AUTHOR Juha-Pekka Pellonpää, Sébastien Designolle and Roope Uola

TITLE Naimark dilations of qubit POVMs and joint measurements

YEAR 2023

URL https://doi.org/10.1088/1751-8121/acc21c

VERSION Final Draft

CITATION **Citation** Juha-Pekka Pellonpää *et al* 2023 *J. Phys. A: Math. Theor.* **56** 155303
DOI 10.1088/1751-8121/acc21c

NAIMARK DILATIONS OF QUBIT POVMS AND JOINT MEASUREMENTS

JUHA-PEKKA PELLONPÄÄ, SÉBASTIEN DESIGNOLLE, AND ROOPE UOLA

ABSTRACT. Measurement incompatibility is one of the cornerstones of quantum theory. This phenomenon appears in many forms, of which the concept of non-joint measurability has received considerable attention in the recent years. In order to characterise this non-classical phenomenon, various analytical and numerical methods have been developed. The analytical approaches have mostly concentrated on the qubit case, as well as to scenarios involving sets of measurements with symmetries, such as position and momentum or sets of mutually unbiased bases. The numerical methods can, in principle, decide any finite-dimensional and discrete joint measurability problem, but they naturally have practical limitations in terms of computational power. These methods exclusively start from a given set of measurements and ask whether the set possesses incompatibility. Here, we take a complementary approach by asking which measurements are compatible with a given measurement. It turns out, that this question can be answered in full generality through a minimal Naimark dilation of the given measurement: the set of interest is exactly those measurements that have a block-diagonal representation in such dilation. We demonstrate the use of the technique through various qubit examples, leading to an alternative characterisation of all compatible pairs of binary qubit measurements, which retrieves the celebrated Busch criterion. We further apply the technique to special examples of trinary and continuous qubit measurements.

PACS numbers: 03.65.Ta, 03.67.–a

1. Introduction

The act of measurement is in the core of any physical theory. In quantum theory, this process provides a window between the microscopic and macroscopic worlds. It is through this fundamental action that one collects new data, verifies and falsifies predictions, and ultimately develops novel theoretical models. Given its fundamental position and strong mathematical grounds, the theory of quantum measurements keeps naturally lending itself to open questions, such as the measurement problem [1], uncertainty relations [2, 3], and practical applications of quantum information theory [4, 5, 6]. In this article, we contribute to a specific open question of characterising a central quantum-to-classical border within quantum measurement theory, that is, the threshold between sets of measurements that allow and do not allow a simultaneous readout.

In quantum theory, the traditional description for the act of measurement is given through Hermitian operators, also called observables. By now, it is self-evident that such presentation differs from classical physics, in that whereas all classical measurements can be performed simultaneously, this is no longer true in the quantum scenario. Such discrepancy sets fundamental limitations on the possibility of coding information about non-commuting quantities into a quantum state, as shown by the famous preparation uncertainty relations of Heisenberg and Robertson.

By now, the quantum information theoretic representation of quantum measurements has deviated from the notion of observables due to the introduction of more general positive operator-valued measures (POVMs for short). POVMs offer various advantages over their predecessors in that they, e.g., better capture realistic measurement implementations [1, 7], can perform better in discriminating quantum states [8, 9], and offer various fine-tuned notions of measurement incompatibility [10, 11, 12, 13, 14, 15].

For our purposes, a central property of POVMs is that of joint measurability. This is a generalisation of the notion of commutativity of observables. In short, joint measurability asks whether the measurement data of a given set of measurements can be classically post-processed from the data of a single measurement. On the conceptual level, joint measurability has found various applications in, e.g., quantum correlations [16, 17, 18, 19, 20], contextuality [21], quantum state discrimination [22, 23, 9, 8], quantum communication [24], and quantum thermodynamics [25]. Hence, the task of characterising sets of measurements that allow a simultaneous readout has become an important and actively investigated problem not only from the foundational, but also from the practical perspective [26, 27, 28, 29], see also [30] for a recent review.

Typically, one is interested in characterising the sets of measurements that allow a joint measurement. Here, we take a slightly complementary approach by using a technique that characterises those measurements that are jointly measurable with a given POVM. The technique is based on a minimal Naimark dilation, i.e., representing POVMs as observables in a larger space, and it connects to the sole notion of incompatibility possessed by observables, i.e., non-commutativity. Namely, the set of POVMs that are jointly measurable with a given POVM turns out to be exactly the set of those measurements that have a block-diagonal representation in a minimal dilation of the given POVM. Such block-diagonal measurements are characterised by the commutant of the relevant observable in the minimal dilation space. We want to stress out that this technique has appeared in the past in conceptual works including some of the authors [31, 32, 33] and that connections between joint measurability and commutativity in some dilation space have been reported independently in [34, 31, 35, 36]. However, the works [32, 33] did not concentrate on explicitly characterising joint measurability of given sets of POVMs and the works [34, 35, 36] did not specify a single generally applicable Naimark dilation. Some of the involved dilations even require one to know some joint measurement before the construction of the dilation, i.e. one gets the dilation only after one has solved the problem of joint measurability. Here, we stress that one can simply use a minimal dilation of the involved measurements in order to solve the problem of joint measurability. We further note that this dilation is fully constructive. The investigation of joint measurability criteria arising from this process is the main contribution of this manuscript.

Although the minimal dilation technique applies even to infinite-dimensional systems, cf. Ref. [31], we concentrate on the qubit case for simplicity. In this setting, the technique provides an alternative way of obtaining the celebrated Busch criterion for joint measurability of two unbiased qubit measurements [37] and gives a full characterisation of those qubit effects that are (pairwisely) compatible with a given qubit effect. Moreover, we compare the technique to that of Ref. [38] in the case of two symmetric three-valued qubit POVMs, and provide examples of compatible pairs and triplets of continuous qubit POVMs.

2. Joint measurability

We describe quantum measurements as POVMs. In the case of a discrete (finite dimensional) measurement, these are essentially sets of positive semi-definite matrices $\mathsf{E} = (E_1, E_2, \dots, E_N)$ that sum to the identity operator, i.e., $\sum_{i=1}^N E_i = \mathbb{1}$. In the continuous case, a POVM is a normalised (weakly) σ-additive map from a σ-algebra $\mathcal{A} \subseteq 2^\Omega$ to the set of positive operators of a Hilbert space. A discrete POVM $\mathsf{B} = (B_1, B_2, \dots, B_M)$ is called jointly measurable with E, if there exists a third POVM $\mathsf{N} = (N_{ij})$ with $i \in 1, ..., N$ and $j \in 1, ..., M$ such that

$$E_i = \sum_{j=1}^{M} N_{ij} \tag{2.1}$$

$$B_j = \sum_{i=1}^{N} N_{ij} \tag{2.2}$$

for all i and j. Otherwise, the POVMs E and B are called incompatible. The POVM N is called a joint or parent POVM. This definition generalises directly to continuous POVMs by replacing the sums in Eq. (2.1) and Eq. (2.2) with (total) sets as well as the singletons i and j with measurable sets, i.e., two POVMs E and B (defined on σ-algebras $\mathcal{A} \subseteq 2^\Omega$ and $\mathcal{B} \subseteq 2^\Xi$)

are jointly measurable if there is a POVM N defined on the product σ-algebra $\mathcal{A}\otimes\mathcal{B}\subseteq 2^{\Omega\times\Xi}$ such that $\mathsf{E}(X)=\mathsf{N}(X\times\Xi)$ for all $X\in\mathcal{A}$ and $\mathsf{B}(Y)=\mathsf{N}(\Omega\times Y)$ for all $Y\in\mathcal{B}$.

For basic examples of jointly measurable POVMs, one can choose E and B commuting, i.e., $[E_i,B_j]=0$ for all i and j, in which case $N_{ij}=E_iB_j=\sqrt{B_j}E_i\sqrt{B_j}$ is clearly a joint POVM. For a non-commuting example one can take noisy spin measurements defined by

$$E_{\pm1}=\frac{1}{2}\Big(\mathbb{1}\pm\frac{1}{\sqrt{2}}\sigma_1\Big),$$
$$B_{\pm1}=\frac{1}{2}\Big(\mathbb{1}\pm\frac{1}{\sqrt{2}}\sigma_3\Big).$$

In this case, a joint POVM is given by

$$N_{ij}=\frac{1}{4}\Big[\mathbb{1}\pm\frac{1}{\sqrt{2}}(i\sigma_1+j\sigma_3)\Big],\qquad i,j=\pm1.$$

One can further show [37] that this joint POVM is optimal in the sense that if one increases the length of the Bloch vectors of $E_{\pm1}$ or $B_{\pm1}$, the measurements become incompatible.

In the above examples, the (optimal) joint POVM is rather simple to find. Some techniques for finding joint POVMs for more complex scenarios have been reported in the literature based on a so-called adaptive strategy [39] and other ansätze [40, 41, 42]. Such techniques typically do not use auxiliary systems. In the following, we map the problem of finding joint measurements into a problem of characterising a commutant in a minimal Naimark dilation space. This provides further natural ansätze for joint measurements. We focus our attention to the qubit case and demonstrate the technique by building optimal and suboptimal joint measurements for various scenarios.

3. Naimark dilations of discrete qubit POVMs

Let $\mathcal{H}$ be a two-dimensional (qubit) Hilbert space. By fixing an orthonormal basis $\{\varphi_1,\varphi_2\}\subset\mathcal{H}$ we may identify $\mathcal{H}$ with $\mathbb{C}^2$ via unitary operator $U:\mathcal{H}\to\mathbb{C}^2$, $U:=|(1,0)\rangle\langle\varphi_1|+|(0,1)\rangle\langle\varphi_2|$, where the vectors $(1,0)$ and $(0,1)$ constitute the standard basis of the Hilbert space $\mathbb{C}^2$. In what follows, we identify any operator $O:\mathcal{H}\to\mathcal{H}$ with the operator (2×2–matrix) UOU^* on $\mathbb{C}^2$ and study only matrices. Clearly, each 2×2-matrix M corresponds to a unique operator U^*MU on $\mathcal{H}$.

We say that a positive semidefinite 2×2–matrix E is an *effect* if $\mathbb{1}-E$ is also positive semidefinite; here $\mathbb{1}$ is the identity matrix. Any effect E can be written in the form

$$E=\frac{1}{2}\sum_{\mu=0}^{3}e^\mu\sigma_\mu=\frac{1}{2}(e^0\mathbb{1}+\boldsymbol{e}\cdot\boldsymbol{\sigma})=\frac{1}{2}\begin{pmatrix}e^0+e^3 & e^1-ie^2\\ e^1+ie^2 & e^0-e^3\end{pmatrix}$$

where $(e^0, e^1, e^2, e^3) = (e^0, \boldsymbol{e}) \in \mathbb{R}^4$, $\|\boldsymbol{e}\| := \sqrt{(e^1)^2 + (e^2)^2 + (e^3)^2} \leqslant \min\{e^0, 2 - e^0\}$, and

$$\sigma_0 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices [1, Chapter 14]. In particular, $e^\mu = \mathrm{tr}\,[E\sigma_\mu]$, $\mu = 0, 1, 2, 3$, $e^0 \in [0, 2]$, and $|e^j| \leqslant \|\boldsymbol{e}\| \leqslant 1$, $j = 1, 2, 3$. The eigenvalues of E are $\frac{1}{2}(e^0 \pm \|\boldsymbol{e}\|) \in [0, 1]$ so that E is of rank 1 if and only if $e^0 = \|\boldsymbol{e}\| \neq 0$. Especially, E is a rank-1 (resp. rank-2) projection exactly when $e^0 = \|\boldsymbol{e}\| = 1$ (resp. $e^0 = 2$ and $\|\boldsymbol{e}\| = 0$).

Define then the parameters

$$\begin{aligned} c^\pm(E) &:= \sqrt{\frac{(e^0 \pm \|\boldsymbol{e}\|)(\|\boldsymbol{e}\| \pm e^3)}{4\|\boldsymbol{e}\|}} \geqslant 0, \\ d^\pm(E) &:= \pm\big(e^1 - ie^2\big)\sqrt{\frac{e^0 \pm \|\boldsymbol{e}\|}{4\|\boldsymbol{e}\|(\|\boldsymbol{e}\| \pm e^3)}} \in \mathbb{C}, \end{aligned}$$

if $\|\boldsymbol{e}\| \neq |e^3|$, and

$$\begin{aligned} c^\pm(E) &:= \frac{1 \pm \operatorname{sgn} e^3}{2}\sqrt{\frac{e^0 \pm |e^3|}{2}} \geqslant 0, \\ d^\pm(E) &:= \frac{1 \mp \operatorname{sgn} e^3}{2}\sqrt{\frac{e^0 \pm |e^3|}{2}} \geqslant 0, \end{aligned} \tag{3.1}$$

if $\|\boldsymbol{e}\| = |e^3|$ (i.e., $e^1 = e^2 = 0$); here $\operatorname{sgn} x := 1$ when $x \geqslant 0$ and -1 otherwise. Clearly, $\overline{c^+(E)}c^-(E) + \overline{d^+(E)}d^-(E) = 0$. We can write the spectral decomposition $E = E^+ + E^-$ where

$$E^\pm := \begin{pmatrix} |c^\pm(E)|^2 & \overline{c^\pm(E)}d^\pm(E) \\ c^\pm(E)\overline{d^\pm(E)} & |d^\pm(E)|^2 \end{pmatrix} = \frac{e^0 \pm \|\boldsymbol{e}\|}{2} \cdot \frac{1}{2\|\boldsymbol{e}\|} \begin{pmatrix} \|\boldsymbol{e}\| \pm e^3 & \pm(e^1 - ie^2) \\ \pm(e^1 + ie^2) & \|\boldsymbol{e}\| \mp e^3 \end{pmatrix}$$

are rank-1 or zero effects. If $\|\boldsymbol{e}\| = 0$ then $E = (e^0/2)\mathbb{1}$ and one has $E^\pm = (e^0/4)(\mathbb{1} \pm \sigma_3)$ by Eq. (3.1); only in this degenerate case the spectral projections are not unique. Note that $E \neq 0$ is of rank 1 if and only if $E^- = 0$ (i.e., $E = E^+$ or $c^-(E) = d^-(E) = 0$). Finally, if M is a positive semidefinite rank-1 matrix then M can be written in the form

$$M = \begin{pmatrix} |c|^2 & \overline{c}d \\ c\overline{d} & |d|^2 \end{pmatrix}$$

where the complex numbers c and d are unique if we assume that either $c > 0$ and $d \in \mathbb{C}$ or $c = 0$ and $d > 0$ (that is, $(c, d) \in \big(\mathbb{R}_+ \times \mathbb{C}\big) \cup \big(\{0\} \times \mathbb{R}_+\big)$ where $\mathbb{R}_+$ is the set of positive reals).

Let $\mathsf{E} = (E_1, E_2, \ldots, E_N)$ be an *N-valued POVM* of $\mathbb{C}^2$, i.e., the *non-zero* effects

$$E_i = \frac{1}{2}\sum_{\mu=0}^{3} e_i^\mu \sigma_\mu = \frac{1}{2}(e_i^0\mathbb{1} + \boldsymbol{e}_i \cdot \boldsymbol{\sigma}), \qquad i = 1, 2, \ldots N,$$

sum to the identity matrix $\mathbb{1}$. Write, as above, $E_i = E_i^+ + E_i^-$ where

$$E_i^\pm := \begin{pmatrix} |c^\pm(E_i)|^2 & \overline{c^\pm(E_i)} d^\pm(E_i) \\ c^\pm(E_i)\overline{d^\pm(E_i)} & |d^\pm(E_i)|^2 \end{pmatrix}.$$

Let $m_i \in \{1,2\}$ be the rank of E_i and form the multiplicity or rank vector of E,

$$\boldsymbol{m} := (m_1, m_2, \dots m_N)$$

whose ℓ^1-norm is $\|\boldsymbol{m}\|_1 = \sum_{i=1}^N m_i$. If E_i is rank-1 ($m_i = 1$) define $\boldsymbol{c}(E_i) := c^+(E_i)$ and $\boldsymbol{d}(E_i) := d^+(E_i)$. If E_i is of rank 2 ($m_i = 2$) define $\boldsymbol{c}(E_i) := \big(c^+(E_i), c^-(E_i)\big)$ and $\boldsymbol{d}(E_i) := \big(d^+(E_i), d^-(E_i)\big)$ which satisfy the orthogonality relation $\overline{c^+(E_i)}c^-(E_i) + \overline{d^+(E_i)}d^-(E_i) = 0$.

Now the vectors

$$\boldsymbol{c} := \big(\boldsymbol{c}(E_1), \boldsymbol{c}(E_2), \dots, \boldsymbol{c}(E_N)\big), \qquad \boldsymbol{d} := \big(\boldsymbol{d}(E_1), \boldsymbol{d}(E_2), \dots, \boldsymbol{d}(E_N)\big)$$

belong to the minimal Naimark dilation space $\mathbb{C}^{\|\boldsymbol{m}\|_1}$. Indeed, define an isometry $J_{\boldsymbol{c},\boldsymbol{d}} : \mathbb{C}^2 \to \mathbb{C}^{\|\boldsymbol{m}\|_1}$ via

$$J_{\boldsymbol{c},\boldsymbol{d}} := |\boldsymbol{c}\rangle\langle(1,0)| + |\boldsymbol{d}\rangle\langle(0,1)| = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ \vdots & \vdots \\ c_{\|\boldsymbol{m}\|_1} & d_{\|\boldsymbol{m}\|_1} \end{pmatrix}$$

where we have denoted briefly $\boldsymbol{c} = (c_1, \dots, c_{\|\boldsymbol{m}\|_1})$ and $\boldsymbol{d} = (d_1, \dots, d_{\|\boldsymbol{m}\|_1})$. Especially, $\boldsymbol{c}$ and $\boldsymbol{d}$ are orthonormal vectors (i.e., $\|\boldsymbol{c}\| = \|\boldsymbol{d}\| = 1$ and $\langle \boldsymbol{c}|\boldsymbol{d}\rangle = 0$), $J_{\boldsymbol{c},\boldsymbol{d}}^* J_{\boldsymbol{c},\boldsymbol{d}} = \mathbb{1}$, and $J_{\boldsymbol{c},\boldsymbol{d}} J_{\boldsymbol{c},\boldsymbol{d}}^* = |\boldsymbol{c}\rangle\langle\boldsymbol{c}| + |\boldsymbol{d}\rangle\langle\boldsymbol{d}|$ is a projection on $\mathbb{C}^{\|\boldsymbol{m}\|_1}$. In addition,

$$(c_i, d_i) \in \big(\mathbb{R}_+ \times \mathbb{C}\big) \cup \big(\{0\} \times \mathbb{R}_+\big)$$

for all $i = 1, 2, \dots, \|\boldsymbol{m}\|_1$. Let $\{b_k\}_{k=1}^{\|\boldsymbol{m}\|_1}$ be the standard (orthonormal) basis of $\mathbb{C}^{\|\boldsymbol{m}\|_1}$. Define $K_0 := 0$, $K_i := \sum_{k=1}^i m_k$, $i \in \{1, \dots, N\}$, and projections

$$P_i := \sum_{k=1+K_{i-1}}^{K_i} |b_k\rangle\langle b_k|$$

so that $\mathsf{P} = (P_1, \dots, P_N)$ is a projection valued measure (PVM). Since $E_i = J_{\boldsymbol{c},\boldsymbol{d}}^* P_i J_{\boldsymbol{c},\boldsymbol{d}}$ the triple $(\mathbb{C}^{\|\boldsymbol{m}\|_1}, \mathsf{P}, J_{\boldsymbol{c},\boldsymbol{d}})$ is a *minimal*[1] *Naimark dilation* of E, see, e.g., Ref. [1].

It should be stressed that any orthonormal vectors $\boldsymbol{c}$, $\boldsymbol{d} \in \mathbb{C}^{\|\boldsymbol{m}\|_1}$ can be used to define an isometry $J_{\boldsymbol{c},\boldsymbol{d}} := |\boldsymbol{c}\rangle\langle(1,0)| + |\boldsymbol{d}\rangle\langle(0,1)|$ and POVM E via $E_i := J_{\boldsymbol{c},\boldsymbol{d}}^* P_i J_{\boldsymbol{c},\boldsymbol{d}}$ but $\boldsymbol{m}$ is not necessarily the multiplicity vector of E. It may happen that some $m_i = 2$ but the rank of E_i is

[1] Clearly, the vectors $P_i\boldsymbol{c}$ and $P_i\boldsymbol{d}$, $i = 1, \dots, N$, span $\mathbb{C}^{\|\boldsymbol{m}\|_1}$.

0 or 1. However, if E is a rank-1 POVM (i.e., $m_i \equiv 1$ so $\|\boldsymbol{m}\|_1 = N$) then one has the following uniqueness result.

Proposition 1. Let N be a positive integer. Then there is a bijection between the set of N-valued rank-1 POVMs and the set of orthonormal vectors $\boldsymbol{c}$, $\boldsymbol{d} \in \mathbb{C}^N$ such that $(c_i, d_i) \in \big(\mathbb{R}_+ \times \mathbb{C}\big) \cup \big(\{0\} \times \mathbb{R}_+\big)$ for all $i = 1, 2, \ldots, N$.

Example 1. In this example, we characterise all N-valued qubit POVMs whose multiplicity vectors $\boldsymbol{m}$ have the same length $\|\boldsymbol{m}\|_1 = 4$ using the above dilation technique. We have the following nontrivial cases $\boldsymbol{m} = (1,1,1,1)$ $[N=4]$, $\boldsymbol{m} = (2,1,1)$, $\boldsymbol{m} = (1,2,1)$, $\boldsymbol{m} = (1,1,2)$ $[N=3]$, and $\boldsymbol{m} = (2,2)$ $[N=2]$. In all these cases, the dilation space is $\mathbb{C}^4$ and the isometry

$$J_{\mathbf{c},\boldsymbol{d}} = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \\ c_4 & d_4 \end{pmatrix}$$

where $(c_i, d_i) \in \big(\mathbb{R}_+ \times \mathbb{C}\big) \cup \big(\{0\} \times \mathbb{R}_+\big)$, $i = 1,2,3,4$, are such that $\|\boldsymbol{c}\| = \|\boldsymbol{d}\| = 1$ and $\langle \boldsymbol{c} | \boldsymbol{d} \rangle = 0$. By varying $\boldsymbol{c}$ and $\boldsymbol{d}$ we get all POVMs with $\|\boldsymbol{m}\|_1 = 4$. The above cases differ on the definition of the projections P_i:

- $\boxed{\boldsymbol{m} = (1,1,1,1)}$ Now $P_i = |b_i\rangle\langle b_i|$ for all i, e.g.,

$$P_3 = |b_3\rangle\langle b_3| = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$E_3 = J_{\mathbf{c},\boldsymbol{d}}^* P_3 J_{\mathbf{c},\boldsymbol{d}} = \begin{pmatrix} \overline{c}_1 & \overline{c}_2 & \overline{c}_3 & \overline{c}_4 \\ \overline{d}_1 & \overline{d}_2 & \overline{d}_3 & \overline{d}_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \\ c_4 & d_4 \end{pmatrix} = \begin{pmatrix} |c_3|^2 & \overline{c}_3 d_3 \\ c_3 \overline{d}_3 & |d_3|^2 \end{pmatrix}.$$

Similarly,

$$E_i = \begin{pmatrix} |c_i|^2 & \overline{c}_i d_i \\ c_i \overline{d}_i & |d_i|^2 \end{pmatrix}$$

for all $i = 1, 2, 3, 4$, and one can check that

$$\sum_{i=1}^{4} E_i = \sum_{i=1}^{4} \begin{pmatrix} |c_i|^2 & \overline{c}_i d_i \\ \overline{d}_i c_i & |d_i|^2 \end{pmatrix} = \begin{pmatrix} \|\boldsymbol{c}\|^2 & \langle \boldsymbol{c}|\boldsymbol{d}\rangle \\ \langle \boldsymbol{d}|\boldsymbol{c}\rangle & \|\boldsymbol{d}\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}.$$

- $\boxed{\boldsymbol{m} = (2,1,1)}$ The cases $\boldsymbol{m} = (2,1,1)$, $\boldsymbol{m} = (1,2,1)$, and $\boldsymbol{m} = (1,1,2)$ are essentially the same, so we study only the first one. Now $P_1 = |b_1\rangle\langle b_1| + |b_2\rangle\langle b_2|$, $P_2 = |b_3\rangle\langle b_3|$, and $P_3 = |b_4\rangle\langle b_4|$ showing that

$$E_1 = \begin{pmatrix} |c_1|^2 & \overline{c}_1 d_1 \\ c_1 \overline{d}_1 & |d_1|^2 \end{pmatrix} + \begin{pmatrix} |c_2|^2 & \overline{c}_2 d_2 \\ c_2 \overline{d}_2 & |d_2|^2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} |c_3|^2 & \overline{c}_3 d_3 \\ c_3 \overline{d}_3 & |d_3|^2 \end{pmatrix}, \quad E_3 = \begin{pmatrix} |c_4|^2 & \overline{c}_4 d_4 \\ c_4 \overline{d}_4 & |d_4|^2 \end{pmatrix}.$$

  Note that E_1 is of rank 2 exactly when $c_1 d_2 \neq c_2 d_1$ (which we must assume).
- $\boxed{\boldsymbol{m} = (2,2)}$ Now $P_1 = |b_1\rangle\langle b_1| + |b_2\rangle\langle b_2|$, $P_2 = |b_3\rangle\langle b_3| + |b_4\rangle\langle b_4|$ so that

$$E_1 = \begin{pmatrix} |c_1|^2 & \overline{c}_1 d_1 \\ c_1 \overline{d}_1 & |d_1|^2 \end{pmatrix} + \begin{pmatrix} |c_2|^2 & \overline{c}_2 d_2 \\ c_2 \overline{d}_2 & |d_2|^2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} |c_3|^2 & \overline{c}_3 d_3 \\ c_3 \overline{d}_3 & |d_3|^2 \end{pmatrix} + \begin{pmatrix} |c_4|^2 & \overline{c}_4 d_4 \\ c_4 \overline{d}_4 & |d_4|^2 \end{pmatrix}.$$

  Further assumptions $c_1 d_2 \neq c_2 d_1$ and $c_3 d_4 \neq c_4 d_3$ yield rank-2 effects E_1 and E_2.

Example 2. In this example, we study 2-valued qubit POVMs. It is easy to see that we have (essentially) the following cases:

- $\boxed{\boldsymbol{m} = (1,1)}$ Now

$$E_1 = \begin{pmatrix} |c_1|^2 & \overline{c}_1 d_1 \\ c_1 \overline{d}_1 & |d_1|^2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} |c_2|^2 & \overline{c}_2 d_2 \\ c_2 \overline{d}_2 & |d_2|^2 \end{pmatrix} = \mathbb{1} - E_1 = \begin{pmatrix} 1 - |c_1|^2 & -\overline{c}_1 d_1 \\ -c_1 \overline{d}_1 & 1 - |d_1|^2 \end{pmatrix}$$

  where $(c_i, d_i) \in \big(\mathbb{R}_+ \times \mathbb{C}\big) \cup \big(\{0\} \times \mathbb{R}_+\big)$, $i = 1, 2$, are such that $\|\boldsymbol{c}\| = \|\boldsymbol{d}\| = 1$ and $\langle \boldsymbol{c}|\boldsymbol{d}\rangle = 0$.
- $\boxed{\boldsymbol{m} = (2,1)}$ Now

$$E_1 = \begin{pmatrix} |c_1|^2 & \overline{c}_1 d_1 \\ c_1 \overline{d}_1 & |d_1|^2 \end{pmatrix} + \begin{pmatrix} |c_2|^2 & \overline{c}_2 d_2 \\ c_2 \overline{d}_2 & |d_2|^2 \end{pmatrix}, \quad E_2 = \begin{pmatrix} |c_3|^2 & \overline{c}_3 d_3 \\ c_3 \overline{d}_3 & |d_3|^2 \end{pmatrix}$$

  where $(c_i, d_i) \in \big(\mathbb{R}_+ \times \mathbb{C}\big) \cup \big(\{0\} \times \mathbb{R}_+\big)$, $i = 1, 2, 3$, are such that $\|\boldsymbol{c}\| = \|\boldsymbol{d}\| = 1$, $\langle \boldsymbol{c}|\boldsymbol{d}\rangle = 0$, and $c_1 d_2 \neq c_2 d_1$.
- $\boxed{\boldsymbol{m} = (2,2)}$ See the preceding example.

4. Jointly measurable discrete qubit POVMs

Let $\mathsf{E} = (E_1, E_2, \ldots, E_N)$ be an N-valued qubit POVM with the multiplicity vector $\boldsymbol{m}$ and the minimal Naimark dilation $(\mathbb{C}^{\|\boldsymbol{m}\|_1}, \mathsf{P}, J_{\mathbf{c},\boldsymbol{d}})$ as before. To characterise *all* qubit POVMs B jointly measurable with E, one can pick any POVM $\mathsf{F} = (F_1, F_2, \ldots, F_M)$ of the dilation space $\mathbb{C}^{\|\boldsymbol{m}\|_1}$ such that each effect F_j is decomposable[2] with respect to P, that is,

$$F_j = \bigoplus_{i=1}^{N} F_{ij}$$

where $\mathsf{F}_i = (F_{i1}, F_{i2}, \ldots, F_{iM})$ is a POVM of $\mathbb{C}^{m_i}$ (whose identity operator is P_i). Now it may happen that an effect F_{ij} is zero. In the case $m_i = 1$ the POVM F_i 'is' just a sequence of numbers $f_{ij} \geqslant 0$ (i.e., $F_{ij} = f_{ij} P_i$) such that and $\sum_{j=1}^{M} f_{ij} = 1$, whereas in the case $m_i = 2$, F_i *is a qubit POVM.* The jointly measurable POVM $\mathsf{B} = (B_1, B_2, \ldots, B_M)$ is of the form

$$B_j = J_{\mathbf{c},\boldsymbol{d}}^* F_j J_{\mathbf{c},\boldsymbol{d}} = \sum_{i=1}^{N} J_{\mathbf{c},\boldsymbol{d}}^* F_{ij} J_{\mathbf{c},\boldsymbol{d}}$$

and the joint POVM is $\mathsf{N} = (N_{ij})$ where

$$N_{ij} = J_{\mathbf{c},\boldsymbol{d}}^* F_{ij} J_{\mathbf{c},\boldsymbol{d}}.$$

Indeed, since $\sum_{j=1}^{M} F_{ij} = P_i$ one sees that $\sum_{j=1}^{M} N_{ij} = J_{\mathbf{c},\boldsymbol{d}}^* P_i J_{\mathbf{c},\boldsymbol{d}} = E_i$ and $\sum_{i=1}^{N} N_{ij} = B_j$. It can be shown that we get all compatible POVMs B by using this *Naimark dilation technique* [31]. This follows easily since any effect N_{ij} of a joint POVM is majorised by $E_i = (P_i J_{\mathbf{c},\boldsymbol{d}})^* P_i J_{\mathbf{c},\boldsymbol{d}}$ so that there exists an effect $F_{ij} \leqslant P_i$ for which $N_{ij} = J_{\mathbf{c},\boldsymbol{d}}^* F_{ij} J_{\mathbf{c},\boldsymbol{d}}$ holds. Next we study the structure of the effects N_{ij}.

If $m_i = 1$ then $N_{ij} = f_{ij} E_i$. If $m_i = 2$ then we can identify the qubit effect F_{ij} with the matrix

$$\frac{1}{2} \sum_{\mu=0}^{3} f_{ij}^{\mu} \sigma_\mu = \frac{1}{2}(f_{ij}^0 \mathbb{1} + \boldsymbol{f}_{ij} \cdot \boldsymbol{\sigma})$$

where $(f_{ij}^0, f_{ij}^1, f_{ij}^2, f_{ij}^3) = (f_{ij}^0, \boldsymbol{f}_{ij}) \in \mathbb{R}^4$ and $\|\boldsymbol{f}_{ij}\| \leqslant \min\{f_{ij}^0, 2 - f_{ij}^0\}$. Indeed, since now $P_i = \sum_{k=K_i-1}^{K_i} |b_k\rangle\langle b_k|$,

$$F_{ij} = Y_i^* \, \frac{1}{2} \sum_{\mu=0}^{3} f_{ij}^{\mu} \sigma_\mu \; Y_i = \frac{1}{2} \sum_{\mu=0}^{3} f_{ij}^{\mu} Y_i^* \sigma_\mu Y_i$$

---

[2] That is, F and P commute: $[F_j, P_i] = 0$.

where $Y_i = |(1,0)\rangle\langle b_{K_i-1}| + |(0,1)\rangle\langle b_{K_i}|$ is an isometry. From the equations

$$J^*_{\boldsymbol{c},\boldsymbol{d}} Y_i^* \sigma_0 Y_i J_{\boldsymbol{c},\boldsymbol{d}} = \begin{pmatrix} |c_{K_i-1}|^2 & \overline{c}_{K_i-1} d_{K_i-1} \\ c_{K_i-1}\overline{d}_{K_i-1} & |d_{K_i-1}|^2 \end{pmatrix} + \begin{pmatrix} |c_{K_i}|^2 & \overline{c}_{K_i} d_{K_i} \\ c_{K_i}\overline{d}_{K_i} & |d_{K_i}|^2 \end{pmatrix},$$

$$J^*_{\boldsymbol{c},\boldsymbol{d}} Y_i^* \sigma_1 Y_i J_{\boldsymbol{c},\boldsymbol{d}} = \begin{pmatrix} c_{K_i}\overline{c}_{K_i-1} + c_{K_i-1}\overline{c}_{K_i} & d_{K_i}\overline{c}_{K_i-1} + d_{K_i-1}\overline{c}_{K_i} \\ c_{K_i}\overline{d}_{K_i-1} + c_{K_i-1}\overline{d}_{K_i} & d_{K_i}\overline{d}_{K_i-1} + d_{K_i-1}\overline{d}_{K_i} \end{pmatrix},$$

$$J^*_{\boldsymbol{c},\boldsymbol{d}} Y_i^* \sigma_2 Y_i J_{\boldsymbol{c},\boldsymbol{d}} = i \begin{pmatrix} -c_{K_i}\overline{c}_{K_i-1} + c_{K_i-1}\overline{c}_{K_i} & -d_{K_i}\overline{c}_{K_i-1} + d_{K_i-1}\overline{c}_{K_i} \\ -c_{K_i}\overline{d}_{K_i-1} + c_{K_i-1}\overline{d}_{K_i} & -d_{K_i}\overline{d}_{K_i-1} + d_{K_i-1}\overline{d}_{K_i} \end{pmatrix},$$

$$J^*_{\boldsymbol{c},\boldsymbol{d}} Y_i^* \sigma_3 Y_i J_{\boldsymbol{c},\boldsymbol{d}} = \begin{pmatrix} |c_{K_i-1}|^2 & \overline{c}_{K_i-1} d_{K_i-1} \\ c_{K_i-1}\overline{d}_{K_i-1} & |d_{K_i-1}|^2 \end{pmatrix} - \begin{pmatrix} |c_{K_i}|^2 & \overline{c}_{K_i} d_{K_i} \\ c_{K_i}\overline{d}_{K_i} & |d_{K_i}|^2 \end{pmatrix}$$

one can calculate

$$N_{ij} = J^*_{\boldsymbol{c},\boldsymbol{d}} F_{ij} J_{\boldsymbol{c},\boldsymbol{d}} = \frac{1}{2}\sum_{\mu=0}^{3} f^{\mu}_{ij} \cdot J^*_{\boldsymbol{c},\boldsymbol{d}} Y_i^* \sigma_\mu Y_i J_{\boldsymbol{c},\boldsymbol{d}}.$$

Next we give some examples.

4.1. **Compatible effects.** In this section we take advantage of the method presented above to derive the criterion on joint measurability of two two-valued (unbiased) qubit POVMs first presented by Paul Busch in 1986 [37].

Fix a two-valued qubit POVMs $\mathsf{E} = (E_1, E_2)$, $E_1 + E_2 = \mathbb{1}$, and all related notions as in Section 3. Note that E is fully determined by the (nontrivial) effect E_1 which we denote briefly by E. Next we characterise all two-valued qubit POVMs $\mathsf{B} = (B, \mathbb{1} - B)$ (i.e., effects B) which are jointly measurable with E. We have (essentially) three cases:

- $\boxed{\boldsymbol{m} = (1,1)}$ Let e_1, $e_2 \in [0,1]$ be arbitrary and

$$B = J^*_{\boldsymbol{c},\boldsymbol{d}} \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} J_{\boldsymbol{c},\boldsymbol{d}} = e_1 E_1 + e_2 E_2 = e_2 \mathbb{1} + (e_1 - e_2) E$$

  showing that the effects E and B commute.
- $\boxed{\boldsymbol{m} = (2,1)}$ Let $F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ be any effect and $e_3 \in [0,1]$. Then

$$B = J^*_{\boldsymbol{c},\boldsymbol{d}} \begin{pmatrix} f_{11} & f_{12} & 0 \\ f_{21} & f_{22} & 0 \\ 0 & 0 & e_3 \end{pmatrix} J_{\boldsymbol{c},\boldsymbol{d}} = \begin{pmatrix} \overline{c}_1 & \overline{c}_2 & \overline{c}_3 \\ \overline{d}_1 & \overline{d}_2 & \overline{d}_3 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & 0 \\ f_{21} & f_{22} & 0 \\ 0 & 0 & e_3 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix}.$$

- $\boxed{\boldsymbol{m} = (2,2)}$

Let $F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ and $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ be any effects so that

$$B = J^*_{\boldsymbol{c},\boldsymbol{d}} \begin{pmatrix} f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & g_{21} & g_{22} \end{pmatrix} J_{\boldsymbol{c},\boldsymbol{d}} = \begin{pmatrix} \bar{c}_1 & \bar{c}_2 & \bar{c}_3 & \bar{c}_4 \\ \bar{d}_1 & \bar{d}_2 & \bar{d}_3 & \bar{d}_4 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \\ c_4 & d_4 \end{pmatrix}.$$

Consider then an *unbiased* two-outcome qubit POVM $(E, \mathbb{1} - E)$. Up to an irrelevant unitary, one can write

$$E = \frac{\mathbb{1} + \boldsymbol{e} \cdot \boldsymbol{\sigma}}{2} = \frac{\mathbb{1} + a\sigma_3}{2},$$

where $\boldsymbol{e} = (0, 0, a)$ and $|a| \leqslant 1$. The case $|a| = 1$ is trivial so that we restrict to $|a| < 1$ in the following. Therefore we have $\boldsymbol{m} = (2, 2)$ and, by using the general procedure introduced previously, we write

$$E = J^* \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J \quad \text{where} \quad J = \begin{pmatrix} \sqrt{\frac{1+a}{2}} & 0 \\ 0 & \sqrt{\frac{1-a}{2}} \\ 0 & \sqrt{\frac{1+a}{2}} \\ \sqrt{\frac{1-a}{2}} & 0 \end{pmatrix}.$$

Now let us consider another (possibly biased) two-outcome qubit POVM $(B, \mathbb{1} - B)$ jointly measurable with $(E, \mathbb{1} - E)$. We know that

$$B = J^* \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix} J, \tag{4.1}$$

where F and G are two qubit effects. In the Pauli basis, equation (4.1) gives

$$\begin{cases} b^0 = \frac{f^0 + g^0}{2} + a \cdot \frac{f^3 + g^3}{2} \\ b^1 = \sqrt{1 - a^2} \cdot \frac{f^1 + g^1}{2} \\ b^2 = \sqrt{1 - a^2} \cdot \frac{f^2 - g^2}{2} \\ b^3 = \frac{f^3 - g^3}{2} + a \cdot \frac{f^0 - g^0}{2} \end{cases}$$

where $B = (b^0 \mathbb{1} + \boldsymbol{b} \cdot \boldsymbol{\sigma})/2$ and similarly for F and G. Define $m(x) := \min\{x, 2 - x\}$, $x \in [0, 2]$. Then we can use $(f^1)^2 + (f^2)^2 + (f^3)^2 \leqslant [m(f^0)]^2$, $(g^1)^2 + (g^2)^2 + (g^3)^2 \leqslant [m(g^0)]^2$, and the

Cauchy–Schwarz inequality to get

$$\boldsymbol{e}^2 + \boldsymbol{b}^2 - (\boldsymbol{e}\cdot\boldsymbol{b})^2 = a^2 + (1-a^2)\left[\left(\frac{f^1+g^1}{2}\right)^2 + \left(\frac{f^2-g^2}{2}\right)^2 + \left(\frac{f^3-g^3}{2} + a\cdot\frac{f^0-g^0}{2}\right)^2\right]$$

$$\leqslant a^2 + (1-a^2)\left[\left(\frac{m(f^0)+m(g^0)}{2}\right)^2 + \left(a\cdot\frac{f^0-g^0}{2}\right)^2 + 2a\cdot\frac{f^3-g^3}{2}\cdot\frac{f^0-g^0}{2}\right]$$

$$\leqslant a^2 + (1-a^2)\left[\left(\frac{m(f^0)+m(g^0)}{2}\right)^2 + \left(a\cdot\frac{f^0-g^0}{2}\right)^2 + 2|a|\cdot\frac{m(f^0)+m(g^0)}{2}\cdot\left|\frac{f^0-g^0}{2}\right|\right]$$

$$\leqslant a^2 + (1-a^2)\left[\frac{m(f^0)+m(g^0)+|f^0-g^0|}{2}\right]^2 \leqslant 1,$$

so that in the end we have the following (equivalent) inequality [1, Prop. 14.1]:

$$\|\boldsymbol{e}+\boldsymbol{b}\| + \|\boldsymbol{e}-\boldsymbol{b}\| = \sqrt{\boldsymbol{e}^2+\boldsymbol{b}^2+2\boldsymbol{e}\cdot\boldsymbol{b}} + \sqrt{\boldsymbol{e}^2+\boldsymbol{b}^2-2\boldsymbol{e}\cdot\boldsymbol{b}} \leqslant |1+\boldsymbol{e}\cdot\boldsymbol{b}| + |1-\boldsymbol{e}\cdot\boldsymbol{b}| = 2.$$

Thus, we have proven that any qubit effect B compatible with E satisfies the above Busch's criterion [37]. Actually, this holds for any compatible pair of (possibly biased) qubit effects [1, Prop. 14.2].

Conversely, if we take any *unbiased* two-outcome qubit POVM such that $\boldsymbol{e}^2+\boldsymbol{b}^2 \leqslant 1+(\boldsymbol{e}\cdot\boldsymbol{b})^2$, the following choice gives rise to valid *positive* F and G satisfying equation (4.1):

$$\begin{cases} f^0 = g^0 = b^0 = 1 \quad \text{(unbiased)} \\ f^1 = g^1 = \frac{b^1}{\sqrt{1-a^2}} \\ f^2 = -g^2 = \frac{b^2}{\sqrt{1-a^2}} \\ f^3 = -g^3 = b^3 \end{cases}$$

that is, E and B are compatible. Finally, we note that there exist incompatible (biased) effects E and B such that $\boldsymbol{e}^2+\boldsymbol{b}^2 \leqslant 1+(\boldsymbol{e}\cdot\boldsymbol{b})^2$ holds. For instance, take $e^0 = \sqrt{15}/4$, $\boldsymbol{e} = (0,0,e^0)$, $b^0 = \frac{1}{4}$, and $\boldsymbol{b} = \frac{1}{4}(1,0,0)$ to get $\boldsymbol{e}^2+\boldsymbol{b}^2-(\boldsymbol{e}\cdot\boldsymbol{b})^2 = 1$. If the corresponding (rank-1) effects

$$E = \frac{\sqrt{15}}{4}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \frac{1}{8}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

had a joint POVM, i.e., $E = N_{11}+N_{12}$ and $B = N_{11}+N_{21}$, then $N_{11} \leqslant E$ and $N_{11} \leqslant B$ yield $N_{11} = 0$ so that

$$N_{22} = \mathbb{1} - E - B = \frac{1}{8}\begin{pmatrix} 7-2\sqrt{15} & -1 \\ -1 & 7 \end{pmatrix}$$

is not positive (since $7-2\sqrt{15} \approx -0.7$).

4.2. **Three-valued symmetric POVM.** Our symmetry group is the (additive) cyclic group $\mathbb{Z}_3 = \{0,1,2\}$ equipped with the addition modulo 3, e.g., $1+2 \equiv 0 \pmod 3$. It operates on itself: any $k \in \mathbb{Z}_3$ corresponds to a permutation (i.e., bijection) $b_k(\ell) := k+\ell \pmod 3$ for all $\ell \in \mathbb{Z}_3$. Furthermore, $\mathbb{Z}_3$ acts in $\mathbb{C}^2$ via the unitary representation $k \mapsto U_k := R(2k\pi/3)$ where

$$R(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2)$$

is the rotation matrix. Hence, we get the mutually commuting unitaries

$$U_0 = \mathbb{1}, \quad U_1 = U_2^* = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad U_2 = U_1^* = (U_1)^2 = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$

Let $|+\rangle = (1,0)$ and $|-\rangle = (0,1)$ denote the eigenvectors of σ_3. Define a 3-valued covariant POVM E with effects $E_k := \frac{2}{3}U_k|+\rangle\langle+|U_k^*$, $k \in \mathbb{Z}_3$, and its noisy version E^λ, $\lambda \in [0,1]$, via $E_k^\lambda := \lambda E_k + (1-\lambda)\mathbb{1}/3$, i.e.,

$$E_0^\lambda = \begin{pmatrix} a_+^\lambda & 0 \\ 0 & a_-^\lambda \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}, \quad E_1^\lambda = \frac{1}{6}\begin{pmatrix} 2-\lambda & -\sqrt{3}\lambda \\ -\sqrt{3}\lambda & 2+\lambda \end{pmatrix}, \quad E_2^\lambda = \frac{1}{6}\begin{pmatrix} 2-\lambda & \sqrt{3}\lambda \\ \sqrt{3}\lambda & 2+\lambda \end{pmatrix}$$

where $a_\pm^\lambda := (1\pm\lambda)/3$. One can also write

$$E_0^\lambda = \frac{1}{3}\mathbb{1} + \frac{\lambda}{3}\sigma_3, \quad E_1^\lambda = \frac{1}{3}\mathbb{1} - \frac{\sqrt{3}\lambda}{6}\sigma_1 - \frac{\lambda}{6}\sigma_3, \quad E_2^\lambda = \frac{1}{3}\mathbb{1} + \frac{\sqrt{3}\lambda}{6}\sigma_1 - \frac{\lambda}{6}\sigma_3.$$

Let $\eta \in [0,1]$ and define a POVM B^η via

$$B_0^\eta = \eta\frac{2}{3}|-\rangle\langle-| + (1-\eta)\frac{1}{3}\mathbb{1}, \qquad B_k^\eta = U_k B_0^\eta U_k^*, \quad k \in \mathbb{Z}_3.$$

It is shown in Appendix A that POVMs E^λ and B^η are jointly measurable when

$$\eta \leqslant \frac{\lambda^2}{2\left(1-\sqrt{1-\lambda^2}\right)}.$$

If we assume that $\eta = \lambda$ then one must have $\lambda \leqslant 4/5 = 0.8$. This, however, is not the optimal value, as can be seen analytically by using the joint measurement characterisation of Ref. [38], which gives $\lambda \lesssim 0.866$.

5. Examples of jointly measurable continuous qubit POVMs

Suppose that two qubit POVMs E and B (defined on σ-algebras $\mathcal{A} \subseteq 2^\Omega$ and $\mathcal{B} \subseteq 2^\Xi$) are jointly measurable with a joint POVM N. Now a Naimark dilation of E can be constructed as follows: Let $\mu : \mathcal{A} \to [0,1]$ be a probability measure such that E is absolutely continuous with respect to it. For example, $\mu(X) = \frac{1}{2}\mathrm{tr}\,[\mathsf{E}(X)]$ is fine. Then, by the Radon-Nikodym theorem,

E has a qubit density D, i.e., one can write $\mathsf{E}(X) = \int_X D(x)\mathrm{d}\mu(x)$, $x \in X$, where each $D(x)$ is positive semidefinite 2×2 matrix and $x \mapsto D(x)$ is μ-measurable [43]. By using the spectral decomposition of $D(x)$, also $x \mapsto \sqrt{D(x)}$ is μ-measurable and one can define an isometry J from $\mathbb{C}^2$ into $L^2(\mu) \otimes \mathbb{C}^2$, where $L^2(\mu)$ is the Lebesgue (Hilbert) space,[3] via

$$\left[J \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right](x) := \sqrt{D(x)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad (c_1, c_2) \in \mathbb{C}^2, \quad x \in \Omega.$$

Now, for all $X \in \mathcal{A}$, one gets $\mathsf{E}(X) = J^*\mathsf{P}(X)J$ where P is the canonical spectral measure defined by

$$\left[\mathsf{P}(X) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right](x) := \begin{pmatrix} \chi_X(x)\psi_1(x) \\ \chi_X(x)\psi_2(x) \end{pmatrix}, \qquad \psi_1,\, \psi_2 \in L^2(\mu),$$

and χ_X is the characteristic function of the set $X \in \mathcal{A}$. The obtained Naimark dilation is not necessarily minimal[4] (since the rank of $D(x)$ can be 0 or 1) but we can still write, for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$,

$$\mathsf{N}(X \times Y) = \int_X \sqrt{D(x)}\mathsf{F}_x(Y)\sqrt{D(x)}\mathrm{d}\mu(x), \qquad \mathsf{B}(Y) = \int_\Omega \sqrt{D(x)}\mathsf{F}_x(Y)\sqrt{D(x)}\mathrm{d}\mu(x),$$

where each F_x is a qubit POVM[5] on $\mathcal{B}$ [31, 33]. In the next examples, E and B are given and we try to find N by assuming that each F_x is absolutely continuous with respect to the probability measure ν of B (e.g., $\nu(Y) = \frac{1}{2}\mathrm{tr}\,[\mathsf{B}(Y)]$); now N is absolutely continuous with respect to the product measure $\mu \times \nu$ and has a qubit density with respect to it. This method is easy to generalise for three (or more) jointly measurable POVMs as follows.

In the rest of this section, we study a single-mode optical field (i.e., a harmonic oscillator) in the case of a single photon. We use the position representation of the position (quadrature) operator Q (with the spectral measure Q) where the number states $|n\rangle$ are represented as the Hermite–Gauss functions

$$h_n(x) := \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{1}{2}x^2} = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{1}{2}x^2} \frac{\mathrm{d}^n e^{-x^2}}{\mathrm{d}x^n}$$

where H_n is a (normalised) Hermite polynomial. Hermite polynomials H_n are given by the recursion relation $H_0(x) = 1$, $H_1(x) = 2x$, and $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ or by the

[3]Consisting of equivalence classes of μ-square integrable complex functions on Ω.

[4]The only drawback of non-minimality is that, for a given N, the POVMs F_x are note necessarily unique.

[5]Actually, one needs some reqularity conditions for the measurable spaces, e.g., they are standard Borel [33].

Rodrigue's formula $H_n(x) = (-1)^n e^{x^2} \mathrm{d}^n e^{-x^2}/\mathrm{d}x^n$. The spectral measure of the rotated (or tilted) quadrature Q_θ is

$$\mathsf{Q}_\theta(X) = e^{i\theta a^* a}\mathsf{Q}(X)e^{-i\theta a^* a} = \sum_{n,m=0}^{\infty} e^{i(n-m)\theta} \int_X h_n(x)h_m(x)\mathrm{d}x|n\rangle\langle m|$$

for all Borel sets $X \subseteq \mathbb{R}$; here a is the lowering operator [1]. Note that the momentum operator $P = Q_{\pi/2}$. The projection of Q_θ onto the single-photon subspace span$\{|0\rangle, |1\rangle\}$ is the (qubit) POVM

$$\int_X \begin{pmatrix} 1 & \sqrt{2}xe^{-i\theta} \\ \sqrt{2}xe^{i\theta} & 2x^2 \end{pmatrix} \frac{e^{-x^2}\mathrm{d}x}{\sqrt{\pi}}$$

whose noisy version is

$$\begin{aligned} \mathsf{Q}_\theta^{\text{prono}}(X) &:= (1-\epsilon_\theta)\underbrace{\int_X \begin{pmatrix} 1 & \sqrt{2}xe^{-i\theta} \\ \sqrt{2}xe^{i\theta} & 2x^2 \end{pmatrix} \frac{e^{-x^2}\mathrm{d}x}{\sqrt{\pi}}}_{\text{noiseless projected } \mathsf{Q}_\theta} + \epsilon_\theta \underbrace{\int_X \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{e^{-x^2}\mathrm{d}x}{\sqrt{\pi}}}_{\text{the random noise}} \\ &= \int_X \begin{pmatrix} 1 & (1-\epsilon_\theta)\sqrt{2}xe^{-i\theta} \\ (1-\epsilon_\theta)\sqrt{2}xe^{i\theta} & (1-\epsilon_\theta)2x^2 + \epsilon_\theta \end{pmatrix} \frac{e^{-x^2}\mathrm{d}x}{\sqrt{\pi}} \end{aligned}$$

where $\epsilon_\theta \in [0,1]$.

Similarly consider energy $H = \hbar\omega\left(a^* a + \frac{1}{2}\right)$ whose spectral measure is $\{n\} \mapsto \mathsf{N}_n := |n\rangle\langle n|$. The projected noisy energy POVM has two non-zero effects:

$$\begin{aligned} \mathsf{N}_0^{\text{prono}} &:= (1-\epsilon)\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{vacuum effect}} + \epsilon\,\frac{1}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{noise effect}} = \begin{pmatrix} 1-\epsilon/2 & 0 \\ 0 & \epsilon/2 \end{pmatrix} \\ \mathsf{N}_1^{\text{prono}} &:= (1-\epsilon)\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{1-photon effect}} + \epsilon\,\frac{1}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{noise effect}} = \begin{pmatrix} \epsilon/2 & 0 \\ 0 & 1-\epsilon/2 \end{pmatrix} \end{aligned}$$

where $\epsilon \in [0,1]$. Next we study joint measurability of POVMs $\mathsf{N}^{\text{prono}}$, $\mathsf{Q}_0^{\text{prono}}$, and $\mathsf{Q}_\theta^{\text{prono}}$ for any $\theta \notin \{0,\pi\}$. It is shown in Appendix B that

- $\mathsf{N}^{\text{prono}}$, $\mathsf{Q}_0^{\text{prono}}$, and $\mathsf{Q}_\theta^{\text{prono}}$ are jointly measurable if

$$\frac{\epsilon}{2-\epsilon} + \epsilon_0 + \epsilon_\theta - 2 \geqslant 0,$$

- $\mathsf{Q}_0^{\text{prono}}$ and $\mathsf{Q}_\theta^{\text{prono}}$ are jointly measurable if

$$\epsilon_0 + \epsilon_\theta - 1 \geqslant 0,$$

- $\mathsf{N}^{\rm prono}$ and $\mathsf{Q}_\theta^{\rm prono}$ are jointly measurable if
$$\frac{2}{2-\epsilon}+\epsilon_\theta-2\geqslant 0,$$
- $\mathsf{N}^{\rm prono}$ and $\mathsf{Q}_0^{\rm prono}$ are jointly measurable if
$$\frac{2}{2-\epsilon}+\epsilon_0-2\geqslant 0.$$

In the case $\epsilon=\epsilon_0=\epsilon_\theta$ we find better limits: for example, $\mathsf{N}^{\rm prono}$, $\mathsf{Q}_0^{\rm prono}$, and $\mathsf{Q}_\theta^{\rm prono}$ are jointly measurable if $\epsilon\geqslant 4/7$.

Finally, we show in Appendix B that the noisy projected canonical phase

$$\Phi(X):=(1-\epsilon)\underbrace{\int_X\begin{pmatrix}1 & e^{-i\theta}\\ e^{i\theta} & 1\end{pmatrix}\frac{\mathrm{d}\theta}{2\pi}}_{\text{the canonical phase}}+\epsilon\underbrace{\int_X\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\frac{\mathrm{d}\theta}{2\pi}}_{\text{the random noise}},\qquad X\subseteq[0,2\pi),$$

and $\mathsf{N}^{\rm prono}$ are jointly measurable if $\epsilon\geqslant 1-1/\sqrt{2}$.

6. Conclusions

We have analysed a method for characterising all measurements that are jointly measurable with a given measurement. The technique maps the problem of deciding joint measurability into the problem of finding suitable block-diagonal POVMs in a minimal Naimark dilation space of one of the involved POVMs. We have demonstrated the use of the technique with heuristic ansätze. Whereas some of these lead to the optimal noise tolerance, such as in the case of the celebrated Busch criterion [37], we have shown that in other scenarios, such as symmetric trinary qubit measurements, an optimal ansatz may be harder to find. We have further presented a full closed-form characterisation of all qubit effects that are jointly measurable with a given qubit effect, and extended our analysis to scenarios involving pairs and triplets of continuous qubit measurements.

References

[1] P. Busch, P. Lahti, J.-P. Pellonpää, and K.Ylinen. *Quantum Measurement (Theoretical and Mathematical Physics)*. Springer, 2016.

[2] Paul Busch, Pekka Lahti, and Reinhard F. Werner. Colloquium: Quantum root-mean-square error and measurement uncertainty relations. *Rev. Mod. Phys.*, 86:1261–1281, 2014.

[3] Patrick J. Coles, Mario Berta, Marco Tomamichel, and Stephanie Wehner. Entropic uncertainty relations and their applications. *Rev. Mod. Phys.*, 89:015002, 2017.

[4] Nicolas Gisin, Grégoire Ribordy, Wolfgang Tittel, and Hugo Zbinden. Quantum cryptography. *Rev. Mod. Phys.*, 74:145–195, 2002.

[5] C. L. Degen, F. Reinhard, and P. Cappellaro. Quantum sensing. *Reviews of Modern Physics*, 89(3), 2017.

[6] Miguel Herrero-Collantes and Juan Carlos Garcia-Escartin. Quantum random number generators. *Rev. Mod. Phys.*, 89:015004, 2017.

[7] Yelena Guryanova, Nicolai Friis, and Marcus Huber. Ideal projective measurements have infinite resource costs. *Quantum*, 4:222, 2020.

[8] Michał Oszmaniec and Tanmoy Biswas. Operational relevance of resource theories of quantum measurements. *Quantum*, 3:133, 2019.

[9] R. Uola, T. Kraft, J. Shang, X.-D. Yu, and O. Gühne. Quantifying quantum resources with conic programming. *Phys. Rev. Lett.*, 122:130404, 2019.

[10] P. Lahti. Coexistence and joint measurability in quantum mechanics. *Int. J. Theor. Phys.*, 42:893, 2003.

[11] T. Heinosaari and M.M. Wolf. Non-disturbing quantum measurements. *J. Math. Phys.*, 51:092201, 2010.

[12] T. Heinosaari. Simultaneous measurement of two quantum observables: compatibility, broadcasting, and in-between. *Phys. Rev. A*, 93:042118, 2016.

[13] M. Oszmaniec, L. Guerini, P. Wittek, and A. Acín. Simulating positive-operator-valued measures with projective measurements. *Phys. Rev. Lett.*, 119:190501, 2017.

[14] Marie Ioannou, Pavel Sekatski, Sébastien Designolle, Benjamin D. M. Jones, Roope Uola, and Nicolas Brunner. Simulability of high-dimensional quantum measurements. arxiv: 2202.12980.

[15] Thomas Cope and Roope Uola. Quantifying the high-dimensionality of quantum devices. arxiv:2207.05722.

[16] M.M. Wolf, D. Perez-Garcia, and C. Fernandez. Measurements incompatible in quantum theory cannot be measured jointly in any other local theory. *Phys. Rev. Lett.*, 103:230402, 2009.

[17] M. T. Quintino, T. Vértesi, and N. Brunner. Joint measurability, einstein-podolsky-rosen steering, and bell nonlocality. *Phys. Rev. Lett.*, 113:160402, 2014.

[18] R. Uola, T. Moroder, and O. Gühne. Joint measurability of generalized measurements implies classicality. *Phys. Rev. Lett.*, 113:160403, 2014.

[19] R. Uola, C. Budroni, O. Gühne, and J.-P. Pellonpää. One-to-one mapping between steering and joint measurability problems. *Phys. Rev. Lett.*, 115:230402, 2015.

[20] J. Kiukas, C. Budroni, R. Uola, and J.-P. Pellonpää. Continuous-variable steering and incompatibility via state-channel duality. *Phys. Rev. A*, 96:042331, 2017.

[21] Armin Tavakoli and Roope Uola. Measurement incompatibility and steering are necessary and sufficient for operational contextuality. *Physical Review Research*, 2(1), 2020.

[22] P. Skrzypczyk, I. Šupić, and D. Cavalcanti. All sets of incompatible measurements give an advantage in quantum state discrimination. *Phys. Rev. Lett.*, 122:130403, 2019.

[23] C. Carmeli, T. Heinosaari, and A. Toigo. Quantum incompatibility witnesses. *Phys. Rev. Lett.*, 122:130402, 2019.

[24] Leonardo Guerini, Marco Túlio Quintino, and Leandro Aolita. Distributed sampling, quantum communication witnesses, and measurement incompatibility. *Phys. Rev. A*, 100:042308, 2019.

[25] Konstantin Beyer, Roope Uola, Kimmo Luoma, and Walter T. Strunz. Joint measurability in nonequilibrium quantum thermodynamics. *Physical Review E*, 106(2), 2022.

[26] Fei Zhou, Leilei Yan, Shijie Gong, Zhihao Ma, Jiuzhou He, Taiping Xiong, Liang Chen, Wanli Yang, Mang Feng, and Vlatko Vedral. Verifying heisenberg's error-disturbance relation using a single trapped ion. *Science Advances*, 2(10):e1600578, 2016.

[27] Sébastien Designolle, Vatshal Srivastav, Roope Uola, Natalia Herrera Valencia, Will McCutcheon, Mehul Malik, and Nicolas Brunner. Genuine high-dimensional quantum steering. *Phys. Rev. Lett.*, 126:200404, 2021.

[28] Hammad Anwer, Sadiq Muhammad, Walid Cherifi, Nikolai Miklin, Armin Tavakoli, and Mohamed Bourennane. Experimental characterization of unsharp qubit observables and sequential measurement incompatibility via quantum random access codes. *Phys. Rev. Lett.*, 125:080403, 2020.

[29] Andrea Smirne, Simone Cialdi, Daniele Cipriani, Claudio Carmeli, Alessandro Toigo, and Bassano Vacchini. Experimentally determining the incompatibility of two qubit measurements. *Quantum Sci. Technol.*, 7:025016, 2022.

[30] Otfried Gühne, Erkka Haapasalo, Tristan Kraft, Juha-Pekka Pellonpää, and Roope Uola. Incompatible measurements in quantum information science. arxiv:2112.06784.

[31] Juha-Pekka Pellonpää. On coexistence and joint measurability of rank-1 quantum observables. *J. Phys. A*, 47(5):052002, 2014.

[32] Erkka Haapasalo, Juha-Pekka Pellonpää, and Roope Uola. Compatibility properties of extreme quantum observables. *Lett. Math. Phys.*, 105:661–673, 2015.

[33] Erkka Haapasalo and Juha-Pekka Pellonpää. Optimal quantum observables. *J. Math. Phys.*, 58(12):122104, 2017.

[34] P. Kruszyński and W. M. de Muynck. Compatibility of observables represented by positive operator valued measures. *Journal of Mathematical Physics*, 28(8):1761–1763, 1987.

[35] Roberto Beneduci. Joint measurability through naimarkâs dilation theorem. *Reports on Mathematical Physics*, 79(2):197–214, 2017.

[36] Arindam Mitra, Sibasish Ghosh, and Prabha Mandayam. Characterizing incompatibility of quantum measurements via their naimark extensions. arxiv:2011.11364.

[37] P. Busch. Unsharp reality and joint measurements for spin observables. *Phys. Rev. D*, 33:2253, 1986.

[38] Jeongwoo Jae, Kyunghyun Baek, Junghee Ryu, and Jinhyoung Lee. Necessary and sufficient condition for joint measurability. *Phys. Rev. A*, 100:032113, 2019.

[39] R. Uola, K. Luoma, T. Moroder, and T. Heinosaari. Adaptive strategy for joint measurements. *Phys. Rev. A*, 94:022109, 2016.

[40] Teiko Heinosaari. A simple sufficient condition for the coexistence of quantum effects. *Journal of Physics A: Mathematical and Theoretical*, 46(15):152002, 2013.

[41] Sébastien Designolle, Máté Farkas, and Jedrzej Kaniewski. Incompatibility robustness of quantum measurements: a unified framework. *New J. Phys.*, 21:113053, 2019.

[42] Jukka Kiukas, Daniel McNulty, and Juha-Pekka Pellonpää. Amount of quantum coherence needed for measurement incompatibility. *Physical Review A*, 105(1), 2022.

[43] Tuomas Hytönen, Juha-Pekka Pellonpää, and Kari Ylinen. Positive sesquilinear form measures and generalized eigenvalue expansions. *J. Math. Anal. Appl.*, 336:1287–1304, 2007.

Appendix A

Let $E_k^\lambda := \lambda\frac{2}{3}U_k|+\rangle\langle+|U_k^* + (1-\lambda)\mathbb{1}/3$, $k \in \mathbb{Z}_3$. If $\lambda \neq 1$, E^λ is of rank 2 so that its minimal covariant Naimark dilation consists of the dilation space $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \cong \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$, with the basis $\{|k\pm\rangle\}$ where $|0\pm\rangle := (|\pm\rangle, 0, 0)$, $|1\pm\rangle := (0, |\pm\rangle, 0)$, $|2\pm\rangle = (0, 0, |\pm\rangle)$, the PVM $P_k := |k+\rangle\langle k+| + |k-\rangle\langle k-|$, $k \in \mathbb{Z}_3$, the isometry

$$J_\lambda = \sum_{k=0}^{2} \sqrt{a_+^\lambda}|k+\rangle\langle+|U_k^* + \sqrt{a_-^\lambda}|k-\rangle\langle-|U_k^*, \qquad a_\pm^\lambda := \frac{1\pm\lambda}{3},$$

and the unitary representation

$$\mathbb{Z}_3 \ni k \mapsto V_k := \sum_{\ell=0}^{2} |b_k(\ell)+\rangle\langle\ell+| + |b_k(\ell)-\rangle\langle\ell-|.$$

Indeed, clearly $P_k = V_k P_0 V_k^*$ is covariant, $J_\lambda U_k = V_k J_\lambda$, and $E_k^\lambda = J_\lambda^* P_k J_\lambda$.

Suppose then that $\mathsf{B} = (B_j)$ is a qubit POVM which is jointly measurable with E^λ. For any[6] joint POVM $\mathsf{N} = (N_{kj})$ (i.e., $\sum_j N_{kj} = E_k^\lambda$ and $\sum_k N_{kj} = B_j$) there exist three unique qubit POVMs $\tilde{\mathsf{A}}^{(k)} = \big(\tilde{A}_j^{(k)}\big)$, $k \in \mathbb{Z}_p$, such that $N_{kj} = J_\lambda^* \tilde{A}_j^{(k)} J_\lambda$; here $\tilde{\mathsf{A}}^{(k)}$ operates in the subspace spanned by $\{|k+\rangle, |k-\rangle\}$. By defining matrices

$$A_j^{(k)} := \begin{pmatrix} \langle k+|\tilde{A}_j^{(k)}|k+\rangle & \langle k+|\tilde{A}_j^{(k)}|k-\rangle \\ \langle k-|\tilde{A}_j^{(k)}|k+\rangle & \langle k-|\tilde{A}_j^{(k)}|k-\rangle \end{pmatrix},$$

we get

$$N_{kj} = J_\lambda^* \tilde{A}_j^{(k)} J_\lambda = U_k\big(M^\lambda \star A_j^{(k)}\big)U_k^*$$

where

$$M^\lambda := \begin{pmatrix} a_+^\lambda & \sqrt{a_+^\lambda a_-^\lambda} \\ \sqrt{a_-^\lambda a_+^\lambda} & a_-^\lambda \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1+\lambda & \sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & 1-\lambda \end{pmatrix} \geqslant 0$$

and $\star$ is the entrywise (Schur) product. One can solve the A–matrices (showing uniqueness):

$$A_j^{(k)} = \big(U_k^* N_{kj} U_k\big) \star N^\lambda$$

where

$$N^\lambda := 3\begin{pmatrix} (1+\lambda)^{-1} & 1/\sqrt{1-\lambda^2} \\ 1/\sqrt{1-\lambda^2} & (1-\lambda)^{-1} \end{pmatrix} \geqslant 0.$$

[6]Of course, the joint POVM of $(\mathsf{E}^\lambda, \mathsf{F})$ needs not be unique.

Note that there may be many A–matrices (i.e., many joint POVMs G) giving the same marginal $B_j = \big(M^\lambda \star A_j^{(0)}\big) + U_1\big(M^\lambda \star A_j^{(k)}\big)U_1^* + U_2\big(M^\lambda \star A_j^{(2)}\big)U_2^*$.

Let $\mathsf{B} = \mathsf{B}^\eta$ above, where ψ is a unit vector, $\eta \in [0,1]$, and

$$B_0^\eta = \eta\frac{2}{3}|\psi\rangle\langle\psi| + (1-\eta)\frac{1}{3}\mathbb{1}, \qquad B_k^\eta = U_k B_0^\eta U_k^*, \quad k \in \mathbb{Z}_3.$$

Since B^η is also covariant we may assume that N is covariant, i.e., satisfies $N_{k+\ell,j+\ell} = U_\ell N_{kj} U_\ell^*$ (note that $\tilde N_{kj} := \frac{1}{3}\sum_{s=0}^{2} U_s^* N_{k+s,j+s} U_s$ gives the same marginals as N and is covariant).

For $\psi = |-\rangle$, we guess that

$$A_0^{(k)} = U_k^* A U_k, \qquad A = \begin{pmatrix} d & 0 \\ 0 & e \end{pmatrix}, \qquad d,\, e \geqslant 0, \quad d + e = \frac{2}{3},$$

could be a good choice for $N_{k0} = U_k\big(M^\lambda \star A_0^{(k)}\big)U_k^*$. This determines the rest of the effects by the covariance condition $N_{k+\ell,j+\ell} = U_\ell N_{kj} U_\ell^*$. Let us solve parameter d from

$$\begin{aligned} B_0^\eta &= \frac{1}{3}\begin{pmatrix} 1-\eta & 0 \\ 0 & 1+\eta \end{pmatrix} = M^\lambda \star A + U_1\big(M^\lambda \star U_1^* A U_1\big)U_1^* + U_2\big(M^\lambda \star U_2^* A U_2\big)U_2^* \\ &= \frac{1}{4}\begin{pmatrix} \big(3+\sqrt{1-\lambda^2}\big)\, d + \big(1-\sqrt{1-\lambda^2}\big)\, e & 0 \\ 0 & \big(3+\sqrt{1-\lambda^2}\big)\, e + \big(1-\sqrt{1-\lambda^2}\big)\, d \end{pmatrix} \end{aligned}$$

where $e = \frac{2}{3} - d$. The solution is

$$d = \frac{\lambda^2 - 2\eta\left(1-\sqrt{1-\lambda^2}\right)}{3\lambda^2} \leqslant \frac{\lambda^2}{3\lambda^2} = \frac{1}{3}$$

so that automatically $e = \frac{2}{3} - d \in [\frac{1}{3}, \frac{2}{3}]$ when $d \in [0, \frac{1}{3}]$. The condition $d \geqslant 0$ is equivalent to $\lambda^2 - 2\eta\left(1-\sqrt{1-\lambda^2}\right) \geqslant 0$ or

$$\eta \leqslant \frac{\lambda^2}{2\left(1-\sqrt{1-\lambda^2}\right)} =: f(\lambda)$$

where $f(\lambda)$ runs from 1 to $\frac{1}{2}$ when λ goes from 0 to 1, i.e., the POVMs are jointly measurable when $\eta \in [0, f(\lambda)]$. In this case, one can easily calculate the joint POVM N.

Appendix B

In this appendix, we study joint measurability of the triplet $\big(\mathsf{N}^{\rm prono}, \mathsf{Q}_0^{\rm prono}, \mathsf{Q}_\theta^{\rm prono}\big)$ for any $\theta \notin \{0, \pi\}$. Using a result of [20, Appendix B] we get a joint normalised operator valued measure (OVM) G whose effects are

$$\mathsf{G}\big(\{0\}\times X\times Y\big)=\int_Y\int_X\frac{\mathrm{d}x\mathrm{d}y}{\pi}e^{-x^2-y^2}\times$$
$$\times\begin{pmatrix}1-\epsilon/2 & (1-\epsilon/2)\big[(1-\epsilon_0)\sqrt{2}x+(1-\epsilon_\theta)\sqrt{2}ye^{-i\theta}\big]\\ (1-\epsilon/2)\big[(1-\epsilon_0)\sqrt{2}x+(1-\epsilon_\theta)\sqrt{2}ye^{i\theta}\big] & \epsilon/2+(1-\epsilon/2)\big[(1-\epsilon_0)2x^2+(1-\epsilon_\theta)2y^2+(1-\epsilon_0)(1-\epsilon_\theta)4xy\cos\theta+\epsilon_0+\epsilon_\theta-2\big]\end{pmatrix},$$
$$\mathsf{G}\big(\{1\}\times X\times Y\big)=\int_Y\int_X\frac{\mathrm{d}x\mathrm{d}y}{\pi}e^{-x^2-y^2}\times$$
$$\times\begin{pmatrix}\epsilon/2 & \epsilon/2\big[(1-\epsilon_0)\sqrt{2}x+(1-\epsilon_\theta)\sqrt{2}ye^{-i\theta}\big]\\ \epsilon/2\big[(1-\epsilon_0)\sqrt{2}x+(1-\epsilon_\theta)\sqrt{2}ye^{i\theta}\big] & 1-\epsilon/2+\epsilon/2\big[(1-\epsilon_0)2x^2+(1-\epsilon_\theta)2y^2+(1-\epsilon_0)(1-\epsilon_\theta)4xy\cos\theta+\epsilon_0+\epsilon_\theta-2\big]\end{pmatrix}.$$

Since $\int_{\mathbb{R}}xe^{-x^2}\mathrm{d}x=0$ and $\int_{\mathbb{R}}2x^2e^{-x^2}\mathrm{d}x/\sqrt{\pi}=1$ one gets a joint OVM for $\mathsf{Q}_0^{\rm prono}$ and $\mathsf{Q}_\theta^{\rm prono}$:

$$\mathsf{G}_1(X\times Y):=\mathsf{G}\big(\{0,1\}\times X\times Y\big)=\mathsf{G}\big(\{0\}\times X\times Y\big)+\mathsf{G}\big(\{1\}\times X\times Y\big)=\int_Y\int_X\frac{\mathrm{d}x\mathrm{d}y}{\pi}e^{-x^2-y^2}\times$$
$$\times\begin{pmatrix}1 & (1-\epsilon_0)\sqrt{2}x+(1-\epsilon_\theta)\sqrt{2}ye^{-i\theta}\\ (1-\epsilon_0)\sqrt{2}x+(1-\epsilon_\theta)\sqrt{2}ye^{i\theta} & (1-\epsilon_0)2x^2+(1-\epsilon_\theta)2y^2+(1-\epsilon_0)(1-\epsilon_\theta)4xy\cos\theta+\epsilon_0+\epsilon_\theta-1\end{pmatrix},$$
$$\mathsf{G}_1(X\times\mathbb{R})=\mathsf{G}\big(\{0,1\}\times X\times\mathbb{R}\big)=\int_X\frac{\mathrm{d}x}{\sqrt{\pi}}e^{-x^2}\begin{pmatrix}1 & (1-\epsilon_0)\sqrt{2}x\\ (1-\epsilon_0)\sqrt{2}x & (1-\epsilon_0)2x^2+\epsilon_0\end{pmatrix}=\mathsf{Q}_0^{\rm prono}(X),$$
$$\mathsf{G}_1(X\times\mathbb{R})=\mathsf{G}\big(\{0,1\}\times X\times\mathbb{R}\big)=\mathsf{Q}_\theta^{\rm prono}(Y).$$

Similarly, a joint OVM of $\mathsf{N}^{\rm prono}$ and $\mathsf{Q}_\theta^{\rm prono}$

$$\mathsf{G}_2\big(\{0\}\times Y\big):=\mathsf{G}\big(\{0\}\times\mathbb{R}\times Y\big)=\int_Y\frac{\mathrm{d}y}{\sqrt{\pi}}e^{-y^2}\begin{pmatrix}1-\epsilon/2 & (1-\epsilon/2)(1-\epsilon_\theta)\sqrt{2}ye^{-i\theta}\\ (1-\epsilon/2)(1-\epsilon_\theta)\sqrt{2}ye^{i\theta} & \epsilon/2+(1-\epsilon/2)\big[(1-\epsilon_\theta)2y^2+\epsilon_\theta-1\big]\end{pmatrix},$$
$$\mathsf{G}_2\big(\{1\}\times Y\big):=\mathsf{G}\big(\{1\}\times\mathbb{R}\times Y\big)=\int_Y\frac{\mathrm{d}y}{\sqrt{\pi}}e^{-y^2}\begin{pmatrix}\epsilon/2 & \epsilon/2(1-\epsilon_\theta)\sqrt{2}ye^{-i\theta}\\ \epsilon/2(1-\epsilon_\theta)\sqrt{2}ye^{i\theta} & (1-\epsilon/2)+\epsilon/2\big[(1-\epsilon_\theta)2y^2+\epsilon_\theta-1\big]\end{pmatrix},$$
$$\mathsf{G}_2\big(\{0\}\times\mathbb{R}\big)=\mathsf{G}\big(\{0\}\times\mathbb{R}\times\mathbb{R}\big)=\begin{pmatrix}1-\epsilon/2 & 0\\ 0 & \epsilon/2\end{pmatrix}=\mathsf{N}_\theta^{\rm prono},\qquad \mathsf{G}_2\big(\{1\}\times\mathbb{R}\big)=\mathsf{G}\big(\{1\}\times\mathbb{R}\times\mathbb{R}\big)=\mathsf{N}_1^{\rm prono}$$

Finally, for $\mathsf{N}^{\rm prono}$ and $\mathsf{Q}_0^{\rm prono}$ we get

$$\mathsf{G}_3\big(\{0\}\times X\big):=\mathsf{G}\big(\{0\}\times X\times\mathbb{R}\big)=\int_X\frac{\mathrm{d}x}{\sqrt{\pi}}e^{-x^2}\begin{pmatrix}1-\epsilon/2 & (1-\epsilon/2)(1-\epsilon_0)\sqrt{2}x\\ (1-\epsilon/2)(1-\epsilon_0)\sqrt{2}x & \epsilon/2+(1-\epsilon/2)\big[(1-\epsilon_0)2x^2+\epsilon_0-1\big]\end{pmatrix},$$
$$\mathsf{G}_3\big(\{1\}\times X\big):=\mathsf{G}\big(\{1\}\times X\times\mathbb{R}\big)=\int_X\frac{\mathrm{d}x}{\sqrt{\pi}}e^{-x^2}\begin{pmatrix}\epsilon/2 & \epsilon/2(1-\epsilon_0)\sqrt{2}x\\ \epsilon/2(1-\epsilon_0)\sqrt{2}x & (1-\epsilon/2)+\epsilon/2\big[(1-\epsilon_0)2x^2+\epsilon_0-1\big]\end{pmatrix}.$$

When OVMs G, G_1, G_2, and G_3 are positive, i.e., POVMs? Since these OVMs contain a matrix inside an integral, the matrix must be positive semidefinite for all values of x and y. But the left upper corners of the matrices are always non-negative so it is enough to check that the determinants are non-negative:

For $\mathsf{G}\big(\{0\}\times X\times Y\big)$ and $\mathsf{G}\big(\{1\}\times X\times Y\big)$ the determinants are

$$\underbrace{\left(1-\frac{\epsilon}{2}\right)^2}_{>0}\left[\frac{\epsilon}{2-\epsilon}+\epsilon_0+\epsilon_\theta-2+\underbrace{\epsilon_0(1-\epsilon_0)2x^2+\epsilon_\theta(1-\epsilon_\theta)2y^2}_{\geqslant 0}\right]$$

and

$$\underbrace{\left(\frac{\epsilon}{2}\right)^2}_{\geqslant 0}\left[\underbrace{\frac{2-\epsilon}{\epsilon}}_{\geqslant\frac{\epsilon}{2-\epsilon}}+\epsilon_0+\epsilon_\theta-2+\underbrace{\epsilon_0(1-\epsilon_0)2x^2+\epsilon_\theta(1-\epsilon_\theta)2y^2}_{\geqslant 0}\right]$$

which are non-negative (i.e., G is a POVM) exactly when

$$\boxed{\frac{\epsilon}{2-\epsilon}+\epsilon_0+\epsilon_\theta-2\geqslant 0.}$$

The determinant related to G_1 is

$$\epsilon_0+\epsilon_\theta-1+(1-\epsilon_0)\epsilon_0 2x^2+(1-\epsilon_\theta)\epsilon_\theta 2y^2\geqslant\epsilon_0+\epsilon_\theta-1$$

so that G_1 is positive exactly when

$$\boxed{\epsilon_0+\epsilon_\theta-1\geqslant 0.}$$

Since $\epsilon_0+\epsilon_\theta-1\geqslant\frac{\epsilon}{2-\epsilon}+\epsilon_0+\epsilon_\theta-2$,[7] the positivity of G implies the positivity of G_1 as expected. If $\epsilon<1$ and $\epsilon_0=\epsilon_\theta=\frac{1}{2}$ then G_1 is positive but G is not.

For G_2 we get (from determinants) the positivity conditions

$$\underbrace{\frac{2}{2-\epsilon}}_{\leqslant 2/\epsilon}+\epsilon_\theta-2+\underbrace{(1-\epsilon_\theta)\epsilon_\theta 2y^2}_{\geqslant 0}\geqslant 0,\qquad \frac{2}{\epsilon}+\epsilon_\theta-2+(1-\epsilon_\theta)\epsilon_\theta 2y^2\geqslant 0$$

showing that G_2 is a POVM if and only if

$$\boxed{\frac{2}{2-\epsilon}+\epsilon_\theta-2\geqslant 0.}$$

Note that, since $\frac{2}{2-\epsilon}+\epsilon_\theta-2\geqslant\frac{\epsilon}{2-\epsilon}+\epsilon_0+\epsilon_\theta-2$,[8] clearly G_2 is positive if G is positive but the converse does not hold (e.g., if $\epsilon=\frac{1}{2}$, $\epsilon_\theta=\frac{2}{3}$, and $\epsilon_0<1$). Similarly, G_3 is a POVM exactly when

$$\boxed{\frac{2}{2-\epsilon}+\epsilon_0-2\geqslant 0.}$$

Example 3. Suppose that $\epsilon=\epsilon_0=\epsilon_\theta$. Now $\mathsf{N}^{\rm prono}$, $\mathsf{Q}_0^{\rm prono}$, and $\mathsf{Q}_\theta^{\rm prono}$ are jointy measurable (G) if $-2\epsilon^2+7\epsilon-4\geqslant 0$, i.e., when

$$\boxed{\epsilon\geqslant\frac{1}{4}\left(7-\sqrt{17}\right)\approx 0.72.}$$

---

[7]Where the equality holds iff $\epsilon=1$ (i.e., the effects $\mathsf{N}_{0,1}^{\rm prono}=\frac{1}{2}\mathbb{1}$ which commute with everything).

[8]Where the equality holds iff $\epsilon_0=1$ (i.e., $\mathsf{Q}_0^{\rm prono}$ is a trivial POVM which commutes with everything).

In addition, $\mathsf{Q}_0^{\rm prono}$ and $\mathsf{Q}_\theta^{\rm prono}$ are compatible (G_1) if

$$\epsilon \geqslant \frac{1}{2},$$

and $\mathsf{N}^{\rm prono}$ and $\mathsf{Q}_\theta^{\rm prono}$ (or $\mathsf{Q}_0^{\rm prono}$) are jointly measurable if $2 \geqslant (2-\epsilon)^2$, i.e., when

$$\epsilon \geqslant 2-\sqrt{2} \approx 0.59.$$

□

Note that G (based on [20]) is not the best possible joint measurement since, e.g., its marginal G_1 gives a joint measurement for $\mathsf{Q}_0^{\rm prono}$ and $\mathsf{Q}_\theta^{\rm prono}$ for any θ only when $\epsilon_0+\epsilon_\theta \geqslant 1$ but we know that, if $\theta=0$ or $\theta=\pi$, then $\mathsf{Q}_0^{\rm prono}$ and $\mathsf{Q}_\theta^{\rm prono}$ are jointly measurable *for all* ϵ_0, $\epsilon_\theta \in [0,1]$. Let us try to find a better joint measurement (i.e., we will modify G).

Suppose for simplicity that $\epsilon=\epsilon_0=\epsilon_\theta$ like in the preceding example. Define an OVM via

$$\mathsf{G}'(\{0\}\times X\times Y) = \int_Y\int_X \frac{{\rm d}x{\rm d}y}{\pi}e^{-x^2-y^2}\begin{pmatrix} 1-\epsilon/2 & f(\epsilon)\big[\sqrt{2}x+\sqrt{2}ye^{-i\theta}\big] \\ f(\epsilon)\big[\sqrt{2}x+\sqrt{2}ye^{i\theta}\big] & f(\epsilon)\big[2x^2+2y^2+4xy\cos\theta\big]+g(\epsilon)\end{pmatrix},$$

$$\mathsf{G}'(\{1\}\times X\times Y) = \int_Y\int_X \frac{{\rm d}x{\rm d}y}{\pi}e^{-x^2-y^2}\begin{pmatrix} \epsilon/2 & h(\epsilon)\big[\sqrt{2}x+\sqrt{2}ye^{-i\theta}\big] \\ h(\epsilon)\big[\sqrt{2}x+\sqrt{2}ye^{i\theta}\big] & h(\epsilon)\big[2x^2+2y^2+4xy\cos\theta\big]+i(\epsilon)\end{pmatrix}$$

where f, g, h, and i are unknown real functions. Since

$$\mathsf{G}'(\{0,1\}\times X\times Y) = \int_Y\int_X \frac{{\rm d}x{\rm d}y}{\pi}e^{-x^2-y^2}\begin{pmatrix} 1 & [f(\epsilon)+h(\epsilon)]\big[\sqrt{2}x+\sqrt{2}ye^{-i\theta}\big] \\ [f(\epsilon)+h(\epsilon)]\big[\sqrt{2}x+\sqrt{2}ye^{i\theta}\big] & [f(\epsilon)+h(\epsilon)]\big[2x^2+2y^2+4xy\cos\theta\big]+g(\epsilon)+i(\epsilon)\end{pmatrix}$$

we must have

$$f(\epsilon)+h(\epsilon)=1-\epsilon, \qquad g(\epsilon)+i(\epsilon)=2\epsilon-1, \qquad 2f(\epsilon)+g(\epsilon)=\epsilon/2,$$

that is,

$$g(\epsilon)=\epsilon/2-2f(\epsilon), \qquad h(\epsilon)=1-\epsilon-f(\epsilon), \qquad i(\epsilon)=3\epsilon/2-1+2f(\epsilon)$$

where $f(\epsilon)$ is free. The determinants of the matrices inside $\mathsf{G}'\big(\{0\}\times X\times Y\big)$ and $\mathsf{G}'\big(\{1\}\times X\times Y\big)$ are

$$\big[(1-\epsilon/2)f(\epsilon)-f(\epsilon)^2\big]\big|\sqrt{2}x+\sqrt{2}ye^{-i\theta}\big|^2+(1-\epsilon/2)g(\epsilon),$$
$$\big[(\epsilon/2)h(\epsilon)-h(\epsilon)^2\big]\big|\sqrt{2}x+\sqrt{2}ye^{-i\theta}\big|^2+(\epsilon/2)i(\epsilon)$$

which must be non-negative for all x, $y\in\mathbb{R}$. Especially, by putting $x=y=0$ (or looking the lower right corners) we get the necessary conditions $g(\epsilon)\geqslant 0$ and $i(\epsilon)\geqslant 0$, that is, $\epsilon/4\geqslant f(\epsilon)\geqslant 1/2-3\epsilon/4$ which can hold if and only if $\epsilon/4\geqslant 1/2-3\epsilon/4$, i.e., $\epsilon\geqslant 1/2$.[9]

[9]Let us see what happens if we put $\epsilon=1/2$ implying

$$f(1/2)=1/8, \qquad g(1/2)=0, \qquad h(1/2)=3/8, \qquad i(1/2)=0,$$

The next conditions are *sufficient* for G' being positive:

$$\begin{aligned}
&\epsilon > 1/2, \qquad \epsilon/4 \geqslant f(\epsilon) \geqslant 1/2 - 3\epsilon/4,\\
&(1-\epsilon/2)f(\epsilon) - f(\epsilon)^2 \geqslant 0 \quad \Longleftrightarrow \quad 0 \leqslant f(\epsilon) \leqslant 1-\epsilon/2,\\
&(\epsilon/2)h(\epsilon) - h(\epsilon)^2 \geqslant 0 \quad \Longleftrightarrow \quad 0 \leqslant h(\epsilon) \leqslant \epsilon/2 \quad \Longleftrightarrow \quad 0 \leqslant 1-\epsilon-f(\epsilon) \leqslant \epsilon/2
\end{aligned}$$

which reduce to $\epsilon > 1/2$ and $\min\{\epsilon/4, 1-\epsilon/2, 1-\epsilon\} \geqslant f(\epsilon) \geqslant \max\{1/2-3\epsilon/4, 0, 1-3\epsilon/2\}$ or, equivalently,

$$\begin{cases}
1-3\epsilon/2 \leqslant f(\epsilon) \leqslant \epsilon/4, & 1/2 < \epsilon \leqslant 2/3,\\
0 \leqslant f(\epsilon) \leqslant \epsilon/4, & 2/3 < \epsilon \leqslant 4/5,\\
0 \leqslant f(\epsilon) \leqslant 1-\epsilon, & 4/5 < \epsilon \leqslant 1.
\end{cases}$$

To conclude, to find some $f(\epsilon)$ (and a joint POVM G') one must have $1-3\epsilon/2 \leqslant \epsilon/4$, i.e.,

$$\boxed{\epsilon \geqslant 4/7 \approx 0.57...}$$

so we have a better limit. Now $f(4/7) = 1/7$, $g(4/7) = 0$, $h(4/7) = 2/7$, $i(4/7) = 1/7$, and the joint POVM is

$$\mathsf{G}'(\{0\}\times X\times Y) = \int_Y\int_X \frac{\mathrm{d}x\mathrm{d}y}{\pi} e^{-x^2-y^2} \begin{pmatrix} 5/7 & (1/7)\big[\sqrt{2}x+\sqrt{2}ye^{-i\theta}\big] \\ (1/7)\big[\sqrt{2}x+\sqrt{2}ye^{i\theta}\big] & (1/7)\big[2x^2+2y^2+4xy\cos\theta\big] \end{pmatrix},$$

$$\mathsf{G}'(\{1\}\times X\times Y) = \int_Y\int_X \frac{\mathrm{d}x\mathrm{d}y}{\pi} e^{-x^2-y^2} \begin{pmatrix} 2/7 & (2/7)\big[\sqrt{2}x+\sqrt{2}ye^{-i\theta}\big] \\ (2/7)\big[\sqrt{2}x+\sqrt{2}ye^{i\theta}\big] & (2/7)\big[2x^2+2y^2+4xy\cos\theta\big]+1/7 \end{pmatrix}.$$

By using similar methods, we find a better limit $\epsilon = \frac{1}{2}$ and a joint POVM for the pair $\mathsf{N}^{\mathrm{prono}}$ and $\mathsf{Q}_\theta^{\mathrm{prono}}$:

$$\mathsf{G}_2'(\{0\}\times Y) = \int_Y \frac{\mathrm{d}y}{\sqrt{\pi}} e^{-y^2} \begin{pmatrix} 3/4 & (1/4)\sqrt{2}ye^{-i\theta} \\ (1/4)\sqrt{2}ye^{i\theta} & (1/4)2y^2 \end{pmatrix},$$

$$\mathsf{G}_2'(\{1\}\times Y) = \int_Y \frac{\mathrm{d}y}{\sqrt{\pi}} e^{-y^2} \begin{pmatrix} 1/4 & (1/4)\sqrt{2}ye^{-i\theta} \\ (1/4)\sqrt{2}ye^{i\theta} & (1/4)2y^2+1/2 \end{pmatrix}.$$

For $\mathsf{Q}_0^{\mathrm{prono}}$ and $\mathsf{Q}_\theta^{\mathrm{prono}}$ this method gives the same limit $\epsilon = 1/2$ as before with a joint POVM

$$\mathsf{G}_1'(X\times Y) = \int_Y\int_X \frac{\mathrm{d}x\mathrm{d}y}{\pi} e^{-x^2-y^2} \frac{1}{2} \begin{pmatrix} 2 & \sqrt{2}x+\sqrt{2}ye^{-i\theta} \\ \sqrt{2}x+\sqrt{2}ye^{i\theta} & 2x^2+2y^2+4xy\cos\theta \end{pmatrix}.$$

Finally, we study compatibility of the projected energy (or number) and the canonical phase.

---

so the second determinant reduce to $-(3/64)\big|\sqrt{2}x+\sqrt{2}ye^{-i\theta}\big|^2$ which is negative if, e.g., $(x,y) = (1,0)$.

Let $\epsilon \in [0,1]$ and denote $\epsilon^\perp := 1-\epsilon$. Define the noisy projected canonical phase [1]: for all Borel $X \subseteq [0,2\pi)$,

$$\Phi(X) := \epsilon^\perp \underbrace{\int_X \begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix} \frac{\mathrm{d}\theta}{2\pi}}_{\text{the canonical phase}} + \epsilon \underbrace{\int_X \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\mathrm{d}\theta}{2\pi}}_{\text{the random noise}} = \int_X \begin{pmatrix} 1 & \epsilon^\perp e^{-i\theta} \\ \epsilon^\perp e^{i\theta} & 1 \end{pmatrix} \frac{\mathrm{d}\theta}{2\pi} \geqslant 0$$

for which $\Phi\big([0,2\pi)\big) = \mathbb{1}$. Let c and d be complex numbers in the unit disc, i.e., $|c| \leqslant 1$ and $|d| \leqslant 1$. Define a POVM via

$$\mathsf{M}\big(\{0\}\times X\big) := \int_X \begin{pmatrix} 1-\epsilon/2 & c\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}e^{-i\theta} \\ \overline{c}\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}e^{i\theta} & \epsilon/2 \end{pmatrix} \frac{\mathrm{d}\theta}{2\pi} \geqslant 0,$$

$$\mathsf{M}\big(\{1\}\times X\big) := \int_X \begin{pmatrix} \epsilon/2 & d\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}e^{-i\theta} \\ \overline{d}\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}e^{i\theta} & 1-\epsilon/2 \end{pmatrix} \frac{\mathrm{d}\theta}{2\pi} \geqslant 0.$$

[For positivity note that, e.g., the determinant of the above matrix is $(\epsilon/2)(1-\epsilon/2)(1-|d^2|) \geqslant 0$.] Now $\mathsf{M}\big(\{0\}\times[0,2\pi)\big) = \mathsf{N}^{\text{prono}}_0$ and $\mathsf{M}\big(\{1\}\times[0,2\pi)\big) = \mathsf{N}^{\text{prono}}_1$. To get a joint measurement of $\mathsf{N}^{\text{prono}}$ and Φ we must have also $\mathsf{M}\big(\{0\}\times X\big) + \mathsf{M}\big(\{1\}\times X\big) = \Phi(X)$ for all X, i.e., for all θ,

$$\begin{pmatrix} 1-\epsilon/2 & c\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}e^{-i\theta} \\ \overline{c}\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}e^{i\theta} & \epsilon/2 \end{pmatrix} + \begin{pmatrix} \epsilon/2 & d\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}e^{-i\theta} \\ \overline{d}\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}e^{i\theta} & 1-\epsilon/2 \end{pmatrix} = \begin{pmatrix} 1 & \epsilon^\perp e^{-i\theta} \\ \epsilon^\perp e^{i\theta} & 1 \end{pmatrix},$$

i.e., $(c+d)\sqrt{\epsilon/2}\sqrt{1-\epsilon/2} = \epsilon^\perp = 1-\epsilon$ or

$$c+d = f(\epsilon) := \frac{1-\epsilon}{\sqrt{\epsilon/2}\sqrt{1-\epsilon/2}}.$$

Since $|c+d| \leqslant |c|+|d| \leqslant 2$, one has to have $f(\epsilon) \leqslant 2$ [i.e., then c and d and M exist; one can choose, e.g., $c = d = f(\epsilon)/2$]. So to get the smallest possible ϵ (see the next figure) we must solve equation $f(\epsilon) = 2$. The solution is $\epsilon = \epsilon_{\min} := 1 - 1/\sqrt{2} \approx 0.292893$.

To conclude, if $\epsilon \geqslant \epsilon_{\min}$ then $\mathsf{N}^{\text{prono}}$ and Φ are jointly measurable.

Email address: juhpello@utu.fi

Turku Centre for Quantum Physics, Department of Physics and Astronomy, University of Turku, FI-20014 Turku, Finland

Email address: sebastien.designolle@unige.ch

Group of Applied Physics, University of Geneva, 1211 Geneva, Switzerland

Email address: roope.uola@unige.ch

University of Geneva, 1211 Geneva, Switzerland

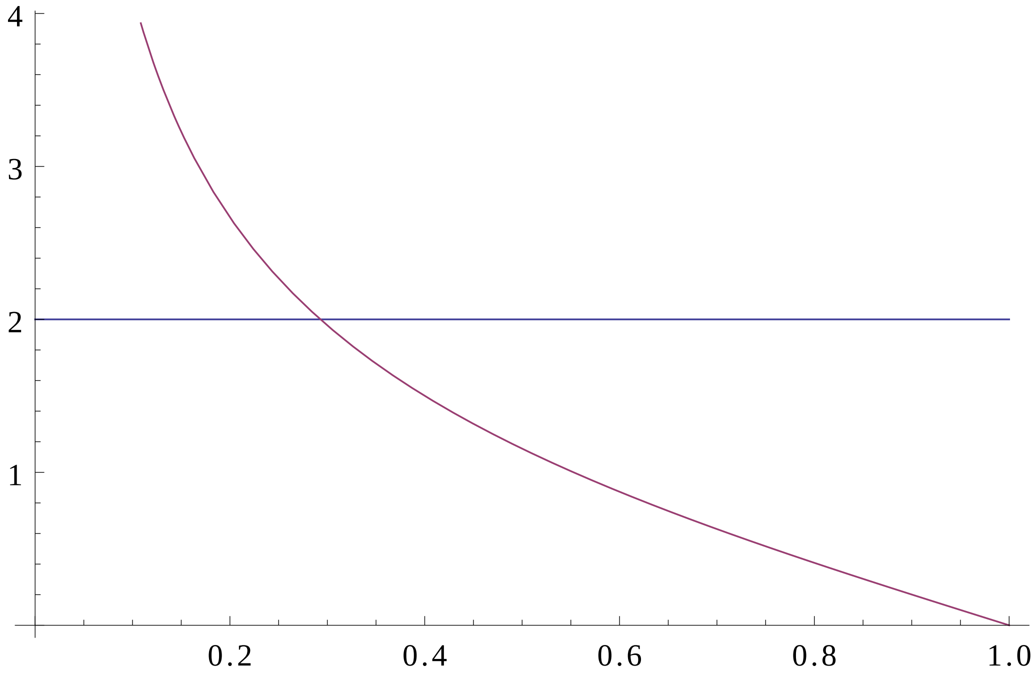

FIGURE 1. The function $\epsilon \mapsto f(\epsilon)$ [the red curve].