TURUN YLIOPISTO
UNIVERSITY OF TURKU

Algebraic and Structural Aspects on Multidimensional Symbolic Dynamics and Delone Sets

Forced Periodicity and Local Complexity

Pyry Herva

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SARJA – SER. AI OSA – TOM. 747 | ASTRONOMICA – CHEMICA – PHYSICA – MATHEMATICA | TURKU 2025

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ABSTRACT

Classically, multidimensional symbolic dynamics studies colorings of the integer grid of different dimensions with finitely many colors. In this thesis, a configuration is a coloring of either the d-dimensional integer grid or the d-dimensional Euclidean space. A configuration is periodic if it is equal to some of its translation by a non-zero vector. Delone sets are certain subsets of Euclidean spaces, and they form mathematical models for crystals and quasicrystals. In particular, Delone sets are identified with certain configurations on the Euclidean space with only two colors.

This thesis focuses on algebraic and structural aspects on multidimensional symbolic dynamics and Delone sets. In particular, the connection between forced periodicity and local complexity is studied. In our considerations, it is usually assumed that the alphabets of the configurations have some algebraic structure. A polynomial is an annihilator of a configuration if their discrete convolution is the zero configuration. We consider configurations that have non-trivial annihilators. In particular, periodic configurations have non-trivial annihilators – they are annihilated by a difference polynomial. It is known that if a configuration on the integer grid with integer coefficients has a non-trivial annihilator, then it is annihilated by a product of finitely many difference polynomials. Consequently, the periodic decomposition theorem states that such a configuration is a sum of finitely many periodic functions. However, these functions may not be configurations, that is, they may get infinitely many distinct values.

In this thesis, a certain family of configurations with non-trivial annihilators motivated by coding and graph theory is studied. We give new proofs for some known results on their forced periodicity. Also, some new results are proved. In addition, improvements of the periodic decomposition theorem are proved under some more involved assumptions. Finally, configurations on the Euclidean space and in particular Delone sets are considered. Known concepts and results are generalized to this setting, and some differences between these two settings are emphasized.

KEYWORDS: multidimensional symbolic dynamics, Delone sets, periodicity, local comlexity, annihilator, periodizer

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TIIVISTELMÄ

Moniulotteisessa symbolidynamiikassa tutkitaan perinteisesti kokonaislukuhilojen värityksiä eri ulottuvuuksissa käyttäen äärellistä määrää eri värejä. Tässä väitöskirjassa konfiguraatioilla tarkoitetaan joko d-ulotteisen kokonaislukuhilan tai Euklidisen avaruuden värityksiä. Konfiguraation sanotaan olevan jaksollinen, jos se saadaan itsestään jollain nollasta erovalla vektorilla siirtämällä. Delone-joukot ovat tietynlaisia Euklidisen avaruuden osajoukkoja. Ne muodostavat matemaattisia malleja kiteille ja kvasikiteille. Delone-joukot samaistetaan tietynlaisten Euklidisen avaruuden konfiguraatioiden kanssa.

Väitöskirjassa keskitytään moniulotteisen symbolidynamiikan ja Delone-joukkojen algebrallisiin ja rakenteellisiin ominaisuuksiin. Erityisesti keskitytään pakotetun jaksollisuuden ja matalan kuviokompleksisuuden väliseen yhteyteen. Tässä työssä värien oletetaan tavallisesti omaavan joitain algebrallisia ominaisuuksia. Polynomin sanotaan olevan konfiguraation annihilaattori, jos näiden diskreetti konvoluutio on nollakonfiguraatio. Väitöskirjassa keskitytään konfiguraatioihin, joilla on annihilaattoreita. Erityisesti jaksollisilla konfiguraatioilla on annihilaattoreita – niillä on erotuspolynomi annihilaattorina. Tiedetään, että jos kokonaislukukertoimisella kokonaislukuhilan konfiguraatiolla on annihilaattori, niin silloin sillä on annihilaattori, joka on äärellisen monen erotuspolynomin tulo. Tästä seuraa niin sanottu jaksollinen hajotelma -teoreema, jonka mukaan tällainen konfiguraatio on äärellisen monen jaksollisen kokonaislukuhilan värityksen summa. Näissä värityksissä saatetaan kuitenkin käyttää äärettömän montaa eri väriä.

Tässä väitöskirjassa tutkitaan erästä joukkoa konfiguraatioita, joilla on annihilaattoreita. Motivaatio näiden konfiguraatioiden tutkimiseen tulee koodaus- ja graafiteoriasta. Uusia todistuksia tunnetuille tuloksille ja joitain uusia tuloksia annetaan. Lisäksi tutkitaan jaksollinen hajotelma -teoreemaa ja parannetaan sitä tietyin oletuksin. Lopuksi käsitellään Euklidisen avaruuden konfiguraatioita ja Delone-joukkoja. Tunnettuja käsitteitä ja tuloksia yleistetään toimimaan myös tässä Delone-joukkojen tapauksessa.

ASIASANAT: moniulotteinen symbolidynamiikka, Delone-joukot, jaksollisuus, paikallinen kuviokompleksisuus, annihilaattori, jaksollistaja

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List of Original Publications

This thesis is based on the following original publications.

1. Elias Heikkilä, **Pyry Herva**, and Jarkko Kari. On Perfect Coverings of Two-Dimensional Grids. In: Developments in Language Theory. DLT 2022. Lecture Notes in Computer Science, vol 13257, pages 152-163.

2. **Pyry Herva** and Jarkko Kari. On Forced Periodicity of Perfect Colorings. Theory of Computing Systems, 2023; Volume 67, pages 732–759.

3. **Pyry Herva** and Jarkko Kari. On the Periodic Decompositions of Multidimensional Configurations. In: SOFSEM 2025: Theory and Practice of Computer Science. Lecture Notes in Computer Science, vol 15539, pages 45-57.

4. **Pyry Herva** and Jarkko Kari. Periodicity and Local Complexity of Delone Sets. Submitted for publication; preprint at https://arxiv.org/abs/2504.20709.

1 Introduction

1.1 Symbolic dynamics

In general, a dynamical system is a pair consisting of a compact metric space and a continuous monoid action, the dynamics. A symbolic dynamical system is a dynamical system where the space consists of functions $M \to \mathcal{A}$ from a monoid M to a non-empty finite set $\mathcal{A}$ — the alphabet. These functions are called M-configurations or just configurations. The set of all M-configurations is denoted by $\mathcal{A}^M$. The dynamics is given by shifts or translations by the elements of M. Often, the alphabet $\mathcal{A}$ is considered as a color set and a configuration as a coloring of the monoid M with colors of the set $\mathcal{A}$. Symbolic dynamics is a branch of the study of discrete dynamical systems. It studies configurations and subsets of configurations that are topologically closed and invariant under shifts — this kind of subsets of configurations are called subshifts.

Let d be a positive integer. In this thesis we study symbolic dynamical systems with $M = \mathbb{Z}^d$, the d-dimensional integer grid. We call $\mathbb{Z}^d$-configurations simply d-dimensional configurations. We call the set $\mathcal{A}^{\mathbb{Z}^d}$ the d-dimensional configuration space over the alphabet $\mathcal{A}$. Sometimes, 1-dimensional configurations $c \in \mathcal{A}^{\mathbb{Z}}$ are also called bi-infinite words. A translation $\tau^{\mathbf{t}}(c)$ of a d-dimensional configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ by a vector $\mathbf{t} \in \mathbb{Z}^d$ is the configuration whose value at $\mathbf{u} \in \mathbb{Z}^d$ is $c(\mathbf{u} - \mathbf{t})$, the value of c at $\mathbf{u} - \mathbf{t}$. Our dynamics consists of all the translations. We say that $c \in \mathcal{A}^{\mathbb{Z}^d}$ is periodic with period vector $\mathbf{v}$ if $c(\mathbf{u} + \mathbf{v}) = c(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{Z}^d$. In other words, c has period $\mathbf{v}$ if the translation $\tau^{\mathbf{t}}(c)$ of c by $\mathbf{v}$ equals c. We say that $c \in \mathcal{A}^{\mathbb{Z}^d}$ is strongly periodic if it has d linearly independent period vectors.

We are interested in setups of forced periodicity. These are settings where some assumptions imply that a configuration satisfying them is necessarily periodic. For example, sufficient local restrictions on the configuration are known to force periodicity. Such restrictions are usually given in terms of the number of different patterns of some finite shape. To be more precise, let us define finite patterns. For a non-empty finite set $D \subseteq \mathbb{Z}^d$, a D-pattern of a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ is a function $p \in \mathcal{A}^D$ such that $p = \tau^{\mathbf{t}}(c) \restriction_D$, that is, $p(\mathbf{d}) = c(\mathbf{d} + \mathbf{t})$ for all $\mathbf{d} \in D$ for some $\mathbf{t} \in \mathbb{Z}^d$. The pattern complexity $P_c(D)$ of c with respect to shape D gives the number of distinct D-patterns of c. For $n, n_1, \ldots, n_d \geq 1$, let us use the shorthand notations $P_c(n_1, \ldots, n_d) = P_c(\{0, \ldots, n_1 - 1\} \times \ldots \times \{0, \ldots, n_d - 1\})$ and $P_c(n) = P_c(n, \ldots, n) = P_c(\{0, \ldots, n - 1\}^d)$.

Nivat's conjecture

In their seminal paper [70] on symbolic dynamics in 1938, Morse and Hedlund proved the following statement of forced periodicity of 1-dimensional configurations. It states that a 1-dimensional configuration is periodic if and only if it contains at most n distinct subwords of length n for some n.

Theorem (Morse-Hedlund theorem [70]). *A one-dimensional configuration $c \in \mathcal{A}^{\mathbb{Z}}$ is periodic if and only if*

$$P_c(n) \leq n$$

for some $n \geq 1$.

More generally, it is quite easily seen that the Morse-Hedlund theorem holds for any shape. In other words, if $P_c(D) \leq |D|$ for any shape $D \subseteq \mathbb{Z}$, then c is periodic. The threshold

$$P_c(D) \leq |D|$$

seems to be relevant in many occasions. Thus, we say that a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ in any dimension d has low complexity with respect to shape D if it satisfies the above inequality, that is, if the number of D-patterns of c is at most $|D|$, the size of D.

The 2-dimensional generalization of the Morse-Hedlund theorem is still an open problem. It states that if a 2-dimensional configuration has low complexity with respect to a rectangle, then it is periodic. It was stated by Nivat at ICALP 1997.

Conjecture (Nivat's conjecture 1997 [73]). *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional configuration and let $m, n \geq 1$. If*

$$P_c(m, n) \leq mn,$$

then c is periodic.

It is also conjectured that Nivat's conjecture holds more generally for convex shapes [13; 49]. In other words, it is conjectured that if a 2-dimensional configuration has low complexity with respect to a convex shape, then it is periodic.

As mentioned, despite of all the interest Nivat's conjecture has received, it still remains an open problem. However, many partial results of the conjecture have been proved. For example, there are proofs for different $\alpha < 1$ saying that if the number of different $m \times n$ rectangular patterns of a 2-dimensional configuration c is at most αmn, then c is periodic. In [23] this was shown for $\alpha = 1/144$ by Epifanio, Koskas, and Mignosi. In [77] the result was improved for $\alpha = 1/16$ by Quas and Zamboni. The best such α so far is $\alpha = 1/2$ established by Cyr and Kra [18]. In [82] Sander and Tijdeman proved that if $P_c(2, m) \leq 2m$ for some $m \geq 1$, then $c \in \mathcal{A}^{\mathbb{Z}^2}$ is periodic. Similarly, in [19] Cyr and Kra proved that if $P_c(3, m) \leq 3m$ for some $m \geq 1$, then c is periodic.

In [55] Kari and Szabados introduced an algebraic approach to make progress in proving Nivat's conjecture. They managed to prove an asymptotic version of the conjecture, that is, they proved that for a non-periodic 2-dimensional configuration c there exist only finitely many numbers m, n such that c has at most mn different $m \times n$-patterns. Kari and Moutot proved using the algebraic approach that if a 2-dimensional configuration c has low complexity with respect to a convex shape, then its orbit closure (*i.e.*, the smallest subshift that contains c) contains a periodic configuration [52; 53]. We shall talk about this algebraic approach more thoroughly a bit later in this introduction.

Nivat's conjecture is not an equivalence since there exist periodic 2-dimensional configurations $c \in \mathcal{A}^{\mathbb{Z}^2}$ that have $P_c(m,n) > mn$ for all $m, n \geq 1$ as the construction by Cassaigne in [13] shows. Note that if Nivat's conjecture is true, then it is tight since there exist non-periodic 2-dimensional configurations $c \in \mathcal{A}^{\mathbb{Z}^2}$ with $P_c(m,n) = mn + 1$ for all $m, n \geq 1$. These were classified in [12] by Cassaigne. Note also that the d-dimensional generalization of Nivat's conjecture does not hold, that is, the condition $P_c(n_1, \ldots, n_d) \leq n_1 \cdots n_d$ does not imply periodicity of $c \in \mathcal{A}^{\mathbb{Z}^d}$ for $d \geq 3$. A 3-dimensional counter-example is provided in [81] by Sander and Tijdeman.

An interesting related question about forced periodicity concerns the relation between the growth rate of $P_c(n) = P_c(n, \ldots, n)$ and the periodicity of c. For example, if $c \in \mathcal{A}^{\mathbb{Z}^2}$ and $P_c(n) = o(n^2)$, then c is necessarily periodic. Indeed, the condition $P_c(n) = o(n^2)$ implies that $P_c(n) = P_c(n,n) \leq n^2/2$ for some n and hence c is periodic. This follows from the result by Cyr and Kra saying that if $P_c(m,n) \leq mn/2$ for some m, n then c is periodic.

Periodic tiling problem

In the following, a tile is any non-empty finite set $D \subseteq \mathbb{Z}^d$. A translational tiling by a tile D or a co-tiler of D is a set $C \subseteq \mathbb{Z}^d$ such that

$$D \oplus C = \mathbb{Z}^d.$$

In other words, every point $\mathbf{u} \in \mathbb{Z}^d$ has a unique expression as a sum $\mathbf{u} = \mathbf{d} + \mathbf{c}$ where $\mathbf{d} \in D$ and $\mathbf{c} \in C$. Visually speaking, a co-tiler C of the tile D gives the positions where the copies of D are placed such that every point of the grid $\mathbb{Z}^d$ belongs to exactly one copy of D.

Typically, we identify translational tilings with their indicator functions. So, we identify $C \subseteq \mathbb{Z}^d$ with the binary configuration $c \in \{0,1\}^{\mathbb{Z}^d}$ defined such that $c(\mathbf{u}) = 1$ if and only if $\mathbf{u} \in C$. Now, a binary configuration is a co-tiler of D if and only if every $(-D)$-pattern of c contains exactly one symbol 1. In fact, Szegedy showed in [87] that any co-tiler of D is also a co-tiler of $-D$ and hence also any

D-pattern of a co-tiler of D contains a single symbol 1. Thus, the set of all co-tilers of D is obtained by forbidding a finite set of D-patterns. This kind of set of configurations is called a subshift of finite type (SFT). Note that any co-tiler c of D has $P_c(D) = |D|$. In particular, c has low complexity with respect to D.

If a tile $D \subseteq \mathbb{Z}^d$ has a co-tiler, we say that it tiles the grid $\mathbb{Z}^d$. We say that it tiles the grid periodically if it has a strongly periodic co-tiler. The periodic tiling problem or the periodic tiling conjecture claims that any tile that tiles the grid, also tiles it periodically. It was explicitly stated by Lagarias and Y. Wang in 1996 in [62], more generally, for translational tilings of $\mathbb{R}^d$. However, the conjecture has also been previously asked, at least implicitly; see *e.g.* [36; 85].

Conjecture (Periodic tiling problem [62])**.** *If a tile D has a co-tiler, then it also has a strongly periodic co-tiler.*

Newman showed already in 1977 using a simple pigeonholing argument that any translational tiling in dimension $d = 1$ is necessarily periodic [72] and hence the periodic tiling problem is true for $d = 1$. For $d \geq 2$, the conjecture is much trickier. Only very recently, Bhattacharya proved in a paper [8] published in 2020 that the periodic tiling problem holds also for $d = 2$. In [32] Greenfeld and Tao proved a quantative version of Bhattacharyas result. Moreover, they obtained some structural results on translational tilings in any dimension. In [35] they showed that the periodic tiling problem fails for sufficiently large d. For recent research around the periodic tiling problem, see *e.g.* [30; 33; 34; 67; 68; 31; 20].

A related problem is the Golomb-Welch conjecture [29; 46]. It concerns translational tilings by the Lee spheres

$$B_r^d = \{(u_1, \ldots, u_d) \in \mathbb{Z}^d \mid \sum_{i=1}^{d} |u_i| \leq r\}$$

where d is the dimension of the space and r is the radius in consideration. If $d \leq 2$ or $r = 1$, there exist strongly periodic translational tilings by B_r^d [29]. The strong Golomb-Welch conjecture claims that for other values of d and r, there are no translational tilings by B_r^d. The weak Golomb-Welch conjecture claims that no strongly periodic tilings by B_r^d exist. The conjectures remain open for $d \geq 6$ and for small radii. If the periodic tiling problem was true, then the strong and weak Golomb-Welch conjectures would be equivalent. This is the case in dimensions $d = 1$ and $d = 2$ where the periodic tiling problem is known to be true.

Perfect colorings and abelian complexity

Any co-tiler $c \in \{0,1\}^{\mathbb{Z}^d}$ of a tile D has the property that any two $(-D)$-patterns of c (and any two D-patterns as noted above) contain the same number of symbols 0 and the same number of symbols 1. Patterns of this type, that is, patterns with the same shape that contain the same number of each symbol are called abelian equivalent. The abelian complexity $A_c(D)$ of a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ with respect to shape D is the number of different D-patterns of c up to abelian equivalence. Clearly, $A_c(D) \leq P_c(D)$. Abelian complexity of 1-dimensional configurations is a widely studied concept in combinatorics on words [65]. See also [24] for a recent survey on abelian complexity in combinatorics on words.

So, any co-tiler c of a tile D has $A_c(D) = 1$. An interesting question is the following. When does the condition $A_c(D) = 1$ imply periodicity of c? As one could expect, the answer to the question depends on the dimension. For $d = 1$, the condition $A_c(D) = 1$ implies always periodicity of c. For $d \geq 2$, this is not the case anymore. Indeed, in [76] it was shown that there exist non-periodic two-dimensional configurations that have abelian complexity $A_c(D) = 1$ for some shape D.

A perfect coloring with respect to a non-empty finite set $D \subseteq \mathbb{Z}^d$ of size at least 2 or a D-perfect coloring is a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ such that any two patterns $c\restriction_{\mathbf{u}+D}$ and $c\restriction_{\mathbf{v}+D}$ of shape D of c are abelian equivalent whenever $c(\mathbf{u}) = c(\mathbf{v})$. In other words, the number of different colors in $c\restriction_{\mathbf{u}+D}$ is determined by the color $c(\mathbf{u})$. Perfect colorings are usually defined for graphs. They form a special case of so-called equitable partitions [27; 28].

In the case of only two colors we may rename the colors and assume that $\mathcal{A} = \{0, 1\}$. Then we identify any D-perfect coloring $c \in \{0, 1\}^{\mathbb{Z}^d}$ with the subset $C \subseteq \mathbb{Z}^d$ which contains all the points $\mathbf{u} \in \mathbb{Z}^d$ with color 1, that is, $c(\mathbf{u}) = 1$. This set has the property that the number of elements of C in the D-neighborhood $\mathbf{u} + D$ of $\mathbf{u}$ depends only on whether $\mathbf{u} \in C$. This kind of sets are called perfect (multiple) coverings [2]. So, translational tilings are perfect coverings, in particular. Again, perfect multiple coverings are defined for arbitrary graphs. They are generalizations of perfect covering codes in graphs. For more on covering codes, see *e.g.* [16; 39].

Algebraic approach

In the algebraic approach, it is typically assumed that $\mathcal{A}$ is a finite subset of integers $\mathbb{Z}$, although this is not always the case. Then any configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ is identified with the formal power series

$$c(X) = \sum_{\mathbf{u} \in \mathbb{Z}^d} c_{\mathbf{u}} X^{\mathbf{u}}$$

in d variables $X = (x_1, \ldots, x_d)$ where the shorthand notations $c_{\mathbf{u}} = c(\mathbf{u})$ and $X^{\mathbf{u}} = x_1^{u_1} \cdots x_d^{u_d}$ for $\mathbf{u} = (u_1, \ldots, u_d)$ are used. We may multiply configurations by complex Laurent polynomials in variables $X = (x_1, \ldots, x_d)$. An annihilator of a configuration c is a Laurent polynomial f such that $fc = 0$. A configuration c is $\mathbf{v}$-periodic if and only if it is annihilated by the difference polynomial $X^{\mathbf{v}} - 1$. A periodizer of c is a Laurent polynomial f such that fc is strongly periodic.

It is known that if this kind of configuration has low complexity with respect to any shape, then it has a non-zero annihilator [55]. Moreover, if a configuration (over a finite alphabet which is a subset of $\mathbb{Z}$) has a non-zero annihilator then it has an annihilator which is a product of finitely many difference polynomials:

Theorem ([55])**.** *If a configuration has a non-zero annihilator, then it has an annihilator of the form*

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1).$$

The proof of the theorem is based on Hilbert's Nullstellensatz. As a corollary we have the following periodic decomposition theorem which says that any configuration with non-zero annihilators can be written as a sum of finitely many periodic functions $\mathbb{Z}^d \to \mathbb{Z}$. However, the functions may get infinitely many distinct values [54] so that they may not be configurations in the classical sense.

Theorem (Periodic decomposition theorem [56])**.** *Assume that a configuration c has a non-zero annihilator. Then there exist periodic functions $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{Z}^d}$ such that*

$$c = c_1 + \cdots + c_m.$$

1.2 Delone sets

In crystalline materials, particles attach to each other to form an ordered structure. Crystallography is a branch of science that studies these materials [1; 10; 83]. For traditional crystals, the structure is periodic while for quasicrystals the structure is non-periodic. It is an outstanding question in crystallography to predict how given particles come together in crystals and to determine the formed crystal structure [79].

For a long time, it was believed that all ordered materials are necessarily periodic and hence crystals. However, in 1982 Shechtman discovered the first quasicrystals by studying the X-ray diffraction pattern of a certain metallic alloy [84]. See Figure 1 for a pictorial illustration of this diffraction pattern. The diffraction pattern has 10-fold rotational symmetry which together with the crystallographic restriction implies that it cannot have any translational symmetries.

The discovery of quasicrystals gave pace to the study of a field of mathematics called the study of aperiodic order [3; 4]. It studies the mathematical foundation of quasicrystals, such as Delone sets and aperiodic tilings. For example, the famous Penrose tilings [74] are closely related to quasicrystals [37].

Delone sets are mathematical models for crystals and quasicrystals. They are uniformly discrete and relatively dense subsets of the Euclidean space. Delone sets as an object of study in crystallography were introduced in the late 1930s and named after Russian mathematician Boris Delone (also Delaunay in some contexts) [21].

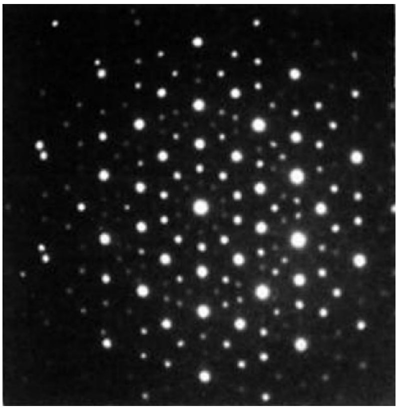

Figure 1. Part of the diffraction pattern of the quasicrystal metallic alloy discovered by Shechtman with a 10-fold rotational symmetry [84].

In this thesis, we study the connection between periodicity and local complexity of Delone sets. The typical measure for local complexity of Delone sets is the patch-complexity. A T-patch of a Delone set S at a point $\mathbf{s} \in S$ is the set of all points of S within distance T from $\mathbf{s}$. The patch-complexity $N_S(T)$ of S gives, for a radius $T > 0$, the number of distinct T-patches of S up to translation. In general, $N_S(T)$ can be infinite or finite. If it is finite for all T, then S is called a Delone set of finite local complexity (FLC). It is known by Lagarias and Pleasants [60] that a small enough patch-complexity implies periodicity (Theorem 5.1.5).

Besides the class of Delone sets of finite local complexity, there are two other well-known classes of Delone sets: Meyer sets and finitely generated Delone sets. Meyer sets are Delone sets S such that also the set $S - S$ is a Delone set. A finitely generated Delone set is a Delone set S such that the abelian group it generates is finitely generated. There is a known hierarchy between these classes: Delone sets of finite local complexity are finitely generated Delone sets, and Meyer sets are Delone sets of finite local complexity.

We identify Delone sets with their indicator functions that are $\mathbb{R}^d$-configurations $\mathbb{R}^d \to \{0, 1\}$. More generally, we study $\mathbb{R}^d$-configurations $\mathbb{R}^d \to \mathcal{A}$ over any non-empty finite alphabet $\mathcal{A}$. Any $\mathbb{R}^d$-configuration with $\mathcal{A} \subseteq \mathbb{C}$ whose support (that is, the set of cells for which the configuration gets non-zero values) is a Delone set can be regarded as a colored Delone set. We generalize some structural results on $\mathbb{Z}^d$-configurations to $\mathbb{R}^d$-configurations. We generalize the concept of annihilators and periodizers. As in the case of $\mathbb{Z}^d$-configurations, we can define $\mathbb{R}^d$-patterns and $\mathbb{R}^d$-pattern complexity. Similarly as for $\mathbb{Z}^d$-configurations, we have annihilators in the low complexity setting. So, $\mathbb{R}^d$-pattern complexity gives another complexity measure for Delone sets along with the classical patch-complexity.

1.3 Our contributions and the structure of the thesis

In Chapter 2 we present the preliminary concepts and notation needed in this thesis. A brief review of algebraic concepts is given in Section 2.3. Also, Hilbert's Nullstellensatz is considered. In Section 2.4 we give a short survey of the relevant concepts on symbolic dynamics and define the basic terminology on $\mathbb{Z}^d$-configurations. We conclude the chapter with Section 2.5 by presenting the algebraic approach to multidimensional symbolic dynamics. The results are known, but some new terminology is introduced. In Subsection 2.5.2 we give a comprehensive survey of known results and past research concerning complex configurations with annihilators.

In Chapter 3 we consider forced periodicity perfect colorings. This is an interesting family of configurations with annihilators. We begin by introducing graphs and defining perfect colorings. Then we study first perfect colorings with only two colors, *i.e.*, perfect coverings. We give sufficient conditions on forced periodicity on three well-known two-dimensional grid graphs in Section 3.3. These results are then generalized to perfect coverings with convex shapes. The higher-dimensional setting is discussed very briefly. After this, we turn to two-dimensional perfect colorings over arbitrary alphabets. We give a sufficient condition on their forced periodicity (Theorem 3.4.5). To prove this result, we need to consider configurations whose coefficients are integer vectors. Then the coefficients of their annihilators are integer matrices. In addition, forced periodicity of configurations of low abelian complexity is discussed in Section 3.5. Finally, we state some algorithmic aspects in Section 3.6.

In Chapter 4 we consider two improvements of the periodic decomposition theorem. The first improvement concerns configurations that have annihilators of a specific type. This result is very much inspired by similar results in [68]. More precisely, we assume that for some $k \in \{1, \ldots, d\}$ and for every $(k-1)$-dimensional linear subspace V of $\mathbb{R}^d$ the configuration has a periodizer which has exactly one non-zero value in V. We show that the configuration is then a sum of finitely many functions which all have k linearly independent periods (Theorem 4.1.4). If $k = 1$, then the statement says just that any configuration with a non-trivial periodizer is a sum of finitely many periodic functions which is exactly the original periodic decomposition theorem. Also, for $k = d$ the statement is known to be true [50]. As a corollary (Corollary 4.1.7) of this result we get a similar result as in [68] concerning translational tilings. This corollary is closely related to the periodic tiling problem. In the second improvement of the periodic decomposition theorem we consider configurations whose non-zero values are located very "sparsely" in the grid $\mathbb{Z}^d$. More precisely, we consider configurations for which the number of non-zero values in patterns grows at most linearly with respect to the diameter of the pattern. We show that if this kind of configuration has a non-zero annihilator, then it is a sum of finitely many periodic configurations whose non-zero values are aligned on unique lines (Theorem 4.2.1 and Corollary 4.2.7). This version of the periodic decomposition theorem is used in the next chapter in the study of colorings of 1-dimensional Delone sets with non-trivial annihilators and in proving that this kind of colorings are necessarily periodic.

In Chapter 5 we study Delone sets and $\mathbb{R}^d$-configurations. We begin by defining Delone sets, the patch-complexity, and the common classes of Delone sets. Also, some new terminology is introduced, such as $\mathbb{R}^d$-polynomials and Delone configurations. Moreover, annihilation is defined for $\mathbb{R}^d$-configurations. In Section 5.2 we consider Meyer sets. We show that if the patch-complexity function of a Meyer set S grows sufficiently slowly, then S has low $\mathbb{R}^d$-pattern complexity (Theorem 5.2.4). Also, a related conjecture (that has already been proven to be false) is discussed. In Section 5.3 we consider $\mathbb{R}^d$-configurations with annihilators. First, we note that if an $\mathbb{R}^d$-configuration has low complexity, then it has a non-trivial annihilator, that is, a non-trivial linear combination of some finitely many of its translations is the zero function (Lemma 5.3.1). This is a direct generalization of a similar result for $\mathbb{Z}^d$-configurations (Lemma 2.5.8). Then we show that if an $\mathbb{R}^d$-configuration c with integer coefficients has a non-trivial annihilator with integer coefficients, then it has an annihilator of a simple form (Theorem 5.3.4). This result is improved for such $\mathbb{R}^d$-configurations whose supports are Delone sets of finite local complexity (Theorem 5.3.11). Also, a periodic decomposition theorem for $\mathbb{R}^d$-configurations is provided (Theorem 5.3.15) as a direct generalization of the original periodic decomposition theorem for $\mathbb{Z}^d$-configurations. Finally, we prove that if a 1-dimensional $\mathbb{R}^d$-configuration whose support is a Delone set has a non-trivial annihilator with integer coefficients, then it is periodic (Theorem 5.3.16). In Section 5.4 we consider forced periodicity of $\mathbb{R}^d$-configurations whose supports are Delone sets of finite local complexity. We give a condition on the existence of particular annihilators to imply periodicity (Theorem 5.4.4).

We conclude with Chapter 6 by discussing some relevant open problems in the field related to the thesis.

2 Preliminaries

In this chapter we present the preliminary concepts, definitions, and basic notation which is used throughout the thesis. Also, a survey on related results is given.

2.1 Notation

We use the notation $|A|$ for the cardinality of any set A. We may denote $A \Subset B$ if $A \subseteq B$ and $|A| < \infty$, that is, if A is a finite subset of B. (However, this notation is not necessarily always used.) By A^B we mean the set of all functions from B to A.

As usual, we denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of integers, rational numbers, real numbers, and complex numbers, respectively. By $\mathbb{Z}_+$ and $\mathbb{R}_+$ we mean the sets of positive integers and positive reals, respectively. Natural numbers is the set $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$ of non-negative integers. By $\mathbb{Z}_p$ we mean the ring of integers modulo p for a positive integer p.

Linear algebra

Let $d \in \mathbb{Z}_+$ be the dimension. We consider the vector space $\mathbb{R}^d$ — the d-dimensional Euclidean space. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel over $\mathbb{F}$ if they are linearly dependent over $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{Q}$ or $\mathbb{R}$. As usual, we denote by $\mathbf{e}_i \in \mathbb{Z}^d$ the the ith natural base vector, that is, the vector whose ith coordinate is 1 and all the other coordinates are 0. The scalar product of two vectors $\mathbf{u} = (u_1, \ldots, u_d), \mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{C}^d$ is denoted by

$$\mathbf{u} \cdot \mathbf{v} = u_1\overline{v_1} + \ldots + u_d\overline{v_d}$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We denote by

$$||\mathbf{v}|| = \sqrt{v_1^2 + \ldots + v_d^2}$$

the Euclidean norm of a vector $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$. By the distance of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ we typically mean their Euclidean distance $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

The linear subspace of $\mathbb{R}^d$ generated by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ is the set

$$\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle = \{a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n \mid a_1, \ldots, a_n \in \mathbb{R}\}.$$

We use also the notation $L(S)$ for the subspace of $\mathbb{R}^d$ generated by a subset $S \subseteq \mathbb{R}^d$. So, we have two notations for the same thing: $\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle = L(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\})$.

For a fixed dimension d, we denote by $\mathbb{G}_k$ the set of all k-dimensional subspaces of $\mathbb{R}^d$. Clearly, if $k > d$, then $\mathbb{G}_k = \emptyset$.

Asymptotic notation

For completeness, let us introduce the commonly known Bachmann-Landau notation concerning the growth rate of functions. In the following we have $f\colon \mathbb{N} \to \mathbb{R}_+ \cup \{0\}$ and $g\colon \mathbb{N} \to \mathbb{R}_+$.

- We write $f(x) = O(g(x))$ if there exist $\alpha \in \mathbb{R}_+$ and $x_0 \in \mathbb{N}$ such that $f(x) \leq \alpha g(x)$ for all $x \geq x_0$. Asymptotically, this means that
$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty.$$
- We write $f(x) = o(g(x))$ if for all $\alpha \in \mathbb{R}_+$ there exists $x_0 \in \mathbb{N}$ such that $f(x) \leq \alpha g(x)$ for all $x \geq x_0$. Asymptotically, this means that
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$
- We write $f(x) = \Omega(g(x))$ if
$$\liminf_{x \to \infty} \frac{f(x)}{g(x)} > 0.$$
- We write $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

2.2 Basic concepts

In the following, $d \in \mathbb{Z}_+$ denotes the dimension, and by G we mean either $\mathbb{Z}^d$ or $\mathbb{R}^d$.

Configurations and periodicity

Let $\mathcal{A}$ be any set. In this thesis we study functions $c\colon G \to \mathcal{A}$, that is, elements of the set $\mathcal{A}^G$. If $c(G) \subseteq \mathcal{A}$ and $c(G) \subseteq \mathcal{B}$, we may consider c both as an element of the set $\mathcal{A}^G$ and the set $\mathcal{B}^G$. We say that $c \in \mathcal{A}^G$ is *finitary* if the image $c(G)$ of c is a finite set. In this case $c \in \mathcal{B}^G$ for some $\mathcal{B} \Subset \mathcal{A}$. We say that c is *integral* if $c(G) \subseteq \mathbb{Z}$. We may use the notation $c_{\mathbf{u}} = c(\mathbf{u})$ for the value of $c \in \mathcal{A}^G$ at position $\mathbf{u}$.

Definition 2.2.1. A finitary function $c \in \mathcal{A}^G$ is called a *G-configuration*.

The set $\mathcal{A}^G$ of all G-configurations over $\mathcal{A}$ is called the *G-configuration space* over $\mathcal{A}$. If it is clear from the context what G is, we may call any G-configuration simply a configuration.

Next, let us define different forms of periodicity. The *translation* $\tau^{\mathbf{t}}(c)$ of $c \in \mathcal{A}^G$ by $\mathbf{t} \in G$ is defined such that $\tau^{\mathbf{t}}(c)_{\mathbf{u}} = c_{\mathbf{u}-\mathbf{t}}$ for all $\mathbf{u} \in G$.

Definition 2.2.2. A function $c \in \mathcal{A}^G$ is $\mathbf{t}$*-periodic* if $\tau^{\mathbf{t}}(c) = c$.

If $c \in \mathcal{A}^G$ is $\mathbf{t}$-periodic for some non-zero vector $\mathbf{t} \in G$, then c is called simply *periodic*. Moreover, if c is $\mathbf{t}$-periodic, we may call $\mathbf{t}$ a *period*, a *vector of periodicity* or a *period vector* of c.

Definition 2.2.3. A function $c \in \mathcal{A}^G$ is *strongly periodic* if it has d linearly independent vectors of periodicity over $\mathbb{R}$.

Note that if a function $c \in \mathcal{A}^{\mathbb{Z}^d}$ has d linearly independent periods, then it is necessarily a configuration. This is not the case for $c \in \mathcal{A}^{\mathbb{R}^d}$.

A function $c \in \mathcal{A}^G$ is *periodic in direction* $\mathbf{v}$ if it is $k\mathbf{v}$-periodic for some non-zero $k \in \mathbb{R}$. For $c \in \mathcal{A}^{\mathbb{Z}^d}$, we say that c is V*-periodic* for a vector space $V \subseteq \mathbb{R}^d$ if it is periodic in direction $\mathbf{v}$ for all $\mathbf{v} \in V \cap \mathbb{Q}^d$. Note that if V is an irrational subspace, that is, if $V \cap \mathbb{Q}^d = \{\mathbf{0}\}$, then any $\mathbb{Z}^d$-configuration is trivially V-periodic since every configuration is $\mathbf{0}$-periodic. A function $c \in \mathcal{A}^{\mathbb{Z}^d}$ is k*-periodic* if it has k linearly independent periods. Clearly, any k-periodic $c \in \mathcal{A}^{\mathbb{Z}^d}$ is V-periodic for some $V \in \mathbb{G}_k$.

The indicator function $\mathbb{1}_S \colon \mathbb{R}^d \to \{0, 1\}$ of a subset $S \subseteq \mathbb{R}^d$ is defined as usual:

$$\mathbb{1}_S(\mathbf{u}) = \begin{cases} 1 & \text{, if } \mathbf{u} \in S \\ 0 & \text{, if } \mathbf{u} \notin S \end{cases}.$$

A set $S \subseteq \mathbb{R}^d$ is $\mathbf{t}$-periodic, periodic, strongly periodic or periodic in direction $\mathbf{v}$ if its indicator function is $\mathbf{t}$-periodic, periodic, strongly periodic or periodic in direction $\mathbf{v}$, respectively.

Remark 2.2.4. Clearly, if c is a $\mathbb{Z}^d$-configuration and strongly periodic, then it is periodic in every direction $\mathbf{v} \in \mathbb{Z}^d$. Indeed, since c is strongly periodic, it has period vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{Z}^d$ that are linearly independent over $\mathbb{Q}$. Any $\mathbf{v} \in \mathbb{Z}^d$ can be presented as a linear combination $\mathbf{v} = q_1\mathbf{v}_1 + \ldots + q_d\mathbf{v}_d$ where $q_i \in \mathbb{Q}$. By multiplying this equation by any common multiple k of the denominators of the rational numbers $q_1, \ldots, q_d$ in the equation we have $k\mathbf{v} = k_1\mathbf{v}_1 + \ldots + k_d\mathbf{v}_d$ where each k_i is an integer. Since c is $\mathbf{v}_i$-periodic for each i and hence $k_i\mathbf{v}_i$-periodic, it is also $k\mathbf{v}$-periodic.

This is not true for $\mathbb{R}^d$-configurations. In other words, a strongly periodic $\mathbb{R}^d$-configuration c is not necessarily periodic in every direction $\mathbf{v} \in \mathbb{R}^d$. Consider for example the configuration $c' \in \{0,1\}^{\mathbb{R}^2}$ defined as

$$c'(\mathbf{u}) = \begin{cases} 1 & \text{, if } \mathbf{u} = k_1(\pi,0) + k_2(0,1) \text{ for some } k_1, k_2 \in \mathbb{Z} \\ 0 & \text{, otherwise} \end{cases},$$

that is, the indicator function of the set $\pi\mathbb{Z}\times\mathbb{Z}$. It is clearly $(\pi,0)$-periodic and $(0,1)$-periodic and hence strongly periodic, but it is not periodic in direction $(1,1)$, for example. However, if c has d linearly independent rational period vectors $\mathbf{v}_1,\ldots,\mathbf{v}_d \in \mathbb{Q}^d$, then it is periodic in every rational direction $\mathbf{v} \in \mathbb{Q}^d$ by the same argument as above.

Pattern complexity

Let us call any non-empty finite set $D \Subset G$ a *G-shape* or just a *shape* if G is clear from the context. In particular, $\mathbb{Z}^d$-shapes are also $\mathbb{R}^d$-shapes.

Definition 2.2.5. Let $D \Subset G$ be a shape and $\mathcal{A}$ be a finite non-empty set. A *D-pattern* or a *pattern* with shape D over $\mathcal{A}$ is a function of the set $\mathcal{A}^D$.

To emphasize what G is, we may call a pattern $p \in \mathcal{A}^D$ with $D \Subset G$ a G-pattern. Moreover, let us denote by $\mathcal{A}^*$ the set of all patterns over the alphabet $\mathcal{A}$ where the dimension and G is known from the context.

For a fixed shape $D \Subset G$, the set of all D-patterns of a configuration $c \in \mathcal{A}^G$ is the set $\mathcal{L}_D(c) = \{\tau^{\mathbf{t}}(c)|_D \mid \mathbf{t} \in G\}$. The set of all patterns of c is denoted by $\mathcal{L}(c)$ which we may call the *language of* c. For a set $\mathcal{S} \subseteq \mathcal{A}^G$ of configurations, we define $\mathcal{L}_D(\mathcal{S})$ and $\mathcal{L}(\mathcal{S})$ as the unions of $\mathcal{L}_D(c)$ and $\mathcal{L}(c)$, respectively, over all $c \in \mathcal{S}$.

Definition 2.2.6. The *pattern complexity* $P_c(D)$ of a configuration $c \in \mathcal{A}^G$ with respect to a shape $D \Subset G$ is the number of distinct D-patterns that c contains, that is,

$$P_c(D) = |\mathcal{L}_D(c)|.$$

Definition 2.2.7. A configuration $c \in \mathcal{A}^G$ has *low complexity* with respect to a shape $D \Subset G$ if

$$P_c(D) \leq |D|.$$

The pattern complexity of a set $S \subseteq \mathbb{R}^d$ with respect to a shape $D \Subset \mathbb{R}^d$ is $P_S(D) = P_c(D)$ where $c = \mathbb{1}_S$ is the indicator function of S. Note that we have

$$P_S(D) = |\{(S \cap (D+\mathbf{t})) - \mathbf{t} \mid \mathbf{t} \in \mathbb{R}^d\}| = |\{(S-\mathbf{t}) \cap D \mid \mathbf{t} \in \mathbb{R}^d\}|.$$

By a slight abuse of terminology we call any set $D' \Subset \mathbb{R}^d$ a *D-pattern of the set S* if $D' = S \cap (D+\mathbf{t})$ for some $\mathbf{t}$. Thus, $P_S(D)$ counts the number of D-patterns of S up to translation. We denote by $\mathcal{L}_D(S) = \{(S \cap (D+\mathbf{t})) - \mathbf{t} \mid \mathbf{t} \in \mathbb{R}^d\}$ the set of all D-patterns of S translated to origin.

Remark 2.2.8. Above, we defined pattern complexity only for configurations. For non-finitary functions $c \in \mathcal{A}^G$, the pattern complexity would always be infinite.

Some geometric concepts

Recall that a subset of $\mathbb{R}^d$ is *convex* if for any two points of the set, also the entire line segment joining them is in the set. The *convex hull* $\text{conv}(D)$ of a set $D \subseteq \mathbb{R}^d$ is the smallest convex set that contains D, that is, the intersection of all convex sets containing D. A subset $D \subseteq \mathbb{Z}^d$ is convex if $D = \text{conv}(D) \cap \mathbb{Z}^d$.

Let $\mathbf{v} \in \mathbb{R}^d$ be a non-zero vector. The open *half space* in direction $\mathbf{v}$ is the set

$$H_{\mathbf{v}} = \{\mathbf{u} \in \mathbb{R}^d \mid \mathbf{u} \cdot \mathbf{v} < 0\}$$

and the closed half space in direction $\mathbf{v}$ is the set

$$\overline{H_{\mathbf{v}}} = \{\mathbf{u} \in \mathbb{R}^d \mid \mathbf{u} \cdot \mathbf{v} \leq 0\}.$$

See Figure 2 for an illustration.

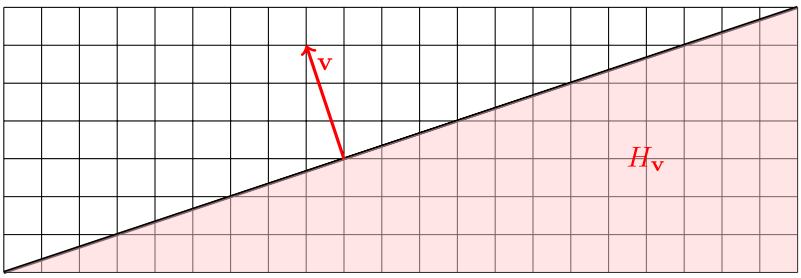

Figure 2. The open half space $H_{\mathbf{v}}$ in direction $\mathbf{v} = (-1, 3)$. The black line corresponds to the set $\overline{H_{\mathbf{v}}} \setminus H_{\mathbf{v}}$.

We say that a non-empty set $D \Subset \mathbb{R}^d$ *has a vertex in direction* $\mathbf{v}$ if there exist $\mathbf{t} \in \mathbb{R}^d$ such that $D \subseteq \overline{H_{\mathbf{v}}} + \mathbf{t}$ and $|D \cap ((\overline{H_{\mathbf{v}}} \setminus H_{\mathbf{v}}) + \mathbf{t})| = 1$. In this case we may call the unique element of the set $D \cap ((\overline{H_{\mathbf{v}}} \setminus H_{\mathbf{v}}) + \mathbf{t})$ a vertex of D. Note that D may have a vertex in direction $\mathbf{v}$ but not in direction $-\mathbf{v}$. Finite non-empty sets $D \Subset \mathbb{R}^d$ (that is, shapes) have always vertices.

2.3 Algebraic concepts

In this section we introduce the necessary algebraic terminology and concepts. For a more thorough introduction to the topic, see *e.g.* [17; 22; 25; 38; 63].

2.3.1 Rings, ideals, and modules

In this thesis we consider basic algebraic structures such as groups, rings and fields and assume that the reader is familiar with the definitions of these concepts. However, let it be noted that in our definition a ring has an identity element. Also, note that our rings contain always at least two elements.

A (left) *ideal* I of a ring R, denoted by $I \leq R$, is an additive subgroup of R such that for all $r \in R$ and $i \in I$ also $ri \in I$, that is, I is closed under (left) multiplication by elements of R. For $a_1, \ldots, a_k \in R$, we denote by

$$\langle a_1, \ldots, a_k \rangle = \{r_1 a_1 + \ldots + r_k a_k \mid r_1, \ldots, r_k \in R\}$$

the ideal of R generated by $a_1, \ldots, a_k$.

Remark 2.3.1. Note that we use the same notations $\langle a_1, \ldots, a_k \rangle$ for the ideal of a ring R generated by $a_1, \ldots, a_k \in R$ and $\langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$ for the vector space over $\mathbb{R}$ generated by vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^d$. These notations should not be confused with each other.

A *principal ideal* is an ideal generated by a single element. A *prime ideal* is an ideal $I \leq R$ such that if $ab \in I$, then necessarily either $a \in I$ or $b \in I$. A proper ideal $I \leq R$ is called a *maximal ideal* if for any ideal $J \leq R$ such that $I \subseteq J$ we have either $J = I$ or $J = R$. Two ideals $I, J \leq R$ are *comaximal* if $I + J = R$, that is, if $1 \in I + J$. The *radical* of an ideal $I \leq R$ is the set

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n\}.$$

A *radical ideal* is an ideal $I \leq R$ such that $I = \sqrt{I}$.

Let R be a ring with identity element 1. A (left) R-*module* is an abelian group M equipped with a module multiplication $\cdot\colon R \times M \to M$ such that

- $r \cdot (a + b) = r \cdot a + r \cdot b$ for all $r \in R$ and $a, b \in M$,
- $(r + r') \cdot a = r \cdot a + r' \cdot a$ for all $r, r' \in R$ and $a \in M$,
- $(rr') \cdot a = r \cdot (r' \cdot a)$ for all $r, r' \in R$ and $a \in M$, and
- $1 \cdot a = a$ for all $a \in M$.

If R is a field, then the R-modules are exactly the vector spaces over R.

Example 2.3.2. Let R be a ring and let $n \geq 1$ be a positive integer. Let us denote by $R^{n \times n}$ and R^n the sets of all $n \times n$ matrices and length n vectors, respectively, with entries in R. Let us interpret the vectors as column vectors. Then the multiplication $\mathbf{Mv}$ of any vector $\mathbf{v} \in R^n$ from the left by a matrix $\mathbf{M} \in R^{n \times n}$ is defined, and it is easily seen that R^n is an $R^{n \times n}$-module.

Also, any ideal $I \leq R$ of a ring R is clearly an R-module.

Let $S \subseteq R$ be a subring of a ring R and let M be an R-module. For finitely many elements $a_1, \ldots, a_n \in M$, we denote by

$$S[a_1, \ldots, a_n] = \{s_1 a_1 + \ldots + s_n a_n \mid s_1, \ldots, s_n \in S\}$$

which is an S-module. Note that for $a_1, \ldots, a_n \in R$ we have

$$R[a_1, \ldots, a_n] = \langle a_1, \ldots, a_n \rangle.$$

2.3.2 Polynomials, Laurent polynomials and Laurent series

Polynomials and Laurent polynomials

Let R be a ring and let d be a positive integer. We consider polynomials and Laurent polynomials in d variables $x_1, \ldots, x_d$ over R. If $d \leq 3$, we usually denote $x = x_1, y = x_2, z = x_3$.

We use the common abbreviations $X = (x_1, \ldots, x_d)$ and $X^{\mathbf{u}} = x_1^{u_1} \cdots x_d^{u_d}$ for a vector $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{Z}^d$. Then any polynomial $p = p(X)$ over R may be written as

$$p(X) = \sum_{\mathbf{u} \in D} p_{\mathbf{u}} X^{\mathbf{u}}$$

where $D \Subset \mathbb{N}^d$ is non-empty and $p_{\mathbf{u}} \in R$. Similarly, any Laurent polynomial $p(X)$ over R may be written as

$$p(X) = \sum_{\mathbf{u} \in D} p_{\mathbf{u}} X^{\mathbf{u}}$$

where $D \Subset \mathbb{Z}^d$ is non-empty and $p_{\mathbf{u}} \in R$. Clearly, any polynomial is also a Laurent polynomial.

As usual, we denote by $R[X] = R[x_1, \ldots, x_d]$ and $R[X^{\pm 1}] = R[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ the sets of all polynomials and Laurent polynomials, respectively, over R. We call these sets the *polynomial ring* and the *Laurent polynomial ring*, respectively, over R. As usual, we call (Laurent) polynomials in one variable *univariate*. Otherwise, they are called *multivariate*. In particular, (Laurent) polynomials in two variables may be called *two-variate*.

Remark 2.3.3. If R is a commutative ring, that is, if $r_1 r_2 = r_2 r_1$ for all $r_1, r_2 \in R$, then also $R[X]$ and $R[X^{\pm 1}]$ are commutative rings. If R is not commutative, then the sets $R[X]$ and $R[X^{\pm 1}]$ are still rings but not commutative.

Formal power series

We consider *Laurent series* or *formal power series* over R-modules M, that is, objects of the form

$$s(X) = \sum_{\mathbf{u} \in \mathbb{Z}^d} s_{\mathbf{u}} X^{\mathbf{u}}$$

where $s_{\mathbf{u}} \in M$. We use the notation $M[[X^{\pm 1}]]$ for the set of all Laurent series over M. Since a ring R is always an R-module itself, above we have defined Laurent series also over any ring. Also, note that any Laurent polynomial over a ring R is also a Laurent series over R.

Convention. From now on, when considering Laurent polynomials $p(X)$ and Laurent series $s(X)$ we may drop the variable vector X from sight and write simply $p = p(X)$ and $s = s(X)$.

Definition 2.3.4. The *support* of a Laurent series $s = s(X) = \sum_{\mathbf{u} \in \mathbb{Z}^d} s_{\mathbf{u}} X^{\mathbf{u}} \in M[[X^{\pm 1}]]$ is the set

$$\text{supp}(s) = \{\mathbf{u} \in \mathbb{Z}^d \mid s_{\mathbf{u}} \neq 0\}$$

of the cells where s has a non-zero coefficient. If supp(s) is a finite set, then s is called finitely supported.

So, the Laurent polynomials over a ring R are exactly the finitely supported Laurent series over R.

In this thesis, the role of the Laurent series is purely formal, that is, we are not interested in analytical properties of the series such as the possible convergence of the series, for example. We may sum up two Laurent series or multiply a Laurent series by a Laurent polynomial, as we shall see in the following.

Indeed, the multiplication of a Laurent series $s = \sum_{\mathbf{u} \in \mathbb{Z}^d} s_{\mathbf{u}} X^{\mathbf{u}} \in M[[X^{\pm 1}]]$ by a Laurent polynomial $p = \sum_{\mathbf{u} \in D} p_{\mathbf{u}} X^{\mathbf{u}} \in R[X^{\pm 1}]$ is well defined as

$$ps = \sum_{\mathbf{u} \in \mathbb{Z}^d} \sum_{\mathbf{v} \in D} p_{\mathbf{v}} s_{\mathbf{u}} X^{\mathbf{u}+\mathbf{v}} = \sum_{\mathbf{u} \in \mathbb{Z}^d} \sum_{\mathbf{v} \in D} p_{\mathbf{v}} s_{\mathbf{u}-\mathbf{v}} X^{\mathbf{u}} \in M[[X^{\pm 1}]]$$

since $D \Subset \mathbb{Z}^d$ is a finite set. In fact, it is easily seen that the set $M[[X^{\pm 1}]]$ is an $R[X^{\pm 1}]$-module.

Annihilators

Definition 2.3.5 (Annihilation)**.** Let R be a commutative ring and let M be an R-module. A Laurent polynomial $p(X) \in R[X^{\pm 1}]$ *annihilates* (or is an *annihilator* of) a Laurent series $s(X) \in M[[X^{\pm 1}]]$ if

$$p(X)s(X) = 0.$$

The set

$$\mathrm{Ann}_R(s) = \{p \in R[X^{\pm 1}] \mid ps = 0\}$$

of all annihilators with coefficients in R of a Laurent series $s \in M[[X^{\pm 1}]]$ is an ideal of $R[X^{\pm 1}]$. We call this set the *annihilator ideal* of s over R.

Hilbert's Nullstellensatz

In the following, K is an algebraically closed field. We consider polynomials and Laurent polynomials over K in d variables $X = (x_1, \ldots, x_d)$. For a proper polynomial ideal $I \leq K[X]$, let

$$\mathbf{V}(I) = \{\mathbf{z} \in K^d \mid p(\mathbf{z}) = 0 \text{ for all } p \in I\}$$

be the *algebraic variety* defined by I. For a set $S \subseteq K^d$, let

$$\mathbf{I}(S) = \{p \in K[X] \mid p(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in S\}$$

be the set of all polynomials whose sets of zeros contain the set S. Hilbert's Nullstellensatz states that for a polynomial ideal $I \leq K[X]$ the radical of I can be expressed using the above sets.

***Theorem* 2.3.6** (Hilbert's Nullstellensatz [45])**.** *For an ideal $I \leq K[x_1, \ldots, x_d]$, the following holds:*

$$\mathbf{IV}(I) = \sqrt{I}.$$

An immediate corollary of the Nullstellensatz is the following weak Nullstellensatz. However, the weak Nullstellensatz can be also proved without the Nullstellensatz. In fact, the Nullstellensatz follows from the weak Nullstellensatz with the Rabinowitsch trick [78].

***Theorem* 2.3.7** (Weak Nullstellensatz)**.** *An ideal $I \leq K[x_1, \ldots, x_d]$ is a proper ideal, that is, $I \neq K[x_1, \ldots, x_d]$ if and only if $\mathbf{V}(I) \neq \emptyset$.*

If an ideal $I \leq K[x_1, \ldots, x_d]$ contains the monomial 1, then it is the whole polynomial ring. Thus, the weak Nullstellensatz states that $1 \in I$ if and only if $\mathbf{V}(I) = \emptyset$.

Let us next state a generalization of the Hilbert's Nullstellensatz to Laurent polynomials with complex coefficients. This generalization was proved in [86]. In the following, we use the notation $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Moreover, for a Laurent polynomial ideal $I \leq \mathbb{C}[X^{\pm 1}]$ and a subset $S \subseteq (\mathbb{C}^*)^d$, we denote

$$\mathbf{V}(I) = \{\mathbf{z} \in (\mathbb{C}^*)^d \mid p(\mathbf{z}) = 0 \text{ for all } p \in I\}$$

and

$$\mathbf{I}(S) = \{p \in \mathbb{C}[X^{\pm 1}] \mid p(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in S\}.$$

***Theorem* 2.3.8** (Hilbert's Nullstellensatz for complex Laurent polynomials [86]). *For an ideal $I \leq \mathbb{C}[X^{\pm 1}]$, the following holds:*

$$\mathbf{IV}(I) = \sqrt{I}.$$

Now, the following generalization of the weak Nullstellensatz follows immediately.

***Corollary* 2.3.9** (Weak Nullstellensatz for complex Laurent polynomials). *An ideal $I \leq \mathbb{C}[X^{\pm 1}]$ is a proper ideal, that is, $I \neq \mathbb{C}[X^{\pm 1}]$ if and only if $\mathbf{V}(I) \neq \emptyset$.*

Proof. Assume first that I is a proper ideal. Then also $\sqrt{I}$ is a proper ideal. By Theorem 2.3.8 we have $\sqrt{I} = \mathbf{IV}(I)$. Since $\sqrt{I}$ is a proper ideal, we have $\mathbf{V}(I) \neq \emptyset$. Otherwise, $\mathbf{IV}(I) = \mathbb{C}[X^{\pm 1}] \neq \sqrt{I}$.

Assume then that $\mathbf{V}(I) \neq \emptyset$. Thus, $\mathbf{IV}(I) \neq \mathbb{C}[X^{\pm 1}]$. By Theorem 2.3.8 we have $\mathbf{IV}(I) = \sqrt{I}$ and hence I has to be a proper ideal. □

From now on, by Nullstellensatz and weak Nullstellensatz we may refer also to Theorem 2.3.8 and Corollary 2.3.9, respectively.

Common factors

Let f and g be (Laurent) polynomials. We say that g is a factor of f if there exists a (Laurent) polynomial h such that $f = gh$. We say that a set of (Laurent) polynomials have no common factors if all their common factors are units. A (Laurent) polynomial f is *irreducible* if it has no non-unit factors.

Univariate polynomials $f_1, \ldots, f_n \in \mathbb{C}[x^{\pm 1}]$ have a common factor if and only if they have a common zero. Thus, by weak Nullstellensatz we have the following lemma.

***Lemma* 2.3.10.** *Let $I \leq \mathbb{C}[x^{\pm 1}]$. Polynomials in I have no common factors if and only if $1 \in I$, that is, $I = \mathbb{C}[x^{\pm 1}]$.*

The above lemma does not work for polynomials in several variables. For example, consider the two-variate polynomials $x-1$ and $xy-1$. They have no common factors but they generate a proper ideal.

Resultants

The x_i-resultant $\mathrm{Res}_{x_i}(f, g)$ of two proper polynomials $f, g \in R[x_1, \ldots, x_d]$ is the determinant of the *Sylvester matrix* of f and g with respect to variable x_i. We omit the details, which the reader can check from [17]. Instead, we consider the resultant

$$\mathrm{Res}_{x_i}(f, g) \in R[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d]$$

for every $i \in \{1, \ldots, d\}$ as a certain proper polynomial that has the following two properties:

- $\mathrm{Res}_{x_i}(f, g)$ is in the ideal generated by f and g, *i.e.*, there exist proper polynomials h and l such that

$$hf + lg = \mathrm{Res}_{x_i}(f, g).$$

- If two proper polynomials f and g have no common factors in $R[x_1, \ldots, x_d]$, then $\mathrm{Res}_{x_i}(f, g) \neq 0$.

2.4 Symbolic dynamics

Let us review the basic concepts of symbolic dynamics for $\mathbb{Z}^d$-configurations. In the following considerations we do not make any additional assumptions about the alphabet $\mathcal{A}$ — it is just a non-empty finite set. For a reference, see *e.g.* [15; 57; 64].

2.4.1 Subshifts

We consider the configuration space $\mathcal{A}^{\mathbb{Z}^d}$ as a metric space. We use the usual metric $d\colon \mathcal{A}^{\mathbb{Z}^d} \times \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{R}_+ \cup \{0\}$ defined such that

$$d(c, e) = \begin{cases} 0, \text{ if } c = e \\ 2^{-\min\{||\mathbf{u}|| \mid c(\mathbf{u}) \neq e(\mathbf{u})\}}, \text{ if } c \neq e \end{cases}.$$

Under this metric, two configurations are close if they agree on a large area around the origin. Moreover, the obtained metric space is compact and hence every sequence of configurations has a converging subsequence.

For a shape $D \Subset \mathbb{Z}^d$ and a pattern $p \in \mathcal{A}^D$, a *cylinder set* determined by p is the set

$$\mathrm{Cyl}(p) = \{c \in \mathcal{A}^{\mathbb{Z}^d} \mid c(\mathbf{u}) = p(\mathbf{u}) \text{ for all } \mathbf{u} \in D\}.$$

Any open ball (under the metric d) is a cylinder. The set of all cylinders forms a clopen base for the configuration space $\mathcal{A}^{\mathbb{Z}^d}$ endowed with the metric d.

Note that the exact form of the metric does not matter since we are more interested on the topology it induces. Indeed, in the above definition of the metric d we could replace the Euclidean norm $||\mathbf{u}||$ by any other norm (for example, by the Manhattan norm) of $\mathbb{Z}^d$. Also, we could consider any other decreasing function with limit 0 defined from $\mathbb{R}$ to $\mathbb{R}_+$ instead of $x \mapsto 2^{-x}$. All of these give the same topology. We could also endow $\mathcal{A}$ with the discrete topology and consider the product topology it induces on $\mathcal{A}^{\mathbb{Z}^d}$ — the *prodiscrete topology*. Again, we would get the same topology as the one induced by the above metric.

A subset $\mathcal{S} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ of the configuration space is a *subshift* if it is topologically closed and translation-invariant which means that if $c \in \mathcal{S}$, then for all $\mathbf{t} \in \mathbb{Z}^d$ also

$\tau^{\mathbf{t}}(c) \in \mathcal{S}$. Equivalently, subshifts can be defined by using forbidden patterns. Given a set $F \subseteq \mathcal{A}^*$ of *forbidden* patterns, the set

$$X_F = \{c \in \mathcal{A}^{\mathbb{Z}^d} \mid \mathcal{L}(c) \cap F = \emptyset\}$$

of configurations that avoid all forbidden patterns is a subshift. Conversely, every subshift is obtained by forbidding some set of finite patterns. If F is a finite set, then X_F is a *subshift of finite type* (SFT).

The *orbit* of a configuration c is the set

$$\mathcal{O}(c) = \{\tau^{\mathbf{t}}(c) \mid \mathbf{t} \in \mathbb{Z}^d\}$$

of all translations of c. The *orbit closure* $\overline{\mathcal{O}(c)}$ of c is the topological closure of its orbit. The orbit closure of a configuration c is the smallest subshift that contains c. It consists of all configurations c' such that $\mathcal{L}(c') \subseteq \mathcal{L}(c)$. Moreover, a configuration c is strongly periodic if and only if its orbit closure is a finite set. Let us provide a short proof for this fact.

***Lemma* 2.4.1.** *A configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ is strongly periodic if and only if its orbit closure is a finite set.*

Proof. If c is not strongly periodic, then there exists a non-zero $\mathbf{v} \in \mathbb{Z}^d$ such that $\tau^{k\mathbf{v}}(c) \neq c$ for all $k \in \mathbb{Z} \setminus \{0\}$. Since $\tau^{k\mathbf{v}}(c) \in \overline{\mathcal{O}(c)}$, it follows that $\overline{\mathcal{O}(c)}$ is an infinite set.

Assume then that c is strongly periodic. There exists $n \in \mathbb{N}$ such that c is $n\mathbf{e}_i$-periodic for each $i \in \{1, \ldots, d\}$. Thus, every $e \in \overline{\mathcal{O}(c)}$ is $n\mathbf{e}_i$-periodic. This means that the content of e in the shape $D = \{0, \ldots, n-1\}^d$ fully determines the whole configuration e. Since there are only finitely many patterns $p \in \mathcal{A}^D$, it follows that $\overline{\mathcal{O}(c)}$ is a finite set. □

In fact, the above lemma holds more generally for any subshift. In other words, every element of a subshift $\mathcal{S} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is strongly periodic if and only if $\mathcal{S}$ is finite. This fact is proved in [5] for two-dimensional subshifts but the proof directly generalizes to any dimension.

Configuration c is called *uniformly recurrent* if every configuration c' in its orbit closure satisfies $\mathcal{L}(c') = \mathcal{L}(c)$. This is equivalent to the condition that the orbit closure of c is a *minimal* subshift, *i.e.*, no proper non-empty subset of $\overline{\mathcal{O}(c)}$ is a subshift. A classical result by Birkhoff on minimal dynamical systems implies that every non-empty subshift contains a uniformly recurrent configuration [9].

A non-empty subshift is *aperiodic* if it does not contain any periodic configurations. There exist aperiodic subshifts in any dimension. For example, the orbit closure of the bi-infinite Fibonacci word is a 1-dimensional aperiodic subshift. However, there are no 1-dimensional aperiodic SFTs. This is due to the fact that for any

1-dimensional SFT there exists a directed finite graph such that the elements of the SFT are exactly the bi-infinite walks of the graph.

Berger showed that there exist 2-dimensional aperiodic SFTs [6; 7] in his proof of the undecidability of the so called domino problem — the emptiness problem of 2-dimensional SFTs. If there were no aperiodic 2-dimensional SFTs, then the domino problem would be decidable as the classical argumentation by H. Wang shows. He proved the following theorem.

***Theorem* 2.4.2** (H. Wang [88])**.** *If a given SFT is known to be either the empty set or to contain a strongly periodic configuration, then its emptiness problem is decidable, that is, there is an algorithm to determine whether there exist any configurations in the SFT.*

Expansivity

Let us use the notation $V^r = \{\mathbf{u} \in \mathbb{R}^d \mid d(\mathbf{u}, V) \leq r\}$ for any set $V \subseteq \mathbb{R}^d$.

Definition 2.4.3 (Boyle and Lind [11])**.** A linear subspace $V \subseteq \mathbb{R}^d$ is *expansive for* a subshift $\mathcal{S} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ if there exists a positive real number r such that for all $c, e \in \mathcal{S}$ we have

$$c \restriction_{V^r} = e \restriction_{V^r} \implies c = e.$$

***Theorem* 2.4.4** (Boyle and Lind [11])**.** *A subshift $\mathcal{S}$ is finite if and only if every $(d-1)$-dimensional subspace $V \subseteq \mathbb{R}^d$ is expansive for $\mathcal{S}$.*

A subshift $\mathcal{S} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is *expansive in direction* $\mathbf{v}$ or *deterministic in direction* $\mathbf{v}$ if for all $c, e \in \mathcal{S}$ we have

$$c \restriction_{H_\mathbf{v}} = e \restriction_{H_\mathbf{v}} \implies c = e.$$

Clearly, the orthogonal space $\langle \mathbf{v} \rangle^\perp$ is expansive for a subshift $\mathcal{S}$ if and only if $\mathcal{S}$ is expansive in both directions $\mathbf{v}$ and $-\mathbf{v}$.

2.4.2 Block maps and cellular automata

In the following we give a brief survey on block maps and cellular automata (CA). For a more thorough survey on cellular automata, see *e.g.* [48].

Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty finite sets. A d-dimensional *block map* determined by a neighborhood vector $N = (\mathbf{t}_1, \ldots, \mathbf{t}_n)$ and a local rule $f \colon \mathcal{A}^n \longrightarrow \mathcal{B}$ is a function $F \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}$ defined such that

$$F(c)(\mathbf{u}) = f(c(\mathbf{u} + \mathbf{t}_1), \ldots, c(\mathbf{u} + \mathbf{t}_n)).$$

We are interested in functions between configuration spaces that are continuous and commute with translations. The following famous theorem by Curtis, Hedlund, and Lyndon states that these functions are exactly the block maps.

***Theorem* 2.4.5** (Curtis-Hedlund-Lyndon [40]). *Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty finite sets. A function $F\colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}$ is continuous and commutes with translations if and only if it is a block map.*

A d-dimensional *cellular automaton* or a *CA* for short over a finite alphabet $\mathcal{A}$ is a block map $F\colon \mathcal{A}^{\mathbb{Z}^d} \longrightarrow \mathcal{A}^{\mathbb{Z}^d}$. A CA is *additive* or *linear* if its local rule is of the form

$$f(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n$$

where $a_1, \dots, a_n \in R$ are elements of some finite ring R and $\mathcal{A}$ is an R-module.

A classical result called the *Garden-of-Eden theorem* proved by Moore and Myhil gives a characterization for surjectivity in terms of injectivity on "finite" configurations. Two configurations $c_1 \in \mathcal{A}^{\mathbb{Z}^d}$ and $c_2 \in \mathcal{A}^{\mathbb{Z}^d}$ are called *asymptotic* if the set

$$\text{diff}(c_1, c_2) = \{\mathbf{u} \in \mathbb{Z}^d \mid c_1(\mathbf{u}) \neq c_2(\mathbf{u})\}$$

of cells where they differ is finite. A cellular automaton F is *pre-injective* if

$$F(c_1) \neq F(c_2)$$

for any distinct asymptotic configurations c_1 and c_2. Clearly, injective CA are pre-injective. The Garden-of-Eden theorem states that pre-injectivity of a CA is equivalent to surjectivity.

***Theorem* 2.4.6** (Garden-of-Eden theorem, [69; 71]). *A CA is surjective if and only if it is pre-injective.*

In the one-dimensional setting the Garden-of-Eden theorem yields the following corollary.

***Corollary* 2.4.7.** *For a one-dimensional surjective CA, every configuration has only a finite number of pre-images.*

2.5 Algebraic approach to multidimensional symbolic dynamics

Throughout this section R is a ring and M an R-module. By 1 we mean the multiplicative identity element of R and by 0 the zero element of R or M.

There is a one-to-one correspondence — a bijection — between the sets $M^{\mathbb{Z}^d}$ and $M[[X^{\pm 1}]]$. Indeed, one may represent any function $c \in M^{\mathbb{Z}^d}$ as a Laurent series

$$c(X) = \sum_{\mathbf{u} \in \mathbb{Z}^d} c_{\mathbf{u}} X^{\mathbf{u}}$$

and vice versa.

Convention. In the following and throughout this thesis by a polynomial we mean also a Laurent polynomial. When dealing with strictly proper polynomials we will use the term "proper". By a non-trivial polynomial we mean a non-zero polynomial.

We define the multiplication fc of a function $c \in M^{\mathbb{Z}^d}$ by a polynomial $f = f(X) \in R[X^{\pm 1}]$ to be the image of $f(X)c(X)$ under the bijection described above between the sets $M^{\mathbb{Z}^d}$ and $M[[X^{\pm 1}]]$.

Similarly as for Laurent series, we define the support of a function $c \in M^{\mathbb{Z}^d}$ as the set

$$\text{supp}(c) = \{\mathbf{u} \in \mathbb{Z}^d \mid c_{\mathbf{u}} \neq 0\}$$

and call c finitely supported if its support is a finite set. If one interprets the polynomial $f \in R[X^{\pm 1}]$ as a finitely supported function of $R^{\mathbb{Z}^d}$, then fc can also be seen as the *discrete convolution* $f * c$ of f and c.

We say that $f \in R[X^{\pm 1}]$ is an annihilator of $c \in M^{\mathbb{Z}^d}$ if f is an annihilator of $c(X)$. Naturally, we then define the annihilator ideal of a function $c \in M^{\mathbb{Z}^d}$ over R as

$$\text{Ann}_R(c) = \text{Ann}_R(c(X)).$$

Clearly, multiplying $c \in M^{\mathbb{Z}^d}$ by the monomial $X^{\mathbf{v}}$ corresponds to translating it by $\mathbf{v}$ and hence c is $\mathbf{v}$-periodic if and only if it is annihilated by the *difference polynomial* $X^{\mathbf{v}} - 1$. Let us state this as a lemma.

***Lemma* 2.5.1.** *A function $c \in M^{\mathbb{Z}^d}$ is $\mathbf{v}$-periodic if and only if it is annihilated by the polynomial $X^{\mathbf{v}} - 1$.*

Thus, the question whether a function $c \in M^{\mathbb{Z}^d}$ is periodic transforms into the question whether the annihilator ideal $\text{Ann}_R(c)$ contains a non-trivial difference polynomial.

A polynomial $f \in R[X^{\pm 1}]$ *periodizes* (or is a *periodizer* of) $c \in M^{\mathbb{Z}^d}$ if fc is strongly periodic. Clearly, c has a non-trivial annihilator if and only if it has a non-trivial periodizer. Indeed, any annihilator is a periodizer and conversely if c has a non-trivial periodizer f, then it has a non-trivial annihilator $(X^{\mathbf{v}} - 1)f$ for some $\mathbf{v} \neq \mathbf{0}$. The *periodizer ideal* of c over R is the set

$$\text{Per}_R(c) = \{f \in R[X^{\pm 1}] \mid fc \text{ is strongly periodic}\}$$

of all the periodizers of c over R.

Multiplying a configuration by a fixed polynomial $g \in R[X^{\pm 1}]$ defines a function between certain configuration spaces. This function is continuous:

***Lemma* 2.5.2.** *Let $g = \sum_{i=1}^n g_i X^{\mathbf{u}} \in R[X^{\pm 1}]$ be a polynomial, and let $\mathcal{A} \Subset M$ be a finite non-empty set. Let us define the set $\mathcal{B} = \{\sum_{i=1}^n g_i a_i \mid a_i \in \mathcal{A}\} \Subset M$. The function*

$$G \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{B}^{\mathbb{Z}^d}, c \mapsto gc$$

is continuous.

Proof. Clearly, G is a block map. Hence, G is continuous by the Curtis-Hedlund-Lyndon theorem. □

In particular, the above lemma says that for any converging sequence (c_i) of configurations over a fixed alphabet and a polynomial g we have

$$g \lim_{i\to\infty} c_i = \lim_{i\to\infty} gc_i.$$

This follows from the continuity of G.

2.5.1 Line polynomials and fibers

A *line polynomial* $f \in R[X^{\pm 1}]$ is a polynomial whose support contains at least two points and the points of the support are aligned on a line, that is, there exist vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ such that $\text{supp}(f) \subseteq \mathbf{u} + \mathbb{Q}\mathbf{v}$. This $\mathbf{v}$ (which is non-zero) is called a *direction* of f. We may say that f is a *line polynomial in direction* $\mathbf{v}$. Clearly, any line polynomial in direction $\mathbf{v}$ is also a line polynomial in any parallel non-zero direction $\mathbf{v}'$ over $\mathbb{Q}$.

For a non-zero vector $\mathbf{v} \in \mathbb{Q}^d$, we say that $c \in M^{\mathbb{Z}^d}$ is a $\mathbf{v}$*-fiber* if its support is contained in a line in direction $\mathbf{v}$, that is, if $\text{supp}(c) \subseteq \mathbf{u} + \mathbb{Q}\mathbf{v}$ for some $\mathbf{u} \in \mathbb{Z}^d$. We call any $\mathbf{v}$-fiber a *fiber*. Note that by interpreting polynomials as functions as discussed earlier, line polynomials are finitely supported fibers whose supports contain at least two points. Hence, fibers generalize the concept of line polynomials. By a $\mathbf{v}$*-fiber of* $c \in M^{\mathbb{Z}^d}$ we mean a function that agrees with c on $\mathbf{u} + \mathbb{Q}\mathbf{v}$ for some $\mathbf{u} \in \mathbb{Z}^d$ and gets value 0 elsewhere. Clearly, any $\mathbf{v}$-fiber of c is a $\mathbf{v}$-fiber.

If $c \in M^{\mathbb{Z}^d}$ is a configuration and R is a field, then the existence of a line polynomial annihilator of c implies that c is periodic. Indeed, the annihilation by a line polynomial in direction $\mathbf{v}$ implies a linear recurrence on every $\mathbf{v}$-fiber of c which due to the finiteness of $c(\mathbb{Z}^d)$ implies periodicity of c. More precisely, we may assume that c is annihilated by the line polynomial

$$f = a_0 + a_1 X^{\mathbf{v}} + \ldots + a_n X^{n\mathbf{v}}$$

where $n \geq 1$, $a_0, \ldots, a_n \in R$ and $a_0, a_n \neq 0$. Then

$$a_0 c_{\mathbf{u}} + a_1 c_{\mathbf{u}-\mathbf{v}} + \ldots + a_n c_{\mathbf{u}-n\mathbf{v}} = 0$$

for all $\mathbf{u} \in \mathbb{Z}^d$ since $fc = 0$. So, we have

$$c_{\mathbf{u}} = -a_0^{-1}(a_1 c_{\mathbf{u}-\mathbf{v}} + \ldots + a_n c_{\mathbf{u}-n\mathbf{v}}) \tag{1}$$

and

$$c_{\mathbf{u}-n\mathbf{v}} = -a_n^{-1}(a_0 c_{\mathbf{u}} + \ldots + a_{n-1} c_{\mathbf{u}-(n-1)\mathbf{v}}) \tag{2}$$

for all $\mathbf{u} \in \mathbb{Z}^d$. Since c is finitary, there exist $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{u} + \mathbb{Z}\mathbf{v}, \mathbf{u}_1 \neq \mathbf{u}_2$ such that

$$c \restriction_{\{\mathbf{u}_1, \mathbf{u}_1+\mathbf{v}, \ldots, \mathbf{u}_1+n\mathbf{v}\}} = c \restriction_{\{\mathbf{u}_2, \mathbf{u}_2+\mathbf{v}, \ldots, \mathbf{u}_2+n\mathbf{v}\}} .$$

It follows from Equations (1) and (2) that the fiber of c whose support is contained in $\mathbf{u} + \mathbb{Z}\mathbf{v}$ is periodic. Moreover, these periods are bounded and hence c is periodic in direction $\mathbf{v}$. In fact, since periodicity means annihilation by a difference polynomial, also the converse holds. So, we have the following lemma.

***Lemma* 2.5.3.** *Let R be a field and M an R-module. Let $c \in M^{\mathbb{Z}^d}$ be finitary, that is, a configuration. Then c is periodic if and only if c is annihilated by a line polynomial $f \in R[X^{\pm 1}]$.*

Remark 2.5.4. In particular, if $d = 1$ and R is a field, then annihilation of $c \in M^{\mathbb{Z}}$ by any non-trivial polynomial $f \in R[x^{\pm 1}]$ implies periodicity of c.

If R is not a field, then annihilation of a configuration $c \in M^{\mathbb{Z}^d}$ by a line polynomial does not imply periodicity. However, if the annihilating line polynomial is a difference polynomial, then we have periodicity for any ring as Lemma 2.5.1 says. Also, a non-finitary and non-periodic $c \in M^{\mathbb{Z}^d}$ may have a line polynomial annihilator. In dimension $d = 1$ this means that a non-finitary (and hence non-periodic) function $c \in M^{\mathbb{Z}}$ may have a non-trivial annihilator. In the following we give examples of these settings.

Example 2.5.5. Let $R = \mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ be the ring of integers modulo 4. Consider the configuration $c \in R^{\mathbb{Z}^d}$ defined such that $c_{\mathbf{0}} = \overline{2}$ and $c_{\mathbf{u}} = \overline{0}$ for all $\mathbf{u} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Clearly, c is non-periodic but has a non-trivial annihilator $f = \overline{2} + \overline{2}x_1$

Example 2.5.6. Consider the function $c \in \mathbb{Z}^{\mathbb{Z}}$ defined such that $c_i = 2^i$ for all $i \in \mathbb{Z}$. Now, c is not finitary and not periodic but has an annihilator $f = 1 - 2x$.

V-fibers

Let us generalize the concept of fibers to higher dimensional subspaces. Let $V \subseteq \mathbb{R}^d$ be a linear subspace of $\mathbb{R}^d$. A function $c \in M^{\mathbb{Z}^d}$ is called a *V-fiber* if

$$\text{supp}(c) \subseteq \mathbf{u} + V$$

for some $\mathbf{u} \in \mathbb{Z}^d$, that is, if the support of c is contained in some translate of V. Consequently, by a V-fiber of $c \in M^{\mathbb{Z}^d}$ we mean a V-fiber $e \in M^{\mathbb{Z}^d}$ whose support is contained in $\mathbf{u} + V$ for some $\mathbf{u}$ and

$$e \restriction_{\mathbf{u}+V} = c \restriction_{\mathbf{u}+V} .$$

Clearly, any fiber is a V-fiber where V is a 1-dimensional linear subspace of $\mathbb{R}^d$, that is, a line that goes through origin. More precisely, a $\mathbf{v}$-fiber is a $\langle \mathbf{v} \rangle$-fiber where $\langle \mathbf{v} \rangle = L(\mathbf{v})$ is the line in direction $\mathbf{v}$ through the origin.

A polynomial $f \in R[X^{\pm 1}]$ is a V-fiber if the corresponding finitely supported function of $R^{\mathbb{Z}^d}$ is a V-fiber. A *normal form* of a V-fiber polynomial $f \in R[X^{\pm 1}]$ is a polynomial of the form $X^{-\mathbf{u}} f$ such that $\text{supp}(f) - \mathbf{u} \subseteq V$. Let us denote by

$$\begin{aligned}\mathcal{N}_R(V) &= \{f \in R[X^{\pm 1}] \mid f \text{ is a normal form of a } V\text{-fiber}\} \\ &= \{f \in R[X^{\pm 1}] \mid \text{supp}(f) \subseteq V\}\end{aligned}$$

the set of all normal forms of V-fibers over R. It is a subring of $R[X^{\pm 1}]$.

A V-fiber of a polynomial

$$f = \sum_{\mathbf{v} \in \text{supp}(f)} f_{\mathbf{v}} X^{\mathbf{v}} \in R[X^{\pm 1}]$$

is a polynomial of the form

$$f \restriction_{\mathbf{u}+V} = \sum_{\mathbf{v} \in \text{supp}(f) \cap (\mathbf{u}+V)} f_{\mathbf{v}} X^{\mathbf{v}}$$

for some $\mathbf{u} \in \mathbb{Z}^d$, that is, the restriction of f to $\mathbf{u} + V$. By $\mathbf{v}$-fibers of a polynomial we mean its $\langle \mathbf{v} \rangle$-fibers. For a (Laurent) polynomial ideal $I \leq R[X^{\pm 1}]$, the set

$$\mathcal{N}_R(V, I) = \{f \in R[X^{\pm 1}] \mid f \text{ is a normal form of a } V\text{-fiber of some } f' \in I\}$$

of all the normal forms of the V-fibers of elements of I is an ideal of the ring $\mathcal{N}_R(V)$.

Example 2.5.7. Consider the polynomial

$$f = f(X) = f(x, y, z) = 1 + 2x + 3xy + z + xyz + z^4 + z^5$$

and the 2-dimensional subspace $V = \mathbb{R}^2 \times \{0\}$. The V-fibers of f are the polynomials $1 + 2x + 3xy$, $z + xyz$, z^4 and z^5. These have normal forms $1 + 2x + 3xy$, $1 + xy$, 1 and 1, respectively, for example.

2.5.2 Complex and integral configurations with annihilators

In the following we consider the setting with $R = M = \mathbb{C}$. Recall that a function $c \in \mathbb{C}^{\mathbb{Z}^d}$ is called integral if $c \in \mathbb{Z}^{\mathbb{Z}^d}$ and finitary if $c \in \mathcal{A}^{\mathbb{Z}^d}$ for some $\mathcal{A} \Subset \mathbb{C}$, that is, if its image set $c(\mathbb{Z}^d)$ is a finite set. Finitary functions are also called configurations.

Configurations with annihilators

First, the low complexity assumption implies the existence of annihilators. More precisely, we have the following result.

***Lemma* 2.5.8** ([55]). *Assume that a configuration $c \in \mathbb{C}^{\mathbb{Z}^d}$ has low complexity with respect to shape $D = \{\mathbf{d}_1, \ldots, \mathbf{d}_m\}$, that is, $P_c(D) \leq |D|$. Then c has a periodizer of the form*

$$a_1 X^{-\mathbf{d}_1} + \ldots + a_m X^{-\mathbf{d}_m}$$

for some non-zero $(a_1, \ldots, a_m) \in \mathbb{C}^m$.

The following lemma states that if an integral configuration has a non-trivial annihilator, then it has a non-trivial integral annihilator with the same support. The lemma was stated as a remark in [50].

***Lemma* 2.5.9** ([50]). *Let $c \in \mathbb{Z}^{\mathbb{Z}^d}$ be an integral configuration and assume that it has a non-trivial annihilator $f \in Ann_{\mathbb{C}}(c)$. Then it has a non-trivial integral annihilator $f' \in Ann_{\mathbb{Z}}(c)$ with $supp(f') = supp(f)$.*

The following theorem is crucial in our considerations. It states that if an integral configuration has a non-trivial integral annihilator, then it has an annihilator which is a product of difference polynomials. Together with Lemma 2.5.9 it states that if an integral configuration has any non-trivial annihilator, then it has an annihilator which is a product of difference polynomials.

***Theorem* 2.5.10** ([55]). *Let $c \in \mathbb{Z}^{\mathbb{Z}^d}$ be an integral configuration and assume that it has a non-trivial integral annihilator f. For all $\mathbf{u} \in supp(f)$, there exist pairwise non-parallel vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{Z}^d$ such that each $\mathbf{v}_i$ is parallel to $\mathbf{u}_i - \mathbf{u}$ for some $\mathbf{u}_i \in supp(f) \setminus \{\mathbf{u}\}$, and c is annihilated by the polynomial*

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1).$$

The proof of above theorem relies on Hilbert's Nullstellensatz. Let us give a short description of the proof since the ideas and tools of the proof are used later in this thesis. A *dilation* of a polynomial $f(X) \in \mathbb{C}[X^{\pm 1}]$ is a polynomial of the form $f(X^k)$ for some integer k. See Figure 3 for an illustration of dilations.

***Lemma* 2.5.11** (Dilation lemma [55]). *Let $c \in \mathcal{A}^{\mathbb{Z}^d}$ be an integral configuration and let f be a non-trivial integral annihilator of c. There exists a positive integer r such that for every positive integer k with $\gcd(k, r) = 1$ also $f(X^k)$ annihilates c.*

Let us call a number r that has the property of the above lemma a *dilation constant* of c with respect to f. From the proof of the dilation lemma we adapt the following result. For completeness, we provide a short description of the proof.

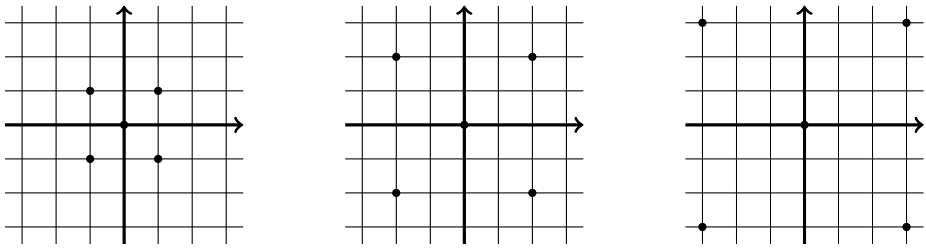

Figure 3. The supports of the polynomial $f(X) = 1 + x^{-1}y^{-1} + x^{-1}y^{1} + x^{1}y^{-1} + x^{1}y^{1}$ and its dilations $f(X^2)$ and $f(X^3)$.

***Lemma* 2.5.12** (Adapted from [55]). *Let $\mathcal{I}$ be an arbitrary index set, and let $\left(c^{(i)}\right)_{i\in\mathcal{I}}$ be a collection of integral configurations over the same alphabet $\mathcal{A} \Subset \mathbb{Z}$. If f is a non-trivial integral annihilator of $c^{(i)}$ for every $i \in \mathcal{I}$, then the configurations in the collection have a common dilation constant with respect to f.*

Proof. Let $c_{\max}$ be the maximum absolute value of the coefficients of the configurations $c^{(i)}$. Since the configurations $c^{(i)}$ are over the same alphabet, such number exists. Let $f = \sum_{\mathbf{v}\in\text{supp}(f)} f_{\mathbf{v}}X^{\mathbf{v}}$ and define $s = c_{\max}\sum_{\mathbf{v}\in\text{supp}(f)} |f_{\mathbf{v}}|$. In the proof of the dilation lemma in [56] it was shown that $r = s!$ is a dilation constant of any $c^{(i)}$. The claim follows. □

It is shown that if an ideal $I \leq \mathbb{C}[X^{\pm 1}]$ contains for some r the dilations $f(X^{1+kr})$ of a non-trivial integral polynomial $f(X)$ for all $k \in \mathbb{N}$, then

$$\prod_{\mathbf{v}\in\text{supp}(f)\setminus\{\mathbf{u}\}} (X^{r(\mathbf{v}-\mathbf{u})} - 1) \in \mathbf{IV}(I)$$

for any $\mathbf{u} \in \text{supp}(f)$. Combining this with the dilation lemma and Hilbert's Nullstellensatz, the following result is proved. Moreover, a simple argument is used to conclude that if $\varphi^m c = 0$ and φ is a line polynomial, then $\varphi c = 0$. In particular, in the proof of Theorem 2.5.10 the following result is proved.

***Theorem* 2.5.13** ([55]). *Let c be an integral configuration and f a non-trivial integral annihilator of c. If r is a dilation constant of c with respect to f, then for all $\mathbf{u} \in$ supp(f) the configuration c is annihilated by the polynomial*

$$\prod_{\mathbf{v}\in\mathit{supp}(f)\setminus\{\mathbf{u}\}} (X^{r(\mathbf{v}-\mathbf{u})} - 1).$$

Note that the polynomial in the above theorem depends only on f and the dilation constant r. Thus, if a family of configurations have a common annihilator f and a common dilation constant with respect to f, then they are all annihilated by the polynomial

$$\prod_{\mathbf{v}\in\text{supp}(f)\setminus\{\mathbf{u}\}} (X^{r(\mathbf{v}-\mathbf{u})} - 1).$$

from the above theorem for any $\mathbf{u} \in \text{supp}(f)$.

Periodic decomposition theorem

The multiplication of c by a difference polynomial can be thought as a "discrete derivation" of c. Theorem 2.5.10 says that if a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ has a non-trivial periodizer, then there is a sequence of derivations which annihilates c. So, by "integrating" step by step we have the following periodic decomposition theorem.

***Theorem* 2.5.14** (Periodic decomposition theorem [56]). *Let c be an integral configuration and assume that it has a non-trivial periodizer. There exist pairwise non-parallel vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ and functions $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{Z}^d}$ such that*

$$c = c_1 + \ldots + c_m$$

and each c_i is $\mathbf{v}_i$-periodic.

The crucial step in proving the above periodic decomposition theorem uses the following "integration lemma".

***Lemma* 2.5.15** (Integration lemma [56]). *Let φ and ψ be line polynomials in non-parallel directions. Assume that $c' \in \mathbb{C}^{\mathbb{Z}^d}$ is annihilated by ψ. Then there exists $c \in \mathbb{C}^{\mathbb{Z}^d}$ such that $\varphi c = c'$ and c is also annihilated by ψ. Moreover, if $c' \in \mathbb{Z}^{\mathbb{Z}^d}$, then we can choose also $c \in \mathbb{Z}^{\mathbb{Z}^d}$.*

Example 2.5.16. The periodic functions c_i in the periodic decomposition of the configuration c may not be configurations, that is, they may get infinitely many different values. Indeed, in [54] the authors considered the *snowflake configuration* $c \in \{0,1\}^{\mathbb{Z}^d}$ defined by

$$c(i,j) = \lfloor (i+j)\alpha \rfloor - \lfloor i\alpha \rfloor - \lfloor j\alpha \rfloor$$

for an irrational $\alpha \in \mathbb{R}$. It is the sum of periodic functions $c_1, c_2, c_3 \in \mathbb{Z}^{\mathbb{Z}^d}$ defined such that

- $c_1(i,j) = -\lfloor i\alpha \rfloor$,
- $c_2(i,j) = -\lfloor j\alpha \rfloor$ and
- $c_3(i,j) = \lfloor (i+j)\alpha \rfloor$.

These functions have periods $(0,1)$, $(1,0)$ and $(1,-1)$, respectively. Thus, the snowflake configuration c is annihilated by

$$(X^{(0,1)} - 1)(X^{(1,0)} - 1)(X^{(1,-1)} - 1).$$

In [54] it was shown that c cannot be expressed as a sum of finitely many periodic finitary functions.

There are also versions of the periodic decomposition theorem where the values of the finitely many periodic functions are bounded but not necessarily integers. For example, the following theorem was obtained using ultrafilter limits.

***Theorem* 2.5.17** (Bounded periodic decomposition theorem [86]). *Let c be an integral configuration and assume that it has a non-trivial periodizer. There exist real numbers $a < b$, pairwise non-parallel vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ and functions $c_1, \ldots, c_m \in [a, b]^{\mathbb{Z}^d}$ such that*

$$c = c_1 + \ldots + c_m$$

and each c_i is $\mathbf{v}_i$-periodic.

Variants of the above theorem have been obtained in studying translational tilings and the periodic tiling problem. For example, see Theorem 1.7 in [32] and Theorem 3.1 in [68].

2-dimensional setting

In the two-dimensional setting there are some more precise results. For example, in [56; 86] using known structure theorems concerning the prime ideals and radical ideals of the two-variate polynomial ring $\mathbb{C}[x, y]$ which are extended to the Laurent polynomial ring $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ it is shown that the annihilator ideal $\text{Ann}_{\mathbb{C}}(c)$ of a two-dimensional integral configuration c is a radical ideal. More precisely, the minimal decomposition of radical ideals and the quite simple structure of two-variate prime ideals are used. The following theorem is proved. It gives a description of the annihilator ideal of a two-dimensional integral configuration and a detailed periodic decomposition in the two-dimensional setting.

***Theorem* 2.5.18** (Corollary 4.2.1 in [86]). *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional integral configuration and assume that it has a non-trivial annihilator. Then there exist a non-negative integer m, line polynomials $\varphi_1, \ldots, \varphi_m$ in pairwise non-parallel directions and an ideal H which is an intersection of maximal ideals such that H and the principal ideal $\langle \varphi_1 \cdots \varphi_m \rangle$ are comaximal and*

$$Ann_{\mathbb{C}}(c) = \varphi_1 \cdots \varphi_m H$$

where m and the ideals H and $\langle \varphi_1 \cdots \varphi_m \rangle$ are determined uniquely. Moreover, there exist functions $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{Z}^2}$ and a strongly periodic configuration $c_H \in \mathbb{Z}^{\mathbb{Z}^2}$ such that

$$c = c_1 + \ldots + c_m + c_H$$

where $Ann_{\mathbb{C}}(c_i) = \langle \varphi_i \rangle$ and $Ann_{\mathbb{C}}(c_H) = H$.

In particular, the above theorem gives the following corollary which says that the periodizer ideal of a two-dimensional integral configuration is a principal ideal generated by a product of line polynomials in pairwise non-parallel directions. The corollary is not explicitly proved and hence we provide a short proof for it.

***Corollary* 2.5.19** ([49]). *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional integral configuration and assume that it has a non-trivial annihilator. Then there exist a non-negative integer m and line polynomials $\varphi_1, \ldots, \varphi_m$ in pairwise non-parallel directions such that*

$$\mathit{Per}_{\mathbb{C}}(c) = \langle \varphi_1 \cdots \varphi_m \rangle.$$

Proof. Let m and $\varphi_1, \ldots, \varphi_m$ be as in Theorem 2.5.18. So, $\varphi_1, \ldots, \varphi_m$ are line polynomials in pairwise non-parallel directions $\mathbf{v}_1, \ldots, \mathbf{v}_m$. By Theorem 2.5.18 we have

$$c = c_1 + \ldots + c_m + c_H$$

for some strongly periodic $c_H \in \mathbb{Z}^{\mathbb{Z}^2}$, and each c_i is annihilated by φ_i. Thus, $\varphi_1 \cdots \varphi_m c = \varphi_1 \cdots \varphi_m c_H$ is strongly periodic and hence

$$\langle \varphi_1 \cdots \varphi_m \rangle \subseteq \mathrm{Per}_{\mathbb{C}}(c).$$

Assume then that $f \in \mathrm{Per}_{\mathbb{C}}(c)$. Since fc is strongly periodic, there exists $\mathbf{v}$ which is pairwise non-parallel to each $\mathbf{v}_i$ such that fc is $\mathbf{v}$-periodic, that is, annihilated by $X^{\mathbf{v}} - 1$. In particular, $(X^{\mathbf{v}} - 1)f \in \mathrm{Ann}_{\mathbb{C}}(c)$. By Theorem 2.5.18 we have

$$\mathrm{Ann}_{\mathbb{C}}(c) = \varphi_1 \cdots \varphi_m H.$$

Hence, $\varphi_1 \cdots \varphi_m$ is a factor of $(X^{\mathbf{v}} - 1)f$. Since $\mathbf{v}$ is non-parallel to each $\mathbf{v}_i$, it follows that $\varphi_1 \cdots \varphi_m$ is a factor of f and hence $f \in \langle \varphi_1 \cdots \varphi_m \rangle$. Thus,

$$\mathrm{Per}_{\mathbb{C}}(c) \subseteq \langle \varphi_1 \cdots \varphi_m \rangle.$$

□

The number $m \in \mathbb{N}$ above is called the *order* of c. We denote it by $\mathrm{ord}(c)$. In the following we make some observations on the connection between the order of c and the periodicity of c.

- If $\mathrm{ord}(c) = 0$, then $\mathrm{Per}_{\mathbb{C}}(c) = \langle 1 \rangle = \mathbb{C}[X^{\pm 1}]$ and hence c is strongly periodic.
- If $\mathrm{ord}(c) = 1$, then $\mathrm{Per}_{\mathbb{C}}(c) = \langle \varphi \rangle$ is generated by a line polynomial φ which means that φc is strongly periodic. Let $\mathbf{v}$ be a direction of φ. Since φc is strongly periodic, it is periodic in direction $\mathbf{v}$ and hence it is annihilated by a line polynomial ψ in direction $\mathbf{v}$. Thus, c is annihilated by $\psi\varphi$ which is a line polynomial in direction $\mathbf{v}$ and hence c is periodic in direction $\mathbf{v}$. However, c cannot be periodic in a direction $\mathbf{v}'$ which is non-parallel with $\mathbf{v}$. Indeed, if c is periodic in direction $\mathbf{v}'$, then it is annihilated by a line polynomial φ' in

direction $\mathbf{v}'$. Since annihilators are periodizers, we have $\varphi' \in \text{Per}_{\mathbb{C}}(c) = \langle \varphi \rangle$. This is a contradiction. So, if $\text{ord}(c) = 1$, then c is periodic and all the periods of c are parallel.

- If $\text{ord}(c) \geq 2$, then c is non-periodic.

In fact, all the above implications are equivalences as proved in [86]:

***Theorem* 2.5.20** ([86])**.** *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional configuration with a non-trivial annihilator. Then*

- *ord$(c) = 0$ if and only if c is strongly periodic,*
- *ord$(c) = 1$ if and only if c is periodic and all the periods of c are parallel, and*
- *ord$(c) \geq 2$ if and only if c is non-periodic.*

In particular, we have the following corollary that gives a tool for studying forced periodicity of two-dimensional integral configurations.

***Corollary* 2.5.21.** *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional integral configuration and let f be a periodizer of c. Then the following conditions hold.*

- *If f does not have any line polynomial factors, then c is strongly periodic.*
- *If all line polynomial factors of f are in direction $\mathbf{v}$, then c is periodic in direction $\mathbf{v}$.*

Proof. If f has no line polynomial factors, then $\text{ord}(c) = 0$ and hence c is strongly periodic. If all line polynomial factors of f are in direction $\mathbf{v}$, then $\text{ord}(c) \in \{0, 1\}$ and hence c is periodic in direction $\mathbf{v}$. □

The above proof relies heavily on the structure of the ideal $\text{Ann}_{\mathbb{C}}(c)$ developed in [55]. We give an alternative proof that mimics the usage of resultants in [51].

Second proof of Corollary 2.5.21. The existence of a non-trivial periodizer f implies by Theorem 2.5.10 that c has a special annihilator $g = \varphi_1 \cdots \varphi_m$ which is a product of line polynomials $\varphi_1, \ldots, \varphi_m$ in pairwise non-parallel directions. All irreducible factors of g are line polynomials. If f does not have any line polynomial factors, then the periodizers f and g do not have common factors. We can assume that both are proper polynomials as they can be multiplied by a suitable monomial if needed. Since $f, g \in \text{Per}_{\mathbb{C}}(c)$, also $\text{Res}_x(f, g) \in \text{Per}_{\mathbb{C}}(c)$. This implies that c has a non-trivial annihilator containing only variable y since $\text{Res}_x(f, g) \neq 0$ because f and g have no common factors. This means that c is periodic in the vertical direction $(0, 1)$. Analogously, the *y-resultant* $\text{Res}_y(f, g)$ shows that c is horizontally periodic, that is, periodic in direction $(1, 0)$, and hence strongly periodic.

The proof for the case that all the line polynomial factors of f are in direction $\mathbf{v}$ goes analogously by considering φc instead of c, where φ is the greatest common line polynomial factor of f and g in direction $\mathbf{v}$. We get that φc is strongly periodic, implying that c is periodic in direction $\mathbf{v}$. □

General setting

Let us now consider d-dimensional integral configurations with non-trivial annihilators for arbitrary d. The following theorem gives a sufficient condition on forced periodicity of such configurations.

Theorem 2.5.22 ([50]). *Let $c \in \mathcal{A}^{\mathbb{Z}^d}$ be a d-dimensional integral configuration and assume that for every $(d-1)$-dimensional subspace $V \in \mathbb{G}_{d-1}$ the configuration c has a periodizer f such that supp$(f) \cap V = \{\mathbf{0}\}$. Then c is strongly periodic.*

The proof of the above theorem is based on showing that the condition implies that every $V \in \mathbb{G}_{d-1}$ is expansive for $\overline{\mathcal{O}(c)}$. Then Theorem 2.4.4 yields that $\overline{\mathcal{O}(c)}$ is finite and hence c is strongly periodic by Lemma 2.4.1.

Remark 2.5.23. The assumption in Theorem 2.5.22 that c has a periodizer f such that supp$(f) \cap V = \{\mathbf{0}\}$ is equivalent to having a periodizer f such that $|\text{supp}(f) \cap V| = 1$ since one can always multiply f by a suitable monomial.

Theorem 2.5.22 can be also formulated in terms of V-fibers. Indeed, the assumption of the theorem that c has a periodizer f such that supp$(f) \cap V = \{\mathbf{0}\}$ means that c has a periodizer f which has the constant polynomial 1 as its V-fiber. This implies that the ideal $\mathcal{N}_{\mathbb{C}}(V, \text{Per}_{\mathbb{C}}(c))$ of all the normal forms of V-fibers of the periodizer ideal $\text{Per}_{\mathbb{C}}(c)$ contains the monomial 1. Thus, the ideal $\mathcal{N}_{\mathbb{C}}(V, \text{Per}_{\mathbb{C}}(c))$ is the complete V-fiber ring $\mathcal{N}_{\mathbb{C}}(V)$. So, any normal form of a V-fiber is generated by the normal forms of V-fibers of periodizers of c.

Reformulation of Theorem 2.5.22 ([50]). *Let $c \in \mathcal{A}^{\mathbb{Z}^d}$ be a d-dimensional integral configuration and assume that for every $(d-1)$-dimensional subspace V it holds that $1 \in \mathcal{N}_{\mathbb{C}}(V, Per_{\mathbb{C}}(c))$, that is,*

$$\mathcal{N}_{\mathbb{C}}(V, Per_{\mathbb{C}}(c)) = \mathcal{N}_{\mathbb{C}}(V).$$

Then c is strongly periodic.

In dimension $d = 2$ the above assumption is equivalent to saying that there are normal forms of f with no common factors. Indeed, in the 2-dimensional setting the assumption is equivalent to saying that for all $\mathbf{v} \in \mathbb{Z}^2$, the normal forms of $\mathbf{v}$-fibers of periodizers of c generate the constant 1. By the weak Nullstellensatz this is equivalent to saying that the normal forms of $\mathbf{v}$-fibers of periodizers of c have no common zeros. One can view normal forms of $\mathbf{v}$-fibers as univariate polynomials, that is, elements of the univariate Laurent polynomial ring $\mathbb{C}[t^{\pm 1}]$ for a variable t. In fact, the set $\mathcal{N}_{\mathbb{C}}(\langle \mathbf{v} \rangle, \mathrm{Per}_{\mathbb{C}}(c))$ of all normal forms of $\mathbf{v}$-fibers of periodizers of c can be viewed as an ideal of the ring $\mathbb{C}[t^{\pm 1}]$. For a univariate polynomial ideal, having no common zeros is equivalent to having no common factors. Thus, for $d = 2$, the above statement is equivalent to the first part of Corollary 2.5.21.

3 Forced Periodicity of Perfect Colorings

In this chapter we consider a certain family of configurations with non-trivial annihilators. This chapter is based on the articles [41; 43]. We concentrate on settings where some conditions on the annihilators force periodicity. The main result of the chapter is Theorem 3.4.5 and the corollaries following it.

3.1 Grid graphs

We consider graphs that are *simple*, *undirected* and *connected*. A graph G that has vertex set V and edge set E is denoted by $G = (V, E)$. The (graphical) *distance* $d_G(u, v)$ of two vertices $u \in V$ and $v \in V$ of a graph $G = (V, E)$ is the length of a shortest path between them in G. The r-neighborhood of $u \in V$ in a graph $G = (V, E)$ is the set

$$N_r(u) = \{v \in V \mid d_G(v, u) \leq r\}$$

of all vertices within distance r from u.

The graphs we consider have vertex set $V = \mathbb{Z}^d$ and a translation invariant edge set $E \subseteq \{\{\mathbf{u}, \mathbf{v}\} \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}^d, \mathbf{u} \neq \mathbf{v}\}$. This means that for all r and for any two points $\mathbf{u} \in \mathbb{Z}^d$ and $\mathbf{v} \in \mathbb{Z}^d$ their r-neighborhoods are the same up to translation, that is, $N_r(\mathbf{u}) = N_r(\mathbf{v}) + \mathbf{u} - \mathbf{v}$. Moreover, we assume that all the vertices of G have only finitely many neighbors, *i.e.*, we assume that the *degree* of G is finite. We call these graphs (d-dimensional) *(infinite) grid graphs* or just *(infinite) grids*.

In a grid graph G, let us call the r-neighborhood of $\mathbf{0}$ the *relative r-neighborhood* of G. It determines the r-neighborhood of any vertex in G. Indeed, for all $\mathbf{u} \in \mathbb{Z}^d$ we have $N_r(\mathbf{u}) = N_r + \mathbf{u}$ where N_r is the relative r-neighborhood of G. Given the edge set of a grid graph, the relative r-neighborhood is determined for every r.

We specify three 2-dimensional infinite grid graphs:

- The (2-dimensional) *square grid* is the infinite grid graph $(\mathbb{Z}^2, E_{\mathcal{S}})$ with

$$E_{\mathcal{S}} = \{\{\mathbf{u}, \mathbf{v}\} \mid \mathbf{u} - \mathbf{v} \in \{(\pm 1, 0), (0, \pm 1)\}\}.$$

- The (2-dimensional) *triangular grid* is the infinite grid graph $(\mathbb{Z}^2, E_{\mathcal{T}})$ with

$$E_{\mathcal{T}} = \{\{\mathbf{u}, \mathbf{v}\} \mid \mathbf{u} - \mathbf{v} \in \{(\pm 1, 0), (0, \pm 1), (1, 1), (-1, -1)\}\}.$$

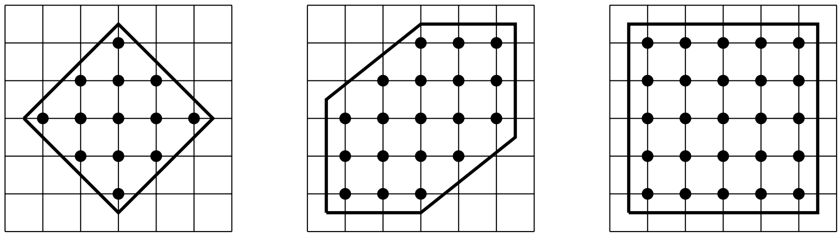

Figure 4. The relative 2-neighborhoods of the square grid, the triangular grid and the king grid, respectively.

- The (2-dimensional) *king grid* is the infinite grid graph $(\mathbb{Z}^2, E_{\mathcal{K}})$ with

$$E_{\mathcal{K}} = \{\{\mathbf{u}, \mathbf{v}\} \mid \mathbf{u} - \mathbf{v} \in \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}\}.$$

The relative 2-neighborhoods of these grid graphs are pictured in Figure 4.

3.2 Perfect colorings

Definition 3.2.1. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a finite alphabet of n colors and let $D \Subset \mathbb{Z}^d$ be a shape with $|D| \geq 2$. A configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ is a *perfect coloring with respect to* D or a D*-perfect coloring* if for all $i, j \in \{1, \ldots, n\}$ there exist numbers b_{ij} such that for all $\mathbf{u} \in \mathbb{Z}^d$ with $c_{\mathbf{u}} = a_j$ the number of occurrences of color a_i in the D-neighborhood of $\mathbf{u}$, *i.e.*, in the pattern $c \upharpoonright_{\mathbf{u}+D}$ is exactly b_{ij}.

The *matrix of a* D*-perfect coloring* c is the matrix $\mathbf{B} = (b_{ij})_{n \times n}$ where the numbers b_{ij} are as above. A D-perfect coloring with matrix $\mathbf{B}$ is called a (perfect) $(D, \mathbf{B})$*-coloring*.

Any D-perfect coloring is called simply a perfect coloring. In other words, a configuration is a perfect coloring if the number of cells of a given color in the given neighborhood of a vertex $\mathbf{u}$ depends only on the color of $\mathbf{u}$.

Perfect colorings are defined also for arbitrary graphs $G = (V, E)$. Again, let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a finite set of n colors. Let r be a non-negative integer. A vertex coloring $\varphi \colon V \to \mathcal{A}$ of G is an r-perfect coloring with matrix $\mathbf{B} = (b_{ij})_{n \times n}$ if the number of vertices of color a_i in the r-neighborhood of a vertex of color a_j is exactly b_{ij}. Clearly, if G is a translation invariant graph with vertex set $\mathbb{Z}^d$, then the r-perfect colorings of G are exactly the D-perfect colorings in $\mathcal{A}^{\mathbb{Z}^d}$ where D is the relative r-neighborhood of the graph G. The definition of perfect colorings of graphs is a special case of the definition of equitable partitions of graphs [27].

3.3 Forced periodicity of perfect coverings

In this section we consider forced periodicity of perfect colorings with only two colors. Without loss of generality we may assume that $\mathcal{A} = \{a_1, a_2\} = \{0, 1\}$

$(a_1 = 0, a_2 = 1)$ and consider perfect colorings $c \in \mathcal{A}^{\mathbb{Z}^d}$ since the names of the colors do not matter in our considerations.

So, let $c \in \{0,1\}^{\mathbb{Z}^d}$ be a perfect coloring with respect to $D \Subset \mathbb{Z}^d$ and let $\mathbf{B} = (b_{ij})_{2\times 2}$ be the matrix of c. Let us define a set

$$C = \{\mathbf{u} \in \mathbb{Z}^d \mid c_{\mathbf{u}} = 1\}.$$

This set has the property that the D-neighborhood $\mathbf{u} + D$ of a point $\mathbf{u} \in \mathbb{Z}^d$ contains exactly $a = b_{21}$ elements of C, *i.e.*, points with color 1 if $\mathbf{u} \notin C$ and exactly $b = b_{22}$ elements of C if $\mathbf{u} \in C$.

In fact, C is called a *perfect (multiple) covering* of the infinite grid G determined by the relative neighborhood D. More precisely, the set C is called a (perfect) (D, b, a)*-covering* of G. This is a variant of the following definition. See [2] for a reference.

Definition 3.3.1. In any graph, a subset C of its vertex set is an (r, b, a)*-covering* if the number of vertices of C in the r-neighborhood of a vertex u is a if $u \notin C$ and b if $u \in C$.

Clearly, for translation invariant graphs G there is a one-to-one correspondence between the (r, b, a)-coverings and the (D, b, a)-coverings where D is the relative r-neighborhood of the graph. Thus, it is natural to call any perfect coloring with only two colors a perfect covering. So, in the following a (D, b, a)-covering is considered as a D-perfect coloring with the matrix

$$\mathbf{B} = \begin{pmatrix} |D| - a & |D| - b \\ a & b \end{pmatrix}.$$

Recall for a shape $D \Subset \mathbb{Z}^d$, its characteristic polynomial

$$f_D = f_D(X) = \sum_{\mathbf{u} \in D} X^{-\mathbf{u}}.$$

We denote by $\mathbb{1}(X)$ the constant power series $\sum_{\mathbf{u}\in\mathbb{Z}^d} X^{\mathbf{u}}$, that is, the power series presentation of the identity map of $\mathbb{Z}^d$. If $c \in \{0,1\}^{\mathbb{Z}^d}$ is a (D, b, a)-covering, then from the definition we get that

$$f_D(X)c(X) = (b - a)c(X) + a\mathbb{1}(X)$$

which is equivalent to

$$(f_D(X) - (b - a))\, c(X) = a\mathbb{1}(X).$$

Thus, if c is a (D, b, a)-covering, then $f_D(X) - (b - a)$ is a periodizer of c. Note that since $|D| \geq 2$, this polynomial is always non-zero.

We study conditions that force every (D, b, a)-covering to be necessarily periodic. Note that in the 1-dimensional setting every (D, b, a)-covering is periodic since any (D, b, a)-covering has a non-trivial annihilator which implies periodicity in the 1-dimensional setting.

2-dimensional perfect coverings

In the following we have $d = 2$ and $X = (x, y)$. Hence, all our grid graphs are two-dimensional. By Corollary 2.5.21 the condition that the polynomial $f_D(x, y) - (b - a)$ has no line polynomial factors is a sufficient condition for forced periodicity of a 2-dimensional (D, b, a)-covering. Hence, we have the following corollary.

***Corollary* 3.3.2.** *Let $D \Subset \mathbb{Z}^2$ be a 2-dimensional shape and let a and b be non-negative integers. If $g = f_D - (b - a)$ has no line polynomial factors, then every (D, b, a)-covering is strongly periodic.*

Next, let us discuss the question how to determine whether a (Laurent) polynomial in two variables has line polynomial factors.

Finding the line polynomial factors of a given two-variate Laurent polynomial

For a vector $\mathbf{v} = (v_1, v_2)$, let us denote $\mathbf{v}^\perp = (v_2, -v_1)$ which is perpendicular to $\mathbf{v}$. See Figure 5 for an illustration. We say that a non-empty finite set $D \Subset \mathbb{Z}^2$ has an *outer edge* perpendicular to $\mathbf{v}$ if there exists a vector $\mathbf{t} \in \mathbb{Z}^2$ such that $D \subseteq \overline{H}_\mathbf{v} + \mathbf{t}$ and $|D \cap (\overline{H}_\mathbf{v} \setminus H_\mathbf{v} + \mathbf{t})| \geq 2$. In this case, we say that D has an outer edge in direction $\mathbf{v}^\perp$. The vector $\mathbf{v}^\perp$ is called an *outer edge direction* of D. We may call $D \cap (\overline{H}_\mathbf{v} \setminus H_\mathbf{v} + \mathbf{t})$ the outer edge of D in direction $\mathbf{v}^\perp$.

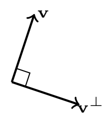

Figure 5. Vectors $\mathbf{v} = (1, 3)$ and $\mathbf{v}^\perp = (3, -1)$.

Remark 3.3.3. Note that D may have an outer edge in direction $\mathbf{u}$ but not in $-\mathbf{u}$. Note also that if D does not have an outer edge in direction $\mathbf{v}^\perp$, then there exists a vector $\mathbf{t} \in \mathbb{Z}^2$ such that $D \subseteq \overline{H}_\mathbf{v} + \mathbf{t}$ and $|D \cap (\overline{H}_\mathbf{v} \setminus H_\mathbf{v} + \mathbf{t})| = 1$. So, in this case D has a vertex in direction $\mathbf{v}$.

Example 3.3.4. Consider the set

$$D = \{(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (1, 3)\}.$$

It has outer edges directions $(-1, 0)$, $(0, 1)$, $(1, 1)$, and $(2, -3)$. The points $(0, 0)$, $(3, 0)$, $(0, 2)$, and $(1, 3)$ are its vertices. See Figure 6 for an illustration of D.

We say that a polynomial f has an outer edge in direction $\mathbf{v}^\perp$ if its support has an outer edge in direction $\mathbf{v}^\perp$. The following lemma says that a polynomial can have line polynomial factors only in directions of its outer edges.

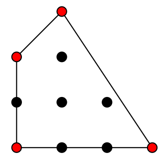

Figure 6. The set $D = \{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(0,2),(1,2),(1,3)\}$ and its outer edges. The red points are the vertices of D.

***Lemma* 3.3.5** ([51])**.** *Let f be a non-zero polynomial with a line polynomial factor in direction $\mathbf{w}$. Then f has outer edges in both directions $\mathbf{w}$ and $-\mathbf{w}$.*

Let $\mathbf{v} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ be a non-zero vector and let $f = \sum f_{\mathbf{u}} X^{\mathbf{u}} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be a polynomial. Recall that a $\mathbf{v}$-*fiber* of f is a polynomial of the form $f \upharpoonright_{\mathbf{u}+\langle \mathbf{v} \rangle}$ for some $\mathbf{u} \in \mathbb{Z}^2$. So, a non-zero $\mathbf{v}$-fiber of a polynomial is either a line polynomial or a monomial.

In the following we assume that $\mathbf{v} = (v_1, v_2)$ is primitive, that is, $\gcd(v_1, v_2) = 1$. Any line polynomial in direction $\mathbf{v}$ is of the form

$$\sum_{k \in \mathbb{Z}} a_k X^{\mathbf{u}+k\mathbf{v}}$$

where finitely many of the coefficients are non-zero. So, a normal form of a $\mathbf{v}$-fiber of any polynomial is of the form $\sum_{k \in \mathbb{Z}} a_k X^{k\mathbf{v}}$ where only finitely many of the coefficients are non-zero. Denote $t = X^{\mathbf{v}}$. The standard normal form of a line polynomial in direction $\mathbf{v}$ or a monomial ϕ is a univariate proper polynomial

$$a_0 + a_1 t + \ldots + a_n t^n \in \mathbb{C}[t]$$

such that

$$\phi = X^{\mathbf{u}}(a_0 + a_1 X^{\mathbf{v}} + \ldots + a_n X^{n\mathbf{v}}) = X^{\mathbf{u}}(a_0 + a_1 t + \ldots + a_n t^n)$$

for some $\mathbf{u} \in \mathbb{Z}^d$ where $n \geq 0, a_0 \neq 0, a_n \neq 0$. The standard normal form of a monomial $aX^{\mathbf{v}}$ is a. Let us denote by $\mathcal{F}_{\mathbf{v}}(f)$ the set of different standard normal forms of all non-zero $\mathbf{v}$-fibers of a polynomial f, which is hence a finite subset of $\mathbb{C}[t]$. The following simple example illustrates the concept of fibers and their standard normal forms.

Example 3.3.6. Let us determine the set $\mathcal{F}_{\mathbf{v}}(f)$ of all standard normal forms of the $\mathbf{v}$-fibers of a polynomial $f = f(X) = f(x,y) = 3x + y + xy^2 + xy + x^3y^3 + x^4y^4$ with $\mathbf{v} = (1,1)$. By grouping the terms we can write

$$\begin{aligned} f &= 3x + y(1 + xy) + xy(1 + x^2y^2 + x^3y^3) \\ &= X^{(1,0)} \cdot 3 + X^{(0,1)}(1 + t) + X^{(1,1)}(1 + t^2 + t^3) \end{aligned}$$

where $t = X^{(1,1)} = xy$. Hence, $\mathcal{F}_{\mathbf{v}}(f) = \{3, 1 + t, 1 + t^2 + t^3\}$. See Figure 7 for a pictorial illustration. □

Remark 3.3.7. Note that whenever we use the notation $\mathcal{F}_{\mathbf{v}}(f)$, we assume that $\mathbf{v}$ is a primitive vector since we defined standard normal forms only for primitive vectors. This is not a problem because for any $\mathbf{v} = (v_1, v_2)$, we may always consider the primitive vector $\mathbf{v}' = (v_1/\gcd(v_1, v_2), v_2/\gcd(v_1, v_2))$.

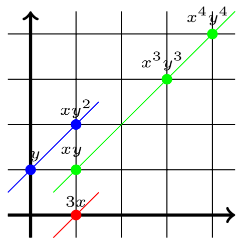

Figure 7. The support of $f = 3x + y + xy^2 + xy + x^3y^3 + x^4y^4$ and its different $(1, 1)$-fibers.

As noticed in the example above, polynomials are linear combinations of their fibers: for any polynomial f and any non-zero primitive vector $\mathbf{v}$ we can write

$$f = X^{\mathbf{u}_1}\psi_1 + \ldots + X^{\mathbf{u}_n}\psi_n$$

for some $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{Z}^2$ where $\psi_1, \ldots, \psi_n \in \mathcal{F}_{\mathbf{v}}(f)$. We use this in the proof of the next lemma.

***Lemma* 3.3.8.** *A polynomial f has a line polynomial factor in direction $\mathbf{v}$ if and only if the polynomials in $\mathcal{F}_{\mathbf{v}}(f)$ have a common factor.*

Proof. For any line polynomial ϕ in direction $\mathbf{v}$ and for any polynomial g, the $\mathbf{v}$-fibers of the product ϕg have a common factor ϕ. In other words, if a polynomial f has a line polynomial factor ϕ in direction $\mathbf{v}$, then the polynomials in $\mathcal{F}_{\mathbf{v}}(f)$ have the standard normal form of ϕ as a common factor.

For the converse direction, assume that the polynomials in $\mathcal{F}_{\mathbf{v}}(f)$ have a common factor ϕ. Then there exist vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{Z}^2$ and polynomials $\phi\psi_1, \ldots, \phi\psi_n \in \mathcal{F}_{\mathbf{v}}(f)$ such that

$$f = X^{\mathbf{u}_1}\phi\psi_1 + \ldots + X^{\mathbf{u}_n}\phi\psi_n.$$

Hence, ϕ is a line polynomial factor of f in direction $\mathbf{v}$. □

Note that Lemma 3.3.5 actually follows immediately from Lemma 3.3.8. Indeed, if f does not have an edge in direction $\mathbf{w}$ or $-\mathbf{w}$, then it has a non-zero monomial $\mathbf{w}$-fiber which implies that the polynomials in $\mathcal{F}_{\mathbf{w}}(f)$ have no common factors.

So, to find out the line polynomial factors of f we first need to find out the possible directions of the line polynomials, that is, a finite set E such that any outer edge direction of f is parallel to some vector of E. Then we need to check for which of these possible directions $\mathbf{v}$ the polynomials in $\mathcal{F}_{\mathbf{v}}(f)$ have a common factor. There are clearly algorithms to find the outer edge directions of a given polynomial and to determine whether finitely many polynomials have a common factor. If such a factor exists, then by Lemma 3.3.8 the polynomial f has a line polynomial factor in this direction. Thus, we have the following lemma.

***Lemma* 3.3.9.** *There is an algorithm to find the line polynomial factors of a given (Laurent) polynomial in two variables.*

Forced periodicity in 2-dimensional grid graphs

The following theorem by Axenovich states that "almost every" $(1, b, a)$-covering in the 2-dimensional square grid is strongly periodic.

***Theorem* 3.3.10** ([2])**.** *If $b - a \neq 1$, then every $(1, b, a)$-covering in the square grid is strongly periodic.*

Using our formulation and the algebraic approach we get a simple proof for this result:

Reformulation of Theorem 3.3.10. *Let D be the relative 1-neighborhood of the square grid and assume that $b - a \neq 1$. Then every (D, b, a)-covering is strongly periodic.*

Proof. Let c be an arbitrary (D, b, a)-covering. The outer edge directions of

$$g = f_D - (b - a) = x^{-1} + y^{-1} + 1 - (b - a) + x + y$$

are parallel to $(1, 1)$ or $(1, -1)$. Hence, by Lemma 3.3.5 any line polynomial factor of g is parallel to $(1, 1)$ or $(1, -1)$. For $\mathbf{v} \in \{(1, 1), (1, -1)\}$, we have $\mathcal{F}_{\mathbf{v}}(g) = \{1 + t, 1 - (b - a)\}$. See Figure 8 for an illustration. Since $1 - (b - a)$ is a non-trivial monomial, by Lemma 3.3.8 the periodizer $g \in \text{Per}(c)$ has no line polynomial factors and hence the claim follows by Corollary 3.3.2. □

We have a similar proof for the following known result concerning the forced periodicity of perfect coverings in the square grid with radius $r \geq 2$.

***Theorem* 3.3.11** ([75])**.** *Let $r \geq 2$ and let D be the relative r-neighborhood of the square grid. Then every (D, b, a)-covering is strongly periodic. In other words, all (r, b, a)-coverings in the square grid are strongly periodic for all $r \geq 2$.*

Proof. Let c be an arbitrary (D, b, a)-covering. By Lemma 3.3.5 any line polynomial factor of $g = f_D - (b - a)$ has direction $(1, 1)$ or $(1, -1)$. So, assume that $\mathbf{v} \in \{(1, 1), (1, -1)\}$. We have that

$$\phi_1 = 1 + t + \ldots + t^r \in \mathcal{F}_{\mathbf{v}}(g)$$

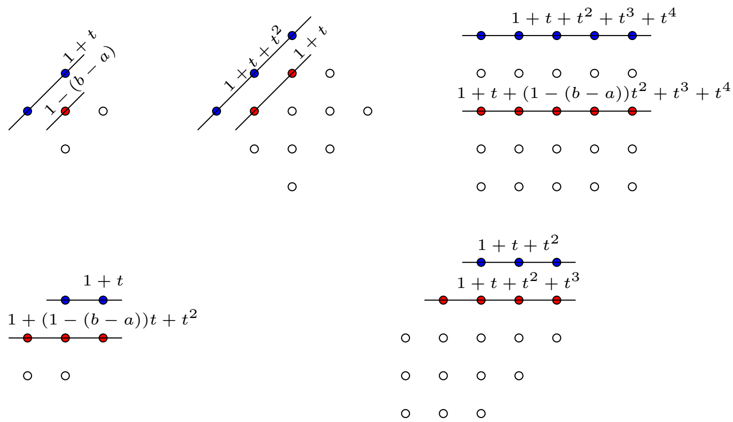

Figure 8. Pictorial illustrations for the proofs of Theorems 3.3.10, 3.3.11, 3.3.12, 3.3.13 and 3.3.14. The constellation on the left of the upper row illustrates the proof of Theorem 3.3.10. The constellation in the center of the upper row illustrates the proof of Theorem 3.3.11 with $r = 2$. The constellation on the right of the upper row illustrates the proof of Theorem 3.3.14 with $r = 2$. The constellation on the left of the lower row illustrates the proof of Theorem 3.3.12. The constellation on the right of the lower row illustrates the proof of Theorem 3.3.13 with $r = 2$. In each of the constellations we have pointed out two normal forms with no common factors in $\mathcal{F}_{\mathbf{v}}(g)$ from the points of $\text{supp}(g)$ for one of the outer edges $\mathbf{v}$ of $\text{supp}(g)$.

and

$$\phi_2 = 1 + t + \ldots + t^{r-1} \in \mathcal{F}_{\mathbf{v}}(g).$$

See Figure 8 for an illustration in the case $r = 2$. Since $\phi_1 - \phi_2 = t^r$, the polynomials ϕ_1 and ϕ_2 have no common factors, and hence by Lemma 3.3.8 the periodizer g has no line polynomial factors. Corollary 3.3.2 gives the claim. □

There are analogous results in the (2-dimensional) triangular grid, and we can prove them similarly with Corollary 3.3.2.

***Theorem* 3.3.12** ([75])**.** *Let D be the relative 1-neighborhood of the triangular grid and assume that $b - a \neq -1$. Then every (D, b, a)-covering in the triangular grid is strongly periodic. In other words, all $(1, b, a)$-coverings in the triangular grid are strongly periodic whenever $b - a \neq -1$.*

Proof. Let c be an arbitrary (D, b, a)-covering. The outer edge directions of

$$g = f_D - (b - a) = x^{-1}y^{-1} + x^{-1} + y^{-1} + 1 - (b - a) + x + y + xy$$

are parallel to $(1, 1)$, $(1, 0)$, or $(0, 1)$ and hence by Lemma 3.3.5 any line polynomial factor of g is parallel to one of these vectors. So, let $\mathbf{v} \in \{(1, 1), (1, 0), (0, 1)\}$. We have

$$\mathcal{F}_{\mathbf{v}}(g) = \{1 + t, 1 + (1 - (b - a))t + t^2\}.$$

See Figure 8 for an illustration. Polynomials $\phi_1 = 1+t$ and $\phi_2 = 1+(1-(b-a))t+t^2$ satisfy $\phi_1^2 - \phi_2 = (1+b-a)t$. Thus, they do not have any common factors if $b - a \neq -1$ and hence by Lemma 3.3.8 the polynomial g has no line polynomial factors. The claim follows by Corollary 3.3.2. □

***Theorem* 3.3.13** ([75]). *Let $r \geq 2$ and let D be the relative r-neighborhood of the triangular grid. Then every (D, b, a)-covering is strongly periodic. In other words, every (r, b, a)-covering in the triangular grid is strongly periodic for all $r \geq 2$.*

Proof. Let c be an arbitrary (D, b, a)-covering. The outer edge directions of $g = f_D - (b - a)$ are parallel to $(1, 1)$, $(1, 0)$, or $(0, 1)$ and hence by Lemma 3.3.5 any line polynomial factor of g is parallel to one of these vectors. So, let $\mathbf{v} \in \{(1, 1), (1, 0), (0, 1)\}$. There exists $n \geq 1$ such that $1 + t + \ldots + t^n \in \mathcal{F}_{\mathbf{v}}(g)$ and $1 + t + \ldots + t^{n+1} \in \mathcal{F}_{\mathbf{v}}(g)$. See Figure 8 for an illustration with $r = 2$. Since these two polynomials have no common factors, by Lemma 3.3.8 the polynomial g has no line polynomial factors. Again, Corollary 3.3.2 yields the claim. □

If $a \neq b$, then for all $r \geq 1$ any (r, b, a)-covering in the king grid is strongly periodic:

***Theorem* 3.3.14.** *Let $r \geq 1$ be arbitrary and let D be the relative r-neighborhood of the king grid and assume that $a \neq b$. Then any (D, b, a)-covering is strongly periodic. In other words, all (r, b, a)-coverings in the king grid are strongly periodic whenever $a \neq b$.*

Proof. Let c be an arbitrary (D, b, a)-covering. The outer edge directions of $g = f_D - (b - a)$ are parallel to $(1, 0)$ or $(0, 1)$ and hence by Lemma 3.3.5 any line polynomial factor of g is parallel to one of these vectors. Let $\mathbf{v} \in \{(1, 0), (0, 1)\}$. We have

$$\phi_1 = 1 + t + \ldots + t^{r-1} + (1 - (b - a))t^r + t^{r+1} + \ldots + t^{2r} \in \mathcal{F}_{\mathbf{v}}(g)$$

and

$$\phi_2 = 1 + t + \ldots + t^{2r} \in \mathcal{F}_{\mathbf{v}}(g).$$

See Figure 8 for an illustration in the case $r = 2$. Since $\phi_2 - \phi_1 = (b - a)t^r$ is a non-trivial monomial, ϕ_1 and ϕ_2 have no common factors. Thus, by Lemma 3.3.8 the polynomial g has no line polynomial factors and the claim follows by Corollary 3.3.2. □

In the above proofs we used the fact that two Laurent polynomials in one variable have no common factors if and only if they have no common zeros. By the weak Nullstellensatz this means that they generate the entire ring $\mathbb{C}[t^{\pm 1}]$, and they do this if and only if they generate a non-zero monomial.

Convex neighborhoods

Let $D \Subset \mathbb{Z}^2$ be a convex shape. Any (D, b, a)-covering has a periodizer

$$g = f_D - (b - a).$$

As earlier, we study whether g has any line polynomial factors, since if it does not, then Corollary 3.3.2 guarantees forced periodicity. For any primitive $\mathbf{v} \neq \mathbf{0}$, the set $\mathcal{F}_\mathbf{v}(f_D)$ contains only polynomials $\phi_n = \sum_{k=0}^{n-1} t^k = 1 + \ldots + t^{n-1}$ for different $n \geq 1$ since D is convex: if D contains two points, then D contains every point between them. Thus, $\mathcal{F}_\mathbf{v}(g)$ contains polynomials ϕ_n for different $n \geq 1$. If $b - a \neq 0$, it contains also one different polynomial. If $b - a = 0$, then $g = f_D$ and thus $\mathcal{F}_\mathbf{v}(g) = \mathcal{F}_\mathbf{v}(f_D)$.

We have $t^n - 1 = (t-1)\phi_n$ and $t^n - 1 = \prod_{1 \leq k \leq n}(t - e^{i \cdot \frac{2\pi k}{n}})$ where i is the imaginary unit. Thus, $\phi_n = \prod_{1 \leq k < n}(t - e^{i \cdot \frac{2\pi k}{n}})$ and hence polynomials ϕ_m and ϕ_n have a common factor if and only if $\gcd(m, n) > 1$. More generally, the polynomials $\phi_{n_1}, \ldots, \phi_{n_r}$ have a common factor if and only if $d = \gcd(n_1, \ldots, n_r) > 1$. Their greatest common factor is the dth *cyclotomic polynomial*

$$\prod_{\substack{1 \leq k \leq d \\ \gcd(k,d)=1}} (t - e^{i \cdot \frac{2\pi k}{d}}) \in \mathbb{C}[t].$$

Let us introduce the following notation. For any polynomial f, we denote by $\mathcal{F}'_\mathbf{v}(f)$ the set of the standard normal forms of the non-zero fibers

$$\sum_{k \in \mathbb{Z}} f_{\mathbf{u}+k\mathbf{v}} X^{\mathbf{u}+k\mathbf{v}}$$

of f for all $\mathbf{u} \notin \mathbb{Z}\mathbf{v}$. In other words, we exclude the fiber through the origin. Let us also denote $\text{fib}_\mathbf{v}(f)$ for the standard normal form of the fiber $\sum_{k \in \mathbb{Z}} f_{k\mathbf{v}} X^{k\mathbf{v}}$ through the origin. We have $\mathcal{F}_\mathbf{v}(f) = \mathcal{F}'_\mathbf{v}(f) \cup \{\text{fib}_\mathbf{v}(f)\}$ if $\text{fib}_\mathbf{v}(f) \neq 0$, and $\mathcal{F}_\mathbf{v}(f) = \mathcal{F}'_\mathbf{v}(f)$ if $\text{fib}_\mathbf{v}(f) = 0$.

Applying Corollary 3.3.2 and Lemma 3.3.8 we have the following theorem that gives sufficient conditions for every (D, b, a)-covering to be periodic for a convex D. This theorem generalizes the results proved above. In fact, they are corollaries of the theorem. The first part of the theorem was also mentioned in [26] in a slightly different context and in a more general form.

***Theorem* 3.3.15.** *Let $D \Subset \mathbb{Z}^2$ be a convex shape and let $g = f_D - (b - a)$ for non-negative integers a and b. Let $E \Subset \mathbb{Z}^2$ be a finite set of primitive vectors such that any outer edge direction of g is parallel to some element of E.*

- *Assume that $b - a = 0$. For any $\mathbf{v} \in E$, denote $d_\mathbf{v} = \gcd(n_1, \ldots, n_r)$ where $\mathcal{F}_\mathbf{v}(g) = \{\phi_{n_1}, \ldots, \phi_{n_r}\}$. If $d_\mathbf{v} = 1$ holds for all $\mathbf{v} \in E$, then every (D, b, a)-covering is strongly periodic. If all vectors $\mathbf{v}$ of E that do not satisfy $d_\mathbf{v} = 1$ are parallel, then every (D, b, a)-covering is periodic.*

- *Assume that $b - a \neq 0$. For any $\mathbf{v} \in E$, denote $d_{\mathbf{v}} = \gcd(n_1, \ldots, n_r)$ where $\mathcal{F}'_{\mathbf{v}}(g) = \{\phi_{n_1}, \ldots, \phi_{n_r}\}$. If the $d_{\mathbf{v}}$'th cyclotomic polynomial and fib$_{\mathbf{v}}(g)$ have no common factors for any $\mathbf{v} \in E$, then every (D, b, a)-covering is strongly periodic. If all vectors of E that do not satisfy the condition are parallel, then every (D, b, a)-covering is periodic. (Note that the condition is satisfied, in particular, if $d_{\mathbf{v}} = 1$.)*

Proof. By Corollary 3.3.2 a (D, b, a)-covering is strongly periodic if g has no line polynomial factors and periodic if all the line polynomial factors of g are in parallel directions. By Lemma 3.3.5 any line polynomial factor of g is in direction $\mathbf{v}$ for some $\mathbf{v} \in E$. Moreover, by Lemma 3.3.8 the polynomial g has a line polynomial factor in a primitive direction $\mathbf{v}$ if and only if the standard normal forms of the $\mathbf{v}$-fibers of g have no common factors. For any primitive $\mathbf{v}$, the set $\mathcal{F}_{\mathbf{v}}(g)$ of the standard normal forms of the $\mathbf{v}$-fibers of g is $\mathcal{F}_{\mathbf{v}}(g) = \mathcal{F}'_{\mathbf{v}}(g) = \{\phi_{n_1}, \ldots, \phi_{n_r}\}$ if $b - a = 0$ and $\mathcal{F}_{\mathbf{v}}(g) = \mathcal{F}'_{\mathbf{v}}(g) \cup \{\mathrm{fib}_{\mathbf{v}}(g)\}$ if $b - a \neq 0$. As mentioned earlier, the greatest common factor of the polynomials $\phi_{n_1}, \ldots, \phi_{n_r}$ is the $d_{\mathbf{v}}$th cyclotomic polynomial. The claim follows. □

Higher dimensions

For dimensions $d \geq 3$, we do not have very general results concerning forced periodicity of perfect colorings. However, applying Theorem 2.5.22 we have the following result giving a sufficient condition on forced strong periodicity.

***Theorem* 3.3.16.** *Let $D \Subset \mathbb{Z}^d$ be a shape, and let $g = f_D - (b - a)$ for non-negative integers a and b. Assume that for all $V \in \mathbb{G}_{d-1}$ the V-fibers of g generate a non-zero monomial, that is, there exist $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^d$ such that*

$$\mathit{supp}(X^{\mathbf{u}_1} g - X^{\mathbf{u}_2} g) \cap V = \{\mathbf{0}\}.$$

Then any (D, b, a)-covering is strongly periodic.

Proof. Let $c \in \{0, 1\}^{\mathbb{Z}^d}$ be an arbitrary (D, b, a)-covering. As seen earlier, g is a periodizer of c. The claim follows from the reformulation of Theorem 2.5.22. □

In the following example we apply the above theorem in the 3-dimensional king grid and generalize Theorem 3.3.14 to dimension $d = 3$.

Example 3.3.17. Let us consider the 3-dimensional king grid. It is the 3-dimensional grid graph with relative r-neighborhood $N_r = \{(i_1, i_2, i_3) \in \mathbb{Z}^3 \mid |i_1|, |i_2|, |i_3| \leq r\}$ — this is the cube of size $(2r + 1)^3$ centered at the origin. We consider (D, b, a)-coverings $c \in \{0, 1\}^{\mathbb{Z}^3}$ with $D = N_r$ for arbitrary positive integer r and non-negative integers b and a such that $b - a \neq 0$. Let us prove that every such c is strongly periodic. We prove this by considering the polynomial $g = f_D - (b - a)$ and by showing that for all $V \in \mathbb{G}_2$ the V-fibers of g generate a non-zero monomial. Then the claim follows from the above theorem.

Consider a 2-dimensional subspace $V \in \mathbb{G}_2$. It is a plane going through the origin. There exists $\mathbf{v} \in \mathbb{Z}^d$ such that $V = \overline{H_{\mathbf{v}}} \setminus H_{\mathbf{v}} = \langle \mathbf{v} \rangle^{\perp}$. Let us translate D with a vector $\mathbf{t}$ such that $D + \mathbf{t} \subseteq \overline{H_{\mathbf{v}}}$ and $(D + \mathbf{t}) \cap \overline{H_{\mathbf{v}}} \setminus H_{\mathbf{v}} \neq \emptyset$. We have three cases depending on the size of the intersection $E = (D + \mathbf{t}) \cap \overline{H_{\mathbf{v}}} \setminus H_{\mathbf{v}} = (D + \mathbf{t}) \cap V$. The intersection can be either a vertex of D, an edge of D, or a face of D. So, we have $|E| \in \{1, 2r + 1, (2r + 1)^2\}$.

- If the intersection E contains a single point $\mathbf{w}$, then $X^{\mathbf{w}}$ is a non-zero monomial V-fiber of g.

- Assume then that E is an edge of g. Consider a normal form

$$h = 1 + \ldots + X^{(2r+1)\mathbf{u}}$$

  of the V-fiber $g \restriction_E$ where $\mathbf{u} \in \{(1,0,0), (0,1,0), (0,0,1)\} \subseteq V$. The V-fiber of g going through the origin has a normal form

$$p + X^{\mathbf{u}_1} h + \ldots X^{\mathbf{u}_m} h$$

  for some $m \geq 0$ and $\mathbf{u}_1, \ldots, \mathbf{u}_m \in V$ where

$$p = 1 + \ldots + X^{(r-1)\mathbf{u}} + (1 - (b - a))X^{r\mathbf{u}} + X^{(r+1)\mathbf{u}} + \ldots + X^{(2r+1)\mathbf{u}}.$$

  Consequently, the V-fibers of g generate the non-trivial monomial

$$(b - a)X^{r\mathbf{u}}.$$

- Finally, assume that the intersection is a face of D. The V-fiber $g \restriction_E$ has a normal form

$$\sum_{k=1}^{2r+1} X^{\mathbf{u}_k} h$$

  for some $\mathbf{u}_1, \ldots, \mathbf{u}_{2r+1} \in V$ where h is as above. The V-fiber of g going through the origin has almost a same normal form except "in the middle" there is a term with coefficient $(1 - (b - a))$. Subtracting this from the above normal form of $g \restriction_E$ we conclude that the V-fibers of g again generate the non-trivial monomial $(b - a)$.

3.4 Forced periodicity of perfect colorings over arbitrarily large alphabets

In this section we consider perfect colorings over arbitrarily large alphabets.

To obtain annihilators and periodizers we rename the alphabet such that all the letters are integer vectors. Also in this setting, we consider multiplication of configurations by polynomials. The coefficients of the polynomials are $n \times n$ integer matrices, *i.e.*, elements of the ring $\mathbb{Z}^{n \times n}$. Since $\mathbb{Z}^n$ is a (left) $\mathbb{Z}^{n \times n}$-module where we consider the vectors of $\mathbb{Z}^n$ as column vectors, the product of a polynomial $f = f(X) \in \mathbb{Z}^{n \times n}[X^{\pm 1}]$ and a function $c \in (\mathbb{Z}^n)^{\mathbb{Z}^d}$ is well defined. So, in the following we have $M = \mathbb{Z}^n$ and $R = \mathbb{Z}^{n \times n}$. Let us call finitary functions of the set $(\mathbb{Z}^n)^{\mathbb{Z}^d}$ *integral vector configurations*.

There is a natural way to present configurations over arbitrary alphabets as integral vector configurations. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a finite alphabet with n elements. The *vector presentation* of a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ is the configuration $c' \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^{\mathbb{Z}^d}$ defined such that $c'_\mathbf{u} = \mathbf{e}_i$ if and only if $c_\mathbf{u} = a_i$. Clearly, c is $\mathbf{t}$-periodic if and only if its vector presentation is $\mathbf{t}$-periodic. Thus, to study the periodicity of a configuration we may as well study the periodicity of its vector presentation.

Remark 3.4.1. Strictly speaking, the vector presentation of a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ is not unique since it depends on the indexing of the alphabet $\mathcal{A} = \{a_1, \ldots, a_n\}$. However, in our considerations we assume that the indexing is fixed because it really does not matter and hence we talk about ***the*** vector presentation (instead of ***a*** vector presentation).

The ith *layer* of $c \in (\mathbb{Z}^n)^{\mathbb{Z}^d}$ is the function $\text{layer}_i(c) \in \mathbb{Z}^{\mathbb{Z}^d}$ defined such that $\text{layer}_i(c)_\mathbf{u} = c_\mathbf{u}^{(i)}$ where $c_\mathbf{u}^{(i)} \in \mathbb{Z}$ is the ith component of $\mathbf{c_u} \in \mathbb{Z}^n$. Clearly, $c \in (\mathbb{Z}^n)^{\mathbb{Z}^d}$ is periodic in direction $\mathbf{v}$ if and only if for all $i \in \{1, \ldots, n\}$ the ith layer of c is periodic in direction $\mathbf{v}$.

3.4.1 Annihilators of perfect colorings

We start by proving some lemmas that work in any dimension. We consider vector presentations of perfect colorings because this way we get a non-trivial annihilators for any such vector presentation as the following lemma shows.

***Lemma* 3.4.2.** *Let c be the vector presentation of a D-perfect coloring over an alphabet of size n with matrix $\mathbf{B} = (b_{ij})_{n \times n}$. Then c is annihilated by the polynomial*

$$f(X) = \sum_{\mathbf{u} \in D} \mathbf{I} X^{-\mathbf{u}} - \mathbf{B}.$$

Remark. Note the similarity of the above annihilator to the periodizer

$$\sum_{\mathbf{u} \in D} X^{-\mathbf{u}} - (b - a)$$

of a (D, b, a)-covering.

Proof. Let $\mathbf{v} \in \mathbb{Z}^d$ be arbitrary and assume that $c_{\mathbf{v}} = \mathbf{e}_j$. Then $(\mathbf{B}c)_{\mathbf{v}} = \mathbf{B}\mathbf{e}_j$ is the jth column of $\mathbf{B}$. On the other hand, from the definition of $\mathbf{B}$ we get

$$\left(\left(\sum_{\mathbf{u}\in D} \mathbf{I}X^{-\mathbf{u}}\right) c\right)_{\mathbf{v}} = \sum_{\mathbf{u}\in D} c_{\mathbf{v}+\mathbf{u}} = \sum_{i=1}^{n} b_{ij}\mathbf{e}_i$$

which is also the jth column of $\mathbf{B}$. Thus, $(fc)_{\mathbf{v}} = 0$ and hence $fc = 0$ since $\mathbf{v}$ was arbitrary. □

The following lemma shows that as in the case of integral configurations with non-trivial annihilators (Theorem 2.5.10), also the vector presentation of a perfect coloring has a special annihilator which is a product of difference polynomials. The proof has some similarities with the proof of Lemma 7 in [56]. By congruence of two polynomials with integer matrix coefficients (mod p) we mean that their corresponding coefficients are congruent (mod p) and by congruence of two integer matrices (mod p) we mean that their corresponding components are congruent (mod p).

***Lemma* 3.4.3.** *Let c be the vector presentation of a D-perfect coloring over an alphabet of size n with matrix $\mathbf{B} = (b_{ij})_{n\times n}$. Then c is annihilated by the polynomial*

$$g(X) = (\mathbf{I}X^{\mathbf{v}_1} - \mathbf{I})\cdots(\mathbf{I}X^{\mathbf{v}_m} - \mathbf{I})$$

for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

Proof. By Lemma 3.4.2 the vector presentation configuration c is annihilated by

$$f(X) = \sum_{\mathbf{u}\in D} \mathbf{I}X^{-\mathbf{u}} - \mathbf{B}.$$

Let $p > 2$ be a prime number larger than $|D|$. Since the coefficients of f commute with each other, we have for any positive integer k using the binomial theorem that

$$f^{p^k} = f^{p^k}(X) \equiv \sum_{\mathbf{u}\in D} \mathbf{I}X^{-p^k\mathbf{u}} - \mathbf{B}^{p^k} \pmod{p}.$$

We have $f^{p^k}(X)c(X) \equiv 0 \pmod{p}$. There are only finitely many distinct matrices $\mathbf{B}^{p^k} \pmod{p}$. So, let k and k' be distinct and such that $\mathbf{B}^{p^k} \equiv \mathbf{B}^{p^{k'}} \pmod{p}$. Then the coefficients of $f^{p^k} - f^{p^{k'}}$ are among $\mathbf{I}$ and $-\mathbf{I}$ when considering (mod p). Let us denote $f' = f'(X) = f(X^{p^k}) - f(X^{p^{k'}})$. Now, we have

$$f^{p^k}(X) - f^{p^{k'}}(X) \equiv f(X^{p^k}) - f(X^{p^{k'}}) = f' \pmod{p}.$$

Since $f^{p^k}c \equiv 0 \pmod{p}$ and $f^{p^{k'}}c \equiv 0 \pmod{p}$, also

$$(f^{p^k}(X) - f^{p^{k'}}(X))c \equiv 0 \pmod{p}$$

and hence

$$f'c \equiv 0 \pmod{p}.$$

The components of the configuration $f'c$ are bounded in absolute value by $|D|$. Since we chose p larger than $|D|$, this implies that

$$f'c = 0.$$

Because $f' = \sum_{\mathbf{u} \in P_1} \mathbf{I}X^{\mathbf{u}} - \sum_{\mathbf{u} \in P_2} \mathbf{I}X^{\mathbf{u}}$ for some finite subsets P_1 and P_2 of $\mathbb{Z}^d$, the annihilation of c by f' is equivalent to the annihilation of every layer of c by $f'' = \sum_{\mathbf{u} \in P_1} X^{\mathbf{u}} - \sum_{\mathbf{u} \in P_2} X^{\mathbf{u}}$. Thus, every layer of c has a non-trivial annihilator and hence by Theorem 2.5.10 every layer of c has a special annihilator which is a product of difference polynomials. Let

$$g' = (X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$$

be the product of all these special annihilators. Since g' annihilates every layer of c, the polynomial

$$g = (\mathbf{I}X^{\mathbf{v}_1} - \mathbf{I}) \cdots (\mathbf{I}X^{\mathbf{v}_m} - \mathbf{I})$$

annihilates c. □

The following technical lemma is useful.

***Lemma* 3.4.4.** *Let p be a prime and let H be an additive CA over $\mathbb{Z}_p^n$ determined by a polynomial $h = \sum_{i=0}^{k} \mathbf{A}_i X^{\mathbf{u}_i} \in \mathbb{Z}_p^{n \times n}[X^{\pm 1}]$ whose coefficients $\mathbf{A}_i$ commute with each other. (In other words, $H(c) = hc$.) Assume that there exist $M \in \mathbb{Z}_p \setminus \{0\}$ and matrices $\mathbf{C}_0, \ldots, \mathbf{C}_k \in \mathbb{Z}_p^{n \times n}$ that commute with each other and with every $\mathbf{A}_i$ such that*

$$\mathbf{C}_0\mathbf{A}_0 + \ldots + \mathbf{C}_k\mathbf{A}_k = M \cdot \mathbf{I}$$

holds in $\mathbb{Z}_p^{n \times n}$. Then H is surjective.

Proof. Assume the contrary that H is not surjective. By the Garden-of-Eden theorem (Theorem 2.4.6) H is not pre-injective and hence there exist two distinct asymptotic configurations c_1 and c_2 such that $H(c_1) = H(c_2)$, that is, $h(X)c_1(X) = h(X)c_2(X)$. Thus, h is an annihilator of $e = c_1 - c_2$. Without loss of generality we may assume that $c_1(\mathbf{0}) \neq c_2(\mathbf{0})$, *i.e.*, that $e(\mathbf{0}) = \mathbf{v} \neq \mathbf{0}$. Let l be such that the support $\text{supp}(e) = \{\mathbf{u} \in \mathbb{Z}^d \mid e(\mathbf{u}) \neq \mathbf{0}\}$ of e is contained in a d-dimensional $p^l \times \ldots \times p^l$ hypercube. Note that in $\mathbb{Z}_p^{n \times n}[X^{\pm 1}]$ we have

$$h^{p^l} = \sum_{i=0}^{k} \mathbf{A}_i^{p^l} X^{p^l \mathbf{u}_i}$$

which is also an annihilator of e. Hence, by the choice of l we have $\mathbf{A}_i^{p^l}\mathbf{v} = \mathbf{0}$ for each $i \in \{0, \dots, k\}$. By raising the identity

$$\mathbf{C}_0\mathbf{A}_0 + \dots + \mathbf{C}_k\mathbf{A}_k = M \cdot \mathbf{I}$$

to power p^l and multiplying the result by the vector $\mathbf{v}$ from the right we get

$$M^{p^l} \cdot \mathbf{v} = \mathbf{C}_0^{p^l}\mathbf{A}_0^{p^l}\mathbf{v} + \dots + \mathbf{C}_k^{p^l}\mathbf{A}_k^{p^l}\mathbf{v} = \mathbf{0} + \dots + \mathbf{0} = \mathbf{0}.$$

However, this is a contradiction because $M^{p^l}\mathbf{v} \neq \mathbf{0}$. Thus, H must be surjective as claimed. □

3.4.2 Forced periodicity of 2-dimensional perfect colorings

***Theorem* 3.4.5.** *Let $D \Subset \mathbb{Z}^2$ be a shape and assume that there exists an integer t_0 such that the polynomial $f_D - t = \sum_{\mathbf{u} \in D} X^{-\mathbf{u}} - t$ has no line polynomial factors whenever $t \neq t_0$. Then any D-perfect coloring with matrix $\mathbf{B}$ is strongly periodic whenever* $\det(\mathbf{B} - t_0\mathbf{I}) \neq 0$. *If $f_D - t$ has no line polynomial factors for any t, then every D-perfect coloring is strongly periodic.*

Proof. Let c be the vector presentation of a D-perfect coloring with matrix $\mathbf{B}$. By Lemmas 3.4.2 and 3.4.3 it has two distinct annihilators:

$$f = \sum_{\mathbf{u} \in D} \mathbf{I}X^{-\mathbf{u}} - \mathbf{B}$$

and

$$g = (\mathbf{I}X^{\mathbf{v}_1} - \mathbf{I}) \cdots (\mathbf{I}X^{\mathbf{v}_m} - \mathbf{I}).$$

Let us replace $\mathbf{I}$ by 1 and $\mathbf{B}$ by a variable t and consider the corresponding integral polynomials $f' = \sum_{\mathbf{u} \in D} X^{-\mathbf{u}} - t = f_D - t$ and $g' = (X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$ in $\mathbb{C}[x, y, t]$. Here $X = (x, y)$.

Without loss of generality we may assume that f' and g' are proper polynomials. Indeed, we can multiply f' and g' by monomials such that the obtained polynomials f'' and g'' are proper polynomials and that they have a common factor if and only if f' and g' have a common factor. So, we may consider f'' and g'' instead of f' and g' if they are not proper polynomials.

We consider the y-resultant $\text{Res}_y(f', g')$ of f' and g', and write

$$\text{Res}_y(f', g') = f_0(t) + f_1(t)x + \dots + f_k(t)x^k.$$

By the properties of resultants, $\text{Res}_y(f', g')$ is in the ideal generated by f' and g', and it can be the zero polynomial only if f' and g' have a common factor. Since g' is a product of line polynomials, any common factor of f' and g' is also a product of line polynomials. In particular, if f' and g' have a common factor, then they have a common line polynomial factor. However, by the assumption f' has no line polynomial factors if $t \neq t_0$. Thus, f' and g' may have a common factor only if $t = t_0$ and hence $\text{Res}_y(f', g')$ can be zero only if $t = t_0$. On the other hand, $\text{Res}_y(f', g') = 0$ if and only if $f_0(t) = \ldots = f_k(t) = 0$. We conclude that $\gcd(f_0(t), \ldots, f_k(t)) = (t - t_0)^m$ for some $m \geq 0$. Thus,

$$\text{Res}_y(f', g') = (t - t_0)^m (f'_0(t) + f'_1(t)x + \ldots + f'_k(t)x^k)$$

where the polynomials $f'_0(t), \ldots, f'_k(t)$ have no common factors.

By the Euclidean algorithm there are polynomials $a_0(t), \ldots, a_k(t)$ such that

$$a_0(t)f'_0(t) + \ldots + a_k(t)f'_k(t) = 1. \tag{3}$$

Moreover, the coefficients of the polynomials $a_0(t), \ldots, a_k(t)$ are rational numbers because the polynomials $f'_0(t), \ldots, f'_k(t)$ are integral. Note that if f' has no line polynomial factors for any t, then $m = 0$ and hence $f'_i(t) = f_i(t)$ for every $i \in \{1, \ldots, k\}$.

Let us now consider the polynomial

$$(\mathbf{B} - t_0\mathbf{I})^m (f'_0(\mathbf{B}) + f'_1(\mathbf{B})x + \ldots + f'_k(\mathbf{B})x^k)$$

which is obtained from $\text{Res}_y(f', g')$ by plugging back $\mathbf{I}$ and $\mathbf{B}$ in the place of 1 and t, respectively. Since $\text{Res}_y(f', g')$ is in the ideal generated by f' and g', the above polynomial is in the ideal generated by f and g. Thus, it is an annihilator of c because both f and g are annihilators of c.

Assume that $\det(\mathbf{B} - t_0\mathbf{I}) \neq 0$ or that $m = 0$. Now also

$$h = f'_0(\mathbf{B}) + f'_1(\mathbf{B})x + \ldots + f'_k(\mathbf{B})x^k$$

is an annihilator of c. Since $f'_0(t), \ldots, f'_k(t)$ have no common factors, h is non-zero, because otherwise it would be $f'_0(\mathbf{B}) = \ldots = f'_k(\mathbf{B}) = 0$ and the minimal polynomial of $\mathbf{B}$ would be a common factor of $f'_0(t), \ldots, f'_k(t)$, a contradiction.

Plugging $t = \mathbf{B}$ to Equation (3) we get

$$a_0(\mathbf{B})f'_0(\mathbf{B}) + \ldots + a_k(\mathbf{B})f'_k(\mathbf{B}) = \mathbf{I}.$$

Let us multiply the above equation by a common multiple M of all the denominators of the rational numbers appearing in the equation and let us consider it (mod p) where p is a prime that does not divide M. We obtain the following identity

$$a'_0(\mathbf{B})f'_0(\mathbf{B}) + \ldots + a'_k(\mathbf{B})f'_k(\mathbf{B}) = M \cdot \mathbf{I} \not\equiv 0 \pmod{p}$$

where all the coefficients in the equation are integer matrices.

By Lemma 3.4.4 the additive CA determined by $h = \sum_{i=0}^{k} f_i'(\mathbf{B})x^i$ is surjective. Since h is a polynomial in variable x only, it defines a 1-dimensional CA H which is surjective and which maps every horizontal fiber of c to 0. Hence, every horizontal fiber of c is a pre-image of 0. Let c' be a horizontal fiber of c. The Garden-of-Eden theorem implies that 0 has finitely many, say N, pre-images under H. Since also every translation of c' is a pre-image of 0, we conclude that $c' = \tau^i(c')$ for some $i \in \{0, \ldots, N-1\}$. Thus, $(N-1)!$ is a common period of all the horizontal fibers of c and hence c is horizontally periodic.

Repeating the same argumentation for the x-resultant of f' and g', we can show that c is also vertically periodic. Thus, c is strongly periodic. □

3.4.3 Forced periodicity of perfect colorings of 2-dimensional grid graphs

As corollaries of Theorem 3.4.5 and theorems from the previous section, we obtain new proofs for known results of forced periodicity of perfect colorings in the square and the triangular grids, and a new result for forced periodicity of perfect colorings in the king grid.

***Corollary* 3.4.6** ([75])**.** *Let D be the relative 1-neighborhood of the square grid. Then any D-perfect coloring with matrix $\mathbf{B}$ is strongly periodic whenever $\det(\mathbf{B} - \mathbf{I}) \neq 0$. In other words, any 1-perfect coloring with matrix $\mathbf{B}$ in the square grid is strongly periodic whenever $\det(\mathbf{B} - \mathbf{I}) \neq 0$.*

Proof. In our proof of the reformulation of Theorem 3.3.10 it was shown that the polynomial $f_D - t$ has no line polynomial factors if $t \neq 1$. Thus, by Theorem 3.4.5 any $(D, \mathbf{B})$-coloring is strongly periodic whenever $\det(\mathbf{B} - \mathbf{I}) \neq 0$. □

***Corollary* 3.4.7** ([75])**.** *Let D be the relative 1-neighborhood of the triangular grid. Then any D-perfect coloring with matrix $\mathbf{B}$ is strongly periodic whenever $\det(\mathbf{B} + \mathbf{I}) \neq 0$. In other words, any 1-perfect coloring with matrix $\mathbf{B}$ in the triangular grid is strongly periodic whenever $\det(\mathbf{B} + \mathbf{I}) \neq 0$.*

Proof. In the proof of Theorem 3.3.12 it was shown that the polynomial $f_D - t$ has no line polynomial factors if $t \neq -1$. Thus, by Theorem 3.4.5 any $(D, \mathbf{B})$-coloring is strongly periodic whenever $\det(\mathbf{B} + \mathbf{I}) \neq 0$. □

***Corollary* 3.4.8** ([75])**.** *Let $r \geq 2$ and let D be the relative r-neighborhood of the square grid. Then every D-perfect coloring is strongly periodic. In other words, any r-perfect coloring in the square grid is strongly periodic for all $r \geq 2$.*

Proof. In the proof of Theorem 3.3.11 it was shown that the polynomial $f_D - t$ has no line polynomial factors for any t. Thus, by Theorem 3.4.5 every D-perfect coloring is strongly periodic. □

***Corollary* 3.4.9** ([75]). *Let $r \geq 2$ and let D be the relative r-neighborhood of the triangular grid. Then every D-perfect coloring is strongly periodic. In other words, any r-perfect coloring in the triangular grid is strongly periodic for all $r \geq 2$.*

Proof. In the proof of Theorem 3.3.13 it was shown that the polynomial $f_D - t$ has no line polynomial factors for any t. Thus, by Theorem 3.4.5 every D-perfect coloring is strongly periodic. □

***Corollary* 3.4.10.** *Let $r \geq 1$ and let D be the relative r-neighborhood of the king grid. Then every D-perfect coloring with matrix $\mathbf{B}$ is strongly periodic whenever* $\det(\mathbf{B}) \neq 0$. *In other words, every r-perfect coloring with matrix $\mathbf{B}$ in the king grid is strongly periodic whenever* $\det(\mathbf{B}) \neq 0$.

Proof. In the proof of Theorem 3.3.14 we showed that the polynomial $f_D - t$ has no line polynomial factors if $t \neq 0$. Thus, by Theorem 3.4.5 any $(D, \mathbf{B})$-coloring is strongly periodic whenever $\det(\mathbf{B}) \neq 0$. □

Remark 3.4.11. The results in Corollaries 3.4.6, 3.4.7, 3.4.8 and 3.4.9 were originally stated and proved in [75] in a slightly more general form. Indeed, in [75] it was proved that if a configuration $c \in \mathcal{A}^{\mathbb{Z}^2}$ is annihilated by

$$\sum_{\mathbf{u} \in D} \mathbf{I} X^{-\mathbf{u}} - \mathbf{B}$$

where $\mathbf{B} \in \mathbb{Z}^{n \times n}$ is an arbitrary integer matrix whose determinant satisfies the conditions in the four corollaries, and D is as in the corollaries, then c is necessarily periodic. This kind of configuration was called a *generalized centered function*. However, in Lemma 3.4.2 we proved that the vector presentation of any D-perfect coloring with matrix $\mathbf{B}$ is annihilated by this polynomial, that is, we proved that the vector presentation of a perfect coloring is a generalized centered function. By analyzing the proof of Theorem 3.4.5 we see that the theorem holds also for generalized centered functions and hence the corollaries following it hold also for generalized centered functions, and thus we have the same results as in [75].

3.5 Forced periodicity of configurations of low abelian complexity

Abelian complexity

Let us recall the definition of *abelian complexity*. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a fixed alphabet and $D \Subset \mathbb{Z}^d$ a shape. For $a \in \mathcal{A}$ and a D-pattern $p \in \mathcal{A}^D$, we denote by $|p|_a$ the number of occurrences of symbol a in p. The *Parikh vector* of $p \in \mathcal{A}^D$ is the length n vector $parikh(p) = (|p|_{a_1}, \ldots, |p|_{a_n})$. In other words, the ith entry of the Parikh vector of a pattern tells the number of occurences of a_i in p. Two D-patterns are abelian equivalent if their Parikh vectors are equal, that is, if they contain the same number of each symbol. The abelian complexity of a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ with respect to D is defined as

$$A_c(D) = |\{parikh(p) \mid p \in \mathcal{L}_D(c)\}|.$$

So, the abelian complexity $A_c(D)$ gives the number of distinct D-patterns of c up to abelian equivalence. We say that a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ has *low abelian complexity* with respect to D if $A_c(D) = 1$. As in the case of low pattern complexity, we have non-trivial annihilator and periodizers in the low abelian complexity setting:

***Lemma* 3.5.1.** *Let $c \in \mathcal{A}^{\mathbb{Z}^d}$ be a configuration over $\mathcal{A} = \{a_1, \ldots, a_n\} \Subset \mathbb{C}$ and assume that it has low abelian complexity with respect to $D = \{\mathbf{d}_1, \ldots, \mathbf{d}_m\}$. Then it is periodized by $f_D = \sum_{i=1}^{m} X^{-\mathbf{d}_i}$.*

Proof. Let $(p_1, \ldots, p_n)$ be the unique Parikh vector of the D-patterns of c. Now, we have

$$(f_D c)_{\mathbf{u}} = \sum_{i=1}^{m} c_{\mathbf{u}+\mathbf{d}_i} = p_1 a_1 + \ldots + p_n a_n$$

for any $\mathbf{u} \in \mathbb{Z}^d$. Thus, $f_D c$ is strongly periodic and hence f_D is a periodizer of c. □

Remark 3.5.2. If a configuration c has low abelian complexity with respect to D, then it is a D-perfect coloring. Conversely, any D-perfect coloring over an alphabet of size n has $A_c(D) \leq n$.

Example 3.5.3. Recall that a co-tiler of a tile, that is, a shape $D \Subset \mathbb{Z}^d$ is a binary configuration c such that

$$\left(\sum_{\mathbf{v} \in D} X^{\mathbf{v}}\right) c = f_{-D} c = \sum_{\mathbf{u} \in \mathbb{Z}^d} X^{\mathbf{u}} = \mathbb{1}$$

and hence any $(-D)$-pattern of c contains exactly one symbol 1. So, any co-tiler of a tile D has low abelian complexity with respect to $-D$. In fact, as mentioned in the introduction and as proved in [87], any co-tiler of D is also a co-tiler of $-D$ and hence c has also low abelian complexity with respect to D.

Let $c \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^{\mathbb{Z}^d}$ and let $D \Subset \mathbb{Z}^d$ be a shape. Consider the polynomial

$$f = \mathbf{I} \cdot f_D(X) = \sum_{\mathbf{u} \in D} \mathbf{I} X^{-\mathbf{u}} \in \mathbb{Z}^{n \times n}[X^{\pm 1}].$$

The ith entry of

$$(fc)_{\mathbf{v}} = \sum_{\mathbf{u} \in D} \mathbf{I} \cdot \mathbf{c}_{\mathbf{v}+\mathbf{u}}$$

tells the number of cells of color $\mathbf{e}_i$ in the D-neighborhood of $\mathbf{v}$ in c. Thus, $(fc)_{\mathbf{v}}$ is the parikh vector of a D-pattern of c and hence the abelian complexity of c with respect to D is exactly the number of distinct coefficients of fc.

We define the abelian complexity $A_c(f)$ of an integral vector configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$, where $\mathcal{A}$ is a finite set of length n integer vectors, *with respect to a polynomial* $f \in \mathbb{Z}^{n\times n}[X^{\pm 1}]$ as

$$A_c(f) = |\{(fc)_{\mathbf{v}} \mid \mathbf{v} \in \mathbb{Z}^d\}|.$$

We extend the above definition for configurations and polynomials. We define the abelian complexity $A_c(f)$ of a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$, where $\mathcal{A}$ is any finite non-empty set, with respect to a polynomial $f = \sum f_i X^{\mathbf{u}_i} \in \mathbb{C}[X^{\pm 1}]$ to be the abelian complexity $A_{c'}(f')$ of the vector presentation c' of c with respect to the polynomial

$$f' = \mathbf{I} \cdot f = \sum f_i \cdot \mathbf{I} \cdot X^{\mathbf{u}_i}.$$

Consequently, we say that c has low abelian complexity with respect to a polynomial f if $A_c(f) = 1$.

Remark 3.5.4. Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ and $f = \sum f_{\mathbf{u}} X^{\mathbf{u}}$. A natural interpretation of the abelian complexity $A_c(f)$ is that it gives the number of different linear combinations

$$\sum f_i a_i \in \mathbb{Z}[a_1, \ldots, a_n]$$

of $(-\mathrm{supp}(f))$-patterns in c.

Clearly, the definition of abelian complexity with respect to polynomials is consistent with the definition of abelian complexity with respect to shapes. Indeed, for a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ over any finite alphabet $\mathcal{A}$, the abelian complexity of c with respect to D is the abelian complexity of c with respect to the polynomial f_D. So, we have $A_c(D) = A_c(f_D)$ for any shape.

Forced periodicity

Note that a configuration of low abelian complexity is not necessarily periodic as mentioned in the introduction. However, in [26] it was shown that if $A_c(f) = 1$ for a two-dimensional configuration c and if the polynomial f has no line polynomial factors, then c is strongly periodic assuming that the support of f is convex. The following theorem strengthens this result and shows that the convexity assumption of the support of the polynomial is not needed. We obtain this result as a direct corollary of Corollary 2.5.21.

***Theorem* 3.5.5.** *Let c be a two-dimensional configuration over an alphabet of size n and assume that it has low abelian complexity with respect to a polynomial $f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$. If f has no line polynomial factors, then c is strongly periodic. If all* *the line polynomial factors of f are in direction $\mathbf{v}$ for some $\mathbf{v}$, then c is periodic in direction $\mathbf{v}$. Thus, if f_D has no line polynomial factors or its line polynomial factors are in parallel directions, then any configuration that has low abelian complexity with respect to D is strongly periodic or periodic, respectively.*

Proof. By the assumption that $A_c(f) = 1$ we have $f'c' = \mathbf{c}_0\mathbb{1}$ for some $\mathbf{c}_0 \in \mathbb{Z}^n$ where c' is the vector presentation of c and $f' = \mathbf{I} \cdot f$. Thus, f periodizes every layer of c'. If f has no line polynomial factors, then by Corollary 2.5.21 every layer of c' is strongly periodic and hence c' is strongly periodic. If all the line polynomial factors of f are in direction $\mathbf{v}$, then by the same corollary every layer of c' is periodic in direction $\mathbf{v}$ and hence also c' is periodic in direction $\mathbf{v}$. Since c is periodic if and only if its vector presentation c' is periodic, the claim follows. □

Remark 3.5.6. In [26] a polynomial $f \in \mathbb{Z}[X^{\pm 1}]$ is called abelian rigid if any configuration c that has low abelian complexity with respect to f is necessarily strongly periodic. In the above theorem we proved that if a two-variate polynomial $f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ has no line polynomial factors then it is abelian rigid. In [26] it was proved that also the converse holds partially. Indeed, the authors showed that if a polynomial $f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ has a line polynomial factor of the simple form

$$\sum_{k=0}^{m} X^{k\mathbf{v}}$$

for a non-zero vector $\mathbf{v}$ and positive integer m, then it is not abelian rigid. This means that if f has a line polynomial factor of the above form, then there exists a configuration which is not strongly periodic but has low abelian complexity with respect to f. In fact, the proof of this direction generalizes to higher dimensions as the authors have mentioned.

Finally, we have a general statement in any dimension.

***Theorem* 3.5.7.** *Let $c \in \mathcal{A}^{\mathbb{Z}^d}$ be a d-dimensional configuration over an alphabet of size n and assume that it has low abelian complexity with respect to a polynomial $f \in \mathbb{Z}[X^{\pm 1}]$. If for all $V \in \mathbb{G}_{d-1}$ there exist $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^d$ such that*

$$\text{supp}(X^{\mathbf{u}_1} f - X^{\mathbf{u}_2} f) \cap V = \{\mathbf{0}\},$$

then c is strongly periodic.

Proof. Follows from Theorem 2.5.22. □

3.6 Algorithmic aspects

All configurations in a subshift are periodic, in particular, if there are no configurations in the subshift at all. It is useful to be able to detect such trivial cases.

Perfect coverings

The set

$$\mathcal{S}(D,b,a) = \{c \in \{0,1\}^{\mathbb{Z}^d} \mid (f_D - (b-a))c = a\mathbb{1}(X)\}$$

of all (D,b,a)-coverings is an SFT for any given shape D with $|D| \geq 2$ and non-negative integers b and a. Hence, the question whether there exist any (D,b,a)-coverings for a given neighborhood D and covering constants b and a is equivalent to the question whether the SFT $\mathcal{S}(D,b,a)$ is non-empty.

Let us assume first that $d = 2$. The question of emptiness of a given two-dimensional SFT is undecidable in general. However, if the SFT is known to be not aperiodic, then the problem becomes decidable by Theorem 2.4.2. In particular, if $g = f_D - (b-a)$ has line polynomial factors in at most one direction, then the question whether there exist any (D,b,a)-coverings is decidable:

***Theorem* 3.6.1.** *There exists an algorithm to determine whether there exist any (D,b,a)-coverings for any given $D \Subset \mathbb{Z}^2$ with $|D| \geq 2$ and non-negative integers b and a such that all the possible line polynomial factors of the polynomial $g = f_D - (b-a) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ are in the same direction.*

Proof. Let $\mathcal{S} = \mathcal{S}(D,b,a)$ be the SFT of all (D,b,a)-coverings. Since all the possible line polynomial factors of g are in the same direction, by Corollary 2.5.21 every element of $\mathcal{S}$ is periodic. Any two-dimensional SFT that contains periodic configurations contains also strongly periodic configurations. Thus, $\mathcal{S}$ is either empty or contains a strongly periodic configuration and hence by Theorem 2.4.2 there is an algorithm to determine whether $\mathcal{S}$ is non-empty. □

One may also want to design a perfect (D,b,a)-covering for given D, b and a. This can be effectively done under the assumptions of Theorem 3.6.1. Indeed, as we have seen, if $\mathcal{S} = \mathcal{S}(D,b,a)$ is non-empty, it contains a strongly periodic configuration. For any strongly periodic configuration c, it is easy to check if c contains a forbidden pattern. By enumerating strongly periodic configurations one-by-one, one is guaranteed to find eventually one that is in $\mathcal{S}$.

If the polynomial g has no line polynomial factors, then the following stronger result holds.

***Theorem* 3.6.2.** *If the polynomial $g = f_D - (b-a)$ has no line polynomial factors for given shape $D \Subset \mathbb{Z}^2$ and non-negative integers b and a, then the SFT $\mathcal{S} = \mathcal{S}(D,b,a)$ is finite. One can then effectively construct all the finitely many elements of $\mathcal{S}$.*

The proof of the first part of above theorem relies on the fact that a subshift is finite if and only if it contains only strongly periodic configurations [5]. If g has no line polynomial factors, then every configuration it periodizes (including every configuration in $\mathcal{S}$) is strongly periodic by Corollary 2.5.21, and hence $\mathcal{S}$ is finite. The second part of the theorem, *i.e.*, the fact that one can effectively produce all the finitely many elements of $\mathcal{S}$ holds generally for finite SFTs in any dimension. For completeness, let us prove this fact.

***Lemma* 3.6.3.** *Given a finite $F \subseteq \mathcal{A}^*$ such that X_F is finite, one can effectively construct the elements of X_F.*

Proof. Given a finite $F \subseteq \mathcal{A}^*$ and a pattern $p \in \mathcal{A}^D$, assuming that strongly periodic configurations are dense in X_F, one can effectively check whether $p \in \mathcal{L}(X_F)$. Indeed, we have a semi-algorithm for the positive instances that guesses a strongly periodic configuration c and verifies that $c \in X_F$ and $p \in \mathcal{L}(c)$. A semi-algorithm for the negative instances exists for any SFT X_F and is a standard compactness argument: guess a finite $E \subseteq \mathbb{Z}^d$ such that $D \subseteq E$ and verify that every $q \in \mathcal{A}^E$ such that $q \restriction_D = p$ contains a forbidden subpattern.

Consequently, given finite $F, G \subseteq \mathcal{A}^*$, assuming that strongly periodic configurations are dense in X_F and X_G, one can effectively determine whether $X_F = X_G$. Indeed, $X_F \subseteq X_G$ if and only if no $p \in G$ is in $\mathcal{L}(X_F)$, a condition that we have shown above to be decidable. Analogously we can test $X_G \subseteq X_F$.

Finally, let a finite $F \subseteq \mathcal{A}^*$ be given such that X_F is known to be finite. All elements of X_F are strongly periodic so that strongly periodic configurations are certainly dense in X_F. One can effectively enumerate all finite sets P of strongly periodic configurations. For each P that is translation invariant (and hence a finite SFT) one can construct a finite set $G \subseteq \mathcal{A}^*$ of forbidden patterns such that $X_G = P$. As shown above, there is an algorithm to test whether $X_F = X_G = P$. Since X_F is finite, a set P is eventually found such that $X_F = P$. □

For perfect coverings in arbitrary dimension, we have the following decidability result.

***Theorem* 3.6.4.** *Let $D \Subset \mathbb{Z}^d$ with $|D| \geq 2$ and non-negative integers b and a be such that for all $V \in \mathbb{G}_{d-1}$ there exist $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^d$ such that*

$$\textit{supp}(X^{\mathbf{u}_1} g - X^{\mathbf{u}_2} g) \cap V = \{\mathbf{0}\}$$

where $g = f_D - (b - a)$. The subshift $\mathcal{S} = \mathcal{S}(D, b, a)$ is finite and one can effectively construct all the elements of $\mathcal{S}$. In particular, the emptiness of $\mathcal{S} = \mathcal{S}(D, b, a)$ is decidable, that is, there is an algorithm to determine whether there exist any (D, b, a)-coverings whenever D, b, a are given as above.

Proof. Let D, b, a be given as in the statement. By Theorem 3.3.16 any (D, b, a)-covering is strongly periodic and hence $\mathcal{S}$ is finite [5]. The claim follows from Lemma 3.6.3. □

Perfect colorings

Let us now turn to the more general question of existence of perfect colorings over alphabets of arbitrary size. We consider only the two-dimensional setting. Let $D \Subset \mathbb{Z}^2$ be a two-dimensional shape with at least two elements and let $\mathbf{B}$ be an $n \times n$ integer matrix. To determine whether there exist any $(D, \mathbf{B})$-colorings is equivalent to asking whether the SFT

$$\mathcal{S}(D, \mathbf{B}) = \{c \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^{\mathbb{Z}^2} \mid gc = 0\}$$

is non-empty where $g = \sum_{\mathbf{u} \in D} \mathbf{I} X^{-\mathbf{u}} - \mathbf{B}$ since it is exactly the set of the vector presentations of all $(D, \mathbf{B})$-colorings.

***Theorem* 3.6.5.** *Let a shape $D \Subset \mathbb{Z}^2$ with $|D| \geq 2$, a non-negative integer matrix $\mathbf{B}$ be such that $\det(\mathbf{B} - t_0\mathbf{I}) \neq 0$, and the polynomial $f_D(x, y) - t \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ has no line polynomial factors whenever $t \neq t_0$ for some t_0. There are only finitely many $(D, \mathbf{B})$-colorings and one can effectively construct them whenever $\mathbf{B}$ is such that $\det(\mathbf{B} - t_0\mathbf{I}) \neq 0$. If $f_D(x, y) - t$ does not have line polynomial factors for any t, then this holds for any $\mathbf{B}$. In particular, there is an algorithm to determine whether there exist any $(D, \mathbf{B})$-colorings whenever D and $\mathbf{B}$ are given as above.*

Proof. Let $\mathcal{S} = \mathcal{S}(D, \mathbf{B})$ be the SFT of the vector presentations of all $(D, \mathbf{B})$-colorings where $\mathbf{B}$ is as in the statement. By Theorem 3.4.5 all elements of $\mathcal{S}$ are strongly periodic. Hence, $\mathcal{S}$ is finite, and the claim follows by Lemma 3.6.3. □

Corollaries 3.4.6, 3.4.7, 3.4.8, 3.4.9 and 3.4.10 together with the above theorem give the following corollary.

***Corollary* 3.6.6.** *The following decision problems are decidable for a given matrix $\mathbf{B}$ satisfying the given conditions.*

- *The existence of $(D, \mathbf{B})$-colorings where D is the relative 1-neighborhood of the square grid and $\det(\mathbf{B} - \mathbf{I}) \neq 0$.*
- *The existence of $(D, \mathbf{B})$-colorings where D is the relative 1-neighborhood of the triangular grid and $\det(\mathbf{B} + \mathbf{I}) \neq 0$.*
- *The existence of $(D, \mathbf{B})$-colorings where D is the relative r-neighborhood of the square grid and $\mathbf{B}$ is arbitrary.*
- *The existence of $(D, \mathbf{B})$-colorings where D is the relative r-neighborhood of the triangular grid and $\mathbf{B}$ is arbitrary.*
- *The existence of $(D, \mathbf{B})$-colorings where D is the relative r-neighborhood of the king grid and $\det(\mathbf{B}) \neq 0$.*

Low abelian complexity configurations

Finally, Theorems 3.5.5 and 3.5.7 give the following decidability results. The first result concerns only two-dimensional configurations while the second result is a general statement in any dimension.

***Theorem* 3.6.7.** *There is an algorithm to determine whether there exist any two-dimensional configurations over an alphabet of size n that have low abelian complexity with respect to f for a given two-variate polynomial $f \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ such that all of its possible line polynomial factors are in the same direction. If f has no line polynomial factors, then there are only finitely many such configurations and one can effectively construct all of them.*

Proof. The set $\mathcal{S} = \{c \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}^{\mathbb{Z}^2} \mid \mathbf{I}fc = 0\}$ of the vector presentations of all configurations over an alphabet of size n with low abelian complexity with respect to f is an SFT. The question of existence configurations that have low abelian complexity with respect to f is clearly equivalent to the question whether $\mathcal{S}$ is non-empty.

If all the line polynomial factors of f are in the same direction and if $\mathcal{S}$ is non-empty, then every element of $\mathcal{S}$ is periodic by Theorem 3.5.5 and hence it contains a strongly periodic configuration. Thus, the emptiness problem of $\mathcal{S}$ is decidable by Theorem 2.4.2.

If f has no line polynomial factors, then by Theorem 3.5.5 the SFT $\mathcal{S}$ contains only strongly periodic configurations and hence it is finite. Thus, by Lemma 3.6.3 we have the claim. □

***Theorem* 3.6.8.** *There is an algorithm to determine if there exist d-dimensional configurations over an alphabet of size n that have low abelian complexity with respect to f for a given polynomial $f \in \mathbb{Z}[X^{\pm 1}]$ in d variables $X = (x_1, \ldots, x_d)$ such that for all $V \in \mathbb{G}_{d-1}$ there exist $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{Z}^d$ such that*

$$\text{supp}(X^{\mathbf{u}_1} f - X^{\mathbf{u}_2} f) \cap V = \{\mathbf{0}\}.$$

In fact, there are only finitely many such configurations and one can effectively construct all of them.

Proof. If f is as in the statement, then any configuration that has low abelian complexity with respect to f is strongly periodic by Theorem 3.5.7. Thus, the SFT $\mathcal{S}$ of the vector presentations of every configuration over an alphabet of size n is finite and hence by Lemma 3.6.3 we have the claim. □

4 Periodic Decomposition Theorems

In this chapter we prove two improvements of the periodic decomposition theorem (Theorem 2.5.14). This chapter is based on the article [44]. The main results of the chapter are Theorem 4.1.4 and Theorem 4.2.1.

4.1 Periodic decomposition of configurations with specific annihilators

The first improvement of the periodic decomposition theorem considers arbitrary configurations with annihilators of particular type. Recall that c is V-periodic for a vector space $V \subseteq \mathbb{R}^d$ if it is periodic in direction $\mathbf{v}$ for all $\mathbf{v} \in V \cap \mathbb{Q}^d$. We begin with some lemmas. The following lemma is a generalization of the integration lemma (Lemma 2.5.15).

***Lemma* 4.1.1.** *Let $V \subseteq \mathbb{R}^d$ be a linear subspace of $\mathbb{R}^d$, and let φ and ψ be line polynomials in non-parallel directions $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Also, assume that*

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \cap V = \{\mathbf{0}\}.$$

If $c' \in \mathbb{C}^{\mathbb{Z}^d}$ is a V-periodic function annihilated by ψ, then there exists $c \in \mathbb{C}^{\mathbb{Z}^d}$ such that the following conditions hold:

- $\varphi c = c'$,
- $\psi c = 0$, *and*
- *c is V-periodic.*

Moreover, if $c' \in \mathbb{Z}^{\mathbb{Z}^d}$ and φ is a difference polynomial, then the coefficients of c can be chosen to be integers, that is, $c \in \mathbb{Z}^{\mathbb{Z}^d}$.

Proof. Without loss of generality we may assume that $\varphi = \alpha_0 + \alpha_1 X^{\mathbf{v}_1} + \ldots + \alpha_n X^{n\mathbf{v}_1}$ for some positive integer n such that $\alpha_0, \alpha_n \neq 0$. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\} \subseteq \mathbb{Z}^d$ be a base of $V \cap \mathbb{Q}^d$.

The space $\mathbb{Z}^d$ is partitioned into cosets modulo

$$\mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{b}_1, \ldots, \mathbf{b}_k] = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + b_1\mathbf{b}_1 + \ldots + b_k\mathbf{b}_k \mid a_1, a_2, b_1, \ldots, b_k \in \mathbb{Z}\}.$$

Let us fix a point $\mathbf{z}_\Lambda \in \Lambda$ for each such coset Λ. Now, we have $\Lambda = \mathbf{z}_\Lambda + \mathbb{Z}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{b}_1, \ldots, \mathbf{b}_k]$. Let us denote

$$\Lambda[a_1, a_2, b_1, \ldots, b_k] = \mathbf{z}_\Lambda + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + b_1\mathbf{b}_1 + \ldots + b_k\mathbf{b}_k.$$

By the assumption $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \cap V = \{\mathbf{0}\}$, each element of Λ has a unique expression as $\Lambda[a_1, a_2, b_1, \ldots, b_k]$. In other words, if

$$\Lambda[a_1, a_2, b_1, \ldots, b_k] = \Lambda[a_1', a_2', b_1', \ldots, b_k'],$$

then $a_1 = a_1', a_2 = a_2', b_1 = b_1', \ldots, b_k = b_k'$.

The equation $\varphi c = c'$ is satisfied in an arbitrary coset Λ if and only if

$$\alpha_n c_{\Lambda[a_1 - n, a_2, b_1, \ldots, b_k]} + \ldots + \alpha_0 c_{\Lambda[a_1, a_2, b_1, \ldots, b_k]} = c'_{\Lambda[a_1, a_2, b_1, \ldots, b_k]} \tag{4}$$

for all $a_1, a_2, b_1, \ldots, b_k \in \mathbb{Z}$. Let us define $c_{\Lambda[a_1, a_2, b_1, \ldots, b_k]} = 0$ for $a_1 \in [0, n)$. The rest of c is then determined by Equation (4) such that $\varphi c = c'$.

Since $\psi c' = 0$, we have

$$\varphi(\psi c) = \psi(\varphi c) = \psi c' = 0,$$

and since we defined $c_{\Lambda[a_1, a_2, b_1, \ldots, b_k]} = 0$ for all $a_1 \in [0, n)$ and $a_2, b_1, \ldots, b_k \in \mathbb{Z}$, it follows that also $(\psi c)_{\Lambda[a_1, a_2, b_1, \ldots, b_k]} = 0$ for all $a_1 \in [0, n)$ and $a_2, b_1, \ldots, b_k \in \mathbb{Z}$ and hence the above recurrence implies that

$$(\psi c)_{\Lambda[a_1, a_2, b_1, \ldots, b_k]} = 0$$

for all $a_1, a_2, b_1, \ldots, b_k \in \mathbb{Z}$. This holds in every coset Λ and hence $\psi c = 0$.

By a similar argument as above, since c' is periodic in direction $\mathbf{b}_i$ for each $i \in \{1, \ldots, k\}$, that is, annihilated by a difference polynomial in direction $\mathbf{b}_i$, it follows that also c is annihilated by this same difference polynomial and hence periodic in direction $\mathbf{b}_i$. Thus, c is periodic in direction $\mathbf{v}$ for every $\mathbf{v} \in V \cap \mathbb{Q}^d$ and hence by the definition c is V-periodic.

Finally, assume that $c' \in \mathbb{Z}^{\mathbb{Z}^d}$ and φ is a difference polynomial, that is, $\varphi = X^{t\mathbf{v}_1} - 1$ for some $t \in \mathbb{Z}$. Now, Equation (4) turns into form

$$c_{\Lambda[a_1 - t, a_2, b_1, \ldots, b_k]} - c_{\Lambda[a_1, a_2, b_1, \ldots, b_k]} = c'_{\Lambda[a_1, a_2, b_1, \ldots, b_k]}.$$

Since the values of c' are integers, it follows that the values of c as defined above are integers as well. □

***Lemma* 4.1.2.** *Let $V \subseteq \mathbb{R}^d$ be a linear subspace of $\mathbb{R}^d$. Assume that $c \in \mathbb{C}^{\mathbb{Z}^d}$ is V-periodic and that there exist $m \geq 1$ and line polynomials $\varphi_1, \ldots, \varphi_m$ in pairwise* *non-parallel directions $\mathbf{v}_1, \ldots, \mathbf{v}_m$, respectively, such that c is annihilated by the product $\varphi_1 \cdots \varphi_m$. Also, assume that for all $i, j \in \{1, \ldots, m\}, i \neq j$ it holds that*

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle \cap V = \{\mathbf{0}\}.$$

Then there exist functions $c_1, \ldots, c_m \in \mathbb{C}^{\mathbb{Z}^d}$ such that

$$c = c_1 + \ldots + c_m$$

where each c_i is annihilated by φ_i and each c_i is V-periodic.

Moreover, if the line polynomials $\varphi_1, \ldots, \varphi_m$ are difference polynomials and $c \in \mathbb{Z}^{\mathbb{Z}^d}$, then we may choose $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{Z}^d}$.

Proof. We prove the claim by induction on m. The case $m = 1$ is clear by choosing $c_1 = c$. Let us assume then that $m > 1$ and that the claim holds for $m - 1$. The function $\varphi_m c$ is annihilated by $\varphi_1 \cdots \varphi_{m-1}$ and hence by the induction hypothesis there exist functions $c'_1, \ldots, c'_{m-1} \in \mathbb{C}^{\mathbb{Z}^d}$ such that

$$\varphi_m c = c'_1 + \ldots + c'_{m-1}$$

where each c'_i is annihilated by φ_i and V-periodic. By Lemma 4.1.1 for each $i \in \{1, \ldots, m-1\}$ there exists $c_i \in \mathbb{C}^{\mathbb{Z}^d}$ such that $\varphi_m c_i = c'_i$ and $\varphi_i c_i = 0$, and c_i is V-periodic. Set $c_m = c - c_1 - \ldots - c_{m-1}$. Then $c = c_1 + \ldots + c_m$ and

$$\begin{aligned} \varphi_m c_m &= \varphi_m c - \varphi_m c_1 - \ldots - \varphi_m c_{m-1} \\ &= c'_1 + \ldots + c'_{m-1} - c'_1 - \ldots - c'_{m-1} \\ &= 0. \end{aligned}$$

Finally, since c and all $c_1, \ldots, c_{m-1}$ are V-periodic, also c_m is V-periodic.

The “moreover” part follows from the “moreover” part of Lemma 4.1.1. □

The following lemma is adapted from the proof of Theorem 3 in [50].

***Lemma* 4.1.3** ([50]). *Let $V \in \mathbb{R}^d$ be a proper linear subspace of $\mathbb{R}^d$ ($\dim(V) \leq d-1$) and let $c \in \mathbb{C}^{\mathbb{Z}^d}$. Assume that c has a periodizer g such that supp$(g) \cap V = \{\mathbf{0}\}$. Then c has also an annihilator f such that supp$(f) \cap V = \{\mathbf{0}\}$.*

Now, we are ready to prove our first improvement of the periodic decomposition theorem.

***Theorem* 4.1.4.** *Let c be a d-dimensional configuration and let $k \in \{1, \ldots, d\}$. Assume that for every $(k-1)$-dimensional linear subspace $V \subseteq \mathbb{R}^d$ the configuration c has a periodizer f such that supp$(f) \cap V = \{\mathbf{0}\}$. Then there exist k-periodic functions $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{Z}^d}$ such that*

$$c = c_1 + \ldots + c_m.$$

Proof. We prove the claim by induction on k. If $k = 1$, then the assumption implies that the configuration c has a non-trivial periodizer and hence by Theorem 2.5.14 it is a sum of periodic functions.

Let $k \in \{1, \ldots, d-1\}$ and assume that the claim holds for k. Let us prove that the claim holds then also for $k+1$. So, assume that for all $V \in \mathbb{G}_k$ the configuration c has a periodizer f such that $\text{supp}(f) \cap V = \{\mathbf{0}\}$. Since every $(k-1)$-dimensional subspace is contained in a k-dimensional subspace, there exists such periodizer, in particular, also for all $V \in \mathbb{G}_{k-1}$. Thus, by the induction hypothesis there exist k-periodic functions $e_1, \ldots, e_l \in \mathbb{Z}^{\mathbb{Z}^d}$ such that

$$c = e_1 + \ldots + e_l.$$

For each $i \in \{1, \ldots, l\}$ let $V_i \in \mathbb{G}_k$ be such that e_i is V_i-periodic. We may assume that for all $i \neq j$ we have $V_i \neq V_j$ since the sum of two V-periodic functions is also V-periodic. Indeed, if for some $i \neq j$ we have $V_i = V_j$, then we replace the functions e_i and e_j by the function $e_i + e_j$ which is also V_i-periodic.

Let us fix an arbitrary $i \in \{1, \ldots, l\}$. By the assumption the configuration c has a periodizer f such that

$$\text{supp}(f) \cap V_i = \{\mathbf{0}\}.$$

By Lemma 4.1.3 we may assume that f is an annihilator of c and hence by Theorem 2.5.10 we conclude that c has an annihilator

$$(X^{\mathbf{u}_1} - 1) \cdots (X^{\mathbf{u}_r} - 1)$$

such that $\mathbf{u}_j \notin V_i$ for all $j \in \{1, \ldots, r\}$ and the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are pairwise non-parallel.

For each $j \in \{1, \ldots, l\} \setminus \{i\}$ let $\mathbf{w}_j \in V_j$ be such that $\mathbf{w}_j \notin V_i$ and e_j is annihilated by $X^{\mathbf{w}_j} - 1$. Since $c = e_1 + \ldots + e_l$ is annihilated by the product $(X^{\mathbf{u}_1} - 1) \cdots (X^{\mathbf{u}_r} - 1)$ and each e_j is annihilated by $X^{\mathbf{w}_j} - 1$, it follows that e_i is annihilated by the polynomial

$$(X^{\mathbf{u}_1} - 1) \cdots (X^{\mathbf{u}_r} - 1) \prod_{j \in \{1, \ldots, l\} \setminus \{i\}} (X^{\mathbf{w}_j} - 1).$$

Thus, we have seen that e_i is annihilated by a polynomial of the form

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_s} - 1)$$

where $\mathbf{v}_j \notin V_i$ for each $j \in \{1, \ldots, s\}$. Let us assume that s is minimal in the sense that if a product $(X^{\mathbf{v}'_1} - 1) \cdots (X^{\mathbf{v}'_t} - 1)$ annihilates e_i such that $\mathbf{v}_j \notin V_i$ for each $j \in \{1, \ldots, t\}$, then $t \geq s$. We claim that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s$ are pairwise non-parallel and $\langle \mathbf{v}_j, \mathbf{v}_{j'} \rangle \cap V_i = \{\mathbf{0}\}$ for all $j, j' \in \{1, \ldots, s\}, j \neq j'$.

(Note that in the above annihilator we may replace any $\mathbf{v}_j$ with $p\mathbf{v}_j$ for any integer p and the obtained polynomial is still an annihilator of e_i. This is due to the fact that if a function is $\mathbf{v}$-periodic, then it is also $p\mathbf{v}$-periodic for any integer p.)

If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s$ are not pairwise non-parallel, then for some $j, j' \in \{1, \ldots, s\}, j \neq j'$ the product $\varphi = (X^{\mathbf{v}_j} - 1)(X^{\mathbf{v}_{j'}} - 1)$ is a line polynomial in direction $\mathbf{v}_j$. Since φ annihilates the function

$$\frac{(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_s} - 1)}{(X^{\mathbf{v}_j} - 1)(X^{\mathbf{v}_{j'}} - 1)} e_i,$$

it is periodic in direction $\mathbf{v}_j$, that is, annihilated by $X^{p\mathbf{v}_j} - 1$ for some non-zero $p \in \mathbb{Z}$. Thus, e_i is annihilated by

$$(X^{p\mathbf{v}_j} - 1) \cdot \frac{(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_s} - 1)}{(X^{\mathbf{v}_j} - 1)(X^{\mathbf{v}_{j'}} - 1)}$$

which is a product of $s-1$ non-trivial difference polynomials. This is a contradiction with the minimality of s and hence the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s$ must be pairwise non-parallel.

Let us then show that $\langle \mathbf{v}_j, \mathbf{v}_{j'} \rangle \cap V_i = \{\mathbf{0}\}$ for all $j, j' \in \{1, \ldots, s\}, j \neq j'$. Assume on the contrary that there exist $j, j' \in \{1, \ldots, s\}, j \neq j'$ such that $\langle \mathbf{v}_j, \mathbf{v}_{j'} \rangle \cap V_i \neq \{\mathbf{0}\}$. Then there exist integers p, p' such that $p'\mathbf{v}_{j'} = p\mathbf{v}_j + \mathbf{v}$ for some $\mathbf{v} \in V_i \setminus \{\mathbf{0}\}$. Since $\mathbf{v}_j, \mathbf{v}_{j'} \notin V_i$ we have $p, p' \neq 0$. Now, we replace the term $X^{\mathbf{v}_{j'}} - 1$ in the annihilator $(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_s} - 1)$ of e_i by $X^{p\mathbf{v}_j + \mathbf{v}} - 1$. Let us denote the obtained annihilator of e_i by g. Since e_i is $\mathbf{v}$-periodic, also the function

$$e = \frac{g}{X^{p\mathbf{v}_j + \mathbf{v}} - 1} e_i$$

is $\mathbf{v}$-periodic, that is, $X^{\mathbf{v}} e = e$. Since it is annihilated by $X^{p\mathbf{v}_j + \mathbf{v}} - 1$, we have

$$0 = (X^{p\mathbf{v}_j + \mathbf{v}} - 1)e = X^{p\mathbf{v}_j + \mathbf{v}} e - e = X^{p\mathbf{v}_j} e - e = (X^{p\mathbf{v}_j} - 1)e$$

and hence it is also annihilated by $X^{p\mathbf{v}_j} - 1$. Consequently, we replace the term $X^{p\mathbf{v}_j + \mathbf{v}} - 1$ in g by $X^{p\mathbf{v}_j} - 1$. Finally, the term $(X^{p\mathbf{v}_j} - 1)(X^{\mathbf{v}_j} - 1)$ is replaced by $X^{q\mathbf{v}_j} - 1$ for suitable $q \in \mathbb{Z}$. Again, we get a contradiction with the minimality of s by obtaining an annihilator of e_i which is a product of $s - 1$ non-trivial difference polynomials.

Thus, by Lemma 4.1.2 there exist V_i-periodic functions $c_1, \ldots, c_s$ such that for each $j \in \{1, \ldots, s\}$ the function c_j is annihilated by $X^{\mathbf{v}_j} - 1$, that is, $\mathbf{v}_j$-periodic and

$$e_i = c_1 + \ldots + c_s.$$

So, each c_j is $\langle \{\mathbf{v}_j\} \cup V_i \rangle$-periodic and hence $(k+1)$-periodic since $\mathbf{v}_j \notin V_i$.

This same reasoning works for any $i \in \{1, \ldots, l\}$. So, we conclude that each e_i is a sum of $(k+1)$-periodic functions and hence $c = e_1 + \ldots + e_l$ is a sum of $(k+1)$-periodic functions. □

Also, the converse of the above theorem holds. In other words, if $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{Z}^d}$ are k-periodic functions, then their sum $c = c_1 + \ldots + c_m$, has for all $V \in \mathbb{G}_{k-1}$ a periodizer f such that $\text{supp}(f) \cap V = \{\mathbf{0}\}$. Let us state this direction as a lemma and provide a proof for it.

***Lemma* 4.1.5.** *Let $k \in \{1, \ldots, d\}$, and let $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{Z}^d}$ be k-periodic functions. Then for all $V \in \mathbb{G}_{k-1}$ the function $c = c_1 + \ldots + c_m$ has a periodizer f such that*

$$\textit{supp}(f) \cap V = \{\mathbf{0}\}.$$

Proof. We prove the claim by induction on m. Assume first that $m = 1$. Let $V \in \mathbb{G}_{k-1}$. Since $c = c_1$ is k-periodic, it has a period vector $\mathbf{v}$ such that $\mathbf{v} \notin V$. Now, $X^{\mathbf{v}} - 1$ is an annihilator (and hence also a periodizer) of c satisfying $\text{supp}(X^{\mathbf{v}} - 1) \cap V = \{\mathbf{0}\}$.

Assume then that $m \geq 2$ and that the claim holds for $m - 1$. Let $V \in \mathbb{G}_{k-1}$ be arbitrary. By the induction hypothesis the function

$$c' = c_2 + \ldots + c_m$$

has a periodizer f' such that $\text{supp}(f') \cap V = \{\mathbf{0}\}$. Again, let $\mathbf{v}$ be such that c_1 is $\mathbf{v}$-periodic and $\mathbf{v} \notin V$. Note that c_1 is also $n\mathbf{v}$-periodic and hence $X^{n\mathbf{v}} - 1$ is an annihilator of c_1 for all $n \in \mathbb{Z}$. Thus, $f = (X^{n\mathbf{v}} - 1)f'$ is a periodizer of c for all n. For large enough n, we have $\text{supp}(f) \cap V = \{\mathbf{0}\}$. □

Remark 4.1.6. If in Theorem 4.1.4 we have $k = d$, then it is easily seen that every $(d-1)$-dimensional subspace is expansive for the orbit closure $\overline{\mathcal{O}(c)}$ of c and hence $\overline{\mathcal{O}(c)}$ is finite by Theorem 2.4.4 Thus, c is strongly periodic in this setting by using a similar argument as in [5]. This case is also stated as Corollary 1 in [50]. For $k \in \{1, \ldots, d-1\}$ it may be that the functions in Theorem 4.1.4 have infinitely many distinct coefficients, that is, they are not configurations as Example 2.5.16 suggests.

Translational tilings

Recall that a translational tiling is a binary configuration $c \in \{0,1\}^{\mathbb{Z}^d}$ such that for some shape D — a tile — and a polynomial $f = f_D = \sum_{\mathbf{u} \in -D} X^{\mathbf{u}}$ we have

$$fc = \mathbb{1} = 1^{\mathbb{Z}^d}.$$

In this case c is called a co-tiler of the tile D.

In [68] the authors say that k tiles $D_1, \ldots, D_k \subseteq \mathbb{Z}^d$ satisfying $\mathbf{0} \in D_i$ for each $i \in \{1, \ldots, k\}$ are *independent* if the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent over $\mathbb{Q}$ with any choice of $\mathbf{v}_i \in D_i \setminus \{\mathbf{0}\}$. It is shown that if k independent tiles have a common co-tiler c, then c is a sum of finitely many k-periodic functions from the set $[a,b]^{\mathbb{Z}^d}$ for some reals $a < b$. In the following we prove a similar result as a corollary of Theorem 4.1.4.

Let us say that k polynomials $f_1, \ldots, f_k$ satisfying $\mathbf{0} \in \text{supp}(f_i)$ for each $i \in \{1, \ldots, k\}$ are independent if their supports are independent.

***Corollary* 4.1.7.** *Let $c \in \mathcal{A}^{\mathbb{Z}^d}$ be a configuration and let $f_1, \ldots, f_k$ be k periodizers of c satisfying $\mathbf{0} \in supp(f_i)$ for each $i \in \{1, \ldots, k\}$ with $k \in \{1, \ldots, d\}$. If $f_1, \ldots, f_k$ are independent, then*

$$c = c_1 + \ldots + c_m$$

where each $c_i \in \mathbb{Z}^{\mathbb{Z}^d}$ is k-periodic.

Proof. Let us prove that for every $V \in \mathbb{G}_{k-1}$ the configuration c has a periodizer f such that $\text{supp}(f) \cap V = \{\mathbf{0}\}$. Then the claim follows from Theorem 4.1.4.

Assume on the contrary that there exists a $V \in \mathbb{G}_{k-1}$ such that $\text{supp}(f) \cap V \neq \{\mathbf{0}\}$ for any periodizer f of c. This implies that for any periodizer f of c we have either $\text{supp}(f) \cap V = \emptyset$ or $|\text{supp}(f) \cap V| \geq 2$. Since $\mathbf{0} \in \text{supp}(f_i)$, we have $|\text{supp}(f_i) \cap V| \geq 2$ for each $i \in \{1, \ldots, k\}$. Thus, there exist non-zero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that $\mathbf{v}_i \in \text{supp}(f_i) \cap V$ for each $i \in \{1, \ldots, k\}$ and hence the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span a linear subspace of dimension at most $k-1$. This means that the sets $\text{supp}(f_1), \ldots, \text{supp}(f_k)$ are not independent and hence the polynomials $f_1, \ldots, f_k$ are not independent. A contradiction. $\square$

For any tile $D \subseteq \mathbb{Z}^d$, the polynomial $f = \sum_{\mathbf{u} \in -D} X^{\mathbf{u}}$ is a periodizer of any co-tiler c of D. Thus, by the above corollary any common co-tiler of k independent tiles is a sum of finitely many k-periodic functions from the set $\mathbb{Z}^{\mathbb{Z}^d}$. So, we have a similar result as the authors of [68] except that the k-periodic functions in the periodic decomposition of the co-tiler are now functions of the set $\mathbb{Z}^{\mathbb{Z}^d}$ instead of the set $[a,b]^{\mathbb{Z}^d}$.

4.2 Periodic decomposition of sparse configurations

Let us denote by $C_m = \{-m, \ldots, m\}^d$ for $m \in \mathbb{N}$ the discrete d-dimensional hypercube of size $(2m+1)^d$ centered at the origin. Any function $c \in \mathbb{C}^{\mathbb{Z}^d}$ is called *sparse* if there exists a positive integer a such that

$$|\text{supp}(c) \cap (C_m + \mathbf{t})| \leq am$$

for all $m \in \mathbb{Z}_+$ and for all $\mathbf{t} \in \mathbb{Z}^d$. Such a is called a sparseness constant of c.

In this section we consider sparse configurations with annihilators and prove that they are sums of finitely many periodic fibers. We prove the following theorem.

***Theorem* 4.2.1.** *Let c be a sparse configuration and assume that it is annihilated by a product $\varphi_1 \cdots \varphi_n$ of line polynomials $\varphi_1, \ldots, \varphi_n$ in pairwise non-parallel directions $\mathbf{v}_1, \ldots, \mathbf{v}_n$, respectively. Then*

$$c = c_1 + \ldots + c_n$$

where each c_i is a sum of finitely many periodic $\mathbf{v}_i$-fibers and $\varphi_i c_i = 0$.

Before presenting the proof of the theorem we state some simple observations concerning sparse configurations.

***Lemma* 4.2.2.** *Let $c_1, \ldots, c_n$ be sparse configurations and let $k_1, \ldots, k_n \in \mathbb{Z}$. Then also $k_1 c_1 + \ldots + k_n c_n$ is sparse. In particular, for a sparse configuration c and a polynomial g also gc is sparse.*

Proof. There exist $a_1, \ldots, a_n$ such that

$$|\text{supp}(c_i) \cap (C_m + \mathbf{t})| \leq a_i m$$

for all $m \in \mathbb{Z}_+$ and $\mathbf{t} \in \mathbb{Z}^d$. Since

$$\text{supp}(k_1 c_1 + \ldots + k_n c_n) \subseteq \cup_{i=1}^n \text{supp}(c_i),$$

we have

$$|\text{supp}(k_1 c_1 + \ldots + k_n c_n) \cap (C_m + \mathbf{t})| \leq (a_1 + \ldots + a_n) m$$

for all $m \in \mathbb{Z}_+$ and $\mathbf{t} \in \mathbb{Z}^d$. Thus, $k_1 c_1 + \ldots + k_n c_n$ is sparse.

For a polynomial $g = \sum_{i=1}^n k_i X^{\mathbf{u}_i}$, we have $gc = k_1 c_1 + \ldots + k_n c_n$ where each $c_i = X^{\mathbf{u}_i} c$ is sparse as a translation, *i.e.*, a monomial multiplication of c. Thus, gc is sparse by the first part. □

***Lemma* 4.2.3.** *Every element of the orbit closure $\overline{\mathcal{O}(c)}$ of a sparse configuration c is also sparse.*

Proof. Let $e \in \overline{\mathcal{O}(c)}$. For all $m \in \mathbb{Z}_+$ any C_m-pattern of e is also a C_m-pattern of c and hence for any $\mathbf{t} \in \mathbb{Z}^d$ there exists $\mathbf{t}' \in \mathbb{Z}^d$ such that

$$\text{supp}(e) \cap (C_m + \mathbf{t}) = \text{supp}(c) \cap (C_m + \mathbf{t}').$$

Thus, if c is sparse, then also e is sparse. □

Remark 4.2.4. To make the above lemma work it is important to define sparse configurations in the way we did. For example, the larger class of configurations c having a constant a such that

$$|\text{supp}(c) \cap C_m| \leq am$$

for all $m \in \mathbb{Z}_+$ does not satisfy the above lemma. Indeed, consider a configuration $c \in \{0,1\}^{\mathbb{Z}^2}$ such that $c(\mathbf{u}) = 1$ if $\mathbf{u} \in C_m + (2^m, 0)$ for some $m \in \mathbb{Z}_+$ and otherwise $c(\mathbf{u}) = 0$. Now, $|\text{supp}(c) \cap C_m| = O(m)$ and hence there exists a such that

$$|\text{supp}(c) \cap C_m| \leq am$$

for all $m \in \mathbb{Z}_+$. However, the constant configuration $1^{\mathbb{Z}^2}$ is in the orbit closure $\overline{\mathcal{O}(c)}$ and does not satisfy the condition.

The following lemma is Theorem 4.2.1 in the case $n = 1$.

***Lemma* 4.2.5.** *Let c be a sparse configuration and assume that it is periodic in direction $\mathbf{v}$ for some non-zero $\mathbf{v}$. Then c is a sum of finitely many periodic $\mathbf{v}$-fibers.*

Proof. Without loss of generality we may assume that c is $\mathbf{v}$-periodic by replacing $\mathbf{v}$ by $m\mathbf{v}$ for suitable $m \in \mathbb{Z}$.

Assume on the contrary that c is not a sum of finitely many periodic $\mathbf{v}$-fibers, that is, it contains infinitely many distinct non-zero periodic $\mathbf{v}$-fibers. Let $m \in \mathbb{Z}_+$ be such that $\mathbf{v} \in C_m$, and let $e_1, \ldots, e_t$ be t distinct non-zero fibers of c where $t = am+1$ and a is a sparseness constant of c. Let $n \in \mathbb{Z}_+$ be such that $\text{supp}(e_j) \cap C_n \neq \emptyset$ for each $j \in \{1, \ldots, t\}$.

Since each e_j is $\mathbf{v}$-periodic, we have $|\text{supp}(e_j) \cap C_{n+km}| > k$ for each $j \in \{1, \ldots, t\}$ and for all $k \in \mathbb{Z}_+$. Let $k = an + 1$. Then

$$\begin{aligned}
|\text{supp}(c) \cap C_{n+km}| &\geq \sum_{j=1}^{t} |\text{supp}(e_j) \cap C_{n+km}| \\
&> kt = (an+1)(am+1) = anam + an + am + 1 \\
&> an + aanm + am = a(n+km).
\end{aligned}$$

This is a contradiction with a being a sparseness constant of c. □

In fact, also the converse of the above lemma holds, that is, if a configuration is a sum of finitely many periodic $\mathbf{v}$-fibers, then it is sparse and periodic in direction $\mathbf{v}$. Indeed, clearly any $\mathbf{v}$-fiber e is sparse since

$$\text{supp}(e) \cap (C_m + \mathbf{t}) \subseteq (\mathbf{u} + \mathbb{Q}\mathbf{v}) \cap (C_m + \mathbf{t})$$

for some $\mathbf{u}$ and $|(\mathbf{u} + \mathbb{Q}\mathbf{v}) \cap (C_m + \mathbf{t})| \leq 2m + 1$. By Lemma 4.2.2 a sum of finitely many sparse configurations is sparse and hence a sum of finitely many periodic $\mathbf{v}$-fibers is sparse and periodic. Thus, a configuration is sparse and periodic in direction $\mathbf{v}$ if and only if it is a sum of finitely many periodic $\mathbf{v}$-fibers.

The following lemma is Theorem 4.2.1 in the case $n = 2$.

***Lemma* 4.2.6.** *Let c be a sparse configuration and assume that it is annihilated by the product $\varphi\psi$ of two line polynomials φ and ψ in non-parallel directions $\mathbf{v}$ and $\mathbf{u}$, respectively. Then*

$$c = c_1 + c_2$$

where c_1 is a sum of finitely many periodic $\mathbf{v}$-fibers and $\varphi c_1 = 0$, and c_2 is a sum of finitely many periodic $\mathbf{u}$-fibers and $\psi c_2 = 0$.

Proof. Consider the configuration $e_1 = \psi c$. It is sparse by Lemma 4.2.2. Moreover, $\varphi e_1 = \varphi\psi c = 0$ and hence e_1 is periodic in direction $\mathbf{v}$. Thus, by Lemma 4.2.5 the configuration e_1 is a sum of finitely many periodic $\mathbf{v}$-fibers. Similarly, we conclude that the configuration $e_2 = \varphi c$ is a sum of finitely many periodic $\mathbf{u}$-fibers.

Let $p \in \mathbb{Z} \setminus \{0\}$ be such that e_1 is $p\mathbf{v}$-periodic. Let c_1 be a limit of a converging subsequence of the sequence

$$c, X^{p\mathbf{v}}c, X^{2p\mathbf{v}}c, X^{3p\mathbf{v}}c, \ldots$$

of translated copies of c by multiples of $p\mathbf{v}$. By compactness of the configuration space such subsequence exists. Since $c_1 \in \overline{\mathcal{O}(c)}$ and c is sparse, by Lemma 4.2.3 also c_1 is sparse. Let $k_1, k_2, k_3, \ldots$ be such that $c_1 = \lim_{i\to\infty} X^{k_i p\mathbf{v}}c$. We have

$$\varphi c_1 = \varphi \lim_{i\to\infty} X^{k_i p\mathbf{v}}c = \lim_{i\to\infty} X^{k_i p\mathbf{v}}\varphi c = \lim_{i\to\infty} X^{k_i p\mathbf{v}}e_2 = 0.$$

Above we used the fact that the function $e \mapsto ge$ is a continuous function in the topology for a configuration e and polynomial g. The final equality holds because e_2 is a sum of finitely many $\mathbf{u}$-fibers. Moreover, we have

$$\psi c_1 = \psi \lim_{i\to\infty} X^{k_i p\mathbf{v}}c = \lim_{i\to\infty} X^{k_i p\mathbf{v}}\psi c = \lim_{i\to\infty} X^{k_i p\mathbf{v}}e_1 = e_1.$$

Again, we used the continuity of the function $e \mapsto ge$. The final equality holds because e_1 is $p\mathbf{v}$-periodic.

Similarly, we take $q \in \mathbb{Z} \setminus \{0\}$ such that e_2 is $q\mathbf{u}$-periodic. Then we define

$$c_2 = \lim_{i\to\infty} X^{t_i q\mathbf{u}}c$$

for a suitable sequence $t_1, t_2, t_3, \ldots$. Again, since $c_2 \in \overline{\mathcal{O}(c)}$ and c is sparse, by Lemma 4.2.3 also c_2 is sparse. By similar arguments as above, we have

$$\psi c_2 = \psi \lim_{i\to\infty} X^{t_i q\mathbf{u}}c = \lim_{i\to\infty} X^{t_i q\mathbf{u}}\psi c = \lim_{i\to\infty} X^{t_i q\mathbf{u}}e_1 = 0$$

and

$$\varphi c_2 = \varphi \lim_{i\to\infty} X^{t_i q\mathbf{u}}c = \lim_{i\to\infty} X^{t_i q\mathbf{u}}\varphi c = \lim_{i\to\infty} X^{t_i q\mathbf{u}}e_2 = e_2.$$

Let us show that $c = c_1 + c_2$. We have

$$\varphi(c - c_1 - c_2) = \varphi c - \varphi c_1 - \varphi c_2 = e_2 - 0 - e_2 = 0.$$

Hence, $c - c_1 - c_2$ is periodic in direction $\mathbf{v}$. By Lemma 4.2.2 it is sparse since c, c_1 and c_2 are all sparse. It follows by Lemma 4.2.5 that $c - c_1 - c_2$ is a sum of finitely many periodic $\mathbf{v}$-fibers. Similarly,

$$\psi(c - c_1 - c_2) = \psi c - \psi c_1 - \psi c_2 = e_1 - e_1 - 0 = 0$$

and hence $c - c_1 - c_2$ is a sum of finitely many periodic $\mathbf{u}$-fibers. So, $c - c_1 - c_2$ is both a sum of finitely many periodic $\mathbf{v}$-fibers and a sum of finitely many periodic $\mathbf{u}$-fibers. Since $\mathbf{v}$ and $\mathbf{u}$ are non-parallel, it follows that $c - c_1 - c_2 = 0$.

We have seen that

$$c = c_1 + c_2$$

where $\varphi c_1 = 0$ and $\psi c_2 = 0$. Moreover, both c_1 and c_2 are sparse. Thus, c_1 is a sum of finitely many periodic $\mathbf{v}$-fibers and c_2 is a sum of finitely many periodic $\mathbf{u}$-fibers by Lemma 4.2.5. The claim follows. □

Now, we are ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. The proof is by induction on n. First, let us consider the case $n = 1$. So, c is periodic in direction $\mathbf{v}_1$. Then by Lemma 4.2.5 it is a sum of finitely many periodic $\mathbf{v}_1$-fibers.

Assume then that $n > 1$ and that the claim holds for $n - 1$. Consider the configuration

$$c' = \varphi_n c.$$

It is sparse and annihilated by $\varphi_1 \cdots \varphi_{n-1}$. By the induction hypothesis

$$c' = c'_1 + \ldots + c'_{n-1}$$

where each c'_i is a sum of finitely many periodic $\mathbf{v}_i$-fibers and annihilated by φ_i.

So, for any $i \in \{1, \ldots, n-1\}$ the configuration c'_i is periodic in direction $\mathbf{v}_i$, that is, $k_i\mathbf{v}_i$-periodic for some $k_i \in \mathbb{Z} \setminus \{0\}$. Consider the sequence

$$c', X^{k_i\mathbf{v}_i}c', X^{2k_i\mathbf{v}_i}c', X^{3k_i\mathbf{v}_i}c', \ldots.$$

It converges to c'_i and hence $c'_i \in \overline{\mathcal{O}(c')}$. Let e be a limit of a converging subsequence of the sequence

$$c, X^{k_i\mathbf{v}_i}c, X^{2k_i\mathbf{v}_i}c, X^{3k_i\mathbf{v}_i}c, \ldots.$$

By compactness of the configuration space such subsequence exists.

We have $\varphi_n e = c_i'$. Since c is sparse and $e \in \overline{\mathcal{O}(c)}$, by Lemma 4.2.3 also e is sparse. Moreover, we have $\varphi_i \varphi_n e = 0$ since $\varphi_i c_i' = 0$. By Lemma 4.2.6 we have

$$e = e_i + e_n$$

where e_i is a sum of finitely many periodic $\mathbf{v}_i$-fibers and $\varphi_i e_i = 0$, and e_n is a sum of finitely many $\mathbf{v}_n$-fibers and $\varphi_n e_n = 0$. It follows that $\varphi_n e_i = \varphi_n e = c_i'$.

Now, we choose $c_i = e_i$ for $i \in \{1, \ldots, n-1\}$ and $c_n = c - c_1 - \ldots - c_{n-1}$. Clearly, $c = c_1 + \ldots + c_n$. Moreover, we have

$$\begin{aligned} \varphi_n c_n &= \varphi_n c - \varphi_n c_1 - \ldots - \varphi_n c_{n-1} \\ &= c' - c_1' - \ldots - c_{n-1}' \\ &= 0. \end{aligned}$$

Since for each $i \in \{1, \ldots, n-1\}$ the configuration c_i is sparse as a sum of finitely many periodic $\mathbf{v}_i$-fibers and c is sparse, also $c_n = c - c_1 - \ldots - c_{n-1}$ is sparse by Lemma 4.2.2. Thus, by Lemma 4.2.5 the configuration c_n is a sum of finitely many periodic $\mathbf{v}_n$-fibers. □

Theorem 4.2.1 together with Theorem 2.5.10 yields the following corollary.

***Corollary* 4.2.7.** *Let c be a sparse configuration and assume that it has a non-trivial annihilator. Then it is a sum of finitely many periodic fibers.*

Proof. By Theorem 2.5.10 the configuration c is annihilated by a polynomial

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$$

where the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are pairwise non-parallel and hence by Theorem 4.2.1 there exist configurations $c_1, \ldots, c_m$ such that each c_i is a sum of finitely many periodic $\mathbf{v}_i$-fibers and $c = c_1 + \ldots + c_m$. Thus, c is a sum of finitely many periodic fibers. □

5 Delone Sets

In this chapter we consider the local complexity and periodicity of Delone sets and $\mathbb{R}^d$-configurations. This chapter is based on [42]. The main results of the chapter are Theorem 5.2.4, Theorem 5.3.4, Theorem 5.3.11, Theorem 5.3.15, Theorem 5.3.16, and Theorem 5.4.4.

5.1 Delone sets

In this section we define Delone sets and some concepts concerning them. Our considerations are mostly adapted from [3; 59; 60; 61; 66].

Let d be a positive integer. A subset $S \subseteq \mathbb{R}^d$ of the d-dimensional Euclidean space is *uniformly discrete* if there exists a positive real number r such that any open ball of radius r in $\mathbb{R}^d$ contains at most one point of S. It is *relatively dense* if there exists a positive real number R such that any closed ball of radius R in $\mathbb{R}^d$ contains at least one point of S. A subset of $\mathbb{R}^d$ is *locally finite* if its intersection with any bounded set is finite. Clearly, any uniformly discrete set is also locally finite. We denote by $B_T(\mathbf{u})$ and $B^\circ_T(\mathbf{u})$ the closed and open balls, respectively, of radius T centered at $\mathbf{u} \in \mathbb{R}^d$ for any real number $T \geq 0$. In addition, we use the shorthand notations $B_T = B_T(\mathbf{0})$ and $B^\circ_T = B^\circ_T(\mathbf{0})$.

Definition 5.1.1. A set $S \subseteq \mathbb{R}^d$ is a (d-dimensional) *Delone set* if it is both uniformly discrete and relatively dense. The largest possible uniform discreteness constant r of a Delone set S is called the *packing radius* of S and the smallest possible relative denseness constant R of S is called the *covering radius* of S.

Example 5.1.2. Let $S \subseteq \mathbb{R}$ be a one-dimensional Delone set with packing radius r and covering radius R. Clearly $r \leq R$. We may have equality. Indeed, for example the Delone set $S = \mathbb{Z}$ has $r = R = \frac{1}{2}$. However, if the equality holds, then S is periodic. In fact, if the equality holds, we have $S = \{t + k \cdot 2R \mid k \in \mathbb{Z}\}$ for some $t \in \mathbb{R}$. However, for $d \geq 2$ we have $r < R$. In fact, $\sqrt{\frac{2d}{d+1}}r \leq R \leq 2r$ [80].

5.1.1 The patch-complexity of Delone sets

In the following we define the classical measure for local complexity of Delone sets.

Definition 5.1.3. Let $S \subseteq \mathbb{R}^d$ be a Delone set and let $T \geq 0$ be a real number. The T*-patch* of S centered at $\mathbf{s} \in S$ is the set

$$\mathcal{P}_S(\mathbf{s}, T) = S \cap B_T(\mathbf{s}).$$

We say that that two T-patches $\mathcal{P}_S(\mathbf{s}_1, T)$ and $\mathcal{P}_S(\mathbf{s}_2, T)$ of S are (translation) *equivalent* if $\mathcal{P}_S(\mathbf{s}_1, T) - \mathbf{s}_1 = \mathcal{P}_S(\mathbf{s}_2, T) - \mathbf{s}_2$ and denote $\mathcal{P}_S(\mathbf{s}_1, T) \sim \mathcal{P}_S(\mathbf{s}_2, T)$. Otherwise, we say that $\mathcal{P}_S(\mathbf{s}_1, T)$ and $\mathcal{P}_S(\mathbf{s}_2, T)$ are *inequivalent* and denote $\mathcal{P}_S(\mathbf{s}_1, T) \not\sim \mathcal{P}_S(\mathbf{s}_2, T)$. Clearly, any T-patch of a Delone set is a finite set since Delone sets are uniformly discrete and hence locally finite.

Definition 5.1.4. The *patch-counting function* $N_S(T)$ of a Delone set $S \subseteq \mathbb{R}^d$ gives for any $T \geq 0$ the number of distinct T-patches of S up to translation equivalence, that is,

$$N_S(T) = |\{(S \cap B_T(\mathbf{s})) - \mathbf{s} \mid \mathbf{s} \in S\}| = |\{(S - \mathbf{s}) \cap B_T \mid \mathbf{s} \in S\}|.$$

In general, $N_S(T)$ may be infinite for sufficiently large T.

It is known that sufficiently small patch-complexity of a Delone set implies its strong periodicity as the following theorem shows. Asymptotically the theorem implies that if $N_S(T) = o(T)$, then S is strongly periodic.

***Theorem* 5.1.5** ([60]). *Let S be a Delone set with covering radius R. If*

$$N_S(T) < \frac{T}{2R}$$

for some T, then S is strongly periodic, that is, it has d linearly independent periods.

Remark 5.1.6. In [60] the authors define T-patches using open balls instead of closed balls. So, for them, a T-patch of a Delone set S centered at $\mathbf{s} \in S$ is the set $S \cap B_T^\circ(\mathbf{s})$ instead of the set $S \cap B_T(\mathbf{s})$. However, all the results we cite from [60] (including the Theorem 5.1.5) work also with our standard definition.

The following two lemmas were used in the proof of Theorem 5.1.5. A *lattice* is a finitely generated discrete additive subgroup of $\mathbb{R}^d$ that spans $\mathbb{R}^d$. The first lemma says that if the patch-counting function of a Delone set is bounded, then it is a union of finitely many cosets of a lattice and hence strongly periodic. The second lemma says that if the patch-counting function gives the same value for some T_1 and $T_2 > T_1 + 2R$, then it remains the same for all $T \geq T_1$ where R is the covering radius of the Delone set in consideration.

***Lemma* 5.1.7** ([60]). *Let $S \subseteq \mathbb{R}^d$ be a Delone set and assume that $N_S(T) \leq N$ for all $T > 0$. If N is minimal, that is, if for some T_0 we have $N_S(T_0) = N$, then S is union of N cosets of a lattice. Moreover, S is strongly periodic.*

***Lemma* 5.1.8** ([60]). *Let $S \subseteq \mathbb{R}^d$ be a Delone set with covering radius R. If there exist real numbers $T_1 > 0$ and $T_2 > T_1 + 2R$ such that $N_S(T_2) = N_S(T_1)$, then*

$$N_S(T) = N_S(T_1)$$

for all $T \geq T_1$ and S is strongly periodic.

Remark 5.1.9. In the literature, and in the statement of Theorem 5.1.5, strongly periodic Delone sets are called *ideal crystals*. The standard definition of ideal crystals looks quite different from our definition of strongly periodic Delone sets. Indeed, in [60] a Delone set $S \subseteq \mathbb{R}^d$ is defined to be an ideal crystal if it has a full rank lattice of translational symmetries, that is, if there exists a finite set $F \Subset \mathbb{R}^d$ such that $S = F + \Lambda_S$ where $\Lambda_S = \{\mathbf{t} \in \mathbb{R}^d \mid S + \mathbf{t} = S\}$ is the set of translational symmetries of S. However, the definitions are equivalent: If a Delone set S is strongly periodic, then it has d linearly independent periods $\mathbf{v}_1, \ldots, \mathbf{v}_d$ over $\mathbb{R}$. Since S is a Delone set, the set

$$F = S \cap \{a_1\mathbf{v}_1 + \ldots + a_d\mathbf{v}_d \mid a_1, \ldots, a_d \in [0, 1)\}$$

is finite. It follows that $S = F + \Lambda_S$ and hence S is an ideal crystal. Conversely, assume that S is an ideal crystal, that is, $S = F + \Lambda_S$ for some $F \Subset \mathbb{R}^d$. If S is not strongly periodic, then Λ_S is contained in some subspace V of $\mathbb{R}^d$ of dimension at most $d - 1$. This implies that $S = F + \Lambda_S$ is not relatively dense. A contradiction.

Remark 5.1.10. In the proof of Theorem 5.1.5 it was shown that the condition that $N_S(T) < \frac{T}{2R}$ for some T implies that $N_S(T) = N_S(T + 2R + \varepsilon)$ for some $\varepsilon > 0$. This implies that $N_S(T') = N_S(T)$ for all $T' \geq T$, that is, the patch-counting function is bounded which implies that S is strongly periodic by Lemmas 5.1.7 and 5.1.8. There is an analogous result in symbolic dynamics — the *Morse-Hedlund theorem* — saying that if a bi-infinite word, that is, a one-dimensional configuration w has for some n the same number of factors of length n and $n+1$, then the number of factors of w of any length $n' \geq n$ remains the same. In fact, this result can be generalized to any dimension: It is straight-forward to verify that if a d-dimensional configuration c has $P_c(\{0, \ldots, n+1\}^d) = P_c(\{0, \ldots, n\}^d)$ for some $n \geq 0$, then $P_c(\{0, \ldots, n+k\}^d) = P_c(\{0, \ldots, n\}^d)$ for all $k \geq 0$ which then implies that c is strongly periodic. This was also proved in [60].

In [60] it was noted that the coefficient $\frac{1}{2R}$ in Theorem 5.1.5 is optimal in the sense that for given $R > 0$ and for any $M > \frac{1}{2R}$ there exist Delone sets $S \subseteq \mathbb{R}^d$ with covering radius R that have $N_S(T) < MT$, for some $T > 0$, and that are not strongly periodic. The construction is described in the following example. Since in [60] patches are defined using open balls, the construction in the example is slightly different from the original construction.

Example 5.1.11. Consider a fixed $R > 0$ and let $M > \frac{1}{2R}$. There exist non-periodic 1-dimensional Delone sets whose packing and covering radii are arbitrarily close to each other. (For example, the non-periodic Delone set $\mathbb{Z} \setminus \{0\} \cup \{\epsilon\}$ has packing packing radius $\frac{1-\epsilon}{2}$ and covering radius $\frac{1+\epsilon}{2}$ which are close for small ϵ.) In particular, there exists a Delone set $S \subseteq \mathbb{R}$ with packing radius r and covering radius R such that $\frac{1}{2R} < \frac{1}{2r} < M$ where R and M are as given above. By the assumption $M \cdot 2r = 1 + \epsilon$ for some $\epsilon > 0$. Take $T = 2r - \delta > 0$ where $0 < \delta < 2r$ is such that $\delta < \epsilon/M$. We have $T < 2r$ and hence $N_S(T) = 1$. Moreover, $MT = M(2r - \delta) = M \cdot 2r - M\delta = 1 + \epsilon - M\delta > 1$. So, S is non-periodic and satisfies $N_S(T) < MT$.

In dimension $d = 1$ we prove the following better bound.

***Theorem* 5.1.12.** *Let $S \subseteq \mathbb{R}$ be a 1-dimensional Delone set with covering radius R and assume that $N_S(T) < \frac{T}{R} - 1$ for some $T > 0$. Then S is periodic.*

Proof. Let $S = \{s_i \mid i \in \mathbb{Z}\}$ where $s_i < s_{i+1}$ for all $i \in \mathbb{Z}$. Define a function $c \in \mathcal{A}^{\mathbb{Z}}$ such that $c_i = s_{i+1} - s_i$ and hence $\mathcal{A} = \{s_{i+1} - s_i \mid i \in \mathbb{Z}\}$.

First, let us show that $\mathcal{A}$ is a finite set, that is, c is a configuration. Notice that from the bound $N_S(T) < \frac{T}{R} - 1$ it follows that $T > 2R$. Indeed, since $N_S(T) \geq 1$ for any $T > 0$, we have $\frac{T}{R} - 1 > 1$ and hence $T > 2R$. Also, note that $s_{i+1} - s_i \leq 2R$ for any $i \in \mathbb{Z}$, that is, the gap between two consecutive points of S is at most $2R$. Thus, if $\mathcal{A}$ was an infinite set, then also $N_S(2R)$ and hence also $N_S(T)$ would be infinite, a contradiction. Thus, $\mathcal{A}$ is a finite set and hence c is a configuration.

Write $T = m \cdot 2R + q$ where m is a positive integer and $0 \leq q < 2R$. Let us consider the number $P_c(2m)$ of factors of c of length $2m$. Any factor $c_k \cdots c_{k+2m-1}$ of c of length $2m$ is determined by the $2m + 1$ consecutive points $s_k, \ldots, s_{k+2m}$ of S. Moreover, since the length of the gap between any two consecutive points of S is at most $2R$, these points are contained in the T-patch of S centered at s_{k+m}. See Figure 9 for illustration. Thus, we have

$$P_c(2m) \leq N_S(T) < \frac{T}{R} - 1 = \frac{m \cdot 2R + q}{R} - 1 = 2m + \frac{q}{R} - 1 < 2m + 1$$

and hence $P_c(2m) \leq 2m$ which implies by the Morse-Hedlund theorem that c is periodic. Thus, also S is periodic. □

In the following example we see that the above bound of Theorem 5.1.12 is optimal in the sense that for any given $R > 0$ there exist arbitrarily large T and a non-periodic Delone set $S \subseteq \mathbb{R}$ with covering radius R such that $N_S(T)$ is arbitrarily close to $\frac{T}{R} - 1$. In other words, the following example shows that for fixed R and a Delone set $S \subseteq \mathbb{R}$ whose covering radius is R the condition that $N_S(T) < N$ for some T does not imply the periodicity of S if $N > \frac{T}{R} - 1$.

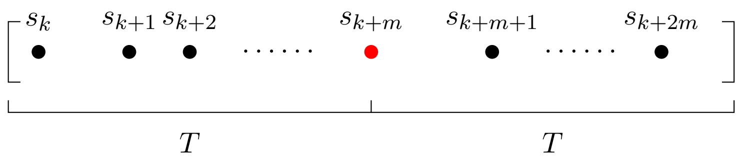

Figure 9. Illustration of the proof of Theorem 5.1.12. The T-patch of S centered at s_{k+m} contains the points $s_k, \ldots, s_{k+2m}$ and hence determines the factor $c_k \cdots c_{k+2m-1}$ of the 1-dimensional configuration c.

Example 5.1.13. Let $w \in \{0,1\}^{\mathbb{Z}}$ be a Sturmian word, that is, w is non-periodic and $P_w(n) = P_w(\{0, \ldots, n-1\}) = n+1$ for all $n \geq 1$. Let us fix a positive real number R and let $\varepsilon < 2R$. Define a set $S = \{s_i \mid i \in \mathbb{Z}\}$ such that

- $s_0 = 0$, and
- for all $i \in \mathbb{Z}$
$$s_i - s_{i-1} = \begin{cases} 2R, & \text{if } w_i = 0 \\ 2R - \varepsilon, & \text{if } w_i = 1 \end{cases}.$$

(A pictorial illustration is given in Figure 10.)

Clearly, S is a Delone set with covering radius R and packing radius $r = R - \frac{\varepsilon}{2}$. Moreover, S is non-periodic since w is non-periodic.

Let $m \in \mathbb{Z}_+$ be arbitrary, and set $T = m \cdot 2R + q$ where $q > R$. Let us assume that ε is so small that any T-patch of S contains exactly $2m+1$ points. We claim that $N_S(T) = 2m+1$.

So, consider the T-patch $\mathcal{P}_S(s_i, T) = \{s_{i-m}, \ldots, s_i, \ldots, s_{i+m}\}$ of S centered at s_i for some i. It is determined by the length $2m$ factor

$$w_{i-m+1} \cdots w_i \cdots w_{i+m}$$

of w. Thus, $N_S(T) \leq P_c(2m)$ and since w is a Sturmian word, we have $P_c(2m) = 2m+1$ and hence $N_S(T) \leq 2m+1$. On the other hand, since S is non-periodic we have

$$N_S(T) \geq \frac{T}{R} - 1 = \frac{m \cdot 2R + q}{R} - 1 = 2m + \frac{q}{R} - 1$$

by Theorem 5.1.12. Since $q > R$, it follows that $N_S(T) \geq 2m+1$. Thus,

$$2m + 1 \leq N_S(T) \leq 2m+1$$

and hence $N_S(T) = 2m+1$.

If $q > R$ is close to $2R$, then $\frac{T}{R} - 1$ is close to $2m+1$, that is, $N_S(T)$ is close to $\frac{T}{R} - 1$.

By Considering the set $S' = S \times \mathbb{Z}^{d-1}$ we may generalize this construction to any dimension $d > 1$, that is, S' has $N_{S'}(T) = 2m+1$ but S' is not strongly periodic since S is not periodic.

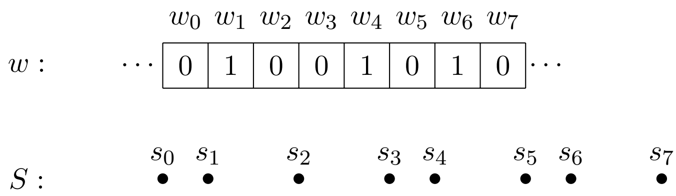

Figure 10. Illustration of the Delone set S of Example 5.1.13 with $R = \varepsilon = 1$.

In fact, Theorem 5.1.12 holds in arbitrary dimension [47].

See Figure 11 for a pictorial illustration of the old and new bounds of forced periodicity for fixed covering radius R.

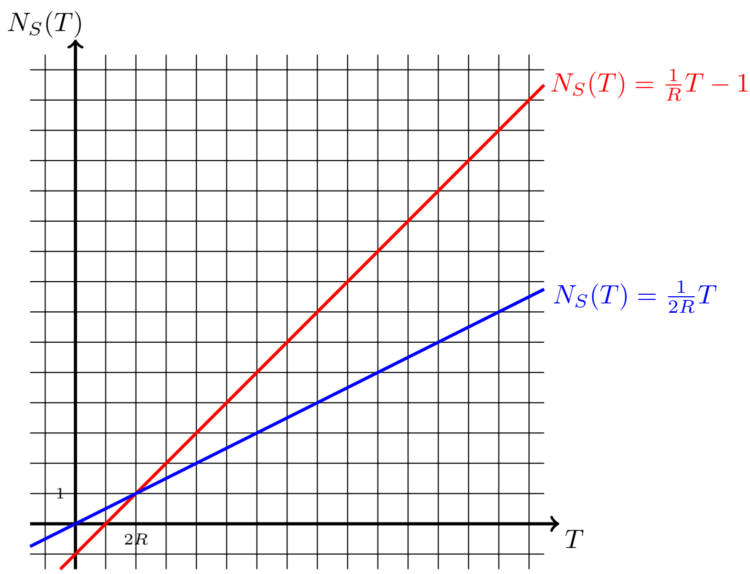

Figure 11. The bounds of forced periodicity of Theorems 5.1.5 and 5.1.12 with blue and red, respectively, for fixed R.

5.1.2 Classes of Delone sets

There are three well known classes of Delone sets.

Definition 5.1.14. A Delone set S is a *Delone set of finite local complexity* (FLC) if $N_S(T)$ is finite for every T.

Sometimes Delone sets of finite local complexity are also called *Delone sets of finite type* since they have only finitely many distinct "local neighborhoods". Equivalently, a Delone set S is a Delone set of finite local complexity if the set $S - S$ is locally finite. In fact, already if the set $(S-S)\cap B_{2R}$ is finite where R is a relative denseness constant of S, then S is a Delone set of finite local complexity [59].

Definition 5.1.15. A Delone set S is a *Meyer set* if also $S - S$ is a Delone set.

Clearly, any Meyer set is also a Delone set of finite local complexity. The class of Meyer sets was originally introduced by Meyer in [66] under the name "quasicrystal" as the set of Delone sets S such that $S - S \subseteq S + F$ for some finite set $F \Subset \mathbb{R}^d$. However, this definition is equivalent to Definition 5.1.15 [58]. Sometimes Meyer sets are called also *almost lattices* [3]. It is quite easily seen that all strongly periodic Delone sets are Meyer sets:

***Lemma* 5.1.16.** *If a Delone set $S \subseteq \mathbb{R}^d$ is strongly periodic, then it is a Meyer set.*

Proof. As noted in Remark 5.1.9 strongly periodic Delone sets are ideal crystals, that is, there exists a finite set $F \Subset \mathbb{R}^d$ such that $S = F + \Lambda_S$ where $\Lambda_S = \{\mathbf{t} \in \mathbb{R}^d \mid S + \mathbf{t} = S\}$. Then

$$\begin{aligned} S - S &= F + \Lambda_S - (F + \Lambda_S) \\ &= F - F + \Lambda_S - \Lambda_S \\ &= F - F + \Lambda_S \\ &= F + \Lambda_S - F \\ &= S + (-F). \end{aligned}$$

Thus, S is a Meyer set by the original definition of Meyer sets. □

The statement in the above lemma is not an equivalence since there are also non-periodic Meyer sets (for example the set $\mathbb{Z} \setminus \{0\}$).

Let us denote by $[S] = \mathbb{Z}[S] = \{a_1\mathbf{s}_1 + \ldots + a_k\mathbf{s}_k \mid \mathbf{s}_i \in S, a_i \in \mathbb{Z}, k \in \mathbb{N}\}$ the abelian group generated by $S \subseteq \mathbb{R}^d$.

Definition 5.1.17. A Delone set S is a *finitely generated Delone set* if the set $[S - S]$ or equivalently the set $[S]$ is a finitely generated abelian group.

The class of finitely generated Delone sets contains the class of Delone sets of finite local complexity, that is, any Delone set of finite local complexity is a finitely generated Delone set [59].

In the following example we see that the mentioned inclusions of different classes of Delone sets are strict.

Example 5.1.18. We give 1-dimensional examples of a Delone set which is not a finitely generated Delone set, a finitely generated Delone set which is not a Delone set of finite local complexity, and a Delone set of finite local complexity which is not a Meyer set. See Figure 12 for pictorial illustrations. In the following, we use the fact that any subgroup of a finitely generated abelian group is also a finitely generated abelian group. This follows from the fundamental theorem of finitely generated abelian groups [22].

- Consider the Delone set
$$S_1 = \{n + \frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\}\}.$$
It can be quite easily verified that $\mathbb{Q} \subseteq [S_1]$. Since $\mathbb{Q}$ is not a finitely generated abelian group, neither is its superset $[S_1]$. So, S_1 is not a finitely generated Delone set.

- Consider the Delone set
$$S_2 = \{n\pi \mid n \in \mathbb{Z}\} \cup \mathbb{Z} \setminus \{\lfloor n\pi \rfloor, \lceil n\pi \rceil \mid n \in \mathbb{Z}\}.$$
We have $[S_2 - S_2] \subseteq \mathbb{Z}[1, \pi]$ which means that S_2 is a finitely generated Delone set since any subgroup of a finitely generated abelian group is finitely generated as noted above. However, the set of gaps between consecutive points of S_2 is infinite which implies that $S_2 - S_2$ is not locally finite. Hence, S_2 is not a Delone set of finite local complexity. So, S_2 is a finitely generated Delone set but not a Delone set of finite local complexity.

- Consider the Delone set
$$S_3 = -\mathbb{N} \cup \{n\pi \mid n \in \mathbb{N}\}.$$
Since there are only two kinds of gaps between two consecutive points of S_3, it is a Delone set of finite local complexity. However, we have $\mathbb{N} \subseteq S_3 - S_3$ and $\{n\pi \mid n \in \mathbb{N}\} \subseteq S_3 - S_3$. The latter set does not contain any integers, but it contains real numbers that are arbitrarily close to positive integers by the irrationality of π. This means that $S_3 - S_3$ is not uniformly discrete and hence not a Delone set implying that S_3 is not a Meyer set.

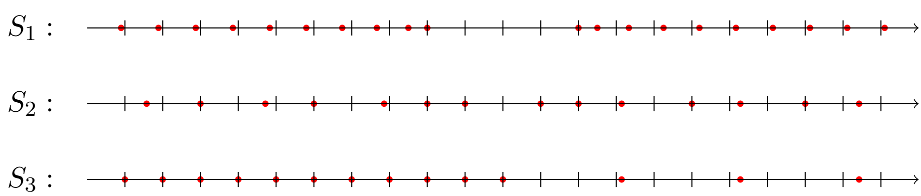

Figure 12. The Delone sets defined in Example 5.1.18.

The following lemma states that the family of Delone sets of finite local complexity is closed under addition with non-empty finite sets. We will use this lemma later in our considerations.

***Lemma* 5.1.19.** *Let $S \subseteq \mathbb{R}^d$ be a Delone set of finite local complexity and let $F \Subset \mathbb{R}^d$ be a non-empty finite set. Then also $S + F$ is a Delone set of finite local complexity.*

Proof. Clearly, $S + F$ is relatively dense since it contains a translation of S as a subset. Let us show that it is uniformly discrete too. For this, let $\mathbf{e}_1, \mathbf{e}_2 \in S + F$ be arbitrary but different. So, we have $\mathbf{e}_1 = \mathbf{s}_1 + \mathbf{f}_1$ and $\mathbf{e}_2 = \mathbf{s}_2 + \mathbf{f}_2$ for some $\mathbf{s}_1, \mathbf{s}_2 \in S$ and $\mathbf{f}_1, \mathbf{f}_2 \in F$ and hence

$$||\mathbf{e}_1 - \mathbf{e}_2|| = ||\mathbf{s}_1 + \mathbf{f}_1 - (\mathbf{s}_2 + \mathbf{f}_2)|| = ||(\mathbf{s}_1 - \mathbf{s}_2) - (\mathbf{f}_2 - \mathbf{f}_1)||.$$

Since $\mathbf{e}_1 \neq \mathbf{e}_2$, we also have $\mathbf{s}_1 - \mathbf{s}_2 \neq \mathbf{f}_2 - \mathbf{f}_1$ and since F is finite, $\mathbf{f}_2 - \mathbf{f}_1$ belongs to a finite set $F - F$. By the fact that $S - S$ is locally finite, $||\mathbf{s}_1 + \mathbf{f}_1 - (\mathbf{s}_2 + \mathbf{f}_2)||$ has a positive lower bound which is also a lower bound of $||\mathbf{e}_1 - \mathbf{e}_2||$ by the above computation. Thus, $S + F$ is uniformly discrete and hence a Delone set.

Finally, let us show that $S + F - (S + F)$ is locally finite, that is, $S + F$ is a Delone set of finite local complexity. For this it is enough to show that $\big(S + F - (S + F)\big) \cap B_\varepsilon(\mathbf{0})$ is finite for arbitrary $\varepsilon > 0$. This fact follows from computations

$$(S + F - (S + F)) \cap B_\varepsilon(\mathbf{0}) \subseteq ((S - S) \cap B_{\varepsilon+T}(\mathbf{0})) + F - F$$

and from the assumption that S is locally finite where $T = \max\{||a|| \mid a \in F - F\}$. Thus, the set $S + F$ is a Delone set of finite local complexity too. □

Remark 5.1.20. Note that the above lemma does not hold for arbitrary Delone sets, that is, the sum of a Delone set and a finite set is not necessarily a Delone set. Indeed, consider the Delone set

$$S = S_1 = \{n + \frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\}\}$$

defined in Example 5.1.18 and let $F = \{0, 1\}$. We have $n + \frac{1}{n} \in S + F$ and $n + \frac{1}{n-1} \in S + F$ for arbitrarily large n. Thus, $S + F$ is not uniformly discrete and hence not a Delone set. (In fact, we could show that the lemma does not hold even for finitely generated Delone sets considering the finitely generated Delone set $S = S_2 = \{n\pi \mid n \in \mathbb{Z}\} \cup \mathbb{Z} \setminus \{\lfloor n\pi \rfloor, \lceil n\pi \rceil \mid n \in \mathbb{Z}\}$ and the finite set $F = \{0, 1\}$ as above.)

5.1.3 Delone configurations and some algebraic concepts

We call a finitary function $c \in \mathbb{C}^{\mathbb{R}^d}$ (that is, an $\mathbb{R}^d$-configuration) a *Delone configuration*, a *finitely generated Delone configuration*, a *Delone configuration of finite local complexity* or a *Meyer configuration* if its support supp(c) is a Delone set, a finitely generated Delone set, a Delone set of finite local complexity or a Meyer set, respectively.

Let us extend our terminology of annihilators and periodizers to Delone sets and $\mathbb{R}^d$-configurations. First, an $\mathbb{R}^d$*-polynomial* is a finitely supported function $f \in \mathbb{C}^{\mathbb{R}^d}$. We use the convenient "polynomial notation" also for $\mathbb{R}^d$-polynomials, that is, we denote

$$f = f(X) = \sum_{\mathbf{u} \in \text{supp}(f)} f_{\mathbf{u}} X^{\mathbf{u}}.$$

An $\mathbb{R}^d$-polynomial $f \in \mathbb{C}^{\mathbb{R}^d}$ annihilates (or is an annihilator of) a function $c \in \mathbb{C}^{\mathbb{R}^d}$ if $f * c = 0$, and it periodizes (or is a periodizer of) c if $f * c$ is strongly periodic. Consequently, f annihilates or periodizes a set $S \subseteq \mathbb{R}^d$ if it annihilates or periodizes, respectively, its indicator function $\mathbb{1}_S$. From now on we denote $fc = f * c$.

We generalize the definition of line polynomials and say that an $\mathbb{R}^d$-polynomial f is a *line* $\mathbb{R}^d$*-polynomial* if its support supp(f) contains at least two points, and supp(f) is contained in a line, that is, supp$(f) \subseteq \mathbf{u} + \mathbb{R}\mathbf{v}$ for some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. Again, the vector $\mathbf{v}$ is called a direction of f. Analogously, we generalize the definition of difference polynomials and say that an $\mathbb{R}^d$-polynomial is a *difference* $\mathbb{R}^d$*-polynomial* if it is of the form $X^{\mathbf{v}} - 1$.

As in the case of $\mathbb{Z}^d$-configurations, an $\mathbb{R}^d$-configuration is periodic if and only if it is annihilated by a non-trivial difference $\mathbb{R}^d$-polynomial. Note that annihilation of an $\mathbb{R}^d$-configuration by a line $\mathbb{R}^d$-polynomial does not necessarily imply periodicity as we will see later in this chapter. This differs from $\mathbb{Z}^d$-configurations for which annihilation by a line polynomial implies periodicity. However, if the $\mathbb{R}^d$-configuration in consideration is a Delone configuration of finite local complexity, then annihilation by a line $\mathbb{R}^d$-polynomial indeed implies periodicity as we will be proved in Lemma 5.3.8.

5.1.4 An example

In this subsection, we consider an example of a family of Delone sets with non-trivial annihilators. These Delone sets are obtained from certain rotations of the *torus*. For a real number $r \in \mathbb{R}$, we denote by $\lfloor r \rfloor = \max\{z \in \mathbb{Z} \mid z \leq r\}$ and $\{r\} = r - \lfloor r \rfloor$ the integer and fractional parts of r, respectively.

The (2-dimensional) torus is the set $\mathbb{T}^2 = [0,1) \times [0,1)$ equipped with the metric d such that

$$d((u_1, u_2), (v_1, v_2)) = \max\{d_1(u_1, v_1), d_1(u_2, v_2)\}$$

for all $(u_1, u_2), (v_1, v_2) \in \mathbb{T}^2$ where $d_1(u, v) = \min\{|u - v|, 1 - |u - v|\}$ for all $u, v \in [0, 1)$. In fact, d_1 is the metric in the 1-dimensional torus, that is, the circle $[0, 1)$. A *rotation* $\rho_{a,b} \colon \mathbb{T}^2 \to \mathbb{T}^2$ of the torus by the parameters $a, b \in \mathbb{R}$ is defined such that

$$\rho_{a,b}(u_1, u_2) = (\{u_1 + a\}, \{u_2 + b\})$$

for all $(u_1, u_2) \in \mathbb{T}^2$. For a reference on torus rotations, see *e.g.* Section 1.11 in [57].

The torus $\mathbb{T}^2$ is partitioned into sets

$$A_0 = \{(u_1, u_2) \in \mathbb{T}^2 \mid 0 \leq u_1 + u_2 < 1\}$$

and

$$A_1 = \{(u_1, u_2) \in \mathbb{T}^2 \mid 1 \leq u_1 + u_2 < 2\}.$$

For a vector $\mathbf{z} = (z_1, z_2) \in \mathbb{T}^2$ and a non-zero real number α, define a binary $\mathbb{Z}^2$-configuration $c_{\mathbf{z},\alpha} \in \{0,1\}^{\mathbb{Z}^2}$ such that $c_{\mathbf{z},\alpha}(i,j) = 1$ if $\rho_{i\alpha,j\alpha}(\mathbf{z}) \in A_1$ and $c_{\mathbf{z},\alpha}(i,j) = 0$ if $\rho_{i\alpha,j\alpha}(\mathbf{z}) \in A_0$. Define a Delone set

$$S_{\mathbf{z},\alpha} = \{\mathbf{z} + (i,j) \mid c_{\mathbf{z},\alpha}(i,j) = 1\} \subseteq \mathbb{R}^2.$$

The configurations $c_{\mathbf{z},\alpha}$ and hence the Delone sets $S_{\mathbf{z},\alpha}$ have non-trivial annihilators. The proof of this fact goes by observing that the configurations $c_{\mathbf{z},\alpha}$ are sums of periodic functions:

***Lemma* 5.1.21.** *For any $\mathbf{z} = (z_1, z_2) \in \mathbb{T}^2$ and non-zero $\alpha \in \mathbb{R}$, we have*

$$c_{\mathbf{z},\alpha}(i,j) = \lfloor z_1 + z_2 + (i+j)\alpha \rfloor - \lfloor z_1 + i\alpha \rfloor - \lfloor z_2 + j\alpha \rfloor$$

for all $(i,j) \in \mathbb{Z}^2$. Consequently, $c_{\mathbf{z},\alpha}$ is annihilated by the polynomial

$$(X^{(1,-1)} - 1)(X^{(0,1)} - 1)(X^{(1,0)} - 1).$$

Proof. The first claim follows from the following computation:

$$\begin{aligned}
&\lfloor z_1 + z_2 + (i+j)\alpha \rfloor - \lfloor z_1 + i\alpha \rfloor - \lfloor z_2 + j\alpha \rfloor \\
&= \{z_1 + i\alpha\} + \{z_2 + j\alpha\} - \{z_1 + z_2 + (i+j)\alpha\} \\
&= \lfloor \{z_1 + i\alpha\} + \{z_2 + j\alpha\} \rfloor + \{\{z_1 + i\alpha\} + \{z_2 + j\alpha\}\} - \{z_1 + z_2 + (i+j)\alpha\} \\
&= \lfloor \{z_1 + i\alpha\} + \{z_2 + j\alpha\} \rfloor \\
&= \begin{cases} 1, \text{ if } \{z_1 + i\alpha\} + \{z_2 + j\alpha\} \in A_1 \\ 0, \text{ if } \{z_1 + i\alpha\} + \{z_2 + j\alpha\} \in A_0 \end{cases} \\
&= c_{\mathbf{z},\alpha}(i,j).
\end{aligned}$$

So, we have $c_{\mathbf{z},\alpha} = c_1 + c_2 + c_3$ where $c_1(i,j) = \lfloor z_1 + z_2 + (i+j)\alpha \rfloor$, $c_2(i,j) = -\lfloor z_1 + i\alpha \rfloor$, and $c_3(i,j) = -\lfloor z_2 + j\alpha \rfloor$ for all $(i,j) \in \mathbb{Z}^2$. These functions are annihilated by the polynomials $X^{(1,-1)} - 1$, $X^{(0,1)} - 1$, and $X^{(1,0)} - 1$, respectively. Thus, $c_{\mathbf{z},\alpha}$ is annihilated by the product of these polynomials. □

Remark 5.1.22. In [55; 54] the authors gave the configuration $c_{\mathbf{0},\alpha}$ for an irrational α as an example of a configuration that has a non-trivial annihilator but cannot be expressed as a sum of periodic finitary functions. This was also discussed in Example 2.5.16 where it was called the snowflake configuration.

5.2 Meyer sets with slow patch-complexity growth

In this section we consider Meyer sets with slow patch-complexity growth. In particular, we show that if a Meyer set S satisfies $\liminf_{T\to\infty} \frac{N_S(T)}{T^d} = 0$, then it has low complexity with respect to some $\mathbb{R}^d$-shape. Consequently, it has a non-trivial annihilator as we will see in Section 5.3. Note that the condition $\liminf_{T\to\infty} \frac{N_S(T)}{T^d} = 0$ means that $N_S(T) = o(T^d)$.

Consider the set $U = (S + S - S) \cap B_R$ for a Meyer set S with covering radius R. The following lemma shows that this set is finite.

***Lemma* 5.2.1.** *Let $S \subseteq \mathbb{R}^d$ be a Meyer set. The set $S + S - S$ is locally finite.*

Proof. Without loss of generality we may assume that S is translated such that $\mathbf{0} \in S$. Since S is a Meyer set, there exists a finite set F such that $S - S \subseteq S + F$. Using this twice and the fact that $\mathbf{0} \in S$ we get

$$\begin{aligned} S + S - S &\subseteq S + S - S - S = S - S - (S - S) \\ &\subseteq S + F - (S + F) = S - S + F - F \\ &\subseteq S + F + F - F. \end{aligned}$$

Lemma 5.1.19 says that the sum of a Delone set of finite local complexity and a non-empty finite set is also a Delone set of finite local complexity. In particular, $S + F + F - F$ is uniformly discrete since $F + F - F$ is a non-empty finite set. This implies that $S + S - S$ is also uniformly discrete as a subset of a uniformly discrete set. The claim follows since uniformly discrete sets are locally finite. □

Define for every $T \geq 0$ a set $H_T = U \cup (S \cap B_T)$. This is a finite set since U is a finite set as seen above and since S is a Delone set (and hence $S \cap B_T$ is a finite set). We start by the observation that any non-empty H_T-pattern of S contains an element of S already in its U-part.

***Lemma* 5.2.2.** *Let $S \subseteq \mathbb{R}^d$ be a Meyer set with covering radius R, and let $U = (S + S - S) \cap B_R$. Define for every $T \geq 0$ a set $H_T = U \cup (S \cap B_T)$. If $S \cap (H_T + \mathbf{t}) \neq \emptyset$, then also $S \cap (U + \mathbf{t}) \neq \emptyset$.*

Proof. Assume that $\mathbf{u} \in S \cap (H_T + \mathbf{t})$. If $\mathbf{u} \in U + \mathbf{t}$, then the claim is valid and we are done. So, assume that $\mathbf{u} \in (H_T \setminus U) + \mathbf{t}$. Since $H_T \setminus U \subseteq S$, we have $\mathbf{u} - \mathbf{t} \in S$. By the definition of R we have $S \cap B_R(\mathbf{t}) \neq \emptyset$. So, let $\mathbf{w} \in S \cap B_R(\mathbf{t})$. We claim that $\mathbf{w} \in U + \mathbf{t}$, *i.e.*, $\mathbf{w} - \mathbf{t} \in U$. This follows from the observation that

$$\mathbf{w} - \mathbf{t} = \mathbf{w} - \mathbf{u} + \mathbf{u} - \mathbf{t} = \mathbf{w} + (\mathbf{u} - \mathbf{t}) - \mathbf{u} \in S + S - S$$

and from the fact that $\mathbf{w} - \mathbf{t} \in B_R$. □

We have the following upper bound for $P_S(H_T)$.

***Lemma* 5.2.3.** *Let $S \subseteq \mathbb{R}^d$ be a Meyer set with covering radius R, and let $U = (S+S-S) \cap B_R$. Define for every $T \geq 0$ a set $H_T = U \cup (S \cap B_T)$. Then*

$$P_S(H_T) \leq 1 + |U| N_S(T+R).$$

Proof. Let P be a non-empty H_T-pattern of S, that is, $P = S \cap (H_T + \mathbf{t}) \neq \emptyset$ for some $\mathbf{t} \in \mathbb{R}^d$. Then by Lemma 5.2.2 it contains a point already in its U-part, that is, $\mathbf{u} + \mathbf{t} \in P$ for some $\mathbf{u} \in U$. Clearly, P is contained in the $(T+R)$-patch of S centered at $\mathbf{u} + \mathbf{t}$. So, the $(T+R)$-patch centered at $\mathbf{u} + \mathbf{t}$ together with $\mathbf{u}$ uniquely determines the H_T-pattern centered at $\mathbf{t}$. Hence, the number of non-empty H_T-patterns up to translation is at most $|U| N_S(T+R)$. Thus,

$$P_S(H_T) \leq 1 + |U| N_S(T+R)$$

since the empty set is always in $\mathcal{L}_S(H_T)$. □

Let us now analyze the size of H_T. It contains the set $S \cap B_T$. Since R is the covering radius of S, any ball of radius R must contain at least one point of S. Hence, the number of points of $S \cap B_T$ is at least the number of separate balls of radius R contained in B_T. We give a rough lower bound for $|S \cap B_T|$ which is also a lower bound for $|H_T|$. Consider the hypercube

$$C = \left\{ (u_1, \ldots, u_d) \mid |u_i| \leq \frac{T}{\sqrt{d}} \right\}$$

contained in B_T with sides of length $\frac{2T}{\sqrt{d}}$. It is contained in B_T. Write

$$\frac{2T}{\sqrt{d}} = k \cdot 2R + q$$

where k is a non-negative integer — the integer part of $\frac{T}{\sqrt{d}R}$ — and $q < 2R$ is a non-negative real number. Clearly, C contains at least k^d separate hypercubes with sides of length $2R$ and hence at least k^d separate balls of radius R. Since R is the covering radius of S, each ball of radius R contains at least one point of S. Thus,

$$|H_T| \geq |S \cap B_T| \geq |S \cap C| \geq k^d = \left(\frac{T}{\sqrt{d}R} - \frac{q}{2R} \right)^d$$

and hence $\liminf_{T \to \infty} \frac{|H_T|}{T^d} > 0$, that is, $|H_T| = \Omega(T^d)$.

Now, we are ready to prove that for Meyer sets sufficiently slow growth of patch-complexity implies low $\mathbb{R}^d$-pattern complexity.

***Theorem* 5.2.4.** *Let $S \subseteq \mathbb{R}^d$ be a Meyer set with covering radius R, and let $U = (S + S - S) \cap B_R$. Define for every $T \geq 0$ a set $H_T = U \cup (S \cap B_T)$. If*

$$\liminf_{T \to \infty} \frac{N_S(T)}{T^d} = 0,$$

then S has low complexity with respect to H_T for some T.

Proof. By Lemma 5.2.3 we have

$$P_S(H_T) \leq 1 + |U| N_S(T + R).$$

Thus, if $\liminf_{T\to\infty} \frac{N_S(T)}{T^d} = 0$, then also $\liminf_{T\to\infty} \frac{P_S(H_T)}{T^d} = 0$. As seen above, we have $\liminf_{T\to\infty} \frac{|H_T|}{T^d} > 0$. Thus, for some T we have

$$P_S(H_T) \leq |H_T|.$$

□

Together with Lemma 5.3.1 and Theorem 5.3.4 which are proved in the following section the above theorem yields the following corollary.

***Corollary* 5.2.5.** *Let $S \subseteq \mathbb{R}^d$ be a Meyer set. If*

$$\liminf_{T \to \infty} \frac{N_S(T)}{T^d} = 0,$$

then S has a non-trivial annihilator. In particular, S has an annihilator of the form

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1).$$

Proof. By Theorem 5.2.4 the Meyer set S has low complexity with respect to some $\mathbb{R}^d$-shape. Consequently, by Lemma 5.3.1 it has a non-trivial annihilator and hence by Theorem 5.3.4 it has an annihilator of the desired form. □

A related conjecture

Finally, let us mention a related conjecture which is already known to be false.

Conjecture (Lagarias and Pleasants 2003 [61])**.** *Any non-periodic Delone set $S \subseteq \mathbb{R}^d$ satisfies*

$$\limsup_{T \to \infty} \frac{N_S(T)}{T^d} > 0.$$

As mentioned, the conjecture has already proven to be false in [14] where it is given for any $d \geq 3$ an example of a non-periodic binary configuration $c \in \{0,1\}^{\mathbb{Z}^d}$ such that

$$\lim_{n\to\infty} \frac{P_c(n)}{n^d} = 0$$

where $P_c(n) = P_c(\{0,\dots,n-1\}^d)$. One can then define a d-dimensional Delone set $S \subseteq \mathbb{Z}^d$ such that $\mathbf{u} \in S$ if and only if $c_{\mathbf{u}} = 1$. It satisfies

$$\limsup_{T\to\infty} \frac{N_S(T)}{T^d} = 0.$$

So, this construction shows that the above conjecture is false.

However, if in the conjecture we strengthen the assumption that the Delone set S is non-periodic and instead assume that it is a Meyer set and has no non-trivial annihilators, then the statement of the conjecture holds by Corollary 5.2.5. Indeed, we have the following theorem.

***Theorem* 5.2.6.** *Any Meyer set $S \subseteq \mathbb{R}^d$ with no non-trivial annihilators satisfies*

$$\liminf_{T\to\infty} \frac{N_S(T)}{T^d} > 0.$$

Proof. Let S be a Meyer set with no non-trivial annihilators. Then by Corollary 5.2.5 we have

$$\liminf_{T\to\infty} \frac{N_S(T)}{T^d} > 0.$$

□

5.3 Delone configurations with annihilators

5.3.1 General setting

We begin with the following direct generalization of Lemma 2.5.8. For completeness, we provide a proof for the claim. The orthogonal complement of a linear subspace U of an inner-product space is denoted by $U^\perp$. In the following, the inner-product is the scalar product of complex vectors. For a complex number $z \in \mathbb{C}$, we denote its complex conjugate by $\overline{z}$, as usual.

***Lemma* 5.3.1.** *Let $c \in \mathbb{C}^{\mathbb{R}^d}$ be an $\mathbb{R}^d$-configuration and let $D = \{\mathbf{d}_1,\dots,\mathbf{d}_m\} \Subset \mathbb{R}^d$ be an $\mathbb{R}^d$-shape. If c has low complexity with respect to D, that is, if $P_c(D) \leq m$, then c has a periodizer of the form*

$$a_1 X^{-\mathbf{d}_1} + \ldots + a_m X^{-\mathbf{d}_m}$$

for some non-zero $(a_1,\dots,a_m) \in \mathbb{C}^m$.

Proof. Consider the set

$$V = \left\{ (1, c_{\mathbf{d}_1+\mathbf{v}}, \ldots, c_{\mathbf{d}_m+\mathbf{v}}) \mid \mathbf{v} \in \mathbb{R}^d \right\}.$$

The subspace $L(V) \subseteq \mathbb{C}^{m+1}$ generated by V has dimension at most m since $|V| = P_c(D) \leq m$. Thus, $L(V)^{\perp}$ is non-trivial. Let $(\overline{a}_0, \ldots, \overline{a}_m) \in L(V)^{\perp} \setminus \{\mathbf{0}\}$. Then

$$a_0 \cdot 1 + a_1 \cdot c_{\mathbf{d}_1+\mathbf{v}} + \ldots + a_m \cdot c_{\mathbf{d}_m+\mathbf{v}} = 0$$

for all $\mathbf{v} \in \mathbb{R}^d$. Thus, $a_1 X^{-\mathbf{d}_1} + \ldots + a_n X^{-\mathbf{d}_m}$ is a non-trivial periodizer of c. □

As noted earlier, annihilation by a line $\mathbb{R}^d$-polynomial does not necessarily imply periodicity. However, if an $\mathbb{R}^d$-configuration has a line $\mathbb{R}^d$-polynomial annihilator of a particularly simple type, namely a power of a difference $\mathbb{R}^d$-polynomial, then it is periodic as the following lemma shows.

Lemma 5.3.2. *Assume that a configuration $c \in \mathcal{A}^{\mathbb{R}^d}$ has an annihilator of the form $g(X) = (X^{\mathbf{v}} - 1)^k$ for some positive integer k and $\mathbf{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. Then c is periodic in direction $\mathbf{v}$. More precisely, c is $p\mathbf{v}$-periodic for some non-zero integer p.*

Proof. By the binomial theorem we have $g(X) = \sum_{j=0}^{k} a_j X^{j\mathbf{v}}$ where $a_0, a_k \neq 0$. Let us define a $\mathbb{Z}$-configuration $e^{(\mathbf{u})} \in \mathcal{A}^{\mathbb{Z}}$ for any $\mathbf{u} \in \mathbb{R}^d$ such that

$$e_i^{(\mathbf{u})} = c_{\mathbf{u}+i\mathbf{v}}$$

for all $i \in \mathbb{Z}$. It follows that $g' = \sum_{j=0}^{k} a_j x^j$ annihilates $e^{(\mathbf{u})}$ since

$$(g' e^{(\mathbf{u})})_i = \sum_{j=0}^{k} a_j e_{i-j}^{(\mathbf{u})} = \sum_{j=0}^{k} a_j c_{\mathbf{u}+(i-j)\mathbf{v}} = (gc)_{\mathbf{u}+i\mathbf{v}} = 0$$

for all $i \in \mathbb{Z}$. Hence, each $e^{(\mathbf{u})}$ is periodic. Moreover, the smallest periods of $e^{(\mathbf{u})}$ are bounded and hence c is periodic in direction $\mathbf{v}$. In fact, c is $p\mathbf{v}$-periodic where p is a common period of the configurations $e^{(\mathbf{u})}$. The claim follows. □

Remark 5.3.3. In the proof of the above lemma we in fact proved that if an $\mathbb{R}^d$-configuration c is annihilated by a line $\mathbb{R}^d$-polynomial f such that

$$\text{supp}(f) \subseteq \mathbf{u} + \mathbb{Z}\mathbf{v}$$

for some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, then it is periodic.

The following theorem is a generalization of Theorem 2.5.10.

Theorem 5.3.4. *Let c be an integral $\mathbb{R}^d$-configuration and assume that it has a non-trivial integral annihilator f. Then for every $\mathbf{u} \in supp(f)$ it has an annihilator of the form*

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$$

where each $\mathbf{v}_i$ is parallel to $\mathbf{u}_i - \mathbf{u}$ over $\mathbb{Q}$ for some $\mathbf{u}_i \in supp(f) \setminus \{\mathbf{u}\}$. Moreover, the vectors $\mathbf{v}_i$ can be chosen to be pairwise linearly independent over $\mathbb{Q}$.

Proof. Assume that $f = \sum_{i=1}^{n} f_i X^{\mathbf{u}_i}$ for some non-zero $f_i \in \mathbb{Z}$ where $\text{supp}(f) = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. Define for any $\mathbf{u} \in \mathbb{R}^d$ a $\mathbb{Z}^n$-configuration $c^{(\mathbf{u})} \in \mathcal{A}^{\mathbb{Z}^n}$ over the same finite alphabet $\mathcal{A}$ as c such that

$$c^{(\mathbf{u})}(i_1, \ldots, i_n) = c(\mathbf{u} + i_1\mathbf{u}_1 + \ldots + i_n\mathbf{u}_n)$$

for all $(i_1, \ldots, i_n) \in \mathbb{Z}^n$.

The polynomial $g = g(T) = g(t_1, \ldots, t_n) = \sum_{i=1}^{n} f_i t_i = \sum_{i=1}^{n} f_i T^{\mathbf{e}_i}$ is a non-trivial annihilator of $c^{(\mathbf{u})}$ for any $\mathbf{u} \in \mathbb{R}^d$ since

$$\begin{aligned}
(gc^{(\mathbf{u})})(i_1, \ldots, i_n) &= f_1 c^{(\mathbf{u})}(i_1 - 1, i_2, \ldots, i_n) + \ldots + f_n c^{(\mathbf{u})}(i_1, \ldots, i_{n-1}, i_n - 1) \\
&= f_1 c(\mathbf{u} + (i_1 - 1)\mathbf{u}_1 + i_2\mathbf{u}_2 + \ldots + i_n\mathbf{u}_n) + \ldots \\
&\quad + f_n c(\mathbf{u} + i_1\mathbf{u}_1 + \ldots + i_{n-1}\mathbf{u}_{n-1} + (i_n - 1)\mathbf{u}_n) \\
&= (fc)(\mathbf{u} + i_1\mathbf{u}_1 + \ldots + i_n\mathbf{u}_n) \\
&= 0
\end{aligned}$$

for all $(i_1, \ldots, i_n) \in \mathbb{Z}^n$.

By Lemma 2.5.12 the $\mathbb{Z}^n$-configurations $c^{(\mathbf{u})}$ have a common dilation constant with respect to g. Moreover, by Theorem 2.5.13 they have for all $\mathbf{e} \in \text{supp}(g) = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ a common annihilator of the form

$$(T^{\mathbf{w}_1} - 1) \cdots (T^{\mathbf{w}_m} - 1)$$

where each $\mathbf{w}_i$ is parallel to $\mathbf{e}_j - \mathbf{e}$ for some $\mathbf{e}_j \in \text{supp}(g) \setminus \{\mathbf{e}\}$, that is, $\mathbf{w}_i = k\mathbf{e}_j - k\mathbf{e}$ for some non-zero integer k.

Let us denote $\mathbf{w}_i = (w_{i,1}, \ldots, w_{i,n})$ for $i \in \{1, \ldots, m\}$ and let $i_0 \in \{1, \ldots, n\}$ be such that $\mathbf{e} = \mathbf{e}_{i_0}$. Note that each $\mathbf{w}_i$ has exactly two non-zero entries, and they are equal to k and $-k$. It follows that the $\mathbb{R}^d$-polynomial

$$(X^{w_{1,1}\mathbf{u}_1 + \ldots + w_{1,n}\mathbf{u}_n} - 1) \cdots (X^{w_{m,1}\mathbf{u}_1 + \ldots + w_{m,n}\mathbf{u}_n} - 1)$$

is an annihilator of c such that each $w_{i,1}\mathbf{u}_1 + \ldots + w_{i,n}\mathbf{u}_n$ is parallel to $\mathbf{u}_j - \mathbf{u}_{i_0}$. This holds for all $i_0 \in \{1, \ldots, n\}$ and hence we have proved the first part of the claim.

For the "moreover" part, assume that

$$g = (X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$$

annihilates c, and assume that for some distinct $i, j \in \{1, \ldots, m\}$ the vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ are parallel over $\mathbb{Q}$, that is, $q\mathbf{v}_i = p\mathbf{v}_j$ for some $p, q \in \mathbb{Z} \setminus \{0\}$. We may replace $X^{\mathbf{v}_i} - 1$ in g by $X^{p\mathbf{v}_j} - 1 = X^{q\mathbf{v}_i} - 1$. Since

$$X^{p\mathbf{v}_j} - 1 = (X^{\mathbf{v}_j} - 1)(X^{(p-1)\mathbf{v}_j} + \ldots + X^{\mathbf{v}_j} + 1),$$

we may replace $(X^{p\mathbf{v}_j} - 1)(X^{\mathbf{v}_j} - 1)$ by $(X^{p\mathbf{v}_j} - 1)^2$. Iterating this argument we conclude that c has an annihilator

$$(X^{\mathbf{w}_1} - 1)^{e_1} \cdots (X^{\mathbf{w}_k} - 1)^{e_k}$$

where the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are pairwise linearly independent over $\mathbb{Q}$ and $e_1, \ldots, e_k$ are positive integers. Consequently, the $\mathbb{R}^d$-configuration

$$(X^{\mathbf{w}_2} - 1)^{e_2} \cdots (X^{\mathbf{w}_k} - 1)^{e_k} c$$

is annihilated by $(X^{\mathbf{w}_1} - 1)^{e_1}$. By Lemma 5.3.2 it is periodic in direction $\mathbf{w}_1$. Repeating this argument we conclude that c is annihilated by

$$(X^{r_1\mathbf{w}_1} - 1) \cdots (X^{r_k\mathbf{w}_k} - 1)$$

for some non-zero $r_1, \ldots, r_k \in \mathbb{Z}$ and hence we have proved the claim. □

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ in the above theorem may not be pairwise linearly independent over $\mathbb{R}$ as the following example shows. However, in the following subsection we see that if c is a Delone configuration of finite local complexity, then they can be chosen to be pairwise linearly independent over $\mathbb{R}$. The following example gives also an example of a non-periodic $\mathbb{R}$-configuration with a non-trivial annihilator.

Example 5.3.5. Consider the binary $\mathbb{R}$-configurations $c_1, c_2 \in \{0, 1\}^{\mathbb{R}}$ defined such that

$$c_1(i) = \begin{cases} 1 & \text{, if } i \in \mathbb{Z} \\ 0 & \text{, otherwise} \end{cases}$$

and

$$c_2(i) = \begin{cases} 1 & \text{, if } i \in \alpha\mathbb{Z} \\ 0 & \text{, otherwise} \end{cases}$$

where α is an irrational number. Both c_1 and c_2 are periodic, that is, annihilated by non-trivial difference $\mathbb{R}$-polynomials. More precisely, c_1 is annihilated by $x - 1$ and c_2 is annihilated by $x^{\alpha} - 1$. Consequently, their sum $c = c_1 + c_2$ has a non-trivial annihilator $(x - 1)(x^{\alpha} - 1)$ but c is non-periodic since α is irrational. If in the above theorem $\mathbf{v}_1, \ldots, \mathbf{v}_m$ could be chosen to be pairwise linearly independent over $\mathbb{R}$, this would mean in the 1-dimensional setting that $m = 1$ and hence c would be necessarily periodic. Note that the support of c is not a Delone set.

5.3.2 Delone configurations of finite local complexity

We have seen that any integral $\mathbb{R}^d$-configuration with non-trivial annihilators has an annihilator which is a product of difference $\mathbb{R}^d$-polynomials. Moreover, the directions of these difference $\mathbb{R}^d$-polynomials can be chosen to be pairwise linearly independent over $\mathbb{Q}$. However, in Example 5.3.5 we noticed that the directions of these difference $\mathbb{R}^d$-polynomials cannot always be chosen to be pairwise linearly independent over $\mathbb{R}$.

In this section we study Delone configurations of finite local complexity and improve Theorem 5.3.4. We show that if the $\mathbb{R}^d$-configuration in consideration is a Delone configuration of finite local complexity, then the directions of the difference $\mathbb{R}^d$-polynomial factors of the special annihilator in fact can be chosen to be pairwise linearly independent over $\mathbb{R}$ instead of just $\mathbb{Q}$.

We begin with some observations concerning Delone configurations of finite local complexity with non-trivial annihilators. The following lemma generalizes the fact that any $\mathbb{Z}$-configuration with a non-trivial annihilator is necessarily periodic.

***Lemma* 5.3.6.** *Let $c \in \mathcal{A}^{\mathbb{R}}$ be a 1-dimensional Delone configuration of finite local complexity and let S be the support of c with uniform discreteness constant r and relative denseness constant R. Assume that c has a non-trivial annihilator f. Then c is periodic. Moreover, a period of c belongs to the finite set $(S-S) \cap B_{2MR}$ where $M \leq |\mathcal{A}|^{|(S-S)\cap B_{2R}|^{\lfloor \frac{k}{2r} \rfloor}}$ and k is such that supp$(f) \subseteq [0,k]$.*

Proof. Without loss of generality we may assume that supp$(f) = \{t_0, t_1, \ldots, t_n\}$ where $0 < t_1 < \ldots < t_n$ and $t_0 = 0$. So, we have $f = \sum_{i=0}^{n} f_i x^{t_i}$ for some non-zero $f_0, \ldots, f_n$. Let k be such that supp$(f) \subseteq [0,k]$. Since S is a Delone set, we can order it and write $S = \{s_i \mid i \in \mathbb{Z}\}$ such that $s_i < s_{i+1}$ for every $i \in \mathbb{Z}$. By the assumption that S is a Delone set of finite local complexity, the set $(S-S) \cap B_{2R}$ is finite and hence the set $\{s_{i+1} - s_i \mid i \in \mathbb{Z}\} \subseteq (S-S) \cap B_{2R}$ of the lengths of intervals between two consecutive points of S is finite. This implies that for some $i < j$ we have $\tau^{-s_i}(c) \restriction_{-[0,k]} = \tau^{-s_j}(c) \restriction_{-[0,k]}$, that is, $c_{s_i - u} = c_{s_j - u}$ for all $u \in [0,k]$.

Since f is an annihilator of c, we have

$$f_0 c_t + f_1 c_{t-t_1} + \ldots + f_n c_{t-t_n} = 0$$

for every $t \in \mathbb{R}$. We can solve c_t and c_{t-t_n} from the equation because f_0 and f_n are non-zero. Thus, the content of c in the interval $t - [0,k]$ determines whole c since supp$(f) \subseteq [0,k]$ and hence the condition $\tau^{-s_i}(c) \restriction_{-[0,k]} = \tau^{-s_j}(c) \restriction_{-[0,k]}$ (or equivalently $\tau^{s_j - s_i}(c) \restriction_{-[0,k]} = c \restriction_{-[0,k]}$) which we obtained above implies that c is $(s_j - s_i)$-periodic.

Let $M = j - i$. The set $s_i - [0,k]$ contains s_i and at most $\lfloor \frac{k}{2r} \rfloor$ other points of S. Moreover, there are at most $|(S-S) \cap B_{2R}|$ different gaps between consecutive points of S. Together, these imply that there are at most $|(S-S) \cap B_{2R}|^{\lfloor \frac{k}{2r} \rfloor}$ different sets $\text{supp}(\tau^{-s_i}(c)) \cap (-[0,k])$. Thus, there are at most $|\mathcal{A}|^{|(S-S)\cap B_{2R}|^{\lfloor \frac{k}{2r} \rfloor}}$ different functions $\tau^{-s_i}(c) \upharpoonright_{-[0,k]}$. So, we may choose $M = j - i \leq |\mathcal{A}|^{|(S-S)\cap B_{2R}|^{\lfloor \frac{k}{2r} \rfloor}}$. This concludes the proof. □

Recall that in Example 5.3.5 we showed that the above lemma does not hold for arbitrary $\mathbb{R}$-configurations. However, the support of the $\mathbb{R}$-configuration in the example is not even a Delone set. Later in this section we will see that the above lemma indeed holds for general Delone configurations, that is, we will see that if a 1-dimensional Delone configuration has a non-trivial annihilator, then it is periodic. However, the proof in this more general case is more complicated than the proof of the above lemma.

Remark 5.3.7. More generally, we can prove that any $\mathbb{R}$-configuration c whose support is a subset of a Delone set of finite local complexity that has a non-trivial annihilator is necessarily periodic. Indeed, if $c = 0$, then it is trivially periodic, and if c is non-zero, then the non-trivial annihilator defines a linear recurrence on c and hence the support of c is relatively dense. This implies that it is a Delone set of finite local complexity since any relatively dense subset of a Delone set of finite local complexity is a Delone set of finite local complexity too. The periodicity of c follows from Lemma 5.3.6.

Using Lemma 5.3.6 we show that if a Delone configuration of finite local complexity has a line $\mathbb{R}^d$-polynomial annihilator, then it is periodic. In the proofs of the following results for a non-zero vector $\mathbf{v} \in \mathbb{R}^d$, by a $\mathbf{v}$-fiber of an $\mathbb{R}^d$-configuration $c \in \mathcal{A}^{\mathbb{R}^d}$ we mean an $\mathbb{R}$-configuration $e^{(\mathbf{u})} \in \mathcal{A}^{\mathbb{R}}$ for some $\mathbf{u} \in \mathbb{R}^d$ defined such that

$$e^{(\mathbf{u})}_r = c_{\mathbf{u}+r\mathbf{v}}$$

for all $r \in \mathbb{R}$.

***Lemma* 5.3.8.** *Let $c \in \mathcal{A}^{\mathbb{R}^d}$ be a Delone configuration of finite local complexity and assume that it is annihilated by a line $\mathbb{R}^d$-polynomial f in direction $\mathbf{v}$. Then c is periodic in direction $\mathbf{v}$.*

Proof. By multiplying f with a suitable monomial we may assume that f is of the form $f(X) = f_0 + f_1 X^{r_1\mathbf{v}} + \ldots + f_n X^{r_n\mathbf{v}}$ where $0 < r_1 < \ldots < r_n$ with $n \geq 1$ and $f_0, \ldots, f_n \neq 0$.

Consider an arbitrary $\mathbf{v}$-fiber e of c. Clearly, e is annihilated by the polynomial $f'(x) = f_0 + f_1 x^{r_1} + \ldots + f_n x^{r_n}$. If $e = 0$, then it is trivially periodic. If $e \neq 0$, then $E = \text{supp}(e)$ must be relatively dense since f' defines a recurrence relation on e. In fact, if $e \neq 0$, then E is a Delone set of finite local complexity since $S = \text{supp}(c)$ is a Delone set of finite local complexity. Thus, e is a Delone configuration of finite local complexity and hence by Lemma 5.3.6 it is periodic.

It remains to show that all the $\mathbf{v}$-fibers of c have a common non-zero period. Then it follows that c is periodic in direction $\mathbf{v}$. Note that the supports of the non-zero fibers of c have a common uniform discreteness constant and a common relative denseness constant. So, the number M from Lemma 5.3.6 can be chosen to be the same for every fiber e. Hence, there exist only finitely many possibilities for the smallest non-zero period (in absolute value) of e. Let e_1 and e_2 be two non-zero $\mathbf{v}$-fibers of c with smallest periods p_1 and p_2, respectively. There exist $\mathbf{u}_1, \mathbf{u}_2 \in S$ such that $\mathbf{u}_1 + np_1\mathbf{v} \in S$ and $\mathbf{u}_2 + mp_2\mathbf{v} \in S$ for all $n, m \in \mathbb{Z}$. Consequently, for any $\varepsilon > 0$ there exist infinitely many n and m such that $np_1\mathbf{v}$ and $mp_2\mathbf{v}$ are within distance ε from each other. Then $\mathbf{u}_1 + np_1\mathbf{v}$ and $\mathbf{u}_2 + mp_2\mathbf{v}$ are within distance $||\mathbf{u}_1 - \mathbf{u}_2|| + \varepsilon$ from each other. We conclude that p_1 and p_2 must be rationally dependent. Otherwise, $(S - S) \cap B_{||\mathbf{u}_1 - \mathbf{u}_2|| + \varepsilon}$ is an infinite set which is a contradiction with the fact that S is a Delone set of finite local complexity, that is, $S - S$ is a locally finite set. So, the finitely many smallest periods of the $\mathbf{v}$-fibers of c are rationally dependent. It follows that the $\mathbf{v}$-fibers of c have a common period. □

Remark 5.3.9. Similarly to Remark 5.3.7 we can prove that the above lemma holds more generally for any $\mathbb{R}^d$-configuration whose support is a subset of a Delone set of finite local complexity. Indeed, if the support of c is a subset of a Delone set of finite local complexity, then the support of each $\mathbf{v}$-fiber of c is a subset of a Delone set of finite local complexity (actually, it is either a Delone set of finite local complexity or the empty set). The claim follows by Remark 5.3.7.

Lemma 5.3.8 does not hold for general Delone configurations, that is, there are non-periodic Delone configurations with line $\mathbb{R}^d$-polynomial annihilators as the following example shows.

Example 5.3.10. Let α be an irrational real number. Consider the Delone configuration $c \in \{0,1\}^{\mathbb{R}^2}$ defined such that

$$c_{\mathbf{u}} = \begin{cases} 1 & \text{, if } \mathbf{u} = (k, i) \text{ for any } k \in \mathbb{Z} \text{ and } i \in \mathbb{Z} \setminus \{0\} \\ 1 & \text{, if } \mathbf{u} = (k\alpha, 0) \text{ for any } k \in \mathbb{Z} \\ 0 & \text{, otherwise} \end{cases} .$$

It is non-periodic but has a line $\mathbb{R}^d$-polynomial annihilator $(x-1)(x^\alpha - 1)$.

Next, we prove our improvement of Theorem 5.3.4.

***Theorem* 5.3.11.** *Let c be an integral Delone configuration of finite local complexity and assume that it has a non-trivial integral annihilator f. Then for every $\mathbf{u} \in$ supp(f) it has an annihilator of the form*

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$$

where the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ *are pairwise linearly independent over* $\mathbb{R}$ *and each* $\mathbf{v}_i$ *is parallel to* $\mathbf{u}_i - \mathbf{u}$ *over* $\mathbb{R}$ *for some* $\mathbf{u}_i \in \text{supp}(f) \setminus \{\mathbf{u}\}$.

Proof. By Theorem 5.3.4 for every $\mathbf{u} \in \text{supp}(f)$ there exist vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ such that

$$g = (X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$$

annihilates c where each $\mathbf{v}_i$ is parallel to $\mathbf{u}_i - \mathbf{u}$ over $\mathbb{Q}$ for some $\mathbf{u}_i \in \text{supp}(f) \setminus \{\mathbf{u}\}$.

Consider the $\mathbb{R}^d$-polynomial g for some fixed $\mathbf{u} \in \text{supp}(f)$. Let $\mathbf{w}_1, \dots, \mathbf{w}_k$ be pairwise linearly independent vectors over $\mathbb{R}$ such that each $\mathbf{v}_i$ is parallel to some $\mathbf{w}_j$ over $\mathbb{R}$. Moreover, assume that k is minimal, that is, each $\mathbf{w}_j$ is parallel to some $\mathbf{v}_i$.

Clearly, $k \leq m$. If $k = m$, then all the difference $\mathbb{R}^d$-polynomials in the product are in pairwise linearly independent directions over $\mathbb{R}$, and we are done. So, we may assume that $k < m$. Define $\mathbb{R}^d$-polynomials $\varphi_1, \dots, \varphi_k$ such that φ_i is the product of every $X^{\mathbf{v}_j} - 1$ that has direction $\mathbf{w}_i$. Note that every φ_i is a line $\mathbb{R}^d$-polynomial since a product of line $\mathbb{R}^d$-polynomials in the same direction is still a line $\mathbb{R}^d$-polynomial in that same direction. Clearly, $g = \varphi_1 \cdots \varphi_k$ and hence φ_1 is a line $\mathbb{R}^d$-polynomial annihilator of $\varphi_2 \cdots \varphi_k c$ since $gc = 0$.

Since the support of c is a Delone set of finite local complexity, the support of $\varphi_2 \cdots \varphi_k c$ must be a subset of a Delone set of finite local complexity. Thus, by Lemma 5.3.8 and Remark 5.3.9, the configuration $\varphi_2 \cdots \varphi_k c$ is periodic in direction $\mathbf{w}_1$.

The argument can be repeated for each $i \in \{2, \dots, k\}$ and hence we conclude that there exist $r_1, \dots, r_k \in \mathbb{R}$ such that the $\mathbb{R}^d$-polynomial

$$(X^{r_1 \mathbf{w}_1} - 1) \cdots (X^{r_k \mathbf{w}_k} - 1)$$

annihilates c. □

Remark 5.3.12. Again, we have, more generally that the above theorem holds for any integral $\mathbb{R}^d$-configuration whose support is a subset of a Delone set of finite local complexity.

5.3.3 Periodic decomposition theorem for $\mathbb{R}^d$-configurations

The periodic decomposition theorem, *i.e.*, Theorem 2.5.14 states that if an integral $\mathbb{Z}^d$-configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ has a non-trivial integral annihilator, then it is a sum of finitely many periodic functions $c_1, \dots, c_m \in \mathbb{Z}^{\mathbb{Z}^d}$. In this section we prove a similar statement for $\mathbb{R}^d$-configurations. In other words, we show that if an integral $\mathbb{R}^d$-configuration has a non-trivial integral annihilator, then it is a sum of finitely many periodic functions $c_1, \dots, c_m \in \mathbb{Z}^{\mathbb{R}^d}$. The proof of this result is a direct generalization of the proof of Theorem 2.5.14 using Theorem 5.3.4.

We begin with two lemmas as in the proof of Theorem 2.5.14 in [56].

***Lemma* 5.3.13.** *Let $\mathbf{v}_1 \in \mathbb{R}^d$ and $\mathbf{v}_2 \in \mathbb{R}^d$ be linearly independent vectors over $\mathbb{Q}$ such that the difference $\mathbb{R}^d$-polynomial $X^{\mathbf{v}_2} - 1$ annihilates a function $c' \in \mathbb{Z}^{\mathbb{R}^d}$. There exists a function $c \in \mathbb{Z}^{\mathbb{R}^d}$ such that $(X^{\mathbf{v}_1} - 1)c = c'$ and $(X^{\mathbf{v}_2} - 1)c = 0$.*

Proof. The space $\mathbb{R}^d$ is partitioned to cosets modulo $\mathbb{Q}[\mathbf{v}_1, \mathbf{v}_2] = \{a\mathbf{v}_1 + b\mathbf{v}_2 \mid a, b \in \mathbb{Q}\}$. Let us fix an arbitrary $\mathbf{z} \in \mathbb{R}^d$ and consider the coset

$$\mathbf{z} + \mathbb{Q}[\mathbf{v}_1, \mathbf{v}_2] = \{\mathbf{z} + a\mathbf{v}_1 + b\mathbf{v}_2 \mid a, b \in \mathbb{Q}\}.$$

The equation $(X^{\mathbf{v}_1} - 1)c = c'$ is satisfied in the coset $\mathbf{z} + \mathbb{Q}[\mathbf{u}, \mathbf{v}]$ if and only if

$$c_{\mathbf{z}+(a-1)\mathbf{v}_1+b\mathbf{v}_2} - c_{\mathbf{z}+a\mathbf{v}_1+b\mathbf{v}_2} = c'_{a\mathbf{v}_1+b\mathbf{v}_2}$$

holds for every $a, b \in \mathbb{Q}$. Set $c_{\mathbf{z}+b\mathbf{v}_2} = 0$. Then the above equation defines the rest for all $a, b \in \mathbb{Q}$. Since $\mathbf{z}$ was arbitrary, we can do this in every coset and obtain c such that $(X^{\mathbf{v}_1} - 1)c = c'$ in every coset and hence $(X^{\mathbf{v}_1} - 1)c = c'$ everywhere.

Finally, let us show that $(X^{\mathbf{v}_2} - 1)c = 0$. It suffices to show that $(X^{\mathbf{v}_2} - 1)c = 0$ in an arbitrary coset $\mathbf{z} + \mathbb{Q}[\mathbf{v}_1, \mathbf{v}_2]$. This follows from the computation

$$(X^{\mathbf{v}_1} - 1)(X^{\mathbf{v}_2} - 1)c = (X^{\mathbf{v}_2} - 1)(X^{\mathbf{v}_1} - 1)c = (X^{\mathbf{v}_2} - 1)c' = 0.$$

Indeed, above we defined $c_{\mathbf{z}+b\mathbf{v}_2} = 0$ for every $b \in \mathbb{Q}$. This implies that also $[(X^{\mathbf{v}_2} - 1)c]_{\mathbf{z}+b\mathbf{v}_2} = 0$ for every $b \in \mathbb{Q}$. From the above recurrence relation it follows that $[(X^{\mathbf{v}_2} - 1)c]_{\mathbf{z}+a\mathbf{v}_1+b\mathbf{v}_2} = 0$ for every $a, b \in \mathbb{Q}$. □

***Lemma* 5.3.14.** *Let c be an integral $\mathbb{R}^d$-configuration and let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^d$ be pairwise linearly independent vectors over $\mathbb{Q}$. If the product $(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$ of difference $\mathbb{R}^d$-polynomials annihilates c, then there exist functions $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{R}^d}$ such that $(X^{\mathbf{v}_i} - 1)c_i = 0$ for each i and*

$$c = c_1 + \ldots + c_m.$$

Proof. The proof is by induction on m. For $m = 1$ the claim is clear. Assume then that $m \geq 2$ and that the claim holds for $m - 1$. Since $(X^{\mathbf{v}_m} - 1)c$ is annihilated by $(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_{m-1}} - 1)$, by the induction hypothesis there exist functions $c'_1, \ldots, c'_{m-1} \in \mathbb{C}^{\mathbb{R}^d}$ such that

$$(X^{\mathbf{v}_m} - 1)c = c'_1 + \ldots + c'_{m-1}$$

and each c'_i is annihilated by $X^{\mathbf{v}_i} - 1$. By Lemma 5.3.13 for each $i \in \{1, \ldots, m-1\}$ there exists a function c_i such that $(X^{\mathbf{v}_m} - 1)c_i = c'_i$ and $(X^{\mathbf{v}_i} - 1)c_i = 0$. Set $c_m = c - c_1 - \ldots - c_{m-1}$. Then clearly $c = c_1 + \ldots + c_m$ and moreover

$$\begin{aligned}
(X^{\mathbf{v}_m} - 1)c_m &= (X^{\mathbf{v}_m} - 1)(c - c_1 - \ldots - c_{m-1}) \\
&= (X^{\mathbf{v}_m} - 1)c - (X^{\mathbf{v}_m} - 1)c_1 - \ldots - (X^{\mathbf{v}_m} - 1)c_{m-1} \\
&= c'_1 + \ldots + c'_{m-1} - c'_1 - \ldots - c'_{m-1} \\
&= 0.
\end{aligned}$$

Hence, $c = c_1 + \ldots + c_m$ and each c_i is annihilated by $(X^{\mathbf{v}_i} - 1)$. The claim follows. □

Now, we can prove our periodic decomposition theorem.

***Theorem* 5.3.15** (Periodic decomposition theorem for $\mathbb{R}^d$-configurations)**.** *Let c be an integral $\mathbb{R}^d$-configuration with a non-trivial integral annihilator. Then there exist periodic functions $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{R}^d}$ such that*

$$c = c_1 + \ldots + c_m.$$

Proof. By Theorem 5.3.4 the $\mathbb{R}^d$-configuration c is annihilated by

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_m} - 1)$$

for some pairwise linearly independent $\mathbf{v}_1, \ldots, \mathbf{v}_m$ over $\mathbb{Q}$. By Lemma 5.3.14 there exist functions $c_1, \ldots, c_m \in \mathbb{Z}^{\mathbb{R}^d}$ such that their sum is c and each c_i is annihilated by $X^{\mathbf{v}_i} - 1$ and hence $\mathbf{v}_i$-periodic. □

5.3.4 One-dimensional Delone configurations

In this subsection we prove that any 1-dimensional Delone configuration with a non-trivial annihilator is periodic.

Recall from the previous chapter that a configuration $c \in \mathcal{A}^{\mathbb{Z}^d}$ is called sparse if there exists a positive integer $a \in \mathbb{Z}_+$ such that

$$|\text{supp}(c) \cap (C_m + \mathbf{t})| \leq am$$

for all $m \in \mathbb{Z}_+$ and $\mathbf{t} \in \mathbb{Z}^d$. In the proof of the following theorem we use Theorem 4.2.1.

***Theorem* 5.3.16.** *Let $c \in \mathcal{A}^{\mathbb{R}}$ be an integral 1-dimensional Delone configuration with a non-trivial integral annihilator. Then c is periodic.*

Proof. Let f be a non-trivial integral annihilator of c and let d be the rank of the additive abelian group $\mathbb{Z}[\text{supp}(f)]$. Let $\{b_1, \ldots, b_d\}$ be a (minimal) generator set of this group. Note that every element of $\mathbb{Z}[\text{supp}(f)]$ has a unique presentation in the form $i_1 b_1 + \ldots + i_d b_d$ where $i_1, \ldots, i_d \in \mathbb{Z}$, that is, the function $(i_1, \ldots, i_d) \mapsto i_1 b_1 + \ldots + i_d b_d$ is injective. This follows from the fundamental theorem of finitely generated abelian groups [22]. Define for all $\alpha \in \mathbb{R}$, a $\mathbb{Z}^d$-configuration $c^{(\alpha)} \in \mathcal{A}^{\mathbb{Z}^d}$ such that

$$c^{(\alpha)}(i_1, \ldots, i_d) = c(\alpha + i_1 b_1 + \ldots + i_d b_d).$$

Let us show that $c^{(\alpha)}$ is sparse for all α. So, consider an arbitrary $c^{(\alpha)}$. Since $\text{supp}(c)$ is a Delone set and hence uniformly discrete, there exists $\delta > 0$ such that each half-open interval

$$I_k = [\alpha + k\delta, \alpha + (k+1)\delta)$$

contains at most one point of $\text{supp}(c)$ for each $k \in \mathbb{Z}$. (Moreover, the intervals I_k where $k \in \mathbb{Z}$ partition the real line $\mathbb{R}$.) Thus, the sets

$$S_k = \{(i_1, \ldots, i_d) \in \mathbb{Z}^d \mid k\delta \leq i_1 b_1 + \ldots + i_d b_d < (k+1)\delta\}$$

contain at most one element of $\text{supp}(c^{(\alpha)})$ by the mentioned injectivity of the function $(i_1, \ldots, i_d) \mapsto i_1 b_1 + \ldots + i_d b_d$. For $m \in \mathbb{Z}_+$ and $\mathbf{t} \in \mathbb{Z}^d$, let $N(m, \mathbf{t})$ denote the number of sets S_k that intersect the set $C_m + \mathbf{t}$. There exists a such that $N(m, \mathbf{t}) \leq am$ for all $m \in \mathbb{Z}_+$ and $\mathbf{t} \in \mathbb{Z}^d$. Thus,

$$|\text{supp}(c^{(\alpha)}) \cap (C_m + \mathbf{t})| \leq N(m, \mathbf{t}) \leq am$$

for all $m \in \mathbb{Z}_+$ and $\mathbf{t} \in \mathbb{Z}^d$ and hence $c^{(\alpha)}$ is sparse.

By the assumption c has a non-trivial annihilator and hence the $\mathbb{Z}^d$-configurations $c^{(\alpha)}$ have a common non-trivial annihilator. By Lemma 2.5.12 and Theorem 2.5.13 there exist pairwise linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that every $c^{(\alpha)}$ is annihilated by the polynomial

$$(X^{\mathbf{v}_1} - 1) \cdots (X^{\mathbf{v}_n} - 1).$$

Thus, by Theorem 4.2.1 we have

$$c^{(\alpha)} = c_1^{(\alpha)} + \ldots + c_n^{(\alpha)}$$

where each $c_i^{(\alpha)}$ is a sum of finitely many $\mathbf{v}_i$-periodic $\mathbf{v}_i$-fibers. Denote $\mathbf{v}_i = (v_{i,1}, \ldots, v_{i,d})$ for each $i \in \{1, \ldots, n\}$.

If for some α we have $c_i^{(\alpha)} \neq 0$ and $c_j^{(\alpha)} \neq 0$ for some $i \neq j$, then there exist $\mathbf{u}_i = (u_{i,1}, \ldots, u_{i,d})$ and $\mathbf{u}_j = (u_{j,1}, \ldots, u_{j,d})$ such that $c_i^{(\alpha)}(\mathbf{u}_i) \neq 0$ and $c_j^{(\alpha)}(\mathbf{u}_j) \neq 0$. Since $c_i^{(\alpha)}$ is $\mathbf{v}_i$-periodic and $c_j^{(\alpha)}$ is $\mathbf{v}_j$-periodic, it follows that $c_i^{(\alpha)}(\mathbf{u}_i + t\mathbf{v}_i) \neq 0$ and $c_j^{(\alpha)}(\mathbf{u}_j + t\mathbf{v}_j) \neq 0$ for all $t \in \mathbb{Z}$. Thus, $c(\alpha_i + t(v_{i,1}b_1 + \ldots + v_{i,d}b_d)) \neq 0$ and $c(\alpha_j + t(v_{j,1}b_1 + \ldots + v_{j,d}b_d)) \neq 0$ for all $t \in \mathbb{Z}$ where $\alpha_i = \alpha + u_{i,1} + \ldots + u_{i,d}$ and $\alpha_j = \alpha + u_{j,1} + \ldots + u_{j,d}$. If

$$m(v_{i,1}b_1 + \ldots + v_{i,d}b_d) = m'(v_{j,1}b_1 + \ldots + v_{j,d}b_d)$$

for some $m, m' \in \mathbb{Z}$, then $mv_{i,1} = m'v_{j,1}, \ldots, mv_{i,d} = m'v_{j,d}$ by the uniqueness of the representation by the minimal generator set $\{b_1, \ldots, b_d\}$. Thus, $m\mathbf{v}_i = m'\mathbf{v}_j$

and hence $m = m' = 0$ since $\mathbf{v}_i$ and $\mathbf{v}_j$ are linearly independent. This means that the numbers $v_{i,1}b_1 + \ldots + v_{i,d}b_d$ and $v_{j,1}b_1 + \ldots + v_{j,d}b_d$ are rationally independent. Consequently, for all $\varepsilon > 0$, there exist $m, m' \in \mathbb{Z}$ such that $\alpha_i + m(v_{i,1}b_1 + \ldots + v_{i,d}b_d)$ and $\alpha_j + m'(v_{j,1}b_1 + \ldots + v_{j,d}b_d)$ are distinct and within distance ε from each other. This is a contradiction with the uniform discreteness of $\text{supp}(c)$.

Thus, for all α, we have $c^{(\alpha)} = c_i^{(\alpha)}$ for some $i \in \{1, \ldots, n\}$. Similarly as above, if for some α and β, we have $c^{(\alpha)} = c_i^{(\alpha)}$ and $c^{(\beta)} = c_j^{(\beta)}$ for $i \neq j$, we get again a contradiction with the uniform discreteness of $\text{supp}(c)$. So, we conclude that for all $\alpha \in \mathbb{R}$, $c^{(\alpha)} = c_i^{(\alpha)}$ for the same $i \in \{1, \ldots, n\}$. It follows that every $c^{(\alpha)}$ is $\mathbf{v}_i$-periodic and hence c is $(v_{i,1}b_1 + \ldots + v_{i,d}b_d)$-periodic. □

***Corollary* 5.3.17.** *Let $c \in \mathcal{A}^{\mathbb{R}}$ be a 1-dimensional Delone configuration with a non-trivial annihilator. Then c is a Meyer configuration.*

Proof. By Theorem 5.3.16 the Delone configuration c is periodic and hence by Lemma 5.1.16 it is a Meyer configuration. □

For $d \geq 2$, there exist Delone configurations with non-trivial annihilators that are not Meyer configurations. For example, consider a Delone set which is periodic but not strongly periodic and not a Meyer set. Its indicator function is a Delone configuration that has a non-trivial annihilator, and it is not a Meyer configuration. Thus, the above corollary does not hold in higher dimensions.

5.4 Forced periodicity of Delone configurations of finite local complexity

In this section we prove a statement of forced periodicity of Delone configurations of finite local complexity.

Recall that a non-empty finite set $F \Subset \mathbb{R}^d$ has a vertex in direction $\mathbf{v}$ if there exists a vector $\mathbf{t} \in \mathbb{R}^d$ such that $F + \mathbf{t} \subseteq \overline{H}_{\mathbf{v}}$ and $(F + \mathbf{t}) \cap \langle \mathbf{v} \rangle^{\perp} = \{\mathbf{0}\}$. An $\mathbb{R}^d$-polynomial has a vertex in direction $\mathbf{v}$ if its support has a vertex in direction $\mathbf{v}$.

For a Delone configuration $c \in \mathcal{A}^{\mathbb{R}^d}$, let us denote by

$$\mathcal{X}_c = \{\tau^{\mathbf{t}}(c) \mid \mathbf{t} \in \mathbb{R}^d, \tau^{\mathbf{t}}(c)(\mathbf{0}) \neq 0\}$$

the set of translations of c with non-zero values at the origin.

***Lemma* 5.4.1.** *Let c be a Delone configuration. If c is not strongly periodic, then $\mathcal{X}_c$ is an infinite set.*

Proof. Assume that c is not strongly periodic. Let S be the support of c.

Assume first that S is strongly periodic. Thus, S has d linearly independent period vectors $\mathbf{t}_1, \ldots, \mathbf{t}_d$. Since c is not strongly periodic, one of these vectors, say

$\mathbf{t}_1$, is not a period of c. Moreover, no multiple of $\mathbf{t}_1$ is a period of c. Let e be a translation of c such that $e \in \mathcal{X}_c$, that is, $e_{\mathbf{0}} \neq 0$. Clearly, any period of c is also a period of e and vice versa. Thus, for all $k \in \mathbb{Z}$ the Delone configurations $\tau^{k\mathbf{t}_1}(e)$ are distinct. Since $\mathbf{t}_1$ is a period of the support of e and $e_{\mathbf{0}} \neq 0$, also $\tau^{k\mathbf{t}_1}(e)(\mathbf{0}) \neq 0$ for all $k \in \mathbb{Z}$. Thus, $\tau^{k\mathbf{t}_1}(e) \in \mathcal{X}_c$ for all $k \in \mathbb{Z}$ and hence X_c is an infinite set.

Assume then that S is not strongly periodic. It follows that the set $\Lambda_S = \{\mathbf{t} \in \mathbb{R}^d \mid S + \mathbf{t} = S\}$ of periods of S spans a vector space of dimension less than d. By the relative denseness of S there are infinitely many vectors $\mathbf{t}_1, \mathbf{t}_2, \ldots \in S$ such that the cosets $\Lambda_S + \mathbf{t}_i$ are distinct. Let us define for every $i \in \mathbb{Z}_+$, a configuration $c_i = \tau^{-\mathbf{t}_i}(c)$. We have $c_i(\mathbf{0}) = c(\mathbf{t}_i) \neq 0$ since $\mathbf{t}_i \in \text{supp}(c)$. So, we have $c_i \in \mathcal{X}_c$. Let us show that for each $i \neq j$, also $c_i \neq c_j$ and hence $\mathcal{X}_c$ is an infinite set. Assume on the contrary that $c_i = c_j$ for some $i \neq j$. Then $\tau^{-\mathbf{t}_i}(c) = \tau^{-\mathbf{t}_j}(c)$ and hence $\tau^{\mathbf{t}_j - \mathbf{t}_i}(c) = c$. So, $\mathbf{t}_j - \mathbf{t}_i$ is a period of c and hence it is also a period of S. Thus, $\mathbf{t}_j - \mathbf{t}_i \in \Lambda_S$ and hence $\mathbf{t}_j \in \Lambda_S + \mathbf{t}_i$. Similarly, we conclude that $\mathbf{t}_i \in \Lambda_S + \mathbf{t}_j$ and hence $\Lambda_S + \mathbf{t}_i = \Lambda_S + \mathbf{t}_j$. This is a contradiction. □

In the proof of our theorem of forced periodicity we need the following technical lemma. We denote by $\mathbf{S}^1 = \{\mathbf{v} \in \mathbb{R}^d \mid ||\mathbf{v}|| = 1\}$ the d-dimensional unit sphere and use the fact that it is a compact subset of $\mathbb{R}^d$ under the usual Euclidean metric which implies that every sequence of elements of $\mathbf{S}^1$ has a converging subsequence.

***Lemma* 5.4.2.** *Let $(\mathbf{v}'_n)_{n \geq 1}$ be a converging sequence of vectors $\mathbf{v}'_1, \mathbf{v}'_2, \ldots \in \mathbf{S}^1$, and let $\mathbf{v}$ be the limit of this sequence. Let $T_0 \in \mathbb{R}_+$ and $\mathbf{u} \in \mathbb{R}^d$ be such that*

$$\mathbf{u} \in B^\circ_{T_0}(-T_0\mathbf{v}).$$

There exists $N_0 \in \mathbb{Z}_+$ such that

$$\mathbf{u} \in B^\circ_T(-T\mathbf{v}'_n)$$

for all $T \geq T_0$ whenever $n \geq N_0$.

Proof. Let $\varepsilon = 1 - \frac{d(-T_0\mathbf{v}, \mathbf{u})}{T_0}$. This is a positive real number by the assumption $\mathbf{u} \in B^\circ_{T_0}(-T_0\mathbf{v})$. Since $\mathbf{v} = \lim_{n\to\infty} \mathbf{v}'_n$, there exists N_0 such that $d(\mathbf{v}'_n, \mathbf{v}) < \varepsilon$ for all $n \geq N_0$. By the triangle inequality we have

$$d(-T_0\mathbf{v}'_n, \mathbf{u}) \leq d(-T_0\mathbf{v}'_n, -T_0\mathbf{v}) + d(-T_0\mathbf{v}, \mathbf{u}) < T_0 \cdot \varepsilon + d(-T_0\mathbf{v}, \mathbf{u}) = T_0$$

for all $n \geq N_0$. Thus, $\mathbf{u} \in B^\circ_{T_0}(-T_0\mathbf{v}'_n)$ and hence $\mathbf{u} \in B^\circ_T(-T\mathbf{v}'_n)$ for all $T \geq T_0$ and $n \geq N_0$ since $B^\circ_{T_0}(-T_0\mathbf{v}'_n) \subseteq B^\circ_T(-T\mathbf{v}'_n)$ for all $T \geq T_0$. □

The following lemma is also needed.

***Lemma* 5.4.3.** *Let $c \in \mathcal{A}^{\mathbb{R}^d}$ be a Delone configuration of finite local complexity and assume that $\mathcal{X}_c$ is an infinite set. Then there exists an unbounded increasing sequence $R_1, R_2, R_3, \ldots$ of positive real numbers such that for each $n \in \mathbb{Z}_+$ there exist $e, e' \in \mathcal{X}_c$ such that*

$$e \upharpoonright_{B^\circ_{R_n}} = e' \upharpoonright_{B^\circ_{R_n}}$$

but

$$e \upharpoonright_{B_{R_n}} \neq e' \upharpoonright_{B_{R_n}} .$$

Proof. Since $S = \text{supp}(c)$ is a Delone set of finite local complexity, $N_S(T)$ is finite for all T. For any $e \in \mathcal{X}_c$, the set $\text{supp}(e) \cap B_T$ is a T-patch of $S = \text{supp}(c)$. Hence, there exist only finitely many different functions $e \upharpoonright_{B_T}$ for any T because $\mathcal{A}$ is a finite set.

Assume on the contrary that the claim does not hold. Then there exists R_0 such that if $e, e' \in \mathcal{X}_c$ with $e \upharpoonright_{B_{R_0}} = e' \upharpoonright_{B_{R_0}}$, then $e = e'$. It follows that $\mathcal{X}_c$ is finite since there are only finitely many functions $e \upharpoonright_{B_{R_0}}$. □

Now, we are ready to state and prove the main result of the section.

***Theorem* 5.4.4.** *Let $c \in \mathcal{A}^{\mathbb{R}^d}$ be a Delone configuration of finite local complexity and assume that for all non-zero $\mathbf{w}$ it has an annihilator that has a vertex in direction $\mathbf{w}$. Then c is strongly periodic.*

Proof. Assume on the contrary that c is not strongly periodic. Thus, by Lemma 5.4.1 the set $\mathcal{X}_c$ is infinite.

Consequently, by Lemma 5.4.3 there exists an unbounded sequence $0 < R_1 < R_2 < R_3 < \ldots$ such that for all $n \in \mathbb{Z}_+$ there exist a vector $\mathbf{v}_n \in \mathbb{R}^d$ with $||\mathbf{v}_n|| = R_n$ and $e, e' \in \mathcal{X}_c$ such that

$$e \upharpoonright_{B^\circ_{R_n}} = e' \upharpoonright_{B^\circ_{R_n}}$$

but

$$e(\mathbf{v}_n) \neq e'(\mathbf{v}_n).$$

Consider the sequence $(\mathbf{v}'_n)_{n \in \mathbb{Z}_+}$ where $\mathbf{v}_n = R_n \mathbf{v}'_n$ and $\mathbf{v}'_n \in \mathbf{S}^1$. By compactness of $\mathbf{S}^1$ we may assume that this sequence converges since we can replace it by a converging subsequence if necessary. Let $\mathbf{v} \in \mathbf{S}^1$ be the limit of this sequence.

By the assumption, c has an annihilator f such that it has a vertex in direction $-\mathbf{v}$. Without loss of generality we may assume that f is multiplied by a suitable monomial $X^{\mathbf{u}}$ such that $\mathbf{0} \in \text{supp}(f)$ and $\text{supp}(f) \setminus \{\mathbf{0}\} \subseteq H_{-\mathbf{v}}$, *i.e.*, $-\text{supp}(f) \setminus \{\mathbf{0}\} \subseteq H_{\mathbf{v}}$. Moreover, we may assume that $f_{\mathbf{0}} = 1$. Let T_0 be such that $-\text{supp}(f) \subseteq B_{T_0}(-T_0\mathbf{v})$ and $-\text{supp}(f) \setminus \{\mathbf{0}\} \subseteq B^\circ_{T_0}(-T_0\mathbf{v})$.

Let then $N_0 \in \mathbb{Z}_+$ be such that

$$-\text{supp}(f) \setminus \{\mathbf{0}\} \subseteq B^\circ_T(-T\mathbf{v}'_n)$$

for all $T \geq T_0$ and $n \geq N_0$. By Lemma 5.4.2 such N_0 exists since $\text{supp}(f) \setminus \{\mathbf{0}\}$ is a finite set. See Figure 13 for an illustration.

Let $n \geq N_0$ be such that $R_n \geq T_0$. So, we have

$$-\text{supp}(f) \setminus \{\mathbf{0}\} \subseteq B^{\circ}_{R_n}(-R_n \mathbf{v}'_n) = B^{\circ}_{R_n}(-\mathbf{v}_n).$$

Now, consider two distinct $e, e' \in \mathcal{X}_c$ such that

$$e \upharpoonright_{B^{\circ}_{R_n}} = e' \upharpoonright_{B^{\circ}_{R_n}}$$

but $e(\mathbf{v}_n) \neq e'(\mathbf{v}_n)$. Since $-\text{supp}(f) \setminus \{\mathbf{0}\} \subseteq B^{\circ}_{R_n}(-\mathbf{v}_n)$ and hence $\mathbf{v}_n - \text{supp}(f) \setminus \{\mathbf{0}\} \subseteq B^{\circ}_{R_n}$, it follows that

$$e \upharpoonright_{\mathbf{v}_n - \text{supp}(f) \setminus \{\mathbf{0}\}} = e' \upharpoonright_{\mathbf{v}_n - \text{supp}(f) \setminus \{\mathbf{0}\}} .$$

We have $fe = 0$ and $fe' = 0$ and hence

$$e(\mathbf{v}_n) = - \sum_{\mathbf{u} \in \text{supp}(f) \setminus \{\mathbf{0}\}} f_{\mathbf{u}} e(\mathbf{v}_n - \mathbf{u}) = - \sum_{\mathbf{u} \in \text{supp}(f) \setminus \{\mathbf{0}\}} f_{\mathbf{u}} e'(\mathbf{v}_n - \mathbf{u}) = e'(\mathbf{v}_n).$$

This is a contradiction. □

Theorem 5.3.4 together with the above theorem gives the following corollary. The proof of the corollary resembles the proof of Lemma 3 in [50].

***Corollary* 5.4.5.** *Let c be an integral Delone configuration of finite local complexity. Assume that for all $V \in \mathbb{G}_{d-1}$ it has a non-trivial annihilator f such that supp$(f) \cap V = \{\mathbf{0}\}$. Then c is strongly periodic.*

Proof. Consider an arbitrary non-zero vector $\mathbf{v}$ and let $V = \langle \mathbf{v} \rangle^{\perp}$. By the assumption c has an annihilator f such that $\text{supp}(f) \cap V = \{\mathbf{0}\}$. By Theorem 5.3.4 the $\mathbb{R}^d$-polynomial

$$g = \prod_{\mathbf{u} \in \text{supp}(f) \setminus \{\mathbf{0}\}} (X^{k_{\mathbf{u}} \mathbf{u}} - 1)$$

annihilates c for some integers $k_{\mathbf{u}}$. Since $(X^{k_{\mathbf{u}} \mathbf{u}} - 1) = -X^{k_{\mathbf{u}} \mathbf{u}}(X^{-k_{\mathbf{u}} \mathbf{u}} - 1)$, we may replace any $(X^{k_{\mathbf{u}} \mathbf{u}} - 1)$ by $(X^{-k_{\mathbf{u}} \mathbf{u}} - 1)$ if necessary and hence we may assume that $k_{\mathbf{u}} \mathbf{u} \in H_{\mathbf{v}}$ for each $\mathbf{u}$. Since the support of g consists of $\mathbf{0}$ and sums of the vectors $k_{\mathbf{u}} \mathbf{u}$ we have $\text{supp}(g) \setminus \{\mathbf{0}\} \subseteq H_{\mathbf{v}}$. Thus, the annihilator g of c has a vertex in direction $\mathbf{v}$ and hence the claim follows by Theorem 5.4.4. □

In particular, we have the following known result as a corollary. The original proof of the result uses Theorem 2.5.10 and Theorem 2.4.4.

***Corollary* 5.4.6** ([50]). *Let c be an integral $\mathbb{Z}^d$-configuration. Assume that for all $V \in \mathbb{G}_{d-1}$ it has a non-trivial annihilator f such that supp$(f) \cap V = \{\mathbf{0}\}$. Then c is strongly periodic.*

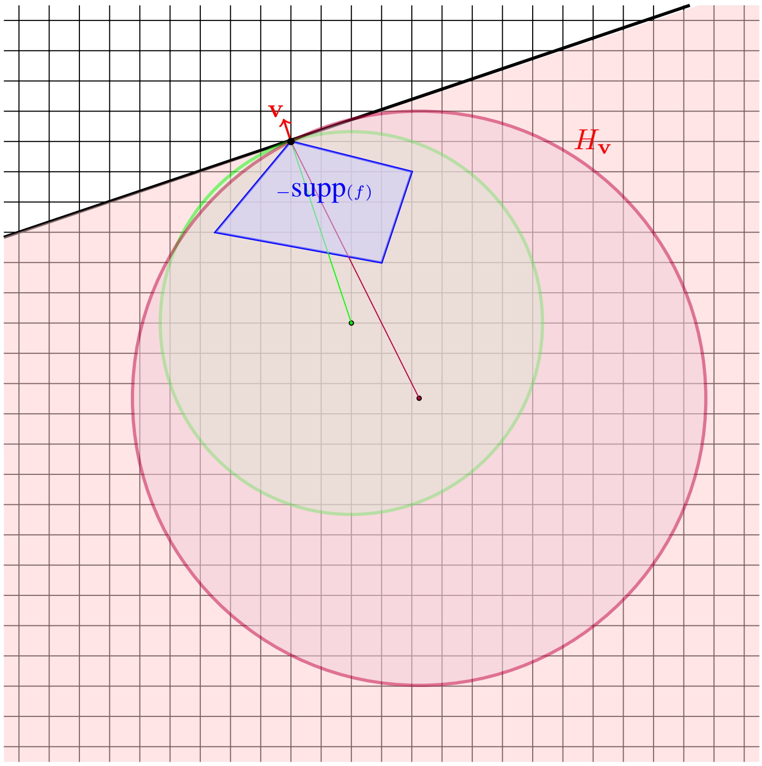

Figure 13. Illustration of the proof of Theorem 5.4.4. The smaller ball is the ball $B_{T_0}(-T_0\mathbf{v})$ and the bigger ball is the ball $B_T(-T\mathbf{v}'_n)$ for some $T \geq T_0$ and $n \geq N_0$.

Remark 5.4.7. The converse direction of Corollary 5.4.5 holds for any $c \in \mathbb{C}^{\mathbb{R}^d}$. In other words, a strongly periodic function $c \in \mathbb{C}^{\mathbb{R}^d}$ has for all $V \in \mathbb{G}_{d-1}$ a periodizer f such that $\mathrm{supp}(f) \cap V = \{\mathbf{0}\}$. Indeed, since c has d linearly independent period vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$, it follows that for any $V \in \mathbb{G}_{d-1}$ some $\mathbf{v}_i$ is not in V and hence $X^{\mathbf{v}_i} - 1$ is an annihilator (and hence a periodizer) of c satisfying $\mathrm{supp}(X^{\mathbf{v}_i} - 1) \cap V = \{\mathbf{0}\}$.

6 Conclusion and Some Open Problems

In this final chapter, we conclude this thesis by stating some relevant open problems. We start by restating the commonly known open problems that we have already discussed in the introduction. After this, some more specific open problems are mentioned that are more closely related to the results of this thesis.

Despite our efforts, Nivat's conjecture still remains an open problem:

***Conjecture* 6.1** (Nivat's conjecture [73])**.** *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional configuration and let $m, n \geq 1$. If*

$$P_c(m, n) \leq mn,$$

then c is periodic.

As mentioned in the introduction, there is a version of Nivat's conjecture for convex shapes:

***Conjecture* 6.2** (Nivat's conjecture for convex shapes [13])**.** *Let $c \in \mathcal{A}^{\mathbb{Z}^2}$ be a two-dimensional configuration and let $D \Subset \mathbb{Z}^2$ be a convex shape. If*

$$P_c(D) \leq |D|,$$

then c is periodic.

Let us then turn to translational tilings. The periodic tiling problem was proved to be false for sufficiently large dimensions d by Greenfeld and Tao [35]. However, the smallest such dimension d was not calculated. An interesting question is to ask what is the smallest d for which the periodic tiling problem fails. Of course $d > 2$ since the periodic tiling problem holds for $d = 1$ and $d = 2$ [72; 8]. In particular, we have the following question:

***Question* 6.3.** *Does the periodic tiling problem hold for $d = 3$?*

Let us state the strong and weak Golomb-Welch conjectures:

***Conjecture* 6.4** (Strong Golomb-Welch conjecture [29])**.** *Let $d \geq 3$, $r \geq 2$, and let B_r^d be the radius r Lee sphere of dimension d. There are no translational tilings by B_r^d.*

***Conjecture* 6.5** (Weak Golomb-Welch conjecture [29])**.** *Let $d \geq 3$, $r \geq 2$, and let B_r^d be the radius r Lee sphere of dimension d. There are no strongly periodic translational tilings by B_r^d.*

In Chapter 3 we considered forced periodicity of two-dimensional perfect coverings (that is, perfect colorings with only two colors). We studied forced periodicity of (D, b, a)-coverings where D is the relative r-neighborhood of 2-dimensional square, triangular, or king grid. In Example 3.3.17 we demonstrated that Theorem 3.3.14 holds also in dimension $d = 3$. We conjecture that it holds for arbitrary dimension:

***Conjecture* 6.6.** *Let $d \geq 2$ and $r \geq 1$ and let D be the relative r-neighborhood of the d-dimensional king grid and assume that $a \neq b$. Then any (D, b, a)-covering is strongly periodic.*

The relative r-neighborhood of the d-dimensional king grid is the set

$$D = \{(u_1, \ldots, u_d) \mid |u_1|, \ldots, |u_d| \leq r\}.$$

We also conjecture that similar results in the 2-dimensional square and triangular grids work for arbitrary dimension. In other words, we conjecture that the analogies of theorems 3.3.10, 3.3.11, 3.3.12 and 3.3.13 hold for arbitrary dimension $d \geq 2$. We skip the exact definitions of d-dimensional square and triangular grids, but they should be clear.

We wonder whether Theorem 5.2.6 works for arbitrary Delone sets, not just for Meyer sets:

***Question* 6.7.** *Does a Delone set $S \subseteq \mathbb{R}^d$ with no non-trivial annihilators satisfy*

$$\liminf_{T \to \infty} \frac{N_S(T)}{T^d} > 0?$$

If the answer to the above question is negative, then we may still wonder whether it holds for Delone sets of finite local complexity:

***Question* 6.8.** *Does a Delone set $S \subseteq \mathbb{R}^d$ of finite local complexity with no non-trivial annihilators satisfy*

$$\liminf_{T \to \infty} \frac{N_S(T)}{T^d} > 0?$$

Finally, we wonder whether an analogy of Nivat's conjecture holds for two-dimensional Delone sets. So, we ask the following question.

***Question* 6.9.** *Let $S \subseteq \mathbb{R}^2$ be a two-dimensional Delone set. Does there exist $a \leq 1$ such that if*

$$N_S(T) \leq aT^2$$

for some T, then S is periodic?

The analogy of the above conjecture does not hold for dimensions $d \geq 3$ as discussed in Section 5.2 of Chapter 5. Indeed, it was mentioned that for $d \geq 3$, there exist non-periodic Delone sets $S \subseteq \mathbb{R}^d$ such that

$$\lim_{T \to \infty} \frac{N_S(T)}{T^d} = 0$$

which means that $N_S(T) = o(T^d)$ and hence for all $a \in \mathbb{R}_+$, we have $N_S(T) \leq aT^d$ for sufficiently large T. In fact, the Delone sets S in the construction were subsets of $\mathbb{Z}$ and hence Meyer sets since Delone subsets of Meyer sets are also Meyer sets [59]. Question 6.9 might be easier for Meyer sets:

***Question* 6.10.** *Let $S \subseteq \mathbb{R}^2$ be a two-dimensional Meyer set. Does there exist $a \leq 1$ such that if*

$$N_S(T) \leq aT^2$$

for some T, then S is periodic?

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