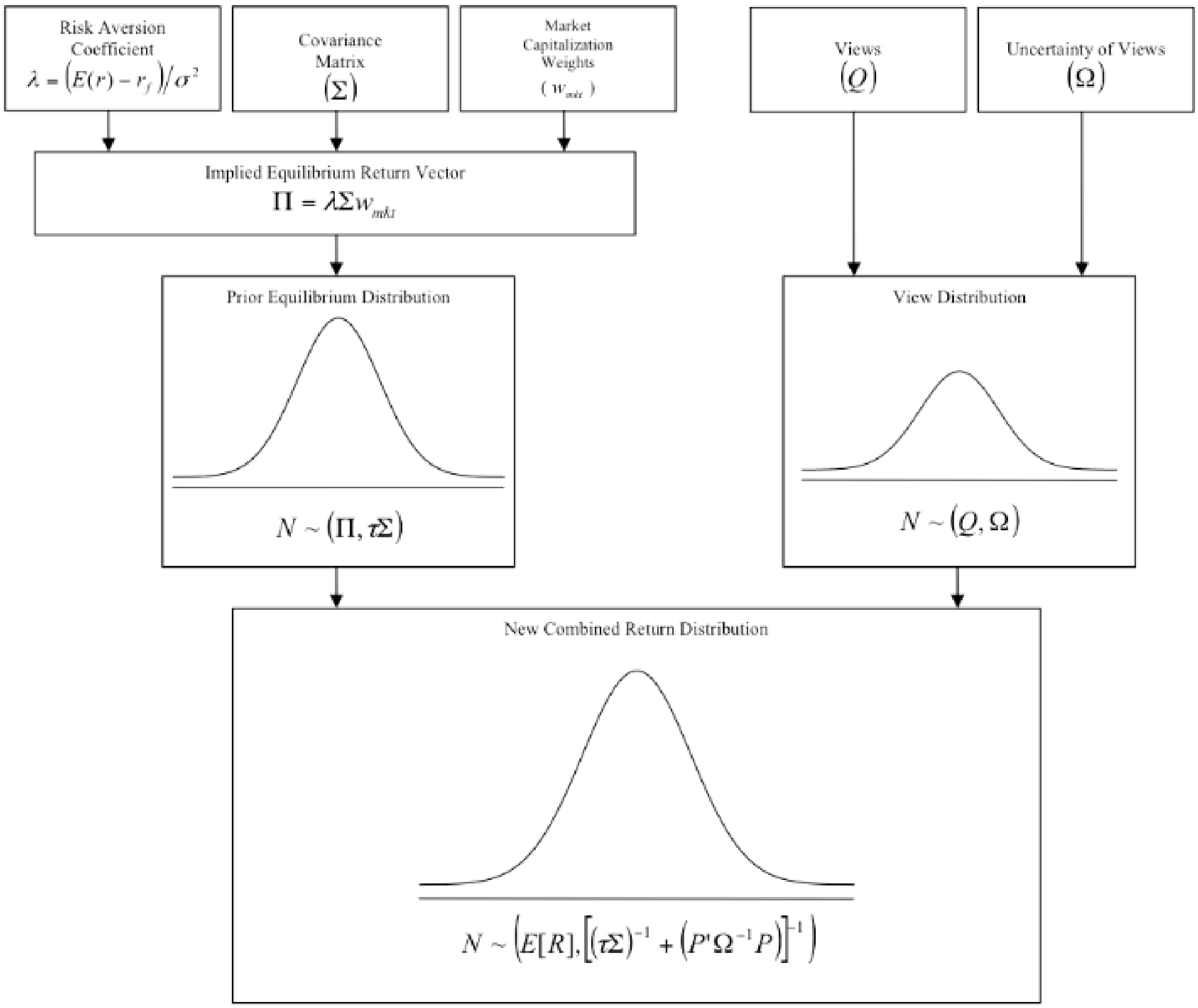

Figure 8: The construction of the Black-Litterman model

(Idzorek, 2005, 16)

On the left side of the figure, the formula for the implied equilibrium return vector is constructed from the risk aversion coefficient λ, covariance matrix $\mathbf{\Sigma}$ and market weights $\boldsymbol{w}_{mkt}$. In fact, the implied equilibrium return vector represents the parameter around which the mean of the expected returns is distributed. The implied equilibrium return vector represents the prior distribution in the Black–Litterman model. The right side of Figure 8 shows the distribution of the investors' views. The view distribution requires the specification of the parameters **Q** and **Ω** where **Q** represents the view-based returns and **Ω** represents the uncertainty relating to them, in other words, variance. Therefore the views are distributed around the mean value of **Q** and variance of **Ω.** Putting together the implied equilibrium return vector and the investors' views in a formula, we get the revised vector of expected returns:

$$\mathrm{E}[\mathbf{R}] = [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \mathbf{P}^{\mathbf{T}}\boldsymbol{\Omega}^{-1}\mathbf{Q}], \tag{33}$$

where $\mathrm{E}[\mathbf{R}]$ is the vector of revised expected returns, $\boldsymbol{\Omega}$ is the confidence matrix, τ is a multiplier related to uncertainty, $\boldsymbol{\Sigma}$ is covariance matrix, $\mathbf{P}$ represents the weights that include views, $\mathbf{Q}$ is a vector for actual views and $\boldsymbol{\Pi}$ is refers to the vector of implied equilibrium returns. (Idzorek 2005)

2.2.3 Implied equilibrium returns

The Black–Litterman model starts by assuming that investors can construct a portfolio of N assets including equities, bonds, currencies etc. This serves as the equilibrium portfolio for the prior distribution. This concept is based on the general equilibrium theory which states that in case the global portfolio lies at the equilibrium all the components must be at the equilibrium as well. The most common case is to use equilibrium model which is based on a quadraric utility function with an assumption of a risk-free rate of return. This leads to the application of CAPM model as an equilibrium market portfolio. (Walters 2011) These assets in the portfolio generate returns that follow the normal distirbution as follows:

$$\boldsymbol{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{34}$$

In Equation (34), the asset returns are denoted as a vector of returns of different assets. They are normally distributed around the mean return $\boldsymbol{\mu}$ and the risk of the portfolio which is represented by variance-covariance matrix $\boldsymbol{\Sigma}$.

As for the expected returns $\boldsymbol{\mu}$, they are unobservable but normally distributed around the equilibrium risk premiums. This is denoted as follows:

$$\boldsymbol{\mu} \sim N(\boldsymbol{\Pi}, \tau\boldsymbol{\Sigma}), \tag{35}$$

where $\boldsymbol{\Pi}$ represents the equilibrium risk premiums and $\tau\boldsymbol{\Sigma}$ is a product of the historical covariance matrix $\boldsymbol{\Sigma}$ and the multiplier τ. This represents the fact that the variance-covariance matrix for the estimated returns is proportional to the historical variance-covariance matrix of the returns. Therefore the parameter τ actually represents this proportionality and hence uncertainty. (Walters 2011) When there is no uncertainty involved, the parameter τ gets value of zero. The parameter $\boldsymbol{\mu}$ can be rewritten as an equation as follows:

$$\boldsymbol{\mu} = \boldsymbol{\Pi} + \varepsilon, \tag{36}$$

where μ is a 1x1 vector of mean return, **Π** is a vector of equilibrium risk premiums and ε is error term. The latter one has a normal distribution with a mean value of zero and standard deviation of **Σ**. Moreover it is assumed that the error term is uncorrelated with the parameter $\boldsymbol{\mu}$ (Walters 2011)

The equilibrium risk premiums are the exceess returns compared to CAPM. They represent the global supply and demand of assets and currencies. The equation for the equilibrium risk premiums is constructed in Equation (37):

$$\boldsymbol{\Pi} = \lambda\boldsymbol{\Sigma}\boldsymbol{w}_{mkt} + \boldsymbol{r_f}, \tag{37}$$

where λ is the investor's risk aversion, $\boldsymbol{\Sigma}$ is the variance-covariance matrix, $\boldsymbol{w}_{mkt}$ represents the weights of equilibrium market portfolio and $\boldsymbol{r_f}$ is 1x1 vector of risk-free rate. (He & Litterman 1999) The concept of the equilibrium portfolio means that in the long-run the expected returns cannot differ significantly from the equilibrum values as the market inbalancies make them move closer to each other. (Black & Litterman 1992) Calculation for the risk aversion factor is shown below:

$$\lambda = \frac{E(R) - R_f}{\sigma}. \tag{38}$$

In fact, this measure is Sharpe Ratio as it has excess returns as a numerator and the measure of risk as denumerator. (Idzorek 2005)

A common approach is to use CAPM as a prior for the Black–Litterman model (He & Litterman 1999). CAPM model can be rewritten as shown in Equation (39) the first one presenting the conditional expected returns for the asset i and the latter one representing that of the market portfolio:

$$E\left[r_{i,t+1} \middle| \boldsymbol{\Omega}_t\right] = \lambda_m Cov(r_{i,t+1}, r_{m,t+1} | \boldsymbol{\Omega}_t)$$

$$E\left[r_{m,t+1} \middle| \boldsymbol{\Omega}_t\right] = \lambda_m Var\left(r_{m,t+1} \middle| \boldsymbol{\Omega}_t\right), \tag{39}$$

where $E\left[r_{i,t+1} \middle| \boldsymbol{\Omega_t}\right]$ is the conditional expected return for the asset *i*, $E\left[r_{m,t+1} \middle| \boldsymbol{\Omega_t}\right]$ is the conditional expected return for the market portfolio, λ_m is the risk aversion factor, $Cov(r_{i,t+1}, r_{m,t+1} | \boldsymbol{\Omega}_t)$ is the conditional covariance between the return of the asset *i* and the market return and $Var(r_{m,t+1} | \boldsymbol{\Omega}_t)$ is the variance of the market portfolio. The returns are conditional on the information at time t and this is repsesented with the symbol $\mathbf{\Omega}_t$. The parameter λ_m and the variance of the market returns can be estimated by using GARCH-in-mean model. As shown in Equation (39) in this case the parameter λ is considered time-variant even though commonly it is assumed to be independent on time. (Antell & Vaihekoski 2007)

2.2.4 Investors' views on asset returns

The investors' views are modelled as a conditional distribution with respect to the estimated excess returns. It is assumed that all the views are uncorrelated with each other. The investors may have either absolute or relative views on the asset returns that differ from the equilibrium returns. Moreover the investors can express their confidence on the views which relate to the Bayesian nature of the investment risk. The confidence or in other words the quality of these investors views are presented as the matrix of uncertainty which is a diagonal matrix presenting the confidence estimates on the diagonal and setting the non-diagonal values zero. (Beach & Orlov 2007) According to Walters (2011) this assumption holds for two main reasons. Firstly the correlation of the views would make the view matrix very complex. Secondly all the views sum up to either 0 or 1 in case of relative and absolute views, respectively.

The views are normally distributed:

$$\boldsymbol{P} \sim N(\boldsymbol{Q}, \boldsymbol{\omega}), \qquad (40)$$

where **P** represents the weights of the assets that are given an active view, **Q** is the expected return and ω is the standard deviation of the view. In relation to the expected returns, the views can be expressed with the following equation:

$$\boldsymbol{P}E[\boldsymbol{R}] = \boldsymbol{Q} + \boldsymbol{e}, \qquad (41)$$

where **P** is a vector of the weights for the assets that are given an active view, $E[\boldsymbol{R}]$ is a vector of the expected returns, **Q** is the absolute or relative view and e includes the error related to the investors' views (He & Litterman 1992). When n represents the number of assets and k is the number of views, we can specify that **P** is a k x n matrix, **Q** is k x 1 matrix and $\mathbf{\Omega}$ is k x k matrix. (Walters 2011) The vector e is normally distributed with the following setting:

$$\boldsymbol{e} \sim N(\mathbf{0}, \boldsymbol{\Omega}), \tag{42}$$

where $\boldsymbol{\Omega}$ is a diagonal matrix that indicates the covariance of the views. The inverse matrix $\boldsymbol{\Omega}^{-1}$ refers to the confidence of the views. As the views are assumed to be independent their covariance is zero. Therefore the off-diagonal values in the matrix are zero and diagonal values represent the variances of the views. The covariance matrix of the error term is formed in Equation (43):

$$\Omega = \begin{pmatrix} \omega_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_k \end{pmatrix}, \tag{43}$$

where ω_k is the variance of the error term of the view k. This assumption of independence of the views results to more stabile and simple results. (Walters 2011) The variance of the error term ω_k has an inverse relationship with the investor's confidence on the view. This means that when the variance of the error term is zero, the confidence of the view is 100 % (Idzorek 2005). A zero value on the diagonal therefore refers to a view of which the investor is fully certain (Walters 2011).

When expressing the confidence matrix in terms of variances of the views, we get the following matrix:

$$\Omega = \begin{pmatrix} \boldsymbol{p}_1 \boldsymbol{\Sigma} \boldsymbol{p}_1' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{p}_k \boldsymbol{\Sigma} \boldsymbol{p}_k' \end{pmatrix}, \tag{44}$$

where p_k is the weight of an individual active view in the **P** matrix and **Σ** is the covariance matrix. This variance of a view portfolio affects the confidence that is given to a specific view. The higher the certainty of the views, the closer the revised expected return vector is to the views. By contrast, when the value for certainty gets lower, the revised expected return vector approaches the implied equilibrium returns. (Idzorek 2005)

The equation for the investor views can be understood by having a look at the example of Walters (2011). In the example, there are four assets and two views. The first view is that the asset 1 will outperform the asset 3 by 2 %. The second view is that the asset 2 will yield 3 %. The first view is relative and the second one is absolute. This can be expressed with the vector notations as follows:

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{Q} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}. \quad (45)$$

Moreover the sum of the weights needs to be consistent so that in case of relative views, the sum equals zero and in case of absolute views, it equals one. (Walters 2011)

In case the investor has no views on a specific asset and there is zero confidence on the views the parameters **P** and τ are zero and it holds that the expected returns are equal to the implied returns from the prior distribution:

$$E(R) = \boldsymbol{\Pi}, \quad (46)$$

where $E(R)$ is the expeceted return and $\boldsymbol{\Pi}$ is the implied return (Beach and Orlov 2007). When combining the implied equilibrium returns and the distribution for the investor views the Black–Litterman returns can be calculated with Equation (33).

2.2.5 The Black–Litterman weights

Once the vector of revised expected returns is consructed the weights can be calculated as follows:

$$\boldsymbol{W}_{\mathbf{i}} = \boldsymbol{\Sigma} E(R), \quad (47)$$

where $\boldsymbol{W}_i$ is 1x1 vector representing weight of the asset i, $\boldsymbol{\Sigma}$ is NxN vector of the historical variance-covariance matrix and $E(R)$ is the vector of revised expected returns. The proportional weights are calculated by dividing the weight of the asset by the sum of all the weights of the assets:

$$w_i = \frac{W_i}{\sum_{i=1}^{n} W_i}. \quad (48)$$

(Beach & Orlov 2007)

According to He and Litterman (1992) the optimal weights are influenced by the investors' views in three ways. Firstly, it is obvious that the stronger the view is the more it impacts the optimal portfolio. Secondly it can be showed that the covariance between the view portfolio and the market portfolio reduces the impact of the view on the optimal portfolio. This is because the prior distribution already includes all the information about the market equilibrium and therefore the view would be considered as a less informative one. Thirdly the covariance between different views diminishes the effect of views on the final portfolio. The covariance between views would mean that the same information is included many times.

2.2.6 Issues related to the Black–Litterman optimization

One of the most controversial issues relating to the Black–Litterman optimization is the estimation of the parameter τ which represents the uncertainty of the mean in the prior distribution. Black and Litterman (1992) recommend to use the value zero because the uncertainty in the returns is considered higher than in the mean. By contrast, Satchell and Scowcroft (2000) suggests the value to be one. The smaller the value for this parameter is the larger is the weight given to the equilibrium implied returns. When τ is closer to zero it gives more weight to the market equilibrium. (Beach & Orlov 2007).

According to Beach and Orlov (2007) a good practice to analyze whether the parameter τ has a reasonable value is to calculate the portfolio variance as follows:

$$\sigma = w'\mathbf{\Sigma}\mathrm{w}, \qquad (49)$$

where w is the weight of the asset and $\mathbf{\Sigma}$ is the historical variance-covariance matrix. When calculating this value it can be easily observed whether the risk of the portfolio is more than allowed. In that case the investor can adjust the parameter τ so that the risk reaches the desired level.

Another problematic parameter to estimate is the confidence matrix $\mathbf{\Omega}$. He and Litterman (1999) suggest that there is a relationship between the confidence matrix $\mathbf{\Omega}$ and the parameter τ. This relationship is defined so that the variance of the view portfolio equals the ratio that divides the variance of the error term ω_k by the parameter τ:

$$p_k\mathbf{\Sigma}p_k' = \frac{\omega_k}{\tau}. \qquad (50)$$

This implies that the confidence matrix gets the following form:

$$\boldsymbol{\Omega} = \begin{pmatrix} (p_1 \boldsymbol{\Sigma} p_1')\tau & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (p_k \boldsymbol{\Sigma} p_k')\tau \end{pmatrix}. \qquad (51)$$

In fact when using this expression the value of the parameter τ becomes insignificant since only the ratio ${}^{\omega_k}/_{\tau}$ is incorporated in the model. (Idzorek 2005)

Walters (2011) presents four different methods to compute the uncertainty matrix $\boldsymbol{\Omega}$. Firstly we can assume that the confidence matrix is proportional to the variance of the prior distribution. Secondly one can use confidence interval to estimate the variance around the mean return estimate. Thirdly the investors may use the variance of the residuals from the factor model used to derive the excess returns. Fourthly one could use the method introduced by Idzorek (2005) in which the confidence of the view is presented as a percentage shift of the weights from 0 % to 100 %.

Satchell and Scowcroft (2000) have studied the Black–Litterman model in the non-Bayesian context. They introduce a model which is based on point estimation in terms of prior distribution and investors' views and they adjust the parameters $\boldsymbol{\tau}$ and $\boldsymbol{\Omega}$ according to shrinkage of views to the prior point estimates. This model was widely used in the literature until the publication of the model by Meucci (2005). The latter one is referred as a shrinkage model in which the parameter $\boldsymbol{\Omega}$ is varying freely and the parameter τ is complex. Idzorek (2005) introduces an approach to specify the value of $\boldsymbol{\Omega}$ so that it gets a shrinkage percentage value between 0 % and 100 %.

3 ECONOMETRIC APPROACH

3.1 Introduction for applying GARCH models in tactical asset allocation

To study the applicability of the Black–Litterman model one needs estimates for the returns and the volatility. There is no common way to predict investors' views but estimates for returns and variances can be jointly estimated with GARCH model. The main reason to apply GARCH models to generate inputs for the investors' views for the Black–Litterman model is that there often exist statistical regularities in asset returns and this impact can be captured when using a time-varying volatility estimation model. (Palomba 2008) This study follows the steps of the articles of Beach and Orlov (2007) and Duqi et al. (2014) which use EGARCH-in-mean model to predict investors' views for the Black–Litterman model.

The return and variance estimates enter the Black–Litterman model in the form of parameters **P**, **Q** and **Ω**. The parameter **P** represents the weights for each view and the parameter **Q** includes the views. By contrast the parameter **Ω** refers to the confidence in views. In this research we apply an exponential GARCH-in-mean model to get these estimates and to test the Black–Litterman model in reality. The estimates for the parameters **P** and **Q** can be generated from the return estimates and the parameter **Ω** can be estimated from the residuals of the results of EGARCH-M. These views are then used as inputs to calculate the revised vector of returns as in Equation (33).

In practice the EGARCH-M model consists of two equation: mean equation and variance equation. When running the model we get estimates for different parameters of these equations and these parameters are used to calculate estimated returns and variances. In this study these estimates are then used when formulating theoretical investors' views for the Black–Litterman model. In practice this means specifying the matrices **P**, **Q** and **Ω.** We form three relative views so that we compare the performance of two different assets at two consecutive periods. The weights of the active assets are included in the matrix **P** and the variance of the returns of the active assets are represented in matrix **Q.**

EGARCH-M model can be considered as a good tool to generate inputs for the investors' views in the Black–Litterman model as it provides simultaneously return and variance forecasts for the future subsequent periods. The model is complex and has a lot of different parameters but it is easy to apply with R program as in this study.

3.2 ARCH and GARCH models

One of the basic assumptions of the ordinary least squares estimation is homoscedasticity, i.e. the expected value of the squared error term gets the same value at all the points. In case the values of these error terms differ the data exhibits heteroscedasticity. As a result one could use ARCH or GARCH models which predict the variance of each error term separately. (Engle 2011)

Momentum refers to persistence in variance, i.e. the current variance is explained by the past variance. This determines the effect of volatility on the stock prices. (Lamoureux & Lastrapes 1990) To estimate this persistence of variance, GARCH model was developed by Engle (1982) and Bollerslev (1986). Engle introduces the concept of ARCH model which captures the effect of momentum in variance to test whether the error term of an autoregressive model (AR) depends on its lagged values. This model was generalized by Bollerslev to autoregressive moving average models (ARMA) so that the effect of the lagged variance is taken into account as well. According to Engle (2001) GARCH models are practical when estimating volatility and the magnitude of errors especially when the time series exhibit strong heteroscedasticity. GARCH models are a common tool for forecasting the volatility as a function of a long-term volatility trend, the estimate of volatility from preceding periods and the information about volatility from past periods.

The first order ARCH model, ARCH(1,1), can be presented as follows:

$$r_t = \boldsymbol{x}'_t\gamma + \varepsilon_t$$

$$\sigma_t^2 = \omega + \alpha{\varepsilon_t}_{t-1}^2, \quad (52)$$

where r_t represents the estimated return, $\boldsymbol{x}'_t$ is a vector of all the independent variables and ε_t is the error term. It is assumed that the error term has a normal distribution so that $\varepsilon_t \sim N(0, \sigma_t^2)$.The latter equation represents how the variance of the error term depends on the mean ω and the preceding error term ε_{t-1}^2. The first order GARCH model, namely GARCH(1,1) model, extends this by adding another regressor to the variance equation. Equation (53) represents this dependence:

$$r_t = x'_t\gamma + \varepsilon_t$$

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \quad (53)$$

where the latter equation has a new parameter σ_{t-1}^2 which represents the variance from the last period. GARCH(1,1) is the simplest form of GARCH model. It refers to a model in which the parameters have only one lagged value so that the values are from the preceding period. The model can be extended so that it includes more lagged values to capture the effect of a longer period of time.

3.3 EGARCH-in-mean model

Introduced by Engle, Lilien and Robins (1987) ARCH-in-mean model extends the original ARCH model so that it adds the effect of variance to the mean equation. As a result the fluctuations in the conditional variance have a direct impact on the return of the portfolio. This is demonstrated with the following equation:

$$r_t = \mu + \delta\hat{\sigma}_t^2 + \varepsilon_t, \tag{54}$$

where r_t represents the mean return at time *t*, $\hat{\sigma}_t^2$ is the estimate of variance, δ is the coefficient that determines the effect of variance on the mean return and ε_t is the error term. When applying this to the original GARCH model, we get the following equations:

$$r_t = x_t'\gamma + \delta\hat{\sigma}_t^2 + \varepsilon_t$$

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \tag{55}$$

where the new paramater δ determines the impact of variance $\hat{\sigma}_t^2$ on the return (risk-return trade-off). The above equations prove that the inclusion of ARCH-in-mean has an effect on the mean equation and the variance equation remains the same as in the original GARCH model.

EGARCH model was developed by Nelson (1991) to take into consideration the tendency of negative shocks to increase the volatility and the positive shocks to decrease it. The ordinary GARCH models give importance only to the size of the shocks regardless their direction. The simple EGARCH model can be expressed as follows:

$$r_t = x_t'\gamma + \varepsilon_t$$

$$\log\sigma_t^2 = \omega + \beta\log\sigma_{t-1}^2 + \alpha\left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + \gamma\frac{\varepsilon_{t-1}}{\sigma_{t-1}}, \tag{56}$$

where ω is the mean, log σ_{t-1}^2 is the logarithmic first-order lagged value of the volatility, $\frac{\varepsilon_{t-1}}{\sigma_{t-1}}$ is the quotient of the past value for the error term and the past value for the volatility and $\left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right|$ is the absolute value of this. As Equation (56) shows the conditional variance depends on three effects. The first one is autoregressive effect captured by the term $\beta \log \sigma_{t-1}^2$. This determines the persistence of volatility over time. The parameter β determines how fast the past shocks have an impact on the future volatility. The persistence coefficient is set to get values between 0 and 1 to be stationary. The second term represents the leverage effect which captures the negative correlation between volatility and investor's reaction to shocks. This can be understood so that a decrease in the stock price makes the firm's debt to equity ratio higher and similarly an increase makes it lower. As a result, this affects the volatility inversely. This is represented with the term $\alpha \left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right|$ in Equation (56). The third effect captures the asymmetry effect, i.e. the fact that negative and positive shocks cannot have the same impact on volatility. Asset returns tend to behave so that negative shocks have larger effect on volatility than equivalent positive shocks. This is the term $\gamma \frac{\varepsilon_{t-1}}{\sigma_{t-1}}$ in Equation (56). (Duqi et al. 2014; Indro, Jiang & Lee 2002)

Putting the two models, exponential GARCH and GARCH-in-mean, together we get the following presentation:

$$r_t = x_t'\gamma + \delta\hat{\sigma}_t^2 + \varepsilon_t$$

$$\log \sigma_t^2 = \omega + \beta \log \sigma_{t-1}^2 + \alpha \left|\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| + \gamma \frac{\varepsilon_{t-1}}{\sigma_{t-1}}. \qquad (57)$$

This model is estimated to obtain inputs for the view vector in this research. However the model is extended with one external regressor, the change in the price of oil.

3.4 Previous studies of GARCH models in the Black–Litterman context

Beach and Orlov (2007) have used monthly data from 15 years to estimate EGARCH-M model to predict volatility forecast for the period of 120 months. They use country-specific data for a set of Morgan Stanley Capital International equity indices. Moreover they add a number of explanatory macroeconomic factors to the model including the growh of industrial production, the return of the US dollar index compared to other currencies, the difference between two different credit ratings from Moody's and the change in the price of oil. The purpose of their research is to select en econometric model which reflects the properties of the asset returns and their volatilities and to apply this model in the context of the Black–Litterman model. They use EGARCH-M model to generate proxies for the views and then calculate the optimal weights for the global portfolio by using these inputs. As a result of their study they find out that by using EGARCH-M method for the investor views we can get higher returns for the assets with the same risk level as in the equilibrium. Moreover, they compare the results from the risk-adjusted Black–Litterman weights to the traditional Markowitz weights and the major finding is that the weights estimated with the Black–Litterman are less extreme in both minimum and maximum values. Duqi et al. (2014) used a similar approach but with a different data set. They use daily data from US markets covering a period of 10 years. The data consists of 30 indices that are included in Dow Jones Industrial Index.

In their research, Beach and Orlov (2007) find out that the deviations between any two subsequent months is correlated with the past periods. Therefore they emphasize the importance of predicting conditional variance and they focus on examining volatility clustering. This makes the use of GARCH models a logic choice. The reason for using EGARCH-M is that the exponential effect takes the asymmetry of the volatility shocks into account and the arch-in-mean effect relates the expected return and the risk. In fact including the exponential term in GARCH model is very reasonable as in equity markets it is common that the volatility reacts more sensitively to negative shocks in markets than to the equivalent positive shocks. According to Duqi et al. (2014) EGARCH models can be used to estimate conditional variance of individual assets and this helps to analyze the risk-return trade-off of the stocks. It allows the estimation of medium-term stock risk as well as the speed of volatility to approach the mean-reverting value.

The articles of Beach and Orlov (2007) and Duqi et al. (2014) start by assuming that the global equilibrium portfolio is built on the actual market capitalization weights. Equilibrium expected returns are generated by using reverse optimization, assets are sorted based on their performance and the views are formulated with the use of EGARCH-M model. These equilibrium expected returns and investor views define the vector of expected returns. This vector together with the covariance matrix is used to predict the Black–Litterman weights. As a measure for analyzing the risk of different assets Duqi et al. (2014) use the half-life index introduced by Bollerslev (1994). Half-life index is a measure for persistance in volatility, i.e. the tendency of volatility to approach the long-term average. It is used to determine the risk of the stock as the higher the absorption speed of past volatility the higher is the risk. Half-life index defines the number of days required to halve the gap between the unconditional volatility and the conditional one. It can be calculated with the following equation:

$$\textit{half-life index} = \frac{\ln(0{,}5)}{\ln(\beta)}, \qquad (58)$$

where β is the parameter in EGARCH-M model that defines the effect of the past variance to the variance equation.

According to Beach and Orlov (2007) the hedging for the foreign currency is not necessary in the context where the purchasing power parity holds as the currency returns would be reversed by impact of inflation. Black (1989) advices to hedge all the foreign investments equally so that the hedging is less than 100 %. In fact there is no specific hedging ratio and it may vary between 0 and 100 %. Black and Litterman (1992) suggest a hedging ratio of 80 %. Beach and Orlov (2007) argue that including the hedging ratio in their research context is not necessary as the benefits are marginal and there are other major issues to overcome when estimating the Black–Litterman model.

4 DATA, RESEARCH METHOD AND DESCRIPTIVE STATISTICS

4.1 Data and research method

The data used in this study covers time series of twelve MSCI total return indices of different market cap sizes. As the study is conducted from an US investor's point of view all the indices are presented in US currency. The data is collected from Thomson Reuters Datastream and it covers daily data from the period of 3.10.2008–3.10.2018. In this study we estimate Black–Litterman weights for two subsequent periods out-of-sample by using the same dataset with different time horizons. The first set covers the period of eight years and the second one nine years. The purpose is to find out the benefit or loss for the investor who is expected to follow a "buy and hold strategy". This strategy is based on the efficient market hypothesis and it means that an investor purchases an asset to hold it for a long period of time (Hui & Yam 2012).

The total return indices are selected randomly from three different categories: developed markets, emerging markets and frontier markets. The purpose is to build a diversified portfolio of assets from different economic backgrounds and geographical areas. For simplicity the study is restricted to cover only equity indices but the same study could be broadened to include other asset classes as well. The classification is done based on MSCI annual market classification review (MSCI Inc 2018) and the indices selected are presented in Table 1.

Table 1:Classification of equity indices

Developed markets	Emerging markets	Frontier markets
MSCI Australia	MSCI Brazil	MSCI Argentina
MSCI Norway	MSCI China	MSCI Kuwait
MSCI Portugal	MSCI India	MSCI Morocco
MSCI United Kingdom	MSCI Russia	MSCI Vietnam

Emerging markets are known to exhibit high volatility, low correlations and high long-term returns. The main contribution of investing in emerging markets is the diversification benefit. They are easily exposed to political conflicts, flunctuations in exchange rates and regulatory changes. In the mean-variance optimization context the inclusion of emerging markets moves the efficient frontier to the right. In fact the mean-variance optimization is not the most optimal tool to be applied for emerging markets as it takes only the mean and the variance of the asset returns into account while the assets in emerging markets exhibit strong skewness and kurtosis. (Bekaert, Erb, Harvey & Viskanta 1998) In this research we concentrate on BRIC countries, namely Brazil, Russia, India and China. These countries are large emerging markets with an average income which are expected to increase the world economic growth. These countries differ from each other for example by their structural attributes, economic practices and geopolitical influence. In China and India the majority of the people live in the countryside while in Brazil and Russia most of the population is focused on urban areas. The capital markets of China and India are more closed and state-oriented than in Brazil and Russia. (Bianconi, Yoshino & Machado de Sousa 2012)

The new emerging markets, namely frontier markets, have grown recently and they are a great opportunity to increase the risk-return trade-off of the portfolio. (Groot, Pang & Swinkels 2012) The term frontier markets was established by the International Finance Corporation in 1992. It describes small economies that are economically less developed than emerging markets. (Fowler 2010) Similarly as emerging markets frontier markets offer great diversification benefits. They can offer even higher expected returns but this is compensated with higher risk. (Pop, Bozdog & Calugaru 2013) Goetzmann, Li and Rouwenhorst (2001) argue that the diversification benefits can be mainly reached when investing on emerging markets. However according to Bekaert et al. (1998) the assets from emerging markets should not be analyzed in the same way as the assets from developed markets because of the high volatility and the non-normality of the returns. For simplicity this assumption is ignored and all the assets are treated equally in this study.

As a risk-free rate of return, 10-Year US Treasury Bond rate is used. This is subtracted from the asset returns to obtain the excess returns. Since the logarithmic returns have better statistical properties, the returns are transformed to logarithmic. The time series of the assets in developed markets, emerging markets and frontier markets during the period of 2008-2018 are presented in Appendix 1.

It can be easily observed that all the time series represent a similar pattern showing stock market crashes in years 2008 and 2011. The crash in 2008 relates to the subprime crisis and the collapse of Lehman Brothers investment bank (The New York Times, 2008). Falling stock prices in 2011 was affected by the debt crisis in Europe and the collapse of the stock market in 2016 was mainly influenced by the referendum about Brexit. The emerging markets were mainly influenced by the increase of the US dollar. (Financial Times 2011; CNBC 2016) When looking at the time series of developed markets the asset returns of Australia and UK seem to have a similar pattern. The asset returns of Norway move similarly but there is a larger relative decrease from 2014 to 2016. The asset returns of Portugal are varying more largely. As for the asset returns of BRIC countries there are clear similarities between China and India and some similarities between Brazil and Russia. These findings match with the expectations about the capital markets and the state orientation discussed previously. The time series of the asset returns of the frontier markets look all different but we can find some similarities at the peaks.

A common way to define the benchmark market portfolio is to use a frequently-used stock market index, e.g. S&P 500 or MSCI World index. However, for simplicity, this study applies naive allocation (1/N) which gives an equal weight to each asset. According to DeMiguel et al. (2009) the naive allocation should be considered as the most attractive option to define the benchmark since it is easy to apply and it has a relatively small error compared to other options.

This study applies quantitative research methods to conduct a scientific research in the context of the Black–Litterman model. The methods applied in the research are econometric. All the calculations and programming are done with R programming language. The codes used are presented in Appendix 2. We use two different methods to calculate the implied returns: CAPM and another method based on risk aversion and covariance between the assets. EGARCH-M model is used to estimate proxies to be used as inputs for the view vector. Finally the revised vector of expected returns is calculated so that the views estimated with EGARCH-M are incorporated. The estimation is done with maximum likelihood method and it is assumed that the errors have a normal distribution.

4.2 Descriptive statistics

Descriptive statistics are calculated to analyze the properties of the time series and to analyze the differences between different MSCI country indices. The descriptive statistics computed in this research are mean, standard deviation, Sharpe ratio, skewness, kurtosis and the first-order autocorrelation AC(1). Table 2 shows descriptive statistics for the twelve assets in the portfolio as well as for the portfolio when using naïve allocation.

Table 2:Descriptive statistics

	Mean (%)	SD (%)	Sharpe	Skewness	Kurtosis	AC(1)
Argentina	0.158	36.673	0.159	-0.348	6.433	0.082
Australia	0.388	25.580	0.123	-0.998	11.064	0.020
Brazil	0.107	35.419	0.086	-0.231	9.683	0.042
China	0.493	25.584	0.123	-0.067	9.138	0.029
India	0.400	25.002	0.124	0.337	14.312	0.056
Kuwait	-0.116	20.283	0.147	-1.037	14.497	0.051
Morocco	-0.087	16.119	0.186	-0.013	4.082	0.104
Norway	0.351	32.431	0.096	-0.363	7.819	-0.016
Portugal	-0.110	25.979	0.115	-0.135	6.155	0.078
Russia	0.177	37.777	0.081	-0.710	15.196	0.076
UK	0.290	23.242	0.132	-0.326	11.491	0.017
Vietnam	0.173	23.480	0.130	-0.214	1.758	0.139
1/N PORTFOLIO	0.185	27.298	0.125	-0.342	9.302	0.057

The statistics are calculated for continuously compounded daily returns. Mean and standard deviation are annualized assuming 260 trading days per year. The descriptive statistics for the total portfolio are computed for the naive allocation 1/N so that each asset has an equal weight 1/12. Based on the results in Table 2 we can observe that China has the largest mean value (0.493 %) while Kuwait has the smallest value (-0.116 %). As for the standard deviation values Russia has the largest deviation (37.777 %) among the return series and Morocco has the smallest deviation (16.119 %).

Sharpe ratio measures the excess return of the investment to it’s standard deviation. It can be calculated as follows:

$$Sharpe\ ratio = \frac{\mu - R_f}{\sigma} \quad (59)$$

(Lo 2002). The risk-free rate used in this case is the expected value of the yearly risk-free rate of return from 10-years Government Treasury Bill time series. Based on the results in Table 2 Sharpe ratio is the largest for Morocco (0.186) and smallest for Russia (0.081). These results reflect that none of the assets have a strong risk-adjusted return. A positive value for Sharpe ratio means that the mean value for the asset returns is larger than the risk-free rate of return and in a case of a negative it is viceversa.

As for the skewness of the asset returns it is easy to observe that most of the assets have a negative skewness. Only India has a positive skewness. Skewness measures asymmetry of the return distribution. Basically this means that for the negative skewed asset returns the tail of the return distribution is on the left and for the positive skewed asset returns the tail is on the right side. Kuwait has the strongest negative skewness (-1.037) and Morocco has the weakest (-0.013). Kurtosis is the measure of tailedness and the values can be similarly positive or negative. The larger the value the larger the less peak the asset distribution has. A large positive value for kurtosis is called leptokurtic and it means that the distribution has a high peak. By contrast a large negative value for the kurtosis means that the asset distribution has a flatter form. As for the results obtained in Table 2 we can conclude that all the assets have a positive kurtosis. Russia has the largest value (15.196) and Vietnam has the smallest (1.759).

AC(1) refers to the first-order autocorrelation. Based on the results in Table 2, Norway is the only one with a negative first-order autocorrelation value (-0.016). It means that the dependence of returns on it's first-order lagged value is negative. This is also the smallest dependence among all the assets in the portfolio. Vietnam has the largest value (0.139) so the returns of this index have the strongest correlation with the past values.

As a part of descriptive statistics also the correlation matrix is calculated to see how the assets are correlated with each other. In the Black–Litterman context we are interested in the excess returns. Therefore the correlation matrix is calculated for excess returns instead of simple asset returns. Table 3 shows the correlation matrix for the twelve test assets:

Table 3: Correlation matrix

	AR	AU	BR	CN	IN	KW	MA	NO	PT	RU	UK	VN
AR	1.000											
AU	0.304	1.000										
BR	0.481	0.466	1.000									
CN	0.271	0.627	0.428	1.000								
IN	0.277	0.489	0.400	0534	1.000							
KW	0.089	0.163	0.120	0.142	0.135	1.000						
MA	0.151	0.271	0.203	0.177	0.161	0.121	1.000					
NO	0.423	0.609	0.623	0.442	0.460	0.138	0.302	1.000				
PT	0.374	0.549	0.566	0.448	0.444	0.168	0.202	0.696	1.000			
RU	0.453	0.633	0.632	0.456	0.470	0.146	0.297	0.818	0.637	1.000		
UK	0.388	0.535	0.543	0.374	0.392	0.141	0.340	0.715	0.540	0.746	1.000	
VN	0.084	0.211	0.080	0.164	0.147	0.109	0.074	0.123	0.147	0.121	0.116	1.000

According to Table 3 the strongest correlation is between Russia and Norway (0.818). Accordingly the weakest correlation is between Brazil and Vietnam (0.080). The correlation between assets tell how the assets move together. When two assets have a large value for the correlation they tend to move together at the same time.

4.3 Prior distribution in the Black–Litterman model

The analysis of the Black–Litterman model starts by defining the prior distribution which is used to estimate the posterior distribution along with the additional information. This additional information is the investors' views that will be estimated in this study with EGARCH-M methodology. (He & Litterman 1999). In this study the prior distribution is naive allocation based on CAPM. This means that each one of the twelve asset is given a weight of (1/12) at the beginning of the estimation.

When estimating the implied returns, a similar approach is used as in the article of Duqi, Franci and Torluccio (2014). The calculation of risk aversion coefficient involves defining the parameters for market return, risk-free return and volatility. As a proxy for market return we use the realized average return of the twelve test assets in the portfolio and as a measure of volatility we use the average variance of the stocks in the portfolio. The current 10-years Government Treasury Bill is used as a proxy for the risk-free rate. Table 4 summarizes the parameters used to calculate the risk aversion coefficient.

Table 4:Parameters for the risk aversion coefficient

Parameter	$E[R_m^g]$	R_f	σ^2
Value	0.185	-0.030	1.341

In this case $E[R_m^g]$ represents global equally weighted portfolio of twelve equity índices. The value for λ obtained with these parameters is 0.173.

The next step is to calculate the historical variance-covariance matrix. It reflects the fact that the assets from the emerging markets have a small covariance with the assets from the developed markets and with the other assets from the emerging markets. (Bekaert et al. 1998) The last parameter needed for implied equilibrium returns is market weights. Table 5 summarizes the results for the implied returns using naïve allocation.

Table 5:Implied returns

	Π
Argentina	23.420 %
Australia	20.139 %
Brazil	29.134 %
China	17.016 %
India	15.991 %
Kuwait	3.895 %
Morocco	4.297 %
Norway	29.868 %
Portugal	20.694 %
Russia	32.518 %
UK	20.496 %
Vietnam	4.998 %
AVERAGE	18.539 %

Russia has the largest value for implied returns (32.518 %) while Kuwait has the smallest value (3.895 %). Alternatively this can be calculated with CAPM. The parameters needed are risk-free rate of return, average rate of return of the market portfolio and beta coefficient. The last one is calculated with the use of variance of the market portfolio and covariance of each asset with the market portfolio. The variance of the market portflio is 1.245 and the covariance is calculated for each asset in the first column in Table 6. The second column shows the beta coefficients that are calculated with these parameters.

Table 6:**Calculation of beta coefficients for CAPM**

	$cov(r_i r_m)$	β
Argentina	1.527	1.227
Australia	1.337	1.074
Brazil	1.857	1.492
China	1.157	0.929
India	1.100	0.882
Kuwait	0.399	0.320
Morocco	0.422	0.339
Norway	1.899	1.526
Portugal	1.370	1.100
Russia	2.053	1.649
UK	1.358	1.091
Vietnam	0.463	0.372

The average return of the market portfolio is 0.185 and the risk-free rate of return at the last observation day is -0.03. The results for the implied returns calculated with CAPM are presented in Table 7.

Table 7:**CAPM results**

	CAPM
Argentina	23.420 %
Australia	20.139 %
Brazil	29.134 %
China	17.016 %
India	15.991 %
Kuwait	3.895 %
Morocco	4.297 %
Norway	29.868 %
Portugal	20.694 %
Russia	32.518 %
UK	20.496 %
Vietnam	4.998 %
AVERAGE	18.539 %

The results in Table 5 and in Table 7 prove that the results for the implied returns are equal with both calculation methods. These results are used in prior distribution when calculating the Black–Litterman returns.

4.4 EGARCH-M modeling

EGARCH-M modeling is done to estimate the two-period forecasts for the asset returns. The periods are specified as *t* and *t+1*. It is assumed that a theoretical investor invests at period *t-1* which refers to the eighth year. The aim is to find out if the investor benefits when using this strategy. The estimation is done with EGARCH-M model with one regressor: the change in the price of oil. The external regressor is a daily time series data of the oil price from the same period as the index returns and represented in USD currency. When adding one external regressor to both the mean equation and the variance equation in EGARCH-M model it can be presented as follows:

$$r_t = \delta \hat{\sigma}_t^2 + \varphi z_t + \varepsilon_t$$

$$\log \sigma_t^2 = \omega + \beta \log \sigma_{t-1}^2 + \alpha \left| \frac{\varepsilon_{t-1}}{\sigma_{t-1}} \right| + \gamma \frac{\varepsilon_{t-1}}{\sigma_{t-1}} + \psi z_{t-1}, \quad (60)$$

where φz_t represents the effect of the external regressor, the change in the price of oil, on the mean equation and ψz_t represents the effect of this on the variance equation. The presentation is similar to the articles of Beach and Orlov (2007) and Duqi et al. (2014). However in their articles there are more external regressors. The inclusion of the oil price is reasonable as the oil price is one of the major factors influencing the economic growth. (Gisser & Goodwin 1986) Moreover all the assets selected for this study are representing oil producing countries except Portugal. Hammoudeh, Dibooglu and Aleisa (2004) have studied the impact of oil volatility on the stock volatility. According to the study investors should invest on stocks which correspond to their risk tolerance and hedge against this risk with a derivative. Table 8 and Table 9 summarize the results of jointly estimating EGARCH-M with one regressor as shown in Equation (60). The estimation is done with an estimation windows of eight years and nine years.

Table 8:EGARCH-M results with one external regressor at period *t*

	μ	δ	φ	ω	α	β	γ	ψ
Argentina	-2.355	1.327	0.274	0.573	-0.051	0.588	0.429	-0.038
	(0.000)	(0.000)	(0.000)	(0.000)	(0.133)	(0.000)	(0.000)	(0.015)
Australia	-0.082	0.081	0.138	0.013	-0.098	0.980	0.146	-0.001
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.831)
Brazil	-0.042	0.023	0.386	0.0186	-0.067	0.986	0.143	-0.010
	(0.027)	(0.204)	(0.000)	(0.054)	(0.000)	(0.000)	(0.210)	(0.128)
China	0.030	0.006	0.100	0.014	-0.054	0.983	0.120	-0.013
	(0.734)	(0.928)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.003)
India	0.019	0.011	0.113	0.017	-0.058	0.984	0.165	-0.012
	(0.582)	(0.699)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Kuwait	-0.013	-0.003	0.031	0.011	-0.036	0.995	0.101	-0.012
	(0.650)	(0.939)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Morocco	0.037	-0.047	0.046	0.002	-0.013	0.996	0.074	-0.010
	(0.177)	(0.115)	(0.000)	(0.141)	(0.114)	(0.000)	(0.000)	(0.001)
Norway	-0.040	0.029	0.327	0.015	-0.087	0.986	0.125	-0.007
	(0.410)	(0.450)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.104)
Portugal	-0.144	0.089	0.222	0.035	-0.073	0.957	0.131	-0.018
	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)
Russia	-0.172	0.094	0.367	0.025	-0.083	0.983	0.128	0.001
	(0.357)	(0.480)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.791)
UK	-0.128	0.133	0.199	0.013	-0.107	0.968	0.172	-0.016
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.004)
Vietnam	-0.242	0.157	0.008	0.048	-0.023	0.941	0.241	0.001
	(0.000)	(0.000)	(0.514)	(0.514)	(0.098)	(0.000)	(0.000)	(0.839)

According to the results presented in Table 8 arch-in-mean (parameter μ) has the strongest effect on Argentina (-2.355) and the weakest effect on Kuwait (-0.013) The effect varies among the assets and is either negative or positive. UK (-0.107) has the largest leverage effect (parameter α) and Morocco has the smallest (-0.013). According to the parameter β Brazil has the strongest persistence in volatility (0.9643) and Morocco has the weakest (0.2684). All the values of the parameter β are considered stationary as they are between the values 0 and 1. The parameter γ defines the asymmetry effect and it is the strongest for Argentina (0.429) and weakest for Morocco (0.074).

When analyzing the effect of the price of the oil it is easily observable that this impact is larger in the mean equation (parameter φ) than in the variance equation (parameter ψ). The mean-effect is the strongest for Brazil (0.386) and weakest for Vietnam (0.008). By contrast the variance-effect is the strongest for Argentina (-0.038) and weakest for Australia (-0.001), Russia (0.001) and Vietnam (0.001).

Table 9: EGARCH-M results with one external regressor at period *t+1*

	μ	δ	φ	ω	α	β	γ	ψ
Argentina	0.015	0.018	0.216	0.137	-0.057	0.910	0.315	-0.028
	(0.927)	(0.845)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Australia	-0.097	0.097	0.119	0.011	-0.087	0.981	0.136	-0.002
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.601)
Brazil	-1.116	0.616	0.347	0.869	-0.182	0.347	0.468	0.011
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.183)
China	0.129	-0.063	0.073	0.011	-0.051	0.986	0.117	-0.013
	(0.000)	(0.006)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
India	0.069	-0.025	0.092	0.015	-0.058	0.985	0.156	-0.014
	(0.042)	(0.380)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)
Kuwait	-0.021	0.024	0.032	0.012	-0.021	0.993	0.103	-0.014
	(0.123)	(0.090)	(0.000)	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)
Morocco	0.108	-0.112	0.038	0.003	-0.002	0.990	0.104	-0.009
	(0.000)	(0.006)	(0.000)	(0.060)	(0.793)	(0.000)	(0.000)	(0.014)
Norway	0.001	0.011	0.286	0.010	-0.073	0.100	0.103	-0.008
	(0.993)	(0.814)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.054)
Portugal	0.073	-0.053	0.199	0.027	-0.072	0.964	0.135	-0.020
	(0.768)	(0.755)	(0.000)	(0.328)	(0.000)	(0.000)	(0.000)	(0.033)
Russia	-0.037	0.021	0.332	0.018	-0.078	0.987	0.127	0.002
	(0.667)	(0.733)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.662)
UK	-0.092	0.110	0.160	0.008	-0.102	0.973	0.168	-0.015
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.006)
Vietnam	-0.083	0.053	0.013	0.027	-0.022	0.965	0.208	-0.001
	(0.049)	(0.142)	(0.326)	(0.000)	(0.049)	(0.000)	(0.000)	(0.927)

shows the impact of different parameters at *t+1*. The arch-in-mean effect is the strongest for Brazil (-1.116) and weakest for Norway (0.001). The leverage effect is strongest for Brazil (-0.182) and weakest for Morocco (-0.002). Similarly the strongest asymmetry effect is for Brazil (0.468) and weakest for Kuwait and Norway (0.103). When looking at the effect of the price of the oil the values are somewhat similar to those at period *t.* As a conclusion from Table 8 and Table 9 Brazil is affected most by all the three effects and the frontier markets are least affected. These effects capture the effect of the volatility of the returns so we can assume that the asset returns of Brazil have high volatility.

4.5 Formulating the investors' views

In their study Beach and Orlov (2007) and Duqi et al. (2014) use EGARCH-M -model to form the investor views, i.e. to get the inputs for the matrices **Q** and **Ω**. Including arch-in-mean effect in the GARCH model allows to predict jointly the expected returns together with the volatility estimates. These expected returns are then used to form relative investor views. The matrix **Ω** is the "uncertainty matrix" which represents the variances of the views. The inputs for this parameter can be obtained with the estimates of variance in the EGARCH-M model. Based on the EGARCH-M results presented in Table 8 and Table 9 we get estimates for the expected return and variances to be used as an input for the investors' views in the Black–Litterman model. Table 10 shows the expected returns that are weighted based on naïve allocation.

Table 10:Equally weighted returns estimated with EGARCH-M

	Expected return t	Expected return t+1
Argentina	11.000 %	0.468 %
Australia	0.274 %	0.274 %
Brazil	0.104 %	0.104 %
China	0.331 %	0.331 %
India	0.311 %	0.311 %
Kuwait	-0.135 %	-0.135 %
Morocco	-0.014 %	-0.001 %
Norway	0.127 %	0.001 %
Portugal	-0.248 %	-0.002 %
Russia	0.510 %	0.005 %
UK	0.282 %	0.003 %
Vietnam	-0.568 %	-0.006 %
TOTAL	11.974 %	1.353 %

The estimates for the expected returns are calculated as a function of different parameters in the mean equation of EGARCH-M model. It includes arch-in-mean term as well as oil as an external regressor. The estimates for the variances are calculated as a function of the variance equation in the EGARCH-M model including the effect of past variance, asymmetry effect, leverage effect as well as the price of the oil. Based on the results presented in Table 10 we see that Argentina has the largest return estimate at time *t* (11.000 %) while Vietnam has the smallest (-0.568 %). In fact the expected return at *t* for Argentina is extremely large compared to the expected returns of the other assets in portfolio. As for the expected returns at *t+1* Argentina has still the largest value (0.468 %) and Kuwait has the smallest (-0.135 %). These values are used when forming the matrices for the views, namely the matrices **Q** and **Ω**.

The method to formulate the views for the Black–Litterman portfolio is similar to the method of Duqi et al. (2014). The assets are sorted according to their implied returns to three different portfolios: high-risk, medium-risk and low risk. In Table 11 both implied returns and the estimated returns are weighted based on naïve allocation so that each asset is given an equal weight (1/12). Based on the results in Table 5 three portfolios were built.

Table 11:Construction of portfolios based on the expected returns

	Index	CAPM	Expected return t	Expected return t+1
Portfolio 1: high risk	Russia	2.710 %	0.510 %	0.054 %
	Norway	2.489 %	0.127 %	0.160 %
	Brazil	2.428 %	0.104 %	15.459 %
	Argentina	1.952 %	11.000 %	0.468 %
Portfolio 2: medium risk	Portugal	1.725 %	-0.248 %	-0.000 %
	UK	1.708 %	0.282 %	0.318 %
	Australia	1.678 %	0.274 %	0.002 %
	China	1.418 %	0.331 %	0.331 %
Portfolio 3: Low risk	India	1.333 %	0.311 %	0.298 %
	Vietnam	0.417 %	-0.568 %	-0.200 %
	Morocco	0.358 %	-0.014 %	-0.108 %
	Kuwait	0.325 %	-0.135 %	0.051 %
TOTAL PORTFOLIO		18.541 %	19.6372 %	16.833 %

In the Black–Litterman context we can form two different kinds of views: absolute or relative views. In this study only relative views are formed. Based on the results in Table 10 we can form the following relative views for the period *t*:

- China will outperform Norway by 0.204 %
- India will outperform Brazil by 0.207 %
- Morocco will outperform Portugal by 0.234 %

Similarly the below views are formed for the period *t+1:*

- China will outperform Norway by 0.171 %
- India will outperform Russia 0.244 %
- Kuwait will outperform Australia by 0.049 %

The comparision is done by comparing the assets in different risk portfolios. These views form the matrix **Q**. Now that we have the results of different parameters of EGARCH-M model as well as the estimated implied equilibrium returns, the Black–Litterman adjusted returns can be calculated. This starts by specifying the matrices **P** and **Q**. In this case there are 12 assets and 3 views. Therefore **P** matrix is 3 x 12. As **Q** matrix includes the actual views it has 3 components. Those assets that are not given any active view will be presented as zero in the matrix. Based on the results obtained in Table 11 we can form these matrices as follows:

$$\boldsymbol{P_t} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{P_{t+1}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (61)$$

$$\boldsymbol{Q_t} = \begin{pmatrix} 0.204 \\ 0.207 \\ 0.234 \end{pmatrix}$$

$$\boldsymbol{Q_{t+1}} = \begin{pmatrix} 0.171 \\ 0.244 \\ 0.049 \end{pmatrix} \quad (62)$$

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}. \quad (63)$$

To form the uncertainty matrix **Ω** we need the variance-covariance matrix for the views. In this study we use the variance-covariance matrix of the residuals of the excess returns estimated with EGARCH-M model. By multiplying this with the single k rows from the view vector we get the following form for the uncertainty matrix:

$$\boldsymbol{\Omega_t} = \begin{pmatrix} 3.850 & 0 & 0 \\ 0 & 4.510 & 0 \\ 0 & 0 & 2.562 \end{pmatrix}$$

$$\boldsymbol{\Omega_{t+1}} = \begin{pmatrix} 3.539 & 0 & 0 \\ 0 & 4.541 & 0 \\ 0 & 0 & 0.693 \end{pmatrix} \quad (64)$$

The values presented in Equation (64) are percentage values.

4.6 The Black–Litterman returns and weights

To calculate the returns based on the Black–Litterman model we need to specify the parameter τ. As there is no consensus on this value in the literature, we perform the equation for multiple values of the parameter, varying from 0.01 to 0.1. When there is no uncertainty, this value would be zero. Table 12 summarizes the results of Black–Litterman returns with different values for the parameter τ.

Table 12: Black–Litterman returns with different values for the parameter τ (% values)

		t			t+1		
	CAPM	**τ=0.01**	**τ=0.05**	**τ=0.1**	**τ =0.1**	**τ=0.05**	**τ=0.1**
Argentina	1.952	0.914	0.527	0.457	-3.277	-5.447	-5.879
Australia	1.678	0.668	0.469	0.431	-3.747	-4.652	-4.796
Brazil	2.428	1.179	0.474	0.344	-3.988	-6.652	-7.157
China	1.418	0.588	0.601	0.612	1.985	3.288	3.485
India	1.333	0.370	0.379	0.389	3.875	9.096	10.818
Kuwait	0.325	-1.841	-1.881	-1.886	0.133	0.050	0.010
Morocco	0.358	-1.556	-1.587	-1.597	-3.411	-4.153	-4.302
Norway	2.489	1.336	0.674	0.556	-8.285	-12.326	-12.932
Portugal	1.725	-0.943	-1.674	-1.729	-6.316	-8.589	-8.878
Russia	2.710	2.630	2.236	0.217	-4.718	-8.881	-9.851
UK	1.708	0.390	0.096	-0.059	-5.880	-8.979	-9.607
Vietnam	0.417	-1.587	-1.604	-1.605	-1.591	-1.242	-1.124
Average	1.545	0.179	-0.110	-0.323	-2.935	-4.041	-4.185
SD	0.826	1.377	1.279	1.037	3.514	6.009	6.626
Max	2.710	2.630	2.236	0.612	3.875	9.096	10.818
Min	0.325	-1.841	-1.881	-1.886	-8.285	-12.326	-12.932
SR	-2.082	-1.789	-1.578	-1.533	0.712	0.986	0.996

According to the results in Table 12 we can see that the parameter τ and Sharpe ratio have a positive correlation so that when the parameter τ gets larger values it increases Sharpe ratio. This dependence is obvious as the parameter τ specifies the uncertainty and Sharpe ratio measures the risk-adjusted returns. Therefore it is clear that when the risk-adjusted returns increase more risk is involved. At period *t* all the asset returns get lower values than at the equilibrium. Kuwait, Morocco, UK, Portugal and Vietnam have negative values while the other countries have positive values. Russia has the largest values but they are decreased from the CAPM based returns. At the period *t+1* all the assets excluding India and China lose. In fact India and China are the only winners in the portfolio with this setting. At this period there are more assets to get negative values. By contrast Kuwait has positive values at *t+1* regardless the value of the parameter τ.

The average return at the period *t* is between -0.323 % to 0.179 % depending on the value of the parameter τ and the equivalent average return at period *t+1* is between -4.185 % to -2.935 %. Therefore it means that the investor loses approximately this much when applying “buy and hold strategy”. In the equilibrium the investor would get an average return of 1.545 % so the strategy is not profitable. When looking at the results in Table 12 it can be observed that the return devitations get larger when the parameter τ increases and when the time passes. In practice this means that the maximum value gets larger while the mínimum value decreases. Therefore even though the investor would benefit from larger returns in some assets the loss in the other assets would clear this benefit and the investor would end up with a smaller return than in the equilbrium.

The results at period *t+1* reflect the fact that the assets that were to outperform other assets, namely China, India and Kuwait, get higher returns while those assets to be outperformed, Norway, Russia and Australia, have smaller returns which are negative in this case. However the same does not apply at period *t.* Moreover it is interesting to notice that the assets have different trends when comparing to the change of the parameter τ. All the other assets have a decreasing trend except China and India at period *t.* The same applies at period *t+1* but in addition Vietnam has a decreasing trend.

The results shown in Table 12 are used to calculate the Black–Litterman weights as shown in Equation (48). The results for the Black–Litterman allocations are presented in Table 13.

Table 13:Equilibrium weights and Black–Litterman weights (% values)

		t			t+1		
	w_{mkt}	τ=0.01	τ=0.05	τ=0.1	τ=0.01	τ=0.05	τ=0.1
Argentina	8.333	12.605	17.987	23.884	10.220	10.635	10.735
Australia	8.333	9.229	11.480	14.071	8.647	8.465	8.434
Brazil	8.333	15.715	20.382	25.440	13.059	13.431	13.515
China	8.333	7.637	10.724	14.319	5.818	5.522	5.445
India	8.333	7.289	9.883	12.920	5.325	4.714	4.505
Kuwait	8.333	-1.505	-10.164	-19.817	1.734	1.702	1.697
Morocco	8.333	1.161	-2.006	-5.584	2.938	2.920	2.910
Norway	8.333	15.426	19.138	23.211	14.623	14.855	14.913
Portugal	8.333	11.836	3.017	-6.744	15.198	15.155	15.146
Russia	8.333	11.710	17.036	22.934	10.018	10.256	10.319
UK	8.333	10.649	13.265	16.109	10.335	10.560	10.630
Vietnam	8.333	-1.746	-10.742	-20.742	2.085	1.795	1.741

As the results show in Table 13 the weights have bigger changes at period *t* than at period *t+1*. Brazil has the largest weights at period *t* while Vietnam has the smallest. As the weights for Vietnam and Kuwait are negative it means that these assets are sold short. The largest weights at the period *t+1* are for Portugal and smallest weights for Vietnam. For most of the assets the weight increases when the parameter τ gets higher values. In the research of Duqi et al. (2014) the weights are influenced most when the parameter τ gets larger values. Moreover in their research only those assets which were given an active view were influenced by a weight change while the other assets stayed in the equilibrium weights. This result was the aim of this study as well but the results were different most probably due to a calculation error. In general we can conclude that the weights increase for the countries from developed economies and emerging markets but they decrease for those countries that are from frontier markets. China is the only exception from the emerging markets that has a decreasing weight at the estimation periods.

To answer the actual research questions we can see based on Table 12 and Table 13 that the investor loses when applying “buy and hold strategy”. The returns have decreasing values and in most cases they get negative values. However the results might change when considering a longer period of time, different dataset or other assumptions. Even though some assets tend to have increasing values the negative impacts are canceling the benefits.

The results of this study were not exactly what was expected and this might be due to a calculation error in one of the calculation steps. After all the Black–Litterman model can be considered as a useful tool to analyze the returns of the portfolio at different periods. When using EGARCH-M method we can get forecasts of how the asset returns behave in the future periods and get a good understanding of how the asset returns change from one period to another. The Black–Litterman model is a complex method that has a lot of steps which makes it exposed to errors. Moreover what makes it complex to study is the fact that the literature is full of different methods and perspectives for the model which makes it hard to make decisions of how to conduct one's study.

CONCLUSION

In the international portfolio management context it is important to understand different methods for analyzing asset returns and risk. This study focused on the Black–Litterman model which provides a new perspective for the global asset allocation. The main contribution is that it allows investors to make subjective views on asset returns and variances. Moreover the model overcomes many issues related to the traditional mean-variance optimization. Overall the model can be very useful in the portfolio management as it takes the perspective of the subjective views into account.

This study used a similar approach as in the articles of Beach and Orlov (2007) and Duqi et al. (2014) with a data consisting of randomly picked MSCI country indices. The data covered 12 MSCI country indices from the period of 3.10.2008-3.10.2018. We sorted assets based on their economic development to three categories: developed markets, emerging markets and frontier markets. Each category had four assets from different geographical areas. We calculated the prior distribution, i.e. the equilibrium implied returns and the historical variance-covariance matrix. We found out that there were clear differences in asset returns between the assets. Russia, Brazil and Argentina had the largest values while Vietnam, Morocco and Kuwait had the smallest. This reflects the fact that the assets were selected from different economic backgrounds. As for the variance-covariance matrix we found clear covariances between the asset classes.

The assets were sorted to three portfolios based on prior CAPM results: high-risk portolio, medium-risk portfolio and low-risk portfolio. Exponential GARCH-in-mean model was used to compute estimates for expected returns and variances. The investors' views were formulated based on these results. Three subjective views were expressed based on the estimates of the expcted returns. Moreover the uncertainty matrix was constructed based on the residuals of the EGARCH-M model. These inputs were used when calculating the returns and weights based on the Black–Litterman model. We used two-period forecast in this study to estimate returns for two consecutive periods. First we used the data from eight years to make forecasts for the asset returns at ninth year and secondly we used data from nine years to make these for the tenth year.

The results of this study were close to what was expected. The aim was to find Black–Litterman weights which would reflect the theoretical investors' views which were derived with EGARCH-M model. The expectations were that the Black–Litterman weights would change for those assets that were given an active view based on the views. The assets that were to outperform other assets would have increased weights while the assets that were being outperformed would have decreased weights. Moreover the effect would increase when the uncertainty parameter τ gets larger values. However in this study the results didn't reflect this. Also the weights of the assets with passive views were affected which was not the purpose of the study. Although this study followed the calculation steps of Beach and Orlov (2007) and Duqi et al. (2014) there might be a calculation error or different calculation methods at different stages of the calculation process. Despite the fact that the results were not fully as expected we can conclude that EGARCH-M model is a useful method to derive the investors' views for the Black–Litterman model. It is relatively simple to apply even though it requires a lot of different steps due to which the results are exposed to calculation errors.

For simplicity this study used the forecast period of two years. However when comparing only two periods it does not provide a whole picture of how the returns and the weights evolve in the Black–Litterman context. This can be used as a reference of how the model works but if we seek to understand the real performance of our assets we would need a longer forecast period. For example in the research of Beach and Orlov (2007) the rolling forecast period was 10 years which gives a broader understanding of the returns and asset allocation. To get the best benefit of the Black–Litterman model

This study was conducted only to equity indices. However it would be interesting to see how the inclusion of different asset classes would change the results. For example, including bonds, currencies and commodities would bring a new inspiring perspective for this study. Another perspective could be to use individual stocks instead of indices. This approach was used by Duqi et al. (2014). Another restriction in this study was that only the change in the price of oil was included as an exgernal regressor in this study. However there are several factors that could be included in future studies. The approach applied in this study was EGARCH-M model but when using a different method one could get different results. There are several return and volatility forecast models available to be applied for the Black–Litterman context. However using EGARCH-M model is reasonable as it takes many statistical properties into account and it is relatively simple to apply.

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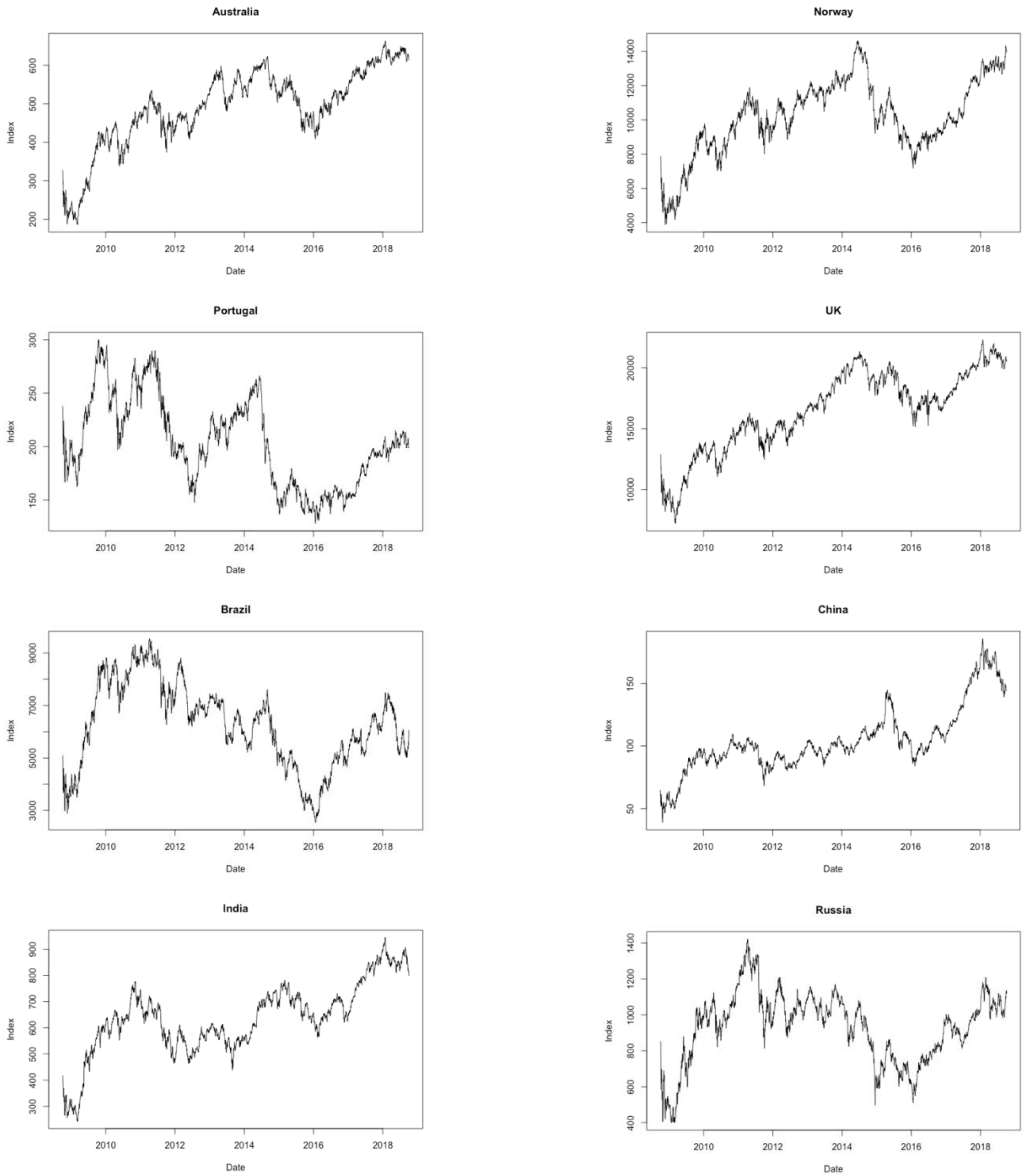
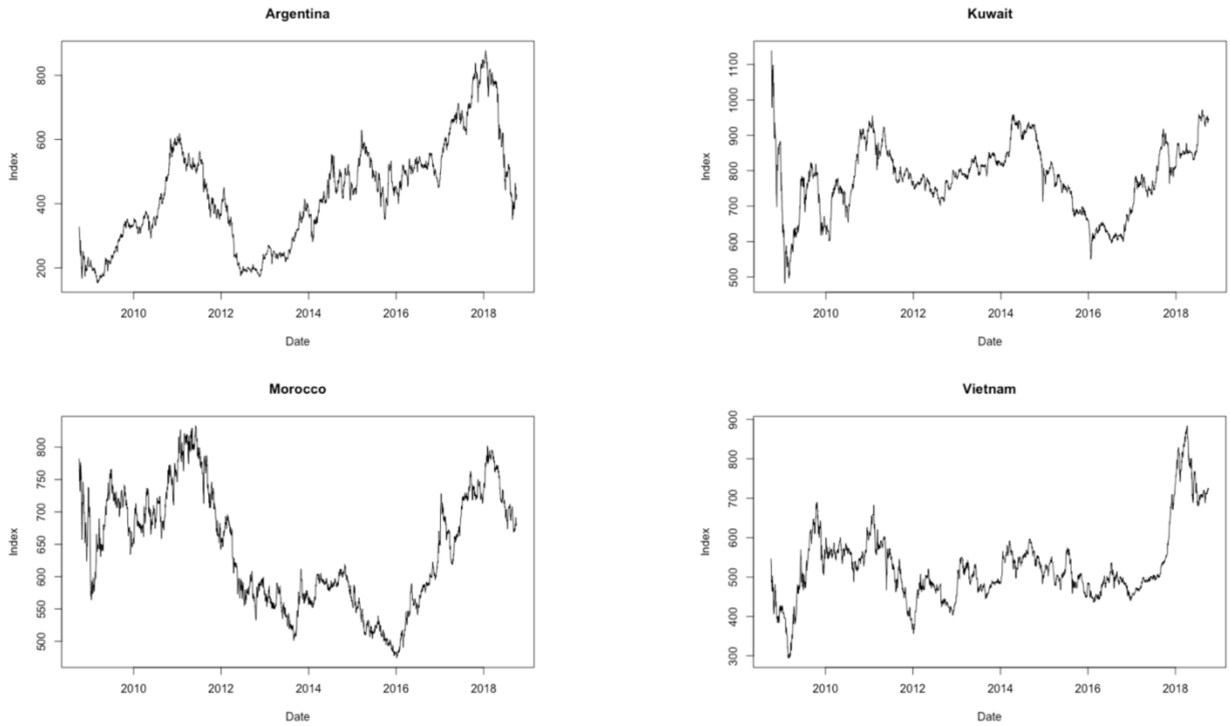
APPENDIX

Appendix 1: Time series of MSCI country indices from the period of 14.3.2008-14.3.2018




Argentina
Index
200
400
600
800
2010
2012
2014
2016
2018
Date
Kuwait
Index
500
600
700
800
900
1000
1100
2010
2012
2014
2016
2018
Date
Morocco
Index
500
550
600
650
700
750
800
2010
2012
2014
2016
2018
Date
Vietnam
Index
300
400
500
600
700
800
900
2010
2012
2014
2016
2018
Date

Appendix 2: R code for the empirical study

#UPLOAD DATA

```
library(fGarch)
library(timeDate)
library(timeSeries)
library(fBasics)
library(forecast)
library(quantmod)
library(TSA)
library(zoo)
library(dplyr)
library(rugarch)
library(MASS)
library(tseries)

data_argentina = read.table("MSCI_ARGENTINA.txt", header = TRUE, fill = TRUE,
dec = ",")

data_australia = read.table("MSCI_AUSTRALIA.txt", header = TRUE, fill = TRUE, dec
= ",")

data_brazil = read.table("MSCI_BRAZIL.txt", header = TRUE, fill = TRUE, dec = ",")

data_china = read.table("MSCI_CHINA.txt", header = TRUE, fill = TRUE, dec = ",")

data_india = read.table("MSCI_INDIA.txt", header = TRUE, fill = TRUE, dec = ",")

data_kuwait = read.table("MSCI_KUWAIT.txt", header = TRUE, fill = TRUE, dec =",")

data_morocco = read.table("MSCI_MOROCCO.txt", header =TRUE, fill = TRUE, dec
=",")

data_norway = read.table("MSCI_NORWAY.txt", header = TRUE, fill =TRUE, dec
=",")

data_portugal = read.table("MSCI_PORTUGAL.txt", header = TRUE, fill = TRUE, dec
=",")

data_russia = read.table("MSCI_RUSSIA.txt", header = TRUE, fill = TRUE, dec =",")

data_uk = read.table("MSCI_UK.txt", header = TRUE, fill = TRUE, dec =",")

data_vietnam = read.table("MSCI_VIETNAM.txt", header = TRUE, fill = TRUE, dec
=",")

data_rf = read.table("RISKFREE.txt", header = TRUE, fill =TRUE, dec =",")

data_oil = read.table ("OIL.txt", header = TRUE, fill = TRUE, dec =",")
```

```
data_argentina_8y = read.table("MSCI_ARGENTINA_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_australia_8y = read.table("MSCI_AUSTRALIA_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_brazil_8y = read.table("MSCI_BRAZIL_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_china_8y = read.table("MSCI_CHINA_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_india_8y = read.table("MSCI_INDIA_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_kuwait_8y = read.table("MSCI_KUWAIT_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_morocco_8y = read.table("MSCI_MOROCCO_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_norway_8y = read.table("MSCI_NORWAY_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_portugal_8y = read.table("MSCI_PORTUGAL_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_russia_8y = read.table("MSCI_RUSSIA_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_uk_8y = read.table("MSCI_UK_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_vietnam_8y = read.table("MSCI_VIETNAM_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_rf_8y = read.table("RISKFREE_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_oil_8y = read.table("OIL_8y.txt", header = TRUE, fill = TRUE, dec = ",")

data_oil_9y = read.table("OIL_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_argentina_9y = read.table("MSCI_ARGENTINA_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_australia_9y = read.table("MSCI_AUSTRALIA_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_brazil_9y = read.table("MSCI_BRAZIL_9y.txt", header = TRUE, fill = TRUE, dec = ",")
```

data_china_9y = read.table("MSCI_CHINA_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_india_9y = read.table("MSCI_INDIA_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_kuwait_9y = read.table("MSCI_KUWAIT_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_morocco_9y = read.table("MSCI_MOROCCO_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_norway_9y = read.table("MSCI_NORWAY_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_portugal_9y = read.table("MSCI_PORTUGAL_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_russia_9y = read.table("MSCI_RUSSIA_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_uk_9y = read.table("MSCI_UK_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_vietnam_9y = read.table("MSCI_VIETNAM_9y.txt", header = TRUE, fill = TRUE, dec = ",")

data_rf_9y = read.table("RISKFREE_9y.txt", header = TRUE, fill = TRUE, dec = ",")

#DEFINE PARAMETERS

argentina = ts(data_argentina$ARGENTINA)

argentina_8y = ts(data_argentina_8y$ARGENTINA)

argentina_9y = ts(data_argentina_9y$ARGENTINA)

date_argentina = as.Date(data_argentina$DATE)

australia = ts(data_australia$AUSTRALIA)

australia_8y = ts(data_australia_8y$AUSTRALIA)

australia_9y = ts(data_australia_9y$AUSTRALIA)

date_australia = as.Date(data_australia$DATE)

brazil = ts(data_brazil$BRAZIL)

brazil_8y = ts(data_brazil_8y$BRAZIL)

brazil_9y = ts(data_brazil_9y$BRAZIL)

```
date_brazil = as.Date(data_brazil$DATE)
china = ts(data_china$CHINA)
china_8y = ts(data_china_8y$CHINA)
china_9y = ts(data_china_9y$CHINA)
date_china = as.Date(data_china$DATE)
india = ts(data_india$INDIA)
india_8y = ts(data_india_8y$INDIA)
india_9y = ts(data_india_9y$INDIA)
date_india = as.Date(data_india$DATE)
kuwait = ts(data_kuwait$KUWAIT)
kuwait_8y = ts(data_kuwait_8y$KUWAIT)
kuwait_9y = ts(data_kuwait_9y$KUWAIT)
date_kuwait = as.Date(data_kuwait$DATE)
morocco = ts(data_morocco$MOROCCO)
morocco_8y = ts(data_morocco_8y$MOROCCO)
morocco_9y = ts(data_morocco_9y$MOROCCO)
date_morocco = as.Date(data_morocco$DATE)
norway = ts(data_norway$NORWAY)
norway_8y = ts(data_norway_8y$NORWAY)
norway_9y = ts(data_norway_9y$NORWAY)
date_norway = as.Date(data_norway$DATE)
portugal = ts(data_portugal$PORTUGAL)
portugal_8y = ts(data_portugal_8y$PORTUGAL)
portugal_9y = ts(data_portugal_9y$PORTUGAL)
date_portugal = as.Date(data_portugal$DATE)
```

```
russia = ts(data_russia$RUSSIA)

russia_8y = ts(data_russia_8y$RUSSIA)

russia_9y = ts(data_russia_9y$RUSSIA)

date_russia = as.Date(data_russia$DATE)

uk = ts(data_uk$UK)

uk_8y = ts(data_uk_8y$UK)

uk_9y = ts(data_uk_9y$UK)

date_uk = as.Date(data_uk$DATE)

vietnam = ts(data_vietnam$VIETNAM)

vietnam_8y = ts(data_vietnam_8y$VIETNAM)

vietnam_9y = ts(data_vietnam_9y$VIETNAM)

date_vietnam = as.Date(data_vietnam$DATE)

rf = ts(data_rf$RISKFREE)

rf = ts(data_rf$RISKFREE)

rf_8y = ts(data_rf_8y$RISKFREE)

rf_9y = ts(data_rf_9y$RISKFREE)

rf_daily = (1+rf)^(1/2600)-1

rf_daily_8y = (1+rf)^(1/2600)-1

rf_daily_9y = (1+rf)^(1/2600)-1

oil = ts(data_oil_8y$OIL)

oil_9y = ts(data_oil_9y$OIL)

combined_vector = cbind(argentina, australia, brazil, china, india, kuwait, morocco,
norway, russia, uk, portugal, vietnam)

combined_vector_9y = cbind(argentina_9y, australia_9y, brazil_9y, china_9y, india_9y,
kuwait_9y, morocco_9y, norway_9y, russia_9y, uk_9y, portugal_9y, vietnam_9y)
```

#PLOT TIME SERIES

plot_argentina = plot(argentina, ylab = "Index", xlab = "Date", x = date_argentina, type = "l", main = "Argentina")

plot_australia = plot(australia, ylab = "Index", xlab = "Date", x = date_australia, type = "l", main = “Australia”)

plot_brazil = plot(brazil, ylab = "Index", xlab = "Date", x = date_brazil, type = "l", main = “Brazil”)

plot_china = plot(china, ylab = "Index", xlab = "Date", x = date_china, type = "l", main = “China”)

plot_india = plot(india, ylab = "Index", xlab = "Date", x = date_india, type = "l", main = “India”)

plot_kuwait = plot(kuwait, ylab = "Index", xlab = "Date", x = date_kuwait, type = "l", main = “Kuwait”)

plot_morocco = plot(morocco, ylab = "Index", xlab = "Date", x = date_morocco, type = "l", main = “Morocco”)

plot_norway = plot(norway, ylab = "Index", xlab = "Date", x = date_norway, type = "l", main = “Norway”)

plot_portugal = plot(portugal, ylab = "Index", xlab = "Date", x = date_portugal, type = "l", main = “Portugal”)

plot_russia = plot(russia, ylab = "Index", xlab = "Date", x = date_russia, type = "l", main = ”Russia”)

plot_uk = plot(uk, ylab = "Index", xlab = "Date", x = date_uk, type = "l", main = “UK”)

#CALCULATE CONTINUOSLY COMPOUNDED RETURNS

log_argentina = log(argentina)

rt_argentina = 100*diff(log_argentina)

log_argentina_8y = log(argentina_8y)

rt_argentina_8y = 100*diff(log_argentina_8y)

log_argentina_9y = log(argentina_9y)

rt_argentina_9y = 100*diff(log_argentina_9y)

log_australia = log(australia)

rt_australia = 100*diff(log_australia)

```
log_australia_8y = log(australia_8y)

rt_australia_8y = 100*diff(log_australia_8y)

log_australia_9y = log(australia_9y)

rt_australia_9y = 100*diff(log_australia_9y)

log_brazil = log(brazil)

rt_brazil = 100*diff(log_brazil)

log_brazil_8y = log(brazil_8y)

rt_brazil_8y = 100*diff(log_brazil_8y)

log_brazil_9y = log(brazil_9y)

rt_brazil_9y = 100*diff(log_brazil_9y)

log_china = log(china)

rt_china = 100*diff(log_china)

log_china_8y = log(china_8y)

rt_china_8y = 100*diff(log_china_8y)

log_china_9y = log(china_9y)

rt_china_9y = 100*diff(log_china_9y)

log_india = log(india)

rt_india = 100*diff(log_india)

log_india_8y = log(india_8y)

rt_india_8y = 100*diff(log_india_8y)

log_india_9y = log(india_9y)

rt_india_9y = 100*diff(log_india_9y)

log_kuwait = log(kuwait)

rt_kuwait = 100*diff(log_kuwait)

log_kuwait_8y = log(kuwait_8y)
```

```
rt_kuwait_8y = 100*diff(log_kuwait_8y)

log_kuwait_9y = log(kuwait_9y)

rt_kuwait_9y = 100*diff(log_kuwait_9y)

log_morocco = log(morocco)

rt_morocco = 100*diff(log_morocco)

log_morocco_8y = log(morocco_8y)

rt_morocco_8y = 100*diff(log_morocco_8y)

log_morocco_9y = log(morocco_9y)

rt_morocco_9y = 100*diff(log_morocco_9y)

log_norway = log(norway)

rt_norway = 100*diff(log_norway)

log_norway_8y = log(norway_8y)

rt_norway_8y = 100*diff(log_norway_8y)

log_norway_9y = log(norway_9y)

rt_norway_9y = 100*diff(log_norway_9y)

log_portugal= log(portugal)

rt_portugal = 100*diff(log_portugal)

log_portugal_8y = log(portugal_8y)

rt_portugal_8y = 100*diff(log_portugal_8y)

log_portugal_9y = log(portugal_9y)

rt_portugal_9y = 100*diff(log_portugal_9y)

log_russia = log(russia)

rt_russia = 100*diff(log_russia)

log_russia_8y = log(russia_8y)

rt_russia_8y = 100*diff(log_russia_8y)

log_russia_9y = log(russia_9y)
```

```
rt_russia_9y = 100*diff(log_russia_9y)
log_uk = log(uk)
rt_uk = 100*diff(log_uk)
log_uk_8y = log(uk_8y)
rt_uk_8y = 100*diff(log_uk_8y)
log_uk_9y = log(uk_9y)
rt_uk_9y = 100*diff(log_uk_9y)
log_vietnam = log(vietnam)
rt_vietnam = 100*diff(log_vietnam)
log_vietnam_8y = log(vietnam_8y)
rt_vietnam_8y = 100*diff(log_vietnam_8y)
log_vietnam_9y = log(vietnam_9y)
rt_vietnam_9y = 100*diff(log_vietnam_9y)
log_rf = log(rf_daily)
rt_rf= 100*diff(log_rf)
mean_rf = mean(rt_rf)*sqrt(260)
log_rf_8y = log(rf_daily_8y)
rt_rf_8y = 100*diff(log_rf_8y)
log_rf_9y = log(rf_daily_9y)
rt_rf_9y = 100*diff(log_rf_9y)
mean_rf_9y = mean(rt_rf_9y)*sqrt(260)
log_oil = log(oil)
rt_oil = 100*diff(log_oil)
matrix_oil = matrix(rt_oil)
log_oil_9y = log(oil_9y)
```

```
rt_oil_9y = 100*diff(log_oil_9y)

matrix_oil_9y = matrix(rt_oil_9y)

naive_portfolio = (1/12) * rt_argentina + (1/12) * rt_australia + (1/12) * rt_brazil+(1/12) * rt_china + (1/12) * rt_india + (1/12) * rt_kuwait + (1/12) * rt_morocco + (1/12) * rt_norway + (1/12) * rt_portugal + (1/12) * rt_russia + (1/12) * rt_uk + (1/12) * rt_vietnam

log_combined_vector = log(combined_vector)

rt_combined_vector = 100*diff(log_combined_vector)

log_combined_vector_9y = log(combined_vector_9y)

rt_combined_vector_9y = 100*diff(log_combined_vector_9y)

#CALCULATE EXCESS RETURNS

E_argentina = rt_argentina - rf_daily

E_australia = rt_australia - rf_daily

E_brazil = rt_brazil - rf_daily

E_china = rt_china - rf_daily

E_india = rt_india - rf_daily

E_kuwait = rt_kuwait - rf_daily

E_morocco = rt_morocco - rf_daily

E_norway = rt_norway - rf_daily

E_portugal = rt_portugal - rf_daily

E_russia = rt_russia - rf_daily

E_uk = rt_uk - rf_daily

E_vietnam = rt_vietnam - rf_daily

E_argentina_8y = rt_argentina_8y - rf_daily_8y

E_australia_8y = rt_australia_8y - rf_daily_8y

E_brazil_8y = rt_brazil_8y - rf_daily_8y

E_china_8y = rt_china_8y - rf_daily_8y
```

E_india_8y = rt_india_8y - rf_daily_8y

E_kuwait_8y = rt_kuwait_8y - rf_daily_8y

E_morocco_8y = rt_morocco_8y - rf_daily_8y

E_norway_8y = rt_norway_8y - rf_daily_8y

E_portugal_8y = rt_portugal_8y - rf_daily_8y

E_russia_8y = rt_russia_8y - rf_daily_8y

E_uk_8y = rt_uk_8y - rf_daily_8y

E_vietnam_8y = rt_vietnam_8y - rf_daily_8y

E_oil = rt_oil - rf_daily_8y

E_argentina_9y = rt_argentina_9y - rf_daily_9y

E_australia_9y = rt_australia_9y - rf_daily_9y

E_brazil_9y = rt_brazil_9y - rf_daily_9y

E_china_9y = rt_china_9y - rf_daily_9y

E_india_9y = rt_india_9y - rf_daily_9y

E_kuwait_9y = rt_kuwait_9y - rf_daily_9y

E_morocco_9y = rt_morocco_9y - rf_daily_9y

E_norway_9y = rt_norway_9y - rf_daily_9y

E_portugal_9y = rt_portugal_9y - rf_daily_9y

E_russia_9y = rt_russia_9y - rf_daily_9y

E_uk_9y = rt_uk_9y - rf_daily_9y

E_vietnam_9y = rt_vietnam_9y - rf_daily_9y

E_oil_9y = rt_oil_9y - rf_daily_9y

E_combined_vector = rt_combined_vector - rf_daily

#DESCRIPTIVE STATISTICS

mean_argentina = mean(rt_argentina)*sqrt(260)

mean_australia = mean(rt_australia) *sqrt(260)

```
mean_brazil = mean(rt_brazil) *sqrt(260)

mean_china = mean(rt_china) *sqrt(260)

mean_india = mean(rt_india) *sqrt(260)

mean_kuwait = mean(rt_kuwait) *sqrt(260)

mean_morocco = mean(rt_morocco) *sqrt(260)

mean_norway = mean(rt_norway) *sqrt(260)

mean_russia = mean(rt_russia) *sqrt(260)

mean_uk = mean(rt_uk) *sqrt(260)

mean_portugal = mean(rt_portugal) *sqrt(260)

mean_vietnam = mean(rt_vietnam) *sqrt(260)

mean_naive_portfolio = mean(naive_portfolio) *sqrt(260)

sd_argentina = sd(rt_argentina) *sqrt(260)

sd_australia = sd(rt_australia) *sqrt(260)

sd_brazil = sd(rt_brazil) *sqrt(260)

sd_china = sd(rt_china) *sqrt(260)

sd_india = sd(rt_india) *sqrt(260)

sd_kuwait = sd(rt_kuwait) *sqrt(260)

sd_morocco = sd(rt_morocco) *sqrt(260)

sd_norway = sd(rt_norway) *sqrt(260)

sd_uk = sd(rt_uk) *sqrt(260)

sd_portugal = sd(rt_portugal) *sqrt(260)

sd_russia = sd(rt_russia) *sqrt(260)

sd_vietnam = sd(rt_vietnam) *sqrt(260)

sd_naive_portfolio = sd(naive_portfolio) *sqrt(260)

sd_argentina_8y = sd(rt_argentina_8y) *sqrt(260)
```

```
sd_argentina_8y_daily = sd(rt_argentina_8y)

sd_australia_8y = sd(rt_australia_8y) *sqrt(260)

sd_australia_8y_daily = sd(rt_australia_8y)

sd_brazil_8y = sd(rt_brazil_8y) *sqrt(260)

sd_brazil_8y_daily = sd(rt_brazil_8y)

sd_china_8y = sd(rt_china_8y) *sqrt(260)

sd_china_8y_daily = sd(rt_china_8y)

sd_brazil_8y_daily = sd(rt_brazil_8y)

sd_india_8y = sd(rt_india_8y) *sqrt(260)

sd_india_8y_daily = sd(rt_india_8y)

sd_kuwait_8y = sd(rt_kuwait_8y) *sqrt(260)

sd_kuwait_8y_daily = sd(rt_kuwait_8y)

sd_morocco_8y = sd(rt_morocco_8y) *sqrt(260)

sd_morocco_8y_daily = sd(rt_morocco_8y)

sd_norway_8y = sd(rt_norway_8y) *sqrt(260)

sd_norway_8y_daily = sd(rt_norway_8y)

sd_portugal_8y = sd(rt_portugal_8y) *sqrt(260)

sd_portugal_8y_daily = sd(rt_portugal_8y)

sd_russia_8y = sd(rt_russia_8y) *sqrt(260)

sd_russia_8y_daily = sd(rt_russia_8y)

sd_uk_8y = sd(rt_uk_8y) *sqrt(260)

sd_uk_8y_daily = sd(rt_uk_8y)

sd_vietnam_8y = sd(rt_vietnam_8y) *sqrt(260)

sd_vietnam_8y_daily = sd(rt_vietnam_8y)

sd_vietnam = sd(rt_vietnam) *sqrt(260)

sd_naive_portfolio = sd(naive_portfolio) *sqrt(260)
```

```
sd_argentina_9y = sd(rt_argentina_9y) *sqrt(260)

sd_argentina_9y_daily = sd(rt_argentina_9y)

sd_australia_9y = sd(rt_australia_9y) *sqrt(260)

sd_australia_9y_daily = sd(rt_australia_9y)

sd_brazil_9y = sd(rt_brazil_9y) *sqrt(260)

sd_brazil_9y_daily = sd(rt_brazil_9y)

sd_china_9y = sd(rt_china_9y) *sqrt(260)

sd_china_9y_daily = sd(rt_china_9y)

sd_brazil_9y_daily = sd(rt_brazil_9y)

sd_india_9y = sd(rt_india_9y) *sqrt(260)

sd_india_9y_daily = sd(rt_india_9y)

sd_kuwait_9y = sd(rt_kuwait_9y) *sqrt(260)

sd_kuwait_9y_daily = sd(rt_kuwait_9y)

sd_morocco_9y = sd(rt_morocco_9y) *sqrt(260)

sd_morocco_9y_daily = sd(rt_morocco_9y)

sd_norway_9y = sd(rt_norway_9y) *sqrt(260)

sd_norway_9y_daily = sd(rt_norway_9y)

sd_portugal_9y = sd(rt_portugal_9y) *sqrt(260)

sd_portugal_9y_daily = sd(rt_portugal_9y)

sd_russia_9y = sd(rt_russia_9y) *sqrt(260)

sd_russia_9y_daily = sd(rt_russia_9y)

sd_uk_9y = sd(rt_uk_9y) *sqrt(260)

sd_uk_9y_daily = sd(rt_uk_9y)

sd_vietnam_9y = sd(rt_vietnam_9y) *sqrt(260)

sd_vietnam_9y_daily = sd(rt_vietnam_9y)
```

```
sharpe_argentina = (mean_argentina - mean_rf/sd_argentina)

sharpe_australia = (mean_australia- mean_rf/sd_australia)

sharpe_brazil = (mean_brazil - mean_rf/sd_brazil)

sharpe_china = (mean_china - mean_rf/sd_china)

sharpe_india = (mean_india - mean_rf/sd_india)

sharpe_kuwait = (mean_kuwait - mean_rf/sd_kuwait)

sharpe_morocco = (mean_morocco - mean_rf/sd_morocco)

sharpe_norway = (mean_norway - mean_rf/sd_norway)

sharpe_portugal = (mean_portugal - mean_rf/sd_portugal)

sharpe_russia = (mean_russia - mean_rf/sd_russia)

sharpe_uk = (mean_uk - mean_rf/sd_uk)

sharpe_vietnam = (mean_vietnam - mean_rf/sd_vietnam)

sharpe_naive_portfolio = (mean_naive_portfolio - mean_rf/sd_naive_portfolio)

skew_argentina = skewness(rt_argentina)

skew_australia = skewness(rt_australia)

skew_brazil = skewness(rt_brazil)

skew_china = skewness(rt_china)

skew_india = skewness(rt_india)

skew_kuwait = skewness(rt_kuwait)

skew_morocco = skewness(rt_morocco)

skew_norway = skewness(rt_norway)

skew_russia = skewness(rt_russia)

skew_uk = skewness(rt_uk)

skew_portugal = skewness(rt_portugal)

skew_vietnam = skewness(rt_vietnam)

skew_naive_portfolio = skewness(naive_portfolio)
```

```
kurt_argentina = kurtosis (rt_argentina)
kurt_australia = kurtosis (rt_australia)
kurt_brazil = kurtosis(rt_brazil)
kurt_china = kurtosis(rt_china)
kurt_india  = kurtosis(rt_india)
kurt_kuwait  = kurtosis(rt_kuwait)
kurt_morocco  = kurtosis(rt_morocco)
kurt_norway  = kurtosis(rt_norway)
kurt_russia  = kurtosis(rt_russia)
kurt_uk  = kurtosis(rt_uk)
kurt_portugal  = kurtosis(rt_portugal)
kurt_vietnam  = kurtosis(rt_vietnam)
kurt_naive_portfolio = kurtosis(naive_portfolio)
acf_argentina = acf(rt_argentina, lag.max=1)
acf_australia = acf(rt_australia, lag.max=1)
acf_brazil = acf(rt_brazil, lag.max=1)
acf_china = acf(rt_china, lag.max=1)
acf_india = acf(rt_india, lag.max=1)
acf_kuwait = acf(rt_kuwait, lag.max=1)
acf_morocco = acf(rt_morocco, lag.max=1)
acf_norway = acf(rt_norway, lag.max=1)
acf_portugal = acf(rt_portugal, lag.max=1)
acf_russia = acf(rt_russia, lag.max=1)
acf_uk = acf(rt_uk, lag.max=1)
acf_vietnam = acf(rt_vietnam, lag.max=1)
```

```
acf_naive_portfolio = acf(naive_portfolio, lag.max=1)

cor_combined_vector = cor(E_combined_vector)
```

#CALCULATE IMPLIED RETURNS

```
mean_naive_portfolio = mean(naive_portfolio)

var_naive_portfolio  = var(naive_portfolio)

lambda = (mean_naive_portfolio-mean_rf)/var(naive_portfolio)

sigma = cov(rt_combined_vector)

sigma_9y = cov(rt_combined_vector_9y)

sigma_matrix = matrix(cov(rt_combined_vector), 12)

weights = c(1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12)

pii1 = weights %*% sigma_matrix

pii2 = lambda * pii1

pii3 = pii2 + mean_rf
```

#CALCULATE EXPECTED RETURNS (CAPM)

```
cov_argentina = cov(rt_argentina, naive_portfolio)

cov_australia = cov(rt_australia, naive_portfolio)

cov_brazil = cov(rt_brazil, naive_portfolio)

cov_china = cov(rt_china, naive_portfolio)

cov_india = cov(rt_india, naive_portfolio)

cov_kuwait = cov(rt_kuwait, naive_portfolio)

cov_morocco = cov(rt_morocco, naive_portfolio)

cov_norway = cov(rt_norway, naive_portfolio)

cov_russia = cov(rt_russia, naive_portfolio)

cov_uk = cov(rt_uk, naive_portfolio)

cov_portugal = cov(rt_portugal, naive_portfolio)

cov_vietnam = cov(rt_vietnam, naive_portfolio)
```

```
beta_argentina = (cov_argentina)/(var_naive_portfolio)

beta_australia = (cov_australia)/(var_naive_portfolio)

beta_brazil = (cov_brazil)/(var_naive_portfolio)

beta_china = (cov_china)/(var_naive_portfolio)

beta_india = (cov_india)/(var_naive_portfolio)

beta_kuwait = (cov_kuwait)/(var_naive_portfolio)

beta_morocco = (cov_morocco)/(var_naive_portfolio)

beta_norway = (cov_norway)/(var_naive_portfolio)

beta_russia = (cov_russia)/(var_naive_portfolio)

beta_uk = (cov_uk)/(var_naive_portfolio)

beta_portugal = (cov_portugal)/(var_naive_portfolio)

beta_vietnam = (cov_vietnam)/(var_naive_portfolio)

capm_argentina = mean_rf + (mean_naive_portfolio - mean_rf) * beta_argentina

capm_argentina_weighted = (1/12) * capm_argentina

capm_australia = mean_rf + (mean_naive_portfolio - mean_rf) * beta_australia

capm_australia_weighted = (1/12) * capm_australia

capm_brazil = mean_rf + (mean_naive_portfolio - mean_rf) * beta_brazil

capm_brazil_weighted = (1/12) * capm_brazil

capm_china = mean_rf + (mean_naive_portfolio - mean_rf) * beta_china

capm_china_weighted = (1/12) * capm_china

capm_india = mean_rf + (mean_naive_portfolio - mean_rf) * beta_india

capm_india_weighted = (1/12) * capm_india

capm_kuwait = mean_rf + (mean_naive_portfolio - mean_rf) * beta_kuwait

capm_kuwait_weighted = (1/12) * capm_kuwait

capm_morocco = mean_rf + (mean_naive_portfolio - mean_rf) * beta_morocco
```

```
capm_morocco_weighted = (1/12) * capm_morocco

capm_norway = mean_rf + (mean_naive_portfolio - mean_rf) * beta_norway

capm_norway_weighted = (1/12) * capm_norway

capm_portugal = mean_rf + (mean_naive_portfolio - mean_rf) * beta_portugal

capm_portugal_weighted = (1/12) * capm_portugal

capm_russia = mean_rf + (mean_naive_portfolio - mean_rf) * beta_russia

capm_russia_weighted = (1/12) * capm_russia

capm_uk = mean_rf + (mean_naive_portfolio - mean_rf) * beta_uk

capm_uk_weighted = (1/12) * capm_uk

capm_vietnam = mean_rf + (mean_naive_portfolio - mean_rf) * beta_vietnam

capm_vietnam_weighted = (1/12) * capm_vietnam
```

#EGARCH-M MODELING WITH ONE REGRESSOR

```
spec = ugarchspec(variance.model=list(model="eGARCH",garchOrder=c(1,1), external.regressors = matrix_oil), mean.model=list(armaOrder=c(0,0), archm = TRUE, include.mean = TRUE, external.regressors = matrix_oil))

spec_9y = ugarchspec(variance.model=list(model="eGARCH",garchOrder=c(1,1), external.regressors = matrix_oil_9y), mean.model=list(armaOrder=c(0,0), archm = TRUE, include.mean = TRUE, external.regressors = matrix_oil_9y))

egarch_argentina  = ugarchfit(spec, E_argentina_8y, solver = 'hybrid')

egarch_australia = ugarchfit(spec, E_australia_8y, solver = 'hybrid')

egarch_brazil = ugarchfit(spec, E_brazil_8y, solver = 'hybrid')

egarch_china = ugarchfit(spec, E_china_8y, solver = 'hybrid')

egarch_india = ugarchfit(spec, E_india_8y, solver = 'hybrid')

egarch_kuwait = ugarchfit(spec, E_kuwait_8y, solver = 'hybrid')

egarch_morocco = ugarchfit(spec, E_morocco_8y, solver = 'hybrid')

egarch_norway = ugarchfit(spec, E_norway_8y, solver = 'hybrid')

egarch_portugal = ugarchfit(spec, E_portugal_8y, solver = 'hybrid')

egarch_russia = ugarchfit(spec, E_russia_8y, solver = 'hybrid')
```

```
egarch_uk = ugarchfit(spec, E_uk_8y, solver = 'hybrid')

egarch_vietnam = ugarchfit(spec, E_vietnam_8y, solver = 'hybrid')

egarch_argentina_9y  = ugarchfit(spec_9y, E_argentina_9y, solver = 'hybrid')

egarch_australia_9y = ugarchfit(spec_9y, E_australia_9y, solver = 'hybrid')

egarch_brazil_9y = ugarchfit(spec_9y, E_brazil_9y, solver = 'hybrid')

egarch_china_9y = ugarchfit(spec_9y, E_china_9y, solver = 'hybrid')

egarch_india_9y = ugarchfit(spec_9y, E_india_9y, solver = 'hybrid')

egarch_kuwait_9y = ugarchfit(spec_9y, E_kuwait_9y, solver = 'hybrid')

egarch_morocco_9y = ugarchfit(spec_9y, E_morocco_9y, solver = 'hybrid')

egarch_norway_9y = ugarchfit(spec_9y, E_norway_9y, solver = 'hybrid')

egarch_portugal_9y = ugarchfit(spec_9y, E_portugal_9y, solver = 'hybrid')

egarch_russia_9y = ugarchfit(spec_9y, E_russia_9y, solver = 'hybrid')

egarch_uk_9y = ugarchfit(spec_9y, E_uk_9y, solver = 'hybrid')

egarch_vietnam_9y = ugarchfit(spec_9y, E_vietnam_9y, solver = 'hybrid')
```

#Coefficients Argentina

```
coef_argentina = coef(egarch_argentina)

coef_argentina_matrix = matrix(coef_argentina)

mu_argentina = coef_argentina_matrix[1, 1]

archm_argentina = coef_argentina_matrix[2, 1]

mxreg1_argentina = coef_argentina_matrix[3, 1]

omega_argentina = coef_argentina_matrix[4, 1]

alpha1_argentina = coef_argentina_matrix[5, 1]

beta1_argentina = coef_argentina_matrix[6, 1]

gamma1_argentina = coef_argentina_matrix[7, 1]

vxreg_1_argentina = coef_argentina_matrix[8, 1]

resid_argentina = residuals(egarch_argentina)
```

resid_argentina_matrix = matrix(tail(resid_argentina, n = 1))

resid_argentina_t = resid_argentina_matrix[1, 1]

coef_argentina_9y = coef(egarch_argentina_9y)

coef_argentina_matrix_9y = matrix(coef_argentina_9y)

mu_argentina_9y = coef_argentina_matrix_9y[1, 1]

archm_argentina_9y = coef_argentina_matrix_9y[2, 1]

mxreg1_argentina_9y = coef_argentina_matrix_9y[3, 1]

omega_argentina_9y = coef_argentina_matrix_9y[4, 1]

alpha1_argentina_9y = coef_argentina_matrix_9y[5, 1]

beta1_argentina_9y = coef_argentina_matrix_9y[6, 1]

gamma1_argentina_9y = coef_argentina_matrix_9y[7, 1]

vxreg_1_argentina_9y = coef_argentina_matrix_9y[8, 1]

resid_argentina_9y = residuals(egarch_argentina_9y)

resid_argentina_matrix_9y = matrix(tail(resid_argentina_9y, n = 1))

resid_argentina_t_9y = resid_argentina_matrix_9y[1, 1]

#Coefficients Australia

coef_australia = coef(egarch_australia)

coef_australia_matrix = matrix(coef_australia)

mu_australia = coef_australia_matrix[1, 1]

archm_australia = coef_australia_matrix[2, 1]

mxreg1_australia = coef_australia_matrix[3, 1]

omega_australia = coef_australia_matrix[4, 1]

alpha1_australia = coef_australia_matrix[5, 1]

beta1_australia = coef_australia_matrix[6, 1]

gamma1_australia = coef_australia_matrix[7, 1]

vxreg_1_australia = coef_australia_matrix[8, 1] resid_australia = residuals(egarch_australia)

resid_australia_matrix = matrix(tail(resid_australia, n = 1))

resid_australia_t = resid_australia_matrix[1, 1]

coef_australia_9y = coef(egarch_australia_9y)

coef_australia_matrix_9y = matrix(coef_australia_9y)

mu_australia_9y = coef_australia_matrix_9y[1, 1]

archm_australia_9y = coef_australia_matrix_9y[2, 1]

mxreg1_australia_9y = coef_australia_matrix_9y[3, 1]

omega_australia_9y = coef_australia_matrix_9y[4, 1]

alpha1_australia_9y = coef_australia_matrix_9y[5, 1]

beta1_australia_9y = coef_australia_matrix_9y[6, 1]

gamma1_australia_9y = coef_australia_matrix_9y[7, 1]

vxreg_1_australia_9y = coef_australia_matrix_9y[8, 1]

resid_australia_9y = residuals(egarch_australia_9y)

resid_australia_matrix_9y = matrix(tail(resid_australia_9y, n = 1))

resid_australia_t_9y = resid_australia_matrix_9y[1, 1]

#Coefficients Brazil

coef_brazil = coef(egarch_brazil)

coef_brazil_matrix = matrix(coef_brazil)

mu_brazil = coef_brazil_matrix[1, 1]

archm_brazil = coef_brazil_matrix[2, 1]

mxreg1_brazil = coef_brazil_matrix[3, 1]

omega_brazil = coef_brazil_matrix[4, 1]

alpha1_brazil = coef_brazil_matrix[5, 1]

beta1_brazil = coef_brazil_matrix[6, 1]

gamma1_brazil = coef_brazil_matrix[7, 1]

```
vxreg_1_brazil = coef_brazil_matrix[8, 1]

resid_brazil = residuals(egarch_brazil)

resid_brazil_matrix = matrix(tail(resid_brazil, n = 1))

resid_brazil_t = resid_brazil_matrix[1, 1]

coef_brazil_9y = coef(egarch_brazil_9y)

coef_brazil_matrix_9y = matrix(coef_brazil_9y)

mu_brazil_9y = coef_brazil_matrix_9y[1, 1]

archm_brazil_9y = coef_brazil_matrix_9y[2, 1]

mxreg1_brazil_9y = coef_brazil_matrix_9y[3, 1]

omega_brazil_9y = coef_brazil_matrix_9y[4, 1]

alpha1_brazil_9y = coef_brazil_matrix_9y[5, 1]

beta1_brazil_9y = coef_brazil_matrix_9y[6, 1]

gamma1_brazil_9y = coef_brazil_matrix_9y[7, 1]

vxreg_1_brazil_9y = coef_brazil_matrix_9y[8, 1]

resid_brazil_9y = residuals(egarch_brazil_9y)

resid_brazil_matrix_9y = matrix(tail(resid_brazil_9y, n = 1))

resid_brazil_t_9y = resid_brazil_matrix_9y[1, 1]
```

#Coefficients China

```
coef_china = coef(egarch_china)

coef_china_matrix = matrix(coef_china)

mu_china = coef_china_matrix[1, 1]

archm_china = coef_china_matrix[2, 1]

mxreg1_china = coef_china_matrix[3, 1]

omega_china = coef_china_matrix[4, 1]

alpha1_china = coef_china_matrix[5, 1]
```

```
beta1_china = coef_china_matrix[6, 1]

gamma1_china = coef_china_matrix[7, 1]

vxreg_1_china = coef_china_matrix[8, 1]

resid_china = residuals(egarch_china)

resid_china_matrix = matrix(tail(resid_china, n = 1))

resid_china_t = resid_china_matrix[1, 1]

coef_china_9y = coef(egarch_china_9y)

coef_china_matrix_9y = matrix(coef_china_9y)

mu_china_9y = coef_china_matrix_9y[1, 1]

archm_china_9y = coef_china_matrix_9y[2, 1]

mxreg1_china_9y = coef_china_matrix_9y[3, 1]

omega_china_9y = coef_china_matrix_9y[4, 1]

alpha1_china_9y = coef_china_matrix_9y[5, 1]

beta1_china_9y = coef_china_matrix_9y[6, 1]

gamma1_china_9y = coef_china_matrix_9y[7, 1]

vxreg_1_china_9y = coef_china_matrix_9y[8, 1]

resid_china_9y = residuals(egarch_china_9y)

resid_china_matrix_9y = matrix(tail(resid_china_9y, n = 1))

resid_china_t_9y = resid_china_matrix_9y[1, 1]
```

#Coefficients India

```
coef_india = coef(egarch_india)

coef_india_matrix = matrix(coef_india)

mu_india = coef_india_matrix[1, 1]

archm_india = coef_india_matrix[2, 1]

mxreg1_india = coef_india_matrix[3, 1]

omega_india = coef_india_matrix[4, 1]
```

```
alpha1_india = coef_india_matrix[5, 1]
beta1_india = coef_india_matrix[6, 1]
gamma1_india = coef_india_matrix[7, 1]
vxreg_1_india = coef_india_matrix[8, 1]
resid_india = residuals(egarch_india)
resid_india_matrix = matrix(tail(resid_india, n = 1))
resid_india_t = resid_india_matrix[1, 1]
coef_india_9y = coef(egarch_india_9y)
coef_india_matrix_9y = matrix(coef_india_9y)
mu_india_9y = coef_india_matrix_9y[1, 1]
archm_india_9y = coef_india_matrix_9y[2, 1]
mxreg1_india_9y = coef_india_matrix_9y[3, 1]
omega_india_9y = coef_india_matrix_9y[4, 1]
alpha1_india_9y = coef_india_matrix_9y[5, 1]
beta1_india_9y = coef_india_matrix_9y[6, 1]
gamma1_india_9y = coef_india_matrix_9y[7, 1]
vxreg_1_india_9y = coef_india_matrix_9y[8, 1]
resid_india_9y = residuals(egarch_india_9y)
resid_india_matrix_9y = matrix(tail(resid_india_9y, n = 1))
resid_india_t_9y = resid_india_matrix_9y[1, 1]
```

#Coefficients Kuwait

```
coef_kuwait = coef(egarch_kuwait)
coef_kuwait_matrix = matrix(coef_kuwait)
mu_kuwait = coef_kuwait_matrix[1, 1]
archm_kuwait = coef_kuwait_matrix[2, 1]
mxreg1_kuwait = coef_kuwait_matrix[3, 1]
```

omega_kuwait = coef_kuwait_matrix[4, 1]

alpha1_kuwait = coef_kuwait_matrix[5, 1]

beta1_kuwait = coef_kuwait_matrix[6, 1]

gamma1_kuwait = coef_kuwait_matrix[7, 1]

vxreg_1_kuwait = coef_kuwait_matrix[8, 1]

resid_kuwait = residuals(egarch_kuwait)

resid_kuwait_matrix = matrix(tail(resid_kuwait, n = 1))

resid_kuwait_t = resid_kuwait_matrix[1, 1]

coef_kuwait_9y = coef(egarch_kuwait_9y)

coef_kuwait_matrix_9y = matrix(coef_kuwait_9y)

mu_kuwait_9y = coef_kuwait_matrix_9y[1, 1]

archm_kuwait_9y = coef_kuwait_matrix_9y[2, 1]

mxreg1_kuwait_9y = coef_kuwait_matrix_9y[3, 1]

omega_kuwait_9y = coef_kuwait_matrix_9y[4, 1]

alpha1_kuwait_9y = coef_kuwait_matrix_9y[5, 1]

beta1_kuwait_9y = coef_kuwait_matrix_9y[6, 1]

gamma1_kuwait_9y = coef_kuwait_matrix_9y[7, 1]

vxreg_1_kuwait_9y = coef_kuwait_matrix_9y[8, 1]

resid_kuwait_9y = residuals(egarch_kuwait_9y)

resid_kuwait_matrix_9y = matrix(tail(resid_kuwait_9y, n = 1))

resid_kuwait_t_9y = resid_kuwait_matrix_9y[1, 1]

#Coefficients Morocco

coef_morocco = coef(egarch_morocco)

coef_morocco_matrix = matrix(coef_morocco)

mu_morocco = coef_morocco_matrix[1, 1]

archm_morocco = coef_morocco_matrix[2, 1] mxreg1_morocco = coef_morocco_matrix[3, 1]

omega_morocco = coef_morocco_matrix[4, 1]

alpha1_morocco = coef_morocco_matrix[5, 1]

beta1_morocco = coef_morocco_matrix[6, 1]

gamma1_morocco = coef_morocco_matrix[7, 1]

vxreg_1_morocco = coef_morocco_matrix[8, 1]

resid_morocco = residuals(egarch_morocco)

resid_morocco_matrix = matrix(tail(resid_morocco, n = 1))

resid_morocco_t = resid_morocco_matrix[1, 1]

coef_morocco_9y = coef(egarch_morocco_9y)

coef_morocco_matrix_9y = matrix(coef_morocco_9y)

mu_morocco_9y = coef_morocco_matrix_9y[1, 1]

archm_morocco_9y = coef_morocco_matrix_9y[2, 1]

mxreg1_morocco_9y = coef_morocco_matrix_9y[3, 1]

omega_morocco_9y = coef_morocco_matrix_9y[4, 1]

alpha1_morocco_9y = coef_morocco_matrix_9y[5, 1]

beta1_morocco_9y = coef_morocco_matrix_9y[6, 1]

gamma1_morocco_9y = coef_morocco_matrix_9y[7, 1]

vxreg_1_morocco_9y = coef_morocco_matrix_9y[8, 1]

resid_morocco_9y = residuals(egarch_morocco_9y)

resid_morocco_matrix_9y = matrix(tail(resid_morocco_9y, n = 1))

resid_morocco_t_9y = resid_morocco_matrix_9y[1, 1]

#Coefficients Norway

coef_norway = coef(egarch_norway)

coef_norway_matrix = matrix(coef_norway)

mu_norway = coef_norway_matrix[1, 1]

```
archm_norway = coef_norway_matrix[2, 1]

mxreg1_norway = coef_norway_matrix[3, 1]

omega_norway = coef_norway_matrix[4, 1]

alpha1_norway = coef_norway_matrix[5, 1]

beta1_norway = coef_norway_matrix[6, 1]

gamma1_norway = coef_norway_matrix[7, 1]

vxreg_1_norway = coef_norway_matrix[8, 1]

resid_norway = residuals(egarch_norway)

resid_norway_matrix = matrix(tail(resid_norway, n = 1))

resid_norway_t = resid_norway_matrix[1, 1]

coef_norway_9y = coef(egarch_norway_9y)

coef_norway_matrix_9y = matrix(coef_norway_9y)

mu_norway_9y = coef_norway_matrix_9y[1, 1]

archm_norway_9y = coef_norway_matrix_9y[2, 1]

mxreg1_norway_9y = coef_norway_matrix_9y[3, 1]

omega_norway_9y = coef_norway_matrix_9y[4, 1]

alpha1_norway_9y = coef_norway_matrix_9y[5, 1]

beta1_norway_9y = coef_norway_matrix_9y[6, 1]

gamma1_norway_9y = coef_norway_matrix_9y[7, 1]

vxreg_1_norway_9y = coef_norway_matrix_9y[8, 1]

resid_norway_9y = residuals(egarch_norway_9y)

resid_norway_matrix_9y = matrix(tail(resid_norway_9y, n = 1))

resid_norway_t_9y = resid_norway_matrix_9y[1, 1]
```

#Coefficients Portugal

```
coef_portugal = coef(egarch_portugal)

coef_portugal_matrix = matrix(coef_portugal)
```

mu_portugal = coef_portugal_matrix[1, 1]

archm_portugal= coef_portugal_matrix[2, 1]

mxreg1_portugal = coef_portugal_matrix[3, 1]

omega_portugal = coef_portugal_matrix[4, 1]

alpha1_portugal = coef_portugal_matrix[5, 1]

beta1_portugal = coef_portugal_matrix[6, 1]

gamma1_portugal = coef_portugal_matrix[7, 1]

vxreg_1_portugal = coef_portugal_matrix[8, 1]

resid_portugal = residuals(egarch_portugal)

resid_portugal_matrix = matrix(tail(resid_portugal, n = 1))

resid_portugal_t = resid_portugal_matrix[1, 1]

coef_portugal_9y = coef(egarch_portugal_9y)

coef_portugal_matrix_9y = matrix(coef_portugal_9y)

mu_portugal_9y = coef_portugal_matrix_9y[1, 1]

archm_portugal_9y = coef_portugal_matrix_9y[2, 1]

mxreg1_portugal_9y = coef_portugal_matrix_9y[3, 1]

omega_portugal_9y = coef_portugal_matrix_9y[4, 1]

alpha1_portugal_9y = coef_portugal_matrix_9y[5, 1]

beta1_portugal_9y = coef_portugal_matrix_9y[6, 1]

gamma1_portugal_9y = coef_portugal_matrix_9y[7, 1]

vxreg_1_portugal_9y = coef_portugal_matrix_9y[8, 1]

resid_portugal_9y = residuals(egarch_portugal_9y)

resid_portugal_matrix_9y = matrix(tail(resid_portugal_9y, n = 1))

resid_portugal_t_9y = resid_portugal_matrix_9y[1, 1]

#Coefficients Russia

coef_russia = coef(egarch_russia)

```
coef_russia_matrix = matrix(coef_russia)
mu_russia = coef_russia_matrix[1, 1]
archm_russia = coef_russia_matrix[2, 1]
mxreg1_russia = coef_russia_matrix[3, 1]
omega_russia  = coef_russia_matrix[4, 1]
alpha1_russia = coef_russia_matrix[5, 1]
beta1_russia = coef_russia_matrix[6, 1]
gamma1_russia = coef_russia_matrix[7, 1]
vxreg_1_russia = coef_russia_matrix[8, 1]
resid_russia = residuals(egarch_russia)
resid_russia_matrix = matrix(tail(resid_russia, n = 1))
resid_russia_t = resid_russia_matrix[1, 1]
coef_russia_9y = coef(egarch_russia_9y)
coef_russia_matrix_9y = matrix(coef_russia_9y)
mu_russia_9y = coef_russia_matrix_9y[1, 1]
archm_russia_9y = coef_russia_matrix_9y[2, 1]
mxreg1_russia_9y = coef_russia_matrix_9y[3, 1]
omega_russia_9y  = coef_russia_matrix_9y[4, 1]
alpha1_russia_9y = coef_russia_matrix_9y[5, 1]
beta1_russia_9y = coef_russia_matrix_9y[6, 1]
gamma1_russia_9y = coef_russia_matrix_9y[7, 1]
vxreg_1_russia_9y = coef_russia_matrix_9y[8, 1]
resid_russia_9y = residuals(egarch_russia_9y)
resid_russia_matrix_9y = matrix(tail(resid_russia_9y, n = 1))
resid_russia_t_9y = resid_russia_matrix_9y[1, 1]
```

#Coefficients UK

```
coef_uk = coef(egarch_uk)

coef_uk_matrix = matrix(coef_uk)

mu_uk = coef_uk_matrix[1, 1]

archm_uk= coef_uk_matrix[2, 1]

mxreg1_uk = coef_uk_matrix[3, 1]

omega_uk = coef_uk_matrix[4, 1]

alpha1_uk = coef_uk_matrix[5, 1]

beta1_uk = coef_uk_matrix[6, 1]

gamma1_uk = coef_uk_matrix[7, 1]

vxreg_1_uk = coef_uk_matrix[8, 1]

resid_uk = residuals(egarch_uk)

resid_uk_matrix = matrix(tail(resid_uk, n = 1))

resid_uk_t = resid_uk_matrix[1, 1]

coef_uk_9y = coef(egarch_uk_9y)

coef_uk_matrix_9y = matrix(coef_uk_9y)

mu_uk_9y = coef_uk_matrix_9y[1, 1]

archm_uk_9y= coef_uk_matrix_9y[2, 1]

mxreg1_uk_9y = coef_uk_matrix_9y[3, 1]

omega_uk_9y = coef_uk_matrix_9y[4, 1]

alpha1_uk_9y = coef_uk_matrix_9y[5, 1]

beta1_uk_9y = coef_uk_matrix_9y[6, 1]

gamma1_uk_9y = coef_uk_matrix_9y[7, 1]

vxreg_1_uk_9y = coef_uk_matrix_9y[8, 1]

resid_uk_9y = residuals(egarch_uk_9y)

resid_uk_matrix_9y = matrix(tail(resid_uk_9y, n = 1))
```

resid_uk_t_9y = resid_uk_matrix_9y[1, 1]

#Coefficients Vietnam

coef_vietnam = coef(egarch_vietnam)

coef_vietnam_matrix = matrix(coef_vietnam)

mu_vietnam = coef_vietnam_matrix[1, 1]

archm_vietnam= coef_vietnam_matrix[2, 1]

mxreg1_vietnam = coef_vietnam_matrix[3, 1]

omega_vietnam = coef_vietnam_matrix[4, 1]

alpha1_vietnam = coef_vietnam_matrix[5, 1]

beta1_vietnam = coef_vietnam_matrix[6, 1]

gamma1_vietnam = coef_vietnam_matrix[7, 1]

vxreg_1_vietnam = coef_vietnam_matrix[8, 1]

resid_vietnam = residuals(egarch_vietnam)

resid_vietnam_matrix = matrix(tail(resid_vietnam, n = 1))

resid_vietnam_t = resid_vietnam_matrix[1, 1]

coef_vietnam_9y = coef(egarch_vietnam_9y)

coef_vietnam_matrix_9y = matrix(coef_vietnam_9y)

mu_vietnam_9y = coef_vietnam_matrix_9y[1, 1]

archm_vietnam_9y = coef_vietnam_matrix_9y[2, 1]

mxreg1_vietnam_9y = coef_vietnam_matrix_9y[3, 1]

omega_vietnam_9y = coef_vietnam_matrix_9y[4, 1]

alpha1_vietnam_9y = coef_vietnam_matrix_9y[5, 1]

beta1_vietnam_9y = coef_vietnam_matrix_9y[6, 1]

gamma1_vietnam_9y = coef_vietnam_matrix_9y[7, 1]

vxreg_1_vietnam_9y = coef_vietnam_matrix_9y[8, 1]

resid_vietnam_9y = residuals(egarch_vietnam_9y) resid_vietnam_matrix_9y = matrix(tail(resid_vietnam_9y, n = 1))

resid_vietnam_t_9y = resid_vietnam_matrix_9y[1, 1]

#EGARCH-M EQUATIONS

oil_t = tail(matrix_oil/100, n=1)

oil_t_9y = tail(matrix_oil_9y/100, n=1)

log_sigma_argentina = omega_argentina+beta1_argentina*log(sd_argentina_8y_daily)^2+alpha1_argentina*abs(resid_argentina_t/sd_argentina_8y_daily)+gamma1_argentina*(resid_argentina_t/sd_argentina_8y_daily)+oil_t*vxreg_1_argentina

sigma_argentina = exp(log_sigma_argentina)

r_argentina = mu_argentina + oil_t * mxreg1_argentina + archm_argentina * sigma_argentina

r_argentina_weighted = (1/12)*r_argentina

log_sigma_argentina_9y = omega_argentina_9y+beta1_argentina_9y*log(sd_argentina_9y_daily)^2+alpha1_argentina_9y*abs(resid_argentina_t_9y/sd_argentina_9y_daily)+gamma1_argentina_9y*(resid_argentina_t_9y/sd_argentina_9y_daily)+oil_t_9y*vxreg_1_argentina_9y

sigma_argentina_9y = exp(log_sigma_argentina_9y)

r_argentina_9y = mu_argentina_9y + oil_t_9y * mxreg1_argentina_9y + archm_argentina_9y * sigma_argentina_9y

r_argentina_weighted_9y = (1/12)*r_argentina_9y

log_sigma_australia = omega_australia+beta1_australia*log(sd_australia_8y_daily)^2+alpha1_australia*abs(resid_australia_t/sd_australia_8y_daily)+gamma1_australia*(resid_australia_t/sd_australia_8y_daily)+oil_t*vxreg_1_australia

sigma_australia = exp(log_sigma_australia)

r_australia = mu_australia + oil_t * mxreg1_australia + archm_australia * sigma_australia

r_australia_weighted = (1/12)*r_australia

log_sigma_australia_9y = omega_australia_9y+beta1_australia_9y*log(sd_australia_9y_daily)^2+alpha1_australia_9y*abs(resid_australia_t_9y/sd_australia_9y_daily)+gamma1_australia_9y*(resid_australia_t_9y/sd_australia_9y_daily)+oil_t_9y*vxreg_1_australia_9y sigma_australia_9y = exp(log_sigma_australia_9y)

r_australia_9y = mu_australia_9y + oil_t_9y * mxreg1_australia_9y + archm_australia_9y * sigma_australia_9y

r_australia_weighted_9y = (1/12)*r_australia_9y
log_sigma_brazil = omega_brazil+beta1_brazil*log(sd_brazil_8y_daily)^2+alpha1_brazil*abs(resid_brazil_t/sd_brazil_8y_daily)+gamma1_brazil*(resid_brazil_t/sd_brazil_8y_daily)+oil_t*vxreg_1_brazil

sigma_brazil = exp(log_sigma_brazil)

r_brazil = mu_brazil + oil_t * mxreg1_brazil + archm_brazil * sigma_brazil

r_brazil_weighted = (1/12)*r_brazil

log_sigma_brazil_9y = omega_brazil_9y+beta1_brazil_9y*log(sd_brazil_9y_daily)^2+alpha1_brazil_9y*abs(resid_brazil_t_9y/sd_brazil_9y_daily)+gamma1_brazil_9y*(resid_brazil_t_9y/sd_brazil_9y_daily)+oil_t_9y*vxreg_1_brazil_9y

sigma_brazil_9y = exp(log_sigma_brazil_9y)

r_brazil_9y = mu_brazil_9y + oil_t_9y * mxreg1_brazil_9y + archm_brazil_9y * sigma_brazil_9y

r_brazil_weighted_9y = (1/12)*r_brazil_9y

log_sigma_china = omega_china+beta1_china*log(sd_china_8y_daily)^2+alpha1_china*abs(resid_china_t/sd_china_8y_daily)+gamma1_china*(resid_china_t/sd_china_8y_daily)+oil_t*vxreg_1_china

sigma_china = exp(log_sigma_china)

r_china = mu_china + oil_t * mxreg1_china + archm_china * sigma_china

r_china_weighted = (1/12)*r_china

log_sigma_china_9y = omega_china_9y+beta1_china_9y*log(sd_china_9y_daily)^2+alpha1_china_9y*abs(resid_china_t_9y/sd_china_9y_daily)+gamma1_china_9y*(resid_china_t_9y/sd_china_9y_daily)+oil_t_9y*vxreg_1_china_9y

sigma_china_9y = exp(log_sigma_china_9y)

r_china_9y = mu_china_9y + oil_t_9y * mxreg1_china_9y + archm_china_9y * sigma_china_9y

r_china_weighted_9y = (1/12)*r_china_9y

log_sigma_india = omega_india+beta1_india*log(sd_india_8y_daily)^2+alpha1_india*abs(resid_india_t/sd_india_8y_daily)+gamma1_india*(resid_india_t/sd_india_8y_daily)+oil_t*vxreg_1_india

sigma_india = exp(log_sigma_india)

r_india = mu_india + oil_t * mxreg1_india + archm_india * sigma_india

r_india_weighted = (1/12)*r_india

log_sigma_india_9y = omega_india_9y+beta1_india_9y*log(sd_india_9y_daily)^2+alpha1_india_9y*abs(resid_india_t_9y/sd_india_9y_daily)+gamma1_india_9y*(resid_india_t_9y/sd_india_9y_daily)+oil_t_9y*vxreg_1_india_9y

sigma_india_9y = exp(log_sigma_india_9y)

r_india_9y = mu_india_9y + oil_t_9y * mxreg1_india_9y + archm_india_9y * sigma_india_9y

r_india_weighted_9y = (1/12)*r_india_9y

log_sigma_kuwait = omega_kuwait+beta1_kuwait*log(sd_kuwait_8y_daily)^2+alpha1_kuwait*abs(resid_kuwait_t/sd_kuwait_8y_daily)+gamma1_kuwait*(resid_kuwait_t/sd_kuwait_8y_daily)+oil_t*vxreg_1_kuwait

sigma_kuwait = exp(log_sigma_kuwait)

r_kuwait = mu_kuwait + oil_t * mxreg1_kuwait + archm_kuwait * sigma_kuwait

r_kuwait_weighted = (1/12)*r_kuwait

log_sigma_kuwait_9y = omega_kuwait_9y+beta1_kuwait_9y*log(sd_kuwait_9y_daily)^2+alpha1_kuwait_9y*abs(resid_kuwait_t_9y/sd_kuwait_9y_daily)+gamma1_kuwait_9y*(resid_kuwait_t_9y/sd_kuwait_9y_daily)+oil_t_9y*vxreg_1_kuwait_9y

sigma_kuwait_9y = exp(log_sigma_kuwait_9y)

r_kuwait_9y = mu_kuwait_9y+ oil_t_9y * mxreg1_kuwait_9y + archm_kuwait_9y * sigma_kuwait_9y

r_kuwait_weighted_9y = (1/12)*r_kuwait_9y

log_sigma_morocco = omega_morocco+beta1_kuwait*log(sd_morocco_8y_daily)^2+alpha1_morocco*abs(resid_morocco_t/sd_morocco_8y_daily)+gamma1_morocco*(resid_morocco_t/sd_morocco_8y_daily)+oil_t*vxreg_1_morocco sigma_morocco = exp(log_sigma_morocco)

r_morocco = mu_morocco + oil_t * mxreg1_morocco + archm_morocco * sigma_morocco
r_morocco_weighted = (1/12)*r_morocco

log_sigma_morocco_9y = omega_morocco_9y+beta1_kuwait_9y*log(sd_morocco_9y_daily)^2+alpha1_morocco_9y*abs(resid_morocco_t_9y/sd_morocco_9y_daily)+gamma1_morocco_9y*(resid_morocco_t_9y/sd_morocco_9y_daily)+oil_t_9y*vxreg_1_morocco_9y

sigma_morocco_9y = exp(log_sigma_morocco_9y)

r_morocco_9y = mu_morocco_9y + oil_t_9y * mxreg1_morocco_9y + archm_morocco_9y * sigma_morocco_9y

r_morocco_weighted_9y = (1/12)*r_morocco_9y

log_sigma_norway = omega_norway+beta1_kuwait*log(sd_norway_8y_daily)^2+alpha1_norway*abs(resid_norway_t/sd_norway_8y_daily)+gamma1_norway*(resid_norway_t/sd_norway_8y_daily)+oil_t*vxreg_1_norway

sigma_norway = exp(log_sigma_norway)

r_norway = mu_norway + oil_t * mxreg1_norway + archm_norway * sigma_norway

r_norway_weighted = (1/12)*r_norway

log_sigma_norway_9y = omega_norway_9y+beta1_kuwait_9y*log(sd_norway_9y_daily)^2+alpha1_norway_9y*abs(resid_norway_t_9y/sd_norway_9y_daily)+gamma1_norway_9y*(resid_norway_t_9y/sd_norway_9y_daily)+oil_t_9y*vxreg_1_norway_9y

sigma_norway_9y = exp(log_sigma_norway_9y)

r_norway_9y = mu_norway_9y + oil_t_9y * mxreg1_norway_9y + archm_norway_9y * sigma_norway_9y

r_norway_weighted_9y = (1/12)*r_norway_9y

log_sigma_portugal = omega_portugal+beta1_portugal*log(sd_portugal_8y_daily)^2+alpha1_portugal*abs(resid_portugal_t/sd_portugal_8y_daily)+gamma1_portugal*(resid_portugal_t/sd_portugal_8y_daily)+oil_t*vxreg_1_portugal

sigma_portugal = exp(log_sigma_portugal)

r_portugal = mu_portugal + oil_t * mxreg1_portugal + archm_portugal * sigma_portugal r_portugal_weighted = (1/12)*r_portugal

log_sigma_portugal_9y = omega_portugal_9y+beta1_portugal_9y*log(sd_portugal_9y_daily)^2+alpha1_portugal_9y*abs(resid_portugal_t_9y/sd_portugal_9y_daily)+gamma1_portugal_9y*(resid_portugal_t_9y/sd_portugal_9y_daily)+oil_t_9y*vxreg_1_portugal_9y

sigma_portugal_9y = exp(log_sigma_portugal_9y)

r_portugal_9y = mu_portugal_9y + oil_t_9y * mxreg1_portugal_9y + archm_portugal_9y * sigma_portugal_9y

r_portugal_weighted_9y = (1/12)*r_portugal_9y

log_sigma_russia = omega_russia+beta1_russia*log(sd_russia_8y_daily)^2+alpha1_russia*abs(resid_russia_t/sd_russia_8y_daily)+gamma1_russia*(resid_russia_t/sd_russia_8y_daily)+oil_t*vxreg_1_russia

sigma_russia = exp(log_sigma_russia)

r_russia = mu_russia + oil_t * mxreg1_russia + archm_russia * sigma_russia

r_russia_weighted = (1/12)*r_russia

log_sigma_russia_9y = omega_russia_9y+beta1_russia_9y*log(sd_russia_9y_daily)^2+alpha1_russia_9y*abs(resid_russia_t_9y/sd_russia_9y_daily)+gamma1_russia_9y*(resid_russia_t_9y/sd_russia_9y_daily)+oil_t_9y*vxreg_1_russia_9y

sigma_russia_9y = exp(log_sigma_russia_9y)

r_russia_9y = mu_russia_9y + oil_t_9y * mxreg1_russia_9y + archm_russia_9y * sigma_russia_9y

r_russia_weighted_9y = (1/12)*r_russia_9y

log_sigma_uk = omega_uk+beta1_uk*log(sd_uk_8y_daily)^2+alpha1_uk*abs(resid_uk_t/sd_uk_8y_daily)+gamma1_uk*(resid_uk_t/sd_uk_8y_daily)+oil_t*vxreg_1_uk

sigma_uk = exp(log_sigma_uk)

r_uk = mu_uk + oil_t * mxreg1_uk + archm_uk * sigma_uk

r_uk_weighted = (1/12)*r_uk

log_sigma_uk_9y = omega_uk_9y+beta1_uk_9y*log(sd_uk_9y_daily)^2+alpha1_uk_9y*abs(resid_uk_t_9y

/sd_uk_9y_daily)+gamma1_uk_9y*(resid_uk_t_9y/sd_uk_9y_daily)+oil_t_9y*vxreg_1_uk_9y

sigma_uk_9y = exp(log_sigma_uk_9y)

r_uk_9y = mu_uk_9y + oil_t_9y * mxreg1_uk_9y + archm_uk_9y * sigma_uk_9y

r_uk_weighted_9y = (1/12)*r_uk_9y

log_sigma_vietnam = omega_vietnam+beta1_vietnam*log(sd_vietnam_8y_daily)^2+alpha1_vietnam*abs(resid_vietnam_t/sd_vietnam_8y_daily)+gamma1_vietnam*(resid_vietnam_t/sd_vietnam_8y_daily)+oil_t*vxreg_1_vietnam

sigma_vietnam = exp(log_sigma_vietnam)

r_vietnam = mu_vietnam + oil_t * mxreg1_vietnam + archm_vietnam * sigma_vietnam

r_vietnam_weighted = (1/12)*r_vietnam

log_sigma_vietnam_9y = omega_vietnam_9y+beta1_vietnam_9y*log(sd_vietnam_9y_daily)^2+alpha1_vietnam_9y*abs(resid_vietnam_t_9y/sd_vietnam_9y_daily)+gamma1_vietnam_9y*(resid_vietnam_t_9y/sd_vietnam_9y_daily)+oil_t_9y*vxreg_1_vietnam_9y

sigma_vietnam_9y = exp(log_sigma_vietnam_9y)

r_vietnam_9y = mu_vietnam_9y + oil_t_9y * mxreg1_vietnam_9y + archm_vietnam_9y * sigma_vietnam_9y

r_vietnam_weighted_9y = (1/12)*r_vietnam_9y

#THE BLACK–LITTERMAN RETURNS

tau1 = 0.01

sigma_tau1 = tau1 * cov(rt_combined_vector)

sigma_tau1_9y = tau1 * cov(rt_combined_vector_9y)

tau2 = 0.05

sigma_tau2 = tau2 * cov(rt_combined_vector)

sigma_tau2_9y = tau2 * cov(rt_combined_vector_9y)

tau3 = 0.1

sigma_tau3 = tau3 * cov(rt_combined_vector)

sigma_tau3_9y = tau3 * cov(rt_combined_vector_9y)

```
views1 = c(0,0,0,1,0,0,0,-1,0,0,0,0)

views2 = c(0,0,-1,0,1,0,0,0,0,0,0,0)

views3 = c(0,0,0,0,0,0,1,0,-1,0,0,0)
views1_9y = c(0,0,0,1,0,0,0,-1,0,0,0,0)

views2_9y = c(0,0,0,0,1,0,0,0,0,-1,0,0)

views3_9y = c(0,-1,0,0,0,1,0,0,0,0,0,0)

views = c(views1, views2, views3)

views_9y = c(views1_9y, views2_9y, views3_9y)

P = matrix(views,12,3)

P_9y = matrix(views_9y,12,3)

P_rotate = t(P)

P_rotate_9y = t(P_9y)

p1 = matrix(views1, 1,12)

p2 = matrix(views2, 1,12)

p3 = matrix(views3, 1,12)

p1_9y = matrix(views1_9y, 1,12)

p2_9y = matrix(views2_9y, 1,12)

p3_9y = matrix(views3_9y, 1,12)

p1_rotate = t(p1)

p2_rotate = t(p2)

p3_rotate = t(p3)

p1_rotate_9y = t(p1_9y)

p2_rotate_9y = t(p2_9y)

p3_rotate_9y = t(p3_9y)

views_percentages = c(0.00204, 0.00207, 0.00234)

views_percentages_9y = c(0.171, 0.244, 0.049)
```

```
Q = matrix(views_percentages, 3,1)

Q_9y = matrix(views_percentages_9y, 3,1)

sigma_matrix = matrix(sigma_9y, 12,12)
sigma_matrix_9y = matrix(sigma_9y, 12,12)
residuals = matrix(cbind(resid_argentina, resid_australia, resid_brazil, resid_china, resid_india, resid_kuwait, resid_morocco, resid_norway, resid_portugal, resid_russia, resid_uk, resid_vietnam),2086,12)

residuals_9y = matrix(cbind(resid_argentina_9y, resid_australia_9y, resid_brazil_9y, resid_china_9y, resid_india_9y, resid_kuwait_9y, resid_morocco_9y, resid_norway_9y, resid_portugal_9y, resid_russia_9y, resid_uk_9y, resid_vietnam_9y),2347,12)

cov_matrix_residuals = cov(residuals)

cov_matrix_residuals_9y = cov(residuals_9y)

omega_1 = p1%*%cov_matrix_residuals%*%p1_rotate

omega_2 = p2%*%cov_matrix_residuals%*%p2_rotate

omega_3 = p3%*%cov_matrix_residuals%*%p3_rotate

omega_1_9y = p1_9y%*%cov_matrix_residuals_9y%*%p1_rotate_9y

omega_2_9y = p2_9y%*%cov_matrix_residuals_9y%*%p2_rotate_9y

omega_3_9y = p3%*%cov_matrix_residuals_9y%*%p3_rotate_9y

omega_combined = matrix(cbind(omega_1, omega_2, omega_3))

omega_combined_9y = matrix(cbind(omega_1_9y, omega_2_9y, omega_3_9y))

omega = matrix(cbind(omega_1, 0,0,0,omega_2,0,0,0,omega_3), 3,3)

omega_9y = matrix(cbind(omega_1_9y, 0,0,0,omega_2_9y,0,0,0,omega_3_9y), 3,3)

omega_percentage = omega/100

omega_percentage_9y = omega_9y/100

capm_matrix = matrix(c(capm_argentina, capm_australia, capm_brazil, capm_china, capm_india, capm_kuwait, capm_morocco, capm_norway, capm_portugal, capm_russia, capm_uk, capm_vietnam), 12,1)

#Black-Litterman returns with tau parameter 0.01

blreturns_step1_tau1 = ginv(tau1*sigma_matrix)
```

blreturns_step2_tau1 = P%*%ginv(omega_percentage)%*%P_rotate

blreturns_step3_tau1 = ginv(blreturns_step1_tau1+blreturns_step2_tau1)

blreturns_step4_tau1 =
blreturns_step1_tau1%*%capm_matrix+P%*%ginv(omega_percentage)%*%Q
blreturns_step5_tau1 = blreturns_step3_tau1%*%blreturns_step4_tau1

blreturns_step1_tau1_9y = ginv(tau1*sigma_matrix_9y)

blreturns_step2_tau1_9y = P_9y%*%ginv(omega_percentage_9y)%*%P_rotate_9y

blreturns_step3_tau1_9y = ginv(blreturns_step1_tau1_9y+blreturns_step2_tau1_9y)

blreturns_step4_tau1_9y =
blreturns_step1_tau1_9y%*%capm_matrix+P_9y%*%ginv(omega_percentage_9y)%*
%Q_9y

blreturns_step5_tau1_9y = blreturns_step3_tau1_9y%*%blreturns_step4_tau1_9y

#Black-Litterman returns with tau parameter 0.05

blreturns_step1_tau2 = ginv(tau2*sigma_matrix)

blreturns_step2_tau2 = P%*%ginv(omega_percentage)%*%P_rotate

blreturns_step3_tau2 = ginv(blreturns_step1_tau2+blreturns_step2_tau2)

blreturns_step4_tau2 =
blreturns_step1_tau2%*%capm_matrix+P%*%ginv(omega_percentage)%*%Q

blreturns_step5_tau2 = blreturns_step3_tau2%*%blreturns_step4_tau2

blreturns_step1_tau2_9y = ginv(tau2*sigma_matrix_9y)

blreturns_step2_tau2_9y = P_9y%*%ginv(omega_percentage_9y)%*%P_rotate_9y

blreturns_step3_tau2_9y = ginv(blreturns_step1_tau2_9y+blreturns_step2_tau2_9y)

blreturns_step4_tau2_9y =
blreturns_step1_tau2_9y%*%capm_matrix+P_9y%*%ginv(omega_percentage_9y)%*
%Q_9y

blreturns_step5_tau2_9y = blreturns_step3_tau2_9y%*%blreturns_step4_tau2_9y

#Black-Litterman returns with tau parameter 0.1

blreturns_step1_tau3 = ginv(tau3*sigma_matrix)

blreturns_step2_tau3 = P%*%ginv(omega_percentage)%*%P_rotate

```
blreturns_step3_tau3 = ginv(blreturns_step1_tau3+blreturns_step2_tau3)

blreturns_step4_tau3 =
blreturns_step1_tau3%*%capm_matrix+P%*%ginv(omega_percentage)%*%Q

blreturns_step5_tau3 = blreturns_step3_tau3%*%blreturns_step4_tau3

blreturns_step1_tau3_9y = ginv(tau3*sigma_matrix_9y)

blreturns_step2_tau3_9y = P_9y%*%ginv(omega_percentage_9y)%*%P_rotate_9y

blreturns_step3_tau3_9y = ginv(blreturns_step1_tau3_9y+blreturns_step2_tau3_9y)

blreturns_step4_tau3_9y =
blreturns_step1_tau3_9y%*%capm_matrix+P_9y%*%ginv(omega_percentage_9y)%*
%Q_9y

blreturns_step5_tau3_9y = blreturns_step3_tau3_9y%*%blreturns_step4_tau3_9y
```

#Black–Litterman weights

```
w_i_tau1 = sigma_matrix%*%blreturns_step5_tau1

sum_weights_tau1 = sum(w_i_tau1)

bl_weights_tau1 = w_i_tau1/sum_weights_tau1

w_i_tau2 = sigma_matrix%*%blreturns_step5_tau2

sum_weights_tau2 = sum(w_i_tau2)

bl_weights_tau2 = w_i_tau2/sum_weights_tau2

w_i_tau3 = sigma_matrix%*%blreturns_step5_tau3

sum_weights_tau3 = sum(w_i_tau3)

bl_weights_tau3 = w_i_tau3/sum_weights_tau3

w_i_tau1_9y = sigma_matrix_9y%*%blreturns_step5_tau1_9y

sum_weights_tau1_9y = sum(w_i_tau1_9y)

bl_weights_tau1_9y = w_i_tau1_9y/sum_weights_tau1_9y

w_i_tau2_9y = sigma_matrix_9y%*%blreturns_step5_tau2_9y

sum_weights_tau2_9y = sum(w_i_tau2_9y)

bl_weights_tau2_9y = w_i_tau2_9y/sum_weights_tau2_9y
```

```
w_i_tau3_9y = sigma_matrix_9y%*%blreturns_step5_tau3_9y

sum_weights_tau3_9y = sum(w_i_tau3_9y)

bl_weights_tau3_9y = w_i_tau3_9y/sum_weights_tau3_9y
```