ESSAYS ON OPTIMAL STOPPING AND CONTROL OF MARKOV PROCESSES

Jukka Lempa

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TURUN KAUPPAKORKEAKOULU
Turku School of Economics

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1 ON MARKOV PROCESSES

1.1 SOME BASIC PROPERTIES

For the results of this section, we refer to Borodin and Salminen 2002 and Revuz 1984.

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space and assume that $\{X_t\}_{t\geq 0}$ is a family of random variables defined on Ω and taking values in the space $(E, \mathscr{B})$, where $E := (e_l, e_r)$ is an interval in $\mathbf{R}$ and $\mathscr{B}$ is the Borel sigma algebra on E. Then $X := \{X_t\}_{t\geq 0}$ is called a *stochastic process* evolving on E. A family $\mathbb{F} := \{\mathscr{F}_t\}_{t\geq 0}$ of σ-algebras on $(\Omega, \mathscr{F})$ is called a *filtration* if $\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}$ for all $0 \leq s \leq t$.

Definition 1.1. *Let $t \geq 0$. Assume that the function $P_t : E \times \mathscr{B} \to \mathbf{R}_+$ satisfies the following conditions:*

(1) $P_t(\cdot, B) : E \to \mathbf{R}_+$ is $\mathscr{B}$-measurable for all $B \in \mathscr{B}$

(2) $P_t(x, \cdot) : \mathscr{B} \to \mathbf{R}_+$ is a measure on E for all $x \in E$

(3) $P_t(x, E) \leq 1$ for all $x \in E$

(4) $\int_E P_s(x, dy)P_t(y, B) = P_{t+s}(x, B)$ for all $s, t \geq 0, x \in E, B \in \mathscr{B}$.

Then the function P_t is called a Markov transition kernel.

If $P_t(x, E) = 1$ for all $t \geq 0$ and $x \in E$, then the kernel P_t is called *conservative* (or *measure-preserving*). Non-conservative kernels can be made conservative by adjoining an isolated extra state Δ, the so-called *cemetery state*. Let $E^\Delta = E \cup \{\Delta\}$ and $\mathscr{B}^\Delta = \sigma\{\mathscr{B}, \{\Delta\}\}$. Then the conservative transition kernel $\{P_t^\Delta\}$ is introduced in the state space $(E^\Delta, \mathscr{B}^\Delta)$ as follows:

(1) $P_t^\Delta(x, B) := P_t(x, B)$ for all $x \in E, B \in \mathscr{B}$

(2) $P_t^\Delta(x, \Delta) := 1 - P_t(x, E)$ for all $x \in E$

(3) $P_t^\Delta(\Delta, B) := \mathbf{1}_B(\Delta)$ for all $B \in \mathscr{B}^\Delta$.

In the sequel, we will assume that all kernels are conservative. Denote the family $\{P_t\}_{t\geq 0}$ simply as $\mathbb{P}$. Utilizing P_t, the associated averaging operator $\mathscr{P}_t^r$ is defined as

$$(\mathscr{P}_t^r f)(x) = \int_E e^{-rt} f(y) P_t(x, dy),$$

where $r \geq 0$. For $r = 0$, we denote simply $\mathscr{P}_t^0 := \mathscr{P}_t$. Condition *(4)* from Definition 1.1, i.e. the *Chapman-Kolmogorov equation*, yields a semigroup structure to the operator family $\{\mathscr{P}_t^r\}_{t\geq 0}$ acting on measurable functions from left. The semigroup $\{\mathscr{P}_t^r\}_{t\geq 0}$ acts from right on measures via the formula

$$(\eta \mathscr{P}_t^r)(B) = \int_E e^{-rt} \eta(dx) P_t(x,B), \tag{1.1}$$

where η is a measure on $\mathscr{B}$ and $B \in \mathscr{B}$.

Definition 1.2. *Let X be a stochastic process on E. If X satisfies*

$$\mathbf{P}[X_{t+s} = y | X_t = x] = \mathbf{P}[X_s = y | X_0 = x]$$

for all $x, y \in E$ and $t, s \geq 0$ and

$$\mathbf{E}\left[e^{-r(t+s)} f(X_{t+s}) | \mathscr{F}_t\right] = (\mathscr{P}_s^r f)(X_t)$$

$\mathbf{P}$*-almost surely for all $t, s \geq 0$ and for any function f bounded and measurable, then X is called a* time-homogeneous Markov process with filtration $\mathbb{F}$ and kernel $\mathbb{P}$.

Intuitively speaking, a stochastic process is Markov if its evolution on a given deterministic time interval $(t, t+s)$ does not depend on the evolution prior to instant t.

Definition 1.3. *Assume that the random variable $\tau : \Omega \to [0, \infty]$ satisfies the condition $\{\omega : \tau(\omega) \leq t\} \in \mathscr{F}_t$. Then τ is called a* stopping time.

A stopping time is a random time whose occurrence can be determined solely on the basis of the history of X. Important examples of such times are

- *first exit times* $\tau_D := \inf\{t \geq 0 : X_t \notin D\}$, $D \in \mathscr{B}$,
- *first hitting times* $\tau_z := \inf\{t \geq 0 : X_t = z\}$, $z \in E$,
- *lifetime* $\tau_\Delta := \inf\{t \geq 0 : X_t = \Delta\}$.

Definition 1.4. *Let τ be a stopping time and $\mathscr{F}_\infty$ the smallest σ-algebra containing the σ-algebras $\mathscr{F}_t$ for $t \geq 0$. Then the σ-algebra $\mathscr{F}_\tau$ consists of all set $F \in \mathscr{F}_\infty$ such that*

$$F \cap \{\omega : \tau(\omega) \leq t\} \in \mathscr{F}_t$$

for all $t \geq 0$.

The σ-algebra $\mathscr{F}_\tau$ is simply the history of the process up to the stopping time τ. This definition finally puts us in position to pose the desired modification of 1.2.

Definition 1.5. *Let X be a time-homogeneous Markov process with filtration $\mathbb{F}$ and kernel $\mathbb{P}$. If X satisfies the* strong Markov property*, in other words the condition*

$$\mathbf{E}\left[e^{-r(\tau+s)}f(X_{\tau+s})|\mathscr{F}_\tau\right] = (\mathscr{P}_s^r f)(X_\tau)$$

$\mathbf{P}$*-almost surely for any almost surely finite $\mathbb{F}$-stopping time τ, bounded and measurable function f and $s \geq 0$, then X is called a* time-homogeneous strong Markov process with filtration $\mathbb{F}$ and kernel $\mathbb{P}$.

Since deterministic times t are also stopping times, we find that strong Markov processes are also Markov processes. However, the contrary is not true in general.

1.2 LINEAR DIFFUSIONS

For all results in this section, we refer to Borodin and Salminen 2002 unless otherwise indicated.

Definition 1.6. *A stochastic process X on E is called a* linear diffusion *if it is a time-homogeneous strong Markov process and has* $\mathbf{P}$*-almost surely continuous sample paths.*

Throughout this thesis, we will restrict our attention on so-called *regular* diffusions, which satisfy the property $\mathbf{P}_x(H_y < \infty) > 0$ for all $x, y \in E$. In other words, a linear diffusion is regular if a sample path started from an arbitrary state x hits any other state y in finite time with positive probability.

Definition 1.7. *Let X be a linear diffusion. Then the operator $\mathscr{A}$ defined as*

$$(\mathscr{A}f)(x) = \lim_{t\to 0+} \frac{(\mathscr{P}_t f)(x) - f(x)}{t}$$

is called the infinitesimal generator *of X.*

The domain $\mathscr{D}(\mathscr{A})$ of $\mathscr{A}$ is the set of all bounded and continuous functions which satisfy the uniform boundedness condition

$$\sup_{t\geq 0}\sup_{x\in E}\left|\frac{(\mathscr{P}_t f)(x) - f(x)}{t}\right| < \infty$$

and for which the limit $(\mathscr{A}f)$ exists pointwise and is also bounded and continuous.

Every linear diffusion X has three basic characteristics: *speed measure*, *killing measure* and *scale function*. These characteristics govern the behavior of its sample paths. The *speed measure* m of X is a positive Radon measure on state space E such that for every $t \geq 0$ and $x \in E$ the measure $B \mapsto P_t(x,B)$, $B \in \mathscr{B}$, is absolutely continuous with respect to m, that is

$$P_t(x,B) = \int_B p_t(x,y)m(dy). \tag{1.2}$$

The *killing measure* k of X is a non-negative Radon measure on E defined as

$$\mathbf{P}_x(X_{\tau_\Delta -} \in B \mid \tau_\Delta < t) = \int_0^t \int_B p_s(x,y)k(dy)ds, \tag{1.3}$$

where τ_Δ is the lifetime of X. The *scale function* S of X is an increasing continuous function from E to $\mathbf{R}$. This function is connected to the drift of X such that for any interval $(a,b) \in E$ of k-measure zero, the condition

$$\mathbf{P}_x(\tau_a < \tau_b) = \frac{S(b)-S(x)}{S(b)-S(a)} \tag{1.4}$$

holds for all $x \in (a,b)$. In other words, the scale function "rescales" the state space such that the hitting probabilities become proportional to actual distances. If $S(x) = x$, diffusion X is said to be in *natural scale*.

In the context of this thesis, an important feature of a linear diffusion is its boundary behavior. We will now describe how boundary behavior can be characterized in terms the basic characteristics.

Definition 1.8. *Assume that $z \in E$ and let*

$$\Sigma(z) := \int_{e_l}^{z} (m((a,z)) + k((a,z)))S(da)$$
$$\Xi(z) := \int_{e_l}^{z} (S(z) - S(a))(m(da) + k(da)).$$

Then the lower boundary e_l is called

$$\text{non-singular } \textit{if } \Sigma(x) < \infty \textit{ and } \Xi(z) < \infty,$$
$$\text{exit-not-entrance } \textit{if } \Sigma(x) < \infty \textit{ and } \Xi(z) = \infty,$$
$$\text{entrance-not-exit } \textit{if } \Sigma(x) = \infty \textit{ and } \Xi(z) < \infty,$$
$$\text{natural } \textit{if } \Sigma(x) = \infty \textit{ and } \Xi(z) = \infty.$$

Completely analogous definitions are made for the upper boundary e_r.

Definition 1.8 is a good definition in the sense that the content of the formal definitions is the same as the intuitive meaning of the names ”exit” and ”entrance”. Indeed, if a boundary point is exit in the sense of 1.8 then the sample paths can leave E through that point. Similarly, if a boundary point is entrance in the sense of 1.8 then sample paths can be started from this point after posing suitable boundary conditions at each non-singular boundary point (for details, see Borodin and Salminen 2002, pp. 15–16).

We will now present some useful implications of Definition 1.8 for a class of linear diffusions where the basic characteristics are absolutely continuous with respect to Lebesgue measure and have smooth derivatives. More precisely, we consider now a diffusion X for which

$$m(dx) = m'(x)dx, \quad k(dx) = k'(x)dx, \quad S(x) = \int^x S'(y)dy,$$

where the functions m' and S' are continuous and positive and k' is continuous and non-negative (see also Karlin and Taylor 1981). Furthermore, we assume that S' is continuously differentiable. Then the infinitesimal generator $\mathscr{A}$ of X can be expressed as

$$(\mathscr{A}f)(x) = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}f(x) + \mu(x)\frac{d}{dx}f(x) - c(x)f(x), \tag{1.5}$$

where the functions σ, μ and c (the *infinitesimal parameters* of X) are related to m, k and S via

$$m(x) = \frac{2}{\sigma^2(x)}e^{B(x)},\ S'(x) = e^{-B(x)},\ k(x) = \frac{2c(x)}{\sigma^2(x)}e^{B(x)}, \tag{1.6}$$

where $B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}dy$. The representation (1.5) can be utilized to find useful implications of the definition 1.8. To this end, let $r > 0$. It is well known that the solutions of the ordinary differential equation $\mathscr{A}f = rf$ are spanned by two linearly independent solutions. Denote now these solutions as ψ and φ and call them *fundamental solutions*. Then ψ and φ can be characterized as the unique (up to a multiplicative constant) solutions of $\mathscr{A}f = rf$ by first demanding that ψ is increasing and φ is decreasing and then posing conditions at the non-singular boundary points of E. If the lower boundary e_l is non-singular, then we pose the following boundary for ψ on e_l:

$$\begin{cases} r\psi(e_l)m(\{e_l\}) = \frac{\psi'(e_l)}{S'(e_l)} - \psi(e_l)k(\{e_l\}), & e_l \in E \\ \lim_{x\to e_l+}\psi(x) = 0, & e_l \notin E. \end{cases}$$

In the latter case, lower boundary e_l is a *killing boundary* meaning that once the sample path hits l, it is immediately sent to the cemetery state Δ. In the case when e_r is non-singular, we have completely analogous conditions for φ. If e_l is singular, then the functions ψ and φ satisfy the following properties at e_l :

if e_l is entrance-not-exit:

$$\lim_{x\to e_l+}\psi(x)>0,\ \lim_{x\to e_l+}\frac{\psi'(x)}{S'(x)}=0,\ \lim_{x\to e_l+}\varphi(x)=\infty,\ \lim_{x\to e_l+}\frac{\varphi'(x)}{S'(x)}>-\infty$$

if e_l is exit-not-entrance:

$$\lim_{x\to e_l+}\psi(x)=0,\ \lim_{x\to e_l+}\frac{\psi'(x)}{S'(x)}=0,\ \lim_{x\to e_l+}\varphi(x)<\infty,\ \lim_{x\to e_l+}\frac{\varphi'(x)}{S'(x)}>-\infty$$

if e_l is natural:

$$\lim_{x\to e_l+}\psi(x)=0,\ \lim_{x\to e_l+}\frac{\psi'(x)}{S'(x)}=0,\ \lim_{x\to e_l+}\varphi(x)=\infty,\ \lim_{x\to e_l+}\frac{\varphi'(x)}{S'(x)}=-\infty.$$

Again, we have analogous properties for e_r is the case when it is singular.

The fundamental solutions ψ and φ carry a load of information of the underlying dynamics needed in this thesis. One important result is the following theorem linking ψ and φ and the distribution of the first hitting times τ_z.

Theorem 1.1. *Let* $\tau_z=\inf\{t\geq 0\ :\ X_t=z\}$. *Then for* $r>0$

$$\mathbf{E}_x\left[e^{-r\tau_z}\right]=\begin{cases}\frac{\psi(x)}{\psi(z)}, & x\leq z\\ \frac{\varphi(x)}{\varphi(z)}, & x\geq z\end{cases}\tag{1.7}$$

where the functions ψ *and* φ *are the fundamental solutions of the ordinary differential equation* $\mathscr{A}f=rf$.

Another important feature of ψ and φ is their connection to potential theory – we will now review shortly some of these connections. Let $r\geq 0$ and denote as $\mathscr{L}_1(E)$ the class of measurable functions f on E satisfying the condition

$$\mathbf{E}_x\left[\int_0^\infty e^{-rt}|f(X_t)|dt\right]<\infty\tag{1.8}$$

for all $x\in E$. Assume that p is the transition density of X with respect to the speed measure (see (1.2)) and introduce the *Green function (or potential kernel) of* X:

$$G_r(x,y):=\int_0^\infty e^{-rt}p_t(x,y)dt.\tag{1.9}$$

The Green function G_r measures the expected time spent by a sample path started from x "near" y. The function G_r has the following useful representation in terms of the fundamental solutions ψ and φ:

$$G_r(x,y) = \begin{cases} B^{-1}\psi(x)\varphi(y), & x \leq y \\ B^{-1}\varphi(x)\psi(y), & y \leq x \end{cases} \tag{1.10}$$

where $B := \frac{\psi'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}\psi(x)$ is the (constant) Wronskian determinant. The Green function G_r constitutes the *resolvent (or potential) operator* R_r acting on $\mathscr{L}$ via the formula

$$(R_r f)(x) := \int_E f(y)G_r(x,y)m'(y)dy = \mathbf{E}_x\left[\int_0^\infty e^{-rt} f(X_t)dt\right]. \tag{1.11}$$

Utilizing the representation (1.10), the resolvent operator can be written as

$$(R_r f)(x) = B^{-1}\varphi(x)\int_{e_l}^{x}\psi(y)f(y)m'(y)dy + B^{-1}\psi(x)\int_{x}^{e_r}\varphi(y)f(y)m'(y)dy.$$

This representation is computationally very handy (see Article *(2)*).

Definition 1.9. *Let X be a linear diffusion and $r \geq 0$. Assume that the function $f : E \to \mathbf{R}_+ \cup \{\infty\}$ is measurable and satisfies the conditions*

(1) $(\mathscr{P}_t^r f)(x) \leq f(x)$ for all $x \in E$, $t \geq 0$

(2) $(\mathscr{P}_t^r f)(x) \to f(x)$ for all $x \in E$ as $t \to 0$.

*Then f is called r-*excessive *for the diffusion X.*

Excessive functions are important in the theory of optimal stopping and optimal control. Definition 1.9 is not the most convenient way of characterizing excessive functions. A more applicable characterization (which is due to Dynkin) goes as follows: A non-negative function f on E is r-excessive if and only if f is continuous and satisfies the inequality

$$(\mathscr{P}_T^r f)(x) \leq f(x) \tag{1.12}$$

for all $x \in E$, where T is the first exit time from an arbitrary compact $\Gamma \subseteq E$. Using this characterization it is easy to see that the fundamental solutions ψ and φ are r-excessive – see Salminen 1985, Theorem 2.7. The functions ψ and φ are in special position among the r-excessive functions. They are *minimal* in the sense that any non-trivial r-excessive function can be expressed in terms of ψ and φ via an integral expression – in the sequel, we will call ψ and φ *the minimal r-excessive functions*. To give a precise statement, assume that $x_0 \in E$ is given and consider a non-negative measurable function f on E. Then f is r-excessive if and only if there exist a finite measure η on $[e_l, e_r]$ such that

$$f(x) = \int_{(e_l,e_r)} \frac{G_r(x,y)}{G_r(x_0,y)} \eta(dy) + \frac{\varphi(x)}{\varphi(x_0)} \eta(\{e_l\}) + \frac{\psi(x)}{\psi(x_0)} \eta(\{e_r\})$$

for all $x \in E$ (see Salminen 1985, Theorem 3.3). The previous statement is a special version of the Martin integral representation theorem for excessive functions. For a through treatise of Martin boundary theory for linear diffusions, see Bass 1995.

To close this section, it is important to note that in addition to the classical approach, linear diffusions have also a second nature. To explain this, consider the process X given as a solution of the Itô equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{1.13}$$

where μ and σ are bounded continuous functions such that $\sigma(x) > 0$ for all $x \in E$. Then X is a linear diffusion and unique in law (i.e. a unique weak solution to the Itô equation (1.13), see Øksendal 2000). This link combines the classical approach and the theory of stochastic calculus into a rich theory which offers a powerful device for dynamic stochastic modelling.

1.3 MARKOV CHAINS

For all results in this section, we refer to Revuz 1984 unless otherwise indicated.

Consider now the following special case of Section 1.1. Assume that $X := \{X_n\}_{n=0}^{\infty}$ is a sequence of random variables defined on Ω and taking values in E. It is now obvious that all the definitions in Section 1.1 have now formally equivalent counterparts in the discrete parameter case. However, we note that in the Definition 1.1 it suffices to assume only parts *(1)-(3)* – the Chapman-Kolmogorov equation is now defined via the recursion

$$P_n(x,B) = \int_E P_{n-1}(x,dy)P(y,B) \tag{1.14}$$

for all $x, y \in E$ and $B \in \mathscr{B}$. The recursion (1.14) yields a semigroup structure into the family $\{\mathscr{P}_n^\beta\}_{n\geq 0}$. As before, the associated averaging Markovian operator $\mathscr{P}_n^\beta$ as

$$(\mathscr{P}_n^\beta f)(x) = \int_E \beta^{-n} f(y) P_n(x,dy),$$

where $\beta^{-1} \leq 1$ is the discount factor.

Example 1.2. *(A general random walk on* $\mathbf{R}$*) Let X be a random variable distributed on entire* $\mathbf{R}$ *with continuous law λ, mean $\mu > 0$ and variance $\sigma^2 < \infty$. Define the general random walk W on* $\mathbf{R}$ *as partial sums of IID random variables $X, X_1, X_2, \ldots$; i.e. let $W_n = X_1 + \cdots + X_n$, where $W_0 = 0$. The continuity of λ implies that the averaging operator $\mathscr{P}^\beta$ can be written as*

$$(\mathscr{P}^\beta f)(x) = \int_{-\infty}^{\infty} \beta^{-1} f(y) \lambda(y-x) dy.$$

The n-step transition density is simply the n-fold convolution of the density λ.

In the previous section we presented some highlights of the theory of linear diffusions. When moving from a continuous time setting into a discrete time framework, it is not surprising that most of those result fail to hold. For example, since the associated semigroup is now simply a discrete semigroup generated by the recursion (1.14), the concept of *infinitesimal* generator is no longer meaningful. This abolishes the differential operator from the discrete time theory. Fortunately, potential theoretic results are retained to a large extent. In particular, the Martin integral representation theorem can be proved also for a class of Markov chains. Our next task is to present this theorem. The proof of the representation theorem is lengthy and relatively tedious. Therefore we will simply refer to Revuz 1984, pp. 241–257, for the detailed treatise. We will proceed by defining some basic concepts.

Definition 1.10. *A non-negative measurable function $u : E \to \mathbf{R}_+ \cup \{\infty\}$ is said to be* β-excessive *if it satisfies the condition $\mathscr{P}^\beta u(x) \leq u(x)$ for all $x \in E$. In the case of an equality, the function u is called* β-harmonic. *A* 1*-excessive function is simply called* excessive *and, similarly,* 1*-harmonic function is called* harmonic.

If a β-harmonic function h has the property that any β-harmonic function u with $u(x) \leq h(x)$ for all $x \in E$ is proportional to h, then h is called β-minimal.

Recall that the operator $\mathscr{P}^\beta$ acts on measures from right. Therefore we can define β*-excessive*, *-harmonic* and *-minimal* measures completely analogously to Definition 1.10.

Definition 1.11. *Let $\beta \geq 1$. The* potential kernel G^β *of X is the kernel*

$$G^\beta(x,A) = \sum_{n=0}^{\infty} \beta^{-n} P_n(x,A).$$

Moreover, the potential operator $\mathscr{G}^\beta$ *of X is defined as*

$$(\mathscr{G}^\beta f)(x) = \int_E f(y) G^\beta(x,dy) = \int_E \sum_{n=0}^{\infty} \beta^{-n} f(y) P_n(x,dy).$$

Using Fubini's theorem and recursion (1.14) we readily check that for a measurable and non-negative f, the function $\mathscr{G}^\beta f$ is β-excessive. Note that the operator $\mathscr{G}^\beta$ acts from right on measures analogously to Definition 1.11. Assume that $B \in \mathscr{B}$ and μ is a measure. Then, again, by Fubini's theorem and recursion (1.14) we find that the measure $\mu\mathscr{G}^\beta$ is β-excessive.

Definition 1.12. *The kernels P and $\hat{P}$ on E are said to be* in duality *relative to the positive sigma-finite measure η, if for every pair (f,g) of measurable functions the condition*

$$\int_E (\mathscr{P}^\beta f)(x) g(x) \eta(dx) = \int_E f(x) (\hat{\mathscr{P}}^\beta g)(x) \eta(dx)$$

holds for all $x \in E$. Moreover, the Markov chains X and $\hat{X}$ are said to be in duality *relative to η, if their transition kernels P and $\hat{P}$ are in duality relative to η.*

All the concepts defined above for X have natural dual counterparts. These dual objects defined in terms of $\hat{X}$ will be designated by the symbol ˆ and the prefix "co". For example, a β-coharmonic function is β-harmonic for the dual chain $\hat{X}$.

Example 1.3. *(A general random walk on* **R***, cont'd) Recall now Example 1.2. We will now derive the dual of the general random walk W. Since $P(x,dy) = \lambda(y-x)dy$ for W, we find that for any measurable functions f and g*

$$\begin{aligned}\int_{-\infty}^{\infty} (\mathscr{P}^\beta f)(x) g(x) dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta^{-1} f(y) \lambda(y-x) dy g(x) dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \beta^{-1} g(x) \lambda(y-x) dx f(y) dy \\ &= \int_{-\infty}^{\infty} (\hat{\mathscr{P}}^\beta g)(x) f(x) dx\end{aligned}$$

where $\hat{\mathscr{P}}^\beta$ is defined via kernel $\hat{P}(x,dy) := \lambda(x-y)dy$. The kernels P and $\hat{P}$ are in duality relative to Lebesgue measure and the associated Markov chain $\hat{W}$ is again a general random walk. Geometrically speaking, the distribution of $\hat{W}$ is obtained by simply taking the mirror image of λ with respect to y-axis. In particular, if λ is symmetric, for example normal with mean zero, then W is its own dual.

We are now in position to present the Martin integral representation theorem of β-harmonic functions for a class of Markov chain. We will next present the assumptions under which the proof can be carried out.

Assumptions 1.1. *(1) The Markov chain X has a dual $\hat{X}$ with respect to a Radon measure η*

(2) The measures $P(x,\cdot)$ and $\hat{P}(x,\cdot)$ are absolutely continuous with respect to η for every $x \in E$

(3) The function $\hat{\mathscr{P}}^\beta f$ is continuous for all bounded and continuous f

(4) The function $\hat{\mathscr{G}} f$ is bounded and continuous for all continuous f with compact support

(5) The constant function 1 is harmonic for X.

Theorem 1.4 (Martin Representation Theorem for β-harmonic Functions). *Let Assumptions 1.1 hold. Then there is a one-to-one correspondence between β-harmonic functions h and Radon measures σ^h on the minimal Martin exit boundary M_e given by the condition*

$$h(x) = \int_{M_e} k(x,\zeta)\sigma^h(d\zeta). \tag{1.15}$$

Conversely, for any probability measure σ on M_e the integral (1.15) *defines a β-harmonic function h and this function is minimal if and only if it is equal to $k(\cdot,\zeta)$ for some $\zeta \in M_e$.*

Proof. See Revuz 1984, Corollary 3.11, pp. 257. □

The key concept in the proof is the so-called *Martin cokernel*, which is essentially a suitably normalized potential cokernel. The state space E is first embedded into an infinite dimensional function space where the Martin cokernels belong. Then it is proved that the ideal Martin completion M_e of E is the one which corresponds β-minimal Martin cokernels.

Example 1.5. *(A general random walk on* **R***, cont'd) Recall Examples 1.2 and 1.3. We will show that the general random walk W satisfies Assumptions 1.1. Conditions* (1) *and* (2) *are already known to hold. Moreover, it is obvious that the constant function* 1 *is harmonic for W. Assume that $f \in C_b(\mathbf{R})$. Then the function*

$$(\hat{\mathscr{P}}^{\beta} f)(x) = \int_{-\infty}^{\infty} \beta^{-1} f(y) \lambda(-(y-x)) dy$$

is clearly continuous. Since $\hat{W}$ is transient, we find that $\hat{G}^{\beta}(x,\cdot)$ is a Radon measure for all $x \in \mathbf{R}$ (see Revuz 1984, Corollary 3.6, pp. 101). This implies that

$$(\hat{\mathscr{G}} f)(x) = \int_{-\infty}^{\infty} f(y) \hat{G}^{\beta}(x, dy)$$

is bounded and continuous for all continuous f with compact support. Therefore, the Martin representation theorem 1.4 holds for the general random walk W.

In Example 1.5 we established that Assumptions 1.1 and, consequently, Theorem 1.4 holds for the general random walk W defined in Example 1.2. Our next task is to present a characterization of β-minimal functions of W. This problem has been studied in Doob *et al.* 1960 in the case where the state space of the general random walk is discrete. Before proving the characterization, we will first introduce some necessary concepts. Firstly, assume that the function h is β-excessive for the general random walk W and define the function $p^h : \mathbf{R}^2 \to \mathbf{R}$ as

$$p^h(x,y) = \beta^{-1} \frac{h(y)}{h(x)} \lambda(y-x)$$

for all $x \in \mathbf{R}$. Since h is β-excessive, the function p^h satisfies $\int_{-\infty}^{\infty} p^h(x,y)dy \leq 1$ and, therefore, constitutes a transition kernel P^h. The associated Markov chain is now denoted as W^h and is called the *(Doob's) h-transform of* W (see e.g. Doob *et al.* 1960, pp. 183–184). In order to give some idea of the applicability of the h-transform, assume that u is β-harmonic for W. Then

$$\left(\mathscr{P}^h \frac{u}{h}\right)(x) = \frac{1}{h(x)} \int_{-\infty}^{\infty} \beta^{-1} u(y) \lambda(y-x) dy = \frac{u}{h}(x).$$

This means that u is β-harmonic for W if and only if (u/h) is harmonic for W^h. In particular, h is β-minimal for W if and only if constant function $x \mapsto 1$ is minimal for W^h (see Doob *et al.* 1960, pp. 184). This observation will be useful later.

We will now prove a generalization of the characterization from Doob *et al.* 1960, where the state space is the real line $\mathbf{R}$.

Lemma 1.6. *Let W be the general random walk defined in Example 1.2 and assume that the function $h : \mathbf{R} \to \mathbf{R}_+$ is non-negative and $h(0) = 1$. Then h is β-minimal for the general random walk W if and only if*

(A) $\mathbf{E}[\beta^{-1}h(X)] = 1$,

(B) $h(x+y) = h(x)h(y)$ *for all* $x, y \in \mathbf{R}$.

Proof. Assume first that h is β-minimal. Then it satisfies Condition *(A)* by definition. In order to prove Condition *(B)*, we first observe that λ is bounded. By dominated convergence, the β-harmonicity of h implies now that h is continuous. Moreover, we find that if $h(x_0) = 0$ for some $x_0 \in \mathbf{R}$, then the β-harmonicity of h implies that $\int_{-\infty}^{\infty} \beta^{-1}h(y)\lambda(y-x_0)dy = 0$, which in turn implies (by positivity of λ and continuity of h) that $h(x) = 0$ for all $x \in \mathbf{R}$. This observation allows us proceed under the assumption $h(x) > 0$ for all $x \in \mathbf{R}$. Since h is β-harmonic, we find that

$$h(x) = \int_{-\infty}^{\infty} \beta^{-1} \frac{h(x+y)}{h(y)} h(y)\lambda(y)dy \tag{1.16}$$

for all $x \in \mathbf{R}$. For an arbitrary $y \in \mathbf{R}$, denote the function $x \mapsto \frac{h(x+y)}{h(y)}$ as u_y. We verify readily that u_y is also β-harmonic with $u_y(0) = 1$ for all $y \in \mathbf{R}$. Rewrite now the expression (1.16) using Condition *(A)* as

$$\int_{-\infty}^{\infty} \beta^{-1} \left(u_y(x) - h(x)\right) h(y)\lambda(y)dy = 0 \tag{1.17}$$

for all $x \in \mathbf{R}$. Assume now that $y \in \mathbf{R}$ is such that $u_y(x) \leq h(x)$ for all $x \in \mathbf{R}$. Since h is β-minimal, there is a constant c_y such that $u_y(x) = c_y h(x)$ for all $x \in \mathbf{R}$. On the other hand, since $h(0) = 1 = u_y(0)$, we find that $c_y = 1$. This finding coupled with the expression (1.17) and the positivity of h and λ yields finally Condition *(B)*.

Assume now that h is non-negative, $h(0) = 1$ and satisfies Conditions *(A)* and *(B)*. Utilizing *(A)* and *(B)*, we find that h is β-harmonic for W; i.e.

$$\mathbf{E}[\beta^{-1}h(x+X)] = h(x)\mathbf{E}[\beta^{-1}h(X)] = h(x)$$

for all $x \in \mathbf{R}$. Denote now as W^h the h-transform of W (recall the discussion prior this theorem). Condition *(B)* now implies that $p^h(x,y) = \beta^{-1}h(y-$

$x)\lambda(y-x)$ for all $x,y \in \mathbf{R}$. Utilizing this and Condition *(A)*, we conclude that W^h is a general random walk, which is driven by a random variable X^h with law $y \mapsto \beta^{-1}h(y)\lambda(y)$. In order to finish the proof, it suffices to show that the constant function $x \mapsto 1$ is minimal for W^h. Clearly, the probability measure $A \mapsto \int_A \beta^{-1}h(y)\lambda(y)dy$, where $A \in \mathscr{B}_{\mathbf{R}}$, is not singular with respect to the Lebesgue measure. By virtue of Revuz 1984, Proposition 1.6, pp. 162 and Theorem 1.3, pp. 161, this implies that all bounded harmonic functions for W^h are constants which in turn implies that the constant function $x \mapsto 1$ is minimal for W^h. □

Utilizing characterization 1.6 we will now explicitly characterize the β-minimal function of the general random walk W. The following lemma is a reproduction Lemma 2.2 in Article *(4)*.

Lemma 1.7. *There exists two real numbers $-a$ and b, $a,b > 0$, such that the functions $\psi(x) = e^{bx}$ and $\varphi(x) = e^{-ax}$ are the only β-minimal functions of the general random walk W.*

Proof. See the proof of Lemma 2.2 in Article *(4)*. □

By coupling Example 1.5 and Lemma 1.7 we find that arbitrary function h, which is β-harmonic for general random walk W can be expressed as $h(x) = c_1\psi(x) + c_2\varphi(x)$ for all $x \in \mathbf{R}$ with unique positive weights c_1 and c_2. We will now formulate and prove a simple extension of Lemma 1.7. To this end, assume that the function $g : \mathbf{R} \to (l,r)$, where $(l,r) \subseteq \mathbf{R}$, is strictly increasing, defined on entire $\mathbf{R}$ and continuously differentiable. Define the process Y on (l,r) as $Y_n := g(W_n)$. The process Y is a homogeneous Markov chain on $\left((l,r), \mathscr{B}|_{(l,r)}\right)$ with transition kernel

$$P^Y(x,A) = P\left(g^{-1}(x), g^{-1}(A)\right).$$

Let A be a Borel set on (l,r) and fix a point $x \in A$. Then

$$\begin{aligned}
P^Y(x,A) &= P\left(g^{-1}(x), g^{-1}(A)\right) \\
&= \int_{g^{-1}(A)} p\left(g^{-1}(x), y\right) dy \\
&= \int_{g^{-1}(A)} \lambda\left(y - g^{-1}(x)\right) dy \\
&= \int_A \lambda\left(g^{-1}(y) - g^{-1}(x)\right) \left(g^{-1}\right)'(y) dy \\
&= \int_A p^Y(x,y) m^Y(dy),
\end{aligned}$$

where $p^Y(x,y) := \lambda\left(g^{-1}(y) - g^{-1}(x)\right)$ and the Radon measure m^Y is defined as $m^Y(dy) := (g^{-1})'(y)dy$. Define the averaging operator $\mathscr{P}_Y$ of the chain Y as

$$(\mathscr{P}_Y f)(x) = \int_l^r \beta^{-1} f(y) p^Y(x,y) m^Y(dy).$$

for a measurable function f on (l,r).

Theorem 1.8. *Let Y be the Markov chain defined above. There exists exactly two real numbers $-a$ and b, $a,b>0$ determined by the condition $\mathbf{E}[\exp(tX))] = \beta$ such that the functions $\psi^Y : (l,r) \to \mathbf{R}_+$ and $\varphi^Y : (l,r) \to \mathbf{R}_+$ defined as $\psi^Y(x) = \exp(bg^{-1}(x))$ and $\varphi^Y(x) = \exp(-ag^{-1}(x))$ are the only β-minimal functions of the Markov chain Y.*

Proof. We will first prove that the function ψ^Y is β-minimal for the chain Y. Since the function ψ is β-harmonic for the chain W and $(g \circ g^{-1}) \equiv \mathrm{id}_{\mathbf{R}}$, we find that

$$\begin{aligned}
(\mathscr{P}_Y \psi^Y)(x) &= \int_l^r \beta^{-1} \exp\left(bg^{-1}(y)\right) \lambda\left(g^{-1}(y) - g^{-1}(x)\right) \left(g^{-1}\right)'(y)dy \\
&= \int_{-\infty}^{\infty} \beta^{-1} \exp(bv) \lambda\left(v - g^{-1}(x)\right) \left(g^{-1}\right)'(g(v))\, g'(v)dv \\
&= \int_{-\infty}^{\infty} \beta^{-1} \exp(bv) \lambda\left(v - g^{-1}(x)\right) dv \\
&= \psi^Y(x).
\end{aligned}$$

In other words, we find that the function ψ^Y is β-harmonic for the chain Y. For minimality, assume that $h : (l,r) \to \mathbf{R}_+$ is β-harmonic for Y such that $h(x) \leq \exp(bg^{-1}(x))$ for all $x \in (l,r)$. The monotonicity of g implies that $h(g(x)) \leq \psi(x)$ for all $x \in \mathbf{R}$. Moreover, the β-harmonicity of h for Y yields

$$\begin{aligned}
(\mathscr{P}(h \circ g))(x) &= \int_{-\infty}^{\infty} \beta^{-1} h(g(y)) \lambda(y - x)dy \\
&= \int_l^r \beta^{-1} h(z) \lambda\left(g^{-1}(z) - g^{-1}(g(x))\right) \left(g^{-1}\right)'(z)dz \\
&= (h \circ g)(x)
\end{aligned}$$

for all $x \in \mathbf{R}$. In other words, the composite $h \circ g : \mathbf{R} \to \mathbf{R}_+$ is β-harmonic for the process W. Since now ψ is β-minimal for W, there is a unique constant c such that $(h \circ g)(x) = c\psi(x)$ for all $x \in \mathbf{R}$. Finally, the monotonicity of the exponential function implies that $h(x) = c\psi(x)$ for all $x \in (l,r)$, that is, ψ is β-minimal for the process Y. The β-minimality of φ is proved completely analogously.

In order to establish that there exist no other β-minimal function of the process Y than ψ and φ, assume that h is β-minimal for the process Y. Then the monotonicity properties of the logarithm function imply that the composite $h \circ g$ is β-minimal for the process W. Therefore $h \circ g$ is either ψ or φ, which in turn implies that h is either ψ^Y or φ^Y. This observation completes the proof. □

Theorem 1.8 establishes that the minimal Martin boundary is retained in continuous and bijective transforms. Moreover, we find that the actual form of the β-minimal functions can be determined via this transform.

The regularity properties of the law λ imply that the chain Y has a dual $\hat{Y}$ with respect to the Radon measure m^Y – the transition kernel $\hat{P}^Y$ of $\hat{Y}$ reads as

$$\hat{P}^Y(x,A) = \int_A \lambda(g^{-1}(x) - g^{-1}(y))m^Y(dy)$$

for all $x \in E$ and $A \in \mathscr{B}|_{(l,r)}$. Given this observation, we verify readily that the Assumptions 1.1 hold also for the transformed random walk Y and, thus, that the Martin integral representation theorem 1.4 is valid for Y.

Consider now the special case $g(x) = e^x$. Now, the chain Y can be written as $Y_n = e^{W_n} = \prod_{i=1}^n e^{X_n}$, where $X \sim X_n$ for all $n \geq 1$. The chain Y is called *geometric random walk on* $\mathbf{R}_+$ for which Theorem 1.8 gives the following result.

Corollary 1.9. *There exists exactly two real numbers $-a$ and b, $a,b > 0$ determined by the condition $\mathbf{E}[X^t] = \beta$ such that the functions $\psi^Y : \mathbf{R}_+ \to \mathbf{R}_+$ and $\varphi^Y : \mathbf{R}_+ \to \mathbf{R}_+$ defined as $\psi^Y(x) = x^b$ and $\varphi^Y(x) = x^{-a}$ are the only β-minimal functions of the geometric random walk Y.*

Corollary 1.9 establishes that the β-minimal functions of a geometric random walk are of the same functional form as the minimal r-harmonic functions of its continuous-time counterpart, geometric Brownian motion. Moreover, we note that if $X \sim N(\mu, \sigma^2)$ (in other words, if $Y \sim \mathrm{LogNormal}(\mu - \frac{1}{2}\sigma^2, \sigma^2)$), then the β-minimal functions of Y coincide with the minimal r-excessive functions of the geometric Brownian motion given as the solution of the Itô equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

see also Article *(4)*.

2 ASPECTS OF OPTIMAL STOPPING

2.1 ON CONTINUOUS TIME PROBLEMS

Many economic decisions are concerned with the timing of an action. As an example, consider an investor who has the opportunity to invest in a project subject to uncertain returns at a given constant sunk cost c. At every instant, the investor chooses between two alternatives: he/she either invests now or postpones the investment into the future. Naturally, it is interesting to know whether the investment time can be chosen optimally. This type of timing problem has a natural formulation as an optimal stopping problem. To fix ideas, assume that the returns evolve according to the geometric Brownian motion given as the solution of the Itô equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{2.1}$$

with $\mu \in \mathbf{R}$ and $\sigma > 0$. Then the expected present value of the returns accrued from the investment at time t reads as

$$\Pi(x,t) = \mathbf{E}_x[e^{-rt}(X_t - c)^+], \tag{2.2}$$

with $r > 0$ and $\mu < r$. Now, a rational investor asks the following question: *I want to maximize the expected present value of my return – when should I invest, if at all?* This question can described formally as follows: *Find a stopping time τ^* such that*

$$G(x) := \Pi(x,\tau^*) = \sup_{\tau} \Pi(x,\tau), \tag{2.3}$$

where τ is an arbitrary stopping time. This optimal stopping problem is a prototype of the problems considered in this thesis. Problem 2.3 is a classic and was first presented in Samuelson 1965 and rigorously formulated and solved in McKean 1965 – the solution reads as

$$G(x) = \begin{cases} x - c, & x > x^* \\ \frac{x^*-c}{x^{*\theta}} x^{\theta}, & x \le x^*, \end{cases} \tag{2.4}$$

where $\theta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$ is the positive root of the characteristic equation $\frac{1}{2}\theta(\theta - 1) + \mu\theta - r = 0$ and the optimal stopping threshold reads as $x^* = \frac{\theta c}{\theta - 1} > c$.

There is a number of different approaches to the problem (2.3). Being a dynamic programming problem in essence, a natural approach is an application of the Bellman principle. A convenient way of writing the Bellman principle is to write it in terms of the well-known Hamilton-Jacobi-Bellman variational inequality

$$\max\left\{((\mathscr{A}-r)G)(x),(x-c)^{+}-G(x)\right\}=0 \tag{2.5}$$

for all $x \in E$. The first component of expression (2.5) represents the classical Bellman's balance condition and the second represents the fact that it is always admissible to exercise immediately. Equation (2.5) offers a concise and general characterization of the optimal value. However, due to its generality it suffers from some downsides as such. In most cases, the expression (2.5) can be difficult to analyze. Moreover, it yields no information on the optimal stopping rule which is crucial information for applications. However, expression (2.5) serves as a good platform to build on when developing solution techniques for optimal stopping problems. A well-known example of such technique is produced by bundling the HJB variational inequality (2.5) with the assumption that the state space E has a partition $\{C,S\}$, where C is called the *continuation region* and S the *stopping region*. These sets are defined as follows: The process X is allowed to evolve as long as it stays in C but as soon as it enters S (leaves C), it is stopped – if X is started in S, then it is stopped immediately. This means that the optimal stopping rule is assumed to be among the exit times $\tau_D =: \inf\{t \geq 0 : X_t \notin D\}$, where $D \subseteq E$ is Borel and the task is to find an optimal partition $\{C^*,S^*\}$ for which the exit time $\tau^* := \tau_{C^*}$ constitutes the optimal value G. Since it is optimal to exercise immediately in S^*, we find that $G(x) = x-c$ for all $x \in S^*$. This leads us to the alternative version of the HJB variational inequality (see Øksendal and Reikvam 1998):

$$\begin{cases} \max\{((\mathscr{A}G)-r)(x),(x-c)^{+}-G(x)\}=0 & x\in E \\ G(x)=x-c & x\in\partial S^{*}. \end{cases} \tag{2.6}$$

Recall from Section 1 that if the value in (2.6) is sufficiently smooth then the infinitesimal generator $\mathscr{A}$ can be expressed as a second-order differential operator which is very handy from a computational point of view. However in many cases, we are not fortunate enough to have this kind of regularity of the value. For these cases, an important technique is the method of *viscosity solutions* originally introduced in Crandall and Lions 1983 for partial differential equations. For related articles, see also Øksendal and Reikvam 1998, Benth and Reikvam 2004.

Equation (2.6) offers also another view of the optimal stopping problem (2.3). Since $G(x) > x - c$ for all $x \in C^*$, we find that $((\mathscr{A} - r)G)(x) = 0$ in C^*. This observation leads us to the so-called *free boundary problem* characterization of the optimal stopping problem (2.3):

$$\begin{cases} ((\mathscr{A} - r)G)(x) = 0 & x \in C^* \\ G(x) = (x-c)^+ & x \in \partial S^*. \end{cases} \tag{2.7}$$

This approach was introduced in McKean 1965. For a thorough exposition of optimal stopping problems and free boundary problems, see the recent textbook Peskir and Shiryaev 2006.

Optimal stopping problems of the form (2.3) can also be studied by means of potential theory. These approaches are based on Dynkin's famous characterization of the optimal value function: *The optimal value function coincides with the smallest r-excessive majorant of the exercise payoff* (see Dynkin 1963). Utilizing this characterization, the theory of excessive functions can be used to analyze optimal stopping problems. Within this theory, an important tool is Martin integral representation theorem of excessive functions. In Salminen 1985, the representing measure of an excessive function of a linear diffusion is determined explicitly in terms of the minimal r-excessive functions and the scale function. This correspondence provides a powerful and general solution method for the considered class of problems. Another, and more recent, potential theoretic approach was proposed in Dayanik and Karatzas 2003. This approach relies on a well-known connection between r-excessivity and a special type of functional concavity originally due to Dynkin 1965 and Dynkin and Yushkevich 1969. The fundamental approach of Articles *(1)-(3)* belongs also in this class. In order to describe the idea of the method, we will now present a sketch proof of the solution (2.4). We start with an assumption that continuation region is of form $(0,z)$ Then Theorem 1.1 implies that

$$\mathbf{E}_x[e^{-r\tau_z}(X_{\tau_z} - c)^+] = \frac{(z-c)^+}{z^\theta} x^\theta, \tag{2.8}$$

where $\theta = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$, for all $x < z$. In order to find a maximal expression of the form (2.8), we observe by ordinary differentiation that the function $x \mapsto \frac{x-c}{x^b}$ attains a global maximum at the state $x^* = \frac{\theta c}{\theta - 1}$. Thus the function $\hat{G}$ defined as

$$\hat{G}(x) = \begin{cases} x - c, & x \geq x^* \\ \frac{x^* - c}{x^{*\theta}} x^\theta, & x < x^* \end{cases}$$

is a potential candidate for the optimal value. The fact that $\hat{G}(x) = G(x)$ for all $x \in \mathbf{R}_+$ is finally proved by variational inequalities. The methods used in Articles *(1)-(3)* are variants and extensions of this idea. In particular, the starting point of Article *(3)* is that the continuation region is an interval (a, b), where $a > 0$.

The Snell envelope (see Snell 1952) is a both practically and historically significant method for solving optimal stopping problems. It adopts a slightly different approach to the problem (2.3) from the methods described above. Define the payoff process Y as

$$Y_t := e^{-rt}(X_t - c)^+$$

and it's *Snell envelope* V as

$$G_t := \operatorname*{ess\,sup}_{t \leq \tau} \mathbf{E}\left[e^{-r\tau}(X_\tau - c)^+ | \mathscr{F}_t\right].$$

It can be proved under mild conditions that the Snell envelope G coincides with the value process of the problem (2.3) and that it can be characterized as the smallest supermartingale dominating the payoff process Y (see Peskir and Shiryaev 2006, Section 2.1).

2.2 ON DISCRETE TIME PROBLEMS

Even though linear diffusions provide a powerful modelling framework, some of their intrinsic properties make them more or less unsatisfactory for many important applications. Linear diffusions evolve in continuous time whereas in real life information typically accumulates in discrete time. In large financial markets where trading typically takes place very densely in time this may not seem to be such a big problem. However, in many economical applications, information updates may happen sparsely. In such cases it is reasonable to question the plausibility and accuracy of continuous time models. Assume now that $X = \{X_n\}_{n=0}^{\infty}$ is a general Markov chain and let $\beta^{-1} \leq 1$ be the discount factor. Analogously to the continuous time setting, define the expected present value of the exercise payoff at time n as

$$\Xi(x,n) = \mathbf{E}_x[\beta^{-n}(X_n - c)^+]. \tag{2.9}$$

The associated optimal stopping problem reads now as

$$H(x) = \sup_{\eta < \eta_\Delta} \Xi(x,\eta). \tag{2.10}$$

where η is a stopping time and η_Δ is the lifetime of X. Some of the methods described in Section 2 have natural formal counterparts in the discrete time setting – the concept of Snell envelope was originally developed for discrete time problems. For example, the balance condition of the Bellman principle can be written as

$$\max\{(\mathscr{P}^\beta H)(x) - H(x), (x-c)^+ - H(z)\} = 0 \tag{2.11}$$

for all $x \in E$ and the free-boundary formulation reads as

$$\begin{cases} (\mathscr{P}^\beta H)(x) - H(x) = 0, & x \in C^* \\ H(x) = (x-c)^+, & x \in \partial S^*. \end{cases} \tag{2.12}$$

In comparison to the continuous time case, these expressions are even more difficult to analyze. This is mainly due to two reasons. First, the averaging operator $\mathscr{P}^\beta$ is an integral operator and, therefore, essentially more difficult to handle than a second order linear differential operator associated to linear diffusions. Moreover, since a Markov chain is a jump process by definition, it is a typical scenario that once the sample path enters the stopping region, it jumps in to an interior state. This ”over- (or under-) shoot” can result into a situation where the optimal value is no longer smooth at the boundary ∂S^* – see Article *(4)*. This makes the determination of the optimal boundary more difficult.

Recently, there has been progress in the development of techniques based on Wiener-Hopf decomposition. Boyarchenko and Levendorskiǐ have written a series of papers (see e.g. Boyarchenko and Levendorskiǐ 2004) where they utilize Wiener-Hopf decomposition in the study of the Bellman equation. Wiener-Hopf decomposition has also been used in the context of continuous time jump processes, see e.g. Alili and Kyprianou 2005 and Mordecki 2002.

3 ASPECTS OF OPTIMAL CONTROL

3.1 FORMULATION OF THE PROBLEMS

Optimal stopping problems form an important subclass of optimal control problems. In optimal stopping, the admissible controls are very crude in the sense that the controller is allowed to influence the evolution of the object only once by using a *take the money and run*-type of policy. This is too restrictive for many applications. For example, if we use an optimal stopping model in cash flow management applications, the board of the firm must liquidate the firm in order to pay dividends to the shareholders. In applications to the management of renewable natural resources, a stopping policy corresponds to the depletion of the entire reserve of the resource. In this light, it is reasonable to pursue more general policies. A natural extension is to allow policies where it is possible to invoke more general, sequential type of controls. Assume now that in the absence of control, the underlying dynamics evolve according to the linear diffusion X given as the solution of the Itô equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

where μ and σ are continuous. In order to simplify the notation, assume that both boundaries of the state space E are unattainable for X. Furthermore, assume that the controller can influence the evolution of X such that at any instant he/she can drive the state to a given regeneration state immediately and start the process anew. This type of control policies are called *(admissible) impulse control policies* and are described by joint sequences of the form $\{(\tau_k, \zeta_k)\}$, where τ_k's stand for the intervention times and ζ_k's for the corresponding impulses (for details, see Articles *(1)* and *(2)*). Now, the controlled dynamics follow the generalized Itô integral

$$X_t^\nu = x + \int_0^t \mu(X_s^\nu)ds + \int_0^t \sigma(X_s^\nu)dW_s - \sum_{\tau_k \le t} \zeta_k \tag{3.1}$$

and the impulse control problem is defined as

$$V(x) = \sup_{\nu \in \mathscr{V}} \mathbf{E}_x \left[\sum_{k=1}^{\infty} e^{-r\tau_k}(\zeta_k - c) \right], \tag{3.2}$$

where $c >$ is the exercise cost and $\mathscr{V}$ is the class of all admissible impulse control policies. The expression (3.1) is a compound of two components. It contains the uncontrolled, ordinary Itô integral component and the control component represented by the sum term. The control process can be described by positive, non-decreasing, finite variation, adapted *càdlàg* processes of pure jump type, i.e. they remain constant between jumps (interventions, that is). An interesting question arises now: What happens when $c \to 0$? As the controlling becomes cheaper, it is intuitively plausible that the controller becomes more inclined to use the control. Indeed, this is exactly what also happens in the model. Moreover, when $c = 0$, the optimal control is still a positive, non-decreasing, finite variation, adapted *càdlàg* process, but the pure jump structure is abolished (see Articles *(1)* and *(2)*). In this limiting case, the controlled dynamics X^Z follow the generalized Itô equation

$$dX_t^Z = \mu(X_t^Z)dt + \sigma(X_t^Z)dW_t - dZ_t \tag{3.3}$$

and the optimal singular control problem reads as

$$K(x) = \sup_{Z \in \Lambda} \mathbf{E}_x \left[\int_0^\infty e^{-rs} dZ_s \right], \tag{3.4}$$

where Λ is the class of all positive, non-decreasing, finite variation, adapted *càdlàg* processes.

It is worth pointing out that we can interpret the stopping policies as degenerate impulse control policies, where the joint sequence $\{(\tau_k, \zeta_k)\}$ has only one element, namely the pair $(\tau, X_{\tau-})$ where τ is a stopping time. This pair corresponds to a control policy where the first intervention takes the state variable into the cemetery state. In other words, we conclude that the admissible stopping policies are admissible impulse control policies, which in turn are admissible singular control policies. In this sense, all these three control problems are realizations of a same problem with different degrees of flexibility of control.

3.2 METHODS OF SOLUTION

We will first focus on the singular control problem (3.4). As the optimal stopping problem (2.3), the singular control problem (3.4) is also a dynamic programming problem in essence. Since the controlled process X^Z is Markovian, it has an infinitesimal generator – denote the generator as $\mathscr{A}^Z$. The Bellman principle of optimality implies now that the value K can be characterized via the functional equation

$$rK(x) = \sup_{Z\in\Lambda}\{(\mathscr{A}^Z K)(x)\}.$$

This is the Hamilton-Jacobi-Bellman equation for the problem (3.4). Similarly to optimal stopping, variational inequalities can be applied for solving the HJB equation. This is due to the close connection between singular stochastic control and optimal stopping problems first discovered by Bather and Chernoff 1966 (see also Benes et al. 1980, Karatzas 1983, Karatzas and Shreve 1984&1985, Boetius and Kohlmann 1998 and Benth and Reikvam 2004). This connection can roughly be described as follows: The derivative of the value of the singular stochastic control problem can be characterized as the value of an associated optimal stopping problem. By adopting notation from Section 2, the variational inequalities for the problem (3.4) can be now written as (see Weerasinghe 2005)

$$\begin{cases} \max\{((\mathscr{A}-r)K')(x), 1-K'(x)\} = 0 & \text{for all } x \in E \\ K'(x) = 1 & \text{for all } x \in \partial S^*. \end{cases} \tag{3.5}$$

The described connection allows us to import also other techniques from optimal stopping. The expression (3.5) gives rise to a free boundary problem formulation of the optimal value K (see Articles *(1)* and *(2)* and references therein):

$$\begin{cases} ((\mathscr{A}-r)K')(x) = 0,\ x \in C^* \\ K'(x) = 1,\ x \in \partial S^*, \end{cases} \tag{3.6}$$

In contrast to optimal stopping, where the associated free boundary problem is a Dirichlet problem, the optimal value K is now characterized by a Von Neumann problem. Snell envelopes have also been utilized in solving singular control problems, see Bank 2005.

The stochastic impulse control problem (3.2) is more difficult to handle. Again, the controlled process X^ν is a Markov process and the Hamilton-Jacobi-Bellman equation for the value V can be written in the same form as in the case of singular control. However, since no useful general links to simpler control problems are not known, we are forced to analyze these problems as such. The most well-known and general method for solving impulse control problems is the so-called *quasi-variational inequalities* that were introduced in Bensoussan and Lions 1984. This technique relies on the same assumption of continuation and stopping regions as variational inequalities and can be written as

$$\begin{cases} \max\{((\mathscr{A}-r)V)(x), (\mathscr{M}V)(x)-V(x)\} = 0 & \text{for all } x \in E \\ V(x) = (\mathscr{M}V)(x) & \text{for all } x \in S^*, \end{cases} \tag{3.7}$$

where the nonlinear operator $\mathscr{M}$ defined as

$$(\mathscr{M}V)(x) := \sup_{\zeta\in[0,x]} [g(\zeta)+V(x-\zeta)]$$

is the so-called *intervention operator*, see Øksendal 1998. In Øksendal, A. 2000 the author presents an interesting approach for solving quasi-variational inequalities by invoking semigroup machinery. The functional concavity approach mentioned in Section 2 has also been utilized recently in impulse control, see Bayraktar and Egami 2007.

The method used in Articles *(1)-(2)* are essentially extensions of the idea presented in the context of optimal stopping. The actual mathematical techniques are naturally more involved. In both articles, the finding of the candidate for the value V can be reduced into solving a non-linear pair of equation. The verification phase is carried using quasi-variational inequalities.

4 SUMMARIES OF INCLUDED PAPERS

(1) A CLASS OF SOLVABLE STOCHASTIC DIVIDEND OPTIMIZATION PROBLEMS: ON THE GENERAL IMPACT OF FLEXIBILITY ON VALUATION

We study a class of stochastic dividend optimization problems where the underlying cash reserve dynamics is modelled on $\mathbf{R}_+$ as a generalized Itô integral

$$X_t^\nu = x + \int_0^t \mu(X_s^\nu)ds + \int_0^t \sigma(X_s^\nu)dW_s - \sum_{\tau_k \leq t} \zeta_k, \tag{4.1}$$

where μ and σ are sufficiently regular functions. The dividend optimization problem is then modelled as the impulse control problem

$$V(x) = \sup_{\nu \in \mathscr{V}} \mathbf{E}_x \left[\sum_{k=1}^{N} e^{-r\tau_k}(\zeta_k - c) \right], \tag{4.2}$$

where $c > 0$ stands for a fixed transaction cost and N is the number of interventions prior to the liquidation of X^ν. The problem (4.2) is studied under the following assumptions:

(1) The upper boundary ∞ is natural and the lower boundary 0 is either natural, exit or regular

(2) The net appreciation rate of the revenues defined as $\rho(x) = \mu(x) - rx$ satisfies the integrability condition (1.8) and the condition $\lim_{x\to\infty} \rho(x) = -\infty$

(3a) If 0 is unattainable, we assume that there is a unique finite state $x^* > 0$, such that ρ is increasing on $(0, x^*)$ and decreasing on (x^*, ∞). Moreover, we assume that $\lim_{x\to 0+} \mu(x) \geq 0$

(3b) If 0 is attainable, we assume that there is a unique finite state $x^* \geq 0$, such that ρ is increasing on $(0, x^*)$ and decreasing on (x^*, ∞). Moreover, we assume that $\lim_{x\to 0+} \mu(x) > 0$.

In the context of cash flow management applications, the assumed behavior of X at the upper boundary ∞ can be interpreted as a requirement that the retained profits from which the dividends are paid out to the shareholders cannot become infinitely large in finite time. On the other hand, the assumed behavior of X at the lower boundary is in line with the concept of liquidation and essentially guarantees that no dividends can be paid out from a negative reserve. The function μ measures the expected net return accrued from postponing the dividend payment into the future instead of paying out dividends instantaneously. This means that the assumed limiting behavior of the net return guarantees that the rate of return earned from retained unit dominates its opportunity cost when the reserves are low and *vica versa* when the reserves are high. Finally, the assumed integrability condition (1.8) means that the expected cumulative present value of the net returns accrued from present up to liquidation is finite (the *absence of speculative bubbles* condition).

We also study two related singular control and optimal stopping problems. The problems are associated to the impulse control problem through the underlying dynamics, which follow the same linear diffusion in the absence of control. This allows us to investigate how increased policy flexibility affects the optimal value. The effect is two-fold: increased policy flexibility does not only increase the value, it also increases the rate at which the value grows. In other words, we prove that the value of the singular stochastic control problem dominates the value of the stochastic impulse control problem which in turn dominates the value of the optimal stopping problem and that this same ordering is valid also for the marginal values (derivatives, that is). The observation of the ordered derivatives is of special interest in capital theory, since the marginal value has an interpretation as the so-called Tobin's marginal q. We find that in all three cases the optimal control policy is a threshold policy. Moreover, if $\lim_{x\to 0+}\psi'(x) < \infty$, we establish that there exists a transaction cost $\hat{c}$ such that when $c \geq \hat{c}$, the optimal dividend payment (impulse control) policy degenerates into a single-opportunity (optimal stopping) policy. In other words, if the transaction cost is sufficiently high, then it is rational to liquidate the firm and pay out the entire reserve as dividends when the reserve exceeds a certain level.

(2) ON THE OPTIMAL STOCHASTIC IMPULSE CONTROL OF LINEAR DIFFUSIONS

In this paper, we extend the treatise of Article *(1)* to a more general setting. The underlying dynamics follow now the generalized Itô integral

$$X_t^\nu = x + \int_0^t \mu(X_s^\nu)ds + \int_0^t \sigma(X_s^\nu)dW_s - \sum_{\tau_k \le t} \beta \zeta_k$$

and the impulse control problem reads as

$$V(x) = \sup_{\nu \in \mathscr{V}} \mathbf{E}_x \left[\int_0^{\tau_0^\nu} e^{-rs}\pi(X_s^\nu)ds + \sum_{k=1}^{N} e^{-r\tau_k}(\lambda \zeta_k - c) \right].$$

The assumptions under which the problem is studied are analogous to Article *(1)*. More precisely, the boundary behavior of X is assumed to be the same as in Article *(1)* and the assumptions *(2)*, *(3a)* and *(3b)* are placed on the function θ defined as $\theta(x) = \beta\pi(x) + \lambda(\mu(x) - rx)$ instead of ρ. Along the impulse control problem, we study the related singular control and optimal stopping problems and prove that in all three cases the optimal policy is a threshold policy. Moreover, we establish the same ordering of the values and the marginal values as in Article *(1)*. In contrast to to Article *(1)*, we find that in the case where $\lim_{x \to 0+} \psi'(x) < \infty$, the critical cost $\hat{c}$ no longer exists. In other words, we find that the optimal impulse control policy is always a non-degenerate sequential policy for the considered class of impulse control problems.

The introduction of the seemingly trivial control parameters λ and β have an interesting implication in cash flow management applications. In this context, we can interpret these parameters as tax rates. We establish that the optimal exercise thresholds of the related singular control and optimal stopping problems are unaffected as long as the ratio λ/β remains constant. In other words, we find that *the harmonization of tax rates imply the tax neutrality of the optimal policy* for the related singular control and optimal stopping problems. However, we establish in the impulse control problem harmonization of taxes does *not* imply the tax neutrality – in this sense the parameters λ and β are non-redundant.

(3) ON TWO-SIDED OPTIMAL STOPPING OF A LINEAR DIFFUSION

We consider a class of optimal stopping problems

$$V(x) = \sup_{\tau < \tau_\Delta} \mathbf{E}_x[e^{-r\tau} g(X_\tau)],$$

where the underlying dynamics X evolve on $\mathbf{R}_+$ according to a regular linear diffusion for which the boundaries 0 and ∞ are either natural or exit-not-entrance. Moreover, it is assumed the diffusion X does not die inside the state space $\mathbf{R}_+$ and that the basic characteristics of X, namely the scale function S, the speed measure m and the killing measure k, are absolutely continuous with respect to the Lebesgue measure, have smooth derivatives and that the scale function S is twice continuously differentiable. The payoff function g is assumed to satisfy the following properties:

(1) The payoff function $g \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus D)$, where D is a finite subset of $\mathbf{R}_+$, satisfies the conditions $\lim_{x \to y\pm} g'(y) < \infty$ and $\lim_{x \to y\pm} g''(y) < \infty$ for all $y \in D$. Moreover, we assume that there is positive real numbers x_1 and x_2 such that g satisfies the strict inequalities

$$((\mathscr{A} - r)g)(x) \begin{cases} \geq 0, & x \in [x_1, x_2] \\ < 0, & x \notin [x_1, x_2] \end{cases} \tag{4.3}$$

and that $D \subset [x_1, x_2]$.

(2) There exists an interval N_1 with compact closure such $x_1 \in \bar{N}_1$ and that the function $x \mapsto \frac{g(x)}{\varphi(x)}$ is decreasing in N_1 and increasing in $\mathbf{R}_+ \setminus N_1$. Similarly, we assume here exists an interval N_2 with compact closure such that $x_2 \in \bar{N}_2$ and that the function $x \mapsto \frac{g(x)}{\psi(x)}$ is increasing in N_2 and decreasing in $\mathbf{R}_+ \setminus N_2$.

We establish that under these assumptions, the resulting optimal stopping rule is a two-sided threshold rule. In the proof, we rely on the fixed point approach utilized also in Article *(2)*. Similarly to papers with one-sided optimal stopping rules (see Alvarez 2001), we find that the behavior of the functions $x \mapsto \frac{g(x)}{\psi(x)}$ and $x \mapsto \frac{g(x)}{\varphi(x)}$ is connected to the whereabouts of the optimal stopping thresholds a^* and b^*. For a wide class of cases with one-sided optimal stopping rules, the optimal stopping threshold coincides with the interior state maximizing globally either $x \mapsto \frac{g(x)}{\psi(x)}$ or $x \mapsto \frac{g(x)}{\varphi(x)}$, depending naturally on the problem formulation. However, we discover that in the setting of the current study, the optimal stopping threshold b^* (a^*) is strictly larger (less) than the interior state $\tilde{x}_2 = \sup N_2$ ($\tilde{x}_1 = \inf N_1$) maximizing locally the function $x \mapsto \frac{g(x)}{\psi(x)}$ ($x \mapsto \frac{g(x)}{\varphi(x)}$). This observation has an interesting implication for the valuation and exercise of a perpetual American straddle. In fact, if we consider determination of the optimal exercise threshold for a perpetual American call option, our results show that the introduction of a put component into the option increases the optimal exercise threshold of the call option – an analogous conclusion holds also for a put option.

(4) ON INFINITE HORIZON OPTIMAL STOPPING OF GENERAL RANDOM WALK

We consider a Black-Scholes-type infinite horizon optimal stopping problem where the underlying process is a general random walk. In more precise terms, let X be a random variable on distributed on entire $\mathbf{R}$ with a continuous law λ, mean $\mu > 0$ and variance $\sigma^2 < \infty$ and define the general random walk W on $\mathbf{R}$ as partial sums of IID random variables $X, X_1, X_2, \ldots$; i.e. let $W_n = X_1 + \cdots + X_n$, where $W_0 = 0$. Now, the optimal stopping problem reads as

$$V(x) = \sup_{\eta} \mathbf{E}\left[\beta^{-\eta}\left(e^{x+W_\eta} - c\right)^+\right],$$

where η is a W-stopping time, $\beta^{-1} < 1$ is the discount factor satisfying the condition $\mathbf{E}\left[\beta^{-1}e^X\right] < 1$ and $c > 0$ is the transaction cost.

We start the analysis by determining the β-minimal functions of W. It turns out that there is two of these functions and that they are of the same functional form (exponential functions of the form $x \mapsto e^{tx}$, $t \in \mathbf{R}$) as the corresponding r-minimal functions of its continuous time counterpart, Brownian motion with drift. Similarly to the diffusion case, we establish that the continuation region reads as $(-\infty, s^*)$, where s^* is the optimal stopping threshold, and that in the continuation region, the optimal value can be expressed as $V(x) = \frac{g(s^*)}{e^{bs^*}} e^{bx}$, where $x \mapsto e^{bx}$ is the increasing β-minimal function. The coefficient b is the unique positive solution of the equation $\mathbf{E}[e^{bX}] = \beta$. Unfortunately, this is where the similarities to the diffusion case ends. The smooth pasting condition does not hold for this problem, which makes the determination of the optimal stopping boundary much more difficult.

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PAPER I

Luis Alvarez - Jukka Virtanen : *A Class of Solvable Stochastic Dividend Optimization Problems: On the General Impact of Flexibility on Valuation*

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A CLASS OF SOLVABLE STOCHASTIC DIVIDEND OPTIMIZATION PROBLEMS: ON THE GENERAL IMPACT OF FLEXIBILITY ON VALUATION

Luis H. R. Alvarez and Jukka Virtanen

ABSTRACT

We consider the determination of an optimal dividend policy in the presence of cash flow uncertainty and transaction costs. We state a set of weak conditions under which the optimal dividend policy can be explicitly characterized for a broad class of diffusions modelling the underlying cash flow dynamics and demonstrate that increased dividend policy flexibility does not only increase the maximal expected cumulative present value of the future dividends, it also increases the rate at which this value grows (i.e. Tobin's marginal q). We also prove that increased transaction costs result into larger but less frequent dividend payments.

Keywords: Optimal dividends, cash flow uncertainty, liquidation, stochastic impulse and singular control.

1 INTRODUCTION

In the classical study Miller and Modigliani 1961 established that dividend policy is irrelevant in a perfect and rational market. As Miller and Modigliani 1961 state (on p. 414):

> ”. . . there are no ”financial illusions” in a rational and perfect economic environment. Values there are determined solely by ”real” considerations – in this case the earning power of the firm’s assets and its investment policy– and not by how the fruits of the earning power are ”packaged” for distribution.”

This irrelevance result (and the related findings on the irrelevance of the capital structure on valuation of Modigliani and Miller 1958) were later extended in the general equilibrium framework by Stiglitz 1974. These to some extent controversial findings based on the perfection of the underlying markets have been subsequently challenged in numerous studies by weakening the assumptions and introducing imperfections into the analysis of the determination of the dividend policy (for example, by introducing economics of information (Ross 1977), agency costs (Easterbrook 1984, Jensen 1986), asymmetric information (Miller and Rock 1985), and taxes (Kose and Williams 1985); see also Chapter 15 in Ross and Westerfield 1988). Moreover, there is empirical evidence indicating that at least in some industries (for example, in the insurance industry; cf. Akhigbe, Borde and Madura 1993) dividend policy does play a role in the valuation of firms and that dividend policy is actually an important strategic element in the decision making process of these corporations.

Given the arguments stated above, we plan to consider in this study the determination of the optimal dividend policy of a rationally managed corporation in the presence of transaction costs for a broad class of diffusions modelling the stochastically fluctuating dynamics of the underlying cash reserves (retained profits) from which the dividends are paid out. Given the recent interest on stochastic impulse control policies, we model the admissible dividend policy as a stochastic lump-sum impulse policy and, therefore, assume that *the objective of the corporation is to determine both the timing and the size of the optimal dividend policy* (cf. Bar-Ilan, Perry, and Stadje 2004, Cadenillas, Sarkar, and Zapatero 2003 and Peura and Keppo 2003; see also Korn 1999 for an excellent survey on stochastic impulse control applications in finance). Instead of relying on ordinary dynamic programming techniques and quasi-variational inequalities we propose an alternative approach based on the classical theory of diffusions. We first derive explicitly the expected cumulative present value of an admissible and potentially suboptimal dividend policy characterized by two constant boundaries. Namely, by the boundary at which dividends are paid out and the boundary at which the underlying stochastic reserve process is subsequently restarted (i.e. the generic initial state). In this way the original problem is transformed into a two-dimensional non-linear programming problem which can be studied by relying on ordinary static optimization techniques. By applying this representation we derive the ordinary first order necessary conditions characterizing the optimal policy within the considered class of admissible dividend policies. We then present a set of relatively weak conditions (which are valid, for example, *for most applied mean reverting diffusion models*) under which both the existence and uniqueness of an optimal pair of boundaries is always guaranteed and under which the proposed dividend policy satisfying the necessary conditions is indeed optimal. Interestingly, we find that *the presence of liquidation risk results into a maximal admissible transaction cost* below which the sequential payment of dividends is optimal. Above this critical cost the sequential payment of dividends is suboptimal and the dividend optimization problem becomes an optimal liquidation problem where the objective of the corporation is to determine the threshold at which it should be irreversibly liquidated. Thus, our results indicate that the combined effect of the risk of potential liquidation and transaction costs on the nature of the optimal dividend policy may be dramatic depending on the size of the transaction costs.

For the sake of comparison, we also consider two associated stochastic cash flow management problems. Namely, a singular stochastic control problem where the optimal dividend policy is characterized by a single exercise threshold at which dividends are paid out in a singular fashion (i.e. the optimal policy typically ranges from periods of intense dividend payments to periods of inactivity; cf. Asmussen and Taksar 1997, Baldursson and Karatzas 1997, Choulli, Taksar, and Zhou 2003, Højgaard and Taksar 1999, 2001, Holt 2003, Jeanblanc-Picqué and Shiryaev 1995, Kobila 1983, Milne and Robertson 1996, Øksendal 2000, and Taksar and Zhou 1998. See also Taksar 2000 for an excellent survey of stochastic dividend optimization models) and an optimal stopping problem where the optimal dividend policy is characterized by a single exercise threshold at which all the reserves are paid out as dividends and the corporation is instantaneously liquidated. We first demonstrate the intuitively clear finding that the value of the associated singular stochastic dividend control problem dominates the value of the stochastic lump-sum (impulse) dividend control problem which, in turn, dominates the value of the associated optimal liquidation problem. However, somewhat surprisingly, we also find that *the marginal values (and, therefore, the Tobin's marginal q associated with the particular cash flow management problem) are ordered in an analogous way*. Therefore, our results clearly support the economically sensible argument that *increased flexibility of the dividend policy does not only increase the value of a rationally managed corporation, it also increases the rate at which this value grows*. It is also worth noticing that our results extend previous results establishing a connection between the marginal value of singular stochastic control problems and the value of associated optimal stopping problems (cf. Alvarez 1999, 2001, Baldursson 1987, Benes, Shepp, and Witsenhausen 1980, Boetius and Kohlmann 1998, Hausmann and Suo 1995, Karatzas 1983, Karatzas and Shreve 1984, 1985, Menaldi and Robin 1983, and Menaldi and Rofman 1983) by showing that the values of the considered cash flow management problems are connected through an associated free boundary value problem as well.

It is worth observing that our results are of importance in the rational management of renewable resources as well, since all the considered cash flow management problems can be interpreted as *the determination of an admissible harvesting strategy maximizing the expected cumulative present value of future harvesting yields in the presence of stochastic value growth*. More specifically, our results demonstrate that typically both the value and the marginal value of the optimal single harvesting strategy are smaller than the value and the marginal value of the optimal ongoing harvesting opportunity, respectively (Alvarez 2003, Sødal 2002, and Willassen 1998 have previously considered the determination of the optimal rotation policy in the presence of stochastic value growth and an exogenously given generic initial state). These values, in turn, are smaller than the value and the marginal value of the singular harvesting opportunity modelling the most flexible harvesting strategy (singular stochastic harvesting strategies have been previously considered, among others, in Alvarez 1998, 2000, Alvarez and Shepp 1998, Lande, Engen, and Sæther 1994, 1995, and Lungu and Øksendal 1996). These observations again imply that the required exercise premia and, consequently, the optimal harvesting thresholds can be ordered accordingly. A natural and economically sensible implication of this observation is that *increased flexibility shortens the expected length of a time interval between two consecutive harvests (i.e. the rotation cycle)*. Put somewhat differently, increased flexibility unambiguously increases the project value by increasing the expected cumulative yield accrued from harvesting and speeds up harvesting by decreasing the optimal harvesting threshold; a finding which is in line with the literature on *real options*.

The contents of this study are as follows. In section 2 the considered stochastic impulse control problem is presented. In section 3 we then present a set of auxiliary results on linear diffusions and associated stochastic control problems needed later in the analysis. In section 4 we then consider the stochastic impulse control problem and present both a set of necessary conditions from which the optimal policy can be derived and a set of general conditions under which the optimal policy exists and is unique. In section 5 our theoretical results are then explicitly illustrated. Finally, section 6 concludes our study.

2 THE STOCHASTIC IMPULSE CONTROL PROBLEM

Consider a value-maximizing competitive corporation operating in the presence of cash flow uncertainty. For the sake of simplicity, assume that the reservoir process measuring the retained profits from which dividends are paid out is exogenous and modelled as a general linear diffusion. More precisely, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions and assume that the dynamics of the controlled cash flow are described by the process characterized by the generalized Itô-equation

$$X_t^\nu = x + \int_0^t \mu(X_s^\nu)ds + \int_0^t \sigma(X_s^\nu)dW(s) - \sum_{\tau_k \leq t} \zeta_k, \quad 0 \leq t \leq \tau_0^\nu, \qquad (2.1)$$

where $\tau_0^\nu = \inf\{t \geq 0 : X_t^\nu \leq 0\}$ denotes the potentially infinite date at which the firm is liquidated and $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and $\sigma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ are known sufficiently smooth (at least continuous) mappings guaranteeing the existence of a solution for the stochastic differential equation (2.1) (cf. Borodin and Salminen 2002, pp. 46–47). A stochastic *lump-sum dividend policy* (i.e. a stochastic impulse control) for the system (2.1) is a possibly finite sequence (cf. Øksendal 1999)

$$\nu = (\tau_1, \tau_2, \dots, \tau_k, \dots; \zeta_1, \zeta_2, \dots, \zeta_k, \dots)_{k \leq N} \quad (N \leq \infty),$$

where $\{\tau_k\}_{k \leq N}$ is an increasing sequence of $\mathscr{F}_t$-stopping times for which $\tau_1 \geq 0$, and $\{\zeta_k\}_{k \leq N}$ denotes a sequence of non-negative dividends (i.e. $\zeta_k \geq 0$ for all k) paid out at the corresponding intervention dates $\{\tau_k\}_{k \leq N}$, respectively. We denote as $\mathscr{V}$ the class of admissible dividend policies ν and assume that $\tau_k \to \tau_0^\nu$ almost surely for all $\nu \in \mathscr{V}$ and $x \in \mathbb{R}_+$. In accordance with most financial and economic applications of stochastic impulse control models, we assume that the upper boundary ∞ is natural (therefore, even though the reserves may be expected to increase, they are never expected to become infinitely high in finite time) and that the lower boundary 0 is either natural, exit, or regular for the controlled diffusion in the absence of interventions. In case it is regular, we assume that it is killing. As usually, we denote as

$$\mathscr{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$$

the differential operator associated to the controlled diffusion X_t.

Given the dynamics in (2.1) and our assumptions on the dynamics of the controlled system, define the expected cumulative present value of the net dividends from the present up to an arbitrarily distant (potentially infinite) future as

$$J_c^\nu(x) = \mathbf{E}_x\left[\sum_{k=1}^{N} e^{-r\tau_k}(\zeta_k - c)\right], \tag{2.2}$$

where $r > 0$ denotes the risk free discount rate and $c > 0$ is a known transaction cost incurred each time dividends are paid out. Given the definition of the expected cumulative dividends $J_c^\nu(x)$ we plan to consider in this study the stochastic impulse control problem

$$V_c(x) = \sup_{\nu \in \mathscr{V}} J_c^\nu(x), \quad x \in \mathbb{R}_+ \tag{2.3}$$

and to determine an admissible lump-sum dividend policy $\nu^* \in \mathscr{V}$ for which $J_c^{\nu^*}(x) = V_c(x)$ for all $x \in \mathbb{R}_+$. Put somewhat differently, we plan to determine an dividend policy $\nu^* \in \mathscr{V}$ maximizing the expected cumulative present value of the paid out dividends from the present up to an arbitrarily distant future. Given our assumptions on the controlled diffusion and the objective function, we now present a verification lemma which is later applied for the verification of optimality of a proposed policy (for related results see, for example, Bensoussan 1982, Brekke and Øksendal 1994, 1996, Harrison, Sellke, and Taylor 1983, Mundaca and Øksendal 1998, Øksendal 1999, and Øksendal 2000).

Lemma 2.1. *Assume that the mapping $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfies the conditions $g \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \backslash \mathscr{D})$, where $\mathscr{D}$ is a set of measure zero and $|g''(x\pm)| < \infty$ for all $x \in \mathscr{D}$. Assume also that $g(x)$ satisfies the quasi-variational inequality*

$$g(x) \geq \sup_{\zeta \in [0,x]} [\zeta - c + g(x-\zeta)] \tag{2.4}$$

for all $x \in \mathbb{R}_+$ and the variational inequality $(\mathscr{A}g)(x) - rg(x) \leq 0$ for all $x \notin \mathscr{D}$. Then, $g(x) \geq V_c(x)$ for all $x \in \mathbb{R}_+$.

Proof. See Appendix A. □

3 AUXILIARY RESULTS

Before proceeding in the analysis of the considered stochastic dividend optimization problem, we first derive some auxiliary results needed later in the analysis of the original problem. For the sake of notational simplicity, denote now as X_t the controlled stochastic cash flow dynamics in the absence of interventions. As usually, we denote as $\mathscr{L}^1$ the collection of cash flows with finite expected cumulative present values. For $f \in \mathscr{L}^1$ we define the functional $(R_r f) : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$(R_r f)(x) = \mathbf{E}_x \int_0^{\tau_0} e^{-rs} f(X_s) ds,$$

where $\tau_0 = \inf\{t \geq 0 : X_t \leq 0\}$. As is well-known from the literature on linear diffusions, if $f \in \mathscr{L}^1$ then

$$(R_r f)(x) = B^{-1}\varphi(x) \int_a^x \psi(y) f(y) m'(y) dy + B^{-1}\psi(x) \int_x^\infty \varphi(y) f(y) m'(y) dy,$$

where $\psi(x)$ denotes the increasing and $\varphi(x)$ the decreasing fundamental solution of the ordinary second order differential equation $(\mathscr{A}u)(x) = ru(x)$ (defined on the domain of the operator of the diffusion; see Borodin and Salminen

2002, pp. 18–20 for a complete characterization of the fundamental solutions and the Green function of a linear diffusion), $B = \frac{\psi'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}\psi(x) > 0$ denotes the constant (with respect to the scale) Wronskian determinant of the fundamental solutions, $S'(x) = \exp\left(-2\int^x (\mu(y)/\sigma^2(y))dy\right)$ denotes the density of the scale function S of X, and $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the density of the speed measure m of X.

For the sake of comparison, consider now the associated singular stochastic dividend control problem

$$K(x) = \sup_{Z\in\Lambda} \mathbf{E}_x \int_0^{\tau^Z(0)} e^{-rs} dZ_s, \tag{3.1}$$

where Λ denotes the class of non-negative, non-decreasing, right-continuous, and $\{\mathscr{F}_t\}$-adapted dividend payment processes, $\tau^Z(0) = \inf\{t \geq 0 : X_t^Z \leq 0\}$ denotes the potentially infinite liquidation date, and the underlying reserve process evolves on $\mathbb{R}_+$ according to the dynamics described by the generalized (Itô) stochastic differential equation

$$dX_t^Z = \mu(X_t^Z)dt + \sigma(X_t^Z)dW_t - dZ_t, \quad X_0^Z = x. \tag{3.2}$$

An important result needed later in the analysis of the dividend optimization problem (2.3) (slightly extending the results originally proved in 3 and 5) is now summarized in our next lemma.

Lemma 3.1. *Assume that the mapping* $\rho(x) = \mu(x) - rx$ *measuring the net appreciation rate of the reserves satisfies the conditions* $\rho \in \mathscr{L}^1(\mathbb{R}_+)$ *and* $\lim_{x\to\infty} \rho(x) < 0$. *Assume also that*

(i) if 0 *is unattainable for* X_t *then there is a unique threshold* $x^* \in (0,\infty)$ *such that* $\rho(x)$ *is increasing on* $(0,x^*)$ *and decreasing on* (x^*,∞) *and* $\lim_{x\downarrow 0} \mu(x) \geq 0$*;*

(ii) if 0 *is attainable for* X_t *then there is a unique threshold* $x^* \in [0,\infty)$ *such that* $\rho(x)$ *is increasing on* $(0,x^*)$ *and decreasing on* (x^*,∞) *and* $\lim_{x\downarrow 0} \mu(x) > 0$.

Then, the value of the optimal dividend payment policy reads as

$$K(x) = \begin{cases} x + \frac{\rho(\hat{x})}{r} & x \geq \hat{x} \\ \frac{\psi(x)}{\psi'(\hat{x})} & x < \hat{x}, \end{cases} \tag{3.3}$$

where $\hat{x}$ is the unique optimal exercise threshold $\hat{x} = \operatorname{argmin}\{\psi'(x)\}$ satisfying the ordinary first order condition $\psi''(\hat{x}) = 0$. The value of the optimal dividend policy is twice continuously differentiable, monotonically increasing, and concave. Moreover, the marginal value (i.e. Tobin's q) of the optimal dividend policy can be expressed as

$$K'(x) = \psi'(x) \sup_{y \geq x} \left[\frac{1}{\psi'(y)} \right] = \begin{cases} 1 & x \geq \hat{x} \\ \frac{\psi'(x)}{\psi'(\hat{x})} & x < \hat{x}. \end{cases} \tag{3.4}$$

Proof. See Appendix B. □

Lemma 3.1 states a set of weak sufficient conditions under which the associated stochastic cash flow management problem (3.1) is solvable and under which the value of the optimal dividend policy can be expressed in terms of the increasing fundamental solution $\psi(x)$. Lemma 3.1 has two important capital theoretic implications. First, since the optimal dividend threshold is attained on the set where net appreciation rate of the underlying reserve is positive, we find that *dividends are paid out on the set where the expected per capita rate at which the reserves are increasing dominate the opportunity cost of investment.* Second, since the optimal dividend threshold is attained on the set where net appreciation rate of the underlying reserve is decreasing, we find that *at the optimum the marginal yield accrued from retaining yet another marginal unit of stock undistributed is smaller than the interest rate r.* Thus, the optimal dividend policy diverges from the deterministic golden rule of capital accumulation (cf. Merton 1990, pp. 594-595, see also Alvarez 2001).

It is at this point worth emphasizing that the value (3.3) of the optimal singular stochastic dividend policy can be re-expressed as

$$K(x) = \begin{cases} x - \hat{x} + \frac{\psi(\hat{x})}{\psi'(\hat{x})} & x \geq \hat{x} \\ \frac{\psi(x)}{\psi'(\hat{x})} & x < \hat{x}. \end{cases} \tag{3.5}$$

As we will later observe in our subsequent analysis, this expression is closely related to the value of the considered stochastic lump-sum dividend optimization problem (2.3). An important inequality emphasizing the importance of the value of the associated singular stochastic dividend policy is now summarized in the following.

Theorem 3.2. *Define the continuously differentiable mapping* $H : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ *as*

$$H(x,y) = \begin{cases} x - y + \frac{\psi(y)}{\psi'(y)} & x \geq y \\ \frac{\psi(x)}{\psi'(y)} & x < y \end{cases}$$

and assume that the conditions of Lemma 3.1 are satisfied. Then $K(x) = H(x,\hat{x}) > H(x,y)$ *and* $K'(x) = H_x(x,\hat{x}) > H_x(x,y)$ *for all* $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{\hat{x}\}$. *Moreover,* $H_y(x,y) < 0$ *for all* $(x,y) \in \mathbb{R}_+ \times (\hat{x},\infty)$.

Proof. Ordinary differentiation of the mapping $H(x,y)$ with respect to the variable y yields

$$H_y(x,y) = -\min(\psi(x),\psi(y))\frac{\psi''(y)}{\psi'^2(y)} \gtreqless 0, \quad y \lesseqgtr \hat{x},$$

since $\psi(x)$ is positive, monotonically increasing, and satisfies the inequality $\psi''(x) \lesseqgtr 0$ for all $x \lesseqgtr \hat{x}$. Combining this observation with the identity $K(x) = H(x,\hat{x})$ then proves that $K(x) = H(x,\hat{x}) > H(x,y)$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{\hat{x}\}$. Inequality $H_x(x,\hat{x}) > H_x(x,y)$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{\hat{x}\}$ then follows from the inequality $\psi'(x) > \psi'(\hat{x})$ for $x \neq \hat{x}$. □

Theorem 3.2 demonstrates that if the conditions of Lemma 3.1 are satisfied then the value $H(x,y)$ attains a unique global maximum as a function of the single dividend threshold y. Interestingly, Theorem 3.2 proves that this value does not only dominate the value of arbitrary single threshold dividend policies, it also grows faster than any other value within this class of dividend policies. An interesting implication of these observations characterizing the relationship between the optimal sequential dividend policy and the value of the optimal liquidation policy is now summarized in the following.

Lemma 3.3. *Assume that the conditions of Lemma 3.1 are satisfied. Then* $K(x) > G_0(x)$ *and* $K'(x) > G_0'(x)$*, where*

$$G_0(x) = \sup_{\tau<\tau_0} \mathbf{E}_x\left[e^{-r\tau}X_\tau\right] = \psi(x)\sup_{y\geq x}\left[\frac{y}{\psi(y)}\right] = \begin{cases} x & x \geq \bar{x}_0 \\ \frac{\psi(x)}{\psi'(\bar{x}_0)} & x < \bar{x}_0, \end{cases} \tag{3.6}$$

denotes the maximal expected present value of the cash reserves and $\bar{x}_0 \in \rho^{-1}(\mathbb{R}_-)$*, denoting the threshold at which this value is attained, is the unique root of equation* $\psi(\bar{x}_0) = \bar{x}_0\psi'(\bar{x}_0)$.

Proof. Proving that the value of the optimal exercise strategy reads as in (3.6) and that an optimal boundary $\bar{x}_0 = \text{argmax}\{x/\psi(x)\}$ satisfying the ordinary first order condition $\psi(\bar{x}_0) = \bar{x}_0\psi'(\bar{x}_0)$ exists and is unique is analogous with the proof of Lemma 2.3. in Alvarez (2004). In order to prove the inequality $K(x) \geq G_0(x)$ we first observe that the value of the associated singular stochastic control problem satisfies the variational inequality $(\mathscr{A}K)(x) \leq rK(x)$ for all $x \in \mathbb{R}_+$. On the other hand, the local concavity of $\psi(x)$ on $(0,\hat{x}]$ implies that $\psi'(x) \geq \psi'(\hat{x})$ and $\psi(x) \geq \psi'(x)x$ for all $x \in (0,\hat{x}]$ and, therefore, that also the condition $K(x) \geq x$ is fulfilled. Thus, $K(x)$ constitutes a r-excessive majorant of the exercise payoff x for the underlying diffusion X_t. Since $G_0(x)$ is the least of these majorants, we observe that $K(x) \geq G_0(x)$. The inequality $K'(x) \geq G_0'(x)$ is now a straightforward consequence of equation (3.4) and Theorem 3.2. □

Lemma 3.3 demonstrates that the value of the optimal singular stochastic dividend policy dominates the maximal expected present value of the future cash reserves. This result is intuitively clear, since the maximal expected present value of the cash reserves can always be attained by choosing the admissible dividend policy $Z_t = X_t 1_{[\bar{x}_0,\infty)}(X_t)$ (i.e. liquidation at the threshold $\bar{x}_0$). Since the class of admissible policies is, however, larger than this single dividend payment strategy, we find that *the value of the optimal singular stochastic dividend policy dominates the maximal expected present value of the cash reserves*. However, a slightly more surprising result is that also the marginal value of the optimal singular stochastic dividend policy dominates the marginal value of the maximal expected present value of the cash reserves. Therefore, *the yield accrued from retaining a marginal unit of stock undistributed is higher in the case where dividends are paid out sequentially than in the case where dividends are paid out only once*. A second important implication of the results of Lemma 3.3 is that $\bar{x}_0 > \hat{x}$. That is, *the required exercise premium is naturally higher in the case where the opportunity to pay out dividends may be exercised only once than in the case where this decision may be subsequently repeated*. Moreover, since $\bar{x}_0$ is attained on the set where the net appreciation rate of the reserves is decreasing we again find that at the optimum the marginal yield accrued from postponing exercise further into the future should be smaller than the opportunity cost of investment measured by the risk free rate r. Hence, the deterministic golden rule of capital accumulation is violated in this case as well.

4 THE OPTIMAL DIVIDEND POLICY

Having presented in the previous section some auxiliary results and an associated singular stochastic dividend optimization problem we now plan to analyze the stochastic lump-sum dividend optimization problem (2.3). In order to present a general detailed treatment of the problem, we first derive a set of necessary conditions which have to be satisfied by a candidate for an optimal policy. We then study the necessary conditions and establish a set of general and typically satisfied conditions under which the necessary conditions admit a unique solution and under which this solution is indeed optimal.

4.1 NECESSARY CONDITIONS

Typically, stochastic impulse control problems of the type (2.3) are solved by relying on dynamic programming techniques and quasi-variational inequalities. Although such an approach is very general and applies in the multidimensional setting as well, it is usually rather difficult to derive expressions independent of the value function with simple and clear economic interpretations. Similarly, marginalistic interpretations providing valuable economic content and general information on the nature of the optimal dividend policy and its value are typically difficult to derive from general approaches based on dynamic programming techniques. Given this argument, we propose in this paper an alternative approach for analyzing and solving the stochastic lump-sum dividend optimization problem (2.3). Instead of considering all admissible dividend policies at once, we follow the approach introduced in 6 and restrict our interest to dividend policies $\nu_{(y,\zeta)} = \{\tau_k^y; \zeta_k^y\}_{k\leq N}$ characterized by the sequence of intervention times $\tau_k^y = \inf\{t \geq \tau_{k-1}^y : X_t^\nu \geq y\}$ (with $\tau_0 = 0$) and the sequence of dividend payments $\zeta_k^y = \zeta + (x-y)^+$. That is, we restrict our attention to the sequence of constant-sized dividends (except for the initial impulse which depends on the current state) which are paid out every time the underlying diffusion hits a given constant threshold y. Given this subclass of admissible dividend strategies, define the value accrued from applying the dividend policy $\nu_{(y,\zeta)}$ as $F_c(x) = J_c^{\nu_{(y,\zeta)}}(x)$. Since $X_{\tau_k+}^\nu = X_{\tau_k-}^\nu - \zeta$ for all k and the underlying controlled diffusion evolves according to the linear diffusion

X_t between two successive intervention dates we observe that for all $x < y$ the value of the considered class of dividend policies reads as

$$F_c(x) = \mathbf{E}_x\left[e^{-r\tau_y}(\zeta - c + F_c(y-\zeta))\right] = (\zeta - c + F_c(y-\zeta))\frac{\psi(x)}{\psi(y)}, \quad (4.1)$$

where $\tau_y = \inf\{t \geq 0 : X_t = y\}$ denotes the first hitting time of X_t to the state y (cf. Borodin and Salminen (2002), p. 18). Letting $x \to y - \zeta$ in (4.1) and solving $F_c(y-\zeta)$ from the resulting equation then implies that the value $F_c(x)$ can be re-expressed on $(0,y)$ as

$$F_c(x) = \frac{(\zeta - c)\psi(x)}{\psi(y) - \psi(y-\zeta)}. \quad (4.2)$$

On the other hand, since the reserves can exceed the dividend threshold y under the proposed dividend policy only at the initial date and $\zeta_1^y = \zeta + (x-y)^+$ we find that on $[y,\infty)$ the value $F_c(x)$ reads as $F_c(x) = x - y + \zeta - c + F_c(y-\zeta)$. Hence, we finally observe that $F_c(x)$ can be re-expressed as

$$F_c(x) = \begin{cases} x - y + \frac{(\zeta-c)\psi(y)}{\psi(y)-\psi(y-\zeta)} & x \geq y \\ \frac{(\zeta-c)\psi(x)}{\psi(y)-\psi(y-\zeta)} & x < y. \end{cases} \quad (4.3)$$

It is worth observing that (4.3) implies the familiar balance identity

$$\zeta + F_c(y-\zeta) = c + F_c(y)$$

stating that *the project value (current revenues + future dividend potential) should be equal to its full costs (transaction costs c + lost option value $F_c(y)$).* This observation is of interest since it clearly indicates that *the balance identity is an intrinsic property of the considered class of admissible policies and, therefore, is independent of the optimality of the proposed policy.*

Given the definition of the value $F_c(x)$, define now the mapping $h : \mathbb{R}_+^2 \mapsto \mathbb{R}$ as

$$h(\zeta, y) = \frac{(\zeta - c)}{\psi(y) - \psi(y-\zeta)} \quad (4.4)$$

and consider the ordinary inequality constrained non-linear programming problem

$$\sup_{\zeta \in [0,y], y \in \mathbb{R}_+} h(\zeta, y). \quad (4.5)$$

If an admissible pair (ζ_c^*, y_c^*) maximizing the mapping $h(\zeta, y)$ exists, denote the value of the admissible dividend strategy characterized by this pair as

$$F_c^*(x) = \begin{cases} x - y_c^* + h(\zeta_c^*, y_c^*)\psi(y_c^*) & x \geq y_c^* \\ h(\zeta_c^*, y_c^*)\psi(x) & x < y_c^*. \end{cases} \tag{4.6}$$

It is now clear that if an admissible interior pair (ζ_c^*, y_c^*) maximizing the mapping $h(\zeta, y)$ exists, then the ordinary first order necessary conditions

$$\psi(y_c^*) - \psi(y_c^* - \zeta_c^*) = \psi'(y_c^* - \zeta_c^*)(\zeta_c^* - c) \tag{4.7}$$
$$\psi'(y_c^*) = \psi'(y_c^* - \zeta_c^*) \tag{4.8}$$

have to be satisfied. Consequently, we find that at the optimum $h(\zeta_c^*, y_c^*) = 1/\psi'(y_c^* - \zeta_c^*) = 1/\psi'(y_c^*)$ and, therefore, that

$$F_c^*(x) = \begin{cases} x - y_c^* + \frac{\psi(y_c^*)}{\psi'(y_c^*)} & x \geq y_c^* \\ \frac{\psi(x)}{\psi'(y_c^*)} & x < y_c^*. \end{cases} \tag{4.9}$$

It is worth observing that the necessary condition (4.8) implies that if an interior solution of the non-linear programming problem (4.5) exists, then by *Rolle's theorem* there has to be at least one state $\hat{x} \in (y_c^* - \zeta_c^*, y_c^*)$ where the marginal value $F^{*\prime}_c(x)$ attains an extreme value and, therefore, where $\psi''(\hat{x}) = 0$. Moreover, it is also clear that $F_c^*(x)$ belongs into the class of mappings considered in Theorem 3.2 and, therefore, that $F_c^*(x) \leq K(x)$ and $F^{*\prime}_c(x) \leq K'(x)$ whenever a unique pair satisfying the necessary conditions (4.7) and (4.8) exists and is unique.

4.2 EXISTENCE AND SUFFICIENCY

Having derived a set of necessary conditions from which the potentially optimal dividend threshold and dividend policy can be derived, we now plan to state a set of general and considerably weak conditions under which these optimal variables exist and are unique, and under which the derived auxiliary mapping indeed constitutes the value of the optimal dividend policy. A set of general conditions under which the necessary conditions (4.7) and (4.8) admit a unique solution is now summarized in the following.

Lemma 4.1. *Assume that* $\rho \in \mathscr{L}^1(\mathbb{R}_+)$ *and that* $\lim_{x\to\infty} \rho(x) = -\infty$. *Assume also that either the conditions (i) or conditions (ii) of Lemma 3.1 are satisfied and that* $\lim_{x\downarrow 0} \psi'(x) = \infty$. *Then there is a unique optimal pair* (ζ_c^*, y_c^*) *satisfying the necessary conditions (4.7) and (4.8) for all* $c \in \mathbb{R}_+$.

Proof. See Appendix C. □

Lemma 4.1 presents a set of typically satisfied conditions under which a pair (ζ_c^*, y_c^*) maximizing the mapping $h(\zeta, y)$ and satisfying the necessary conditions (4.7) and (4.8) exists and is unique for all $c \in \mathbb{R}_+$. It is worth pointing out that the conditions of Lemma 4.1 are typically satisfied in the cases where exogenous liquidation is impossible (i.e. when the lower boundary 0 is unattainable for X_t). In the presence of potential liquidation risk (i.e. whenever 0 is attainable for X_t) we typically have that $\psi'(0) < \infty$ and, therefore, that the conditions of Lemma 4.1 are no longer satisfied. A set of conditions extending the results of Lemma 4.1 to that case as well are presented in the following.

Lemma 4.2. *Assume that* $\rho \in \mathscr{L}^1(\mathbb{R}_+)$ *and that* $\lim_{x\to\infty} \rho(x) = -\infty$. *Assume also that the conditions (ii) of Lemma 3.1 are satisfied and that* $\lim_{x\downarrow 0} \psi'(x) < \infty$. *Then, there is a critical cost* $\hat{c}$ *such that there is a unique optimal pair* (ζ_c^*, y_c^*) *satisfying the necessary conditions (4.7) and (4.8) whenever* $0 < c < \hat{c}$.

Proof. See Appendix D. □

As is now clear from Lemma 4.2, there is a maximal admissible transaction cost $\hat{c}$ under which the necessary conditions (4.7) and (4.8) are satisfied whenever $\psi'(0) < \infty$. Since this condition arises typically in cases where the underlying boundary is attainable for the underlying diffusion modelling the reserves, we find that *the risk of potential liquidation results into a maximal admissible transaction cost*. As we will later observe, this critical cost can be interpreted as the maximal cost the firm is prepared to incur in order to pay out dividends sequentially in first place. Our first result characterizing the optimal dividend policy and its value is now stated in the following.

Theorem 4.3. *Assume that the conditions of either Lemma 4.1 or Lemma 4.2 are satisfied. Then, the optimal lump sum dividend policy is* $\nu^* = \nu_{(y_c^*, \zeta_c^*)}$ *and its value reads as* $V_c(x) = F_c^*(x)$.

Proof. See Appendix E. □

Theorem 4.3 demonstrates that the conditions of both Lemma 4.1 and Lemma 4.2 are actually sufficient for guaranteeing that the mapping $F_c^*(x)$ indeed constitutes the maximal attainable expected cumulative present value of the future dividend payments. As intuitively is clear, the optimal dividend policy is completely characterized by the optimal threshold y_c^* at which a lump-sum dividend ζ_c^* is paid out. Hence, the state $y_c^* - \zeta_c^*$ can be viewed as a generic initial state at which the diffusion process modelling the retained profits is restarted after the dividends have been paid out. An important set of results characterizing the relationship between the associated optimal dividend problems is now presented in the following.

Theorem 4.4. *Assume that the conditions of either Lemma 4.1 or Lemma 4.2 are satisfied. Then,*

$$K(x) \geq V_c(x) \geq G_c(x) \quad \textit{and} \quad K'(x) \geq V_c'(x) \geq G_c'(x),$$

where

$$G_c(x) = \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau}(X_\tau - c) \right] = \psi(x) \sup_{y \geq x} \left[\frac{y-c}{\psi(y)} \right] = \begin{cases} x - c & x \geq \overline{x}_c \\ \frac{\psi(x)}{\psi'(\overline{x}_c)} & x < \overline{x}_c, \end{cases} \tag{4.10}$$

denotes the value of an associated optimal stopping problem and $\overline{x}_c > c$ is the unique root of equation $\psi(\overline{x}_c) = (\overline{x}_c - c)\psi'(\overline{x}_c)$. Moreover, $\overline{x}_c > y_c^* > \hat{x}$ for all admissible costs $c > 0$.

Proof. See Appendix F. □

Theorem 4.4 demonstrates that given the conditions of either Lemma 4.1 or Lemma 4.2 both the values and the marginal values of the considered dividend optimization problems are completely ordered. More precisely, Theorem 4.4 proves that the value of the associated singular stochastic control problem dominates the value of the stochastic impulse control problem which, in turn, dominates the value of the associated optimal stopping problem (single dividend payment). An important implication of this finding is that the optimal dividend threshold associated with the single dividend payment dominates the optimal dividend threshold of the sequential lump-sum dividend policy which, in turn, dominates the optimal dividend threshold of the optimal singular dividend policy. Hence, Theorem 4.4 shows that the required exercise premium is highest in the single dividend payment case and lowest in the singular dividends case. Along the lines of our Theorem 3.2, Theorem 4.4 also proves that the marginal value of the reserves in the presence of a singular dividend policy is higher than in the presence of a sequential lump-sum dividend policy which, in turn, dominates the marginal value of the reserves in the single dividend payment case. Since the marginal value of the optimal dividend policy can be interpreted as the (marginal) *Tobin's q* associated to the particular cash flow management problem, we find that according to the findings of Theorem 4.4 *increased dividend payment flexibility does not only increase the value of the optimal policy, it also increases the marginal benefits (and, therefore, Tobin's marginal q) associated to the increased flexibility*. Another important result illustrating the importance of the risk of potential liquidation is now summarized in the following.

Theorem 4.5. *Assume that the conditions of Lemma 4.2 are satisfied and assume that $c \geq \hat{c}$, where the critical cost $\hat{c}$ satisfies the condition $\hat{c} = -\theta(\hat{y}(0))$, where $\hat{y}(0) > \hat{x}$ satisfies the equation $\psi'(y) = \psi'(0)$. Then, the optimal policy is $\nu^* = \nu_{(\bar{x}_c, \bar{x}_c)}$ (i.e. instantaneous liquidation at $\bar{x}_c$) and its value reads as $V_c(x) = G_c(x)$.*

Proof. As was established in the proof of Theorem 4.4, the value of the optimal stopping policy reads can be expressed as in (4.10). Since $\bar{x}_c$ is attained on the set where $\psi(x)$ is convex and $\psi'(0) < \psi'(\bar{x}_c)$ we find that $G_c'(x) \leq 1$ for all $x \in \mathbb{R}_+$. Consequently, we observe that $G_c(x)$ satisfies the quasi-variational inequality $G_c(x) \geq (x-c) + \sup_{y \in [0,x]}[G_c(y) - y] = x - c$. Since $G_c(x)$ satisfies the condition $(\mathscr{A}G_c)(x) \leq rG_c(x)$ for all $x \in \mathbb{R}_+ \setminus \{\bar{x}_c\}$ as well, we find that $G_c(x) \geq V_c(x)$ proving that $G_c(x) = V_c(x)$. Finally, since the policy $(\zeta_c^*, y_c^*) = (\bar{x}_c, \bar{x}_c)$ and the stopping time $\tau^* = \inf\{t \geq 0 : X_t \geq \bar{x}_c\}$ are admissible, and $V_c(x)$ is attained by implementing this policy, we find that $\nu_{(\bar{x}_c, \bar{x}_c)}$ is optimal. □

Theorem 4.5 demonstrates that the presence of potential liquidation risk (in the sense that the underlying reserve may vanish in finite time even in the absence of a dividend strategy) results into a maximal admissible cost at which the sequential payment of dividends becomes suboptimal. In that case, the problem can actually be interpreted as an optimal liquidation problem where the objective of a rationally managed corporation is only to determine the optimal exercise threshold at which the firm should be liquidated and all the retained profits should be instantaneously paid out as dividends. An interesting special case where liquidation is also the optimal policy is presented in the next corollary.

Corollary 4.6. *Assume that* $\lim_{x\downarrow 0}\mu(x) \leq 0$*, that the net appreciation rate* $\rho(x)$ *is non-increasing, and that* $\lim_{x\to\infty}\rho(x) < -rc$*. Then, the optimal policy is* $\nu^* = \nu_{(\bar{x}_c,\bar{x}_c)}$ *(i.e. instantaneous liquidation at* $\bar{x}_c$*) and its value reads as* $V_c(x) = G_c(x)$*.*

Proof. See Appendix G. □

Corollary 4.6 states a set of conditions under which the sequential payment of dividends is suboptimal and, therefore, under which the value of the considered stochastic impulse control problem coincides with the value of the associated optimal stopping problem corresponding to the optimal liquidation of the firm. It is worth observing that this case arises in situations where the net appreciation rate is negative and, therefore, in cases where the optimal dividend strategy is to liquidate the corporation immediately and pay out all the reserves instantaneously (the so-called *take the money and run-strategy*).

Our main results on the sensitivity of the optimal policy and its value to changes in the transaction cost c are now summarized in the following.

Theorem 4.7. *Assume that the conditions of either Lemma 4.1 or Lemma 4.2 are satisfied. Then,*

$$\begin{aligned}
\frac{dy_c^*}{dc} &= \frac{\psi'(y_c^*)}{\psi''(y_c^*)(\zeta_c^* - c)} > 0 \\
\frac{d\zeta_c^*}{dc} &= \frac{\psi'(y_c^*)(\psi''(y_c^* - \zeta_c^*) - \psi''(y_c^*))}{\psi''(y_c^* - \zeta_c^*)\psi''(y_c^*)(\zeta_c^* - c)} > 0.
\end{aligned}$$

Moreover, $\lim_{c\downarrow 0} y_c^* = \hat{x}$, $\lim_{c\downarrow 0}\zeta_c^* = 0$, $\lim_{c\downarrow 0} dy_c^*/dc = \infty$*, and*

$$\lim_{c\downarrow 0}\frac{\partial V_c}{\partial c}(x) = -\infty \tag{4.11}$$

for all $x \in \mathbb{R}_+$*.*

Proof. The comparative statics of the optimal variables y_c^* and ζ_c^* can be obtained from the ordinary first order conditions (4.7) and (4.8) by implicit differentiation. The limits $\lim_{c\downarrow 0} y_c^* = \hat{x}$ and $\lim_{c\downarrow 0}\zeta_c^* = 0$ follow directly from the proofs of Lemma 4.1 and Lemma 4.2. The continuity of $\psi'(x)$, $\psi''(x)$, y_c^*, and ζ_c^* then imply that $\lim_{c\downarrow 0} dy_c^*/dc = \infty$ since $\lim_{c\downarrow 0}\psi''(y_c^*) = \psi''(\lim_{c\downarrow 0} y_c^*) =$

$\psi''(\hat{x}) = 0$, $\lim_{c\downarrow 0} \zeta_c^* - c = 0$, and $\lim_{c\downarrow 0} \psi'(y_c^*) = \psi'(\lim_{c\downarrow 0} y_c^*) = \psi'(\hat{x}) > 0$. It remains to establish the limit (4.11). Standard differentiation yields that

$$\frac{\partial V_c}{\partial c}(x) = -\frac{\min(\psi(x), \psi(y_c^*))}{\psi'(y_c^*)(\zeta_c^* - c)}$$

which finally implies (4.11). □

Theorem 4.7 establishes the intuitively clear result that increased transaction costs do not only increase the required exercise premium by increasing the optimal threshold at which dividends should optimally be paid out but simultaneously increase the size of the optimal dividend policy. An interesting implication of this conclusion is that *increased transaction costs should result into larger but less frequent dividends*. Moreover, the impact of the transaction costs on the value of the optimal policy is dramatic in the sense that the sensitivity of the value function with respect to changes in the costs becomes unbounded as the transaction costs tend to zero (see, for example, Øksendal 1999 and Øksendal 2000).

5 ILLUSTRATION

5.1 BROWNIAN MOTION WITH DRIFT

In order to illustrate our results in the presence of potential liquidation risk, we now assume that in the absence of interventions the underlying diffusion evolves according to a Brownian motion with drift characterized by the stochastic differential equation

$$dX_t = \mu dt + \sigma dW_t \quad X_0 = x.$$

In this case, $\psi(x) = e^{\kappa x} - e^{\lambda x}$ where

$$\kappa = -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} > 0$$

and

$$\lambda = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0$$

denote the positive and the negative root of the characteristic equation $\sigma^2 b^2 + 2\mu b - 2r = 0$, respectively. In this case the conditions of Lemma 4.2 are satisfied and, therefore, there is a critical cost $\hat{c}$ such that there is a unique optimal pair (ζ_c^*, y_c^*) satisfying the necessary conditions (4.7) and (4.8) whenever

$0 < c < \hat{c}$. In this case, the critical threshold at which $\psi''(x)$ vanishes reads as

$$\hat{x} = \frac{1}{(\kappa - \lambda)} \ln\left(\frac{\lambda^2}{\kappa^2}\right).$$

This example is illustrated numerically in Table 1 for various values of the volatility coefficient σ (with $\mu = 0.1$, $r = 0.025$, and $c = 0.1$).

σ	0.1	0.2	0.3	0.4	0.5
$\hat{x}$	0.43	1.13	1.78	2.30	2.68
$\overline{x}_c$	4.15	4.29	4.51	4.77	5.03
y_c^*	1.26	1.93	2.64	3.25	3.74
ζ_c^*	0.99	1.15	1.39	1.64	1.88
ζ_c^*/y_c^*	0.79	0.60	0.53	0.50	0.50
$y_c^* - \zeta_c^*$	0.27	0.78	1.25	1.61	1.86

Table 1 The Optimal Thresholds, Intervention Size, and Generic Initial State

Along the lines of previous studies considering the determination of a rational dividend strategy Table 1 clearly indicates that increased volatility increases the required exercise premium in all cases and, therefore, that increased volatility leads to the postponement of dividends. However, it is also worth noticing that our numerical results seem to indicate that increased volatility increases the optimal size of the paid out dividends at a lower rate than it increases the optimal exercise threshold. Thus, our findings show that *increased volatility increases the generic initial state and, therefore, leads to a higher capital requirement in terms of the reserves*. Although this result is intuitively clear, it is of importance since it demonstrates that *the presence of liquidation risk should result into greater capital buffers*. The critical cost $\hat{c}$ at which liquidation becomes optimal is illustrated numerically in Table 2 for various values of the volatility coefficient σ (with $\mu = 0.1$ and $r = 0.025$).

σ	0.1	0.2	0.3	0.4	0.5
$\hat{c}$	13.84	8.94	6.47	4.97	3.96

Table 2 The critical cost $\hat{c}$ as a function of volatility

The results of Table 2 indicate that increased volatility decreases the critical cost $\hat{c}$. Therefore, our numerical results support the intuitively clear result that increased liquidation risk decrease the maximal admissible transaction cost under which the sequential payment of dividends can be sustained.

In order to illustrate the results of Theorem 4.4 as well, we illustrate the values of the optimal dividend policies in Figure 1 and the marginal values of these policies in Figure 2 under the assumption that $\mu = 0.1$, $r = 0.025$, $c = 0.1$, and $\sigma = 0.3$. As was established in Theorem 4.4 we observe from these figures that both $K(x) \geq V_c(x) \geq G_c(x)$ and $K'(x) \geq V_c'(x) \geq G_c'(x)$.

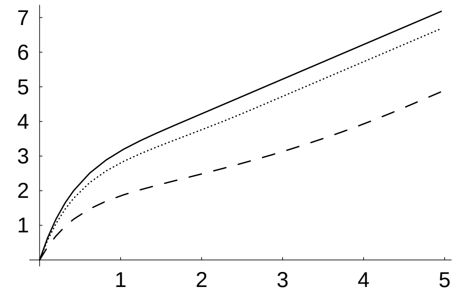

Figure 1. The Value Functions $K(x)$, $V_c(x)$, and $G_c(x)$

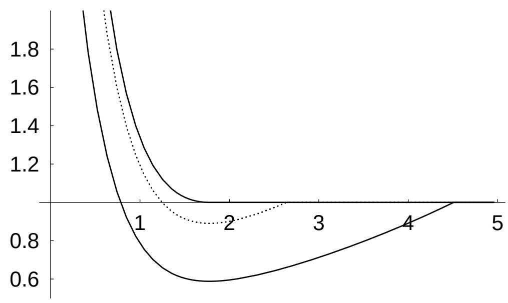

Figure 2. The Marginal Values $K'(x)$, $V_c'(x)$, and $G_c'(x)$

5.2 LOGISTIC DIFFUSION

In order to illustrate our results in the case where exogenous liquidation is impossible, we now assume that in the absence of interventions the cash flow evolves according to a logistic mean reverting diffusion characterized by the stochastic differential equation

$$dX_t = \mu X_t(1-\gamma X_t)dt + \sigma X_t dW_t \quad X_0 = x.$$

In this case, $\psi(x) = x^{\eta} M(\eta, 1+\eta-\alpha, 2\mu\gamma x/\sigma^2)$, where M denotes the Kummer confluent hypergeometric function and

$$\eta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0$$

denotes the positive and

$$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0$$

denotes the negative root of the quadratic characteristic equation $\sigma^2 a(a-1) + 2\mu a - 2r = 0$. It is well-known that if $\mu > r$ then $\eta < 1$ and the conditions of our Theorem 4.1 are satisfied. Given the functional complexity of the increasing fundamental solution in this example we illustrate this example numerically in Table 3 for the parameter values $\mu = 0.1$, $r = 0.025$, $c = 1$, and $\gamma = 0.5$.

σ	0.1	0.2	0.3	0.4	0.5
$\hat{x}$	0.8	0.93	1.08	1.22	1.32
$\overline{x}_c$	2.16	2.72	3.31	3.84	4.31
y_c^*	2.05	2.54	3.08	3.62	4.13
ζ_c^*	1.96	2.41	2.90	3.42	3.93
ζ_c^*/y_c^*	0.96	0.95	0.94	0.94	0.95
$y_c^* - \zeta_c^*$	0.09	0.13	0.18	0.2	0.2

Table 3 The Optimal Thresholds, Intervention Size, and Generic Initial State

Table 3 shows that increased volatility decelerates the dividend policy by increasing the required exercise premium in all the cases. Comparing the results of Table 3 with the results of Table 1 leads to two interesting and intuitively clear conclusions. First, the presence of liquidation risk naturally leads to a higher generic initial state and, therefore, to a larger capital buffer (even in proportional terms). Second, the generic initial state increases as a function of the volatility of the underlying reserves at a faster rate in the presence of liquidation risk than in its absence. More precisely, the ratio between the optimal size of the paid out dividends and the optimal dividend threshold decreases at a relatively significant rate in the presence of liquidation risk while it remains relatively stable in its absence (i.e. the impact of increased volatility on the ratio between the optimal size of the paid out dividends and the optimal dividend threshold is ambiguous). Again, these observations support the view that the combination of potential liquidation risk and cash flow uncertainty is a significant factor affecting rational cash flow management policies.

6 CONCLUDING COMMENTS

In this study we considered the determination of the optimal lump-sum stochastic dividend payment policy for a broad class of linear diffusion modelling the random dynamics of the underlying cash reserves. Instead of tackling the dividend optimization problem directly via ordinary dynamic programming techniques we proposed an alternative approach based on the classical theory of diffusions. We first derived the expected cumulative present value of an arbitrary admissible dividend policy characterized by two constant boundaries. Namely, the boundary at which dividends are paid out and the boundary at which the cash flow process is subsequently restarted. In this way the original stochastic control problem was shown to become an ordinary two-dimensional static non-linear programming problem. We then presented in terms of the net appreciation rate of the underlying reserves a set of general conditions under which the existence and uniqueness of an optimal pair of boundaries characterizing the admissible dividend policy is always guaranteed. Interestingly, our results showed that the presence of liquidation risk results into a maximal admissible transaction cost below which the sequential payment of dividends is optimal. Above this critical cost the objective of the corporation was show to be the determination of the threshold at which it should be irreversibly liquidated.

We also considered two associated stochastic cash flow management problems and established that these values are ordered in an exceptionally strong way. More precisely, we found that the value of the associated singular stochastic control problem dominates the value of the impulse (lump-sum) control problem which, in turn, dominates the value of the associated optimal stopping problem. However, we also demonstrated that the marginal values (and, therefore, Tobin's q associated with these particular problems) are ordered in an analogous way. Hence, our results clearly indicate that increased flexibility does not only increase the value of a rationally managed corporation it also increases the rate at which this value grows.

While our results are considerably general, they are based on a model where the cash flow process is completely exogenous. Thus, our analysis overlooks the capital accumulation dynamics and financing decisions of a corporation. A natural way to extend our analysis would be to introduce endogenous capital accumulation and financial constraints into the model. Unfortunately, such an extension is out of the scope of the present study and is, therefore, left for future research.

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A PROOF OF LEMMA 2.1

Proof. As was established in Theorem D.1. in Øksendal 1999 (pp. 299-302) the conditions of our corollary guarantee that there is a sequence $\{g_n\}_{n=1}^{\infty}$ of mappings $g_n \in C^2(\mathbb{R}_+)$ such that (i) $g_n \to g$ uniformly on compact subsets of $\mathbb{R}_+$ as $n \to \infty$, (ii) $(\mathscr{A} g_n) - rg_n \to (\mathscr{A} g) - rg$ uniformly on compact subsets of $\mathbb{R}_+ \setminus \mathscr{D}$ as $n \to \infty$, and (iii) $\{(\mathscr{A} g_n) - rg_n\}_{n=1}^{\infty}$ is locally bounded on $\mathbb{R}_+$. Applying Itô's theorem to the mapping $(t,x) \mapsto e^{-rt} g_n(x)$, taking expectations, and reordering terms yields

$$e^{-r\tau_j} g_n(X_{\tau_j}^{\nu}) = \mathbf{E}\left[e^{-r\tau_{j+1}} g_n(X_{\tau_{j+1}-}^{\nu}) - \int_{\tau_j}^{\tau_{j+1}-} e^{-rs} \Phi_n(X_s^{\nu}) ds \middle| \mathscr{F}_{\tau_j} \right],$$

where $\Phi_n(x) = (\mathscr{A} g_n)(x) - rg_n(x)$. Letting $n \to \infty$, applying Fatou's theorem, and invoking the variational inequality $(\mathscr{A} g)(x) - rg(x) \leq 0$ yields

$$e^{-r\tau_j} g(X_{\tau_j}^{\nu}) \geq \mathbf{E}\left[e^{-r\tau_{j+1}} g(X_{\tau_{j+1}-}^{\nu}) \middle| \mathscr{F}_{\tau_j} \right].$$

Letting $\tau_0 = 0$ and summing terms from $j = 0$ to $j = n$ results in

$$g(x) \geq \mathbf{E}_x \left[e^{-r\tau_{n+1}} g(X_{\tau_{n+1}-}^{\nu}) \right] + \mathbf{E}_x \sum_{j=1}^{n} e^{-r\tau_j} \left[g(X_{\tau_j -}^{\nu}) - g(X_{\tau_j}^{\nu}) \right].$$

Since $X_{\tau_j}^{\nu} = X_{\tau_j -}^{\nu} - \zeta_j$ for any admissible strategy $\nu \in \mathscr{V}$ and the mapping $g(x)$ is non-negative and satisfies the quasi-variational inequality

$$g(x) \geq \sup_{\zeta \in [0,x]} [\zeta - c + g(x - \zeta)] \geq \zeta - c + g(x - \zeta)$$

we find that

$$g(x) \geq \mathbf{E}_x \sum_{j=1}^{n} e^{-r\tau_j} \left[g(X_{\tau_j -}^{\nu}) - g(X_{\tau_j -}^{\nu} - \zeta_j) \right] \geq \mathbf{E}_x \sum_{j=1}^{n} e^{-r\tau_j} (\zeta_j - c).$$

Since this inequality is valid for any admissible impulse control $\nu \in \mathscr{V}$, it has to be valid for the optimal as well proving that $g(x) \geq V_c(x)$. □

B PROOF OF LEMMA 3.1

Proof. As was established in Lemma 2.1 Alvarez (2004) our assumptions imply that

$$\frac{1}{2}\sigma^2(x) \frac{\psi''(x)}{S'(x)} = r \int_0^x \psi(y)\rho(y)m'(y)dy - \rho(x)\frac{\psi'(x)}{S'(x)}. \tag{B.1}$$

Denote now the right-hand side of equation (B.1) as $I(x)$. It is clear that under our assumptions $I(0) \leq 0$ and

$$I(x) \leq \rho(x) r \int_0^x \psi(y) m'(y) dy - \rho(x) \frac{\psi'(x)}{S'(x)} = -\rho(x) \frac{\psi'(0)}{S'(0)} \leq 0$$

for all $x \in (0, x^*)$. On the other hand, the assumed monotonicity of $\rho(x)$ on (x^*, ∞) and the assumption $\lim_{x\to\infty} \rho(x) < 0$ imply that there is a threshold $x_0 > x^*$ at which $\rho(x_0) = 0$ and, therefore, at which $I(x_0) > 0$. Combining this observation with the continuity and monotonicity of $I(x)$ on (x^*, ∞) then finally implies that equation $I(x) = 0$ and, therefore, that equation $\psi''(x) = 0$ has a unique root $\hat{x} \in (x^*, x_0)$ and that $\hat{x} = \text{argmin}\{\psi'(x)\}$. Consequently, we find that the proposed value function is monotonically increasing, concave, and satisfies the variational inequalities $\min\{J'(x) - 1, rJ(x) - (\mathscr{A}J)(x)\} = 0$ proving that it dominates the value of the singular stochastic control problem (3.1). However, since the proposed value can be attained by applying a *local time push-type* dividend strategy and the solution of the stochastic differential equation (3.2) subject to reflection at $\hat{x}$ exists and is unique (cf. Freidlin 1985, Section 1.6), we find that the proposed value function is indeed the value of the singular stochastic control problem (3.1). Moreover, since

$$\frac{d}{dx}\left[\frac{1}{\psi'(x)}\right] = -\frac{\psi''(x)}{\psi'^2(x)} \gtreqless 0, \quad x \lesseqgtr \hat{x}$$

we find that $K'(x)$ can be expressed as in (3.4). □

C PROOF OF LEMMA 4.1

Proof. Consider now the mappings $L_1 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ and $L_2 : \mathbb{R}_+^2 \mapsto \mathbb{R}$ defined as $L_1(z,y) = \theta(y) - \theta(z) + c$ and $L_2(z,y) = \psi'(y) - \psi'(z)$, where the continuously differentiable mapping $\theta : \mathbb{R}_+ \mapsto \mathbb{R}$ is defined as $\theta(x) = \psi(x)/\psi'(x) - x$. As was established in Lemma 3.1 and in Theorem 3.2 our assumptions imply that there is a unique threshold $\hat{x} = \text{argmin}\{\psi'(x)\} = \text{argmax}\{\theta(x)\} \in \mathbb{R}_+$ such that $\psi''(x) \lesseqgtr 0$ and $\theta'(x) \gtreqless 0$ for $x \lesseqgtr \hat{x}$. Moreover, as was established in Lemma 2.3 of Alvarez (2004) we now have

$$\frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)} = \int_0^x \psi(y) \rho(y) m'(y) dy$$

implying that $\theta(x)$ can be re-expressed as

$$\theta(x) = \left(\frac{\psi'(x)}{S'(x)}\right)^{-1} \int_0^x \psi(y)\rho(y)m'(y)dy.$$

The assumed boundary behavior of the underlying diffusion at ∞ implies that $\psi'(x)/S'(x) \to \infty$ and $\int_0^x \psi(y)\rho(y)m'(y)dy \downarrow -\infty$ as $x \to \infty$. Hence, L'Hospital's rule implies that $\lim_{x\to\infty} \theta(x) = \lim_{x\to\infty} \rho(x)/r = -\infty$. If 0 is attainable for the underlying diffusion X_t then $\lim_{x\downarrow 0} \psi'(x)/S'(x) > 0$ implying that $\lim_{x\downarrow 0} \theta(x) = 0$ in that case. On the other hand, if 0 is unattainable for X_t then $\lim_{x\downarrow 0} \frac{\psi'(x)}{S'(x)} = 0$ and *L'Hospital's rule* implies that $\lim_{x\downarrow 0} \theta(x) = \lim_{x\downarrow 0} \mu(x)/r \geq 0$. Consequently, we observe that for all $z \leq \hat{x}$ the mapping $L_1(z,y)$ satisfies the conditions $L_1(z,z) = c > 0$, $\lim_{y\to\infty} L_1(z,y) = -\infty$, and

$$\frac{\partial L_1}{\partial y}(z,y) = \theta'(y) \gtreqqless 0 \quad y \lesseqqgtr \hat{x}.$$

Therefore, we find that for all $z \leq \hat{x}$ there is a unique $\tilde{y}_c(z) \in (\hat{x},\infty)$ satisfying the equation $L_1(z,\tilde{y}_c(z)) = 0$. Moreover, we also find that $\tilde{y}_c(0+) < \infty$, $\tilde{y}_c(\hat{x}) > \hat{x}$, and

$$\tilde{y}_c'(z) = \frac{\theta'(z)}{\theta'(\tilde{y}_c(z))} = \frac{\psi(z)\psi'^2(\tilde{y}_c(z))\psi''(z)}{\psi(\tilde{y}_c(z))\psi'^2(z)\psi''(\tilde{y}_c(z))} < 0$$

Consider now, in turn, the mapping $L_2(z,y)$. The strict convexity of $\psi(x)$ on $(\hat{x},\infty)$ and the mean value theorem imply that $\psi'(x) \to \infty$ as $x \to \infty$. Consequently, we find that for all $z \in (0,\hat{x})$ we have $L_2(z,z) = 0$, $\lim_{y\to\infty} L_2(z,y) = \infty$, and

$$\frac{\partial L_2}{\partial y}(z,y) = \psi''(y) \lesseqqgtr 0 \quad y \lesseqqgtr \hat{x}.$$

Therefore, we again find that for all $z \leq \hat{x}$ there is a unique $\hat{y}(z) \in [\hat{x},\infty)$ satisfying the equation $L_2(z,\hat{y}(z)) = 0$. Moreover, we also find that $\hat{y}(\hat{x}) = \hat{x} < \tilde{y}_c(\hat{x})$, and

$$\hat{y}'(z) = \frac{\psi''(z)}{\psi''(\hat{y}(z))} < 0.$$

Given these findings, we observe that if $\psi'(0+) = \infty$ then $\hat{y}(0+) = \infty > \tilde{y}_c(0+)$ and, therefore, that equation $\hat{y}(z) - \tilde{y}_c(z) = 0$ has at least at one root $z^* \in (0,\hat{x})$. Since

$$\tilde{y}_c'(z^*) = \frac{\psi(z^*)\psi''(z^*)}{\psi(\tilde{y}_c(z^*))\psi''(\tilde{y}_c(z^*))} > \frac{\psi''(z^*)}{\psi''(\tilde{y}_c(z^*))} = \hat{y}'(z^*)$$

we find that z^* is unique. □

D PROOF OF LEMMA 4.2

Proof. To establish the existence of the critical cost $\hat{c}$ we first consider the mappings

$$\begin{aligned} f_1(x) &= \psi(x) - \psi'(x)x \\ f_2(x) &= \psi'(x) - \psi'(0). \end{aligned}$$

It is now clear that our assumptions and the results of Lemma 3.1 imply that $f_1(0) = f_2(0) = 0$ and that $\hat{x} = \operatorname{argmax}\{f_1(x)\} = \operatorname{argmin}\{f_2(x)\}$. Moreover, standard differentiation implies that $f_1'(x) = -x f_2'(x)$. Integrating this equation from 0 to x and applying integration by parts then yields

$$f_1(x) = \int_0^x f_2(y)dy - x f_2(x).$$

As in the proof of Lemma 4.1 denote now as $\hat{y}(0) < \infty$ (since $\psi'(0) < \infty$) the interior root of equation $f_2(x) = 0$. Then

$$f_1(\hat{y}(0)) = \int_0^{\hat{y}(0)} f_2(y)dy < 0$$

implying that $\hat{y}(0) > \tilde{y}_0(0)$, where $\tilde{y}_0(0)$ denotes the root of the interior root of equation $\theta(x) = 0$ (which, by definition, coincides with the interior root of equation $f_1(x) = 0$). Since $\partial\tilde{y}_c(0)/\partial c > 0$ we finally observe that there is a critical $\hat{c} > 0$ such that $\hat{y}(0) > \tilde{y}_c(0)$ for all $c < \hat{c}$. However, since $\hat{y}(\hat{x}) < \tilde{y}_c(\hat{x})$ the existence and uniqueness of the root z^* of equation $\hat{y}(z) - \tilde{y}_c(z) = 0$ follows from the proof of Lemma 4.1. □

E PROOF OF THEOREM 4.3

Proof. It is now sufficient to establish that the proposed value satisfies the sufficient quasi-variational inequalities, since the admissibility of the considered class of impulse controls naturally implies that $V_c(x) \geq F_c^*(x)$. We first observe that $F_c^* \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{y_c^*\})$, that $F_c^{*\prime\prime}(y_c^*+) = 0$, and that $F_c^{*\prime\prime}(y_c^*-) = h(\zeta_c^*, y_c^*)\psi''(y_c^*) < \infty$. Moreover, since continuous mappings are bounded on compacts and $X_t^\nu \in (0, y_c^*)$ except for a t-set of measure zero we find that $\lim_{t\to\infty} \mathbf{E}_x[e^{-rt} F_c^*(X_t^\nu)] = 0$ for all $x \in \mathbb{R}_+$. Define the mapping $A_1 : \mathbb{R}_+ \setminus \{y_c^*\} \mapsto$

$\mathbb{R}$ as $A_1(x) = (\mathscr{A}F_c^*)(x) - rF_c^*(x)$. It is clear that

$$A_1(x) = \begin{cases} \mu(x) - r\left(x - y_c^* + \frac{\psi(y_c^*)}{\psi'(y_c^*)}\right) & x > y_c^* \\ 0 & x < y_c^* \end{cases}$$

implying that

$$\lim_{x \downarrow y_c^*} A_1(x) = \frac{1}{\psi'(y_c^*)}\left[\mu(y_c^*)\psi'(y_c^*) - r\psi(y_c^*)\right] = -\frac{1}{2}\sigma^2(y_c^*)\frac{\psi''(y_c^*)}{\psi'(y_c^*)} < 0$$

since y_c^* is attained on the set where $\psi(x)$ is convex. However, since $A_1(x) = \rho(x) - r\left(\psi(y_c^*)/\psi'(y_c^*) - y_c^*\right)$ for all $x > y_c^*$ and y_c^* is on the set where the net appreciation rate $\rho(x)$ is decreasing, we find that $A_1(x) \leq 0$ for all $x \in \mathbb{R}_+ \setminus \{y_c^*\}$. It now remains to establish that $F_c^*(x)$ satisfies the quasi-variational inequality $F_c^*(x) \geq \sup_{\zeta \in [0,x]}[\zeta - c + F_c^*(x - \zeta)]$ for all $x \in \mathbb{R}_+$. To accomplish this task, we first re-express this quasi-variational inequality as $F_c^*(x) \geq x - c + \sup_{y \in [0,x]}[F_c^*(y) - y]$. Define now the mapping $A_2 : \mathbb{R}_+ \mapsto \mathbb{R}$ as

$$A_2(x) = F_c^*(x) - (x - c) - \sup_{y \in [0,x]}[F_c^*(y) - y].$$

Since $\psi'(x)/\psi'(y_c^*) < 1$ for all $x \in (y_c^* - \zeta_c^*, y_c^*)$ we first observe that

$$\sup_{y \in [0,x]}[F_c^*(y) - y] = \begin{cases} F_c^*(y_c^* - \zeta_c^*) - (y_c^* - \zeta_c^*) & x > y_c^* - \zeta_c^* \\ F_c^*(x) - x & x \leq y_c^* - \zeta_c^*. \end{cases}$$

Consequently, we find that

$$A_2(x) = \begin{cases} 0 & x \geq y_c^* \\ \frac{\psi(x) - \psi(y_c^*)}{\psi'(y_c^*)} + y_c^* - x & x \in (y_c^* - \zeta_c^*, y_c^*) \\ c & x \leq y_c^* - \zeta_c^*. \end{cases}$$

Since $\lim_{x \to y_c^* -} A_2(x) = 0$ and $A_2'(x) = \frac{\psi'(x)}{\psi'(y_c^*)} - 1 < 0$ on $(y_c^* - \zeta_c^*, y_c^*)$ we find that $A_2(x) > 0$ on $(y_c^* - \zeta_c^*, y_c^*)$ and, therefore, that $A_2(x) \geq 0$ for all $x \in \mathbb{R}_+$. Thus, $F_c^*(x) \geq V_c(x)$ implying that $F_c^*(x) = V_c(x)$ and, therefore, that $\nu^* = \nu_{(y_c^*, \zeta_c^*)}$. □

F PROOF OF THEOREM 4.4

Proof. Inequality $K(x) \geq V_c(x)$ follows directly from Theorem 3.2 and the representation (4.9). On the other hand, as was established in the proof of

Theorem 4.3, the value function $V_c(x)$ is continuously differentiable on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_+ \setminus \{y_c^*\}$ and satisfies the variational inequality $(\mathscr{A}V_c)(x) - rV_c(x) \leq 0$ for all $x \in \mathbb{R}_+ \setminus \{y_c^*\}$. Moreover, since $V_c(x) \geq \sup_{\zeta \leq x}[\zeta - c + V_c(x-\zeta)] \geq x - c$ we observe that $V_c(x)$ constitutes a r-excessive majorant of the exercise payoff $x - c$ and, therefore, that

$$V_c(x) \geq \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau}(X_\tau - c) \right].$$

Establishing equation (4.10) is analogous with the proof of Lemma 3.3. It is now clear from the proof of Lemma 4.1 and Lemma 4.2 that $y_c^* > \hat{x}$. Moreover, for all $x \in (0, \min(y_c^*, \bar{x}_c))$ the inequality

$$V_c(x) - G_c(x) = \frac{(\psi'(\bar{x}_c) - \psi'(y_c^*))\psi(x)}{\psi'(y_c^*)\psi'(\bar{x}_c)} \geq 0$$

implies that $\bar{x}_c > y_c^*$ since both thresholds are attained on the set where $\psi(x)$ is convex. It remains to establish that $K'(x) \geq V_c'(x) \geq G_c'(x)$. The inequality $K'(x) \geq V_c'(x)$ follows directly from Theorem 3.2. Since $\bar{x}_c > y_c^* > \hat{x}$ we find that

$$V_c'(x) - G_c'(x) = \begin{cases} 0 & y_c^* < \bar{x}_c \leq x \\ (\psi'(\bar{x}_c) - \psi'(x))/\psi'(\bar{x}_c) & y_c^* \leq x < \bar{x}_c \\ (\psi'(\bar{x}_c) - \psi'(y_c^*))\psi'(x)/(\psi'(y_c^*)\psi'(\bar{x}_c)) & x < y_c^* < \bar{x}_c \end{cases}$$

which is non-negative since the thresholds $\bar{x}_c$ and y_c^* are attained on the set where $\psi(x)$ is convex. □

G PROOF OF COROLLARY 4.6

Proof. We first observe that under the assumptions of our Corollary $\rho(x) \leq 0$ for all $x \in \mathbb{R}_+$ since $\rho(0+) = \mu(0+) \leq 0$ and $\rho(x)$ is non-increasing. On the other hand, the assumed monotonicity of the net appreciation rate $\rho(x)$ and equation (B.1) imply that

$$\frac{1}{2}\sigma^2(x)\frac{\psi''(x)}{S'(x)} \geq \rho(x)\left(\frac{\psi'(x)}{S'(x)} - \frac{\psi'(0)}{S'(0)}\right) - \rho(x)\frac{\psi'(x)}{S'(x)} = -\rho(x)\frac{\psi'(0)}{S'(0)} \geq 0$$

proving that the convexity of the increasing fundamental solution $\psi(x)$. Establishing that the value of the optimal stopping policy reads as in (4.10) is now analogous with the proof of Lemma 3.3. Combining this observation with the convexity of the increasing fundamental solution implies that $G_c(x)$ is convex and satisfies the inequality $G_c'(x) \leq 1$ for all $x \in \mathbb{R}_+$. Consequently, we notice that $G_c(x)$ satisfies the quasi-variational inequality $G_c(x) \geq (x-c) + \sup_{y\in[0,x]}[G_c(y) - y] = x - c$. Since $G_c(x)$ satisfies the condition $(\mathscr{A} G_c)(x) \leq rG_c(x)$ for all $x \in \mathbb{R}_+ \setminus \{\bar{x}_c\}$ as well, we find that $G_c(x) \geq V_c(x)$ proving that $G_c(x) = V_c(x)$. Finally, since the policy $(\zeta_c^*, y_c^*) = (\bar{x}_c, \bar{x}_c)$ and the stopping time $\tau^* = \inf\{t \geq 0 : X_t \geq \bar{x}_c\}$ are admissible, and $V_c(x)$ is attained by implementing this policy, we find that $\nu_{(\bar{x}_c,\bar{x}_c)}$ is optimal. □

PAPER II

Luis Alvarez - Jukka Lempa: *On the Optimal Stochastic Impulse Control of Linear Diffusions*, 2006

ON THE OPTIMAL STOCHASTIC IMPULSE CONTROL OF LINEAR DIFFUSIONS

Luis H. R. Alvarez and Jukka Lempa

ABSTRACT

We consider a class of stochastic impulse control problems of linear diffusions arising in studies considering the determination of optimal dividend policies and in studies analyzing the optimal management of renewable resources. We state a set of weak conditions guaranteeing both the existence and uniqueness of the boundary characterizing the optimal policy and its value. We also analyze two associated stochastic control problems and establish a general ordering for both the values and the marginal values of the considered stochastic control problems. In this way we extend previous findings obtained by relying on linear payoff characterizations.

Keywords: Stochastic impulse and singular control, optimal stopping, diffusions.

1 INTRODUCTION

A stochastic impulse control policy can be characterized by two factors. These factors are the sequence of random dates at which the policy is exercised and the sequence of impulses describing the magnitude of the applied policies. Thus, solving an impulse control problem typically involves the consideration of two endogenously determined variables: the timing and size of an impulse policy. For example, in most forest economic applications of stochastic impulse control the implemented impulse size is constrained by an exogenously determined generic initial state at which the underlying stochastic process is restarted after the forest has been harvested (see, for example, Alvarez 2004a, Alvarez 2004b, Alvarez and Koskela 2006a, Alvarez and Koskela 2006b, Sødal 2002, and Willassen 1998). Hence, in those models the only endogenous variable determining the size of the optimal policy is the single boundary at which the irreversible policy is optimally exercised. On the other hand, most capital theoretic and cash flow management applications of impulse control are based on models where both the exercise boundary at which the impulse policy is exercised and the generic initial state at which the controlled process is restarted after the irreversible policy has been exerted have to be simultaneously determined (see, for example, Alvarez and Virtanen 2006, Bar-Ilan et al. 2004, Bayraktar and Egami 2006, Cadenillas et al. 2005, Mundaca and Øksendal 1998, Peura and Keppo 2006; see also Korn 1999 for an excellent survey on stochastic impulse control applications in finance). Given the general applicability of stochastic impulse control models in various fields, it is not surprising that the mathematical analysis of such problems is well-established (see, for example, Brekke and Øksendal 1994, Brekke and Øksendal 1996, Harrison et al. 1983, Menaldi and Rofman 1983, Øksendal A. 2000a, Øksendal 1999; see also Bensoussan and Lions 1984 for a seminal textbook on quasi-variational inequalities and impulse control). In most cases the impulse control problem is studied by relying on a combination of variational and quasi-variational inequalities. Even though that approach is general and applies in the multidimensional setting as well, it typically results into functional inequalities which, depending naturally on the explicit form of the considered problem, may be relatively difficult to analyze and in that way difficult to interpret in terms of the particular application.

Given the arguments mentioned above, we consider in this study a class of stochastic impulse control problems of linear time-homogenous diffusion processes arising, among others, in various cash flow management applications and in studies on the rational management of renewable resources. As usually, we assume that the decision maker has to choose both the timing and the size of the optimal policy affecting the dynamics of the underlying diffusion. We generalize the analysis of the study Alvarez and Virtanen 2006 in two ways. First, instead of relying on a simple linear and state-independent exercise payoff, we introduce a state-dependent and potentially non-linear cash flow term measuring the revenue flow accrued from continuing operation (in capital theoretic applications of impulse control this flow can be interpreted either as the short-run profit flow or as a continuous dividend stream and in forest economics this flow term is typically interpreted as the flow of revenues accrued from amenity services; cf. Alvarez and Koskela 2006a). This extension is of interest, since as our analysis clearly demonstrates, in the presence of a state-dependent and potentially non-linear cash flow no strong concavity requirements are needed in order to guarantee both the existence and uniqueness of an optimal policy. This is a result which is in sharp contrast with the findings of the linear state-independent exercise payoff case studied in Alvarez and Virtanen 2006. Second, in order to model the potential imperfect controllability of the underlying stochastic dynamics, we also consider situations where an arbitrary admissible impulse may result into a jump discontinuity which is either greater or smaller than the size of the actual impulse (such configurations typically arise in models considering the effects of taxation or other financial frictions on rational cash flow management). Although this imperfection is modelled as a linear function of the applied impulse control policy, it has a profound impact on both the optimal policy and its value since it affects the required rate of return and, therefore, the marginal value of the optimal policy in a non-linear fashion.

Instead of analyzing the considered class of stochastic impulse control problems directly by relying on the ordinary Hamilton-Jacobi-Bellman approach, we follow the approach introduced in Alvarez 2004a and Alvarez 2004b and first derive the value accrued from applying a potentially suboptimal stochastic impulse control policy characterized by a sequence of constant-sized impulses exerted every time the underlying diffusion hits a predetermined and constant exercise threshold. By relying on standard nonlinear programming techniques we then state the ordinary first order necessary conditions characterizing both the exercise threshold and the impulse size of a potentially optimal policy maximizing the value of the associated class of Markovian functionals (for a recent study utilizing a similar idea, see Bayraktar and Egami 2006). The advantage of this approach is that it simplifies the economic analysis of the optimal policy and its value by admitting the application of standard marginalistic interpretations familiar from ordinary microeconomic theory. We present a set of relatively weak sufficient conditions under which an optimal pair satisfying the necessary conditions exists and is unique and under which this solution actually characterizes both the size of the optimal impulse and the threshold at which the irreversible policy should be optimally exerted. In accordance with these observations, we then find that given the policy mentioned above the iteratively defined Markovian functional actually constitutes the value of the optimal stochastic impulse policy.

We also consider two associated stochastic control problems (namely, a singular stochastic control and an optimal stopping problem) and study the boundary value problem connecting the values (including the value of the stochastic impulse control problem). Both of these associated classical problems have been studied extensively (for singular control cf. e.g. Beneš et al. 1980, Karatzas 1983, Baldursson 1987, Kobila 1993, Alvarez 2004a and Weerasinghe 2005. For optimal stopping we refer to the recent textbook Peskir and Shiryaev 2006) and it is well-known from this literature that singular stochastic control problems and optimal stopping problems are closely connected. More precisely, for a large class of singular stochastic control problems the derivative of the optimal value (i.e. the marginal value) coincides with the value of an associated optimal stopping problem of a certain transformed diffusion (cf. Karatzas and Shreve 1984, Karatzas and Shreve 1985, Boetius and Kohlmann 1998, Benth and Reikvam 2004, see also Alvarez 2001). The results of the current study connecting the problems differ from the previous characterizations in two ways. First, in all the considered three separate stochastic control problems the underlying dynamics are the same in the absence of control. Second, we study how, on the one hand, the optimal values and, on the other hand, the marginal values are interrelated. As intuitively is clear, we find that the value of the associated singular stochastic control problem dominates the value of the stochastic impulse control problem which, in turn, dominates the value of the associated optimal stopping problem. Somewhat surprisingly, we also find that the same ordering is satisfied by the marginal values of the optimal policies as well. More precisely, we establish that the marginal value of the associated singular stochastic control problem dominates the marginal value of the stochastic impulse control problem which, in turn, dominates the marginal value of the associated optimal stopping problem. This finding is important from the point of view of economic and financial applications, since our results demonstrate that increased policy flexibility unambiguously increases the *Tobin's marginal q* (i.e. marginal value) associated with the considered stochastic control problems as well. In this way our results extend the findings of Alvarez and Virtanen 2006 by demonstrating that the positivity of the relationship between the (marginal) value and the flexibility of the admissible policy is satisfied in the presence of a state-dependent and potentially non-linear cash flow as well.

The contents of this study are as follows. In section two we present the considered class of stochastic impulse control problems. In section three we then state a set of auxiliary results and analyze the two associated stochastic control problems. In section four we then analyze the considered stochastic impulse control problem and state our main results. Finally, our results are explicitly illustrated in section five in a model based on geometric Brownian motion.

2 THE IMPULSE CONTROL PROBLEM

2.1 GENERAL SETUP

It is our purpose in this study to analyze a class of stochastic impulse control problems of linear diffusions arising in many financial and economical applications of stochastic control theory. In order to accomplish this task, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ denote a complete filtered probability space satisfying the usual conditions and assume that the dynamics of the underlying controlled diffusion process are given by the generalized Itô equation

$$X_t^\nu = x + \int_0^t \mu(X_s^\nu)ds + \int_0^t \sigma(X_s^\nu)dW_s - \sum_{\tau_k \leq t} \beta\zeta_k,\ 0 \leq t \leq H_0^\nu, \tag{2.1}$$

where $\beta > 0$ is an exogenously given constant, $H_0^\nu = \inf\{t \geq 0 : X_t^\nu \leq 0\}$ denotes the possibly finite first exit time of the controlled diffusion X^ν from the state-space $\mathbf{R}_+$ and $\mu : \mathbf{R}_+ \to \mathbf{R}$ and $\sigma : \mathbf{R}_+ \to \mathbf{R}_+$ are known sufficiently smooth mappings (at least continuous) guaranteeing the existence of a solution for the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{2.2}$$

characterizing the dynamics of the underlying diffusion in the absence of interventions (see Borodin and Salminen 2002, 46–47). In a rational cash flow management application, the uncontrolled diffusion X represents the evolution of cash reserve in the absence of dividend payments and the impulse ζ_k represent the amount paid out as dividends to stock holders at the corresponding time τ_k – hence the sum $\sum_{\tau_k \le t} \beta \zeta_k$ represents the total amount of dividends paid out until time t (or the cumulative portfolio wealth consumed by the decision maker up to time t). The parameter β can be interpreted as a measure of the *imperfect controllability* of the underlying stochastic dynamics, since whenever $\beta \neq 1$ an arbitrary admissible impulse results into a jump discontinuity which is either greater or smaller than the size of the actual impulse ζ_k. As in Øksendal 1999, an admissible impulse control policy for the system (2.1) is a potentially infinite joint sequence $\nu = \{(\tau_k, \zeta_k)\}_{k=1}^N, N \le \infty$, where $\{\tau_k\}_{k=1}^N$ denotes an increasing sequence of $\mathscr{F}_t$-stopping times for which $\tau_1 \ge 0$ and $\{\zeta_k\}_{k=1}^N$ denotes a sequence of non-negative, $\mathscr{F}_{\tau_k -}$-measurable impulses exerted at the corresponding intervention dates $\{\tau_k\}_{k=1}^N$, respectively. We denote as $\mathscr{V}$ the class of admissible impulse controls ν and assume that $\tau_k \to H_0^\nu$ almost surely for all $\nu \in \mathscr{V}$ and $x \in \mathbf{R}_+$. This convergence should be understood as follows: If $N = \infty$, then the convergence is typical almost sure convergence. However, if $N < \infty$ then we augment the control ν with a pair $(\tau_{N+1}, \zeta_{N+1}) := (H_0^\nu, 0)$. As usually, we denote as $\mathscr{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$ the differential operator associated with X.

Denote as $\mathscr{L}_1$ the class of measurable mappings $f : \mathbf{R}_+ \to \mathbf{R}$ satisfying the condition

$$\mathbf{E}_x \left[\int_0^{H_0} e^{-rs} |f(X_s)| ds \right] < \infty,$$

where $r > 0$ is a given constant and $H_0 = \inf\{t \ge 0 : X_t \le 0\}$ is the first, potentially infinite, exit time for the uncontrolled diffusion X from $\mathbf{R}_+$. For a given $f \in \mathscr{L}_1$, define the resolvent $(R_r f) : \mathbf{R}_+ \to \mathbf{R}$ as

$$(R_r f)(x) = \mathbf{E}_x \left[\int_0^{H_0} e^{-rs} f(X_s) ds \right].$$

The resolvent $(R_r f)$ measures the expected cumulative present value of the cash flow $f(X_t)$ from the present up to H_0. It is well-known from the literature on linear diffusions that $(R_r f)$ can be rewritten as (see Øksendal A. 2000b)

$$\begin{aligned}(R_r f)(x) = B^{-1}\varphi(x) \int_0^x \psi(y) f(y) m'(y) dy \\ + B^{-1}\psi(x) \int_x^\infty \varphi(y) f(y) m'(y) dy,\end{aligned} \tag{2.3}$$

where ψ denotes the increasing and φ the decreasing fundamental solution of the ordinary second-order linear differential equation $\mathscr{A}u = ru$ defined on the domain of the characteristic operator of X, $B = (\psi'(x)\varphi(x) - \varphi'(x)\psi(x))/S'(x)$

denotes the Wronskian determinant, $S'(x) = \exp\left(-\int^x \frac{2\mu(y)}{\sigma^2(y)}dy\right)$ denotes the density of the scale function S, and $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the density of the speed measure m of X (for a characterization of the fundamental solutions and the Green function of a linear diffusion, see Borodin and Salminen 2002, 18–20).

2.2 THE IMPULSE CONTROL PROBLEM

Given the stochastic dynamics in (2.1) and the assumptions presented above on the dynamics of the controlled system, define the *expected cumulative net present value of the revenues from the present up to a potentially infinite future* as

$$J_c^\nu(x) = \mathbf{E}_x\left[\int_0^{H_0^\nu} e^{-rs}\pi(X_s^\nu)ds + \sum_{k=1}^{N} e^{-r\tau_k}(\lambda\zeta_k - c)\right], \tag{2.4}$$

where $r > 0$ is the discount rate, $\lambda > 0$ is an exogenously given constant, $c > 0$ is a known constant measuring a lump-sum sunk cost associated with the irreversible policy, and $\pi : \mathbf{R}_+ \to \mathbf{R}_+$ is a given continuous, non-decreasing and non-negative mapping measuring the *revenue flow accrued from continuing the operation*. This type of objective functional arise frequently in studies considering rational cash flow management (optimal dividend policy) and in studies considering the rational harvesting of renewable resources. In cash flow management application, the right hand side of (2.4) represents the expected cumulative present value of the revenues accrued from the present up to the potentially infinite horizon at which the firm is liquidated. Along the lines of our interpretation of β, the parameter λ can be interpreted as another measure of imperfect controllability, since whenever $\lambda \neq 1$ the realized revenue is strictly smaller or larger that the size of the actual impulse ζ_k (in most economic and financial applications this parameter arises due to either taxes or subsidies).

Given the definition of J_c^ν we plan to study the stochastic impulse control problem

$$V_c(x) = \sup_{\nu\in\mathscr{V}} J_c^\nu(x),\; x \in \mathbf{R}_+ \tag{2.5}$$

and to determine an admissible impulse control ν^* for which the maximum $J_c^{\nu^*}(x) = V_c(x)$ is attained for all $x \in \mathbf{R}_+$. We will analyze the problem (2.5) under the following standing assumptions.

Assumptions 2.1. *(1) We assume that the upper boundary ∞ is natural and that the lower boundary 0 is either natural, exit or regular for the uncontrolled diffusion X in the absence of interventions. In the case when 0 is regular, we assume that it is killing.*

(2) Define the mapping $\theta : \mathbf{R}_+ \to \mathbf{R}$ as

$$\theta(x) = \beta\pi(x) + \lambda\rho(x) \tag{2.6}$$

where $\rho(x) = \mu(x) - rx$. Throughout the study, we assume that $\theta \in \mathscr{L}_1(\mathbf{R}_+)$ and that there is a unique state $x^* \geq 0$ for which θ is increasing on $(0, x^*)$ and decreasing on (x^*, ∞). Moreover, we assume that $0 \leq \lim_{x\to 0+} \theta(x) < \infty$ and that $\lim_{x\to\infty} \theta(x) < 0$.

The assumptions *(1)* characterize the boundary behavior of the underlying diffusion. In line with most financial and economical applications the upper boundary is assumed to be natural. Hence, even though the underlying controlled diffusion process may be expected to drift towards infinity, it is never expected to attain it in finite time. In the context of cash flow management applications this assumption can be interpreted as a requirement that the retained profits from which dividends are paid out to the shareholders cannot become infinitely large in finite time. On the other hand, the assumed behavior of the underlying diffusion at the lower boundary is in line with the concept of liquidation and essentially guarantees that no dividends can be paid from negative reserves (since the time horizon is defined up to H_0^ν). From a mathematical point of view, the minimal r-harmonic functions ψ and φ satisfy useful limiting conditions depending on the boundary behavior of X (see Borodin and Salminen 2002, 19). More precisely, if the lower boundary 0 is natural, then $\lim_{x\to 0+} \psi(x) = 0$, $\lim_{x\to 0+} \psi'(x)/S'(x) = 0$, $\lim_{x\to 0+} \varphi(x) = \infty$ and $\lim_{x\to 0+} \varphi'(x)/S'(x) = -\infty$. Analogous conditions hold also for upper boundary ∞ (which was assumed to be natural). If 0 is an exit boundary, then second and third condition are replaced by $\lim_{x\to 0+} \psi'(x)/S'(x) > 0$ and $\lim_{x\to 0+} \varphi(x) < \infty$. In the case of a killing boundary we have limiting condition only for ψ, namely that $\lim_{x\to 0+} \psi(x) = 0$.

The assumptions *(2)* are also quite reasonable from the point of view of financial and economical applications. In cash flow management application, the function θ measures the expected net return (the sum of the continuous dividend flow and the expected capital gain) accrued from postponing the dividend payment into the future instead of paying out dividends instantaneously. Hence, the assumed limiting behavior of the net return guarantees that the rate of return earned from a retained unit dominates its opportunity cost (i.e. the return from a safe investment) when the reserves are low and that the opposite argument is valid when the reserves are large. In this respect our assumption *(2)* characterize a set of sufficient conditions under which the decision maker has incentives to distribute part of the reserves when they become sufficiently large without liquidating the corporation instantaneously. The absence of speculative bubbles condition $\theta \in \mathscr{L}_1$ guarantees that the expected cumulative present value of the net returns accrued from the present up to the liquidation date is finite.

In order to proceed in the analysis of the considered class of stochastic control problems, we first establish the following verification theorem.

Lemma 2.1. *Assume that there is a mapping $F : \mathbf{R}_+ \mapsto \mathbf{R}_+$ satisfying the conditions*

(a) the function $F(x) - (R_r\pi)(x)$ is nonnegative and r-superharmonic for X_t, and

(b) F satisfies the inequality

$$F(x) \geq \sup_{\beta\zeta \in [0,x]} [\lambda\zeta - c + F(x - \beta\zeta)]. \tag{2.7}$$

for all $x \in \mathbf{R}_+$. Then, $F(x) \geq V_c(x)$ for all $x \in \mathbf{R}_+$.

Proof. Let $\nu \in \mathscr{V}$ be an admissible stochastic impulse control. Since $\{\tau_j\}_{j=1}^N$ is an increasing sequence of stopping times, we first observe that the assumed r-superharmonicity of the function $F(x) - (R_r\pi)(x)$ implies that (see Øksendal 2003, 207, Lemma 10.1.3)

$$\begin{aligned} &\mathbf{E}_{\mathscr{F}_{\tau_j}}\left[e^{-r\tau_{j+1}}\left(F(X^\nu_{\tau_{j+1}-}) - (R_r\pi)(X^\nu_{\tau_{j+1}-})\right)\right] \leq \\ &e^{-r\tau_j}\left(F(X^\nu_{\tau_j}) - (R_r\pi)(X^\nu_{\tau_j})\right). \end{aligned} \tag{2.8}$$

Since $((\mathscr{A} - r)(R_r\pi))(x) = -\pi(x)$ for all $x \in \mathbf{R}_+$, application of Dynkin's theorem to $(R_r\pi)$ yields

$$\mathbf{E}_{\mathscr{F}_{\tau_j}}\left[e^{-r\tau_{j+1}}(R_r\pi)(X^\nu_{\tau_{j+1}-})\right] = e^{-r\tau_j}(R_r\pi)(X^\nu_{\tau_j}) - \mathbf{E}_{\mathscr{F}_{\tau_j}}\left[\int_{\tau_j}^{\tau_{j+1}-} e^{-rs}\pi(X^\nu_s)ds\right]$$

implying that inequality (2.8) can be re-expressed as

$$e^{-r\tau_j}F(X^{\nu}_{\tau_j}) - \mathbf{E}_{\mathscr{F}_{\tau_j}}\left[e^{-r\tau_{j+1}}F(X^{\nu}_{\tau_{j+1}-})\right] \geq \mathbf{E}_{\mathscr{F}_{\tau_j}}\left[\int_{\tau_j}^{\tau_{j+1}-} e^{-rs}\pi(X^{\nu}_s)ds\right].$$

Taking expectations and invoking the tower property of conditional expectations then yields

$$\mathbf{E}_x\left[e^{-r\tau_j}F(X^{\nu}_{\tau_j})\right] - \mathbf{E}_x\left[e^{-r\tau_{j+1}}F(X^{\nu}_{\tau_{j+1}-})\right] \geq \mathbf{E}_x\left[\int_{\tau_j}^{\tau_{j+1}-} e^{-rs}\pi(X^{\nu}_s)ds\right].$$

Letting $\tau_0 = 0$, summing terms from $j = 0$ to $j = n \wedge N$, and applying the nonnegativity of the mapping $F(x)$ results in

$$F(x) \geq \sum_{j=1}^{n\wedge N} \mathbf{E}_x\left[e^{-r\tau_j}F(X^{\nu}_{\tau_{j-}}) - F(X^{\nu}_{\tau_j})\right] + \mathbf{E}_x\left[\int_0^{\tau_{n\wedge N+1-}} e^{-rs}\pi(X^{\nu}_s)ds\right].$$

Since $X_{\tau_j} = X_{\tau_{j-}} - \beta\zeta_j$ for any admissible strategy and F satisfies the quasi-variational inequality $F(x) \geq \sup_{\beta\zeta\in[0,x]}[\lambda\zeta - c + F(x - \beta\zeta)]$ for all $\in \mathbf{R}_+$ we find that

$$F(x) \geq \mathbf{E}_x\left[\int_0^{\tau_{n\wedge N+1-}} e^{-rs}\pi(X^{\nu}_s)ds + \sum_{j=1}^{n\wedge N} e^{-r\tau_j}(\lambda\zeta_j - c)\right].$$

Letting $n \to \infty$ and invoking dominated convergence then finally implies that

$$F(x) \geq \mathbf{E}_x\left[\int_0^{H_0^{\nu}} e^{-rs}\pi(X^{\nu}_s)ds + \sum_{j=1}^{N} e^{-r\tau_j}(\lambda\zeta_j - c)\right].$$

Since this inequality is valid for any admissible impulse control, it has to be valid for the optimal as well from which the alleged result follows. □

Lemma 2.1 states a set of considerably weak sufficient conditions which can be applied in the verification of the optimality of a value attained by applying an admissible policy. An interesting implication of Lemma 2.1 stating a set of more easily applicable sufficient conditions is now summarized in the following.

Corollary 2.2. *Assume that the mapping $F : \mathbf{R}_+ \mapsto \mathbf{R}_+$ satisfies the conditions $F \in C^1(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus \mathscr{D})$, where $\mathscr{D}$ is a set of measure zero and $F''(x\pm) < \infty$ for all $x \in \mathscr{D}$. Assume also that F satisfies the quasi-variational inequality (2.7) for all $x \in \mathbf{R}_+$ and the variational inequality $(\mathscr{A}F)(x) - rF(x) + \pi(x) \leq 0$ for all $x \notin \mathscr{D}$. Then, $F(x) \geq V_c(x)$ for all $x \in \mathbf{R}_+$.*

Proof. As was established in Øksendal 2003, 315 – 318, Theorem D.1, the conditions of our corollary guarantee that there a sequence $\{F_n\}_{n=1}^{\infty}$ of mappings $F_n \in C^2(\mathbf{R}_+)$ such that

(i) $F_n \to F$ uniformly on compact subsets of $\mathbf{R}_+$, as $n \to \infty$;

(ii) $(\mathscr{A}F_n) - rF_n \to (\mathscr{A}F) - rF$ uniformly on compact subsets of $\mathbf{R}_+ \backslash \mathscr{D}$, as $n \to \infty$;

(iii) $\{(\mathscr{A}F_n) - rF_n\}_{n=1}^{\infty}$ is locally bounded on $\mathbf{R}_+$.

Applying Itô's theorem to the mapping $(t,x) \mapsto e^{-rt}\Delta_n(x)$, where $\Delta_n(x) = F_n(x) - (R_r\pi)(x)$, taking expectations, and reordering terms yields

$$e^{-r\tau_j}\Delta_n(X^{\nu}_{\tau_j}) = \mathbf{E}_{\mathscr{F}_{\tau_j}}\left[e^{-r\tau_{j+1}}\Delta_n(X^{\nu}_{\tau_{j+1}-}) - \int_{\tau_j}^{\tau_{j+1}-} e^{-rs}\Phi_n(X^{\nu}_s)ds\right],$$

where $\Phi_n(x) := (\mathscr{A}F_n)(x) - rF_n(x) + \pi(x)$. Letting $n \to \infty$, applying Fatou's theorem, and invoking the variational inequality $(\mathscr{A}F)(x) - rF(x) + \pi(x) \leq 0$ then results in inequality (2.8). The alleged result now follows from Lemma 2.1. □

3 AUXILIARY RESULTS

3.1 SOME ASSOCIATED FUNCTIONALS

In subsection 2.2 we made a standing assumption that the upper boundary ∞ is natural and the lower boundary 0 is either natural, exit, or killing. It is important to point out that in all of these cases we have that $\lim_{x\to 0+}\psi(x) = \lim_{x\to\infty}\varphi(x) = 0$ (see Borodin and Salminen 2002, 19). These conditions will be used in this subsection without explicit indication. Recall the definition (2.6) of θ and consider the expected cumulative present value $(R_r\theta)$. By invoking the representation (2.3), differentiating the equation sidewise, and dividing the resulting identity with $\psi'(x)$ we find that

$$\frac{(R_r\theta)'(x)}{\psi'(x)} = B^{-1}\frac{\varphi'(x)}{\psi'(x)}\int_0^x \psi(y)\theta(y)m'(y)dy + B^{-1}\int_x^{\infty}\varphi(y)\theta(y)m'(y)dy. \tag{3.1}$$

Since $\varphi''(x)\psi'(x) - \varphi'(x)\psi''(x) = 2rBS'(x)/\sigma^2(x)$, differentiation of the expression (3.1) yields that

$$\frac{d}{dx}\left[\frac{(R_r\theta)'(x)}{\psi'(x)}\right] = \frac{2S'(x)}{\sigma^2(x)\psi'^2(x)}L(x),$$

where the functional $L : \mathbf{R}_+ \to \mathbf{R}$ is defined as

$$L(x) = r\int_0^x \psi(y)\theta(y)m'(y)dy - \theta(x)\frac{\psi'(x)}{S'(x)}.$$

The functional L will prove to be the principal determinant of the optimal policies in all the considered stochastic control problems. Our first auxiliary result is now summarized in the following.

Lemma 3.1. *Let Assumptions 2.1 be satisfied. Then there is a unique state* $\hat{x} = \operatorname{argmin}\left\{\frac{(R_r\theta)'(x)}{\psi'(x)}\right\} \in (x^*,\infty)$ *satisfying the condition* $L(\hat{x}) = 0$.

Proof. Assume first that $x^* < x < z$. Since the mapping θ is decreasing on (x^*,∞) and

$$r\int_a^b \psi(y)m'(y)dy = \frac{\psi'(b)}{S'(b)} - \frac{\psi'(a)}{S'(a)}$$

for any $0 < a < b < \infty$, we have that

$$\begin{aligned}
\frac{1}{r}[L(z)-L(x)] &= \int_x^z \psi(y)\theta(y)m'(y)dy - \frac{\theta(z)}{r}\frac{\psi'(z)}{S'(z)} + \frac{\theta(x)}{r}\frac{\psi'(x)}{S'(x)} \\
&> \frac{\theta(z)}{r}\left[\frac{\psi'(z)}{S'(z)} - \frac{\psi'(x)}{S'(x)}\right] - \frac{\theta(z)}{r}\frac{\psi'(z)}{S'(z)} + \frac{\theta(x)}{r}\frac{\psi'(x)}{S'(x)} \\
&= \frac{[\theta(x)-\theta(z)]}{r}\frac{\psi'(x)}{S'(x)} > 0
\end{aligned}$$

proving that L is monotonically increasing on (x^*,∞). Analogously, we find that whenever $z < x < x^*$

$$\begin{aligned}
\frac{1}{r}[L(x)-L(z)] &= \int_z^x \psi(y)\theta(y)m'(y)dy - \frac{\theta(x)}{r}\frac{\psi'(x)}{S'(x)} + \frac{\theta(z)}{r}\frac{\psi'(z)}{S'(z)} \\
&< \frac{[\theta(z)-\theta(x)]}{r}\frac{\psi'(z)}{S'(z)} < 0
\end{aligned}$$

showing that L is monotonically decreasing on $(0,x^*)$.

Since the boundary 0 is assumed to be natural, exit or killing, we find that $\lim_{x\to 0+} L(x) \le 0$. Moreover, since x^* is the global maximum of θ, we find that

$$\begin{aligned}
L(x^*) &= r\int_0^{x^*} \psi(y)\theta(y)m'(y)dy - \theta(x^*)\frac{\psi'(x^*)}{S'(x^*)} \\
&< \theta(x^*)\left[\frac{\psi'(x^*)}{S'(x^*)} - \frac{\psi'(0)}{S'(0)}\right] - \theta(x^*)\frac{\psi'(x^*)}{S'(x^*)} \le 0.
\end{aligned}$$

Assumptions 2.1 imply that there is a unique state $x_0 \in (x^*, \infty)$ such that $\theta(x_0) = 0$ and that $\theta(x) > 0$ on $(0, x_0)$. Hence,

$$L(x_0) = r\int_0^{x_0} \psi(y)\theta(y)m'(y)dy > 0.$$

Since L is increasing on (x^*, ∞), we find there is a unique state $\hat{x} \in (x^*, \infty)$ such that $L(\hat{x}) = 0$. □

Along with the functional L, two additional functionals will be important in the subsequent study of the considered stochastic control problems. These functionals, which are denoted as $I : \mathbf{R}_+ \to \mathbf{R}$ and $J : \mathbf{R}_+ \to \mathbf{R}$, are defined as

$$I(x) = \frac{\beta(R_r\pi)'(x) - \lambda}{\psi'(x)} \tag{3.2}$$

and as

$$J(x) = \beta(R_r\pi)(x) - \lambda x - I(x)\psi(x). \tag{3.3}$$

The functional J has an alternative representation which will be used later. More precisely, since

$$\frac{\psi(x)}{\psi'(x)} - x = \frac{S'(x)}{\psi'(x)}\int_0^x \psi(y)\rho(y)m'(y)dy$$

(see Alvarez and Virtanen 2006, Lemma 3.3), we find by using the representation (2.3) for $(R_r\pi)$ that

$$J(x) = \frac{S'(x)}{\psi'(x)}\int_0^x \psi(y)\theta(y)m'(y)dy. \tag{3.4}$$

We will now establish that the behavior of L dictates the monotonicity properties of the associated functionals I and J. This is accomplished in the following.

Lemma 3.2. *Let Assumptions 2.1 be satisfied. Then* $I'(x) \gtreqqless 0$ *and* $J'(x) \lesseqqgtr 0$ *when* $x \gtreqqless \hat{x}$.

Proof. First note that since $\rho \in \mathscr{L}_1$, the expression

$$\frac{d}{dx}\left[\frac{(R_r\rho)'(x)}{\psi'(x)}\right] = \frac{2S'(x)}{\sigma^2(x)\psi'^2(x)}\left[r\int_0^x \psi(y)\rho(y)m'(y)dy - \rho(x)\frac{\psi'(x)}{S'(x)}\right]$$

holds. On the other hand, we have that

$$\psi''(x) = \frac{2S'(x)}{\sigma^2(x)}\left[r\int_0^x \psi(y)\rho(y)m'(y)dy - \rho(x)\frac{\psi'(x)}{S'(x)}\right]$$

(see Alvarez 2004a, Lemma 2.1) which in turn implies that

$$\frac{d}{dx}\left[\frac{(R_r\rho)'(x)}{\psi'(x)}\right] = \frac{\psi''(x)}{\psi'^2(x)}.$$

Combining this observation with the findings of Lemma 3.1 now yields

$$I'(x) = \frac{d}{dx}\left[\frac{\beta(R_r\pi)'(x)-\lambda}{\psi'(x)}\right] = \frac{d}{dx}\left[\frac{(R_r\theta)'(x)}{\psi'(x)}\right] = \frac{2S'(x)}{\sigma^2(x)\psi'^2(x)}L(x)$$

from which the first alleged inequality follows. Second, ordinary differentiation yields $J'(x) = -\psi(x)I'(x)$ which completes the proof of our lemma. □

Lemma 3.2 essentially demonstrates that under Assumptions 2.1 the threshold $\hat{x}$ constitutes the global minimum of the functional I and the global maximum of the functional J. To close the subsection, we present a lemma determining the limiting properties of I and J.

Lemma 3.3. *Let Assumptions 2.1 be satisfied. Then the following limiting properties hold:* $\lim_{x\to 0+} J(x) \geq 0$, $\lim_{x\to 0+} I(x) \geq 0$, $\lim_{x\to\infty} J(x) = -\infty$ *and* $\lim_{x\to\infty} I(x) \leq 0$.

Proof. If 0 is attainable, then $\lim_{x\to 0+} \frac{\psi'(x)}{S'(x)} > 0$. Therefore the expression (3.4) implies that $J(0) = 0$. If 0 is unattainable, then $\lim_{x\to 0+} \frac{\psi'(x)}{S'(x)} = 0$. In this case L'Hospitals rule yields

$$\lim_{x\to 0+} J(x) = \lim_{x\to 0+} \frac{\psi(x)\theta(x)m'(x)}{\frac{d}{dx}\left[\frac{\psi'(x)}{S'(x)}\right]} = \lim_{x\to 0+} \frac{\psi(x)\theta(x)m'(x)}{r\psi(x)m'(x)} = \lim_{x\to 0+} \frac{\theta(x)}{r} \geq 0.$$

To prove the alleged behavior of J at infinity, note that

$$\lim_{x\to\infty} \int_0^x \psi(y)\theta(y)m'(y)dy = -\infty$$

and that $\lim_{x\to\infty} \frac{\psi'(x)}{S'(x)} = \infty$. Thus $\lim_{x\to\infty} J(x) = \lim_{x\to\infty} \frac{\theta(x)}{r} = -\infty$.

We showed in Lemma 3.1 that the state $\hat{x} = \text{argmax}\{J(x)\} = \text{argmin}\{I(x)\}$ lies in the interval (x^*,∞), i.e. where θ is monotonically decreasing. Hence for $x > \hat{x}$, we have that

$$\begin{aligned} I(x) &= B^{-1}\frac{\varphi'(x)}{\psi'(x)}\left[\int_0^{\hat{x}} \psi(y)\theta(y)m'(y)dy + \int_{\hat{x}}^x \psi(y)\theta(y)m'(y)dy\right] + \Gamma(x) \\ &= B^{-1}\frac{\varphi'(x)}{\psi'(x)}\left[\frac{\theta(\hat{x})}{r}\frac{\psi'(\hat{x})}{S'(\hat{x})} + \int_{\hat{x}}^x \psi(y)\theta(y)m'(y)dy\right] + \Gamma(x) \\ &\leq B^{-1}\frac{\varphi'(x)}{\psi'(x)}\left[\frac{\theta(\hat{x})}{r}\frac{\psi'(\hat{x})}{S'(\hat{x})} + \frac{\theta(\hat{x})}{r}\left(\frac{\psi'(x)}{S'(x)} - \frac{\psi'(\hat{x})}{S'(\hat{x})}\right)\right] + \Gamma(x) \\ &= B^{-1}\frac{\varphi'(x)}{S'(x)}\frac{\theta(\hat{x})}{r} + \Gamma(x), \end{aligned}$$

where $\Gamma(x) := B^{-1}\int_x^\infty \varphi(y)\theta(y)m'(y)dy$. By letting x tend to infinity in the inequality above, we find that $\lim_{x\to\infty} I(x) \leq 0$, since $\lim_{x\to\infty} \frac{\varphi'(x)}{S'(x)} = 0$. The property $\lim_{x\to 0+} I(x) \geq 0$ is still left to prove. To prove this, observe that the condition $I(x) \geq I(\hat{x})$ implies that

$$\lim_{x\to 0+} \frac{(R_r\theta)'(x)}{S'(x)} = \lim_{x\to 0+} I(x)\frac{\psi'(x)}{S'(x)} \geq \lim_{x\to 0+} I(\hat{x})\frac{\psi'(x)}{S'(x)} = 0;$$

hence $\lim_{x\to 0+}(R_r\theta)'(x) \geq 0$. Now the desired result

$$\lim_{x\to 0+} I(x) = \lim_{x\to 0+} \frac{(R_r\theta)'(x)}{\psi'(x)} \geq 0$$

follows, since $\psi'(x) > 0$. □

3.2 THE ASSOCIATED SINGULAR CONTROL PROBLEM

Before proceeding to the analysis of the stochastic impulse control problem, we first consider an associated singular stochastic control problem. To this end, consider the associated controlled diffusion process X^Z on $\mathbf{R}_+$ given by the generalized Itô stochastic differential equation

$$dX_t^Z = \mu(X_t^Z)dt + \sigma(X_t^Z)dW_t - \beta dZ_t,\ X_0^Z = x, \tag{3.5}$$

where the process Z_t is an *admissible control*, meaning a non-negative, non-decreasing, right-continuous and $\{\mathscr{F}_t\}$-adapted process. We denote the class of such processes as Λ and assume that μ and σ satisfy the same regularity conditions as in the impulse control case. Given these assumptions, we will consider the associated singular control problem

$$K(x) = \sup_{Z\in\Lambda} \mathbf{E}_x\left[\int_0^{H_0^Z} e^{-rs}\left(\pi(X_s^Z)ds + \lambda dZ_s\right)\right], \tag{3.6}$$

where $H_0^Z = \inf\{t \geq 0 : X_t^Z \leq 0\}$ denotes the first exit time of the controlled diffusion X^Z from $\mathbf{R}_+$. It is worth observing that applying the generalized Itô theorem to the linear mapping $x \mapsto \lambda x/\beta$ yields

$$\mathbf{E}_x\int_0^{\tau_N} e^{-rs}\lambda dZ_s = \frac{\lambda x}{\beta} + \mathbf{E}_x\int_0^{\tau_N} e^{-rs}\frac{\lambda}{\beta}\rho(X_s^Z)ds - \mathbf{E}_x\left[e^{-r\tau_N}\frac{\lambda}{\beta}X_{\tau_N}^Z\right]$$

where $\rho(x) = \mu(x) - rx$ and $\tau_N = H_0^Z \wedge N \wedge \inf\{t \geq 0 : X_t^Z \geq N\}$ is an almost surely finite stopping time. The non-negativity of the controlled process then results by letting N tend to infinity and invoking monotone convergence to the inequality

$$K(x) \leq \beta^{-1}\left[\lambda x + \sup_{Z\in\Lambda} \mathbf{E}_x \int_0^{\tau_0^Z} e^{-rs}\theta(X_s^Z)ds\right]. \tag{3.7}$$

It is clear that the inequality (3.7) becomes an equality if the implemented admissible policy is such that $\lim_{N\to\infty}\mathbf{E}_x\left[e^{-r\tau_N}X_{\tau_N}^Z\right]=0$. In that case the value of the optimal policy can be decomposed into a part measuring the value of the instantaneous liquidation policy and the expected cumulative present value of the excess return accrued from following the optimal policy and postponing the immediate liquidation of the underlying process.

In the next lemma we will establish the value and the optimal policy for the problem (3.6). These results will later turn out to be useful in the analysis of the impulse control problem (2.5) as well.

Lemma 3.4. *Let Assumptions 2.1 be satisfied. Then the optimal singular stochastic control exists and reads as*

$$Z_t^* = \begin{cases} (x-\hat{x})^+ & t=0 \\ \mathscr{L}(t,\hat{x}) & t>0, \end{cases} \tag{3.8}$$

where the threshold $\hat{x}\in(x^*,\infty)$ is the unique root of the first-order condition $L(\hat{x})=0$ and $\mathscr{L}(t,\hat{x})$ is the local time of X at $\hat{x}$ (see Borodin and Salminen 2002, 21–24). Moreover, the value function K reads as

$$K(x) = \begin{cases} \beta^{-1}\left(\lambda x + \frac{\theta(\hat{x})}{r}\right) & x\geq\hat{x} \\ (R_r\pi)(x) - \beta^{-1}I(\hat{x})\psi(x) & x<\hat{x} \end{cases} \tag{3.9}$$

implying that the marginal value K' can be expressed as

$$\begin{aligned} K'(x) &= (R_r\pi)'(x) + \beta^{-1}\psi'(x)\sup_{y\geq x}\left[\frac{\lambda-\beta(R_r\pi)'(y)}{\psi'(y)}\right] \\ &= \begin{cases} \lambda\beta^{-1} & x\geq\hat{x} \\ (R_r\pi)'(x) - \beta^{-1}I(\hat{x})\psi'(x) & x<\hat{x}. \end{cases} \end{aligned} \tag{3.10}$$

The value function K satisfies also the smooth pasting condition $\lim_{x\to\hat{x}}K''(x)=0$.

Proof. Denote the function defined in (3.9) as $\hat{K}(x)$. We will now demonstrate that $K(x) = \hat{K}(x)$ for all $x \in \mathbf{R}_+$. Since $\hat{K}(x)$ is attained by the admissible local time push policy (3.8) (i.e. reflection at $\hat{x}$), it is clear that $\hat{K}(x) \leq K(x)$ for all $x \in \mathbf{R}_+$ (cf. Section 1.6 in Friedlin 1985). In order to establish the opposite inequality, we first observe by ordinary differentiation that

$$\hat{K}'(x) = \begin{cases} \lambda\beta^{-1} & x \geq \hat{x} \\ \beta^{-1}\left[\lambda + \psi'(x)\left(I(x) - I(\hat{x})\right)\right] & x < \hat{x}. \end{cases}$$

Since the state $\hat{x}$ is the global minimum of the functional I, we find that $\hat{K}'(x) \geq \lambda\beta^{-1}$ for all $x \in \mathbf{R}_+$. Moreover,

$$(\mathscr{A}\hat{K})(x) - r\hat{K}(x) + \pi(x) = \begin{cases} \beta^{-1}\left(\theta(x) - \theta(\hat{x})\right) & x \geq \hat{x} \\ 0 & x < \hat{x}. \end{cases}$$

Since the state $\hat{x}$ is on the set where θ is strictly decreasing, we find that $(\mathscr{A}\hat{K})(x) - r\hat{K}(x) + \pi(x) \leq 0$ for all $x \in \mathbf{R}_+$. Finally, since $\hat{x}$ minimizes I, first order optimality conditions imply that $\psi''(\hat{x})\beta(R_r\pi)''(\hat{x}) = \psi''(\hat{x})(\beta(R_r\pi)'(\hat{x}) - \lambda)$. Therefore $\lim_{x\to\hat{x}-}\hat{K}''(x) = 0$ implying that $\hat{K} \in C^2(\mathbf{R}_+)$. The function K satisfies now the conditions of Lemma 1 in Alvarez 1999. Thus $\hat{K}(x) \geq K(x)$ for all $x \in \mathbf{R}_+$. □

Lemma 3.4 demonstrates that under Assumptions 2.1 the associated singular control problem (3.6) is solvable and that the optimal value is attained by utilizing a local time push policy at the threshold $\hat{x}$ (see Harrison 1985). Moreover, Lemma 3.4 also shows that the smooth pasting property holds for (3.6) and we notice from the proof that it is an implication of our approach to the problem. A set of interesting comparative static results implied by Lemma 3.4 are now summarized in the following.

Corollary 3.5. *Let Assumptions 2.1 be satisfied. Then*

(i) the value K and the marginal value K' of the optimal policy are decreasing functions of the parameter β;

(ii) the value K and the marginal value K' of the optimal policy are increasing functions of the parameter λ;

(iii) the optimal exercise threshold $\hat{x}$ is an increasing mapping of the parameter β and a decreasing mapping of the parameter λ.

(iv) if $\lambda = \beta$ then the optimal exercise threshold $\hat{x}$ is independent of λ and β.

Proof. (i) Denote the value associated with the parameter β_i as K_i, $i = 1,2$. It is now clear from the proof of Lemma 3.4 that K_2 satisfies the sufficient variational inequalities $(\mathscr{A}K_2)(x) - rK_2(x) + \pi(x) \leq 0$ and $K_2'(x) \geq \lambda/\beta_2 > \lambda/\beta_1$ for all $x \in \mathbf{R}_+$. Hence, $K_2(x) \geq K_1(x)$ for all $x \in \mathbf{R}_+$. In order to establish that $K_2'(x) \geq K_1'(x)$ for all $x \in \mathbf{R}_+$ we observe that the mapping $x \mapsto (\lambda - \beta(R_r\pi)'(x))/(\beta\psi'(x))$ is a decreasing function of the parameter β from which the alleged result follows by invoking the representation (3.10). Proving part (ii) is entirely analogous. It remains to consider the sensitivity of the optimal exercise threshold $\hat{x}$ with respect to parametric changes. To this end, consider the mapping

$$\bar{L}(x,\lambda,\beta) = r\int_0^x \psi(y)(\beta\pi(y) + \lambda\rho(y))m'(y)dy - (\beta\pi(x) + \lambda\rho(x))\frac{\psi'(x)}{S'(x)}.$$

If $\beta_1 > \beta_2$ then

$$\bar{L}(x,\lambda,\beta_1) - \bar{L}(x,\lambda,\beta_2) = (\beta_1 - \beta_2)\left[r\int_0^x \psi(y)\pi(y)m'(y)dy - \pi(x)\frac{\psi'(x)}{S'(x)}\right] \leq 0$$

since

$$r\int_0^x \psi(y)\pi(y)m'(y)dy - \pi(x)\frac{\psi'(x)}{S'(x)} \leq -\pi(x)\frac{\psi'(0)}{S'(0)} \leq 0$$

by the assumed monotonicity and non-negativity of π. Therefore, if $\hat{x}_i$ denotes the optimal exercise threshold associated with β_i, $i = 1,2$, we observe that $0 = \bar{L}(\hat{x}_1,\lambda,\beta_1) \leq \bar{L}(\hat{x}_1,\lambda,\beta_2)$ which, in turn, implies that $\hat{x}_1 \geq \hat{x}_2$. Establishing that $\hat{x}$ is a decreasing mapping of the parameter λ is entirely analogous. Finally, if $\lambda = \beta$ then

$$L(x) = \beta\left[r\int_0^x \psi(y)(\pi(y) + \rho(y))m'(y)dy - (\pi(x) + \rho(x))\frac{\psi'(x)}{S'(x)}\right]$$

from which the alleged result follows. □

Corollary 3.5 characterizes the impact of parametric changes on the value, the marginal value, and the optimal exercise threshold of the irreversible policy. We observe that an increase in β decreases both the value and the marginal value of the optimal policy and, therefore, postpones exercise by increasing the optimal exercise threshold and, therefore, expanding the continuation region where waiting is optimal. The contrary happens when the parameter λ

increases. An interesting implication of these comparative static results is that parametric changes are neutral (i.e. do not affect the optimal exercise threshold $\hat{x}$) as long as the ratio λ/β is held constant. As usually in models considering cash flow management in the presence of taxation, Corollary 3.5 shows that when $\lambda = \beta$ the optimal policy is independent of the parameters λ and β (i.e. the *harmonization of tax rates imply the tax neutrality of the optimal policy*).

3.3 THE ASSOCIATED OPTIMAL STOPPING PROBLEM

In this subsection we consider another associated control problem, namely an optimal stopping problem. Let X be the diffusion evolving on $\mathbf{R}_+$ according to the ordinary Itô stochastic differential equation (2.2) and assume that the infinitesimal coefficients μ and σ satisfy the same regularity conditions as in the impulse control case. Given these assumptions, consider the corresponding optimal stopping problem

$$G_c(x) = \sup_{\tau < H_0} \mathbf{E}_x \left[\int_0^\tau e^{-rs}\pi(X_s)ds + e^{-r\tau}\left(\lambda\beta^{-1}X_\tau - c\right)\right], \tag{3.11}$$

where $c \geq 0$ is an arbitrary constant and τ is an arbitrary $\mathscr{F}_t$-stopping time satisfying the constraint $\tau < H_0$ stating that the stopping time problem is defined up to the first exit time from $\mathbf{R}_+$. Along the lines indicated by (3.7) we find by applying Dynkin's theorem to the mapping $x \mapsto \lambda x/\beta - c$ (or by applying Itô's theorem to the process $t \mapsto e^{-rt}(\lambda\beta^{-1}X_t - c)$) that

$$G_c(x) = \frac{\lambda}{\beta}x - c + \frac{1}{\beta}\sup_{\tau < H_0} \mathbf{E}_x \int_0^\tau e^{-rs}(\theta(X_s) + \beta cr)ds$$

demonstrating how the value of the optimal policy can in this case be decomposed into the sum of the exercise payoff and the early exercise premium. Our main findings on this associated stopping problem are now summarized in the following.

Lemma 3.6. *Let Assumptions 2.1 be satisfied. Then the optimal stopping policy is to stop at the Markov time* $\tau_{\bar{x}_c} = \inf\{t \geq 0 : X_t \geq \bar{x}_c\}$ *where* $\bar{x}_c$*, denoting the optimal stopping threshold, is the unique root of the equation* $J(\bar{x}_c) = -\beta c$*,* *where J is defined in* (3.3)*. Moreover, the value can be written as*

$$
\begin{aligned}
G_c(x) &= (R_r\pi)(x) + \beta^{-1}\psi(x)\sup_{y\geq x}\left[\frac{\lambda y - \beta(R_r\pi)(y) - \beta c}{\psi(y)}\right] \\
&= \begin{cases} \beta^{-1}\lambda x - c & x \geq \bar{x}_c \\ (R_r\pi)(x) - \beta^{-1}I(\bar{x}_c)\psi(x) & x < \bar{x}_c \end{cases}
\end{aligned}
\tag{3.12}
$$

and it satisfies the smooth-pasting condition $\lim_{x\to\bar{x}_c-} G_c'(x) = \beta^{-1}\lambda$.

Proof. In order to establish (3.12), denote as x_0 the unique interior state at which $\theta(x_0) = 0$. The expression (3.4) implies that

$$
\frac{d}{dx}\left[\frac{\psi'(x)}{S'(x)}J(x)\right] = \psi(x)\theta(x)m'(x) \gtreqless 0,\ x \lesseqgtr x_0.
$$

Thus $J(x) > 0$ for all $x \in (0, x_0)$, since $\lim_{x\to 0+}\frac{\psi'(x)}{S'(x)}J(x) \geq 0$. On the other hand, we proved in Lemma 3.3 that $\lim_{x\to\infty} J(x) = -\infty$. Together with the monotonicity properties of J, this implies that there is an unique state $\bar{x}_c \in \theta^{-1}(\mathbf{R}_-)$ at which the condition $J(\bar{x}_c) = -\beta c$ is satisfied. Since

$$
\frac{d}{dx}\left[\frac{\lambda x - \beta(R_r\pi)(x) - \beta c}{\psi(x)}\right] = \frac{\psi'(x)}{\psi^2(x)}\left(J(x) + \beta c\right),
$$

we find that $\bar{x}_c = \operatorname{argmax}\{(\lambda x - \beta(R_r\pi)(x) - \beta c)/\psi(x)\}$. The first order optimality condition now implies that

$$
\frac{\lambda\bar{x}_c - \beta(R_r\pi)(\bar{x}_c) - \beta c}{\psi(\bar{x}_c)} = \frac{\lambda - \beta(R_r\pi)'(\bar{x}_c)}{\psi'(\bar{x}_c)} = -I(\bar{x}_c)
$$

which gives us the expression (3.12) and proves the smooth-pasting condition $\lim_{x\to\bar{x}_c} G_c'(x) = \beta^{-1}\lambda$.

Given these observations, denote the function defined in (3.12) as $\hat{G}_c(x)$. Since

$$
\hat{G}_c(x) = \mathbf{E}_x\left[\int_0^{\tau_{\bar{x}_c}} e^{-rs}\pi(X_s)ds + e^{-r\tau_{\bar{x}_c}}\left(\lambda\beta^{-1}X_{\tau_{\bar{x}_c}} - c\right)\right],
$$

where $\tau_{\bar{x}_c} = \inf\{t \geq 0 : X_t \geq \bar{x}_c\}$, we find that $\hat{G}_c(x) \leq G_c(x)$ for all $x \in \mathbf{R}_+$. On the other hand, we find that $\hat{G}_c$ is continuously differentiable on $\mathbf{R}_+$, twice continuously differentiable on $\mathbf{R}_+ \setminus \{\bar{x}_c\}$, satisfies the inequalities $|\hat{G}_c''(\bar{x}_c\pm)| < \infty$, and satisfies the variational inequality $\min\{r\hat{G}_c(x) - (\mathscr{A}\hat{G}_c)(x) - \pi(x), \hat{G}_c(x) - \lambda\beta^{-1}x + c\} = 0$. Thus $\hat{G}_c(x) \geq G_c(x)$ for all $x \in \mathbf{R}_+$. □

Lemma 3.6 establishes that under Assumptions 2.1 the optimal stopping problem (3.11) is solvable and that the optimal stopping policy is a threshold policy requiring that the underlying process should be stopped once it hits the constant boundary $\bar{x}_c$ at which the expected present value of the exercise payoff is maximized. At the optimal exercise threshold $\bar{x}_c$ the standard balance identity holds and the value of the project $\bar{x}_c$ coincides with the sum of the sunk cost c and the lost option value $G_c(\bar{x}_c)$. Moreover, along the lines with our findings on the associated singular stochastic control problem, we find that the smooth-pasting principle holds for the problem (3.11) as well. A set of interesting comparative static results implied by Lemma 3.6 are now summarized in the following.

Corollary 3.7. *Let Assumptions 2.1 be satisfied. Then*

(i) the value G_c is a decreasing function of both the parameter β and the sunk cost c and an increasing function of the parameter λ;

(ii) the optimal exercise threshold $\bar{x}_c$ is an increasing mapping of both the parameter β and the sunk cost c and a decreasing mapping of the parameter λ.

(iii) if $\lambda = \beta$ then the optimal exercise threshold $\bar{x}_c$ is independent of λ and β.

Proof. The claim of part (i) of our corollary follows directly from the definition of the exercise payoff. Thus, it is sufficient to consider the sensitivity of the optimal threshold $\bar{x}_c$ to changes in either λ, β, or c. To this end, consider the mapping

$$\tilde{L}(x,\lambda,\beta,c) = \int_0^x \psi(y)(\beta\pi(y)+\lambda\rho(y))m'(y)dy + \beta c\frac{\psi'(x)}{S'(x)}$$

and denote as $\bar{x}_c(\beta_i)$ the optimal exercise threshold associated with the parameter β_i. If $\beta_1 > \beta_2$ then

$$\tilde{L}(x,\lambda,\beta_1,c) - \tilde{L}(x,\lambda,\beta_2,c) = (\beta_1-\beta_2)\left[\int_0^x \psi(y)\pi(y)m'(y)dy + c\frac{\psi'(x)}{S'(x)}\right] > 0$$

which implies that $0 = \tilde{L}(\bar{x}_c(\beta_1),\lambda,\beta_1,c) > \tilde{L}(\bar{x}_c(\beta_1),\lambda,\beta_2,c)$ and, therefore, that $\bar{x}_c(\beta_1) > \bar{x}_c(\beta_2)$. The analysis of the impact of changes in either λ or c on the optimal exercise threshold is entirely analogous. Finally, if $\lambda = \beta$ then $\bar{x}_c = \operatorname{argmax}\{(x-(R_r\pi)(x)-c)/\psi(x)\}$ from which the alleged result follows. □

Corollary 3.7 extends the findings of Corollary 3.5 to the present example. More precisely, we observe that an increases in β postpones rational exercise by expanding the continuation region where stopping is suboptimal. The opposite is shown to happen when λ increases. Interestingly, we again find that parametric changes are neutral (i.e. do not affect the optimal exercise threshold $\bar{x}_c$) as long as the ratio λ/β is held constant. Along the lines of our findings on the associated singular control problem we again find that if $\lambda = \beta$ the optimal exercise strategy is independent of the parameters λ and β (i.e. *harmonization results into neutrality*). Moreover, as intuitively is clear, our findings indicate that increased sunk costs decrease the value and postpone rational exercise by expanding the continuation region.

4 OPTIMAL IMPULSE CONTROL POLICY

4.1 NECESSARY CONDITIONS

The stochastic impulse control problems of type (2.5) are typically tackled by relying on a combination of the classical Hamilton-Jacobi-Bellman approach and quasi-variational inequalities. In this study, we plan to adopt an alternative approach which results into more easily interpretable conditions characterizing a potentially optimal policy. Instead of considering all the admissible impulse controls at once, we restrict our attention to the subclass $\nu_{(\zeta,y)}$ of impulse controls characterized by the sequence of intervention times $\tau_0^y = 0$, $\tau_k^y = \inf\{t \geq \tau_{k-1}^y : X_t^{\nu(\zeta,y)} \geq y\}$ and the sequence of interventions $\zeta_k^y = \zeta + (x-y)^+$, for all $k \geq 1$. That is, we restrict our attention interest to control policies consisting of sequence of constant-sized impulses (with the exception of the initial impulse which depends on the initial state) exerted every time the underlying diffusion hits a predetermined, constant exercise threshold y. Given this class of admissible impulse controls, define the value $F_c : \mathbf{R}_+ \to \mathbf{R}$ accrued from applying the impulse control $\nu_{(\zeta,y)}$ as $F_c(x) = J_c^{(\zeta,y)}(x)$. Since $X_{\tau_k+}^{\nu(\zeta,y)} = X_{\tau_k-}^{\nu(\zeta,y)} - \beta\zeta = y - \beta\zeta$ for all k and the controlled diffusion evolves as the linear diffusion X between any two successive intervention dates, we observe that for all $x < y$ the value satisfies the functional relation (a so-called *running present value formulation*)

$$F_c(x) = \mathbf{E}_x\left[\int_0^{\tau_y} e^{-rs}\pi(X_s)ds\right] \\ + \mathbf{E}_x\left[e^{-r\tau_y}\left(\lambda\left(X_{\tau_y} - (y-\zeta)\right) - c + F_c(y-\beta\zeta)\right)\right], \tag{4.1}$$

where $\tau_y = \inf\{t \geq 0 : X_t \geq y\}$. Invoking the strong Markov property of diffusions now implies that the value $F_c(x)$ can be represented as

$$F_c(x) = \begin{cases} F_c(y-\beta\zeta) + \lambda(x-y+\zeta) - c & x \geq y \\ (R_r\pi)(x) + (\lambda\zeta - c - (R_r\pi)(y) + F_c(y-\beta\zeta))\frac{\psi(x)}{\psi(y)} & x < y. \end{cases} \tag{4.2}$$

First, note that letting x tend to y in (4.2) yields the *value-matching condition* $F_c(y) = F_c(y-\beta\zeta) + \lambda\zeta - c$ which can be re-expressed in the more familiar form $F_c(y-\beta\zeta) + \lambda\zeta = F_c(y) + c$ stating that *the value of the investment opportunity has to coincide with its full costs (lost option value + sunk cost)*. On the other hand, letting x tend to $y-\beta\zeta$ yields

$$F_c(y-\beta\zeta) = \frac{\psi(y)(R_r\pi)(y-\beta\zeta) + [\lambda\zeta - c - (R_r\pi)(y)]\,\psi(y-\beta\zeta)}{\psi(y) - \psi(y-\beta\zeta)}. \tag{4.3}$$

Now, inserting (4.3) into (4.2) implies that the value can be expressed as

$$F_c(x) = \begin{cases} (R_r\pi)(y-\beta\zeta) + h(\zeta,y)\psi(y-\beta\zeta) + \lambda(x-y+\zeta) - c & x \geq y \\ (R_r\pi)(x) + h(\zeta,y)\psi(x) & x < y, \end{cases} \tag{4.4}$$

where the mapping $h : \mathbf{R}_+^2 \to \mathbf{R}$ is defined as

$$h(\zeta,y) = \frac{(R_r\pi)(y-\beta\zeta) - (R_r\pi)(y) + \lambda\zeta - c}{\psi(y) - \psi(y-\beta\zeta)}. \tag{4.5}$$

In order to prove the existence and uniqueness of the optimal impulse control policy, we will consider the ordinary inequality constrained non-linear programming problem

$$\sup_{\substack{\beta\zeta\in[0,y],\\ y\in\mathbf{R}_+}} \frac{(R_r\pi)(y-\beta\zeta) - (R_r\pi)(y) + \lambda\zeta - c}{\psi(y) - \psi(y-\beta\zeta)}. \tag{4.6}$$

To ease up the subsequent analysis, introduce a linear change of variables $z := y - \beta\zeta$. Thus $\zeta = \beta^{-1}(y-z)$. Since the parameter β is assumed to be positive, the programming problem (4.6) can now be re-written as

$$\sup_{\substack{z\in[0,y],\\ y\in\mathbf{R}_+}} \frac{(R_r\pi)(z) - (R_r\pi)(y) + \lambda\beta^{-1}(y-z) - c}{\psi(y) - \psi(z)}. \tag{4.7}$$

If an interior pair maximizing the mapping h exists, denote the associated mapping of the form (4.1) as F_c^*. More precisely, if an interior pair (z_c^*, y_c^*) satisfying the problem (4.7) exists, define the mapping $F_c^* : \mathbf{R} \to \mathbf{R}$ as

$$F_c^*(x) = \begin{cases} (R_r\pi)(z_c^*) + h(z_c^*, y_c^*)\psi(z_c^*) + \gamma(x) - c & x \geq y_c^* \\ (R_r\pi)(x) + h(z_c^*, y_c^*)\psi(x) & x < y_c^*, \end{cases} \tag{4.8}$$

where $\gamma(x) := \lambda\left(x - y_c^* - \beta^{-1}(y_c^* - z_c^*)\right)$. Since h is differentiable, it is clear that if an interior pair (z_c^*, y_c^*) satisfying the problem (4.7) exists, then this pair satisfies the ordinary necessary first-order conditions $\frac{\partial h}{\partial z}(z_c^*, y_c^*) = \frac{\partial h}{\partial y}(z_c^*, y_c^*) = 0$. More precisely, if an optimal pair exists, it must satisfy the conditions

$$\begin{cases} (\psi(y_c^*) - \psi(z_c^*))\left(\lambda\beta^{-1} - (R_r\pi)'(y_c^*)\right) = r(z_c^*, y_c^*)\psi'(y_c^*) \\ (\psi(y_c^*) - \psi(z_c^*))\left(\lambda\beta^{-1} - (R_r\pi)'(z_c^*)\right) = r(z_c^*, y_c^*)\psi'(z_c^*), \end{cases} \tag{4.9}$$

where $r(z,y) = (R_r\pi)(z) - (R_r\pi)(y) + \lambda\beta^{-1}(y-z) - c$. This yields immediately the condition

$$\frac{\lambda - \beta(R_r\pi)'(z_c^*)}{\psi'(z_c^*)} = \frac{\lambda - \beta(R_r\pi)'(y_c^*)}{\psi'(y_c^*)}.$$

Using the definition (3.2), this can be rewritten as

$$I(y_c^*) - I(z_c^*) = 0. \tag{4.10}$$

On the other hand, since

$$\frac{\psi'(z_c^*)}{\psi(y_c^*) - \psi(z_c^*)} = \frac{(R_r\pi)'(z_c^*) - \lambda\beta^{-1}}{r(z_c^*, y_c^*)},$$

we find by invoking condition (4.10) and reordering the terms that

$$[\beta(R_r\pi)(y_c^*) - I(y_c^*)\psi(y_c^*) - \lambda y_c^*] - [\beta(R_r\pi)(z_c^*) - I(z_c^*)\psi(z_c^*) - \lambda z_c^*] = -\beta c.$$

Using the definition (3.3), this can be expressed as

$$J(y_c^*) - J(z_c^*) = -\beta c. \tag{4.11}$$

Conditions (4.10) and (4.11) are standard necessary first-order conditions for the existence of the solution of the problem (4.7). In Bayraktar and Egami 2006, a similar idea is used to solve another impulse control problem, where first order optimality conditions are derived for an associated functional reminiscent of (4.8) (see Bayraktar and Egami 2006, Equation 2.17). However, our control problem differs from the problem of Bayraktar and Egami 2006 on a fundamental level (we will comment on this in the next subsection). Moreover, the actual solution methods are also different.

4.2 EXISTENCE AND SUFFICIENCY

Having the necessary conditions (4.10) and (4.11) at our disposal, we will now study their solvability under Assumptions 2.1.

Lemma 4.1. *Let Assumptions 2.1 be satisfied. Then there exists an unique interior pair* (z_c^*, y_c^*) *for which the necessary conditions* (4.10) *and* (4.11) *are satisfied.*

Proof. Existence. Define the mappings $\check{J}:(0,\hat{x})\to(\check{J}(0),\check{J}(\hat{x}))$ and $\hat{J}:[\hat{x},\infty)\to(-\infty,\hat{J}(\hat{x})]$ as restrictions of the mapping J and the mapping $k:\mathbf{R}\to\mathbf{R}$ as $k(x)=x-\beta c$. It is clear that both $\check{J}$ and $\hat{J}$ are continuous. Moreover, $\check{J}$ is increasing and $\hat{J}$ is decreasing. Define now the mapping $\hat{y}:(0,\hat{x})\to(\hat{y}(\hat{x}),\hat{y}(0))$ as $\hat{y}(x)=(\hat{J}^{-1}\circ k\circ\check{J})(x)$. By the definitions of $\check{J}$, $\hat{J}$ and k, we observe that $\hat{y}$ is well-defined. Moreover, the mapping $\hat{y}$ is continuous as a composition of continuous mappings and decreasing. Finally, since $\hat{y}(z)=\hat{J}^{-1}(\check{J}(z)-\beta c)$, we find that the equation $J(\hat{y}(z))-J(z)=-\beta c$ holds for all $z\in(0,\hat{x})$.

Analogously, define the mappings $\check{I}:(0,\hat{x})\to(\check{I}(\hat{x}),\check{I}(0))$ and $\hat{I}:[\hat{x},\infty)\to[\hat{I}(\hat{x}),\hat{I}(0))$ as restrictions of the mapping I. It is clear that both $\check{I}$ and $\hat{I}$ are continuous. Moreover, $\check{I}$ is decreasing and $\hat{I}$ is increasing. Define the mapping $Y:(0,\hat{x})\to(\check{I}^{-1}(\hat{I}(\hat{y}(\hat{x}))),\check{I}^{-1}(\hat{I}(\hat{y}(0))))$ as $Y(x)=(\check{I}^{-1}\circ\hat{I}\circ\hat{y})(x)$. Since $0<\beta c<\infty$ and $\hat{J}$ is decreasing, we find that $\hat{y}(\hat{x})=\hat{J}^{-1}(\check{J}(\hat{x})-\beta c)>\hat{J}^{-1}(\check{J}(\hat{x}))=\hat{x}$ and $\hat{y}(0)=\hat{J}^{-1}(\check{J}(0)-\beta c)\leq\hat{J}^{-1}(-\beta c)<\infty$. Firstly, these inequalities together with monotocity of $\check{I}$ and $\hat{I}$ guarantee that Y is well-defined, continuous and increasing. Moreover, coupled with Lemma 3.3 they imply that

$$\left(\check{I}^{-1}(\hat{I}(\hat{y}(\hat{x}))),\check{I}^{-1}(\hat{I}(\hat{y}(0)))\right)\subsetneq\left(\check{I}^{-1}(\hat{I}(\hat{x})),\check{I}^{-1}(\hat{I}(x))\right)\big|_{x=\infty}\subseteq(0,\hat{x}).$$

In other words, we find that the image of Y is strictly included in the domain of Y. This observation coupled with the fact that Y is continuous implies that Y has a fixed point. In other words, there is a state $z_c^*\in(0,\hat{x})$ for which $I(z_c^*)=\check{I}(z_c^*)=\hat{I}(\hat{y}(z_c^*))=I(\hat{y}(z_c^*))$. Moreover, since $z_c^*\in(0,\hat{x})$ the equation $J(\hat{y}(z_c^*))-J(z_c^*)=-\beta c$ also holds.

Uniqueness. Assume that z^* is a fixed point of the mapping Y. Since $J'(x)=-\psi(x)I'(x)$, we find that

$$Y'(z^*)=\frac{J'(z^*)}{I'(z^*)}\frac{I'(\hat{y}(z^*))}{J'(\hat{y}(z^*))}=\frac{\psi(z^*)}{\psi(\hat{y}(z^*))}<1.$$

In other words we find that the curve $Y(x)$ intersects the diagonal always from above. Continuity of Y yields now the desired uniqueness. □

Lemma 4.1 demonstrates that Assumptions 2.1 are sufficient for both the existence and uniqueness of a solution for the typically highly nonlinear necessary conditions (4.10) and (4.11). It is worth pointing out that since the existence result of Lemma 4.1 is based on a fixed point argument, the existence of a potentially optimal pair is guaranteed for a considerably broad class of problems. In comparison to Bayraktar and Egami 2006, Lemma 4.1 is analogous to the main result of Bayraktar and Egami 2006, where the original impulse control problem is defined over threshold policies described by single open interval. We will now proceed by proving that the optimal threshold policy determined by Lemma 4.1 is optimal over the class $\mathscr{V}$ defined in (2.5), which is much wider than the class of threshold policies.

Theorem 4.2. *Let Assumptions 2.1 be satisfied. Then the optimal impulse control policy is to instantaneously take the controlled diffusion* X_t^ν *to the state* $y_c^* - \beta\zeta_c^*$ *whenever it hits the state* y_c^* *(i.e. the size of the impulse is* $\beta\zeta_c^*$*). If the initial state* $x \geq y_c^*$*, then* $\tau_1 = 0$ *and* $\zeta_1 = \beta^{-1}(x - (y_c^* - \beta\zeta_c^*))$*. Moreover, the value of the optimal impulse control policy reads as*

$$V_c(x) = F_c^*(x) = \begin{cases} \beta^{-1}(\lambda x + J(y_c^*)) & x \geq y_c^* \\ (R_r\pi)(x) - \beta^{-1} I(y_c^*)\psi(x) & x < y_c^* \end{cases} \tag{4.12}$$

and the optimal intervention times reads as $\tau_{i+1} = \inf\{t \geq \tau_i : X_t^\nu \geq y_c^*\}$, $i \geq 1$.

Proof. Since the policy described above is admissible, it is clear that $F_c^*(x) \leq V_c(x)$ for all $x \in \mathbf{R}_+$. To prove the opposite inequality, we first observe that $F_c^* \in C^1(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus \{y_c^*\})$ and that

$$F_c^{*\prime\prime}(y_c^*+) = 0 \leq |(R_r\pi)''(y_c^*) - \beta^{-1}I(y_c^*)\psi''(y_c^*)| = |F_c^{*\prime\prime}(y_c^*-)| < \infty$$

implying that F_c^* is stochastically $C^2(\mathbf{R}_+)$. Moreover, we find that $((\mathscr{A} - r)F_c^*)(x) + \pi(x) = 0$ on $(0, y_c^*)$ and that $((\mathscr{A} - r)F_c^*)(x) + \pi(x) = \beta^{-1}(\theta(x) - rJ(y_c^*))$ on (y_c^*, ∞). On the other hand, (3.4) implies that

$$I'(x) = \frac{2(rJ(x) - \theta(x))}{\sigma^2(x)\psi'(x)}.$$

Since I is non-decreasing on $(\hat{x}, \infty)$ and $\hat{x} < y_c^*$, we find that $\theta(x) \leq rJ(x)$ on (y_c^*, ∞). This implies that $((\mathscr{A} - r)F_c^*)(x) + \pi(x) \leq r\beta^{-1}(J(x) - J(y_c^*)) \leq 0$ for all $x \in (y_c^*, \infty)$, since J is non-increasing on (y_c^*, ∞). Hence $((\mathscr{A} - r)F_c^*)(x) + \pi(x) \leq 0$ for all $x \in \mathbf{R}_+ \setminus \{y_c^*\}$.

Our next task is to show that F_c^* satisfies the quasi-variational inequality

$$F_c^*(x) \geq \sup_{\beta\zeta\in[0,x]} [F_c^*(x-\beta\zeta)+\lambda\zeta - c]$$

for all $x \in \mathbf{R}_+$. Note that this quasi-variational inequality can also be written in the form $F_c^*(x) \geq \beta^{-1}(\lambda x - \beta c) + \sup_{y\in[0,x]}[F_c^*(y) - \lambda\beta^{-1}y]$. Define now the mapping $A : \mathbf{R}_+ \to \mathbf{R}$ as

$$A(x) = F_c^*(x) - \beta^{-1}(\lambda x - \beta c) - \sup_{y\in[0,x]} [F_c^*(y) - \lambda\beta^{-1}y].$$

Utilizing (4.10) we find that

$$F_c^{*\prime}(x) = \begin{cases} \lambda\beta^{-1} & x \geq y_c^* \\ \beta^{-1}\left(\lambda + \psi'(x)\left(I(x) - I(y_c^* - \beta\zeta_c^*)\right)\right) & x < y_c^*. \end{cases}$$

Since I is non-increasing on $(0,\hat{x})$ and $y_c^* - \beta\zeta_c^* < \hat{x}$, we find that $F_c^{*\prime}(x) > \lambda\beta^{-1}$ on $(0, y_c^* - \beta\zeta_c^*)$. Moreover, the condition (4.10) implies that $I(x) - I(y_c^*) < 0$ for all $x \in (y_c^* - \beta\zeta_c^*, y_c^*)$. Therefore $F_c^{*\prime}(x) \leq \lambda\beta^{-1}$ on $(y_c^* - \beta\zeta_c^*, y_c^*)$. Finally, since $F_c^{*\prime}(x) = \lambda\beta^{-1}$ on (y_c^*, ∞), we find that the function $x \mapsto F_c^*(x) - \lambda\beta^{-1}x$ attains a global maximum at $y_c^* - \beta\zeta_c^*$. Therefore

$$\sup_{y\in[0,x]} [F_c^*(y) - \lambda\beta^{-1}y] = \begin{cases} F_c^*(y_c^* - \beta\zeta_c^*) - \lambda\beta^{-1}(y_c^* - \beta\zeta_c^*) & x > y_c^* - \beta\zeta_c^* \\ F_c^*(x) - \lambda\beta^{-1}x & x \leq y_c^* - \beta\zeta_c^*. \end{cases}$$

Using (4.11) the mapping A can now be written in the form

$$A(x) = \begin{cases} 0 & x \geq y_c^* \\ (R_r\pi)(x) - I(y_c^*)\psi(x) - \lambda\beta^{-1}x - J(y_c^*) & x \in (y_c^* - \beta\zeta_c^*, y_c^*) \\ c & x \leq y_c^* - \beta\zeta_c^*. \end{cases}$$

Since $\lim_{x\to y_c^*-} A(x) = 0$ and $A'(x) = \psi'(x)(I(x) - I(y_c^*)) < 0$ for all $x \in (y_c^* - \beta\zeta_c^*, y_c^*)$, we find that $A(x) \geq 0$ on $(y_c^* - \beta\zeta_c^*, y_c^*)$; hence $A(x) \geq 0$ for all $x \in \mathbf{R}_+$.

Finally, given the continuity of F_c^* and the fact that the state-space $(0, y_c^*)$ of the controlled diffusion X_t^ν is bounded, we observe that $\mathbf{E}_x[e^{-rt}F_c^*(X_t^\nu)] \to 0$ for all $x \in \mathbf{R}_+$ as $t \to \infty$. Thus $F_c^*(x) \geq V_c(x)$ for all $x \in \mathbf{R}_+$ and $\nu^* = \nu_{(\zeta_c^*, y_c^*)}$. □

Theorem 4.2 demonstrates that the admissible policy $\nu^* = \nu_{(\zeta_c^*, y_c^*)}$ is optimal and F_c^* is the value of the optimal policy under Assumptions 2.1. This observation is of interest since it emphasizes the role of the mapping θ as the principal determinant of both the existence and uniqueness of an optimal policy. In comparison to Alvarez and Virtanen 2006, it is worth emphasizing that the conditions of Theorem 4.2 are relatively weak since no concavity assumptions are needed and only the monotonicity and continuity properties of the mapping θ are required for guaranteeing the validity of our results.

Having studied the existence and uniqueness of an optimal impulse control policy, we now plan to analyze the comparative static properties of the optimal policy and its value. In accordance with our earlier findings in Corollary 3.5 and Corollary 3.7 we can now establish the following

Corollary 4.3. *Let Assumptions 2.1 be satisfied. Then the value V_c is a decreasing function of both the parameter β and the sunk cost c and an increasing function of the parameter λ. Especially, if $\lambda = \beta$ then $\partial \zeta_c^*/\partial \lambda = -\zeta_c^*/\lambda < 0$ and both the optimal exercise boundary y_c^* and the optimal generic initial state $z_c^* = y_c^* - \lambda \zeta_c^*$ are independent of λ and β.*

Proof. Denote as V_{c,λ_i} the value of the optimal policy associated with the parameter λ_i, $i = 1,2$, and assume that $\lambda_1 > \lambda_2$. It is now clear from the proof of Theorem 4.2 that the value V_{c,λ_1} satisfies the variational inequality $(\mathscr{A}V_{c,\lambda_1})(x) - rV_{c,\lambda_1}(x) + \pi(x) \leq 0$ for all $x \in \mathbf{R}_+ \setminus \{y^*_{c,\lambda_1}\}$, where y^*_{c,λ_1} denotes the optimal exercise threshold associated with the parameter λ_1. Moreover, since V_{c,λ_1} satisfies also the sufficient quasi-variational inequality

$$V_{c,\lambda_1}(x) \geq \sup_{y\in[0,x]} \left[V_{c,\lambda_1}(y) + \frac{\lambda_1}{\beta}(x-y)\right] - c \geq \sup_{y\in[0,x]} \left[V_{c,\lambda_1}(y) + \frac{\lambda_2}{\beta}(x-y)\right] - c$$

we find that $V_{c,\lambda_1}(x) \geq V_{c,\lambda_2}(x)$. Proving that V_c is a decreasing function of both the parameter β and the sunk cost c is entirely analogous. Finally, if $\lambda = \beta$ then the necessary conditions (4.9) imply that y_c^* and z_c^* are independent of λ and β. However, since $z_c^* = y_c^* - \lambda \zeta_c^*$ we find that $\partial \zeta_c^*/\partial \lambda = -\zeta_c^*/\lambda < 0$ by partial differentiation. □

Corollary 4.3 summarizes the impact of parametric changes on the value of the optimal policy. Interestingly, and in contrast to our findings on the associated control problems, Corollary 4.3 proves that even though the optimal exercise boundary and generic initial state are independent of the parameters λ and β whenever $\lambda = \beta$ it also shows that the optimal impulse ζ_c^* is a decreasing function of λ. Thus, the harmonization of the parameters λ and β does not result into the neutrality of the optimal policy. Unfortunately, it is difficult to characterize explicitly the impact of parametric changes in either λ or β on the optimal exercise boundary y_c^* and the optimal generic initial state $y_c^* - \beta\zeta_c^*$. Fortunately, the impact of changes in the sunk cost c can be explicitly characterized by studying the behavior of the implicit curves $I(y_c^*) - I(y_c^* - \beta\zeta_c^*) = 0$ and $J(y_c^*) - J(y_c^* - \beta\zeta_c^*) = -\beta c$. Implicit differentiation of these curves with respect to c together with the relation $J'(x) = -\psi(x)I'(x)$ yield the conditions

$$\frac{d(y_c^* - \beta\zeta_c^*)}{dc} = -\frac{\beta}{I'(y_c^* - \beta\zeta_c^*)[\psi(y_c^* - \beta\zeta_c^*) - \psi(y_c^*)]} < 0 \tag{4.13}$$

and

$$\frac{dy_c^*}{dc} = -\frac{\beta}{I'(y_c^*)[\psi(y_c^* - \beta\zeta_c^*) - \psi(y_c^*)]} > 0. \tag{4.14}$$

In other words, the optimal threshold y_c^* decreases, the regeneration state $y_c^* - \beta\zeta_c^*$ increases and, therefore, the optimal impulse ζ_c^* decreases as the fixed intervention cost c decreases. This observation is intuitively clear, since the proof of Lemma 4.1 implies that $\lim_{c\to 0+} y_c^* = \hat{x}$ and $\lim_{c\to 0+} \zeta_c^* = 0$. Moreover, by continuity of the increasing fundamental solution ψ, we discover that $\lim_{c\to 0+} \frac{dy_c^*}{dc} = \infty$ and $\lim_{c\to 0+} \frac{d\zeta_c^*}{dc} = -\infty$. Finally, by ordinary differentiation we find that

$$\frac{dV_c}{dc}(x) = \begin{cases} \beta^{-1}J'(y_c^*)\frac{dy_c^*}{dc} & x \geq y_c^* \\ -\beta^{-1}\psi(x)I'(y_c^*)\frac{dy_c^*}{dc} & x < y_c^* \end{cases} < 0.$$

Summarizing, we formulate the following lemma characterizing the impact of the transaction cost c on the value of the optimal policy (see, for example, Øksendal 1999 for a similar observation).

Lemma 4.4. *Let Assumptions 2.1 be satisfied. Then,* $d(y_c^* - \beta\zeta_c^*)/dc < 0$, $dy_c^*/dc > 0$, *and* $d\zeta_c^*/dc > 0$. *Moreover,* $\lim_{c\to 0+} y_c^* = \hat{x}$, $\lim_{c\to 0+} \zeta_c^* = 0$, $\lim_{c\to 0+} \frac{dy_c^*}{dc} = \infty$, $\lim_{c\to 0+} \frac{d\zeta_c^*}{dc} = -\infty$ *and*

$$\lim_{c\to 0+} \frac{dV_c}{dc}(x) = -\infty \tag{4.15}$$

for all $x \in \mathbf{R}_+$.

In light of our general findings and the explicit characterization of the value of the optimal impulse policy, it would be of interest to analyze how increased volatility affects the optimal boundary y_c^* and the optimal impulse ζ_c^*. Unfortunately, as our results indicated the value function is neither concave nor convex on the entire state space of the controlled process. Thus, presenting a set of easily verifiable general conditions under which the sign of the relationship between increased volatility and the optimal policy could be unambiguously characterized is extremely difficult, if possible at all.

4.3 ORDERING OF THE VALUES

We have established in Lemmas 3.4, 3.6 and in Theorem 4.2 that all three control problems (3.6), (3.11) and (2.5) are solvable that under Assumptions 2.1. In this subsection we study how the value functions as well as the marginal values of these associated stochastic control problems can be ordered. To this end, recall first the expression (3.9) for the value of the singular control problem (3.6). This value can be rewritten as

$$K(x) = \begin{cases} \beta^{-1}(\lambda x + J(\hat{x})) & x \geq \hat{x} \\ (R_r\pi)(x) - \beta^{-1}I(\hat{x})\psi(x) & x < \hat{x}. \end{cases} \tag{4.16}$$

In the next lemma we show that the value (4.16) has an interesting maximality property. This lemma extends the results obtained in Alvarez and Virtanen 2006 in a model subject to a linear exercise payoff.

Lemma 4.5. *Define the continuously differentiable mapping $H : \mathbf{R}_+^2 \to \mathbf{R}_+$ as*

$$H(x,y) = \begin{cases} \beta^{-1}(\lambda x + J(y)) & x \geq y \\ (R_r\pi)(x) - \beta^{-1}I(y)\psi(x) & x < y \end{cases}$$

and let Assumptions 2.1 be satisfied. Then $K(x) = H(x,\hat{x}) > H(x,y)$ and $K'(x) = H_x(x,\hat{x}) > H_x(x,y)$ for all $(x,y) \in \mathbf{R}_+ \times \mathbf{R} \setminus \{\hat{x}\}$. Moreover, $H_y(x,y) < 0$, for all $(x,y) \in \mathbf{R}_+ \times (\hat{x},\infty)$.

Proof. By the monotonicity properties of the function I we find that

$$H_y(x,y) = -\beta^{-1}I'(y)\min(\psi(x),\psi(y)) \gtreqqless 0,\ y \lesseqqgtr \hat{x}.$$

This observation coupled with the identity $K(x) = H(x,\hat{x})$ proves that $K(x) = H(x,\hat{x}) > H(x,y)$ for all $(x,y) \in \mathbf{R}_+ \times \mathbf{R}_+ \setminus \{\hat{x}\}$. Moreover, since

$$H_{xy}(x,y) = \begin{cases} 0 & x \geq y \\ -\beta^{-1}I'(y)\psi'(x) & x < y, \end{cases}$$

the monotonicity properties of I imply that $K'(x) = H_x(x,\hat{x}) > H_x(x,y)$ for all $(x,y) \in \mathbf{R}_+ \times \mathbf{R} \setminus \{\hat{x}\}$. □

Lemma 4.5 shows that the value of the associated singular stochastic control problem does not only dominate but also grows faster than any other solution of the associated free boundary value problem

$$(\mathscr{A}u)(x) - ru(x) + \pi(x) = 0, \quad x < y$$
$$u'(x) = \lambda/\beta, \quad x \geq y.$$

This result is of interest since it emphasizes the role of the flexibility of the admissible policy as the main determinant of both the actual value and its growth rate. As we will later observe, it is these variational inequalities which relate the considered stochastic impulse control problem to both the associated singular stochastic control problem and to the associated optimal stopping problem.

Our main characterization of the impact of the flexibility of the applied policy on the values and the marginal values of the considered stochastic control problems are now summarized in the following (see Alvarez and Virtanen 2006 for a similar observation in the linear payoff case).

Theorem 4.6. *Let Assumptions 2.1 be satisfied. Then*

$$K(x) \geq V_c(x) \geq G_c(x) \quad \textit{and} \quad K'(x) \geq V_c'(x) \geq G_c'(x)$$

for all $x \in \mathbf{R}_+$. Moreover, $\bar{x}_c > y_c^* > \hat{x}$ for all $c > 0$.

Proof. In order to prove that $K(x) \geq G_c(x)$ for all $x \in \mathbf{R}_+$, observe that K satisfies the variational inequality $(\mathscr{A}K)(x) - rK(x) + \pi(x) \leq 0$ for all $x \in \mathbf{R}_+$. Moreover, since I is decreasing on $(0,\hat{x})$, we find that the inequality $K(x) - (\lambda\beta^{-1}x - c) \geq \beta^{-1}J(\min(x,\hat{x})) \geq 0$ holds for all $x \in \mathbf{R}_+$. Thus K satisfies the sufficient variational inequalities guaranteeing that $K(x) \geq G_c(x)$ for all $x \in \mathbf{R}_+$. The inequality $K'(x) \geq G_c'(x)$ for all $x \in \mathbf{R}_+$ is now a straightforward consequence of the representation (3.10) and Lemma 4.5.

Inequality $K(x) \geq V_c(x)$ for all $x \in \mathbf{R}_+$ follows directly from Lemma 4.5 and the representation (4.12). On the other hand, as was established in the proof of Theorem 4.2, the value function V_c is continuously differentiable on whole of $\mathbf{R}_+$, twice continuously differentiable on $\mathbf{R}_+ \setminus \{y_c^*\}$ and satisfies the variational inequality $(\mathscr{A}V_c)(x) - rV_c(x) + \pi(x) \leq 0$ for all $x \in \mathbf{R}_+ \setminus \{y_c^*\}$.

Moreover, since

$$V_c(x) \geq \sup_{\beta\zeta \leq x} [\lambda\zeta - c + V_c(x - \beta\zeta)] \geq \lambda\beta^{-1}x - c$$

for all $x \in \mathbf{R}_+$, we observe that V_c satisfies the sufficient quasi-variational inequalities guaranteeing that $V_c(x) \geq G_c(x)$ for all $x \in \mathbf{R}_+$.

It is clear from the proof of Lemma 4.1 that $y_c^* > \hat{x}$. Moreover, since

$$0 \leq V_c(x) - G_c(x) = \beta^{-1}\psi(x)\,(I(\bar{x}_c) - I(y_c^*))$$

for all $x \in (0, \min(y_c^*, \bar{x}_c))$ and both of the thresholds $\bar{x}_c$ and y^* are attained on the set where $I(x)$ is non-decreasing, we find that $\bar{x}_c \geq y_c^*$.

It remains to establish that $K'(x) \geq V_c'(x) \geq G_c'(x)$ for all $x \in \mathbf{R}_+$. Again, the inequality $K'(x) \geq V_c'(x)$ for all $x \in \mathbf{R}_+$ follows directly from Lemma 4.5. Since $\bar{x}_c \geq y_c^* \geq \hat{x}$, we find that

$$V_c'(x) - G_c'(x) \geq \begin{cases} \beta^{-1}J(y_c^* - \beta\zeta_c^*) & y_c^* < \bar{x}_c \leq x \\ \beta^{-1}\psi(x)[I(\bar{x}_c) - I(x)] & y_c^* \leq x < \bar{x}_c \text{ or } x < y_c^* < \bar{x}_c. \end{cases}$$

Consequently, we find that $V_c'(x) - G_c'(x) \geq 0$ for all $x \in \mathbf{R}_+$ since both of the thresholds y_c^* and $\bar{x}_c$ are attained on the set where I is non-decreasing. □

Theorem 4.6 states a strong ordering for the stochastic control problems in terms of their values, marginal values, and continuation regions. Interestingly, Theorem 4.6 proves that increased policy flexibility does not only increase the value of the optimal stochastic control, it increases its marginal value as well. This observation is interesting from the point of view of financial and economical applications since essentially it implies that increased cash flow management flexibility does not only increase the value of a rationally managed corporation it also increases the rate at which this value grows and, therefore, *Tobin's marginal q* associated to the particular cash flow management problem.

The proof of Theorem 4.6 relies on Lemma 4.5. However, it is of interest to notice that the ordering of the optimal values can be justified by a direct inclusion argument. More precisely, it is clear that within our problem specifications, an admissible stopping policy is also an admissible impulse control policy corresponding to a single impulse $\zeta = (X_\tau/\beta)\chi_{\tau<H_0}$ which takes the underlying diffusion to 0 at the stopping time. On the other hand, since an admissible impulse control policy is non-negative, non-decreasing, right-continuous and

$\{\mathscr{F}_t\}$-adapted, we find that since the value of the associated singular stochastic control problem constitutes the largest attainable value within this class of policies it, in turn, dominates the value of an admissible impulse control policy thus resulting into the desired ordering of the values. However, we emphasize that this simple argument cannot be used for the ordering of the marginal values, which is an essentially more delicate result.

5 ILLUSTRATION: CONTROLLED GEOMETRIC BROWNIAN MOTION

In order to illustrate our results explicitly, we now assume that the underlying controlled geometric Brownian motion evolves according to the dynamics characterized by the stochastic differential equation

$$X_t^\nu = x + \int_0^t \mu X_s^\nu ds + \int_0^t \sigma X_s^\nu dW_s - \sum_{\tau_k \le t} \beta \zeta_k,\ 0 \le t \le \tau_0^\nu, \tag{5.1}$$

where $\mu > 0$ and $\sigma > 0$ are exogenously determined known parameters. For the sake of the finiteness of the value of the considered stochastic control problems, we assume that $r > \mu$, that is, that the discount rate dominates the expected per capita growth rate of the controlled GBM. It is well-known that in this case the fundamental solutions read as $\psi(x) = x^\kappa$ and $\varphi(x) = x^\phi$, where

$$\kappa = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$$

and

$$\phi = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0.$$

Given the considered controlled process, we now assume that the revenue flow accrued from continuing operation reads as $\pi(x) = x^\alpha$, where $\alpha \in (0,1)$. Hence, we observe that $\theta(x) = \beta x^\alpha - (r-\mu)\lambda x$ implying that the conditions of Lemma 3.1 are satisfied and that

$$x^* = \operatorname{argmax}\{\theta(x)\} = \left(\frac{\alpha\beta}{(r-\mu)\lambda}\right)^{1/(1-\alpha)}.$$

Moreover, standard integration implies that $(R_r\pi)(x) = x^\alpha/(r-\delta(\alpha))$, where $\delta(\alpha) = \alpha\mu + \sigma^2\alpha(\alpha-1)/2$.

The value of the optimal singular stochastic control policy reads as

$$K(x) = \begin{cases} \frac{\lambda}{\beta}(x-\hat{x}) + \frac{1}{r}\left(\hat{x}^{\alpha} + \frac{\lambda\mu}{\beta}\hat{x}\right) & x \geq \hat{x} \\ \frac{x^{\alpha}}{(r-\delta(\alpha))} + \frac{1}{\kappa}\left(\frac{\lambda}{\beta}\hat{x} - \frac{\alpha\hat{x}^{\alpha}}{(r-\delta(\alpha))}\right)\left(\frac{x}{\hat{x}}\right)^{\kappa} & x < \hat{x}, \end{cases} \tag{5.2}$$

where the optimal threshold $\hat{x}$ reads as

$$\hat{x} = \left(\frac{\alpha\beta(\kappa-\alpha)}{(r-\delta(\alpha))(\kappa-1)\lambda}\right)^{1/(1-\alpha)} = \left(1 + \frac{1-\alpha}{\alpha-\phi}\right)^{1/(1-\alpha)}.$$

Since

$$\frac{\partial\phi}{\partial\sigma} = \frac{2\phi(\phi-1)}{\sigma(\kappa-\phi)} > 0$$

we immediately find that

$$\frac{\partial\hat{x}}{\partial\sigma} = \left(\frac{1-\phi}{\alpha-\phi}\right)^{\alpha/(1-\alpha)} \frac{x^{*}}{(\alpha-\phi)^{2}}\frac{\partial\phi}{\partial\sigma} > 0.$$

Hence, we find that increased volatility increases the optimal threshold at which the irreversible policy should be exercised. Moreover, standard differentiation also yields that

$$\frac{\partial\hat{x}}{\partial\beta} = \frac{\hat{x}}{(1-\alpha)\beta} > 0 \quad \text{and} \quad \frac{\partial\hat{x}}{\partial\lambda} = -\frac{\hat{x}}{(1-\alpha)\lambda} < 0$$

demonstrating along the lines of our Corollary 3.5 that the optimal exercise threshold is an increasing function of the parameter β and a decreasing function of the parameter λ. The optimal exercise boundary is illustrated as a function of the underlying volatility in Figure 1 for $\beta = 0.9, 1, 1.1$ under the assumption that $r = 0.045, \mu = 0.025, \alpha = 0.5$, and $\lambda = 10$.

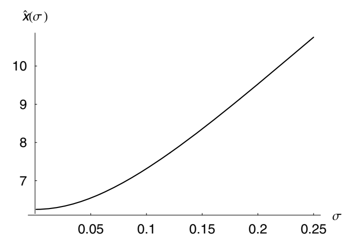

Figure 1. The optimal exercise boundary $\hat{x}(\sigma)$

The value of the associated optimal stopping problem reads as

$$G_c(x) = \begin{cases} \frac{\lambda}{\beta}x - c & x \geq \bar{x}_c \\ \frac{x^{\alpha}}{(r-\delta(\alpha))} + \left(\frac{\lambda}{\beta}\bar{x}_c - \frac{\bar{x}_c^{\alpha}}{(r-\delta(\alpha))} - c\right)\left(\frac{x}{\bar{x}_c}\right)^{\kappa} & x < \bar{x}_c, \end{cases} \tag{5.3}$$

where the optimal stopping boundary $\bar{x}_c > \hat{x}$ is the unique root of the equation

$$\bar{x}_c^{\alpha} - \frac{(\kappa-1)(r-\delta(\alpha))\lambda}{\beta(\kappa-\alpha)}\bar{x}_c + \frac{\kappa c(r-\delta(\alpha))}{\kappa-\alpha} = 0.$$

The optimal exercise boundary $\bar{x}_c$ is illustrated as a function of the underlying volatility in Figure 2 for $\beta = 0.9, 1, 1.1$ under the assumption that $r = 0.045, \mu = 0.025, \alpha = 0.5, c = 1$, and $\lambda = 10$.

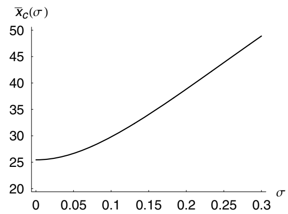

Figure 2. The optimal exercise boundary $\bar{x}_c(\sigma)$

The value of the considered stochastic impulse control problem reads as

$$V_c(x) = \begin{cases} \frac{\lambda}{\beta}(x - y_c^*) + \frac{1}{\kappa}\left(\frac{(\kappa-\alpha)y_c^{*\alpha}}{(r-\delta(\alpha))} + \frac{\lambda}{\beta}y_c^*\right) & x \geq y_c^* \\ \frac{x^{\alpha}}{(r-\delta(\alpha))} - \frac{1}{\kappa}\left(\frac{\alpha y_c^{*\alpha}}{r-\delta(\alpha)} - \frac{\lambda}{\beta}y_c^*\right)\left(\frac{x}{y_c^*}\right)^{\kappa} & x < y_c^*, \end{cases} \tag{5.4}$$

where the optimal impulse threshold y_c^* and generic initial state $z_c^* = y_c^* - \beta\zeta_c^*$ are the unique roots of the optimality conditions

$$\alpha\beta(y_c^{*\alpha-\kappa} - z_c^{*\alpha-\kappa}) = (r - \delta(\alpha))\lambda(y_c^{*1-\kappa} - z_c^{*1-\kappa})$$

and

$$\beta(\kappa-\alpha)(y_c^{*\alpha} - z_c^{*\alpha}) - \lambda(r-\delta(\alpha))(\kappa-1)(y_c^* - z_c^*) = -\kappa\beta(r-\delta(\alpha))c.$$

Unfortunately, solving these non-linear equations explicitly is difficult (if possible at all). Hence, we illustrate numerically the optimal exercise threshold y_c^*, the optimal impulse ζ_c^*, the ratio ζ_c^*/y_c^*, and the optimal generic initial state $y_c^* - \zeta_c^*$ in Table 1 under the assumption that $r = 0.045, \mu = 0.025, \alpha = 0.5, c = 1, \beta = 1$, and $\lambda = 10$. Table 1 clearly indicates that increased volatility does not only increase the optimal threshold at which the impulse policy is irreversibly exercised. It simultaneously increases both the size of the optimal policy and the optimal generic initial state. This result is of interest from the point of view

σ	0.01	0.05	0.1	0.15	0.2	0.25
y_c^*	9.697	10.168	11.443	13.196	15.222	17.399
ζ_c^*	5.215	5.511	6.319	7.447	8.780	10.252
ζ_c^*/y_c^*	0.5377	0.5420	0.5522	0.5643	0.5768	0.5892
$y_c^* - \zeta_c^*$	4.482	4.656	5.124	5.749	6.443	7.147

Table 1. The Impact of Increased Volatility

of risk management since it clearly demonstrates that increased volatility will result both into larger but less frequent dividends and a larger generic initial capital protecting the rationally managed corporation from future unfavorable yet uncertain events (i.e. a larger capital buffer). It is also worth emphasizing that our results indicate that the optimal dividend-capital-ratio ζ_c^*/y_c^* is also an increasing function of volatility. Consequently, even though increased volatility increases the generic initial state $y_c^* - \zeta_c^*$, it simultaneously decreases the ratio between the buffers and the optimal capital.

In order to analyze numerically the impact of a change in the parameter β on the risk sensitivity of the optimal impulse policy numerically illustrated in Table 2 under the assumption that $r = 0.045, \mu = 0.025, \alpha = 0.5, c = 1, \beta = 1.1$, and $\lambda = 10$ and in Table 3 under the assumption that $r = 0.045, \mu = 0.025, \alpha = 0.5, c = 1, \beta = 0.9$, and $\lambda = 10$. Along the lines of our previous findings on both the associated optimal stopping problem and the associated singular stochastic control problem, our numerical illustrations seem to indicate that the optimal variables are increasing as functions of the parameter β and decreasing as functions of the parameter λ.

σ	0.01	0.05	0.1	0.15	0.2	0.25
y_c^*	11.562	12.122	13.641	15.727	18.138	20.725
ζ_c^*	6.091	6.437	7.382	8.699	10.257	11.977
ζ_c^*/y_c^*	0.5268	0.5310	0.5412	0.5532	0.5655	0.5779
$y_c^* - \zeta_c^*$	5.471	5.685	6.259	7.027	7.881	8.747

Table 2. The Impact of Increased Volatility

σ	0.01	0.05	0.1	0.15	0.2	0.25
y_c^*	7.989	8.377	9.429	10.876	12.55	14.35
ζ_c^*	4.394	4.643	5.323	6.272	7.394	8.635
ζ_c^*/y_c^*	0.55	0.5542	0.5646	0.5767	0.5892	0.6017
$y_c^* - \zeta_c^*$	3.595	3.734	4.106	4.604	5.155	5.715

Table 3. The Impact of Increased Volatility

6 CONCLUDING COMMENTS

In this study, we considered a broad class of stochastic impulse control problems arising in the literature on rational cash flow management and optimal harvesting. We presented a set of weak conditions guaranteeing the existence and uniqueness of an optimal pair characterizing the state at which the impulse policy should be exerted and the size of the optimal impulse policy. We derived the value of the optimal policy and characterized its sensitivity with respect to changes in the cost parameters. We also studied two associated stochastic control problems and presented a general ordering for the values as well as for the marginal values of the considered problems. In line with economic intuition, our findings supported the view according to which increased policy flexibility should increase the value of a rationally managed corporation. Moreover, our results indicated that the sign of the relationship between policy flexibility and the rate at which the value of the optimal policy is increasing as a function of the controlled diffusion is unambiguously positive as well.

Even though our results are relatively general in the sense that the controlled diffusion was only assumed to be one-dimensional, they are based on the idea that the admissible bounded variation control policy is one-sided. It would, therefore, be of interest to study whether our findings could be extended to a more general setting where the applied impulse control policy can drive the underlying diffusion both upwards and downwards (as in the recent study by Weerasinghe (2005)). Unfortunately, such extension is out of the scope of the present study and, therefore, left for future research.

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PAPER III

Jukka Lempa: *On Two-sided Optimal Stopping of a Linear Diffusion*, 2007

ON TWO-SIDED OPTIMAL STOPPING OF A LINEAR DIFFUSION

Jukka Lempa

ABSTRACT

We consider a class of two-sided optimal stopping problems of a linear diffusion. We propose an alternative approach to these problems combining results from classical theory of diffusions, fixed point theory and ordinary static optimization. We establish that the optimal stopping rule can be associated with the unique fixed point of a certain auxiliary function. The results of the study are illustrated with the perpetual American straddle option.

Keywords: optimal stopping, linear diffusion, fixed point theory, two-sided stopping rules, perpetual American straddle option

1 INTRODUCTION

A typical approach to optimal stopping problems begins with a guess. Relying on intuition of some type, one first attempts to guess the stopping rule which yields the optimal value (see Alvarez 2001, Guo and Shepp 2001 and Shepp and Shiryaev 1993). The next phase is then to present a set of conditions under which this intuitive guess can be proved to be correct – usual tricks of the trade in the proving phase include variational inequalities, various transformation techniques (random time changes, measure transformations, etc.) and free boundary formulations (for a recent exposition on solution techniques for optimal stopping, see Peskir and Shiryaev 2006). In many cases the optimal stopping turns out to be a threshold rule. There is a myriad of studies where the resulting optimal stopping rule is a one-sided threshold rule (see e.g. Alvarez 2001, Jacka 1991, McKean 1965, Myneni 1992, Salminen 1999 and

Shepp and Shiryaev 1993). This is not surprising since there is a wide variety of problems with practically interesting applications, for example problems of American call and put option-type, where the optimal decision making rules are one-sided. In the recent years there has been an increasing interest in problems where the optimal rules turn out to be two-sided (see Alobaidi and Mallier 2002, Alobaidi and Mallier 2006, Beibel and Lerche 1997, Beibel and Lerche 2002, Chiarella and Ziogas 2005, Dayanik 2006 and Gerber and Shiu 1994). Such problems include the optimal stopping problems of American straddle and strangle (i.e. combined put and call) option-type.

The analysis of the current study begins also with a guess. More precisely, we set up a class of optimal stopping problems of linear diffusions, including the perpetual American straddles and strangles, and guess that the resulting unique optimal stopping rule is a two-sided threshold rule. We then formulate this class of problems as a class of free boundary problems and utilize a combination of results from classical theory of linear diffusions, fixed point theory and static optimization to prove that our initial guess is correct. Finally, an extra emphasis will be put on the perpetual American straddle option by means of explicit examples.

The study is organized as follows. In Section 2 we present the class of optimal stopping problems which we plan to study. In Section 3 we derive our main results on the optimal stopping rule and optimal value. The main results are illustrated with explicit examples in Section 4 and the study is concluded in Section 5.

2 THE OPTIMAL STOPPING PROLEM

As is customary, denote as $(\Omega, \mathscr{F}_t, \mathbf{P})$ the complete probability space satisfying the usual conditions. Assume that the underlying dynamics defined on $(\Omega, \mathscr{F}_t, \mathbf{P})$ are evolving on the state-space $\mathbf{R}_+$ according to a regular linear diffusion X (see Borodin and Salminen 2002, 12–13) for which the boundaries 0 and ∞ are either natural or exit-not-entrance (see Borodin and Salminen 2002, 14–15). The assumption that the state space is $\mathbf{R}_+$ is done for reasons of convenience. In fact, we could assume that the state space is any interval $\mathscr{I}$ in $\mathbf{R}$ and all our subsequent analysis would hold true. Moreover, we assume that the diffusion X does not die inside the state space $\mathbf{R}_+$ and that the basic characteristics of X, namely the scale function S, the speed measure m and the killing measure k (see Borodin and Salminen 2002, 13–14), are absolutely continuous with respect to the Lebesgue measure, have smooth derivatives and that the scale function S is twice continuously differentiable. Under these assumptions, it is known (see Borodin and Salminen 2002, 17) that the infinitesimal generator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \to C_b(\mathbf{R}_+)$ of X can be expressed as

$$\mathscr{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - c(x),$$

where the functions σ, μ and c (the *infinitesimal parameters* of X) are related to m, k and S via the formulae

$$m(x) = \frac{2}{\sigma^2(x)}e^{B(x)},\ S'(x) = e^{-B(x)},\ k(x) = \frac{2c(x)}{\sigma^2(x)}e^{B(x)},$$

where $B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}dy$. Furthermore, we will denote, respectively, as ψ and φ the increasing and the decreasing fundamental solution of the ordinary second-order linear differential equation $\mathscr{A}u = ru$, where $r > 0$ (for the characterization and fundamental properties of ψ and φ, see Borodin and Salminen 2002, 18–20). It is important to emphasize that the assumed boundary classification of X implies the following limiting properties: $\lim_{x\to\infty} \psi'(x)/S'(x) = \infty$ and $\lim_{x\to 0+} \varphi'(x)/S'(x) = -\infty$. Finally, $B = (\psi'(x)\varphi(x) - \varphi'(x)\psi(x))/S'(x)$ denotes the (constant) Wronskian determinant of the underlying X.

Having the underlying dynamics set up, denote the expected present value of exercise payoff at instant t as

$$\Pi(x,t) = \mathbf{E}_x\left[e^{-rt}g(X_t)\right] \tag{2.1}$$

and consider the infinite horizon optimal stopping problem

$$V(x) = \sup_{\tau<\tau_\Delta} \Pi(x,\tau), \tag{2.2}$$

where τ is an arbitrary stopping time, τ_Δ is the first exit time of X from the state space $\mathbf{R}_+$, $r > 0$ is the discount rate and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function representing the exercise payoff. In the sequel, the function $x \mapsto \Pi(x,\tau)$ will be called the value function constituted by the stopping rule *"stop at the time τ"*, the function $x \mapsto V(x)$ will be called the optimal value function and the stopping rule *"stop at the time τ^*"* for which $\Pi(x,\tau^*) = V(x)$ for all $x \in \mathbf{R}_+$ will be called the optimal stopping rule. To fix a class of stopping problems (2.2) to study, we will make the following standing assumptions.

Assumptions 2.1. *(1) We assume that the payoff* $g \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus D)$*, where* D *is a finite subset of* $\mathbf{R}_+$ *and that the quantities* $\lim_{x\to y\pm} g'(y)$ *and* $\lim_{x\to y\pm} g''(y)$ *are finite for all* $y \in D$*. Moreover, we assume that there is positive real numbers* x_1 *and* x_2 *such that* g *satisfies the strict inequalities*

$$((\mathscr{A}-r)g)(x) \begin{cases} \geq 0, & x \in [x_1,x_2] \\ < 0, & x \notin [x_1,x_2] \end{cases} \tag{2.3}$$

and that $D \subset [x_1,x_2]$.

(2) We assume that there exists an interval N_1 *with compact closure such* $x_1 \in \bar{N}_1$ *and that the function* $x \mapsto \frac{g(x)}{\varphi(x)}$ *is decreasing in* N_1 *and increasing in* $\mathbf{R}_+ \setminus N_1$*. Similarly, we assume here exists an interval* N_2 *with compact closure such that* $x_2 \in \bar{N}_2$ *and that the function* $x \mapsto \frac{g(x)}{\psi(x)}$ *is increasing in* N_2 *and decreasing in* $\mathbf{R}_+ \setminus N_2$.

The key implication of Part *(1)* is that the continuation region of the problem (2.2) is non-empty and actually contains the interval $[x_1,x_2]$. In other words, the optimal stopping rule is never *"stop immediately"* or *"wait forever"* for the considered class of problems. Moreover, we assume that the angle points D of the payoff g, given that they exist, are also contained in the continuation region. Many payoffs with option characteristics satisfy these assumptions – an explicit illustration of such case will be presented in section 4. Finally, Part *(2)* means that both the function $x \mapsto \frac{g(x)}{\psi(x)}$ and $x \mapsto \frac{g(x)}{\varphi(x)}$ attains a *local* maximum above x_2 and below x_1, respectively. More precisely, part *(2)* implies that the interior local maximum of $x \mapsto \frac{g(x)}{\psi(x)}$ is attained in $\tilde{x}_2 = \sup N_2$ and the interior local maximum of $x \mapsto \frac{g(x)}{\varphi(x)}$ is attained in $\tilde{x}_1 = \inf N_1$. It is known from studies where resulting optimal rules are one-sided (see e.g. Alvarez 2001) that the monotonicity properties of either $x \mapsto \frac{g(x)}{\psi(x)}$ or $x \mapsto \frac{g(x)}{\varphi(x)}$, depending on the problem formulation, govern the optimal stopping thresholds. In particular, if the function $x \mapsto \frac{g(x)}{\psi(x)}$ or $x \mapsto \frac{g(x)}{\varphi(x)}$ is such that it attains a *global* interior maximum, then it can be established under additional technical assumptions similar to (2.3) that the state at which the maximum is attained is actually the optimal stopping threshold. In this light, a naive guess would be that under Assumptions 2.1, the optimal stopping thresholds coincide with the states $\tilde{x}_i$, $i = 1,2$. However, it turns out that $\tilde{x}_i$'s do not coincide with the optimal stopping thresholds but rather give some indication on the whereabouts of the optimal thresholds.

3 DERIVATION OF THE VALUE

Before proceeding into the analysis of the considered class of problems (2.2), we will first establish some auxiliary results needed in the analysis. Along the lines of Salminen 1985 (see also Alvarez 2006), we define the functions $I:\mathbf{R}_+\to\mathbf{R}$ and $J:\mathbf{R}_+\to\mathbf{R}$ as

$$\begin{aligned} I(x) &= \frac{\psi'(x)}{S'(x)}g(x)-\frac{g'(x)}{S'(x)}\psi(x),\\ J(x) &= \frac{g'(x)}{S'(x)}\varphi(x)-\frac{\varphi'(x)}{S'(x)}g(x). \end{aligned} \tag{3.1}$$

These quantities turn out to be the key ingredients of determination of the optimal stopping rule. We will now summarize some properties of I and J in the following lemma.

Lemma 3.1. *Let Assumptions 2.1 hold. Then the functions I and J defined in* (3.1) *are continuous on* $(0,x_1)\cup(x_2,\infty)$. *Moreover, they satisfy the following monotonicity and limiting properties:*

(1) the function I is decreasing in (x_1,x_2) *and increasing in the complement of* (x_1,x_2). *In addition,* $\lim_{x\to\infty}I(x)=\infty$, $\lim_{x\to 0+}I(x)\geq 0$, $I(x_1)>0$ *and* $I(x_2)<0$.

(2) the function J is increasing in (x_1,x_2) *and decreasing in the complement of* (x_1,x_2). *In addition,* $\lim_{x\to 0+}J(x)=\infty$, $\lim_{x\to\infty}J(x)\geq 0$, $J(x_2)>0$ *and* $J(x_1)<0$.

Proof. The claimed continuity properties of I and J follow from the assumed differentiability properties of g. Since the functions ψ and φ are solutions of the differential equation $\mathscr{A}u=ru$, we find that

$$J'(x)=\varphi(x)((\mathscr{A}-r)g)(x)m'(x)=-\frac{\varphi(x)}{\psi(x)}I'(x). \tag{3.2}$$

The desired monotonicity properties follow now from the positivity of ψ and φ and the condition (2.3). In order to prove the remaining limiting properties, we find that

$$I(x)=-\frac{\psi(x)^2}{S'(x)}\frac{d}{dx}\left[\frac{g(x)}{\psi(x)}\right] \tag{3.3}$$

and

$$J(x) = \frac{\varphi(x)^2}{S'(x)} \frac{d}{dx} \left[\frac{g(x)}{\varphi(x)} \right]. \tag{3.4}$$

Utilizing Part *(2)* of Assumptions (2.1), these expressions imply that $I(x_2) < 0$, $J(x_1) < 0$, $\lim_{x \to 0+} I(x) \geq 0$ and $\lim_{x \to \infty} J(x) \geq 0$. By coupling these last two limiting properties with the conditions (2.3) and (3.2), we find that $I(x_1) > 0$ and $J(x_2) > 0$. In particular, this implies that $x_2 \notin N_1$ and $x_1 \notin N_2$. Moreover, we find readily that Part *(2)* of Assumptions (2.1) imply that I attains a a local maximum in $\tilde{x}_2 = \sup N_2$. By coupling this with expression (3.3) we find that $I(\tilde{x}_2) = 0$ – analogously, we find that $J(\tilde{x}_1) = 0$. Fix now $y > \tilde{x}_2$. Since the function $(\mathscr{A} - r)g$ is continuous on $[\tilde{x}_2, y]$, the mean value theorem for integrals implies that

$$\begin{aligned} I(y) &= -\int_{\tilde{x}_2}^{y} \psi(t)((\mathscr{A} - r)g)(t)m'(t)dt \\ &= -\frac{((\mathscr{A} - r)g)(\xi_2)}{r} \left[\frac{\psi'(y)}{S'(y)} - \frac{\psi'(\tilde{x}_2)}{S'(\tilde{x}_2)} \right], \end{aligned}$$

where $\xi_2 \in (\tilde{x}_2, y)$. Since $\tilde{x}_2 > x_2$, we find that $((\mathscr{A} - r)g)(\xi_2) < 0$. The desired result $\lim_{y \to \infty} I(y) = \infty$ follows now from the limiting property $\lim_{y \to \infty} \frac{\psi'(y)}{S'(y)} = \infty$. The proof of the condition $\lim_{x \to 0+} J(x) = \infty$ is analogous. □

Lemma 3.1 provides us with useful monotonicity and limiting properties of the functions I and J. We find from the proof that Part *(1)* of Assumptions 2.1 results into the monotonicity properties of I and J, whereas Parts *(2)* and the boundary classification of X constitute the ordering of the limiting values at the boundaries and critical interior values at the points x_i, $i = 1, 2$. Having the necessary auxiliary results at our disposal, we are now in position derive the optimal value and the optimal stopping rule for the considered class of problems (2.2). To this end, we restrict our attention to Markov times of the form $\tau_{(a,b)} = \inf\{t \geq 0 : X_t \notin (a, b)\}$, where $a < x_1 < x_2 < b$ (recall Part *(1)* of Assumptions 2.1 for the definition of x_1 and x_2). The reason for this particular choice lies in the assumption (2.3). As we have noted before, the assumption (2.3) implies that the interval $[x_1, x_2]$ is included in the continuation region of (2.2). In other words, the assumption (2.3) implies the admissible thresholds rules are constituted by thresholds lying in $(0, x_1) \cup (x_2, \infty)$. It is the aim of this section to prove that the optimal stopping rule is a two-sided threshold rule, where the thresholds a^* and b^* are such that $a^* < x_1$ and $b^* > x_2$. Let

$x \in (a,b)$, where $a < x_1$ and $b > x_2$. Then it is well known that the function

$$u_1(x) := \mathbf{E}_x \left[e^{-r\tau_b}; \tau_b < \tau_a \right]$$

is the unique positive solution of $((\mathscr{A} - r)u)(x) = 0$, where $x \in (a,b)$ satisfying the boundary conditions $u(b) = 1$ and $u(a) = 0$. This implies that there exists unique non-negative constants c_1 and c_2 such that $u_1(x) = c_1\psi(x) + c_2\varphi(x)$ for all $x \in (a,b)$ (see Borodin and Salminen 2002, 33). By invoking the boundary conditions $u(b) = 1$ and $u(a) = 0$ we find that

$$u_1(x) = \frac{\psi(a)\varphi(x) - \varphi(a)\psi(x)}{\psi(a)\varphi(b) - \varphi(a)\psi(b)}$$

for all $x \in (a,b)$. It is completely analogous to establish that

$$u_2(x) := \mathbf{E}_x \left[e^{-r\tau_a}; \tau_a < \tau_b \right] = \frac{\psi(x)\varphi(b) - \varphi(x)\psi(b)}{\psi(a)\varphi(b) - \varphi(a)\psi(b)}$$

for all $x \in (a,b)$. Define now an auxiliary function $F : \mathbf{R}_+^3 \to \mathbf{R}_+$ as

$$F(x,a,b) = \mathbf{E}_x \left[e^{-r\tau_{(a,b)}} g(X_{\tau_{(a,b)}}) \right].$$

The function F is a value function constituted by a threshold stopping rule *"stop at time $\tau_{(a,b)}$"* with free boundaries a and b. Since $X_{\tau_{(a,b)}}$ is either a or b almost surely, we find that

$$\begin{aligned} F(x,a,b) &= \mathbf{E}_x \left[e^{-r\tau_b}; \tau_b < \tau_a \right] g(b) + \mathbf{E}_x \left[e^{-r\tau_a}; \tau_a < \tau_b \right] g(a) \\ &= \frac{g(b)\left(\psi(a)\varphi(x) - \varphi(a)\psi(x)\right) + g(a)\left(\psi(x)\varphi(b) - \varphi(x)\psi(b)\right)}{\psi(a)\varphi(b) - \varphi(a)\psi(b)} \\ &= \frac{\varphi(b)g(a) - g(b)\varphi(a)}{\psi(a)\varphi(b) - \varphi(a)\psi(b)}\psi(x) + \frac{g(b)\psi(a) - \psi(b)g(a)}{\psi(a)\varphi(b) - \varphi(a)\psi(b)}\varphi(x) \\ :&= h_1(a,b)\psi(x) + h_2(a,b)\varphi(x) \end{aligned} \tag{3.5}$$

for all $x \in (a,b)$. Note that $\lim_{x\to b-} F(x,a,b) = g(b)$ and $\lim_{x\to a+} F(x,a,b) = g(a)$, in other words that value-matching condition is satisfied on both boundaries a and b. Assume now that there exist a pair (a^*,b^*) maximizing the expression (3.5). After ordinary differentiation and some simple manipulations, we find that the ordinary first order conditions $\frac{\partial h_1}{\partial a}(a^*,b^*) = \frac{\partial h_1}{\partial b}(a^*,b^*) = \frac{\partial h_2}{\partial a}(a^*,b^*) = \frac{\partial h_2}{\partial b}(a^*,b^*) = 0$ for the optimality imply that a^* and b^* must satisfy the conditions

$$\begin{cases} J(a^*)\psi(b^*) + I(a^*)\varphi(b^*) = Bg(b^*) \\ J(b^*)\psi(a^*) + I(b^*)\varphi(a^*) = Bg(a^*), \end{cases} \tag{3.6}$$

where the functions I and J are defined in (3.1) and B is the (constant) Wronskian. We readily verify that given the existence of the maximizing pair (a^*, b^*), the resulting function $x \mapsto F(x, a^*, b^*)$ satisfies the smooth pasting conditions $\lim_{x \to b^*-} \frac{\partial F}{\partial x}(x, a^*, b^*) = g'(b^*)$ and $\lim_{x \to a^*+} \frac{\partial F}{\partial x}(x, a^*, b^*) = g'(a^*)$. Since the function $x \mapsto F(x, a^*, b^*)$ is r-harmonic in (a^*, b^*), the conditions (3.6) imply that

$$\begin{cases} I(b^*) - I(a^*) = 0 \\ J(b^*) - J(a^*) = 0. \end{cases} \tag{3.7}$$

The solutions of the pair (3.7) of non-linear equations give rise to two-sided stopping rules which constitute smooth value functions (see Salminen 1985, Theorem 4.7). The next lemma is our main result on the solvability of the pair (3.7).

Lemma 3.2. *Let Assumptions 2.1 hold. Then the necessary conditions* (3.7) *have a unique solution* (a^*, b^*)*, such that* $a^* < \tilde{x}_1 = \inf N_1 \leq x_1$ *and* $b^* > \tilde{x}_2 = \sup N_2 \geq x_2$.

Proof. Denote the restrictions of functions I and J to sets $(0, \tilde{x}_1)$ and $(\tilde{x}_2, \infty)$ as follows: $\check{I} = I|_{(0,\tilde{x}_1)}$, $\hat{I} = I|_{(\tilde{x}_2,\infty)}$, $\check{J} = J|_{(0,\tilde{x}_1)}$ and $\hat{J} = J|_{(\tilde{x}_2,\infty)}$. We will now proceed by proving the existence of a solution in two steps.

Step 1. Denote as $A_1 = (0, \tilde{x}_1)$. Since $\check{I}$ is increasing and $\hat{I}(\tilde{x}_2) = 0$, we find that $A_2 := \check{I}(A) = (0, \check{I}(\tilde{x}_1))$ and $A_2 \subseteq \operatorname{Dom} \hat{I}^{-1} = (0, \infty)$. Let $A_3 = \hat{I}^{-1}(A_2)$. We find that $A_3 \subsetneq (\tilde{x}_2, \infty) = \operatorname{Dom} \hat{J}$. Moreover,the monotonicity properties of $\check{I}$ and $\hat{I}$ imply that for every $z \in A_1$ there exist a unique $y_z \in A_3$ such that $I(z) = \check{I}(z) = \hat{I}(y_z) = I(y_z)$.

Step 2. Let $A_4 = \hat{J}(A_3)$. Since $\check{J}(\tilde{x}_1) = 0$, we find that $A_4 \subsetneq \operatorname{Dom} \check{J}^{-1} = (0, \infty)$. Finally, let $A_5 = \check{J}^{-1}(A_4)$. Since $A_5 \subsetneq \check{J}^{-1}((0, \infty)) = A_1$, we find that the composite $\Phi := \check{J}^{-1} \circ \hat{J} \circ \hat{I}^{-1} \circ \check{I}$ is a well-defined continuous monotonic function from $(0, \tilde{x}_1)$ onto its strict subset. Therefore Φ has a fixed point $z^* \in (0, \tilde{x}_1)$ which generates, by construction, a solution $(z^*, \hat{I}^{-1}(\check{I}(z^*)))$ for the pair (3.7).

Since Φ is continuous, it suffices to establish that $\Phi'(z^*) < 1$ for a given fixed point z^* in order to prove the uniqueness. Utilizing the fixed point property $\Phi(z^*) = z^*$ and the monotonicity properties of fundamental solutions ψ and φ, ordinary differentiation yields

$$\Phi'(z^*) = \frac{\psi(z^*)}{\psi(\hat{I}^{-1}(\check{I}(z^*)))} \cdot \frac{\varphi(\hat{I}^{-1}(\check{I}(z^*)))}{\varphi(z^*)} < 1.$$

In other words, we find that whenever the curve $\Phi(x)$ intersects the diagonal of $\mathbf{R}^2_+$, the intersection is from above. This observation completes the proof. $\square$

Lemma 3.2 shows that under Assumptions 2.1, the pair (3.7) of non-linear equations has a unique solution (a^*,b^*) such that $a^* < \tilde{x}_1$ and $\tilde{x}_2 < b^*$. Analogously to Alvarez and Lempa 2007, we find that the unique solution of the pair (3.7) can be associated to the unique fixed point of a certain auxiliary function. We also find from the proof of Lemma 3.2 how the result relies on Assumptions 2.1. In fact, Part *(1)* of 2.1 enables us to invert the functions I and J on intervals $(0,x_1)$ and (x_2,∞) and, therefore, to construct the composite function Φ, whereas Part *(2)* and boundary classification of X essentially guarantee that Φ is well-defined. It is important to note that the method of the proof of Lemma 3.2 does not rely directly on the differentiability properties assumed in part *(1)* of 2.1 but rather on the monotonicity and limiting properties these assumptions imply. Therefore the method is potentially useful in more general cases where the optimality conditions may not be as smooth as in the current case. The pair (a^*,b^*) gives rise to a unique stopping rule *"stop at time $\tau_{(a^*,b^*)}$"* which is, by Lemma 3.2, optimal for the considered class of problems (2.2) among the admissible threshold rules.

We are now in position to prove our main result on the solvability of the considered class of problems (2.2).

Theorem 3.3. *Let Assumptions 2.1 hold. Then the optimal stopping rule is* "stop at time $\tau^* := \tau_{(a^*,b^*)} = \inf\{t \geq 0 : X_t \notin (a^*,b^*)\}$" *and the optimal value reads as*

$$V(x) = \begin{cases} g(x), & x \notin (a^*,b^*) \\ \frac{\varphi(b^*)g(a^*)-g(b^*)\varphi(a^*)}{\psi(a^*)\varphi(b^*)-\varphi(a^*)\psi(b^*)}\psi(x) + \frac{g(b^*)\psi(a^*)-\psi(b^*)g(a^*)}{\psi(a^*)\varphi(b^*)-\varphi(a^*)\psi(b^*)}\varphi(x), & x \in (a^*,b^*), \end{cases}$$

where the optimal stopping boundaries $a^* < \tilde{x}_1 = \inf N_1$ and $b^* > \tilde{x}_2 = \sup N_2$ are uniquely determined by the conditions (3.7).

Proof. Denote the function defined in Theorem 3.3 as $\hat{V}$. Since

$$\hat{V}(x) = \mathbf{E}_x[e^{-r\tau_{(a^*,b^*)}} g(X_{\tau_{(a^*,b^*)}})],$$

we observe that $V(x) \geq \hat{V}(x)$ for all $x \in \mathbf{R}_+$. In order to prove the opposite inequality, we first note that since the pair (a^*,b^*) maximizes the expression (3.5), the expression $\hat{V}(x) = F(x,a^*,b^*) > F(x,a^*,x) = g(x)$ holds for all $x \in$

(a^*,b^*). In other words, $\hat{V}(x) \geq g(x)$ for all $x \in \mathbf{R}_+$. Moreover, the function $\hat{V}$ is continuous. To prove that $\hat{V}$ is r-excessive, it is now enough to show that

$$\mathbf{E}_x[e^{-r\tau}\hat{V}(X_\tau)] \leq \hat{V}(x), \tag{3.8}$$

where $\tau = \inf\{t \geq 0 : X_t \in [c,d]\}$ for arbitrary $0 < c < d < \infty$ and $x \in \mathbf{R}_+$. Note, that condition (3.8) is trivially satisfied as an equality if $x \in [c,d]$. We will now proceed by proving the condition (3.8) for arbitrary x and c such that $x < c$ – the case $x > d$ for arbitrary x and d is proved analogously. Since $x < c$, the continuity of sample paths implies that $\tau = \tau_c = \inf\{t \geq 0 : X_t = c\}$ almost surely.

We will divide the proof of the case $x > c$ into four prototype cases.

Case 1: $x < c < a^*$. Recall from the proof of Lemma 3.1 that $x_1 \notin N_2$. This implies that $a^* \notin N_2$ and, consequently, that the function $x \mapsto \frac{g(x)}{\psi(x)}$ is decreasing on $(0,a^*)$. Since $\hat{V}(x) = g(x)$ for all $x < a^*$, we find that

$$\mathbf{E}_x[e^{-r\tau}\hat{V}(X_\tau)] = \mathbf{E}_x[e^{-r\tau}g(X_\tau)] = \mathbf{E}_x[e^{-r\tau_c}]g(c) = \frac{g(c)}{\psi(c)}\psi(x) \leq g(x) = \hat{V}(x).$$

The case $b^* < x < c$ is proved completely analogously using the result that the function $x \mapsto \frac{g(x)}{\psi(x)}$ is decreasing on (b^*,∞)

Case 2: $x < a^* < c < b^*$. Recall the definition of the functions h_1 and h_2 from the expression (3.5). Since the function $x \mapsto \frac{\varphi(x)}{\psi(x)}$ is decreasing, we find that

$$\begin{aligned}
\mathbf{E}_x[e^{-r\tau}\hat{V}(X_\tau)] &= \frac{\hat{V}(c)}{\psi(c)}\psi(x) = \left(h_1(a^*,b^*) + h_2(a^*,b^*)\frac{\varphi(c)}{\psi(c)}\right)\psi(x) \\
&\leq \left(h_1(a^*,b^*) + h_2(a^*,b^*)\frac{\varphi(a^*)}{\psi(a^*)}\right)\psi(x) = \frac{g(a^*)}{\varphi(a^*)}\psi(x) \\
&\leq g(x) = \hat{V}(x).
\end{aligned}$$

The case $a^* < x < b^* < c$ is proved completely analogously.

Case 3: $x < a^* < b^* < c$. The continuity of $\hat{V}$ implies that

$$\begin{aligned}
\frac{g(b^*)}{\psi(b^*)} &= h_1(a^*,b^*) + h_2(a^*,b^*)\frac{\varphi(b^*)}{\psi(b^*)} \\
&\leq h_1(a^*,b^*) + h_2(a^*,b^*)\frac{\varphi(a^*)}{\psi(a^*)} = \frac{g(a^*)}{\psi(a^*)}.
\end{aligned}$$

The monotonicity properties of the function $x \mapsto \frac{g(x)}{\psi(x)}$ imply now that $\frac{g(x)}{\psi(x)} \geq \frac{g(c)}{\psi(c)}$, which in turn implies that

$$\mathbf{E}_x[e^{-r\tau}\hat{V}(X_\tau)] = \frac{g(c)}{\psi(c)}\psi(x) \leq g(x) = \hat{V}(x).$$

Case 4: $a^* < x < c < b^*$. This case is an immediate consequence of the r-harmonicity of $\hat{V}$ in (a^*, b^*).

We have now established that $\hat{V}$ is a r-excessive majorant of g. Since the optimal value is the smallest of such majorants, we conclude that $\hat{V}(x) \geq V(x)$ for all $x \in \mathbf{R}$. This observation completes the proof. □

Theorem 3.3 establishes under Assumptions 2.1 the two-sided threshold rule *"stop at the time $\tau_{(a^*,b^*)}$"*, where thresholds $a^* < \tilde{x}_1$ and $b^* > \tilde{x}_2$ are uniquely determined by conditions (3.7), is optimal not only among the threshold rules but among all admissible stopping rules for the considered class of optimal stopping problems. In addition to Lemma 3.2, the proof relies a classical characterization of r-excessive functions. Recall that $\tilde{x}_1$ and $\tilde{x}_2$ are the states at which the functions $x \mapsto \frac{g(x)}{\varphi(x)}$ and $x \mapsto \frac{g(x)}{\psi(x)}$ attain local maximums, respectively. In contrast to optimal stopping problems of linear diffusions with one-sided optimal stopping rules (see e.g. Alvarez 2001), it is interesting to note that the optimal boundary b^* (a^*) is strictly higher than the corresponding locally maximizing state $\tilde{x}_2$ ($\tilde{x}_1$). This results into an interesting interpretation for certain option-type payoffs which we will discuss in the next section.

4 EXPLICIT ILLUSTRATIONS

4.1 GEOMETRIC BROWNIAN MOTION

As an illustration of the main theorem 3.3, we will consider an explicit example where the linear diffusion X whose infinitesimal generator is

$$\mathscr{A} = \mu x \frac{d}{dx} + \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2},$$

where $\mu \in \mathbf{R}$ and $\sigma > 0$ are exogenously given constants. The process X is typically called a *geometric Brownian motion*. Define the payoff function $g : \mathbf{R} \to \mathbf{R}$ as

$$g(x) = \begin{cases} c - x, & x < c \\ x - c, & x \geq c, \end{cases} \tag{4.1}$$

where $c > 0$ is fixed, exogenously given transaction cost. In financial literature, the function g is typically called the payoff of the *straddle option with strike* *prices c* (see Beibel and Lerche 1997). We find that

$$((\mathscr{A}-r)g)(x)=\begin{cases}(r-\mu)x-rc, & x<c\\(\mu-r)x+rc, & x\geq c.\end{cases}$$

Assume that the constant $r>0$ satisfies $\mu<r$. Then the part *(1)* of Assumptions 2.1 is valid for $x_1=c$ and $x_2=\frac{rc}{r-\mu}$ if $\mu>0$; if $\mu<0$, then naturally $x_1=\frac{rc}{r-\mu}$ and $x_2=c$. The minimal r-excessive functions ψ and φ of X are $\psi(x)=x^{\theta^+}$ and $\varphi(x)=x^{\theta^-}$, where

$$\theta^{\pm}=\frac{1}{2}-\frac{\mu}{\sigma^2}\pm\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}} \tag{4.2}$$

are the solutions of the characteristic equation

$$\frac{1}{2}\sigma^2\theta(\theta-1)+\mu\theta-r=0. \tag{4.3}$$

To fix ideas, assume now that $\mu>0$; the case $\mu<0$ is handled analogously. Since $\mu<r$, we find that $\theta^+>1$. Coupled with (4.3), this implies that $r>\mu\theta^+$ and, consequently, that $\frac{r}{r-\mu}<\frac{\theta^+}{\theta^+-1}$. Define the auxiliary functions $g_\pm:\mathbf{R}_+\to\mathbf{R}$ as $g_\pm(x)=x^{-\theta^\pm}g(x)$. Ordinary differentiation yields

$$g'_\pm(x)=\begin{cases}x^{-\theta^\pm-1}\left[(\theta^\pm-1)x-\theta^\pm c\right], & x<c\\x^{-\theta^\pm-1}\left[(1-\theta^\pm)x+\theta^\pm c\right], & x\geq c.\end{cases}$$

Since $\frac{\theta^\pm}{\theta^\pm-1}\gtrless 1$, we find that the part *(2)* of Assumptions 2.1 is valid for

$$N_1=\left(\frac{\theta^-c}{\theta^--1},c\right),\ N_2=\left(c,\frac{\theta^+c}{\theta^+-1}\right). \tag{4.4}$$

For geometric Brownian motion, the scale density S' reads as $S'(x)=x^{-\frac{2\mu}{\sigma^2}}$ for all $x\in\mathbf{R}_+$. Therefore the optimality conditions (3.7) can now be written as

$$\begin{cases}\theta^+a^{*-\theta^-}(c-a^*)-a^{*\theta^++\frac{2\mu}{\sigma^2}}=\theta^+b^{*-\theta^-}(b^*-c)-b^{*\theta^++\frac{2\mu}{\sigma^2}}\\a^{*\theta^-+\frac{2\mu}{\sigma^2}}-\theta^-a^{*-\theta^+}(c-a^*)=b^{*\theta^-+\frac{2\mu}{\sigma^2}}-\theta^-b^{*-\theta^+}(b^*-c).\end{cases} \tag{4.5}$$

The expressions (4.4) have an interesting implication. To explain this, recall that we used $\tilde{x}_2$ to denote a local maximum of the function $x\mapsto\frac{g(x)}{\psi(x)}$. In the current example, we find that $\tilde{x}_2=\frac{\theta^+c}{\theta^+-1}$. Consider for a moment the call component of the straddle (4.1), i.e. the payoff $x\mapsto(x-c)^+$. It is a classical result that for the call component, the optimal stopping threshold coincides with $\tilde{x}_2$. On the other hand, Theorem 3.3 shows that the $b^* > \tilde{x}_2$. This phenomena has the following interpretation: introduction of a put component into a call option increases the optimal exercise threshold of the call option and, therefore, postpones the rational exercise. We also note that an analogous conclusion is valid also for the put component, in other words that addition of a call component in to a put option postpones the exercise of the put option by decreasing the optimal exercise threshold – for a recent paper with similar observations in a finite-horizon setting, see Chiarella and Ziogas 2005. It is important to stress that the same conclusions hold for a wide class of underlying diffusions. In fact, if we consider the valuation of the call component $x \mapsto (x-c)^+$ for a general underlying diffusion, it can be established in that the optimal exercise threshold of the call component coincides with the locally maximizing state $\tilde{x}_2$ (see Alvarez 2001).

We have verified that the Assumptions 2.1 are satisfied and, therefore, that the results of main theorem 3.3 are valid for our example. In table 1 we illustrate numerically the impact of increased volatility to the optimal thresholds of the straddle option and the associated call and put options under the parameter configuration $\mu = 0.03$, $r = 0.05$ and $c = 3$. The quantities a^* and b^* have been solved numerically from the pair (4.5).

σ	0.05	0.1	0.15	0.2	0.25
$\tilde{x}_1$	2.88	2.60	2.27	1.95	1.66
a^*	2.24	1.97	1.67	1.40	1.16
$\tilde{x}_2$	7.80	8.65	9.92	11.56	13.52
b^*	7.80	8.65	9.95	11.75	14.08
$\frac{\tilde{x}_2-\tilde{x}_1}{b^*-a^*}$	0.88	0.91	0.92	0.93	0.92

Table 1. Impact of increased volatility to the optimal thresholds of the straddle option and the associated call and put options for parameters $\mu = 0.03$, $r = 0.05$ and $c = 3$.

Table 1 clearly indicates that increased volatility expands the continuation region and, therefore, postpones the decision to exercise. Moreover, we find that the distance between the lower boundary a^* and the threshold $\tilde{x}_1$ is relatively high whereas the distance between the upper boundary b^* and the threshold $\tilde{x}_2$ is vanishing for small volatilities – for $\sigma = 0.05$, the absolute distance between b^* and $\tilde{x}_2$ is of magnitude 10^{-12}. The last line of Table 1 illustrates the relative error made if the straddle option is valuated using a naive procedure where both of the components are valued separately and then simply added. For the current parameter configuration, this error is relatively high, around 10%. It is important to stress that this simple measure of error is relative and that even small errors in relative scale can add up into substantial losses in absolute scale.

Finally, we illustrate graphically the shape of the optimal value function V for various volatilities under the parameter configuration $\mu = 0.03$, $r = 0.05$ and $c = 3$.

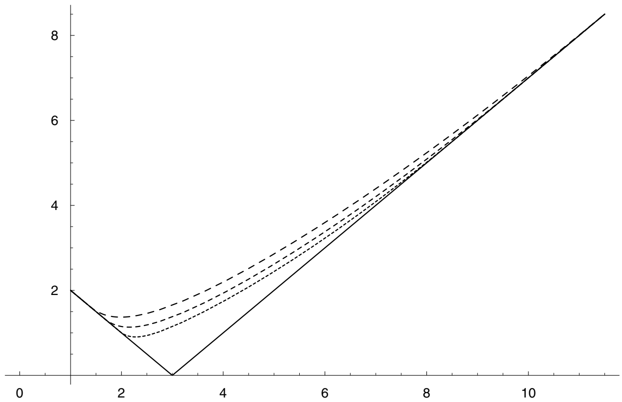

Figure 1. Optimal value function V for volatilities $\sigma = 0.1$ (lowest dashed curve), $\sigma = 0.15$ (middle dashed curve) and $\sigma = 0.2$ (upmost dashed curve) under parameter configuration $\mu = 0.03$, $r = 0.05$ and $c = 3$.

4.2 MEAN-REVERTING DIFFUSION

As an illustration in a slightly more general setting, consider still the payoff function (4.1) and assume that the underlying dynamics follow a linear diffusion X whose infinitesimal generator is

$$\mathscr{A} = \frac{1}{2}\sigma^2 x^2 \frac{d}{dx^2} + \mu x(1 - \gamma x)\frac{d}{dx^2},$$

where $\mu, \gamma, \sigma \in \mathbf{R}_+$ are exogenously determined constants. The process X is typically called a *mean-reverting diffusion*. Note that by choosing $\gamma = 0$ the diffusion X is a geometric Brownian motion. We find that

$$((\mathscr{A}-r)g)(x)=\begin{cases}\mu\gamma x^2+(r-\mu)x-rc, & x<c\\ -\mu\gamma x^2+(\mu-r)x+rc, & x\geq c.\end{cases} \tag{4.6}$$

To fix ideas, assume that $\mu>0$ and $\mu<r$; again the case $\mu<0$ is handled analogously. We find that part *(1)* of Assumptions 2.1 is fulfilled for

$$x_1=\frac{\mu-r}{2\mu\gamma}+\sqrt{\left(\frac{\mu-r}{2\mu\gamma}\right)^2+\frac{rc}{\mu\gamma}}>0,\ x_2=c.$$

It is known from the literature (see e.g. Dayanik and Karatzas 2003, Section 6.5) that the minimal r-excessive functions ψ and φ for the mean-reverting diffusion X can be expressed as

$$\begin{cases}\psi(x)=x^{\theta^+}M(\theta^+,2\theta^++\frac{2\mu}{\sigma^2},\frac{2\mu\gamma}{\sigma^2}x)\\ \varphi(x)=x^{\theta^+}U(\theta^+,2\theta^++\frac{2\mu}{\sigma^2},\frac{2\mu\gamma}{\sigma^2}x),\end{cases}$$

where θ^+ is defined in (4.2) and the functions $M:\mathbf{R}_+\to\mathbf{R}_+$ and $U:\mathbf{R}_+\to\mathbf{R}_+$ are the two linearly independent solutions of the Kummer's equation (i.e. the *confluent hypergeometric functions of first and second kind*, see Abramowitz and Stegun 1968, 504). Define the functions $g_\psi:\mathbf{R}_+\to\mathbf{R}$ and $g_\varphi:\mathbf{R}_+\to\mathbf{R}$ as $g_\psi(x)=\frac{g(x)}{\psi(x)}$ and $g_\varphi(x)=\frac{g(x)}{\varphi(x)}$. Due to the complicated nature of ψ and φ, it is next to impossible analyze the quantities g'_ψ and g'_φ analytically. Therefore we will now fix a parameter configuration $\mu=0.03$, $r=0.05$, $\gamma=0.5$ and $c=3$ and check numerically the validity of part *(2)* of Assumptions 2.1 and compute numerically the optimal boundaries a^* and b^* from the pair (3.7) and the locally maximizing states $\tilde{x}_i$, $i=1,2$, for various volatilities. The results are listed in Table 2.

We find from Table 2 that the Assumptions 2.1 are satisfied and, therefore, that the results of main theorem 3.3 are valid for our present example. Table 2 indicates also that increased volatility expands the continuation region by decreasing the lower boundary a^* and increasing the upper boundary b^*. In comparison to geometric Brownian motion, the numerics indicate that the effect of parameter γ (*the degree of mean-reversion*) is what one might expect. In fact, since γ is positive, it is not as likely that the paths of the mean-reverting diffusion attain large values as in it is the case of a geometric Brownian motion. Therefore, as is also intuitively plausible, the introduction of γ results

σ	0.05	0.1	0.15	0.2	0.25
$\tilde{x}_1$	2.36	2.07	1.81	1.58	1.38
a^*	2.35	2.04	1.74	1.48	1.26
$\tilde{x}_2$	3.20	3.59	4.03	4.51	5.02
b^*	3.44	3.94	4.51	5.15	5.83
$\frac{\tilde{x}_2-\tilde{x}_1}{b^*-a^*}$	0.77	0.80	0.82	0.80	0.80

Table 2. Impact of increased volatility to the optimal boundaries a^* and b^* and the locally maximizing states $\tilde{x}_i$, $i = 1,2$, under parameter configuration $\mu = 0.03$, $r = 0.05$, $\gamma = 0.5$, $c = 3$.

into lowering of the upper boundary b^*. Similarly to the case of geometric Brownian motion, we find that the locally maximizing states $\tilde{x}_i$, $i = 1,2$, lie both in the continuation region (a^*, b^*). Thus we come to the same conclusion that introduction of a put (call) component into a call (put) option postpones the rational decision to exercise the call (put) option by increasing (decreasing) the optimal exercise threshold. The last line of Table 2 indicates that the error made by using a naive, separated valuation of the straddle option is actually even larger than it is in the case of geometric Brownian motion – our simple relative measure gives an error around 20%.

Finally, the value function V is illustrated graphically for various volatilities under parameter configuration $\mu = 0.03$, $r = 0.05$, $\gamma = 0.5$ and $c = 3$.

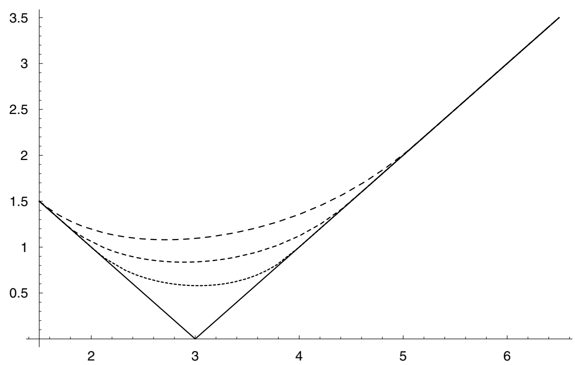

Figure 2. The optimal value function for volatilities $\sigma = 0.1$ (lowest dashed curve), $\sigma = 0.15$ (middle dashed curve) and $\sigma = 0.2$ (upmost dashed curve) under parameter configuration $\mu = 0.03$, $r = 0.05$, $\gamma = 0.5$ and $c = 3$

5 CONCLUDING COMMENTS

In this paper we studied optimal stopping of a linear diffusion. We started by setting up a class of problems including the perpetual American straddles and strangles and guessing that the resulting unique optimal stopping rule is a two-sided threshold rule. Under this hypothesis, we formulated the class of optimal stopping problems as a class of free boundary problems. By utilizing a combination of results from classical theory of linear diffusions, fixed point theory and static optimization, we proved in Theorem 3.3 that our initial hypothesis was correct. The attained results on the solvability are not new themself, but the approach, in particular Lemma 3.2, offers a new insight to the problem – at least to the authors best knowledge. Analogously to Alvarez and Lempa 2007, we established that the optimal stopping rule can be associated with the unique fixed point of a certain auxiliary function. The fixed point approach of Lemma 3.2 relied only on monotonicity and limiting properties of the optimality conditions and, therefore, is potentially useful in more general problems as well where the optimality conditions are not necessarily as smooth as in this study. Similarly to papers with one-sided optimal stopping rules (see Alvarez 2001), we found that the behavior of the functions $x \mapsto \frac{g(x)}{\psi(x)}$ and $x \mapsto \frac{g(x)}{\varphi(x)}$ is connected to the whereabouts of the optimal stopping thresholds a^* and b^*. For a wide class of cases with one-sided optimal stopping rules, the optimal stopping threshold coincides with the interior state maximizing globally either $x \mapsto \frac{g(x)}{\psi(x)}$ or $x \mapsto \frac{g(x)}{\varphi(x)}$, depending on the problem formulation. However, we discovered that in the setting of the current study, the optimal stopping threshold b^* (a^*) is strictly larger (less) than the interior state $\tilde{x}_2$ ($\tilde{x}_1$) maximizing locally the function $x \mapsto \frac{g(x)}{\psi(x)}$ ($x \mapsto \frac{g(x)}{\varphi(x)}$). This observation has an interesting implication for perpetual American straddle. In fact, if we consider determination of the optimal exercise threshold for a perpetual American call option, our results show that the introduction of a put component into the option increases the optimal exercise threshold of the call option – an analogous conclusion holds also for a put option. This observation in turn gives rise to the claim that a naive pricing procedure of a straddle option where both components are priced separately can lead into substantial losses.

This study has several potential generalizations. Firstly, it would be interesting to acquire similar results for more general stochastic control problems, namely for singular stochastic control problems and stochastic impulse control problems. Also, given the infinite horizon setting, generalizations with stochastic interest rate structures would also be of interest. However, these problems are out of the scope of this study and are left for future research.

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PAPER IV

Jukka Lempa: *On Infinite Horizon Optimal Stopping of General Random Walk*, 2007, To appear in *Mathematical Methods of Operations Research*

ON INFINITE HORIZON OPTIMAL STOPPING OF GENERAL RANDOM WALK

Jukka Lempa

ABSTRACT

The objective of this study is to provide an alternative characterization of the optimal value function of a certain Black-Scholes-type optimal stopping problem where the driving process is a general random walk, i.e. the process constituted by partial sums of an IID sequence of random variables. Furthermore, the pasting principle of this optimal stopping problem is studied.

Keywords: General random walk, optimal stopping, minimal functions, continuous pasting

1 INTRODUCTION

The purpose of this paper is to study a certain Black-Scholes-type infinite horizon optimal stopping problem where the driving process is a general random walk. In more precise terms, let X be a random variable distributed on entire $\mathbf{R}$ with a continuous law λ, mean $\mu > 0$ and variance $\sigma^2 < \infty$ and define the general random walk W on $\mathbf{R}$ as partial sums of IID random variables $X, X_1, X_2, \ldots$; i.e. let $W_n = X_1 + \cdots + X_n$, where $W_0 = 0$. Note that in the case where $X \sim N(\mu, \sigma^2)$, the random variable W_n is equal in law to $\mu n + \sigma \hat{W} \sqrt{n}$, where $\hat{W} \sim N(0,1)$. Given the process W, define the expected present value of exercise payoff gained after potentially infinite waiting period as

$$J(n,x) = \mathbf{E}\left[\beta^{-n}\left(e^{x+W_n} - c\right)^+\right] \tag{1.1}$$

and pose the optimal stopping problem

$$V(x) = \sup_{\eta \in \mathcal{N}} J(\eta, x), \tag{1.2}$$

where $\mathcal{N}$ is the set of all W-stopping times, $\beta > 1$ is the discount factor satisfying the condition $\mathbf{E}\left[\beta^{-1}e^{X}\right] < 1$ and $c > 0$ is the exercise cost. The formulation (1.2) is well-established in mathematical finance. In particular, it is closely related to finding the value and exercise policy of an American call option in a Black-Scholes-type market driven in this case by a general random walk. Using this analogy, the increment $e^{W_{n+1}-W_n} = e^{X_{n+1}}$ can be interpreted as the relative price change in the period $n+1$ and η is the date when the option is immediately and irreversibly exercised.

When studying maximization problems of the form (1.2) (possibly for a more general payoff structure) there is a number of different approaches to adopt. Perhaps the most general and fundamental approach is a direct application of principle of dynamic programming; for a recent treatment of dynamic programming in discrete time stochastic control, see Bertsekas and Shreve 1996. Another straightforward approach is to derive a set of conditions under which value function satisfies suitable monotonicity properties and growth rate restrictions and then utilize general martingale or other probabilistic techniques to establish the existence of a unique optimal stopping rule; see e.g. McKean 1965 and Dubins and Teicher 1967. This set of conditions typically include convexity assumptions on the payoff. Yet another possible approach is the utilization of a powerful technique known as the Snell envelope; see e.g. Snell 1952 and Dalang and Hongler 2004. However, these approaches suffer from a downside, namely that they yield very little tangible information on the optimal characteristics, i.e. the optimal stopping rule or the value and typically they are accompanied by complementary techniques in order to gain more detailed information on the problem. In Darling et al. 1972, the authors solve the problem (1.2) and present a probabilistic characterization of the optimal characteristics in terms of the historical maximum of the driving random up to a certain independent, geometrically distributed random time. The characterizations by Darling et al. will be used as the starting point of the current study.

The content of the paper is organized as follows. In section 2 the mathematical apparatus required by our analysis is presented. In section 3 closed-form representations of the optimal characteristics of the problem (1.2) are presented. In section 4 the pasting principle of the optimal value function is investigated. In section 5 the results are illustrated numerically and the study is concluded in section 6.

2 ON THE MINIMAL FUNCTIONS OF W

Denote as $L_1(W)$ the set of all functions $f : \mathbf{R} \to \mathbf{R}$ such that the expectation $\mathbf{E}\left[\beta^{-1} f(x+X)\right]$ exists and define the averaging Markov operator $\mathscr{P}_W$ on $L_1(W)$ as

$$(\mathscr{P}_W f)(x) := \mathbf{E}\left[\beta^{-1} f(x+X)\right] = \int_{-\infty}^{\infty} \beta^{-1} f(z) p(x,z) dz,$$

where $p(x,z) := \lambda(z-x)$ is the single-step transition density. A measurable function $u : \mathbf{R} \to \mathbf{R}_+ \cup \{\infty\}$ satisfying the condition $\mathscr{P}_W u(x) \leq u(x)$ for all $x \in \mathbf{R}$ is called β-*excessive*; in the case of an equality, the function u is called β-*harmonic*. A 1-excessive function is simply called *excessive* and, similarly, 1-harmonic function is called *harmonic*. Moreover, if a β-harmonic function h has the property that any β-harmonic function u with $u(x) \leq h(x)$ for all $x \in \mathbf{R}$ is proportional to h, then h is called β-*minimal*. Assume that h is β-excessive and define the function $p^h : \mathbf{R}^2 \to \mathbf{R}$ as

$$p^h(x,y) = \frac{h(y)}{\beta h(x)} p(x,y).$$

Since h is β-excessive, the function p^h is a transition density. Thus it constitutes a stochastic process. This process will be denoted as W^h and called the *h-process* of W.

The purpose of this section is to present a characterization of the minimal functions of the general random walk W and then utilize this characterization to actually determine the minimal functions. The characterization formulated in Theorem 2.1 is essentially due to Doob et al. 1960. In Doob et al. 1960, the case where the driving general random walk is spatially discrete is considered in the absence of discounting. However, the proof they present can be straightforwardly generalized to cover the present case by simply replacing their corresponding definitions with the ones presented above and carrying out the exactly same computations.

Theorem 2.1. *Assume, that the function* $h : \mathbf{R} \to \mathbf{R}_+$ *satisfies the condition* $h(0) = 1$. *Then* h *is* β-*minimal for the general random walk* W *if and only if it satisfies condition*

(A) $\mathbf{E}[\beta^{-1}h(X)] = 1$,

(B) $h(x+y) = h(x)h(y)$, *for all* $x, y \in \mathbf{R}$.

Theorem 2.1 is a forceful result on a general mathematical level. It is known from the theory of Martin boundaries that there is a fundamental connection between the minimal functions of a stochastic process and the minimal Martin compactification of the state space of the process (see e.g. Revuz 1984, Chapter 7). Roughly speaking, this compactification is attained by embedding the state space in a suitable way in to a certain infinite-dimensional function space. In this light, there is no guarantee *a priori* that the minimal Martin compactification concurs with the elementary two-point compactification of the state-space. However, theorem 2.1 implies that in the case of a general random walk W, these two compactifications concur. This is equivalent to saying that there exist exactly two β-minimal functions of the general random walk W. This statement is now proved by utilizing Theorem 2.1.

Lemma 2.2. *There exists exactly two real numbers* $-a$ *and* b*,* $a, b > 0$*, determined by the condition* $\mathbf{E}[e^{tX}] = \beta$ *such that the functions* $\psi : \mathbf{R} \to \mathbf{R}_+$ *and* $\varphi : \mathbf{R} \to \mathbf{R}_+$ *defined as* $\psi(x) = e^{bx}$ *and* $\varphi(x) = e^{-ax}$ *are the only* β*-minimal functions of the general random walk* W*.*

Proof. It is well known that all positive measurable solutions of the functional equation $h(x+y) = h(x)h(y)$ can be expressed in the form $h(x) = e^{tx}$ for $t \in \mathbf{R}$. For β-harmonicity the condition $\mathbf{E}[e^{tX}] = \beta$ must also be satisfied. Let M be the moment-generating function of X. Then $h(x) = e^{tx}$ is β-minimal if and only if $M(t) = \beta$. Since $M(0) = 1$ and M is convex (and therefore continuous), the domain of M is an open neighborhood of the origin – denote the domain as (l, r), where $-\infty \leq l < 0 < r \leq \infty$. Define the function $\theta : (l, r) \to \mathbf{R}$ as $\theta(t) = M(t) - \beta$. First note that θ is also convex and $\theta(0) = 1 - \beta < 0$. On the other hand, since $\lambda(x) > 0$ for all $x \in \mathbf{R}$, it is clear that $\lim_{t \to l+} M(t) = \lim_{t \to r-} M(t) = \infty$. This implies that also $\lim_{t \to l+} \theta(t) = \lim_{t \to r-} \theta(t) = \infty$. This observation completes the proof. □

Lemma 2.2 has a nice analogue in the theory of continuous time Markov processes. To point this out, consider the continuous time counterpart of W, namely the drifting Brownian motion $B_t^{(\mu)} = \exp\left\{\left(\mu + \sigma^2/2\right)t + \sigma B_t\right\}$ satisfying the stochastic differential equation $dB_t^{(\mu)} = \mu dt + \sigma dB_t$, where B is a standard Brownian motion. It is well known that the β-minimal functions of the process $B_t^{(\mu)}$ are the so-called fundamental solutions of the ordinary differential equation $\frac{1}{2}\sigma^2 h''(x) + \mu h'(x) - \ln\beta h(x) = 0$, in other words the increasing fundamental solution $\psi_B(x) = e^{\gamma x}$ and the decreasing fundamental solution $\varphi_B(x) = e^{-\delta x}$, where γ and $-\delta$ are the positive and the negative root of the characteristic equation $\frac{1}{2}\sigma^2 t^2 + \mu t - \ln\beta = 0$, respectively (see e.g. Borodin and Salminen 2002, 17-18). Two interesting observations can now be made. First, for any particular choice of random variable X, the β-minimal functions of the processes $B^{(\mu)}$ and W are of the same functional form $x \mapsto e^{tx}$ for some parameter $t \in \mathbf{R}$. Moreover, if $X \sim N(\mu, \sigma^2)$, then the condition $\mathbf{E}\left[e^{bX}\right] = \beta$ from Theorem 2.2 can be written as $e^{b\mu + \frac{1}{2}\sigma^2 b^2} = \beta$, which implies that $\gamma = b$ and $\delta = a$. In other words, if $X \sim N(\mu, \sigma^2)$, then the β-minimal functions of the processes $B^{(\mu)}$ and W are the same.

To close the section, a scaled-down version of the integral representation theorem for harmonic functions of a general Markov chain is presented. For the complete formulation of the result and the proof, see Revuz 1984, Corollary 3.11, pp. 257.

Theorem 2.3. *Assume that the process W has exactly two β-minimal functions, say ψ and φ, and that h is β-harmonic. Then there exists a unique pair (c_1, c_2) of non-negative constants such that $c_1 + c_2 = 1$ and $h(x) = c_1\psi(x) + c_2\varphi(x)$ for all $x \in \mathbf{R}$.*

3 ON THE OPTIMAL STOPPING RULE AND VALUE FUNCTION

In many cases, optimal stopping rules for maximization problems of the form (1.2) can be characterized as passage times of either the underlying stochastic process itself or some random variable closely related to it, for example the historical maximum or minimum in to the so-called stopping region. First of these cases cases appears to be connected with processes having almost surely continuous sample paths (see e.g. Alvarez 2001, Dayanik and Karatzas 2003, Øksendal 2000) and second with processes exhibiting jump behavior (see e.g. Alili and Kyprianou 2005, Boyarchenko and Levendorskiǐ 2004, Darling et al. 1972, Mordecki 2002).

This set is potentially very complex and its boundary, the optimal stopping threshold, can be virtually impossible to determine. However, in a number of practically meaningful cases it can be established that the stopping region is of the form (s^*,∞), where $s^* \in \mathbf{R}$. In many cases, the threshold s depends on the properties Given this observation, the study of the problem (1.2) is now continued by defining the random variable M as

$$M = \max_{0 \le k < \tau} W_k, \tag{3.1}$$

where τ is a random time which is independent of $\{X_i\}$ and geometrically distributed with $\mathbf{P}\{\tau > k\} = \beta^{-k}$ for $k \ge 0$. In other words, the random variable M is the historical maximum of the general random walk W up to a certain independent random time. Note that $M \ge 0$, since $W_0 = 0$. The information of the random variable M required by our analysis is now formulated in the following two results.

Theorem 3.1. *Let* $H_0^+ = \inf\{n \ge 0 : W_n > 0\}$. *Then the following are equivalent:*

(A) $\mathbf{E}\left[\beta^{-1}e^X\right] < 1$

(B) $\mathbf{E}\left[\beta^{-H_0^+}\exp\left(W_{H_0^+}\right)\right] < 1$

(C) $\mathbf{E}\left[e^M\right] < \infty$.

Proof. See Darling et al. 1972, page 1367. □

Corollary 3.2. *The random variable* M *has an atom at origin; i.e.* $\mathbf{P}\{M = 0\} > 0$.

Proof. Recall the definition of random time H_0^+ from Theorem 3.1. Since $\exp\left(W_{H_0^+}\right) > 1$ almost surely, the part *(B)* of Theorem 3.1 implies that $\mathbf{P}\{M = 0\} = 1 - \mathbf{P}\{\tau > H_0^+\} = 1 - \mathbf{E}[\beta^{-H_0^+}] > 0$. □

The next theorem gives a probabilistic characterization of the optimal characteristics of the problem (1.2). This theorem is essentially due to Darling et al. 1972, where they consider the case $c = 1$ on pages 1367-8. However, their treatment generalizes easily to the case of general c, see also Mordecki 2002, Theorem 1.

Theorem 3.3. *The optimal stopping rule is to stop at time* $H_{s^*} = \min\{n \geq 0 : x + W_n \geq s^*\}$*, where* $s^* = \ln\left(c\mathbf{E}\left[e^M\right]\right) < \infty$ *and* M *is the random variable defined in* (3.1)*. Moreover, the optimal value reads as*

$$V(x) = \mathbf{E}\left[\beta^{-H_{s^*}}\left(e^{x+W_{H_{s^*}}} - c\right)^+\right] = \frac{\mathbf{E}\left[\left(e^{x+M} - c\mathbf{E}\left[e^M\right]\right)^+\right]}{\mathbf{E}\left[e^M\right]}. \tag{3.2}$$

Useful information can be extracted from the representation (3.2). First of all, notice that V is nondecreasing and convex and that the first equation in the expression (3.2) implies that $V(x) = e^x - c$ for all $x \geq s^*$. On the other hand, the latter equation in the expression (3.2) implies that $V(x) = 0$ if and only if $e^M \leq e^{-x}c\mathbf{E}[e^M]$ almost surely which does not hold for any finite $x \leq s^*$. By combining this observation with the fact that the value V satisfies the principle of dynamic programming, i.e. that $V(x) = \max\{e^x - c, (\mathscr{P}_W V)(x)\}$, yields the condition $V(x) = (\mathscr{P}_W V)(x)$ for all $x \leq s^*$. Theorem 2.3 implies now that there exists a unique constant $K > 0$ such that $V(x) = Ke^{bx}$ for all $x \leq s^*$. Finally, since $V(x)$ is continuous in s^*, the constant $K = \frac{e^{s^*}-c}{e^{bs^*}}$. These results are now summarized in the following theorem, which is the first of the main results.

Theorem 3.4. *The optimal value function* $V(x)$ *reads as*

$$V(x) = \begin{cases} e^x - c, & x \geq s^* \\ \frac{e^{s^*}-c}{e^{bs^*}} e^{bx}, & x \leq s^*, \end{cases} \tag{3.3}$$

where b *is the unique positive solution of the equation* $\mathbf{E}\left[e^{bX}\right] = \beta$.

The function $V(x)$ is constructed from the functions $x \mapsto \frac{e^{s^*}-c}{e^{bs^*}} e^{bx}$ and $x \mapsto e^x - c$ by pasting them (possibly smoothly) together in the threshold s^*. Generalize now this function with respect to both the threshold s^* and the exponent b. More precisely, generate a whole family $\{G_{y,\alpha}(x)\}_{y\in\mathbf{R}_+,\, \alpha\in\mathbf{R}}$ of functions of form (3.3) by first replacing the optimal stopping threshold s^* with a free boundary y and the critical exponent b with an arbitrary exponent α and defining the function $G_{y,\alpha} : \mathbf{R} \to \mathbf{R}_+$ as

$$G_{y,\alpha}(x) = \begin{cases} e^x - c, & x \geq y \\ \frac{e^y - c}{e^{\alpha y}} e^{\alpha x}, & x \leq y. \end{cases} \tag{3.4}$$

Using this notation, $V(x) = G_{s^*,b}(x)$. It is now natural to ask the question when the function $G_{y,\alpha}$ is continuously differentiable in y for a given $\alpha > 1$. Elementary differentiation yields that the function $G_{x^*,\alpha} =: G_{x^*_\alpha}$ is continuously differentiable in $x_\alpha^* \in \mathbf{R}_+$ if and only if $e^{x_\alpha^*} = \frac{\alpha c}{\alpha - 1}$. If now $\alpha = b$, this condition is the smooth pasting principle of the problem (1.2) and the threshold $x_b^* =: x^*$ is called the smooth pasting threshold. With this information, it is natural to pose the following question about the pasting principle: Is $s^* = x^*$?

For the sake of comparison to the continuous time setting, consider again the particular problem (1.2) where $X \sim N(\mu, \sigma^2)$ and define the continuous time version of the problem (1.2) where the underlying process is the drifting Brownian motion $B^{(\mu)}$, in other words the problem

$$V_B(x) = \sup_{\rho \in \mathscr{R}} \mathbf{E} \left[\beta^{-\rho} \left(e^{x + B_\rho^{(\mu)}} - c \right)^+ \right], \tag{3.5}$$

where $B_t^{(\mu)}$ is the continuous time process introduced in Section 2 and $\mathscr{R}$ is the set of all $B^{(\mu)}$-stopping times. It has been established already in McKean 1965, Section 3, that in this case the optimal value V_B reads as $V_B(x) = G_{x_\gamma^*}(x)$, where the constant $\gamma > 1$ is positive solution of the equation $\frac{1}{2}\sigma^2 t^2 + \mu t - \ln \beta = 0$. Recall from Section 2 that $b = \gamma$. This implies that optimal stopping threshold x_γ^* of the continuous time problem (3.5) coincides with the smooth pasting threshold x^* of the discrete time problem (1.2) when $X \sim N(\mu, \sigma^2)$. Moreover, given that the problem (1.2) satisfies the smooth pasting principle, this would imply that values V and V_B satisfy the (rather counter-intuitive) condition $V_B(x) = V(x)$ for al $x \in \mathbf{R}$.

The representation (3.3) is not completely unfamiliar to the literature of temporally discrete optimal stopping. In Taylor 1972, the author considers essentially the same stopping problem as (1.2) and proves that the optimal value is bounded from above by a function of the form (3.3). However, he makes no comment on whether or not the actual value of the problem is of the same form or on its connection to the Martin boundary theory.

4 CONTINUOUS PASTING VS. SMOOTH PASTING

Consider again the optimal stopping problem (3.5). It is a classical result (see McKean 1965, Section 3) that for this problem the optimal value function is continuously differentiable on the optimal stopping threshold. During the recent years, many authors have discussed the pasting principles of optimal stopping problems of various form and there is an increasing number of research articles reporting a failure of smooth pasting in the optimal value function, see e.g. Alili and Kyprianou 2005, Asmussen et al. 2004, Boyarchenko and Levendorskiǐ 2002, Dalang and Hongler 2004, Peskir and Shiryaev 2000. While going through these articles, one observes that the breakdown of smooth pasting appears to be connected to the cases when there is a chance that the underlying process can jump discontinuously into the stopping region. In this light, it is reasonable to guess that the smooth pasting fails also in the present case. In Alili and Kyprianou 2005, Theorem 6, the authors present a handy characterization of smooth pasting for a general Lévy process in terms of the random variable M defined in (3.1). Conveniently, this characterization holds also for (1.2).

Theorem 4.1. *The optimal value function $V(x)$ exhibits smooth-pasting if and only if $M \neq 0$ almost surely.*

Proof. First, recall the expression (3.2) for the optimal value $V(x)$ from Theorem 3.3. Elementary manipulations yield

$$\begin{aligned} V(x) &= c\mathbf{E}\left[\left(e^{-(s^*-x-M)}-1\right); M > s^*-x\right] \\ &= c\left(e^{-(s^*-x)}-1\right)\mathbf{E}\left[e^{M}; M > s^*-x\right] \\ &\quad + c\mathbf{E}\left[\left(e^{M}-1\right); M > s^*-x\right]. \end{aligned}$$

The last equality implies that

$$\begin{aligned} \frac{\left(e^{s^*}-c\right)-V(x)}{s^*-x} &= \frac{c\left(\mathbf{E}\left[e^{M}\right]-1\right)-V(x)}{s^*-x} \\ &= -c\frac{e^{-(s^*-x)}-1}{s^*-x}\mathbf{E}\left[e^{M}; M > s^*-x\right] \\ &\quad + c\frac{\mathbf{E}\left[\left(e^{M}-1\right); M \leq s^*-x\right]}{s^*-x}. \end{aligned}$$

In order to simplify the notation, denote last two terms on the right hand side as $A(x)$ and $B(x)$ respectively. Application of L'Hospitals rule yields

$$\lim_{x \to s^*-} A(x) = c\mathbf{E}\left[e^{M}; M > 0\right] = e^{s^*} - c\mathbf{E}\left[e^{M}; M \leq 0\right].$$

On the other hand, partial integration yields

$$\begin{aligned} B(x) =& c\frac{\mathbf{E}\left[\left(e^{M}-1\right); 0 < M \leq s^*-x\right]}{s^*-x} = c\int_{0+}^{s^*-x}\frac{e^{z}-1}{s^*-x}m(dz) \\ =& c\frac{e^{s^*-x}-1}{s^*-x}m\{(0,s^*-x)\} + \frac{c}{s^*-x}\int_{0+}^{s^*-x}e^{z}m\{(0,z)\}dz \to 0, \end{aligned}$$

as $x \longrightarrow s^*-$, where in the first expectation the atom at origin is removed, this can be done because the integrand $e^M - 1 = 0$ when $M = 0$. Combination of these results yields

$$\lim_{x \to s^*-} V'(x) = c\mathbf{E}\left[e^M; M > 0\right] = e^{s^*} - c\mathbf{E}\left[e^M; M = 0\right] = e^{s^*} - c\mathbf{P}\{M = 0\},$$

which is clearly equivalent to the claim. □

Coupled with Corollary 3.2, Theorem 4.1 yields immediately that smooth pasting fails in the problem (1.2). This result is the second main theorem.

Theorem 4.2. *The optimal value function V exhibits only continuous pasting on the optimal stopping threshold s^*. In other words, $s^* < x^*$.*

As was indicated earlier, the characterization of smooth pasting in Theorem 4.1 holds for a general Lévy processes, in particular for the drifting Brownian motion $B^{(\mu)}$. It is a well-known fact from the theory of Brownian motion that the sample paths of $B^{(\mu)}$ are regular in the sense that $\mathbf{P}\{H_0^+ = 0\} = 1$, where $H_0^+ = \inf\{t \geq 0 \mid B_t^{(\mu)} > 0\}$. This condition is clearly equivalent to the statement $\mathbf{P}\{\hat{M} = 0\} = 0$, where the random variable $\hat{M}$ is defined analogously to (3.1) for $B_t^{(\mu)}$. It is also quite clear that the sample paths of the general random walk W are not regular in the previous sense. This is simply because of the fact that the random variable X admits negative values with positive probability.

5 AN ILLUSTRATION

In Section 4 it was established that the optimal value $V(x)$ does not exhibit smooth pasting in the optimal stopping boundary. The aim of this section is to illustrate to size of the error being made if the smooth pasting principle is used as a basis of decision making in the case where $X \sim N(\mu, \sigma^2)$. This implies that the increments in the geometric random walk $Y_n =: e^{x+W_n}$ are log-normally distributed, which is a typical assumption in investment theoretical applications. Recall from the section 3 that this error can be seen also as a difference between a discrete time and a continuous time model. This error is illustrated using two simple quantities, namely the relative distance $D := \frac{x^*}{s^*}$ of the smooth pasting threshold x^* and the optimal stopping threshold s^* and the relative point-wise distance $D_V(x) := \frac{V_B(x)}{V(x)}$ of the optimal value functions V_B

(see (3.5)) and V (see (1.2)). In the sequel, these quantities are called relative errors. For simplicity, assume that $c = 1$. Then it is known from Darling et al. 1972, page 1368 that the threshold s^* can be expressed as

$$s^* = \sum_{n=1}^{\infty} \frac{\beta^{-n}}{n} \mathbf{E}\left[e^{W_n^+} - 1\right] =: \sum_{n=1}^{\infty} \frac{\beta^{-n}}{n} s_n. \tag{5.1}$$

Since the random variables W_n and $\mu n + \sigma Y \sqrt{n}$, where $Y \sim N(0,1)$, are equal in law, the coefficient s_n can be expressed as

$$\begin{aligned} s_n &= \frac{1}{\sqrt{2n\pi\sigma^2}} \int_0^{\infty} (e^z - 1) e^{\frac{-(z-n\mu)^2}{2n\sigma^2}} dz \\ &= \frac{1}{2}\left(\exp\left[\frac{n}{2\sigma^2}\left(2\mu\sigma^2 + \sigma^4\right)\right] \operatorname{erfc}\left[-\frac{\mu+\sigma^2}{2\sigma}\sqrt{2n}\right] - \operatorname{erfc}\left[-\frac{\mu}{2\sigma}\sqrt{2n}\right]\right), \end{aligned}$$

where $\operatorname{erfc} : \mathbf{R} \to \mathbf{R}_+$ is the complementary error function defined as $\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-y^2} dy$. The thresholds x^* and s^* and the relative error D are now illustrated in Figure 1 as functions of standard deviance σ under the assumption that $\mu = 0.03$ and $\beta = 1.07$. The approximations of s^* are computed from series (5.1) such that the reminder term $R < 10^{-7}$.

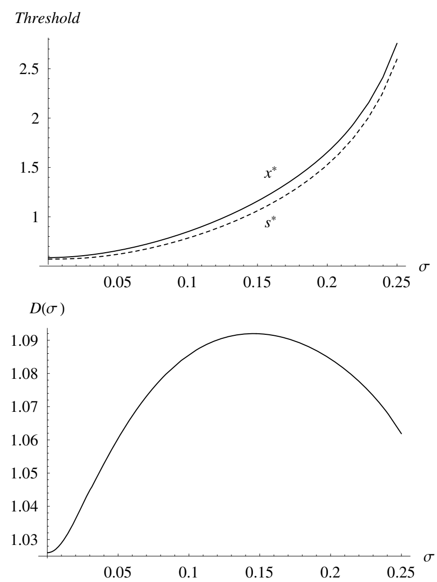

Figure 1. The smooth-fit threshold x^* (continuous curve), optimal stopping threshold s^* (dashed curve) and the relative error D as functions of volatility σ under the assumption $\mu = 0.03$ and $\beta = 1.07$.

The left hand side of Figure 1 indicates that for this specific example the thresholds x^* and s^* are both convex as functions of standard deviance σ but interestingly they are not equally "convex", as the right hand side clearly shows. In other words, the right hand side indicates that for small values of σ, the threshold x^* grows with a faster rate increasing the relative error D until σ reaches a critical value σ^* at which the error D is maximal; for this specific parameter configuration the critical $\sigma^* = 0.147$ at which $D(\sigma^*) = 1.092$. Above this critical value, the threshold s^* starts to gain on x^* and the relative error D starts to decrease.

The behavior of the optimal value functions $V_B(x)$ and $V(x)$ and the relative error $D_V(x) = \frac{V_B(x)}{V(x)}$ are now illustrated in Figure 2 under the assumption $\mu = 0.03$ and $\beta = 1.07$ in the case of maximal relative error $D(\sigma^*)$.

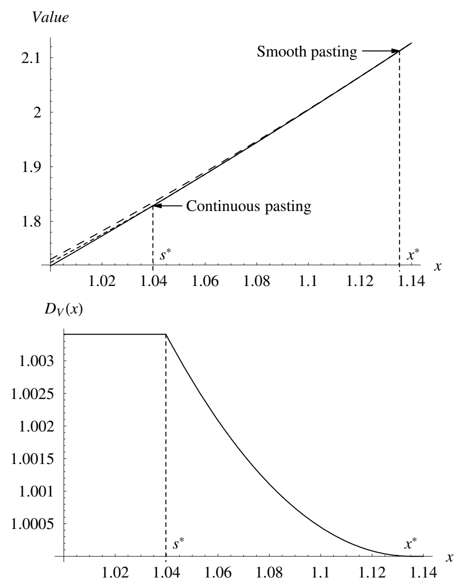

Figure 2. The optimal value function V_B of the problem (3.5) (upper dashed curve) and V of the problem (1.2) (lower dashed curve) and the reward $x \mapsto e^x - 1$ (continuous curve) under the assumption $\mu = 0.03$, $\sigma = 0.147$ and $\beta = 1.07$.

Figure 2 indicates that even though the optimal stopping thresholds x^* and s^* are relatively far away from each other ($D(\sigma^*) = 1.092$), the curves $V_B(x)$ and $V(x)$ are quite close by ($D_V(x) < 1.004$ for all $x \in \mathbf{R}$). More precisely, when $x < s^*$ the error $\frac{V_B(x)}{V(x)}$ is actually independent of x, in other words $\frac{V_B(x)}{V(x)} =$

$\frac{(e^{x^*}-1)e^{bs^*}}{(e^{s^*}-1)e^{bx^*}}$. On the interval (s^*, x^*) the error $\frac{V_B(x)}{V(x)}$ starts to decrease as the curve $x \mapsto e^x - 1$ gains on the curve $x \mapsto \frac{e^{s^*}-c}{e^{bs^*}} e^{bx}$ until it hits 1 at x^*.

To close the section, recall that in Section 3 it was established that the optimal stopping threshold x^*_γ of the continuous time problem (3.5) coincides with the smooth pasting threshold x^* of the discrete time problem (1.2) when $X \sim N(\mu, \sigma^2)$. Combination of this result with Theorem 4.1 yields now an intuitively natural result that when $X \sim N(\mu, \sigma^2)$, the optimal stopping problems 1.2 and 3.5 are intrinsically different problems, i.e. they possess different optimal characteristics. Moreover, in light of the computations of this section it is reasonable to argue the difference between the characteristics of these problems can be substantial, especially in absolute scale. This is an interesting observation, since there is a wide variety of applications, especially in finance, where an intrinsically discrete time phenomenon is modelled using a continuous time model, for example Black-Scholes model.

6 CONCLUDING REMARKS

The presented paper considers an infinite horizon optimal stopping of general random walk and it contains two main results. First, it presents a closed-form formula of the optimal value function in terms of β-minimal functions of the driving random walk. This representation appears to be new in the discrete-time setting at least to some degree. In Taylor 1972, the author analyzes a stopping problem equivalent to (1.2) and proves that the optimal value is dominated by a function of the form (3.3). However, he makes no comment about the actual value of the problem or its connections to the Martin boundary theory. The representation (3.3) is analogous to the one presented in Alvarez 2001 in the case where the driving process is a linear diffusion. In the case of a linear diffusion, where sample paths exhibit almost sure continuity, Alvarez's representation of the value is particularly convenient, since it gives smooth pasting as a simple consequence and therefore simplifies the characterization of the optimal stopping rule significantly. However, in the context of the current study, this representation does not yield any implication on the pasting principle of the problem. Therefore other techniques must be used in order to investigate the pasting principle. To this end, a characterization of smooth pasting is adopted from Alili and Kyprianou 2005 and utilized to prove that the optimal value (3.3) is not differentiable in the optimal stopping threshold s^*.

The analysis of this study has a number of possible interesting extensions. First, a natural extension would be to consider a wider class of admissible control policies than just single stopping policies. More precisely, it would be of interest to extend the results of this paper to the sequential stopping problems appearing in the class of impulse control policies. Impulse control policies are interesting from viewpoint of various applications, for example economics of renewable resources. Given the infinite horizon setting, a second natural extension would be introduction of stochastic interest rate structure. There is also room for extension with respect to underlying stochastic dynamic structure. One possible way of extending the results in to this direction could be a development of some transformation technique of the underlying stochastic process (with respect to either time or scale) in order to be able to tackle more complicated dynamical systems, for example mean-reverting dynamics. However, these investigations are out of the scope of this study and are therefore left for future research.

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