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The originality of this publication has been checked in accordance with the University of Turku quality assurance system using the Turnitin OriginalityCheck service.

ISBN 978-951-29-8055-0 (PRINT)
ISBN 978-951-29-8056-7 (PDF)
ISSN 2343-3159 (Painettu/Print)
ISSN 2343-3167 (Verkkojulkaisu/Online)
Painosalama Oy, Turku, Finland 2020

ABSTRACT

This dissertation discusses various aspects of strategic trading using both analytical modeling and numerical methods. Strategic trading, in short, encompasses models of trading, most notably models of optimal execution and portfolio selection, in which one seeks to rigorously consider various—both explicit and implicit—costs stemming from the act of trading itself. The strategic trading approach, rooted in the market microstructure literature, contrasts with many classical finance models in which markets are assumed to be frictionless and traders can, for the most part, take prices as given.

Introducing trading costs to dynamic models of financial markets tend to complicate matters. First, the objectives of the traders become more nuanced since now overtrading leads to poor outcomes due to increased trading costs. Second, when trades affect prices and there are multiple traders in the market, the traders start to behave in a more calculated fashion, taking into account both their own objectives and the perceived actions of others. Acknowledging this strategic behavior is especially important when the traders are asymmetrically informed. These new features allow the models discussed to better reflect aspects real-world trading, for instance, intraday trading patterns, and enable one to ask and answer new questions, for instance, related to the interactions between different traders.

To efficiently analyze the models put forth, numerical methods must be utilized. This is, as is to be expected, the price one must pay from added complexity. However, it also opens an opportunity to have a closer look at the numerical approaches themselves. This opportunity is capitalized on and various new and novel computational procedures influenced by the growing field of numerical real algebraic geometry are introduced and employed. These procedures are utilizable beyond the scope of this dissertation and enable one to sharpen the analysis of dynamic equilibrium models.

Keywords: Strategic trading, optimal execution, asymmetric information, market microstructure, algebraic geometry

TIIVISTELMÄ

Tämä väitöskirja käsittelee strategista kaupankäyntiä hyödyntäen sekä analyyttisiä että numeerisia menetelmiä. Strategisen kaupankäynnin mallit, erityisesti optimaalinen kauppojen toteutus ja portfolion valinta, pyrkivät tarkasti huomioimaan kaupankäynnistä itsestään aiheutuvat eksplisiittiset ja implisiittiset kustannukset. Tämä erottaa strategisen kaupankäynnin mallit klassisista kitkattomista malleista.

Kustannusten huomioiminen rahoitusmarkkinoiden dynaamisessa tarkastelussa monimutkaistaa malleja. Ensinnäkin kaupankävijöiden tavoitteet muuttuvat hienovaraisemmiksi, koska liian aktiivinen kaupankäynti johtaa korkeisiin kaupankäyntikuluihin ja heikkoon tuottoon. Toiseksi oletus siitä, että kaupankävijöiden valitsemat toimet vaikuttavat hintoihin, johtaa pelikäyttäytymiseen silloin, kun markkinoilla on useampia kaupankävijöitä. Pelikäyttäytymisen huomioiminen on ensiarvoisen tärkeää, mikäli informaatio kaupankävijöiden kesken on asymmetristä. Näiden piirteiden johdosta tässä väitöskirjassa käsitellyt mallit mahdollistavat abstrahoitujen rahoitusmarkkinoiden aiempaa täsmällisemmän tarkastelun esimerkiksi päivänsisäisen kaupankäynnin osalta. Tämän lisäksi mallien avulla voidaan löytää vastauksia uusiin kysymyksiin, kuten esimerkiksi siihen, millaisia ovat kaupankävijöiden keskinäiset vuorovaikutussuhteet dynaamisilla markkinoilla.

Monimutkaisten mallien analysointiin hyödynnetään numeerisia menetelmiä. Tämä avaa mahdollisuuden näiden menetelmien yksityiskohtaisempaan tarkasteluun, ja tätä mahdollisuutta hyödynnetään pohtimalla laskennallisia ratkaisuja tuoreesta numeerista reaalista algebrallista geometriaa hyödyntävästä näkökulmasta. Väitöskirjassa esitellyt uudet laskennalliset ratkaisut ovat laajalti hyödynnettävissä, ja niiden avulla on mahdollista terävöittää dynaamisten tasapainomallien analysointia.

Asiasanat: Strateginen kaupankäynti, optimaalinen toteutus, asymmetrinen informaatio, markkinoiden mikrostruktuuri, algebrallinen geometria

ACKNOWLEDGEMENTS

> *I've worked in so many areas—I'm sort of a dilettante. Basically, I'm not interested in doing research and I never have been. I'm interested in understanding, which is quite a different thing. And often to understand something you have to work it out yourself because no one else has done it.*
>
> **David Blackwell** (1919 – 2010)

As usual, a number of thank-you's are in order. First, I extend my gratitude for Prof. Mika Vaihekoski and Prof. Luis Alvarez for guidance and support throughout this process. I also want to thank the pre-examiners, Prof. Juho Kanniainen and Prof. Tommi Sottinen, for valuable feedback.

Next, I wholeheartedly thank the Department of Accounting and Finance for having me all these years. Special thanks goes to Matti Heikkonen, Mikko Kepsu, Matti Niinikoski, Paavo Okko, and Tuula Vilja for interesting conversations around the coffee table.

Working on this dissertation would have been a great deal more difficult without financial support from OP-Pohjola Group Research Foundation, Turku University Foundation, Finnish Foundation for Advancement of Securities Markets, and Turun Liikemiesyhdistys. This support is most gratefully acknowledged.

Naturally, everyone in the home front deserves a special thanks. I would not have undertaken this project without the support of my parents. A huge thanks also belongs to my better half Eva. I guess you have seen both Jekyll and Hyde—at least after this nonsense is over, it is Dr. Jekyll.

Finally, in an impersonal manner I would like to express my sincerest gratitude towards the long list of authors whose papers have influenced my thinking and this dissertation. The intellectual debt owed is immense and I truly hope that in the following pages I am able to, at least in part, to repay that debt.

TO MY PARENTS

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Part I

SYNTHESIS

1 INTRODUCTION

1.1 Motivation

Market microstructure research delves deep into the inner workings of financial markets, contemplating—among many other topics—issues such as price discovery, intraday trading patterns, trading strategies, and transaction costs. It is one of the most rapidly growing fields in finance, especially considering the research landscape in finance post the 2007 financial crisis (see, Guéant 2016). One of the main reasons for the strong development of this research area is the rise of both electronic and algorithmic trading, both of which have shaped financial markets tremendously, often with outcomes that are in stark contrast with classical market microstructure models in financial economics.[1]

Topics and questions in the field of market microstructure have also captured the imagination of researchers from statistics, mathematics, and physics and due to this the field is vibrant and constantly evolving. As such, there is a myriad of methods and approaches that one comes across in microstructure research starting from game theory and standard (low frequency) econometrics and continuing on to stochastic control theory, high frequency econometrics, and many others. Moreover, with provocative topics such as market manipulation or ominous terms such as dark pools, it is unsurprising that popularized depictions of financial markets and market microstructure have also found their way in to bookstores and even on to bestseller lists, most notably perhaps the 2014 book *Flash Boys* by Michael Lewis.

It should also be noted that even though it is easy to equate the term market in market microstructure to the stock market this would in fact give an erroneously narrow scope for the field. Indeed, questions relating to market microstructure are great of interest in, for instance, the bond markets and the foreign exchange markets. For a (very) recent survey on the former the reader is referred to Biais and Green (2019) and for a (very) recent survey on the latter the reader is referred to Evans and Rime (2019).

Based on the above, it comes as no surprise that there are numerous important research questions and topics in the area of market microstructure that are in need of further inquiry. From the point of view of this dissertation, the most relevant are the topics related to (optimal) trading strategies and intraday trading dynamics under asymmetric information, endogenous learning, and various forms of price processes

[1] For an overview of the classical literature see, for example, O'Hara (1995) or Madhavan (2000).

and transaction costs.

These questions and topics are exceedingly important from both academic and practical perspectives. Considering the academic perspective, understanding the dynamics of financial markets and, for instance, how differential information possessed by various agents affects these dynamics is one of the most challenging and important questions in financial economics. As for the practical perspective, increasing competition and regulation—e.g., *Regulation National Market System* (Reg. NMS) and *Markets in Financial Instruments Directive* II (MiFID II)—in the financial markets means that ever more smaller details—micro-details, if you will—have become essential in ensuring survival and the ability to operate profitably.

Furthermore, all these topics naturally fit under the umbrella of strategic trading. Strategic trading, in this thesis, is in broad terms taken to mean trading activities which take into account price impact and other possible trading costs, endogenous or exogenous, in an explicit fashion.[2] This entails, for instance, optimal execution programs where the goal is to minimize the expected execution costs of a given trade, or portfolio selection under transaction costs where one constructs optimal portfolios in a multiperiod setting, taking into account that portfolio rebalancing is costly.

1.2 Research Questions

Building on the large extant literature, this dissertation focuses on strategic trading models from a mostly *normative* perspective, utilizing both theoretical and numerical methods. The models discussed feature both game-theoretic and non-game-theoretic settings with an emphasis on the former. Special attention is paid to the computational approaches utilized in solving and analyzing various models. Moreover, empirical implications as well as practical aspects arising are emphasized when deemed appropriate. In sum, the main goals of the dissertation are:

- ★ To study how risk preferences, asymmetric information, and endogenous learning—in previously overlooked non-cooperative settings with heterogenous players—affect equilibrium trading and intraday patterns in financial markets.
- ★ To examine how different functional specifications of price impact change optimal trading strategies and the nature of equilibrium trading.
- ★ To put forth new numerical approaches for analyzing questions related to optimal execution, portfolio optimization, and strategic trading.

[2] Price impact can stem, for instance, from revelation of private information or from the impact of orders on the limit order book, while other sorts of transactions fees can be thought of as, for instance, slippage costs or transaction taxes.

The key unifying question throughout the dissertation is how a trader should optimally navigate through different market conditions. To exemplify, the first essay depicts, on one hand, a trader with mean-variance-skewness preferences, facing a myopic multiperiod portfolio optimization problem under quadratic transaction costs and, on the other hand, a trader facing different types of nonstandard optimal execution problems. The scope of changing market conditions, however, goes well beyond just tweaking how price impact or transaction costs are specified.

More fundamental questions, addressed in the second and the third essay, involve markets with asymmetric information and learning as well as non-cooperative strategic interactions between traders with possibly different risk preferences. By examining optimal trading on these types of complicated markets one hopes to shed light on important issues not discussed in earlier literature. Further, pushing the boundaries of existing frameworks is also important in the sense that it is the best way to reveal the limitations of various modeling choices and potential deficiencies in the way these models are typically analyzed.

In the midst of exploring the aforementioned questions various numerical methods, many of which a relatively new to the field of finance, are utilized. A number of programming languages (e.g., Julia and Python) as well as computing platforms (e.g., MATLAB® and MATHEMATICA®) are employed in the implementation of these numerical methods both for performance and robustness reasons.[3] In addition, two freeware packages written for MATLAB® and designed for specialized calculations are extensively used. PHCpack (cf. Verschelde 1999, Guan and Verschelde 2008), which implements polynomial homotopy continuation, is used in the first and the second essay. GloptiPoly 3 (cf. Henrion et al. 2009), which solves (approximates) generalized moment problems and which, consequently, can be used to solve polynomial optimization problems, is used in the first essay.

1.3 Structure

The first, introductory, part of the dissertation proceeds as follows. Section 2 gives an overview of strategic trading and optimal execution problems. Section 3 introduces the field of numerical real algebraic geometry, a relatively young field of mathematics, which forms the basis for the numerical procedures used in the essays. Section 4 provides an overview of the included essays. There is a general guideline adopted in the first part: as there are plenty of equations in the second part of this dissertation—in an effort to balance things out—there will be none in the first part.

The second part consists of three distinct, independent essays, focusing on various aspects of strategic trading. The ordering of the essays is partially arbitrary.

[3] MATLAB® is a registered trademark of The MathWorks, Inc. MATHEMATICA® is a registered trademark of Wolfram Research, Inc.

However, the ordering of the first and second essay can be justified by noting that the first essay gives a general depiction and detailed background information on some of the methods used in the second essay.

2 STRATEGIC TRADING, OPTIMAL EXECUTION, AND TRANSACTION COSTS

2.1 Background

Financial markets and the behavior of agents in these markets is difficult to model and it is not hard to see why. Even if one accepts the classical premise of a Bayes-rational, utility maximizing market participants, one still faces a problem of describing the behavior of a (large) number of potentially heterogenous individuals, making decisions independently (yet acknowledging the presence of others) based on potentially heterogenous, correlated, and possibly overlapping information.[4]

Nevertheless, in a series of seminal papers, a few tractable, and from a modeling perspective rather fruitful, approaches to tackle this problem have been suggested. A good place to start is to consider frictionless (i.e., no transaction costs) and complete or (perfectly) competitive markets with infinite liquidity. In the context of financial market modeling, a convenient feature obtained using this approach is that all market participants may be considered as *price takers*, i.e., the trades of the market participants do not affect the market price at which the trades are executed.

Assuming traders are price takers is a modelling choice made, for instance, in a myriad of (competitive) *rational expectations equilibrium* (REE) models. By assuming the number of traders in the market is infinite, competitive REE models can generate many valuable insights about the workings of financial markets. In general, this modeling approach can be considered to produce a (macro) approximation for a market featuring only very liquid stocks and characterized by the absence of large traders (cf. Cetin et al. 2004). Even after accepting this approximation point of view, the outlined approach is not completely unproblematic.

To exemplify, Vives (2010) distinguishes two problems with competitive rational expectations equilibrium (REE). First, in a fully revealing, competitive REE no one has an incentive to acquire information since all the information is already included in the market price. This instead results in the equilibrium collapsing (the Grossman and Stiglitz 1980 paradox). Second, there is the *schizophrenia* problem introduced by Hellwig (1980). In general, this refers to the phenomenon where a trader takes advantage of the information content of market prices but ignores the price impact of trades. This latter problem only pertains to finite economies—i.e., economies

[4] Even if the market participants have identical information, they may differ, for instance, in their initial wealth endowment or risk aversion.

(or markets) where the number of traders is finite—and vanishes as the size of the economy approaches infinity, since then individual trades no longer have a price impact. Finally, competitive models are unable to capture the nuances brought about by introducing large traders in the market.

To avoid these issues and to direct focus more on the micro features of financial markets, the approach taken in this dissertation is to explicitly model strategic trading behavior along the lines of Kyle (1985).[5] Conversely to the competitive REE models, the backbone of strategic trading models is the notion that trades—or at least the trades of some traders, e.g., the aforementioned large traders—indeed have price impact and traders take this price impact into account in designing their trading strategies.[6] Generally speaking, this yields a class of optimal execution problems in which the additional strategic dimension adds to the complexity of the question under study—especially if one is interested in a model where price impact rises endogenously—but at the same time enables one to study a wealth of topics that are beyond the reach of frictionless competitive models. Differences within this class usually stem from how price impact, or, more generally, price formation, and other trading costs are specified.

2.2 Curious Case of Price Impact

Price formation is a central topic in the area of market microstructure and this question has been looked at from various perspectives. Classical works in this area, often adopting the viewpoint of a dealership market with inventory concerns, include Garman (1976), Stoll (1978), Amihud and Mendelson (1980), and Ho and Stoll (1981); Roll (1984) discusses order processing costs. Further, Glosten and Milgrom (1985) provided a seminal framework to study the relation between bid-ask spread and adverse selection and Kyle (1985) studied price formation and strategic trading under asymmetric information.

Asymmetric information and trading strategies are elevated to a prominent role in this dissertation and hence the most relevant paper from above is Kyle (1985), in which a privately informed insider trades in a market with noise traders and competing market makers. The market makers seek to infer over time the private information of the insider from noisy order flow signals. These endeavors of the market makers give rise to endogenous price impact and the market price is gradually di-

[5] Generally speaking, this dissertation adopts a slightly wider view on strategic trading as compared to classical market microstructure literature. Strategic trading, in this dissertation, is in broad terms taken to mean trading which takes into account price impact and other possible costs explicitly. This entails, for instance, optimal execution programs where the goal is to minimize expected execution costs or portfolio selection under transaction costs. Price impact can be taken to stem, for instance, from revelation of private information or from the impact of orders on the limit order book.

[6] Kyle (1989) avoids the schizophrenia problem by looking at a non-competitive market where traders submit demand schedules instead of market orders.

rected towards the valuation indicated by the insider's private information. A central observation here is that price impact in Kyle (1985) exists because of private information and, for example, uninformed trades in the model have no price impact. This line of thinking is adopted in the second essay.

While price impact stemming from private fundamental information hidden in aggregate order flows is a compelling story, it is not the whole story. This view is endorsed in, for instance, Bouchaud (2010), in which the author characterizes price impact as being the correlation between an incoming (buy or sell) order and the following price change. In many instances, the information content of order flow is low or nonexistent and incoming orders still cause immediate price shifts. One way to explain this is to note that not all traders are always present in the market and liquidity providing intermediaries must be compensated for their efforts (cf. Grossman and Miller 1988).[7]

Alternatively, one could purport that price impact stems from the dynamics of the limit order book as in Obizhaeva and Wang (2013).[8] The authors develop a model between security prices and supply-demand dynamics and use it to study optimal trading strategies in the resulting market.[9] Subsequent literature (cf. Schied et al. 2010, Schied et al. 2017, and Schied and Zhang 2018) has often used the Obizhaeva and Wang (2013) model to justify certain assumptions about the (mathematical) properties of price impact with less emphasis on explicitly modeling limit order book dynamics—the third essay of this dissertation utilizes a similar approach.

Figure 1 summarizes how price impact is modeled in this dissertation.

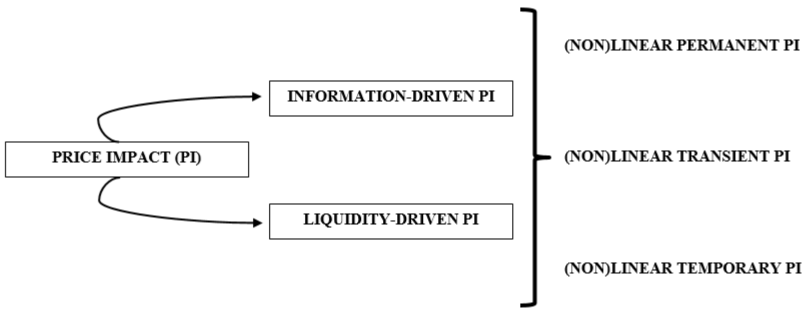

Figure 1: (**Price impact decomposed**) This figure presents an overview of how price impact is "decomposed" in this dissertation.

The first thing to note from Figure 1 is how the general price impact is decomposed

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[7] Liquidity demand/provision and liquidity in general is a closely related, albeit arguably somewhat wider, topic to the themes discussed in this section. Amihud et al. (2006)—in addition to providing an excellent survey of the liquidity literature—defines *liquidity* as the ease of trading a security and lists, e.g., demand pressure, inventory concerns, and information symmetry as well as adverse selection as potential drivers of *illiquidity*.

[8] Most financial markets nowadays are organized around a double auction limit order book.

[9] The link between supply-demand dynamics and price changes is well established in the empirical literature. See, for example, Bouchaud et al. (2009).

to *information-driven* (as in, e.g., Kyle 1985) price impact and *liquidity-driven* (as in, e.g., Obizhaeva and Wang 2013) price impact. This distinction has notable implications from a modeling point of view as discussed in the second and third essay.[10]

The second key notion in Figure 1 is the functional form and the duration of price impact. Starting with the functional form, the price impact function used to transmit the effect of trades on market prices, can be either linear or nonlinear, the former offering more tractability and the latter some added realism with respect to empirical results.[11] In Kyle (1985) price impact is linear and permanent, a choice that facilitates efficient modeling of learning from noisy order flow signals, while Obizhaeva and Wang (2013) features linear transient price impact.

Moving on, the differences between various price impact duration assumptions can be summarized as follows (see also, Alfonsi et al. 2012).

- PERMANENT PRICE IMPACT: Impact is (perfectly) persistent and thus affects the current and all future trades; can stem from, e.g., new fundamental information being incorporated into prices, i.e., information-based price impact.
- TRANSIENT PRICE IMPACT: Impact is greater immediately after the trade but vanishes over time; can stem from, e.g., limit order book effects, i.e., liquidity-driven price impact.
- TEMPORARY PRICE IMPACT: Impact is limited to the current trade; can stem from, e.g., transaction taxes.[12]

Figure 2 provides a further illustration of price impact stemming from a single sell (market) order.[13]

[10] Papers featuring combinations of different types of price impact functions are scarce. Park and Van Roy (2015) is one example.

[11] See, for example, Bouchaud (2010).

[12] Temporary price impact is included in, for instance, Almgren and Chriss (2001) and Huberman and Stanzl (2005).

[13] A buy (market) order could be illustrated analogously.

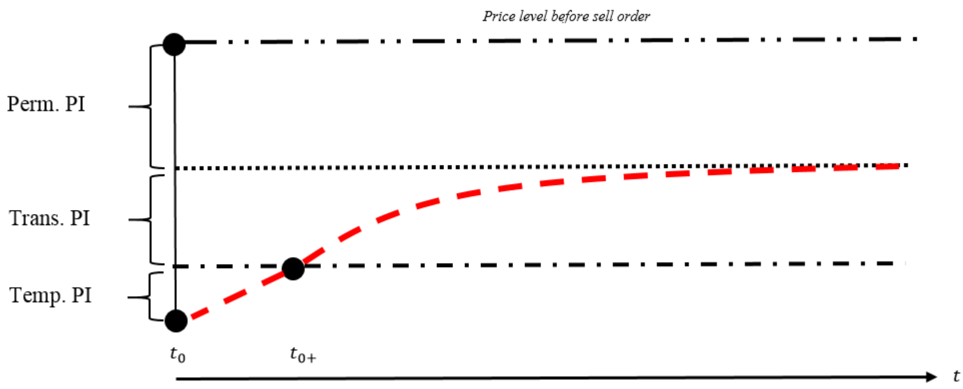

Figure 2: (**Price impact of a sell order**) This figure illustrates the effect of different components of price impact for a single market sell order.

In Figure 2, a sell order first causes an instantaneous drop in the price level at t_0, followed by an (almost) immediate recovery as the temporary price impact has dissipated at t_{0+}.[14] As time moves on, the transient price impact of the sell order slowly fades and the price approaches a new, post-trade, price level as illustrated by the middle (dotted) horizontal line.[15]

In Figure 2, the post trade price level is lower than the original price level due to permanent price impact. In some cases, permanent price impact can, however, be taken to be close to negligible in which case the price eventually recovers close to the original level. A case in point would be a situation where there is a large order, which consumes plenty of liquidity but is not believed to contain any fundamental—i.e., price relevant—information.

The way price impact is specified has important implications on the viability of the modeled financial market. For instance, it turns out that the form of price impact is important from the point of view of market manipulation. In an important paper, Huberman and Stanzl (2004) demonstrate that with permanent, time-independent price impact, manipulation and arbitrage can only be ruled out when the price impact function is *linear*. Similarly, Gatheral (2010) studies how the shape of the price impact function is related to dynamic arbitrage when price impact is allowed to be transient.[16] These previous works are extended by Alfonsi et al. (2012) who study

[14] Note that in this dissertation the focus in mainly on intraday trading, i.e., $t_{0+} - t_0$ is assumed to be small in relative terms; in a discrete time model one can assume that temporary price impact affects only current period trades. Naturally, it can take more time for the price to recover but the effect of longer horizon price dynamics is not considered. Mainly because in the models considered there is an urgency component which prescribes that trading must be completed within a given time-interval. Moreover, with longer horizons it becomes increasingly likely that prices are affected by various outside shocks and these additional shocks should be incorporated into the models considered. Finally, if one interprets temporary costs as, e.g., transaction taxes, it is obvious that these costs are not actually visible in market prices but instead affect only the actual prices incurred by the trader.

[15] See also, Duffie (2010).

[16] Manipulation using the Kyle (1985) model with permanent linear price impact is studied in, for example, Van Bommel (2003), Chakraborty and Yilmaz (2004b), and Chakraborty and Yilmaz (2004a). In these mod-

transaction-triggered price manipulation strategies.

Nevertheless, while (implicit) price impact costs often constitute a lion's share of total transaction costs, there are other costs which might be of interest as different costs can have different implications on equilibrium outcomes. A case in point of costs separate from price impact are explicit transaction costs.

2.3 Explicit Transaction Costs

As noted by Amihud et al. (2006), exogenous transactions costs such as brokerage fees and transaction taxes are an important source of illiquidity. Fabozzi et al. (2010) (pp. 421–426) characterize these types of costs as *explicit*, as compared to the *implicit* price impact costs discussed above, and this categorization is also adopted in this dissertation.[17] Further, explicit transaction (or, trading) costs, whenever present, are taken to be purely exogenous.[18]

By incorporating explicit trading costs in addition to implicit trading costs, one is able to delve deeper into aspects of strategic trading than would be possible if just implicit costs were considered. For example, portfolio optimization with implicit costs is exceedingly complicated, but one can still—in a rather straightforward manner—utilize explicit costs to introduce financial market frictions into portfolio models (cf. Gârleanu and Pedersen 2013). This approach is taken in the first essay.

Moreover, Schied and Zhang (2018) study a price impact game with both implicit and explicit costs and propose that (quadratic) transaction costs may in fact provide protection against predatory trading and even decrease the total (implicit plus explicit) trading costs experienced by traders. This protection feature of transaction costs is examined further in the third essay.

2.4 Other Modeling Choices

In addition to the way (implicit and explicit) transaction costs are modeled in this dissertation, there are also a few additional modeling choices, which are worth a short discussion. First, in this dissertation attention is not on capturing the dynamics of a limit order book and, for instance, only a single market price is utilized, instead of separate bid and ask prices. This choice is made both for simplicity and, due to the fact that, even if one, for example, models a two-sided limit order book often only

els, additional assumptions are added to the original framework to make manipulation feasible even under linear permanent price impact.

[17] For additional information about transaction costs see, for instance, Johnson (2010) (Section 2.5) or Kissell (2014) (Chapter 3.).

[18] More specifically, the functional form of explicit transaction costs is taken to be quadratic in line with, e.g., Gârleanu and Pedersen (2013).

one side of the book is utilized (cf. Obizhaeva and Wang 2013). [19]

Second, differences between order types, most notably market orders and limits orders, are generally not emphasized. Common wisdom dictates that traders with greater urgency utilize market orders while more patient or less time-constrained traders utilize limit orders. Since this dissertation focuses for the most part on optimal execution problems to be carried out within a fixed time-frame, one can think of the traders in the models as being (highly) impatient and thus choosing to submit market orders.[20]

2.5 Recent Research

Much of the theory in this dissertation, especially in the second and third essay, owes to two recent and closely related papers: Moallemi et al. (2012) and Choi et al. (2019). The former leans more towards the optimal execution literature in quantitative finance while the latter builds on the strategic trading foundations provided by Kyle (1985).[21] Starting with Moallemi et al. (2012), a quick overview of the optimal execution literature is in place.

Influential early works in this area include Bertsimas and Lo (1998), Almgren and Chriss (1999), and Almgren and Chriss (2001) with more recent contributions by, for example, Alfonsi et al. (2010) and Obizhaeva and Wang (2013).[22] One of the key developments moving from the "classical" papers towards the more recent work is the way in which price impact is modeled. Earlier papers, such as Bertsimas and Lo (1998), Almgren and Chriss (1999), utilized permanent and temporary price impact while in, e.g., Obizhaeva and Wang (2013) price impact is transient.[23] Typical to the models presented in these papers is that they focus on the trading strategy of a lone trader isolated from other traders potentially present in the market.

Indeed, historically, the key difference between the strategic trading and optimal execution approaches in the literature has been that strategic trading often refers to game-theoretic modeling while standard optimal execution models abstract over the interactions between different traders and concentrate only on formulating optimal strategies under a given, *exogenous*, price impact function. Recently, however, this

[19] Recent reviews related to limit order book modeling are given in, for example, Parlour and Seppi (2008), Bouchaud et al. (2009), and Gould et al. (2013).

[20] It should also be emphasized that strategic trading with limit orders under asymmetric information is highly complicated from a modeling point of view. See, for example, Roşu (2012).

[21] There are numerous extensions of the original Kyle (1985) model: Foster and Viswanathan (1990), Subrahmanyam (1991), Holden and Subrahmanyam (1992), Foster and Vishwanathan (1993), Holden and Subrahmanyam (1994), Caballe and Krishnan (1994), Rochet and Vila (1994), Holden and Subrahmanyam (1996), Huddart et al. (2001), and Bernhardt and Miao (2004) to name some.

[22] See also, e.g., Predoiu et al. (2011) and Gatheral and Schied (2011).

[23] Finite market resilience and replenishing liquidity is described empirically in, for example, Biais et al. (1995), Ranaldo (2004), Degryse et al. (2005), and Large (2007). Other relevant statistical contributions include: Cont (2011), Hautsch and Huang (2012), Cont et al. (2014), and Lo and Hall (2015).

distinction between the two has become more blurred. For instance, Schied and Zhang (2017), Strehle (2017), Schied and Zhang (2018), and Huang et al. (2019) study optimal execution in a game-theoretic framework. One of the goals of this dissertation, especially the third essay, is to blur the distinction even further.

However, a feature even rarer than strategic interactions in optimal execution models is the presence of asymmetric information. In Moallemi et al. (2012) both strategic interactions and asymmetric information have a key role as the paper studies a game between two traders, one of which, say trader A, has to liquidate a given position, the size of which is private information of trader A, while the other trader, say trader B, is actively seeking profitable trading opportunities. Because B does not know how large a position A is going to liquidate, B tries to learn this information from price changes. The quicker B is able to infer the trading needs of A, the better B is able to use this information in formulating an adversary (predatory) trading strategy, taking advantage of predicted future price movements.

The presence of asymmetric information is what distinguishes Moallemi et al. (2012) from, for example, earlier work on predatory trading such as such as Brunnermeier (2005), Carlin et al. (2007), and Schied and Schöneborn (2009). The combination of asymmetric information and strategic interactions also separate Moallemi et al. (2012) from most of the optimal execution literature, even though the foundations of the paper are build on the classical (exogenous) liquidity-driven, linear permanent price impact market. The third essay examines a version of Moallemi et al. (2012) featuring transient price impact and quadratic transaction costs.

Moving on to Choi et al. (2019), the first thing to note is that now, due to the influence of Kyle (1985), the price impact is of the linear permanent form and, more importantly, information-driven and *endogenous*. More specifically, price impact rises due to the efforts of the market (makers) to learn the private (fundamental) information hidden in aggregate order flows. Choi et al. (2019) builds on the works of Foster and Viswanathan (1994), Foster and Viswanathan (1996), and Back et al. (2000) in which the authors study strategic trading by competing, differentially informed traders.[24]

On top of endogenous price impact, the more complex information structure in Choi et al. (2019) is the main distinction that separates the paper from Moallemi et al. (2012). Indeed, in Choi et al. (2019) this complex information structure gives rise to the so-called "forecasting the forecasts of others" problem which complicates the model analysis. The upside, however, is that one is able to: *(a)* form a clearer picture of how (non-nested) differential information affects the intraday dynamics and equilibrium trading strategies and *(b)* complement the results obtained via the Moallemi et al. (2012) framework.

[24] In contrast to the other papers, Back et al. (2000) utilizes a continuous time framework. Continuous time version of the Kyle (1985) model is also examined in, e.g., Back (1992), Back et al. (1998), Baruch (2002), Back and Baruch (2004), Collin-Dufresne and Fos (2016).

While the theoretical results obtained in this dissertation are normative in nature, there are also important linkages to questions and challenges put forth in recent empirical papers. For example, Van Kervel and Menkveld (2019) bring forth the need for models dealing with contemporaneous trading by a number of heterogenous traders.[25] This makes sense as these types of interactions drive the intraday patterns observed in financial markets, and hence, the better one is able to explain the interactions, the better one is able to explain the observed patterns. In addition, the models studied in this dissertation—supplemented with novel and detailed data sets—can be used in hypothesis formation in empirical inquiries to the trading strategies of corporate insiders, as in Klein et al. (2017), or professional asset managers, as in Di Mascio et al. (2017).[26]

Finally, understanding optimal execution of trades from a normative perspective in various settings is also valuable from a practical point of view. On one hand, best execution practices are embedded in regulations, such as Reg. NMS, and on the other hand, there is empirical evidence (see, Anand et al. 2011) which supports the economic significance of the trade implementation process in evaluating institutional trading desks. Hence, motives are clearly present for institutions to seek improvements to these practices and processes.

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[25] See also, for instance, Khan et al. (2012), Dyakov and Verbeek (2013), and Busse et al. (2018).

[26] See also, Dufour and Engle (2000) who use a large transaction level data set to study price impact and market activity, finding a positive relation between activity and price impact.

3 NUMERICAL REAL ALGEBRAIC GEOMETRY

3.1 A Quick Overview of the Field

In dealing with problems pertaining to strategic trading and optimal execution, methods and topics from the—relatively young—area of *numerical real algebraic geometry* (NRAG) are prominently featured in this dissertation. For this reason, this section gives a sententious summary of this ponderable and quickly evolving field from the point of view of applications.

The links between real algebraic geometry and the methods discussed below are neatly summarized in the introductory notes of Bochnak et al. (1998). Algebraic geometry is concerned with the set of solutions to polynomial systems and in real algebraic geometry one is mainly interested in the real solutions, i.e., the set of real solutions. In addition, questions related to sign changes of polynomials and to domains where a given polynomial has constant sign are of great interest in (real) algebraic geometry.

To understand what are the links alluded to above, it suffices to note that the methods in Section 3.2 and 3.4 are designed to find all solutions to a polynomial system or a certain type of polynomial, whereas the methods in Section 3.3 take advantage of so-called certificates of positivity, i.e., theoretical results that guarantee that a given polynomial remains positive when restricted to a certain set.

Having made the distinction between algebraic geometry and its real counterpart, what is left to address is the quantifier *numerical*. It is indeed the case that many questions in real algebraic geometry are such that they are accessible to an algorithmic way of thinking (cf. Basu et al. 2016).[27] This coupled with the rise in computational power and the development of several high quality software packages implementing various algorithms pertinent to algebraic geometry (cf. Stillman et al. 2008), has enabled the rise of the field now known as numerical real algebraic geometry (NRAG). It is precisely this rise of NRAG that has granted access to numerous, novel and valuable, numerical approaches with plenty of untapped potential in the field of finance and economics.

[27] An example of such a question would be the classical root counting problem.

3.2 Gröbner Bases and Polynomial Homotopy Continuation

The methods discussed in this section are all about finding *all* solutions to systems of polynomial equations. This is in contrast to many standard numerical root finding algorithms which typically, given some initial parameters, only produce a single root at a time. This difference is significant, as it is often the case with, for instance, game-theoretic models that: (1) the equilibrium conditions are given by a (square) system of polynomial equations which has many (real) roots (2) potential equilibrium multiplicity, stemming from the existence of multiple roots, may cause serious ambiguities in interpreting the model predictions.[28] To take these issues into consideration and to better understand equilibrium behavior two numerical all-solutions methods, namely, Gröbner bases and polynomial homotopy continuation, are employed in the first and the second essay.

The first all-solutions method to be discussed here is Gröbner bases. The insight behind this method, based on deep results in algebraic geometry, is to simplify the task of finding all solutions to a given polynomial system to finding all solutions to a much simpler system, which nevertheless has exactly the same solution set as the original system. In its core, Gröbner bases enable one to focus attention on finding the roots of a single univariate polynomial instead of a (square) system of polynomials. It is worth pointing out here that univariate root finding represents a significant simplification as compared to multivariate root finding and many efficient univariate root finding algorithms exist in practically any programming language.[29] The computational complexity with respect to Gröbner bases stems from the fact that obtaining a simpler version of the original system requires some work. This simplification process is often thought of as a generalization of the classical Gaussian elimination in linear algebra.

The second method, namely, polynomial homotopy continuation, takes an alternative route to finding all solutions to a polynomial system. This alternative route involves utilizing a special system (the so-called start system), with an appropriate number of ex ante known zeros, in conjunction with the original system to form a continuation system, which can then be used to find all solutions to the original system. Roughly speaking, the idea is to take a convex combination of the start system and the original system, set this new system to zero, and, starting from the known solutions of the start system, to "slide" (i.e., to put more weight on the original system)—little by little—towards the case where all the weight is on the original system and as output one obtains the desired solution set. While polynomial homotopy methods draw heavily from complex analysis, results from algebraic geometry are crucial in finding efficient bounds on the number of (complex) solutions for the polynomial system under study. These bounds are needed in constructing the best

[28] Some examples are listed in Kubler et al. (2014).

[29] The ease of univariate root finding is also the reason why all-solutions methods are not needed in the third essay.

possible start system on a case-by-case basis.

A good question at this point is: Why would one need to utilize two methods? The logic behind this choice is related to the inherent weaknesses present in both Gröbner bases and homotopy continuation methods. On one hand, Gröbner bases are computationally expensive, not naturally parallelizable, and may run into numerical difficulties, especially when polynomial coefficients are real numbers.[30] Thus, Gröbner bases are best suited for studying the solutions of relatively simple and small systems. On the other hand, polynomial homotopy continuation methods can be applied to larger systems and are naturally parallelizable, but can potentially waste a considerable amount of computational effort to non-essential calculations (i.e., tracking diverging paths; cf. Judd et al. 2012) and are also subject to numerical issues (e.g., singularities and rounding errors; cf. Datta 2010).[31] By utilizing both methods, one gets the best of both worlds combined with an additional layer of robustness.

3.3 Polynomial Optimization

Polynomial optimization—as the name implies—deals with optimization problems where the objective functions as well as the constraints take the form of polynomial equations. Recent work (cf. Anjos and Lasserre 2012) has shown that there exist efficient numerical algorithms to solve these types of problems. Furthermore, existing algorithms: (1) have generically finite convergence and (2) are able to numerically verify global optimal solutions. Both features are important from the perspective of practical usability. Finite convergence means that one is actually able to solve these problems numerically, usually within a sensible time frame. Certificates for global optimality instead are profoundly useful when dealing with, for example, problems which are beyond the scope of classical optimization theory and standard optimality conditions.

Generally speaking, the central feature of the polynomial optimization approach (cf. Lasserre 2015, pg. 6) is that it is not tied to the standard setting of convex versus nonconvex, but is instead able, within the family of polynomial problems, to offer a unified treatment for numerous relevant problems belonging to this family. More importantly—at least from the point of view of this dissertation—it turns out that the aforementioned family is large enough to include various interesting problems from the field of finance.

First, problems involving moments of random variables often give rise to polynomial objective functions. A case in point is a portfolio optimization problem with, for instance, mean-variance or mean-variance-skewness preferences. Second, it is well-

[30] When polynomial coefficients are rational numbers of parameters, Gröbner bases computations are exact. See, for example, Kubler and Schmedders (2010).

[31] Polynomial homotopy continuation methods are purely numerical in nature.

known that continuous functions can be accurately approximated with polynomials. Hence, one is often able to utilize polynomial optimization methods even to cases where the original problem is not directly stated in terms of polynomial functions. Therefore, it is safe to conclude that the scope of potential applications of polynomial optimization is vast.

3.4 Interval Polynomials

An interval polynomial is, in essence, an algebraic equation with perturbed coefficients.[32] Here, perturbed coefficients are taken to mean that each coefficient of the polynomial is allowed to take values from an (non-degenerate) appropriate interval. Therefore, one can view an interval polynomial as a family of polynomials, each of which has the same degree and the same variables (unknowns), but at the same time featuring distinct coefficient values.

Interval polynomials arise naturally in situations where, for instance, the coefficients are subject to noise due to inaccurate measurements. A case in point would an equation with a fixed (algebraic) functional form but whose coefficients are estimated from data. Alternatively, one could justify the use of interval polynomials by the properties of floating-point arithmetic ubiquitous in computational applications. Interval polynomials, and interval analysis methods in general, also have notable applications in robust optimization and control (Vehí et al. 2002 and Mansour et al. 2012).

This dissertation shortly touches on the theory of interval polynomials by means of a short example in the context of a dynamic optimization problem. In this problem, the relevant optimality conditions simplify to an algebraic equation whose coefficients are determined by a set of exogenous parameter values as well as known solutions from earlier rounds of the dynamic programming algorithm. The resulting algebraic equation can be treated as an interval polynomial and to this polynomial one can apply the recent results by Zhang and Deng (2013) which, for a fixed interval, allow one to determine the number of associated interval zeros. The purpose of the short example is to bring forth a novel—at least to the best knowledge of the author—approach to study equilibrium existence and uniqueness in dynamic models, featuring algebraic equilibrium conditions.

32 Cf. Ferreira et al. (2001), Ferreira et al. (2005), and Zhang and Deng (2013).

3.5 Applications of NRAG in Finance and Economics

Many of the numerical approaches touched upon in this dissertation have presently a limited number of well-documented applications in finance and related fields. This is reflected in the fact that following overview of related references is close to exhaustive. This does not, however, mean that there would not exist many potential future applications, some of which are discussed in this dissertation.[33]

Starting with the homotopy continuation approach, recent applications include, for instance, dynamic general equilibrium models (see, for example, Amisano and Tristani 2009 and Amisano and Tristani 2011), game theory (see, for example, Bajari et al. 2010 and Judd et al. 2012), and industrial organization (see, for example, Besanko et al. 2010). For a review on usage of homotopy continuation in game theory, the reader is referred to Herings and Peeters (2010), and for a guide on using the methodology to solve dynamic games, the reader is referred to Borkovsky et al. (2010). Some less recent applications include Schmedders (1998) and Schmedders (1999).

Moving on to Gröbner bases, recent applications can be found in, for instance, Kubler and Schmedders (2010) and Robatto (2019). The former is more of a survey paper, while the latter utilizes Gröbner bases to rule out bad outcomes among the multiple equilibria supported by the monetary economics model introduced in the paper. Similarly, Marinovic and Varas (2018) resort to methods from computational algebraic geometry to tackle multiplicity of equilibria in a continuous time model of strategic trading and activism. Gröbner bases complement homotopy methods in dealing with the determination of the solution sets for complicated polynomial systems. As noted earlier, it is for this reason that polynomial homotopy continuation and Gröbner bases are often utilized side-by-side in this paper.

Pertaining to polynomial optimization, one prominent example application is Renner and Schmedders (2015), who study the principal-agent problem—the PA-problem, for short.[34] PA-problems typically fall in to the category of *bilevel optimization problems*. In these types of problems one optimization problem is a part of the constraint set of another, upper-level, optimization problem (see, for instance, Colson et al. 2007). Bilevel programs are, by and large, very difficult to solve and standard approaches to solve PA-problems feature strict assumptions. Renner and Schmedders (2015) tackle this issue by reformulating the PA-problem, under suitable yet relatively mild conditions, as a polynomial optimization problem. This approach enables one to avoid the strict assumptions related to alternative approaches. Renner and Schmedders (2017) extend the methods to a dynamic PA setting, obtaining

[33] Further support for the view that more applications of NRAG in finance and economics are likely to emerge in the near future is provided by the success of these methods in other fields. See, for instance, Josz et al. (2015) and Ghaddar et al. (2017).

[34] See also, Couzoudis and Renner (2013).

numerical approximations for the policy and value functions arising in the proposed model.

Finally, recent applications featuring interval methods include Stradi and Haven (2005) and Stradi-Granados and Haven (2010).[35] The authors emphasize the benefits stemming from interval methods in conjunction with well-known optimization algorithms in solving highly nonlinear, multiperiod (rational expectations) models. One of these benefits is the ability to locate multiple roots without changing the initial conditions and to determine unique roots by iteratively narrowing the intervals (bisection) around points of interest. The application of interval methods introduced in this dissertation is similar in spirit, but focuses more on interval polynomials and recent theoretical results pertaining to *the number of zeros* of the objects, instead of offering purely numerical recipes for actually locating these zeros.

[35] See also, Zilinskas and Bogle (2006).

4 OVERVIEW OF INCLUDED ESSAYS

4.1 Applications of Numerical Algebraic Geometry in Strategic Trading and Portfolio Optimization

The first essay deals with novel numerical approaches to, both old and new, problems in (non-game-theoretic) optimal execution and portfolio optimization as well as (game-theoretic) strategic trading. The approaches utilized rely on the techniques from real algebraic geometry introduced in Section 3. The essay starts by offering a concise overview of polynomial optimization, polynomial homotopy continuation, and Gröbner bases, and then proceeds to a comprehensive exploration of various example applications.

The first example application is related to the classical problem of optimal execution of trades discussed in Section 2. The key observation is that these problems often involve settings, which can be modeled using polynomial objective functions and polynomial constraints. Therefore, polynomial optimization techniques offer an effective way to go about solving these types of problems. Numerical examples show that polynomial optimization can indeed be utilized to analyze (in a global sense) some problems related to optimal execution, which have previously been considered too complicated to solve or solved using, more or less, heuristic methods.

The second example application also takes advantage of polynomial optimization techniques. This time the goal is to solve a mean-variance-skewness (MVS)—instead of the well-known mean-variance (MV)—portfolio optimization problem under quadratic transaction costs. As noted in earlier, this type of problem sits comfortably under the umbrella of strategic trading in this dissertation.

What makes MVS portfolio optimization interesting is the fact that adding skewness to the objective function of the problem breaks the standard quadratic formulation. This in turn means that classical approaches typically cannot guarantee the global optimality of the solutions obtained. This is not, however, a problem for the versatile modern polynomial optimization techniques. Examples are provided to illustrate some noteworthy similarities and differences between the standard MV optimal portfolios and the MVS portfolios. It is worth pointing out that the MV portfolio problem can also be described by a set of polynomial equations and hence solved using polynomial optimization. Therefore, via the use of the polynomial optimization approach, one can implement a unified optimization procedure capable of handling portfolio problems featuring two or more moments.

The final example application presented in the first essay concentrates on a recent dynamic strategic trading model introduced in Choi et al. (2019) and extended in the second essay of this dissertation. The goal is to utilize numerical algebraic geometry (Gröbner bases and polynomial homotopy continuation) to numerically study equilibrium uniqueness in the model. The main motivation for this is the lack of analytical tools—stemming from the complexity of the equilibrium system of equations—to properly answer this question. It is shown that general uniqueness, without additional conditions imposed on the equilibrium properties, is questionable. Nevertheless, all-solutions methods can be used to identify and rule out problematic equilibria and to generally sharpen the equilibrium analysis.

The first essay also covers various worthwhile topics for additional research. Furthermore, an algorithmic viewpoint pertaining to the different example applications is provided.

4.2 Risk Preferences, Dynamic Equilibrium, and Trading Targets

In the second essay, the recent Choi et al. (2019) model, featuring two strategic traders in a dynamic financial market, is extended to allow for more general risk preferences. In addition to this extension, a detailed look at the numerical methods used in solving the model is given, especially from the point of view of equilibrium existence and uniqueness. The numerical methods are again based on numerical algebraic geometry and the theory concerning polynomial equations.

The main theoretical contribution of the paper is to provide a solution to the negative exponential utility version of Choi et al. (2019). One of the reasons why this contribution is important is that is allows one to narrow down the instances where risk neutral traders are good proxies for, e.g., risk averse traders. Indeed, the model introduced allows one to examine both risk averse and risk seeking traders, while at the same time nesting the original risk neutral model, and hence facilitates multifaceted comparisons between various model equilibria under different risk preferences.

Among these comparisons, several interesting features pertaining to the equilibrium values of the model constants and the resulting intraday dynamics arise. For instance, in a high risk aversion market, price impact is monotonically decreasing towards the end of the day, whereas in the risk neutral market price impact exhibits an S-shaped pattern, decreasing more sharply in the beginning and the end of the day, while remaining more or less unchanged during the middle of the day. Further, it is observed that risk aversion increases mutually beneficial liquidity provision between the two traders in the model. Finally, it is shown that the risk preferences also have a marked impact on the intraday autocorrelation patterns observed in equilibrium.

On the methodological front, a computational approach using polynomial homotopy continuation (cf. Verschelde 1999) to tackle the question of equilibrium multiplicity is put forth. Using this approach, combined with key analytic observations, a meticulous discussion pertaining to the equilibrium existence and uniqueness properties as well as techniques to numerically solve for the dynamic equilibrium is presented. It is noted that, for example, equilibrium uniqueness is closely linked to the feasible ranges for the initial parameters in the model and that strict uniqueness may require additional constraints beyond those stemming from explicit equilibrium equations. All in all, the goal is to advocate the use of more insightful and versatile computational means to investigate dynamic strategic models in finance.

4.3 Order Execution Game with Transient Price Impact and Asymmetric Information

The third essay introduces an optimal order execution game with transient price impact between two traders possessing different (ex ante) information endowments. The framework is based on the recent Moallemi et al. (2012) paper. The model developed in the third essay bears resemblance to models of predatory trading (cf. Carlin et al. 2007 and Brunnermeier and Pedersen 2009), but at the same time deviates from these models in the choice of information structure (asymmetric information) and the assumption that price impact is transient instead of permanent.

The third essay contributes to the literature by analyzing in depth the equilibrium implications—i.e., changes in trading strategies, expected execution costs/profits, etc.—of moving from permanent to transient price impact. This analysis is carried out using both theoretical and numerical methods. Regarding the former, the first part of the paper walks the reader through the steps involved in constructing the model solution (equilibrium), while at the same time exemplifying some key model properties. Equilibrium existence and uniqueness are discussed with a view on interval polynomials.

Regarding the latter, the tail end of the essay focuses on the numerical analysis of the model. A sundry of interesting observations is uncovered. First, a simplified version of the model developed in the paper in contrasted against Obizhaeva and Wang (2013) to show that the trading strategies obtained are in line with earlier literature. Next, through various numerical examples, it is illustrated how moving from permanent to transient price impact markedly changes the equilibrium of the order execution game. Most notably, it is discovered that it is no more optimal for trader B (cf. Section 2.5) to pursue a strictly predatory strategy. Finally, the impact of additional transaction fees is examined. It is found that while transaction costs are detrimental from the viewpoint of optimal strategies, they can alleviate the problem of predatory trading when the liquidating player, referred to as player A in Section 2.5, utilizes suboptimal strategies.

The essay concludes with a detailed look at possible extensions and topics for further research. Appendices give an account of the numerical solution and an example, featuring a novel interval polynomial approach, pertaining to the question of existence and uniqueness of the model equilibrium.

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Part II

ESSAYS

MIKA HANNULA

Applications of Numerical Algebraic Geometry in Strategic Trading and Portfolio Optimization

Preprint

I

Applications of Numerical Algebraic Geometry in Strategic Trading and Portfolio Optimization

Mika Hannula*

Abstract

This paper gives an overview of developments in (numerical) algebraic geometry with novel applications to finance. Previously problematic theoretical and practical problems related to strategic trading and portfolio optimization are solved as examples. Multiple topics for further research as well as new application areas within the field of finance are discussed.

Keywords: Algebraic geometry, polynomial optimization, polynomial homotopy continuation, Gröbner bases, optimal execution, portfolio optimization, strategic trading

JEL Classification Numbers: G10, G11, G12, G14

1 INTRODUCTION

This paper discusses the use of numerical methods arising from algebraic geometry in finance applications. More specifically, methods stemming from subfields of algebraic geometry such as real algebraic geometry and numerical algebraic geometry and their potential applications in models of strategic trading and portfolio optimization.[1] In general, the field of algebraic geometry has evolved strongly during the last few decades and many theoretical results have been successfully implemented algorithmically (see, for instance, Basu et al. 2016). Due to this, one is able to tackle, with fresh and efficient computational methods, interesting problems that were previously considered to be out of reach.

One explicit example of these computational methods is *(polynomial) homotopy continuation* (or, PHC), which can be used to find *all* solutions to a given polynomial

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[1] Since it is often impractical to strictly place a given approach to a subfield, the term algebraic geometry is instead utilized as an umbrella term encompassing all relevant subfields.

system.[2] An example application is polynomial systems encountered in game theory. More specifically, in a game setting, one is often interested in determining uniqueness of a given equilibrium and for this purpose one must usually check all possible solutions in order to verify uniqueness. Depending on the application, the interest may also be in, for instance, finding some subset of solutions, potentially even just one solution, or conversely finding all real solutions to a given system.

It is good to note that while the basic theory of (polynomial) homotopy continuation has been known for quite some time, more recent theoretical developments and increases in computational power combined with the availability of professionally implemented programs have made it possible to effectively utilize these methods in practical applications. In general, this side-by-side advancement of theory, algorithms, and processing power has paved the way for the surge in applications of algebraic geometry.

Unsurprisingly then, it is not hard to find noteworthy applications—see, for instance, Schmedders (1998) and Schmedders (1999). Homotopy continuation methods have recently been found useful, for instance, in the context of dynamic general equilibrium models (see, for example, Amisano and Tristani 2009 and Amisano and Tristani 2011), game theory (see, for example, Bajari et al. 2010 and Judd et al. 2012), and industrial organization (see, for example, Besanko et al. 2010). For a review on usage in game theory, see Herings and Peeters (2010), and for a guide on using the methodology to solve dynamic games, see Borkovsky et al. (2010). In this paper homotopy methods are utilized to numerically study equilibrium properties and uniqueness in a recent dynamic strategic trading model of Choi et al. (2019). In addition to describing the numerical procedure, the discussion below sheds light on some overlooked features of the Choi et al. (2019) model.

Gröbner bases (cf. Becker and Weispfenning 1993) represent an alternative to homotopy continuation methods with a different, algebraic geometry based, angle to finding all solutions to polynomial systems of equations. Gröbner bases have also benefited from increased computational power and the diffusion of easy-to-utilize algebraic geometry software. Overall, Gröbner bases complement homotopy methods in dealing with the determination of solution sets for complicated polynomial systems. For this reason, polynomial homotopy continuation and Gröbner bases are utilized side-by-side in this paper.

Recent applications of Gröbner bases can be found in, for instance, Kubler and Schmedders (2010) and Robatto (2019). While the former is more of a survey paper, the latter utilizes Gröbner bases to rule out bad outcomes among the multiple equilibria supported by the novel model introduced in the paper. Similarly, Marinovic and Varas (2018) resort to methods from computational algebraic geometry to tackle multiplicity of equilibria in a continuous-time model of strategic trading and

[2] Homotopy continuation can be used generally when dealing with nonlinear equations or systems but in this paper polynomial systems specifically are of interest.

activism.

The final example is *polynomial optimization* which, in short, is about minimizing/maximizing general polynomial equations over a feasible set described by polynomial inequalities. An example of this sort of problem is the standard mean-variance (MV) portfolio optimization problem with (or without) a short-selling constraint (see, for example, Markowitz 2010). The MV portfolio problem has a quadratic objective function and can typically, depending on how the problem is constrained, be handled in the standard convex optimization framework. Polynomial optimization discussed in this paper, however, goes much further than this. Namely, cubic and quartic polynomial objectives under some given, polynomially constrained, feasible set are analyzed.[3]

A few comments on why this represents a notable advancement in numerical methods are in order. First, for higher degree polynomials, determining whether the objective is convex (or concave) can be NP-hard (see, Ahmadi et al. 2013). This is problematic as standard optimization approaches heavily rely on the convexity (concavity) of objective functions. Second, directly minimizing a polynomial function over a set defined by polynomial equalities and inequalities is also, in general, NP-hard (Laurent 2009).

The solution to this predicament is to utilize (approximative) numerical methods. Numerical methods, however, can sometimes produce ambiguous results due to the lack of general theory. Luckily, the proposed polynomial optimization approach, drawing again from algebraic geometry, is able to provide a numerical certificate of the *global* optimality of the solution obtained. This certificate represents a marked improvement over any sort of case-specific, pseudo-global (heuristic) approach often utilized in the literature.[4]

Most advances in polynomial optimization are more recent than those in polynomial homotopy continuation or Gröbner bases. At the same time, the number of potential applications is numerous from finance and economics to robotics and from probability theory to computer science. Recent applications in game theoretic settings include Couzoudis and Renner (2013), Renner and Schmedders (2015), and Renner and Schmedders (2017).

In this paper, polynomial optimization is exemplified via an optimal execution problem, i.e., a problem of executing a sequence of trades so that price impact is minimized, and a portfolio optimization example. Since polynomial optimization problems are often encountered in situations where approximations—for example, (orthogonal) polynomials such as *Chebyshev polynomials* or (truncated) *Taylor series expansion*—are utilized, two of the problems discussed are in fact based on

[3] Higher degree objectives are certainly possible but not considered in this paper.

[4] An example of a pseudo-global approach is a method based on multiple runs of a local optimization algorithm, with variations in the initial conditions, in hopes that the best solution obtained via this procedure is indeed the global optimum. Typically, there is no way to verify the claim.

approximations. One problem relating to optimal execution of contingent claims and another relating to mean-variance-skewness (MVS) portfolio optimization. The examples considered reveal many interesting, previously unstudied, aspects of some key problems in finance and illustrate broadly the intuition and applicability of polynomial optimization.

The rest of the paper is organized as follows. Section 2 introduces the theoretical background to the methods utilized. In Section 3 polynomial optimization techniques are employed to a classical optimal execution problem. Section 4 illustrates an application of polynomial optimization to mean-variance-skewness portfolio optimization and reviews a number of recent developments in the field of portfolio optimization, which could be tackled using the polynomial optimization approach. Section 5 provides an example of finding all solutions to a polynomial system via homotopy continuation and Gröbner bases. Section 6 concludes the paper and Appendices A–E provide supplementary details.

2 TOOLS FROM ALGEBRAIC GEOMETRY

2.1 Preliminaries

In this section, a bulk of the notation and definitions used below as well as some key underlying results are presented. All definitions and results are standard and hence no references are provided. Further, for conciseness, the presentation omits several details and the interested reading hoping for further details is referred to, for example, Bochnak et al. (1998), Cox et al. (2006), and Lasserre (2015).

In this paper, it suffices to concentrate on real polynomials which in turn are constructed using monomials. Hence, the following is a good starting point.

Definition 1 (Monomial). *A monomial for variables $x_1, ..., x_n$ is given by the product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. If $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, then the monomial is standard and if $\alpha \in \mathbb{Z}^n$, then x^α is called the Laurent monomial.*

As noted above, monomials are used to form polynomials as established in Definition 2.

Definition 2 (Polynomial). *Finite linear combinations:*

$$f = \sum_{\alpha \in S} c_\alpha x^\alpha,$$

where c_α's are constants, are called polynomials if $S \subset \mathbb{Z}^n_{\geq 0}$ or Laurent polynomials if $S \subset \mathbb{Z}^n$.

The degree of a polynomial, i.e., the highest sum $\alpha_1 + ... + \alpha_n$ for a component monomial, is given by $\deg(\cdot)$. The collection of polynomials in the variables $\mathbf{x} = (x_1, ..., x_n)$ having coefficients in some field $\mathbb{K}$—in this paper $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$—is denoted by $\mathbb{K}[\mathbf{x}]$. The following special class of polynomials is central to the methods discussed below.

Definition 3 (Sum of squares polynomial). *A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is a sum of squares polynomial (SOS for short) if it has a representation:*

$$\mathbf{x} \mapsto f(\mathbf{x}) = \sum_{j \in J} p_j(\mathbf{x})^2, \quad \mathbf{x} \in \mathbb{R}^n,$$

where

$$p_j \in \mathbb{R}[\mathbf{x}], \ \forall j \in J,$$

and where J is a finite index set.

The significance of the SOS polynomials from Definition 3 stems from the fact that these polynomials can be used to construct certificates of nonnegativity for polynomials over sets defined by other polynomials.[5]

Indeed, polynomials are pivotal in defining a new, special class of subsets, utilized heavily in, for instance, polynomial optimization. To exemplify, consider the following definition.

Definition 4 (Ideal). *A subset $I \subseteq \mathbb{R}[x_1, ..., x_n] = \mathbb{R}[\mathbf{x}]$ is an ideal if:*

1. $0 \in I$,
2. *If $f, g \in I$, then $f + g \in I$,*
3. *If $f \in I$ and $h \in \mathbb{R}[x_1, ..., x_n]$, then $fh \in I$.*

Ideals are typically constructed using a set of basis polynomials. To acknowledge this, denote by $I_f := \langle \mathbf{f} \rangle = \langle f_1, ..., f_n \rangle$ the ideal generated by the set of polynomials $f_1, ..., f_n$.

Ideals are essential in formally examining the solution (zero) sets of polynomial systems. Another important set defined using polynomials is the basic *semialgebraic set*.

Definition 5 (Semialgebraic set). *A set $\mathbb{S} \subset \mathbb{R}^n$ is called semialgebraic if:*

$$\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^n \ : \ g_j(\mathbf{x}) \geq 0, \quad j = 1, ..., m\},$$

for polynomials $g_j \in \mathbb{R}[\mathbf{x}]$.

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[5] These certificates are a key building block in constructing the duality theory between positive polynomials and moment problems, which is at the heart of polynomial optimization. For further details, the reader is referred to Lasserre (2015).

Definition 5 gives a simple description of the basic semialgebraic set. A bit more generally, semialgebraic subsets of $\mathbb{R}^n$ can be defined using Boolean (disjunction, conjunction, and negation) operations on the *sign conditions* $g > 0$, $g = 0$, and $g < 0$ over a finite number of polynomials (cf. Bochnak et al. 1998, pp. 23–26). For the purposes of this paper it suffices to summarize that semialgebraic sets of $\mathbb{R}^n$ are obtained by invoking a finite number of polynomial sign conditions. In the field of finance, perhaps the simplest example would be a short-selling constraint imposed on portfolio weights $\mathbf{w}$. More specifically, it is required that portfolio weights belong to the set $\{\mathbf{w} \in \mathbb{R}^n \ : \ w_j \geq 0, \ \forall j = 1, ...n\}$.

Another important concept related to sets is the *convex hull* defined below.

Definition 6 (Convex hull, cone & polytope). *For $S \subset \mathbb{R}^n$ the convex hull is given by the set of convex combinations of points in S:*

$$\text{Conv}(S) = \left\{ \sum_{i=1}^{n} \lambda_i s_i \ : \ s_i \in S, \ \lambda_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Alternatively, suppose $C^n_{\mathbb{R}}$ represents the convex sets $C \subset \mathbb{R}^n$, where for each $C \in C^n_{\mathbb{R}}$ it holds that:

$$\forall x, y \in C \text{ and } \lambda \in [0,1] \implies \lambda x + (1-\lambda) y \in C.$$

Then the convex hull of S can be defined as the minimal convex set in $C^n_{\mathbb{R}}$ which contains the set S.

Moreover, a polytope P is the convex hull of a finite set in $\mathbb{R}^n$. Finally, a set $S \subset \mathbb{R}^n$ is a cone if $\mathbf{x} \in S \implies \alpha\mathbf{x} \in S$, $\forall \alpha > 0$. If S is also convex, i.e., $S \in C^n_{\mathbb{R}}$ then is it called a convex cone.

As seen from Definition 6, the convex hull gives a natural way to define the concept of a polytope. Furthermore, on the basis of Definition 6 it is easy to characterize a bit more specialized polytope, namely, the *Newton polytope*.

Definition 7 (Newton Polytope). *For a polynomial $f \in \mathbb{R}[\mathbf{x}]$, as in Definition 2, the Newton polytope is given by:*

$$\text{NP(f)} = \text{Conv}\Big(\{\alpha \in \mathbb{Z}^n_{\geq 0} \ : \ c_\alpha \neq 0\}\Big).$$

Newton polytopes are essential in capturing the sparsity structure of a polynomial. The following example gives a simple illustration.

Example 1. (Newton polytope) Compare $f = (ax + by + cz)^3$ to $f' = dx^3 + ey^2 + yz$, where $a, b, c, d, e \neq 0$. Clearly, it then follows that:

$$\text{NP}(f') = \text{Conv}(\{(3,0,0), \ (0,2,0), \ (0,1,1)\}) \subset \text{NP}(f).$$

Now, denote by $\mathcal{V}_n(\cdot)$ the Euclidean volume in $\mathbb{R}^n$.[6] For a collection $P_1,...,P_m$ of polytopes in $\mathbb{R}^n$, it can be shown that $\mathcal{V}_n(\lambda_1 P_1 + \cdots + \lambda_m P_m)$ is a homogenous polynomial in the λ_i's with degree n. Thus, the following important definition is obtained.

Definition 8 (Mixed volume). *For a collection of polytopes $P_1,...,P_m$, the n-dimensional mixed volume, denoted by* $\mathrm{MV}_n(P_1,...,P_m)$*, is given by the coefficient of the monomial $\lambda_1 \cdots \lambda_m$ in the polynomial expression of:*

$$\mathcal{V}_n(\lambda_1 P_1 + \cdots + \lambda_m P_m).$$

Example 2 demonstrates mixed volume calculations in practice.

Example 2. (Calculating the mixed volume) Consider a 2-tuple of Newton polytopes (P_1, P_2). For $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ it holds that:

$$\mathcal{V}_2(\lambda_1 P_1 + \lambda_2 P_2) = \lambda_1^2 \mathrm{MV}(P_1,P_1) + \lambda_1\lambda_2 \mathrm{MV}(P_1,P_2) + \lambda_2^2 \mathrm{MV}(P_2,P_2).$$

Because $\mathrm{MV}(P_i,P_i) = \mathcal{V}_2(P_i)$, $i \in \{1,2\}$, and as the above holds for all nonnegative λ_1 and λ_2, one obtains:

$$\mathrm{MV}(P_1,P_2) = -\mathcal{V}_2(P_1) - \mathcal{V}_2(P_2) + \mathcal{V}_2(P_1+P_2).$$

For an arbitrary collection of polytopes $(P_1,...,P_n)$, one has (see, Cox et al. 2006, pp. 337–339):

$$\mathrm{MV}(P_1,...,P_n) = \sum_{k=1}^{n} (-1)^{n-k} \sum_{I \subset \{1,...,n\},\ |I|=k} \mathcal{V}_n\Big(\sum_{i \in I} P_i\Big).$$

Newton polytopes and mixed volumes are utilized in establishing an upper bound on the number of solutions for a given polynomial system. Tight upper bounds are crucial from the point of view of efficient numerics as they enable one to design algorithms that are more efficient. This issue is revisited in Section 2.3

The final step in the preliminary preparations is an important (classical) theorem which verifies that one can, associated with the roots of a given polynomial, find a solution path, in the homotopy sense, formulated by an analytic function.[7] To proceed, for $h : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, such that $(\mathbf{x},t) \mapsto h(\mathbf{x},t)$, denote by $\mathbf{J} = \left[\frac{\partial h}{\partial x_1} \cdots \frac{\partial h}{\partial x_n}\ \frac{\partial h}{\partial t}\right]$ the *Jacobian matrix* and by $\mathbf{J_x}$ the restriction of the Jacobian to the partial derivatives of h with respect to $\mathbf{x} = (x_1,...,x_n)$.

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[6] Note that, for instance, $\mathcal{V}_2(P)$ is interpreted as the area of P.

[7] Analytic functions are those functions which are locally equivalent to a convergent power series. Polynomials are naturally analytic.

The theorem presented next has many forms and is often presented with varying levels of generality. As in Judd et al. (2012), the following version is sufficient for the purposes of this paper.

Theorem 1 (Implicit Function Theorem (IFT)). *Let* $(\mathbf{x}_0, t_0) \in \mathbb{C}^n \times \mathbb{C}$ *be such that* $h(\mathbf{x}_0, t_0) = 0$ *and suppose* $\text{rank}(\mathbf{J}_\mathbf{x})(\mathbf{x}_0, t_0) = n$. *Then there exists neighborhoods* $N_\mathbf{x} \subset \mathbb{C}^n$ *of* $\mathbf{x}_0$ *and* $N_t \subset \mathbb{C}$ *of* t_0 *and an analytic function* $x : N_t \to N_\mathbf{x}$ *such that* $h(x(t), t) = 0$.

For more details and a proof see, for example, Munkres (1991) *Theorem 9.2*.

2.2 Polynomial Optimization Problems

2.2.1 Theory

The first main application of algebraic geometry discussed in this paper is polynomial optimization over semialgebraic sets. This approach is utilized to solve optimal execution and portfolio selection problems in Sections 3 and 4 respectively. Since a proper introduction to polynomial optimization is reasonably technical and lengthy, this section covers only the basic idea behind the general approach, utilizing only the language of standard optimization theory. For the sake of completeness, a slightly more detailed account of some aspects of polynomial optimization, focusing especially on implementation, is given in Appendix C. For a complete treatment of the subject, the reader is referred to Lasserre (2015).

Now, consider the following *global* optimization problem:

$$f^* := \inf_{\mathbf{x}} \Big\{ f(\mathbf{x}) : \mathbf{x} \in \mathbb{S} \Big\}, \tag{1}$$

where f is a polynomial and $\mathbb{S}$ is a semialgebraic (constraint) set as in Definition 5. It will become evident that many problems in finance naturally fall into this category of problems.

At this point, one might be troubled about the fact that—even with these restrictions, i.e., a polynomial objective function and a semialgebraic constraint set—it is not at all obvious as to how to go about solving the optimization problem, let alone how to go about solving it in a global sense. The key idea will be to construct a sequence of convex semidefinite relaxations of (1). These relaxations are increasing in size and the sequence of optimal values obtained from the relaxations converges—under certain conditions—to the global optimal value.

To exemplify, note first that problem (1), assuming $\deg(f) \le d$, has an equivalent representation:

$$f^* = \sup_{\lambda} \{\lambda \ : \ f(\mathbf{x}) - \lambda \in C_d(\mathbb{S})\}, \tag{2}$$

where $C_d(\mathbb{S})$ is a convex cone (see, Definition 6) such that the polynomials $p \in C_d(\mathbb{S})$ have a degree of at most d and are nonnegative. Note that the optimization problem is now with respect to a single variable (λ) and that it is a finite dimensional convex optimization problem. Nevertheless, it is not in general a tractable problem since the convex cone of polynomials is difficult to characterize in practice.

The main idea in overcoming this difficulty is to introduce a sequence of—or more precisely, an increasing family of—convex cones:

$$C_d^l(\mathbb{S}) \subset C_d^{l+1}(\mathbb{S}) \subset \cdots \subset C_d(\mathbb{S})$$

and associated optimization problems:

$$\rho_l = \sup_{\lambda} \{\lambda \ : \ f(\mathbf{x}) - \lambda \in C_d^l(\mathbb{S})\}, \quad l = 0, 1, ...$$

which constitute a *hierarchy* of increasing linear or semidefinite programs, converging to the global optimum f^*. In other words, one obtains a sequence of lower bounds $\rho_l \leq \rho_{l+1} \leq ... \leq f^*$ such that $\rho_l \to f^*$ as $l \to \infty$ (cf. Lasserre 2015, p. 4). Tools from (real) algebraic geometry are essential in proving this convergence result.

Finally, by considering the dual of (2), also a finite dimensional convex optimization problem, one obtains a formulation of the original problem (1) which is an instance of the generalized moment problem (GMP; see, Lasserre 2010, Ch. 5). Together the primal and dual can be utilized, under appropriate conditions, to numerically verify the global optimality of the obtained results.[8]

2.2.2 Implementation and Earlier Applications

The numerical results in this paper regarding polynomial optimization are obtained utilizing GloptiPoly 3 (see, Henrion et al. 2009). GloptiPoly is a MATLAB freeware that implements methods to solve or approximate the generalized moment problem.[9]

Indeed, as noted above, a polynomial optimization problem has a representation in terms of the generalized moment problem, i.e., a polynomial optimization problem can be viewed as a special instance of the GMP. Lasserre (2008) develops an approach, taking advantage of semidefinite programming, to tackle GMPs with polynomial data. To solve the semidefinite programs, arising as a by-product of this approach, the semidefinite programming solver SeDuMi (see, Sturm 1999), also a MATLAB add-on, is utilized (GloptiPoly uses SeDuMi by default). Details on this approach can be found in Appendix A.

In many instances, practical applications of polynomial optimization methods are constrained by the size of the problem in question. Namely, general-purpose algorithms, not to mention software, for solving large-scale polynomial optimization

[8] It depends on the viewpoint and context whether the formulation given in (2) is referred to as the primal or the dual of the original global optimization problem (1).

[9] MATLAB is a registered trademark of The MathWorks, Inc.

problems are as of writing this paper yet to be developed. Much of the work related to solving large scale polynomial optimization problems involves improving the performance of existing semidefinite solvers. Even though methods for specific important problems do already exists, this must be considered as both a key venue for future research and also as the single most impactful drawback when considering applications of polynomial optimization methods.

In Section 3.4 problems relating to increasing degree of the polynomial objective function are discussed. Although presently high degree and high "dimension" (large number of variables) are likely to cause some computational issues, there are methods that have been developed to deal with these issues. One example is the method of Mevissen and Kojima (2010) which utilizes a change of variables (cf. Section 3.4). Other methods utilize (see, Waki et al. 2008), for example, sparsity patterns in the objective polynomial to develop more efficient numerical algorithms.

Despite the challenges, several applications, outside the scope of a detailed exploration in this paper, of polynomial optimization have emerged. For instance, in Lasserre (2010) an application to pricing exotic derivatives is detailed. The approach turns out to be rather versatile, being able to handle various models for the underlying asset price such the geometric Brownian motion, the Ornstein-Uhlenbeck process, and the mean-reverting square-root process.

Another potentially fruitful area from the point of view of polynomial optimization is game theoretic modeling. A case in point is Couzoudis and Renner (2013). Furthermore, Renner and Schmedders (2015) and Renner and Schmedders (2017) demonstrate the use of the method to solve moral hazard principal-agent problems, a class of problems which is highly relevant in finance. Particular examples of these problems can be found in theoretical corporate finance (executive compensation) and portfolio delegation (portfolio manager compensation). In addition to the novel solution procedure, the authors illustrate how one can approximate non-polynomial objective functions so that the approximated problem can again be treated as a POP, even when the original problem does not share this feature. This is an important notion as, by the *Stone–Weierstrass theorem*, continuous functions on a bounded interval can be uniformly approximated by polynomials. Combining polynomial approximation and optimization will significantly increase the number of potential application areas.

2.3 All-Solutions Using Homotopy Methods

2.3.1 Theory

The second main application of algebraic geometry is finding all solutions to systems of polynomial equations. As a leading example, the method of *all-solutions* *homotopy continuation* is discussed. This entirely numerical method is well suited for solving problems involving high degree and high dimension polynomial systems. Homotopy methods are utilized in Section 5 to find all solutions to a polynomial system arising in conjunction with a dynamic strategic trading model.

The concise description of homotopy continuation methods provided in this section is largely based on Cox et al. (2006, chs. 7.4 & 7.5). The main reason for this is that the approach taken by the authors is closely related to the numerical implementation of homotopy continuation in the PHCpack (cf. Verschelde 1999)—the numerical software utilized in this paper.

In general, the goal is to solve a (square) system $f(\mathbf{x}) = 0$, where $f = (f_1, ..., f_n)$ and $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{C}^n$. To achieve this, a new system, say $g(\mathbf{x}) = 0$, with known solutions, is introduced together with a parameters $t \in [0,1]$, $c \in \mathbb{C}$, and a continuous family:

$$0 = h(\mathbf{x}, t) = c(1-t)g(\mathbf{x}) + tf(\mathbf{x}).$$

The purpose of the constant c is to act as a scaler to avoid numerical issues and t is allowed to continuously change from 0 to 1. Here, $g(\mathbf{x})$ is referred to as the *start system* and $h(\mathbf{x}, t)$ is referred to as the *continuation system*.

To be able to use Theorem 1 to characterize and to guarantee the existence of the solution curves $x(t)$, with $x(0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is a solution to $g(\mathbf{x}) = 0$, it is required that:

$$\operatorname{rank}(\mathbf{J_x})(\mathbf{x}_0, t_0) = \operatorname{rank}\left(\left[\frac{\partial h}{\partial x_1} \cdots \frac{\partial h}{\partial x_n}\right]\right)(\mathbf{x}_0, t_0) = n, \tag{3}$$

i.e., the Jacobian matrix with respect to $x_1, ..., x_n$ must be invertible at $(\mathbf{x}_0, t_0)$.[10] When this condition is satisfied, one can use the solution curves with the aim of eventually finding $x(1)$, which instead will yield the solution(s) to $f(\mathbf{x}) = 0$.

Noting that $h(x(t), t) \equiv 0$ implies $\frac{d}{dt}h(x(t), t) \equiv 0$, one obtains, via an application of the multivariate chain rule, the following system of ordinary differential equations (ODEs) for functions $x(t)$ (cf. Cox et al. 2006, p. 354):

$$\mathbf{J_x}(x(t), t)\frac{dx(t)}{dt} = -\frac{\partial h}{\partial t}(x(t), t), \tag{4}$$

with initial value $x(0) = \mathbf{x}_0$. One can the solve this *initial value problem* using standard numerical methods developed for ODEs (see, for example, Butcher 2008).[11]

Numerical methods designed for initial value problems are not, however, generally the most effective way to track the solution curves (see, Allgower and Georg

[10] Equivalent conditions include: [1] $\mathbf{J_x}$ has a nonzero determinant [2] 0 is not an eigenvalue of $\mathbf{J_x}$ [3] the columns of $\mathbf{J_x}$ form a linearly independent set.

[11] Another way to approach solving the problem is to start from the known solution with $t = 0$ and to proceed in steps of Δt towards $t = 1$. Namely, one can use $x(0) = \mathbf{x}_0$ as an initial guess for solving $h(\mathbf{x}(\Delta t), \Delta t) = 0$ and solve the problem using some locally convergent root finding method such as Newton-Raphson. Having obtained $\mathbf{x}(\Delta t)$, one can again use it as an initial point for solving the problem in the next step thus progressing towards $h(\mathbf{x}(1), 1) = 0$.

2003). Instead, one typically seeks to iteratively track the solution curve from (4) using the so-called *predictor-corrector* approach. The predictor gives a new approximate point along the solution curve given the last point utilized, and the corrector, typically an application of Newton's method, which has strong local convergence properties, shifts this new point closer to the tracked solution curve.

There are some cases, which cause trouble for the predictor-corrector approach and for homotopy continuation methods in general. The first one is crossing paths. At nonregular points the rank condition (3) for the Jacobian matrix does not hold and thus the method is likely to run into difficulties. This is illustrated in Figure 1.

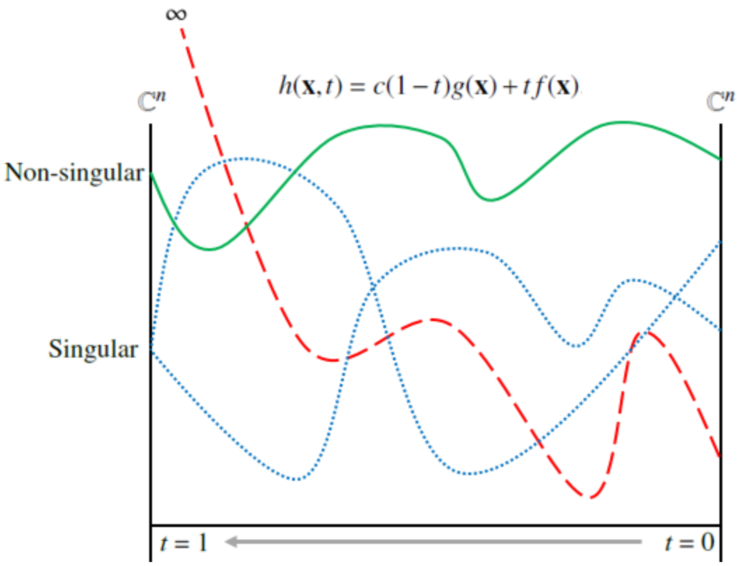

Figure 1: (**Homotopy paths**) This figure illustrates different homotopy path realizations.

To demonstrate another possible trouble area it is good to note that a key aspect of the homotopy continuation method is the relation between the original system $f(\mathbf{x}) = 0$ and the start system $g(\mathbf{x}) = 0$. For example, what can be said about the number of solutions to the start system in relation to the number of solutions to the original system? If the number of solutions to the start system is much larger than the number of solutions to the (possibly sparse) original system, then plenty of unnecessary computations will ensue from tracking diverging solution paths, i.e., solution paths for which $x(t) \to \infty$ as $t \to 1$ (cf. Figure 1).

These instances feature points $(\mathbf{x}, t^*)$ for which no roots for the continuation system can be found. The points "interrupt" the solution path from $t = 0$ to $t = 1$. To deal with this issue one can utilize the so-called *gamma-trick*, i.e., multiplying the start system with an additional term $\exp\{\gamma i\}$. It holds that for almost all choices of the complex constant γ, the obtained solution paths are regular meaning that the

Jacobian remains regular and no path diverges.

Based on the above discussion it is safe to say that careful choice of the start system plays an important role in an effective implementation of the homotopy continuation method. Determining an appropriate start system is closely related to bounds—as tight as possible—on the number of (complex) solutions for the polynomial system under study. To illustrate this, two such bounds—starting with the simpler one—are considered.

Definition 9 (Bezout number). *For a polynomial system $f = (f_1, ..., f_n) : \mathbb{C}^n \to \mathbb{C}^n$, the Bezout number is given by:*

$$\mathcal{B} := \prod_{i=1}^{n} \deg(f_i)$$

The Bezout number given in Definition 9 can immediately be used to determine an upper bound on the number of isolated solutions.

Theorem 2 (Bezout bound). *If $\mathcal{B}$ is the Bezout number of $f = (f_1, ..., f_n)$, then the number of isolated solutions, taking into account multiplicities, of $f(\mathbf{x}) = 0$, $\mathbf{x} = (x_1, ..., x_n)$, is bounded above by $\mathcal{B}$.*

As noted, the Bezout bound is a simple bound for the number of solutions for a square polynomial system. However, if the system has additional structure, such as sparsity, this bound can be improved.[12]

A more efficient bound for the number of solutions for the original system is obtained from *Bernshtein's Theorem*. The theorem below is given in a more general form than is actually needed here (cf. Cox et al. 2007, Thm. 5.4).

Theorem 3 (Bernshtein's Theorem, BKK bound). *Suppose the n-variate Laurent polynomials $f = (f_1, ..., f_n)$ over $\mathbb{C}$ have a finite number of common zeroes in $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \{\mathbf{0}\}$ and denote $P_i = \mathrm{NP}(f_i) \subset \mathbb{R}^n$. The number of common zeroes of f in $(\mathbb{C}^*)^n$ is bounded above by the mixed volume $\mathrm{MV}_n(P_1, ..., P_n)$, and for generic choices of coefficients in the f_i, the number of common zeroes is exactly equal to $\mathrm{MV}_n(P_1, ..., P_n)$.*

Three contemporaneous papers relate to the mixed volume bound given in Theorem 3: Bernshtein (1975), Kushnirenko (1976), and Khovanskii (1977). Thus, the bound is often called the BKK bound.

Example 3. (Bezout versus the BKK bound) Consider the system $(f_1, f_2) = 0$, where

$$\begin{aligned} f_1(x_1, x_2) &= 4x_1^2 x_2^3 - 2 \\ f_2(x_1, x_2) &= 3x_1^3 x_2^2 - x_1 x_2. \end{aligned}$$

[12] See also, Verschelde et al. (1994), Huber and Sturmfels (1995), and Li (2003).

It is immediate for the system above that $\mathcal{B} = 5 \times 5$. Now, let $P_i := \text{NP}(f_i)$, for $i = 1,2$, and recall from Example 2 that:

$$\text{MV}(P_1, P_2) = \mathcal{V}_2(P_1 + P_2) - \mathcal{V}_2(P_1) - \mathcal{V}_2(P_2) = \mathcal{V}_2(P_1 + P_2),$$

where the last equality follows since line segments have a Lebesgue measure of zero in $\mathbb{R}^2$. Finally, one can verify, either by direct calculation or via the use of the PHC-pack, that $\text{MV}(P_1, P_2) = 4 < 25$ which obviously represents a significant improvement on the upper bound.

2.3.2 Implementation and Earlier Applications

The numerical results in this paper are obtained utilizing the MATLAB version of the Polynomial Homotopy Continuation pack, PHCpack for short (see, Verschelde 1999, Guan and Verschelde 2008). The advantages of the PHCpack are active development, clear and comprehensive documentation, and easy initialization. Other alternatives include, for example, the software package Bertini (cf. Bates et al. 2013).

The PHC solver features four steps (see, Verschelde 1999, Fig. 4): **(1)** *Preconditioning* **(2)** *Root counting* **(3)** *Homotopy continuation* **(4)** *Validation*. Precondition refers to, for instance, coefficient scaling and seeks to ensure that the system is of a suitable form. The second step is essential in constructing the start system. This is due to the fact that root counting is crucial from the point of view of computational complexity. For this reason, extra attention is devoted above to this step.

Indeed, one may estimate the required computational time by multiplying the (estimated) time to follow one solution path with the (estimated) number of roots. PHC-pack utilizes multiple root counting methods among which the BKK bound (mixed volume) given in Theorem 3—if available—always returns the lowest bound. The start system $g(\cdot)$ is constructed so that it corresponds to the minimal root count available. The approach that uses mixed volume computations in constructing the start system is called *polyhedral homotopy*.

The third step involves the actual homotopy continuation phase. This involves adjustment of the continuation parameters and the choice the path following methods. Finally, the fourth step consists of validation procedures such as evaluation of local condition numbers and analysis of path directions.

There are a couple applications of homotopy methods worth mentioning. First, Besanko et al. (2010) apply homotopy continuation to a dynamic industrial organization model and show that the homotopy path algorithm put forth in the paper is able identify equilibria which have eluded the standard algorithms utilized to analyze the problem earlier. For some parametrizations, even *nine distinct equilibria* are found in a model thought to have a unique equilibrium. As noted by the authors, examining the question of equilibrium multiplicity and verifying it numerically are steps along the way to solve the multiplicity problem.

Second, it comes as no surprise at this point that the all-solutions methods, again similarly to polynomial optimization, have ample applications in the field of game theory. Many games, especially dynamic stochastic games, are susceptible to the equilibrium multiplicity problem. Recent applications of homotopy continuation to game-theoretic settings include Herings and Peeters (2010), Bajari et al. (2010) Judd et al. (2012).

2.4 All-Solutions Using Gröbner Bases

2.4.1 Theory

The final method to be discussed, namely *Gröbner bases*, offers an alternative to homotopy methods in the quest to find all solutions to a given polynomial system. The presentation closely follows Kubler and Schmedders (2010) and it quickly becomes apparent that Gröbner bases are as intimately tied to insights emerging from algebraic geometry as the two applications introduce above. Indeed, Gröbner bases are often cited as being a (nonlinear) generalization of the familiar Gaussian elimination algorithm from linear algebra.

To understand how exactly this is done it is useful to introduce some new concepts. Recall that a given set of polynomials $f_1,...,f_m$ generate, i.e. form a basis for, an ideal (Definition 4):

$$I_f := \left\langle f_1,...,f_m \right\rangle = \left\{ \sum_{i=1}^{m} h_i f_i : h_i \in \mathbb{C}[\mathbf{x}] \right\}.$$

Now, it should be noted that if $\mathbf{x} \in \mathbb{C}^n$ satisfies $f_i(\mathbf{x}) = 0,\ \forall\ i = 1,...,m$, then it holds that $g(\mathbf{x}) = 0,\ \forall\ g \in I_f$. Hence, what one wishes to accomplish is to determine whether there is alternative set of *simpler* polynomials, say, $g_1,...,g_k$ such that:

$$\left\langle f_1,...,f_m \right\rangle = \left\langle g_1,...,g_k \right\rangle,$$

because if this simpler basis is found, one can use *it* instead of $f_1,...,f_m$ to find (all) solutions to the system $f_i(\mathbf{x}) = 0,\ \forall\ i = 1,...,m$.

The following lemma gives the existence of this kind of simpler basis.[13]

Lemma 4 (Shape Lemma). *Let* $\left\langle f_1,...,f_m \right\rangle$ *be a regular ideal in* $\mathbb{Q}[x_1,...,x_m]$*, having* d *isolated roots with distinct* x_m *coordinates. Then* $\exists$ *a basis with the shape:*

$$\mathcal{S} = \left\{ x_1 - q_1(x_m),\ x_2 - q_2(x_m), ..., x_{m-1} - q_{m-1}(x_m),\ r(x_m) \right\},$$

where r *is a polynomial of degree* d *and* $\deg(q_i) \le d-1$.

[13] Suppose $\mathbf{f} = (f_1,...,f_m) : \mathbb{C}^m \to \mathbb{C}^m$. Regularity of an ideal generated by $\mathbf{f}$ means that the Jacobian matrix has full rank m at all of the complex solutions of $\mathbf{f}$.

From Lemma 4 it can be seen that with the new basis, a reduced Gröbner basis (under lexicographic order), the task of finding every solution to the system of polynomial equations reduces to finding every solution of a single univariate polynomial.[14] For a more detailed account of the Shape Lemma, the reader is referred to Becker and Weispfenning (1993) and Becker et al. (1994).

2.4.2 Implementation, Usage, and Earlier Applications

Two different implementations of Gröbner bases are utilized. The first one is free and open-source under the *GNU General Public Licence* and the second one is proprietary. Starting with the open-source implementation, SINGULAR is described by its developers as a computer algebra system for polynomial computations with core algorithms able to deal with, for example, Gröbner bases, polynomial factorization, and numerical root finding.[15] One can use SINGULAR via a specialized web-interface, making it a very low threshold alternative for learning how to take advantage of numerical algebraic geometry methods. For computational examples and further information, the reader is referred to Greuel and Pfister (2012) and Kubler et al. (2014).

The proprietary implementation is *NSolve* of MATHEMATICA.[16] In case of polynomial systems, NSolve is able to find all solutions using MATHEMATICA's internal implementation of numerical Gröbner bases. Indeed, if one is interested in having a look at the actual Gröbner basis for a given problem, one can use the *GroebnerBasis* command to do just this. NSolve is an obvious choice for researchers already actively using MATHEMATICA.

In the numerical implementation of Gröbner bases there are some additional things worth pointing out. First, in actual applications, the coefficients of the polynomials $f_1,...,f_m$ from Lemma 4 will be real, not rational, scalars. Due to this, the solutions obtained are not exact. However, due to scaling invariance of the solutions, it is easy to transform the coefficients to approximately rational. Second, regarding the conditions given in Lemma 4, verifying them for a small system ($m \in \{2,3\}$) is not overly taxing. Moreover, Kubler and Schmedders (2010) describe means to deal with standard cases where the conditions do not hold.

Third, as Gröbner bases can be seen as a substitute for homotopy methods, it is wise to ask are there any guidelines for which one should be preferred. This question is naturally case-dependent but, as a rule-of-thumb, homotopy methods are preferable for larger, more computationally demanding, systems and for repetitive computations (e.g., dynamic programming applications). Gröbner bases instead enable one to examine in more detail why certain solutions arise and how the solution set is

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14 Lexicographic order means that $x^\alpha \overset{\text{lex.}}{>} x^\beta$, where $\alpha,\beta \in \mathbb{Z}^n$, when the first non-zero element of $\alpha - \beta$ is positive.
15 See, www.singular.uni-kl.de.
16 MATHEMATICA is a registered trademark of Wolfram Research, Inc.

affected by changes in the coefficients of the polynomials.[17]

There are, however, clear benefits from looking at the two methods as complements rather than substitutes. This view stems from the fact that the numerical implementations of the two methods feature non-overlapping problem areas. On one hand, Gröbner bases, implemented using the standard Buchberger's Algorithm (cf. Cox et al. 2007), may run into numerical instabilities due to the fact the coefficients of polynomials introduced in the algorithm blow up to tens of thousands of digits. Hence, one may not always be able to calculate Gröbner bases for a given system, especially a complex system with real coefficients. On the other hand, as opposed to Gröbner bases, there are no theoretical guarantees that homotopy methods, with probability one, produce all solutions to a given system (cf. Kubler et al. 2014). This is due to the use of floating-point arithmetic, instead of exact calculations. Below, the complementary view to the two methods is adopted and both are used to ensure the quality of the numerical results.

2.5 Connection Between Polynomial Optimization and All-Solutions Methods

Polynomial optimization and all-solutions methods are treated above as separate and, from a strictly theoretical perspective, this is true. However, from a practical viewpoint the two methods are, loosely speaking, *substitutes*. On one hand, one can utilize polynomial optimization methods to find (all) solutions to a system of polynomial equations (see, Lasserre 2010, pp. 147–162). On the other hand, one can take advantage of, for example, the homotopy continuation method in finding global solutions to constrained polynomial optimization problems.

This "duality" between the two methods is the main motivation to cover both methods simultaneously. If one of the approaches is infeasible in the context of some pressing problem, one still has the other approach in reserve. Moreover, the majority of existing literature keeps polynomial optimization and all-solutions methods as distinctly separate, which is not likely to be helpful from the point of view of those seeking to apply these methods to topical problems in various fields. Indeed, as neither approach can be considered as fully developed, it is valuable to have alternative solution methods in stock when a particular problem of interest proves to be problematic for one of the approaches. In addition, potential numerical issues encountered with different problems might be specific to the solution method used and thus having an alternative could turn out to be immensely useful from a robustness perspective.

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[17] For more details, the reader is referred to Kubler et al. (2014).

3 OPTIMAL EXECUTION

3.1 Problem Statement

This section describes—using a concise but relatively self-contained style—an optimal execution problem based on Almgren and Chriss (2001) and Almgren (2003).[18] The goal is then to study various modifications of these baseline models. In particular, variations which are problematic to solve using standard optimization machinery, but which can be readily dealt with using polynomial optimization methods. The focus is on discrete time models as the benefits of polynomial optimization are highest with this class of models.

Now, suppose the goal is to liquidate a position of $X > 0$ units of security during T (discrete) points of trade, i.e., $t = 1, \ldots, T$.[19] Trading is thus exogenously motivated and the trading horizon (or, trading window) is taken as given. Relaxing these assumptions is possible but not without complicating the model to a large extent (see, for example, Easley et al. 2015). Denote by x_t the remaining security holdings in period t. Further, denote $n_t := x_{t-1} - x_t$. Since the goal is to liquidate the entire position, there are two natural boundary conditions: $x_0 = X$ and $x_T = 0$; there is no trading at $t = 0$.

The security price in the market follows an *arithmetic random walk*:

$$\begin{aligned} S_t &= S_{t-1} + \sigma\tilde{\xi}_t - g(n_t) \\ &= S_0 + \sum_{j=1}^{t} \left[\sigma\tilde{\xi}_j - g(n_j)\right], \end{aligned}$$

where $\tilde{\xi}_j \overset{\text{i.i.d.}}{\sim} N(0,1)$, $\sigma > 0$, S_0 is the initial price, and $g(\cdot)$ is a yet to be determined *permanent price impact function*.[20] In what follows it is assumed that g is a linear function of n_t. In addition to the added benefit of tractability, this assumption is supported by Huberman and Stanzl (2004), who show that only linear permanent price impact functions rule out what the authors call "quasi-arbitrage".[21]

Permanent price impact is not the only source of trading costs. More specifically, let $h(\cdot)$ denote the *temporary price impact function* and $f(\cdot)$ denote the *uncertainty of trade execution*. Hence, the actual price for the t^{th} trade is given by:

$$\hat{S}_t = S_{t-1} - h(n_t) + f(n_t)\tilde{\xi}_t, \quad t = 1, \ldots, T.$$

[18] See also Perold (1988), Bertsimas and Lo (1998), and Huberman and Stanzl (2005).

[19] The case with $X < 0$ is similar.

[20] Permanent price impact means that the impact of trades is persistent, i.e., it affects current and all future trades. Conversely, temporary price impact is limited only to the current trade

[21] In a market where trades affect prices, Huberman and Stanzl (2004) define quasi-arbitrage as the possibility to carry out a sequence of trades which generates boundless profits in expectation.

A central performance indicator in the model is the *implementation cost*:

$$C := XS_0 - \sum_{t=1}^{T} n_t \hat{S}_t. \tag{5}$$

Clearly the implementation cost is a random variable which depends on $x_1, \ldots, x_{T-1}$. Letting $\lambda > 0$ denote the "risk aversion" parameter, the goal is to solve:

$$\inf_{\mathbf{x} \in \mathcal{A}_X} \mathbb{E}[C] + \lambda \mathbb{V}[C], \tag{6}$$

where

$$\mathbb{E}[C] = \sum_{t=1}^{T} \left[x_t g(n_t) + n_t h(n_t) \right] \quad \text{and} \tag{7}$$

$$\mathbb{V}[C] = \sum_{t=1}^{T} \left[\sigma^2 x_t^2 + n_t^2 f(_t)^2 \right], \tag{8}$$

are the expected implementation cost and the variance of the implementation costs respectively, and

$$\mathcal{A}_X := \left\{ (x_0, \ldots, x_T) \in \mathbb{R}^{T+1} : x_0 = X > 0 \ \ \& \ \ x_T = 0 \right\}.$$

The set $\mathcal{A}_X$ is the set of feasible *trading curves* $(x_t)_{t=0,\ldots,T}$ for a liquidation problem.[22] It should be emphasized that the goal is to solve (6) in a static sense, i.e., the optimal trading program is solved in its entirety before any trading takes place. The optimality of static (deterministic) strategies in this setting is discussed in, e.g., Almgren and Chriss (2001).

3.2 Nonlinear Temporary Price Impact

This section focuses on the temporary price impact function as it allows one to fluently operate on both sides of the "tractability-boundary". Namely, when the temporary price impact function is assumed linear one is able to find—on top of obviously being able to solve the model numerically—exact solutions to the optimal execution problem. However, assuming, for instance, polynomial (nonlinear) temporary price impact, only numerical solution methods are left on the table.

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22 The corresponding set of (deterministic) admissible liquidation strategies is given by:

$$\mathcal{A}_{X,n} := \left\{ (n_1, \ldots, n_T) \in \mathbb{R}^T : \sum_{t=1}^{T} n_t = -X \right\}.$$

Theoretical work has often focused on the linear price impact case for tractability reasons. Yet there is ample evidence in favor of nonlinear price impact (see, for example, Bouchaud et al. 2009). Motivated by this empirical evidence, for instance, Chen et al. (2015) have recently studied deleveraging under nonlinear temporary price impact. Here, the task to be completed next is to provide an illustration of how polynomial optimization methods can be utilized to solve optimal execution problems with varying assumptions on the functional form of price impact functions. The first illustration deals with nonlinear temporary price impact.

For this purpose, in this section the following functional forms are adopted:

$$
\begin{aligned}
&g(z) = \gamma z, \quad \gamma > 0\\
&h(z) = \eta z^k, \quad \eta > 0 \text{ and } k \in \mathbb{N} \setminus \{1\}\\
&f(z) = 0.
\end{aligned}
$$

Thus, the quantities in (7) and (8) take the forms:

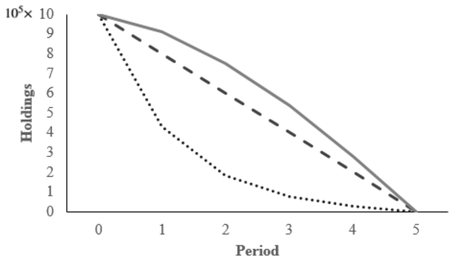
$$
\begin{aligned}
\mathbb{E}[C] &= \sum_{t=1}^{T} \left[\gamma x_t n_t + \eta n_t^{k+1}\right] \text{ and}\\
\mathbb{V}[C] &= \sum_{t=1}^{T} \left[\sigma^2 x_t^2\right],
\end{aligned}
$$

where it is good to recall that $n_t = -\Delta x_t$. Similarly, it is good to note that both the permanent and temporary price impact terms are included solely in the expected implementation cost term and do not appear in the variance term.

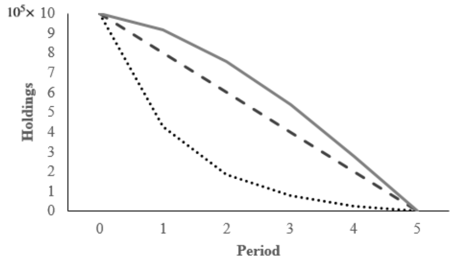
With these specifications it is evident that (6) is a *polynomial optimization problem* (POP) of degree $k+1$. In contrast, Almgren and Chriss (2001) use $g(x) = \gamma x$, $h(x) = \epsilon \operatorname{sgn}(x) + \eta x$, and $f(x) = 0$, thus obtaining a quadratic optimization problem. Here, it is assumed, for simplicity, that $\epsilon = 0$; this assumption is inconsequential for the examples below.

To illustrate the equivalence of the exact and numerical methods in the linear temporary price impact case, Figure 2 illustrates asset holdings evolution for three different risk aversion specifications.[23]

[23] Parameter values, e.g., γ and η are not calibrated to data in these simple examples. Instead they are chosen to closely match the values used in Almgren and Chriss (2001).



(a) Almgren and Chriss (2001) exact solution

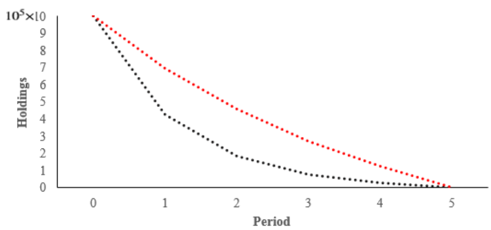


(b) GloptiPoly numerical solution

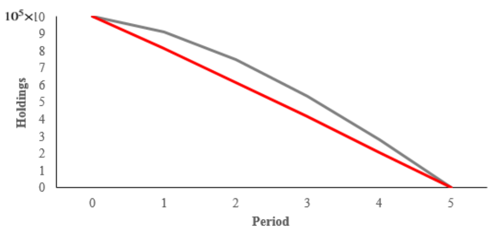
Figure 2: (**Linear temporary price impact**) The round dotted (lowest) line represents $\lambda = 2 \times 10^{-6}$, the dashed line represents $\lambda = 0$, and the solid line represents $\lambda = -2 \times 10^{-7}$. Further, $\sigma = 0.95$, $\gamma = 2.5 \times 10^{-7}$ and $\eta = 2.5 \times 10^{-6}$.

As seen from the figure, numerical polynomial optimization is able to replicate the Almgren and Chriss (2001) exact solution. This illustration should be considered as a baseline verification, because if it were the case that the exact and numerical methods would differ here, one would have to think twice about proceeding to the case where explicit solutions are unavailable.

Moving forward, Figure 3 juxtaposes asset holdings evolution under linear and nonlinear ($k = 2$) temporary price impact. The nonlinear case is solved numerically using GloptiPoly.



(a) $\lambda = 2 \times 10^{-6}$



(b) $\lambda = -2 \times 10^{-7}$

Figure 3: (**Nonlinear temporary price impact**) The black line represents the linear temporary price impact case and the red line represents the nonlinear temporary price impact case ($k = 2$). Further, $\sigma = 0.95$, $\gamma = 2.5 \times 10^{-7}$ and $\eta = 2.5 \times 10^{-6}$.

Contrary to what one might expect, Figure 3 shows that the holding trajectory with nonlinear price impact closely resembles the risk-neutral case ($\lambda = 0$) with linear temporary price impact. For this reason, the instance $\lambda = 0$ is omitted as the holdings evolution would be nearly indistinguishable from the other cases. This reflects the fact that a strategy with large individual trades is not optimal when temporary price impact is large relative to the permanent price impact. In other words, nonlinear temporary price impact pushes the trading strategy towards equal splitting (cf. Almgren 2003). This pushing effect is more emphasized when temporary price impact is nonlinear instead of linear.

3.3 Trading-Enhanced Risk

In this section an alternative specification of the basic optimal execution problem based on Almgren (2003) is illustrated. More specifically, it is assumed that the model features *trading-enhanced risk*: swift execution must be considered more risky as one must take into account the risk that, during the aspired execution window, there is not enough traders present willing to provide liquidity. This issue is easiest to understand by considering snapshots of the composition of traders in a given market throughout some period. One would expect that among these snapshots there are those with high activity (many traders present) and those with low activity (few traders present). If one is impatient and prefers rapid execution—for instance, due to risk aversion—there is an elevated risk that the trading program is executed during a more costly period of low trading activity.

From a modeling point of view, it is easily seen that this extension generally leads to intractable nonlinear problems. To see this and to make matters more explicit, the functional forms utilized in this section are:[24]

$$
\begin{aligned}
g(z) &= \gamma z, \quad \gamma > 0 \\
h(z) &= \eta z, \quad \eta > 0 \\
f(z) &= \alpha + \beta z^k, \quad \alpha \in \mathbb{R},\ \beta > 0, \text{ and } k \in \mathbb{N}.
\end{aligned}
$$

Said differently, permanent and temporary price impact take a linear form while the function representing uncertainty of trade execution is allowed to be polynomial. Thus, the quantities in (7) and (8) take the forms:

$$
\begin{aligned}
\mathbb{E}[C] &= \sum_{t=1}^{T} \left[\gamma x_t n_t + \eta n_t^2 \right] \\
\mathbb{V}[C] &= \sum_{t=1}^{T} \left[\sigma^2 x_t^2 + \alpha^2 n_t^2 + 2\alpha\beta n_t^{k+2} + \beta^2 n_t^{2(k+1)} \right].
\end{aligned}
$$

[24] One can interpret α as representing fixed uncertainty, independent of trades, and β as representing trading related uncertainty in realized trade prices. See, Almgren (2003).

Clearly, the objective function is again in a polynomial form, this time with degree $2(k+1)$. An important distinction related to trading-enhanced risk above—in contrast to the permanent and temporary price impact terms—is that it only appears in the term $\mathbb{V}[C]$. It is illustrated below that due to this distinction the impact of trading-enhanced risk can be somewhat attenuated as compared to, for instance, the impact stemming from temporary price impact.

Figure 4 displays the numerical results under trading-enhanced risk with $k = 1$ and with four different risk aversion levels.

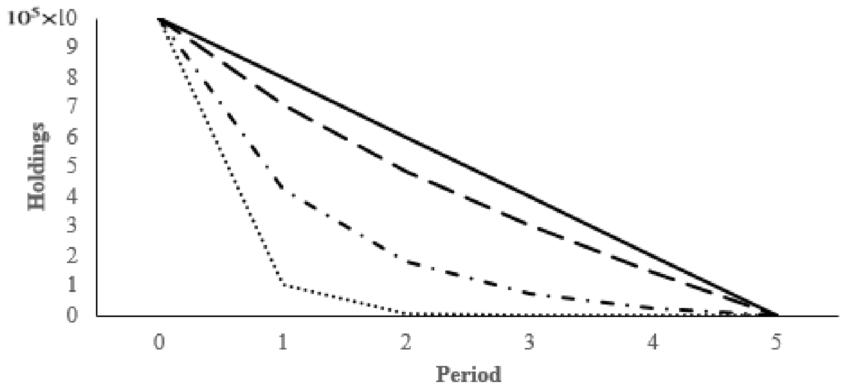


Figure 4: (**Trading-enhanced risk**) The solid line represents the $\lambda = 0$ case, the dashed liner represents $\lambda = 2 \times 10^{-7}$, the dash-dot line represents $\lambda = 2 \times 10^{-6}$, and finally the dotted line represents $\lambda = 2 \times 10^{-5}$. Further, $\sigma = 0.95$, $\gamma = 2.5 \times 10^{-7}$, $\eta = 2.5 \times 10^{-6}$, $\alpha = 2.5 \times 10^{-5}$, and $\beta = 0.5 \times \eta$.

It is observed from the above figure that more risk averse traders choose to execute rapidly under the chosen conditions. This stems from the, relatively speaking, pre-eminent impact of the volatility risk and the risk averse trader's desire to mitigate this risk. More specifically, as the variance of the implementation cost ($\mathbb{V}[C]$) is multiplied by λ, and α as well as β—which are of the same magnitude as η (the temporary price impact "intensity') are raised to the second power, the effect of trading-enhanced risk above is negligible and the term $\sigma^2 x_t^2$ dominates. The volatility risk stemming from $\sigma^2 x_t^2$ is indeed the key reason that a more risk averse trader chooses to trade fast (in large chunks), thus quickly decreasing x_t and the risk related to maintaining a large asset position.

The above example illustrates how trading-enhanced risk—which essentially has the same effect as temporary price impact, i.e., it pushes trading strategies towards equal splitting (see, Almgren 2003)—can be inconsequential under some parametrizations when compared to other sources of trading costs. Naturally, if σ is smaller and α and β are larger in magnitude, one starts to see a more pronounced impact from trading-enhanced risk. From a practical viewpoint, close attention is required in calibrating the model parameters to ensure that the model solutions match the risk assessment of the model user.

3.4 Optimal Execution of Contingent Claims

As the final optimal execution exemplification this section discusses a highly stylized model of contingent claim execution.[25] Here the trader faces a task of *acquiring* a position X_d in some contingent claim D (as in derivative) with underlying S (as in Sections 3.2 and 3.3). Let D_t denote the market price of the contingent claim at time t, more specifically, $D_t := f(S_t)$, $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, and D_0 fixed, so that the implementation cost is given by:

$$C_d := \sum_{t=1}^{T} n_t D_t - X_d D_0. \tag{9}$$

Note, that the above now relates to a buying program instead of a selling program as in (5).

Price dynamics for S in this section are assumed to be of the form:

$$\begin{aligned} S_t &= S_{t-1} + \sigma\tilde{\xi}_t - \gamma n_t, \\ \hat{S}_t &= S_{t-1} - \eta n_t. \end{aligned}$$

Put differently, $S_t = S_0 + \sigma\left(\sum_{j=1}^{t} \tilde{\xi}_j\right) - \gamma\left(\sum_{j=1}^{t} n_j\right)$. Notice that the above implies that both the permanent and temporary price impact are linear. This assumption is made for simplicity and could be relaxed with the cost of somewhat complicating the numerical procedure.

Before moving forward, it is good to take a moment to consider the intuition behind the proposed execution framework. In the previous examples the price impact of trades has been direct, i.e., one is directly selling or buying the underlying asset. Hence, given the specified price dynamics, it is clear that these trades should indeed influence the price of the traded asset. Here, however, the setting is different as one is buying contingent claims instead of the underlying asset and yet changes in the price of the underlying are used to capture the execution (implementation) costs. Implicit in this formulation is then the assumption that the contingent claim trades affect the price of the underlying.

This assumption is backed by both anecdotal evidence as well as academic research. First, one may interpret option trades as a reflection of private information and thus there can be information-based price impact. A case in point is Lowry et al. (2018), in which the authors discover evidence of advisor banks utilizing the private information obtained from dealings with clients by placing trades in the client firm options ahead of important corporate events such as merger announcements. In addition, hedging considerations related to options trading is another example of trades in

[25] The approach discussed draws from Hernandez-del Valle and Sun (2012) albeit utilizing a novel POP representation of the ensuing problem. Due to the POP representation, the contingent claim execution problem can be solved in a global optimization framework.

the derivatives market can influence the price of the underlying. Pearson et al. (2007) and Henderson and Pearson (2010) study the issue empirically, finding evidence that derivatives trading can affect the prices of the underlying assets in a statistically and economically significant manner.

Finally, it is good to note that the parameters γ and η should be interpreted as representing "indirect" impact of contingent claim trading. This interpretation can be thought of as a reduced form approach to contingent claim execution, which enables one to construct a parsimonious model. What magnitudes should γ and η take in this setting is ultimately an empirical question not pursued further here.

Coming back to the contingent claim execution model, the following proposition describes an approximate form of the problem at hand, which can then be solved using the polynomial optimization machinery. It is worth emphasizing that an approximation is required as the function f is not defined explicitly but instead just assumed to be some smooth function of the price of the underlying asset. Indeed, there are plenty of such f's that if one were to plug them explicitly into the objective function, the resulting problem would no longer be polynomial.

Proposition 1 (**Contingent claim execution problem as a POP**[26]). *Assume $f \in C^\infty(\mathbb{R})$, i.e., f is smooth in the underlying S, and denote by $\mathcal{T}^{(2)}_{C_d}$ the second order Taylor expansion of (9) with respect to S_0.*[27] *Then the approximate objective in the contingent claim execution problem:*

$$\mathbb{E}[\mathcal{T}^{(2)}_{C_d}] + \lambda \mathbb{V}[\mathcal{T}^{(2)}_{C_d}],$$

with risk aversion parameter $\lambda > 0$, has the following representation:

$$\mathcal{P} := \sum_{\deg=1,2,3,4} \mathcal{P}_{\deg}$$

where $\mathcal{P}_{\deg} : \mathbb{R}^T \to \mathbb{R}$ are homogeneous polynomials of degree $\deg = 1,2,3,4$, and the optimization problem (6) with $\mathcal{T}^{(2)}_{C_d}$ is equivalent to the following POP:

$$\inf_{\mathbf{n}} \left\{ \mathcal{P}(\mathbf{n}) : \mathbf{n} \in \mathbb{S} \right\} \tag{10}$$
$$\mathbb{S} = \left\{ \mathbf{n} \in \mathbb{R}^T : \; n_t \geq 0, \forall t = 1, ..., T, \text{and} \sum_{t=1,...,T} n_t = X_d \right\}.$$

[26] Proof of this result is available from the author by request.

[27] The second order Taylor expansion $\mathcal{T}^{(2)}_{C_d}$ is obtained by plugging:

$$D = f(S_0) + f'(S_0)(\hat{S}_t - S_0) + \frac{1}{2} f''(S_0)(\hat{S}_t - S_0)^2,$$

into equation (9) and rearranging the obtained expression (see, Hernandez-del Valle and Sun 2012) Higher order terms, denoted by $\mathcal{R}_3$, are ignored in the above approximation; it is possible to include, e.g., additional third order terms but this would increase the degree of the resulting polynomial objective function and complicate the optimization procedure. Moreover, one could also add a partial derivative with respect to time without issues. This term is omitted here for simplicity.

Since the objective function in (10), stemming from a second order approximation, is a dense quartic polynomial, the direct approach to solving the optimization problem is unlikely to be the most effective. Indeed, due to the higher degree of the polynomial, convergence to the global solution will require a higher relaxation order and thus will induce a higher computational cost. For these reasons, an alternative approach attributed to Mevissen and Kojima (2010) is utilized to tackle this problem.

The key idea in Mevissen and Kojima (2010) is to transform the original quartic problem to a more manageable quadratic problem. One way to accomplish this is to introduce new variables:

$$m_t := n_t^2, \quad m_{t+l+1} := n_t n_l, \text{ for } t, l = 1, ..., T, t \neq l$$

together with an extended objective function $\hat{\mathcal{P}} : \mathbb{R}^{2T+3} \to \mathbb{R}$ and an augmented semi-algebraic feasible set:

$$\hat{\mathbb{S}} := \Big\{ (\mathbf{n}, \mathbf{m}) \in \mathbb{R}^{2T+3} : n_t \geq 0, \forall t = 1, ..., T, \text{and} \sum_{t=1,...,T} n_t = X_d; \quad m_t := n_t^2, \quad m_{t+l+1} := n_t n_l, \text{ for } t, l = 1, ..., T, t \neq l \Big\}.$$

Letting $\mathbf{k} := (\mathbf{n}, \mathbf{m})$, it is observed that the new problem:

$$\inf_{\mathbf{k}} \Big\{ \hat{\mathcal{P}}(\mathbf{n}) : \mathbf{k} \in \hat{\mathbb{S}} \Big\}$$

is indeed a *quadratic optimization problem*. The key benefit from this transformation exercise is that it helps to reduce the size of the required SDP relaxations, thus decreasing computational burden. For additional information, the reader is referred to Mevissen and Kojima (2010).

Having formulated the transformed problem, one can now examine, for example, how risk aversion affects the optimal contingent claim trading program. Since volatility σ, as discussed in Section 3.3, is the main parameter capturing the risk involved in executing a given trade, three different volatility levels are considered together with three different risk aversion levels. The numerical results obtained are presented in Table 1.[28]

On one hand, one may observe from Table 1 that, as is to be expected, risk aversion typically causes the trader to execute the trading program faster and this result is more pronounced when the price volatility of the underlying asset is higher. This is a standard result documented in, for example, Almgren and Chriss (1999) as well as Almgren and Chriss (2001).

On the other hand, a phenomenon one would not necessarily expect—at least when dealing with the standard quadratic optimal execution problem—is that, for

[28] In determining values for the model parameters one should acknowledge that the scaling of parameters has important implications on the model outputs. This is especially true when the objective and constraints are nonlinear. Below, n_t's represent percentage proportions of the total position to be acquired.

Table 1: (**Contingent claim execution**) This table shows the optimal contingent claim execution program for three different risk aversion levels. In each run, the global optimality of the solution is verified numerically. n_t's represent percentage proportions of the total position to be acquired. The parameter values utilized are: $\eta = 0.400$ (temporary price impact intensity), $\gamma = 0.100$ (permanent price impact intensity), $f'(S_0) = -0.250$, $f''(S_0) = 0.005$.

$\sigma = 0.500$			
Risk aversion	n_1	n_2	n_3
$\lambda = 0.000$	0.369	0.333	0.298
$\lambda = 0.250$	0.398	0.329	0.273
$\lambda = 0.500$	0.000	0.000	1.000
$\lambda = 1.000$	0.000	0.000	1.000
$\sigma = 0.750$			
Risk aversion	n_1	n_2	n_3
$\lambda = 0.000$	0.412	0.333	0.255
$\lambda = 0.250$	0.597	0.391	0.012
$\lambda = 0.500$	1.000	0.000	0.000
$\lambda = 1.000$	0.000	0.000	1.000
$\sigma = 1.000$			
Risk aversion	n_1	n_2	n_3
$\lambda = 0.000$	0.471	0.333	0.196
$\lambda = 0.250$	1.000	0.000	0.000
$\lambda = 0.500$	1.000	0.000	0.000
$\lambda = 1.000$	1.000	0.000	0.000

lower levels of volatility, raising risk aversion can induce a kind of "waiting game" where the trader holds trades until the *last* period in which the target position is obtained via a single trade. This strategy is the polar opposite of the typical risk aversion result where the trader seeks to minimize any effect of volatility by using the single trade strategy in the *first* round. This result effectively illustrates the potential "non-monotonicities" that may arise when dealing with highly nonlinear polynomial objective functions but are absent in the standard affine or quadratic models.

A piece of intuition behind this observation goes as follows. When volatility is relatively low and when the price of the contingent claim is a nonlinear function of the price of the underlying, a speculative motive can arise to offset the effect of risk aversion.[29] Namely, if the price of the derivative evolves in a befitting manner with the price of the underlying, one can, in an effort to minimize implementation costs, benefit from waiting to allow for the possibility of a price decrease. Naturally, there is a possibility that the price of the underlying evolves to the wrong direction, but

[29] An avid reader may notice that from a hedging perspective the objective function of the trader is somewhat incomplete as it does not capture the notion that the trader might in fact be trying to protect herself against, e.g., a price decrease.

this risk does not out-weight the upside from waiting when volatility is low. When volatility is high ($\sigma = 1.000$), the waiting strategy ceases to be optimal and the standard risk aversion result prevails. Determining how prevalent the above results are and what other outcomes may arise when looking at the entire, possibly calibrated, parameter space is an interesting open question left for future research.

3.5 Additional Considerations in Optimal Execution Models

To conclude the discussion on optimal execution models and polynomial optimization some additional points are worth emphasizing. First, while it is for the most part justified to assume that the permanent price impact function $g(\cdot)$ is linear, the exact functional form of the temporary price impact function $h(\cdot)$ is not as clear-cut.

Above, the form $h(x) = \eta x^k$, for $k \in \mathbb{N}$, was utilized. However, there is a strong case for utilizing a power law function $h(x) = \eta|x|^\psi$ such that $\psi \in (0, 1)$ (see, for instance, Almgren et al. 2005 and Bouchaud et al. 2009)[30] Now, even though under the power law temporary price impact one would not directly end up with a polynomial objective function, it is still possible to utilize the methodology outlined above. This can be achieved, with slight assumptions on ψ, via extending the standard polynomial optimization approach to optimization of semialgebraic functions as discussed in Lasserre (2015) or in a straightforward fashion, in applicable instances, by applying an appropriate change of variables in the objective function.

Another key point to bring up is that while the optimal execution models examined above belong to the class of classical price impact models, one could also consider more recent models seeking to capture the empirical properties of limit order books (LOBs), i.e., models which feature transient price impact (see, for example, Obizhaeva and Wang 2013 and Alfonsi et al. 2010).[31] Notwithstanding the fact that these models are often deliberately constructed in such a way that unique optimal solutions can be constructed explicitly, it would be an interesting venue for further research to apply polynomial optimization methods to cases where numerical optimization is mandatory due to the inherent complexity of the problem.

[30] Note that the absolute values in $h(x) = \eta|x|^\psi$ are redundant if x is positive and does not change sign.

[31] A transient price impact is greatest immediately after the trade but then starts to vanish over time as new orders arrive to the LOB. This property is called the resiliency of the LOB. See, for example, Large (2007) and Lo and Hall (2015).

4 HIGHER MOMENT PORTFOLIO OPTIMIZATION UNDER QUADRATIC TRANSACTION COSTS

4.1 Portfolio Optimization Model

In this section, mean-variance-skewness (MVS) portfolio optimization, as opposed to the standard mean-variance (MV) framework, is discussed (cf. Lai 1991, Konno and Suzuki 1995, and Briec et al. 2007).[32] The motivation for studying portfolio selection with higher moments, such as skewness and kurtosis, stems from the empirically documented issue that returns are not generally normally distributed and thus anchoring investor utility purely to the first (mean) and the second (variance) moments of the return distribution is not completely innocuous. Indeed, considering investors with a preference for positive skewness may notably change the composition of the resulting optimal portfolios as well as help to better understand phenomena such as under-diversification of investment portfolios (see, Briec et al. 2007).

Given the theoretical and practical importance of portfolio selection with higher moments it is not surprising that there exists a wide literature addressing this issue. Examples include Lai (1991), de Athayde and Flôres Jr (2004), Jondeau and Rockinger (2006), Briec et al. (2007), Briec and Kerstens (2010), and Harvey et al. (2010). This wealth of papers means that various methods to tackle the question the portfolio optimization involving higher moments have been proposed in earlier works. There are, however, some complications that should be addressed. Indeed, one of the main issues one is forced to deal with in higher moment portfolio optimization is the inherent nonconvexity of the optimization problem. As a result, a majority of the methods introduced in the literature cannot guarantee the global optimality of the solutions obtained or do so in an ad hoc fashion.[33]

Conversely, by formulating the higher moment portfolio problem as a POP, one is able to frame the problem in such a way that general global optimization machinery is readily available. Kleniati et al. (2009) utilize the POP formulation for a mean-variance-skewness-kurtosis (MVSK) portfolio selection problem with and without parameter uncertainty. The below introduced problem is related to their work but differs on a number of dimensions.

First, Kleniati and Rustem (2009) utilize monthly returns whereas the numerical results obtained in Section 4.2 seek to emulate portfolio optimization based on daily data. This is noteworthy, as it is well known that monthly returns are closer to normally distributed than daily returns. Second, the problem formulated below involves

[32] Note to the reader: notation in this section is independent of the notation in the previous section.

[33] Exceptions include, e.g., Briec et al. (2007).

MVS portfolio selection subject to quadratic transaction costs instead of MVSK portfolio without transaction costs. For clarity, Kleniati and Rustem (2009) ignore transaction costs as their focus is on a one-shot, i.e., static, problem. Conversely, and perhaps most importantly, the numerical example below features dynamic portfolio rebalancing and for this reason it is reasonable to consider the influence of transaction costs.

Moving on to the notation and the specific problem to be solved, suppose the market of interest consists of n risky assets with random simple returns $\mathbf{R} = (R_1, ..., R_n)^\top$ and let x_i denote the portfolio weight in asset i such that a portfolio is given by $\mathbf{x} = (x_1, ..., x_n)^\top$.[34] Feasible portfolios belong to the set:

$$\mathcal{P}_{\mathbf{x}} := \Big\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ \mathbf{x} \geq \boldsymbol{\omega} \Big\}, \tag{11}$$

where $\boldsymbol{\omega} = (\omega_1, ..., \omega_n)^\top$ with $\omega_i \geq 0$, for all i, captures a minimum weight constraint.[35] In other words, the feasible portfolios in the polyhedral—or, more generally, semialgebraic—set $\mathcal{P}_{\mathbf{x}}$ are required to satisfy the so-called budget constraint (*all available funds are allocated to available assets*) and a minimum weight constraint. The latter constraint also means here that short-selling is not allowed. It is, however, noted that one could easily either relax some constraints, e.g., the minimum weight constraint, or introduce more constraints, e.g., additional thresholds regarding allowed asset positions (cf. Boyd et al. 2017).

The simple notation for the moments in the MVS portfolio problem is adopted from Kleniati and Rustem (2009).[36]

Moment	**Definition**	**(Sample) Formula**
μ_i	$\mathbb{E}[R_i]$	$\sum_{m=1}^{M} R_{i,m}/M$
σ_{ij}	$\mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)]$	$\sum_{m=1}^{M} (R_{i,m} - \mu_i)(R_{j,m} - \mu_j)/(M-1)$
s_{ijk}	$\mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)]$	$\sum_{m=1}^{M} (R_{i,m} - \mu_i)(R_{j,m} - \mu_j)(R_{k,m} - \mu_k)/M$

Related to the above moments is the co-skewness matrix $\mathcal{M}_s \in \mathbb{R}^{n \times n^2}$ given, for the case $n = 3$, by $\mathcal{M}_s = [S_{1jk}\ S_{2jk}\ S_{3jk}]$, where

$$S_{i'jk} = \begin{bmatrix} s_{i'11} & s_{i'12} & s_{i'13} \\ s_{i'21} & s_{i'22} & s_{i'23} \\ s_{i'31} & s_{i'32} & s_{i'33} \end{bmatrix},$$

34 It is assumed, for simplicity, that a risk-free asset is not available.

35 A minimum (similarly, maximum) weight constraint can be seen as a device to ensure better diversification. Other types of weight constraints, e.g., those following from UCITS-compliance can easily be handled.

36 Note that $\sigma_{ii} = \sigma_i^2$.

for $i' \in \{1,2,3\}$.

As usual, the portfolio return and expected portfolio return are given by:

$$R_p(\mathbf{x}) := \sum_{i=1}^{n} R_i x_i,$$

$$\mu_p(\mathbf{x}) := \mathbb{E}[R_p(\mathbf{x})] = \sum_{i=1}^{n} \mu_i x_i.$$

Further, as noted earlier, it is assumed that trading is costly. In more detail, quadratic transaction costs are specified similarly to Gârleanu and Pedersen (2013), namely:

$$TC_t = \frac{\lambda}{2} \sum_{i,j=1}^{n} \sigma_{ij} \Delta x_{i,t} \Delta x_{j,t},$$

where $\lambda > 0$ is a scaling constant and $\mathbf{\Lambda} := (\lambda \sigma_{ij})_{i,j} \in \mathbb{R}^{n \times n}$ is a symmetric and positive-definite matrix, and $\Delta x_{l,t} = x_{l,t} - x_{l,t-1}$, $l = 1, \ldots, n$. The matrix $\mathbf{\Lambda}$ can be thought of as a multidimensional version of Kyle's lambda (see, Kyle 1985). Gârleanu and Pedersen (2013) offer the following interpretation for transaction costs of this form: trading $\Delta \mathbf{x}_t$ shifts average prices by $\frac{1}{2}\mathbf{\Lambda}\Delta\mathbf{x}_t$ and total trading costs are given by multiplying this shift in prices by $\Delta \mathbf{x}_t$.[37]

With the relevant notation in place, it is time to state the portfolio optimization problem. The problem in question can be written as:

$$\sup_{(x_1,\ldots,x_n) \in \mathcal{P}_{\mathbf{x}}} \phi_1 \sum_{i=1}^{n} \mu_i x_i - (\phi_2 + \frac{\lambda}{2}) \sum_{i,j=1}^{n} \sigma_{ij} x_i x_j + \phi_3 \sum_{i,j,k=1}^{n} s_{ijk} x_i x_j x_k, \tag{12}$$

where it holds that $\phi_1 + \phi_2 + \phi_3 = 1$. The above problem is expressed in an explicit summation form to emphasize the polynomial nature of the problem. Indeed, from (11) and (12) one can immediately infer that the resulting problem is a POP with total degree 3.[38] In the numerical examples theoretical moments are replaced by their empirical (sample) counterparts estimated from (simulated) historical data. Sample moments are denoted by $\hat{\mu}_i$, $\hat{\sigma}_{ij}$, and $\hat{s}_{ijk}$.

Finally, one might wonder the relation between investor's utility maximization problem and problem (12). To shed light on this, let $u : \mathbb{R} \to \mathbb{R}$ be a utility function satisfying the usual properties.[39] Further, suppose u has bounded derivatives of sufficient order so that the following third order Taylor approximation around expected

[37] Quadratic transaction costs can also be seen as a tool to combat estimation error. See, Olivares-Nadal and DeMiguel (2018).

[38] Due to the way the quadratic transaction costs are specified, the objective function in (12) has a relatively simple form. One could also easily consider other p-norm transaction costs as in Mei et al. (2016) or quadratic transaction cost of the form $\boldsymbol{\beta}^\top |\Delta \mathbf{x}| + |\Delta \mathbf{x}|^\top \mathbf{\Gamma} |\Delta \mathbf{x}|$ for $\boldsymbol{\beta} \in \mathbb{R}^n$ and $\mathbf{\Gamma} \in \mathbb{R}^{n \times n}$ while still being able to treat the problem as a POP. Moreover, one could consider, e.g., additional constraints such as the self-financing constraint. For more on portfolio selection models see, for instance, Kolm et al. (2014).

[39] It should be noted that it is assumed implicitly in this formulation that the relevant wealth of the investor is entirely determined by the portfolio return.

portfolio return is well-defined:

$$\begin{aligned}\mathbb{E}[u(R_p(\mathbf{x})] \approx & u(\mathbb{E}[R_p(\mathbf{x})]) + \frac{u''(\mathbb{E}[R_p(\mathbf{x})])}{2!}\mathbb{E}\Big[(R_p(\mathbf{x}) - \mu_p(\mathbf{x}))^2\Big] \\ & + \frac{u'''(\mathbb{E}[R_p(\mathbf{x})])}{3!}\mathbb{E}\Big[(R_p(\mathbf{x}) - \mu_p(\mathbf{x}))^3\Big] \\ = & u(\mathbb{E}[R_p(\mathbf{x})]) + \frac{u''(\mathbb{E}[R_p(\mathbf{x})])}{2!}\mathbb{V}[R_p(\mathbf{x})] + \frac{u'''(\mathbb{E}[R_p(\mathbf{x})])}{3!}\mathbb{SKEW}[R_p(\mathbf{x})],\end{aligned} \tag{13}$$

where $\mathbb{SKEW}[\cdot]$ is used to denote the third moment, i.e., skewness. Under the standard assumptions concerning the utility function u, it holds that the second derivative $u'' < 0$ captures *variance aversion* and the third derivative $u''' > 0$ captures a *preference for positive skewness*.

Therefore, similar to the way the second order Taylor approximation can be used to motivate the standard mean-variance portfolio optimization problem (cf. Markowitz 2010), the approximation (13) motivates the following optimization problem:

$$\sup_{\mathbf{x} \in \mathcal{P}_\mathbf{x}} \ \psi\mathbb{E}[R_p(\mathbf{x})] + \rho\mathbb{V}[R_p(\mathbf{x})] + \kappa\mathbb{SKEW}[R_p(\mathbf{x})], \tag{14}$$

where $\psi > 0$ captures the preference for higher expected returns, $\rho < 0$ captures the aversion to risk measured by (co)variance, and $\kappa > 0$ captures the preference for positive skewness.

As a concluding note, the similarities between the objective function in (6) and in (14) are worth noticing. In particular, both objectives exhibit weighted moments of some underlying random variable. Problems of this form are frequently encountered in finance and can usually be formulated as a POP, subject to some case specific considerations. This observation speaks on behalf of the usefulness of polynomial optimization in finance.

4.2 Numerical Examples

In this section simple numerical examples illustrating the POP approach to portfolio optimization with higher moments are given. As a starting point and to fix ideas, a short static (one-period) example is discussed first. The static example is followed by a dynamic one. In the dynamic example the purpose is to study myopically rebalanced, i.e., not dynamically optimized, MV and MVS optimal portfolios during a one year evaluation period with monthly rebalancing.[40] Implementing and solving

[40] *Myopically rebalanced* here means that the portfolios are not dynamically optimized but instead are obtained by solving a sequence of one period problems. In fact, a dynamic programming approach often turns out to be intractable in a general multiperiod portfolio selection setting with non-trivial transaction costs. See, for example, Boyd et al. (2014) and Boyd et al. (2017).

this problem utilizing polynomial optimization is straightforward as illustrated below and detailed in Appendix B.

To get started with the numerics and to provide a quick example how portfolio allocations stemming from the standard MV optimization and from the MVS optimization can be quite different. Fix $\lambda = 5 \times 10^{-4}$ and consider three assets with a sample drawn from a Gaussian distribution. The mean vector for the sample is $\boldsymbol{\mu} = (0.6003\%,\ 0.6903\%,\ 0.9275\%)$ and variance-covariance matrix:

$$10^2 \times \boldsymbol{\Sigma} = \begin{bmatrix} 0.0838 & -0.0369 & -0.0076 \\ -0.0369 & 0.1363 & 0.0283 \\ -0.0076 & 0.0283 & 0.6317 \end{bmatrix}.$$

Then, the global optimal solution for problem (12), with $\phi_1 = \phi_2 = 1/2$ and $\phi_3 = 0$, is $\mathbf{x}^*_{mv} = (31.61\%,\ 42.65\%,\ 25.74\%)$. However, shifting to the MVS formulation with $\phi_1 = 1/2$ and $\phi_2 = \phi_3 = 1/4$, one obtains the globally optimal solution $\mathbf{x}^*_{mvs} = (6.47\%,\ 46.37\%,\ 47.15\%)$.

So, while both solutions give a relatively large weight on asset 2, there are quite large differences for the weights of assets 1 and 3. Asset 3 has the highest expected return and highest risk. The MV portfolio seeks to find a balance between these two aspects and assigns a weight 25.74% to the third asset. Conversely, the first asset has the lowest expected return and lowest risk. In addition, the first asset is negatively correlated with both the second and the third asset, thus providing additional diversification benefits and obtains a weight 31.61% in the MV portfolio.

The MVS portfolio, however, assigns a weight of only 6.47% to the first asset. To understand why, it is worth noting that in the MVS portfolio (co)variance and (co)skewness have an equal, although opposite in sign, weight in the objective function. Further, the first asset exhibits negative coskewness with both the second and the third asset. Consequently, from the MVS viewpoint, the diversification benefits of the first asset are diminished. Hence, the weight of the first asset is reduced in the MVS portfolio and, contrariwise, the weight of the third asset grows.

Moving on to the dynamic example, a quick look at how portfolio performance is measured is in order. In the MV framework a natural performance measure is the ratio of realized excess returns to volatility of these returns.[41] This performance indicator is commonly known as the (ex post) Sharpe ratio (SR; see, Sharpe 1966). In the MVS framework, the standard Sharpe ratio is an incomplete performance measure.

By adding a preference for positive skewness, to put it plainly, the investor assigns larger weights on average to assets, which exhibit positive (co)skewness. Therefore, portfolio returns are also more likely to exhibit positive skewness. To take this into account one would want to utilize a performance measure which rewards positive

[41] Excess realized returns, generally speaking, are taken to mean returns in excess of transaction costs and some risk free return. Transaction costs are modeled as discussed above and the risk free return is fixed at zero for simplicity.

skewness of portfolio returns. For this purpose, the skewness adjusted Sharpe ratio, as in Zakamouline and Koekebakker (2009), is given below.

Definition 10 (Adjusted Sharpe ratio). *The skewness adjusted Sharpe ratio, assuming CARA-utility and provided that the quantity obtained is a positive real number, is given by:*

$$\text{ASSR} = \text{SR}\sqrt{1 + \frac{\mathbb{SKEW}[R_p(\mathbf{x})]}{3}\text{SR}}, \tag{15}$$

where $R_p(\mathbf{x})$ *is the portfolio excess return and* SR *is the Sharpe ratio:*

$$\text{SR} = \frac{\mathbb{E}[R_p(\mathbf{x})]}{\sqrt{\mathbb{V}[R_p(\mathbf{x})]}}. \tag{16}$$

Armed with the appropriate performance measures, it is time to move on to the results. In the above static example, it was illustrated how portfolio weights between the MV and the MVS strategy can differ substantially even when the sample data is drawn from a Gaussian distribution. A natural follow-up question is: what if the data is drawn from a non-normal distribution?

To shed light on this, the data utilized in the next example is drawn from a multivariate Gaussian mixture distribution. This distribution type is used to capture the empirically verified fact of non-normal daily returns. The results below exemplify one year of trading with monthly portfolio rebalancing. In the example there are three risky assets—X, Y, and Z—and it is assumed that a *rolling window* of 252 (one year) daily return observations for each asset is used as training data in every iteration of the portfolio rebalancing/optimization procedure. Portfolio returns are determined on an out-of-sample basis. Appendix B gives a more formal account of the procedure.

Figure 5 illustrates the evolution of portfolio weights over the rebalancing periods $t = 1, 2, ..., 12$.

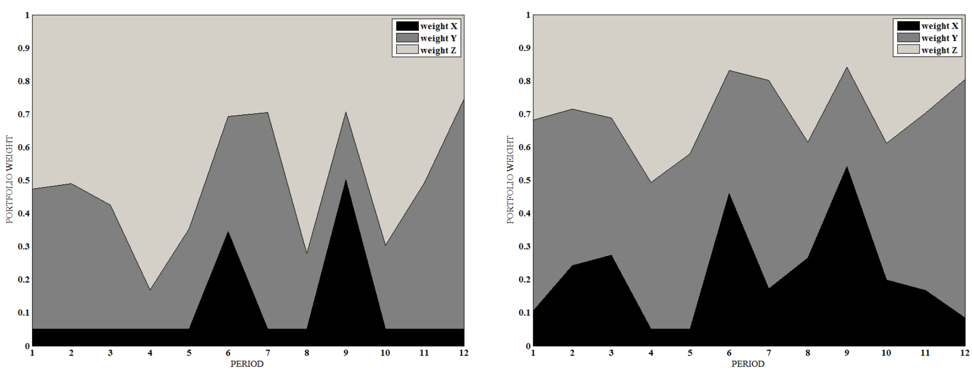


Figure 5: **(Portfolio weights)** This figure illustrates one instance of the evolution of the portfolio weights for the MV portfolio (left) and the MVS portfolio (right) over rebalancing periods $t = 1, 2, ..., 12$.

Looking at Figure 5, one may observe a glimpse of how the MV procedure tends to favor more extreme allocations as compared to the MVS procedure. Despite this, it is also noticeable that the MV and MVS allocations are quite correlated. This, of course, is no surprise as the two allocations are after all based on quite similar preferences.

Table 2 reports the mean SR, the mean ASSR, and the mean turnover (TO) for MV and MVS portfolios over 10000 simulations.[42]

Table 2: (**Performance and turnover**) This table presents the mean performance ratios and mean turnover for both the MV and MVS portfolios.

Strategy	mean SR	mean ASSR	mean Turnover
MV	0.806	0.874	3.988
MVS	0.824	0.900	3.323

It is evident from Table 2 that the mean SR and mean ASSR are relatively close to each other although the MVS portfolio seems to have a slight advantage in terms of performance measures. Figure 6 further illustrates the strong positive correlation in the performance measures for the two portfolio allocations. Evidently, the two figures are virtually identical. Portfolio turnover, however, is more nuanced. The results indicate that the turnover of the MV portfolio is notably higher than the turnover of the MVS portfolio.

Two things are worth noting when examining portfolio turnovers. First, the results in Table 2 indicate that optimizing the portfolio over more than two moments could have a stabilizing effect on the portfolio weights. In other words, as more moments are taken into account when making the rebalancing decision, one may end up adjusting the portfolio weights less compared to the case where fewer moments are utilized. Second, one could conjecture that the MVS portfolio dominates in terms of the performance ratios simply due to the reduced transaction costs. If this were the case, then one would expect that the MVS portfolio would consistently dominate the MV portfolio over different simulation runs. This is not strictly speaking the case.

In fact, while the observation that the MVS portfolio has a lower turnover than the MV portfolio seems very robust over a barrage of numerical runs, the edge of the MVS portfolio over the MV portfolio in terms of performance is much more am-

[42] Regarding the mean ASSR, 13 complex valued sample points for the MV portfolio and 25 complex valued sample points for the MVS strategy were discarded. Sharpe ratios are calculated using (16) and asymmetric Sharpe ratios are calculated using (15). Moreover, portfolio turnover for each individual simulation round is obtained from:

$$\mathrm{TO} = \sum_{i=1}^{3} \sum_{t=1}^{12} |\Delta x_{t,i}|.$$

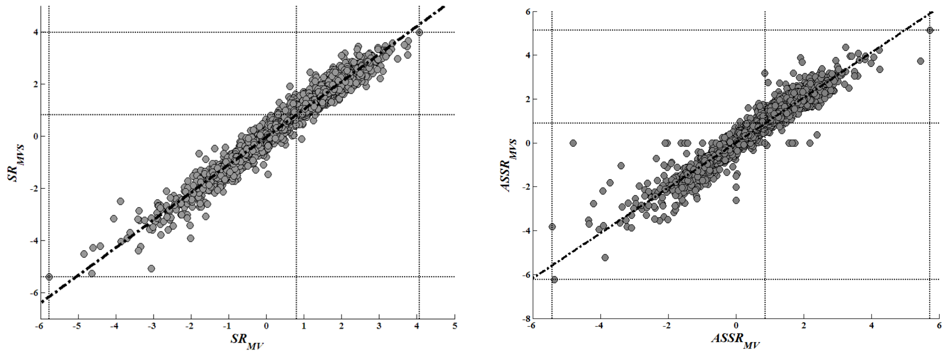


Figure 6: (**Performance ratios**) This figure presents scatter plots for the performance ratios of the MV and MVS portfolios. Sharpe ratio scatter is presented on the left and asymmetric Sharpe ratio scatter is presented on the right. The dash-dotted line represents the trend while the dotted lines represent the average and maximum observations on both axes.

biguous. In some instances, the more extreme allocations or more extreme changes pay off. This observation mirrors the idea of basic diversification in that reducing risk also generally means lower expected returns.

To have an illustrative look at the turnover comparison, Figure 7 highlights the turnover differences between the two portfolios. The two horizontal lines in the figure represent the x-axis (where turnover difference is zero) and the mean of the turnover difference.

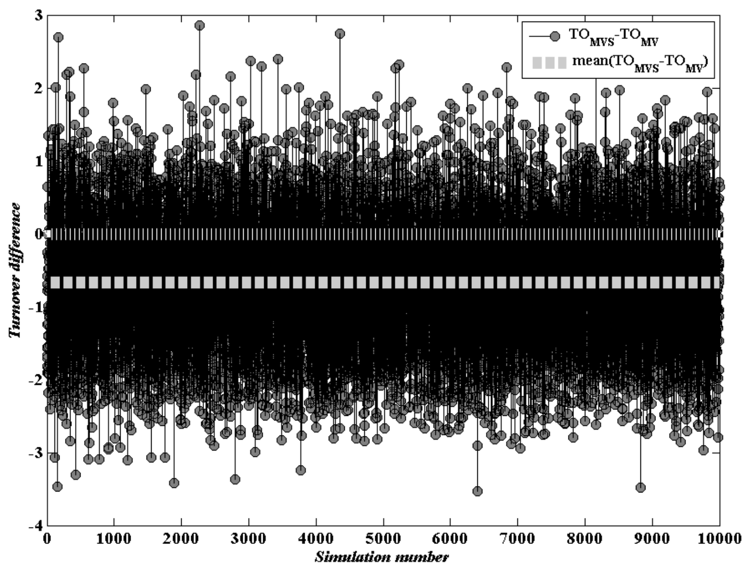


Figure 7: (**Turnover difference**) This figure depicts the turnover differences between the MV and MVS portfolios. The upper horizontal line highlights the x-axis where turnover difference is zero and the lower horizontal line highlights the average of the turnover differences.

The fact that the mean of the turnover difference is negative in Figure 7 indicates that, on average, the MV portfolio generates higher portfolio turnover than the MVS portfolio. As noted earlier, this observation, contrary to the relative order of the performance ratios, holds true in each simulation run.

One reason for the difference in portfolio turnover between the two portfolios could be, as alluded to earlier, that the decision-making rule in the MVS portfolio is more "diversified" as compared to the MV portfolio. Hence, focusing on three moments instead of two could produce smaller adjustments to the portfolio weights moving from one period to another, thus reducing total portfolio turnover. The implications of this observation can be particularly useful in cases where one is trading in a market with high or potentially time-varying transaction costs.

4.3 Portfolio Optimization Extensions

There are various ways to refine the above simple numerical examples. First, one could consider adding even higher moments, e.g., the fourth moment (*kurtosis*). This would result in a POP of a higher yet still manageable degree. Second, one could use real data and focus more on the empirical estimation of moments to gain a better understanding of the impact of estimation errors on portfolio performance. Alternatively, one could utilize a robust formulation of the problem (as in Kleniati and Rustem 2009) which would still boil down to a POP; the interplay of robust optimization and transaction costs is surely worth examining. Third, one could utilize econometric forecasting models for the different moments and compare the performance gains stemming from different forecasting models for different moments or even combined forecasts for multiple moments. Finally, one could consider adding more practically relevant constrains, e.g. position limits and turnover limits, or conversely, one might consider removing existing constraints such as the short-sales constraint.

There are also several recent developments in portfolio optimization outside the examples given above which are compatible with the polynomial optimization approach. A case in point is Ban et al. (2018) who propose to exploit machine learning techniques to tackle the poor out-of-sample performance of classical portfolio optimization combined with simple sample-average-approximation (SAA)—an approach which was also utilized in the above example. The authors propose using *performance-based regularization* (PBR) to constrain the sample variances of the estimated quantities for covariances and means (in the Markowitz model), thus seeking to mitigate the effects of estimation error on the optimal portfolio solution.

Taking advantage of the PBR approach in the mean-variance framework results in an addition of a *quartic* polynomial constraint. Since the original problem can already be stated as a POP, adding another algebraic constraint is readily handled in the polynomial optimization framework. The only real difference is that the new problem has a (total) degree of 4 instead of the degree 2 in the original mean-variance problem.

Indeed, the PBR-MV problem can be stated as:

$$\begin{aligned}\hat{\mathbf{x}}_{n,MV} \in \underset{\mathbf{x}\in\mathbb{R}^p}{\operatorname{argmin}} \ & \mathbf{x}^\top \hat{\boldsymbol{\Sigma}}_n \mathbf{x} \\ \text{s.t.} \ & \mathbf{x}^\top \mathbf{1}_p = 1, \\ & \mathbb{SVAR}[\mathbf{x}^\top \hat{\boldsymbol{\Sigma}}_n \mathbf{x}] \leq \mathcal{U},\end{aligned}$$

and *the sample variance operator* $\mathbb{SVAR}[\cdot]$ is given by:

$$\mathbb{SVAR}[\mathbf{x}^\top \hat{\boldsymbol{\Sigma}}_n \mathbf{x}] = \sum_{i,j,k,l=1}^{p} x_i x_j x_k x_l \hat{Q}_{ijkl},$$

with

$$\hat{Q}_{ijkl} = \frac{1}{n}\left(\hat{k}_{ijkl} - \hat{\sigma}_{ij}^2 \hat{\sigma}_{kl}^2\right) + \frac{1}{n(n-1)}\left(\hat{\sigma}_{ik}^2 \hat{\sigma}_{jl}^2 + \hat{\sigma}_{il}^2 \hat{\sigma}_{jk}^2\right),$$

where $\hat{k}_{ijkl}$ and $\hat{\sigma}_{ij}^2$ is the sample average estimator of the fourth central moment (kurtosis) and the second central moment ((co)variance). Moreover, $\mathcal{U}$ is a regularization parameter which is used to discard solutions with large estimation error.

Another example is Maillard et al. (2010) who, similarly to Ban et al. (2018) above, start by noting the problems stemming from estimation errors in the classical Markowitz portfolio selection framework, and remarking that due to these issues many market participants prefer to employ heuristic approaches that strike a balance between easy implementability and “robustness”, where robustness mostly refers to the fact that these heuristic approaches are typically independent of expected returns. Among these heuristic approaches are, for example, equally weighted and minimum variance portfolios.

Maillard et al. (2010) propose an alternative approach, which seeks to equalize the risk contributions of different portfolio components. The authors state that their approach lies somewhere in between equally-weighted and minimum variance portfolios and they call the resulting portfolio a *equally-weighted risk contributions* (ERC) portfolio. The portfolio weights of an ERC portfolio with no short-selling can be found by solving:

$$\begin{aligned}\mathbf{x}^* \in \operatorname{argmin} \ & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i(\boldsymbol{\Sigma}\mathbf{x})_i - x_j(\boldsymbol{\Sigma}\mathbf{x})_j\right)^2 \\ \text{such that } & \mathbf{1}^\top \mathbf{x} = 1 \ \text{ and } \ x_i \in [0,1]^n \ \ \forall i = 1, \dots, n.\end{aligned}$$

Again, the problem can be straightforwardly formulated as a POP. Similarly, additional polynomial constraints are easily implemented. Developing methods to handle large-scale instances (large n) of the above problem is a research question with notable practical relevance.

Finally, as far as static portfolio optimization is considered, adding higher moments to the standard mean-variance framework is crucial only under rather extreme situations (high leverage portfolios or large deviations from normality). This may no longer be true in dynamic context. At the very least, additional considerations should be made in a dynamic setting as illustrated by the example above. A look at multiperiod portfolio optimization with higher moments using *model predictive control* (MPC, see Boyd et al. 2017) combined with polynomial optimization is one interesting topic to consider.

Moreover, as argued by Jondeau and Rockinger (2012), higher moments may also have a more pronounced role when the distribution of asset returns is time-varying. Further inquiries along these lines are likely to benefit from modelling the portfolio selection problem with higher moments as a POP. In general, dynamic portfolio selection with (potentially time-varying) higher moments is an interesting but highly challenging venue for further research.

5 ALL SOLUTIONS IN DYNAMIC MARKET EQUILIBRIUM

5.1 Strategic Trading Model

The final example presented in this paper is related to strategic trading. In short, strategic trading here refers to a situation where a trader aims to carry out a sequence of trades such that the trades themselves, in addition to other potential factors, influence the price of the traded asset. In other words, trades have a price impact.[43] Another way to think about this is that trading costs are endogenous compared to the portfolio optimization example where trading costs were exogenous.

The starting point for this section is a recent paper by Choi et al. (2019). The paper introduces a game theoretic model of strategic trading based on the seminal work by Kyle (1985). The model introduced in the paper is solved via a dynamic programming approach, complemented by an assumption that the equilibrium strategies are Markovian in nature. The dynamic programming approach leads one to, at each period $n = 1, ..., N$, solve a system of recursive equations which—after repeated

[43] In finance, and more specifically in market microstructure theory, price impact is a concept generally associated with markets with incomplete competition, i.e., it is assumed that there is a finite number of (large) traders in the market instead of (or in addition to) a continuum of (small) traders.

substitutions—can be shown to boil down to solving a system of two coupled high degree polynomial equations.

This part of the model solution is crucial as the coupled system is likely to have a large number of different (real) solutions, which instead may imply that the model has multiple equilibria. Problem is that standard (numerical) approaches in finance and economics tend to search only a single solution while ignoring the others. Alternatively, even if methods capable of producing multiple solutions (roots) simultaneously are used, authors often resort to ad hoc heuristics to trim down the solution set until only a single (or none) solution remains.

An alternative approach, advocated by, e.g., Kubler and Schmedders (2010), Judd et al. (2012), and Kubler et al. (2014), is to utilize advances in numerical algebraic geometry, i.e., to use either *all-solutions homotopy continuation* or *Gröbner bases* methods to tackle the problem of determining all solutions to the given polynomial system. Below these approaches are implemented in the context of the Choi et al. (2019) model. Before that, the goal is to briefly introduce the model to be studied.

5.1.1 Market structure and trader types

For the sake of easy comparability, notation closely resembling the one used in Choi et al. (2019) is utilized. Consider a market with one risky asset that has a (Gaussian) terminal payoff $\tilde{v} \sim N(0, \sigma_v^2)$ for some $\sigma_v^2 > 0$. In addition to the risky asset, there is a riskless asset that pays zero interest rate. The realization of $\tilde{v}$ becomes public information after N rounds of trading, i.e., at time $N+1$ and trading takes place at discrete time points $n \in \{1, 2, ..., N\}$ with evenly spaced time steps.

The market is populated by four types of traders:

(1) A risk neutral *informed trader* who observes the realization of $\tilde{v}$ at $n = 0$ and trades to exploit *his* informational advantage. The letter I is used to refer to the informed trader.

(2) A risk neutral constrained trader (*she*), referred to as the *rebalancer* , who trades to reach a hard terminal trading target, $\tilde{a} \sim N(0, \sigma_a^2)$ with $\sigma_a^2 > 0$, which is jointly normally distributed with $\tilde{v}$ with correlation $\mathbb{CORR}[\tilde{v}, \tilde{a}] =: \rho_{av} \in [0, 1]$. The realization of $\tilde{a}$ is privately learned by the rebalancer at $n = 0$.[44] The letter R is used to refer to the rebalancer.

(3) Non-strategic *liquidity* (or *noise*) *traders* whose aggregate demand at each $n = 1, ..., N$ is given by $\tilde{u}_n \overset{i.i.d.}{\sim} N(0, \sigma_u^2)$ for some $\sigma_u > 0$.

[44] The notion of a hard trading target is interpreted so that the rebalancer must reach the specific position stipulated by the target $\tilde{a}$ during the N rounds of trading. A soft trading target with an associated penalty function is a possible extension not pursued here. In case of a hard trading target, one could simply assume that the trader faces infinite penalty if she deviates from the target.

(4) A number of risk-neutral *market makers* who set prices competitively, using all information available to them, such that the resulting financial market can be considered *semi-strong form efficient*. The letter M is used to refer to the (representative) market maker.

All information is costless to those who observe it and (strategic) traders in the model are assumed perfectly attentive and rational in the sense that they do not ignore any relevant information. Private information, for those who are endowed with it, is learned at time $n = 0$, before trading commences, and no additional private information is received in later periods $n = 1, ..., N$. Furthermore, private information is available only to the trader who observes it and information cannot be bought or sold.

Order flow information is observed by the market makers at each $n = 1, ..., N$. At period n, the information generated by the order flow process is denoted by $\boldsymbol{\sigma}(y_1, ..., y_n)$, where y_n is used to denote the period n aggregate order flow. This describes the information available to the market makers. The market makers utilize the order flow information to set prices and to update beliefs regarding the private information possessed by I and R. The aggregate order flow y_n is available only to the market makers before the price p_n is determined and quoted to the traders.

However, due to the fact that there is—at least in the type of equilibria studied here—a one-to-one mapping from aggregate order flows to prices, the strategic traders learn the realized aggregate order flow *after* observing p_n. Nonetheless, the traders do not have this information available upon placing orders for period n. Hence, at any $n = 1, ..., N$, no one type of market participant is in possession of all the price relevant information. More specifically, denoting I's and R's information sets at time n by $\boldsymbol{\sigma}(\tilde{v}, y_1, ..., y_{n-1})$ and $\boldsymbol{\sigma}(\tilde{a}, y_1, ..., y_{n-1})$ respectively, it can be seen that the information sets of the different market participants are not nested.[45]

All in all, the model proceeds as follows. Periods $n = 0$ (the initial period) and $n = N + 1$ (the terminal period) are special. At the initial period, private information is observed, and no trading occurs, and at $N + 1$ all uncertainty is lifted and the terminal value of the risky asset is revealed. Otherwise, all trading rounds, $n = 1, ..., N$, are identical regarding the sequence of actions: the traders submit market orders, after which the market makers determine the price for the current period.

5.1.2 Strategies and equilibrium concept

Let $\Delta\theta_n^I$ and $\Delta\theta_n^R$ denote respectively the informed trader's and the rebalancer's order at time n. Similarly, let θ_n^I and θ_n^R represent the total risky asset position at time n for the informed trader and the rebalancer respectively. Observe that, $\sum_{n=1}^{N} \Delta\theta_n^R =: \theta_N^R =$

[45] Generally speaking, nested information sets simplify the analysis. See, for example, Foster and Viswanathan (1994).

$\tilde{a}$, where the last equality follows from the terminal constraint of the rebalancer. All initial positions are assumed to be zero, i.e., $\Delta\theta_0^I = \Delta\theta_0^R = 0$.

Utilizing the notation introduced above, the aggregate order flow can be written $y_n = \Delta\theta_n^I + \Delta\theta_n^R + \tilde{u}_n$, with $y_0 = 0$. Due to the risk neutrality of the market makers and the assumption of competitive pricing, the market makers earns zero expected profits and sets prices according to:[46]

$$p_n = \mathbb{E}\Big[\tilde{v} \mid \boldsymbol{\sigma}_n^M\Big] = \mathbb{E}_n^M\Big[\tilde{v}\Big], \text{ for all } n = 1, 2, ..., N, \tag{17}$$

where $\boldsymbol{\sigma}_n^M := \boldsymbol{\sigma}(y_1, ..., y_n)$ is the information set of M, along with the initial condition $p_0 = 0$. As with the traders' orders, define $\Delta p_n := p_n - p_{n-1}$.

The informed trader seeks to maximize:

$$\mathbb{E}\Big[U^I\Big(\sum_{n=1}^{N}(\tilde{v} - p_n)\Delta\theta_n^I\Big) \mid \boldsymbol{\sigma}(\tilde{v})\Big] = \mathbb{E}_0^I\Big[U^I\Big(\sum_{n=1}^{N}(\tilde{v} - p_n)\Delta\theta_n^I\Big)\Big], \tag{18}$$

where $\Delta\theta_n^I$ is $\boldsymbol{\sigma}(\tilde{v}, y_1, ..., y_{n-1}) =: \boldsymbol{\sigma}_n^I$ measurable for all $n = 1, 2, ..., N$. Note especially that $\boldsymbol{\sigma}_0^I := \boldsymbol{\sigma}(\tilde{v})$. The measurability constraint is the only explicit running constraint imposed on I's (similarly R's) optimal orders (controls).

Similarly, the rebalancer seeks to maximize:

$$\begin{aligned}
&\mathbb{E}\Big[U^R\Big((\tilde{a} - \theta_{N-1}^R)(\tilde{v} - p_N) + \Delta\theta_{N-1}^R(\tilde{v} - p_{N-1}) + ... + \Delta\theta_1^R(\tilde{v} - p_1)\Big) \mid \boldsymbol{\sigma}(\tilde{a})\Big] \\
&= \mathbb{E}\Big[U^R\Big(\tilde{a}(\tilde{v} - p_N) - \theta_{N-1}^R(\tilde{v} - p_N) + \Delta\theta_{N-1}^R(\tilde{v} - p_{N-1}) + ... + \Delta\theta_1^R(\tilde{v} - p_1)\Big) \mid \boldsymbol{\sigma}(\tilde{a})\Big] \\
&= \mathbb{E}\Big[U^R\Big(\tilde{a}\tilde{v} - \sum_{n=1}^{N}(\tilde{a} - \theta_{n-1}^R)\Delta p_n\Big) \mid \boldsymbol{\sigma}(\tilde{a})\Big] = \mathbb{E}_0^R\Big[U^R\Big(\tilde{a}\tilde{v} - \sum_{n=1}^{N}(\tilde{a} - \theta_{n-1}^R)\Delta p_n\Big)\Big],
\end{aligned} \tag{19}$$

where the last equality follows once it is noted that $p_N = \sum_{n=1}^{N} \Delta p_n$, with $p_0 = 0$, $\theta_L^R = \sum_{l=1}^{L} \Delta\theta_n^R$ for $L \leq N$, and where the measurability condition dictates that $\Delta\theta_n^R \in \boldsymbol{\sigma}(\tilde{a}, y_1, ..., y_{n-1}) =: \boldsymbol{\sigma}_n^R$ with $\boldsymbol{\sigma}_0^R := \boldsymbol{\sigma}(\tilde{a})$. Only risk neutral traders are examined here, so $U^K(x) = x$, for $K \in \{I, R\}$. It is possible to extend the analysis to a risk averse setting via negative exponential utility functions; this extension is pursued in Hannula (2019).

The equilibrium concept, following Kyle (1985) and the references therein, utilized is *Bayesian Nash* equilibrium. The formal definition is:

Definition 11 (Bayesian Nash Equilibrium (BNE)). *A collection of functions* $(\theta_n^I, \theta_n^R, p_n)$ *is a BNE if:*

(i) p_n *satisfies (17) given* (θ_n^I, θ_n^R),

(ii) θ_n^I *maximizes (18) (informed trader's utility) given* (θ_n^R, p_n),

[46] One may, as is usual in the literature, assume that the market makers are subject to Bertrand competition and, as a result, (17) holds.

(iii) θ_n^R *maximizes (19) (rebalancer's utility) given* (θ_n^I, p_n).

Choi et al. (2019) show that equilibrium strategies have a simple linear structure:

$$\Delta\theta_n^I = \beta_n^I(\tilde{v} - p_{n-1}), \tag{20}$$
$$\Delta\theta_n^R = \alpha_n^R q_{n-1} + \beta_n^R(\tilde{a} - \theta_{n-1}^R). \tag{21}$$

5.1.3 Information structure and filtering

A key component of the Choi et al. (2019) model is endogenous learning of the strategic traders and the market makers, and hence the next task is to briefly describe how this endogenous learning or filtering takes place in the model and how the beliefs of the market participants are formed. Starting with the market makers, the essential quantities are:

$$p_n = \mathbb{E}_n^M[\tilde{v}], \tag{22}$$
$$q_n = \mathbb{E}_n^M[\tilde{a} - \theta_n^R], \tag{23}$$

where equation (22) represents the market makers' period n risky asset valuation and equation (23) their belief about the remaining trading demand of the rebalancer. These quantities, for all $n = 1, ..., N$, adhere to the following dynamics:

$$\Delta p_n = \lambda_n\Big(y_n - \mathbb{E}_{n-1}^M[y_n]\Big), \tag{24}$$
$$\Delta q_n = r_n\Big(y_n - \mathbb{E}_{n-1}^M[y_n]\Big) - \alpha_n^R q_{n-1}, \tag{25}$$

where $\{\lambda_n\}_{n=1,...,N}$ and $\{r_n\}_{n=1,...,N}$ are projection coefficients. Moreover, straightforward manipulations show that:[47]

$$\mathbb{E}_{n-1}^M[y_n] = \underbrace{Q_n^R q_{n-1},}_{\text{Sunshine trading component}} \tag{26}$$

where $Q_n^R := (\alpha_n^R + \beta_n^R)$, $n = 1, ..., N$, is a constant stemming from the orders of R. Clearly, prices are unaffected by the (predictable) *sunshine trading component*.

Moving on to the strategic traders, one should note that due to the presence of heterogenous (asymmetric) information, both R and I try to learn about the information possessed by the other. The relevant quantity for I is:

$$\begin{aligned}\mathbb{E}_n^I[\tilde{a} - \theta_{n-1}^R] &= q_{n-1} + \mathbb{E}_n^I[\tilde{a} - \theta_{n-1}^R - q_{n-1}] \\ &= q_{n-1} + \mathbb{E}[\tilde{a} - \theta_{n-1}^R - q_{n-1} \mid \boldsymbol{\sigma}(\tilde{v} - p_{n-1}],\end{aligned} \tag{27}$$

and for R:

$$\mathbb{E}_n^R[\tilde{v} - p_{n-1}] = \mathbb{E}[\tilde{v} - p_{n-1} \mid \boldsymbol{\sigma}(\tilde{a} - \theta_{n-1}^R - q_{n-1})]. \tag{28}$$

[47] Note that the expectation is conditional on $\boldsymbol{\sigma}(y_1, ..., y_{n-1})$.

The quantity given in (27) is I's expectation of R's remaining trading demand and (28) gives R's view on the misvaluation of the risky asset. Therefore, the privately informed traders can utilize their private information to endogenously filter new information from observed prices in a superior fashion as compared to the market makers. This learning mechanism is the main channel driving the trading dynamics in the model.

The above filtering problems give rise to the following (co)variances:

$$\Sigma_0^{(1)} = \sigma_a^2, \qquad \Sigma_n^{(1)} = \mathbb{V}\Big[\tilde{a} - \hat{\theta}_n^R - \hat{q}_n\Big], \tag{29}$$

$$\Sigma_0^{(2)} = \sigma_v^2, \qquad \Sigma_n^{(2)} = \mathbb{V}\Big[\tilde{v} - \hat{p}_n\Big], \tag{30}$$

$$\Sigma_0^{(3)} = \rho_{av}\sigma_a\sigma_v, \quad \Sigma_n^{(3)} = \mathbb{COV}\Big[\tilde{a} - \hat{\theta}_n^R - \hat{q}_n,\ \tilde{v} - \hat{p}_n\Big], \quad n = 1, ..., N, \tag{31}$$

where $\mathbb{V}[\,\cdot\,]$ represents variance and $\mathbb{COV}[\,\cdot\,]$ represents covariance. The constants λ_n and r_n appearing above are the projection coefficients of the market makers given by:

$$\lambda_n = \frac{\mathbb{COV}\Big[\tilde{v} - \hat{p}_{n-1}, \hat{z}_n^M\Big]}{\mathbb{V}\Big[\hat{z}_n^M\Big]}, \tag{32}$$

$$r_n = \frac{\mathbb{COV}\Big[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}, \hat{z}_n^M\Big]}{\mathbb{V}\Big[\hat{z}_n^M\Big]} \times \Big(1 - \beta_n^R\Big), \tag{33}$$

where $\hat{z}_n^M := \hat{y}_n - \mathbb{E}\Big[\hat{y}_n \,\Big|\, \boldsymbol{\sigma}(\hat{y}_1, ..., \hat{y}_{n-1})\Big]$. The projection coefficients of I and R are given by $\frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}$ and $\frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}$ respectively.

One might wonder about the appearance of "hatted" expressions, e.g. $\hat{\theta}_n^R$, in equations (29)-(33). Before proceeding to discuss the meaning of these new expressions, it is good to note that heterogeneous information introduces an additional layer of complexity to learning dynamics and belief formation. Further, the description of the relevant learning quantities given above implicitly assumes that all traders strictly obey their equilibrium strategies. This need not be the case, and in order to obtain a complete view regarding the equilibrium learning dynamics, one must consider the possibility that a trader deviates from the equilibrium path. Matters are, however, somewhat simplified by the fact (see, Foster and Viswanathan 1996) that since all order flows are observed with non-zero probability, there is no need to describe off-equilibrium-path beliefs.

To account for deviations from the equilibrium path, suppose R, instead of applying the equilibrium strategy, has submitted arbitrary orders during the l initial trading rounds, and define hatted versions of y_n, $\Delta\theta_n^R$, and $\Delta\theta_n^I$ as well as other quantities given above, i.e., $\hat{y}_n = \Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R + \tilde{u}_n$. These hatted processes describe what would have happened if R had instead followed her equilibrium strategy.[48] In other words,

[48] When solving the optimization problem of R, the same hatted processes are not used as inputs in the strategy

the hatted processes capture the beliefs of R regarding what I is able to learn from the trading history under the assumption that R utilizes her equilibrium strategy. A standard rational expectations argument dictates that the hatted processes must equal the unhatted ones in equilibrium, i.e., $\hat{y}_n = y_n$.

In order for the hatted processes to be useful, it is essential to verify that $\hat{y}_l \in \boldsymbol{\sigma}(\tilde{\xi}, y_1, ..., y_l)$ for all $1 \le l \le N$, where $\tilde{\xi} \in \{\tilde{a}, \tilde{v}\}$. That is to say, it should hold that $\boldsymbol{\sigma}(\tilde{\xi}, \hat{y}_1, ..., \hat{y}_n) = \boldsymbol{\sigma}(\tilde{\xi}, y_1, ..., y_n)$. This property is important and useful as it guarantees the viability of certain orthogonality properties which are utilized, for instance, in determining the state variable dynamics for both R and I. For completeness, Appendix C gives additional details.

Concerning the aforementioned state variables, it is shown in Choi et al. (2019) that the following, for $n = 1, ..., N$, are enough to capture the equilibrium relevant state information:

$$
\begin{aligned}
X_n^{(1)} =& \tilde{v} - p_n, \\
X_n^{(2)} =& (\hat{\theta}_n^R - \theta_n^R) + (\hat{q}_n - q_n) + \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}(\tilde{v} - \hat{p}_n) \overset{\text{in eq.}}{=} \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}} X_n^{(1)}, \\
Y_n^{(1)} =& \tilde{a} - \theta_n^R, \\
Y_n^{(2)} =& (\hat{p}_n - p_n) + \frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}(\tilde{a} - \hat{\theta}_n^R - \hat{q}_n) \overset{\text{in eq.}}{=} \frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}(Y_n^{(1)} - Y_n^{(3)}) \\
Y_n^{(3)} =& q_n,
\end{aligned}
$$

where $X_n^{(i)}$, $i = 1, 2$ are state variables for I and $Y_n^{(j)}$, $j = 1, 2, 3$ are state variables for R, and the latter equalities in the second and the fourth equation hold in equilibrium ("in eq."). Using these state variables one can show that the value functions of the strategic traders are quadratic in nature, and that from the first order conditions for the traders' optimal orders one obtains two coupled equations (recall equations (20) and (21)):

$$
\beta_n^I = \gamma_n^{(1)} + \gamma_n^{(2)} \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}}, \tag{34}
$$

$$
\beta_n^R = \delta_n^{(1)} + \delta_n^{(2)} \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}}. \tag{35}
$$

These two equations (34) and (35) can be manipulated via repeated substitutions so that one obtains a bivariate polynomial system in (β_n^I, β_n^R), at each $n = 1, ..., N-1$. Solving this bivariate β-equation system constitutes the part of the model solution to

of I as this would mean that the strategy chosen by I would be independent of the actual strategy of R. This contradicts the premise that the traders act strategically and utilize all the information available to them (cf. Foster and Viswanathan 1996 and Choi et al. 2019).

which all-solutions methods are applied. In particular, the above system has multiple (real) solutions and in order to find out how many of these solutions satisfy the equilibrium conditions (see, Choi et al. 2019, Thm. 1), one first needs to locate all of them.

5.2 Examining Equilibrium Uniqueness Numerically

To fix ideas it is reasonable to start with, what is actually meant by equilibrium uniqueness. First, it is important to emphasize that the discussion is limited to the linear strategies (20)-(21). There is some reason to expect that the linear strategies are unique even if nonlinear strategies are allowed but proving this is highly involved.[49] Thus, one is left with the question of uniqueness under linear strategies.

To better understand this question, it is useful to first digress slightly and consider the uniqueness of the dynamic linear strategy equilibrium in Kyle (1985). Related to the above, iterating the Kyle (1985) model backwards, subject to a single initial guess Σ_N and some boundary conditions, also boils down to solving a polynomial equation.[50] This *univariate* polynomial equation can be formulated using λ_n and it is easy to see that the resulting equation is cubic. Among the three real solutions to this cubic equation, only one meets the second order condition of the Kyle (1985) model and hence λ_n is determined uniquely. Substituting the unique λ_n to other model equations directly gives the values for the remaining equilibrium constants. Further, it is shown in Kyle (1985) that there is a proportional relationship between the initial guess Σ_N (model input) and initial variance Σ_0 (model constant) so that the resulting equilibrium path indeed is unique.

This simple second order condition approach to determining the uniqueness of the equilibrium path is insufficient for the Choi et al. (2019) model even in the case of linear strategies. The reason for this lies in equations (34)-(35) which can produce multiple solution pairs able to pass the second order condition test along with other equilibrium conditions.[51] In the rest of this section, using a two period version of Choi et al. (2019), the insufficiency of the second order condition is demonstrated numerically.

The reason for choosing $N = 2$ is straightforward. As the last period N is special, in the two period model one is required to solve the β-equations only once. Hence, this simple case it the easiest to discuss from the point of view of uniqueness of equilibria.

[49] For a discussion pertaining to the uniqueness of the linear equilibrium in the static (one period) Kyle (1985) model see, for example, Boulatov et al. (2013) and McLennan et al. (2017).

[50] It is worth emphasizing that in this case one has to deal with a *single* polynomial equation not a system which simplifies matters considerably in terms of both analytical and numerical methods.

[51] Note that potential repeated roots are counted only once, i.e., only distinct roots are taken into account when studying equilibrium multiplicity.

The numerical algorithm for finding a Bayesian-Nash equilibrium for the above described model is given in Appendix D. In short, the algorithm takes as inputs initial guesses for $\Sigma^{(1)}_{N-1}$, $\Sigma^{(2)}_{N-1}$, and $\Sigma^{(3)}_{N-1}$ as well as the preset values for a number of model constants and then proceeds via backward induction from the last period $n = N$ towards period $n = 1$. At each period, except the last (N), the pair of equations (34)-(35) is solved and the solved values are plugged into auxiliary equations to obtain the numerical values for the other model parameters. The algorithm terminates when the moments $\Sigma^{(i)}_0$, $i = 1,2,3$, produced by the algorithm match the values specified for these constants ex ante (see, (29)-(31)).

Having covered the basics, it is time to move on to an explicit example. To start, fix parameter values:

Parameter	**Numerical value**
σ^2_a	1.000
σ^2_v	1.000
σ^2_u	0.100
ρ_{av}	0.000

The next step is to look at the uniqueness of the solutions in period $n = N = 2$, i.e., the last period.

To obtain the period $n = 2$ unknowns (β^I_N, λ_N) one solves a pair of equations of a much simpler form compared to the β-equations. To see this, first plug the expression for β^I_N obtained from I's period N problem, namely:

$$\beta^I_N = \frac{1}{2}\Big(\frac{1}{\lambda_N} - \Sigma^{(3/2)}_{N-1}\Big), \tag{36}$$

to the expression for λ_N, namely:

$$\lambda_N = \frac{\beta^I_N \Sigma^{(2)}_{N-1} + \Sigma^{(3)}_{N-1}}{(\beta^I_N)^2 \Sigma^{(2)}_{N-1} + \Sigma^{(1)}_{N-1} + 2\beta^I_N \Sigma^{(3)}_{N-1} + \sigma^2_u},$$

whence one obtains a standard form quadratic equation:

$$a\lambda^2_N + b\lambda_N + c = 0,$$

with $b = 0$ and $c = -\frac{\Sigma^{(2)}_{N-1}}{4}$. The expression for a is slightly more complicated and less relevant for the discussion at hand. Combining the above observations, one attains:

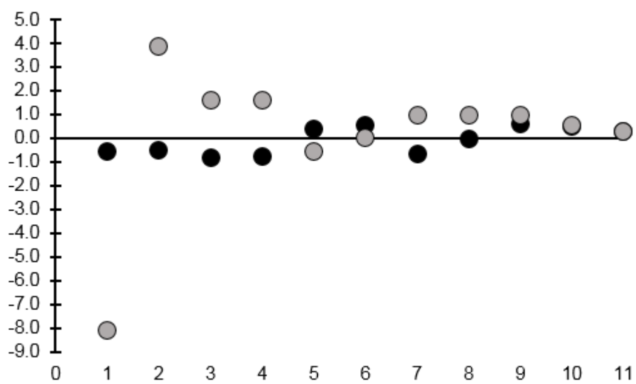
$$\lambda_N = \pm\frac{1}{2}\sqrt{\frac{\Sigma^{(2)}_{N-1}}{a}}.$$

By the second order condition $\lambda_N \in \mathbb{R}_{>0}$, and therefore one can deduce that λ_N is solved uniquely. Furthermore, since λ_N uniquely determines β_N^I via (36), the pair (λ_N, β_N^I) is uniquely determined. Appendix E gives an alternative take on the period N solution.

Note also that if λ_N is allowed to be negative, then it may hold that $\beta_N^I < 0$ which would mean that I would trade in the direction opposite to his private information.[52] Pure strategy trades of this kind cannot be used to manipulate the beliefs of the market makers due to the fact that the market could perfectly anticipate these manipulation efforts. Therefore, this type of equilibrium would lack any sort of economic intuition.

Then, moving to period $n = 1$ one faces the task of finding all (real) solutions to the full bivariate β-equations obtained using (34) and (35). For this purpose, the MATLAB version of the PHCpack, taking advantage of polynomial homotopy continuation, is utilized. To double-check results, the β-equations are also solved using MATHEMATICA's *NSolve*, which tackles the problem by computing the numerical Gröbner basis and exploiting eigensystem methods to extract all roots.

The first task after obtaining the entire solution set $\mathcal{S} \subset \mathbb{C}^2$ is to isolate the real solution pairs, after which the feasibility of the remaining solution pairs is checked against the equilibrium conditions. Those pairs that do not meet the equilibrium conditions are discarded. In terms of uniqueness one wishes to end up with a single solution pair or none if the initial guess is incorrect. The set of real solution pairs for the present case, obtained by simply removing all complex solutions, is presented in Figure 8.



	β_1^I	β_1^R
1.	−0.535	−8.058
2.	−0.503	3.880
3.	−0.790	1.597
4.	−0.758	1.624
5.	0.392	−0.539
6.	0.574	0.011
7.	−0.633	1.000
8.	0.633	1.000
9.	−0.002	1.000
10.	0.519	0.542
11.	0.307	0.319

Figure 8: (**Real solutions to the β-equations**) This figure presents the real β-solutions for the two period example with $\sigma_u^2 = 0.1$.

From the solutions pairs obtained, pairs 1–6 fail the second order condition and

[52] It is typically the case in numerical tests that $\Sigma_n^{(3)} \leq 0$ for $n = 1, ..., N$.

pairs 7–9 fails the proviso that $\beta_n^R \neq 1$, for $n = 1, ..., N-1$ (see, Choi et al. 2019). In order to gain better intuition regarding the proviso, Figure 9 illustrates what happens to the market makers' projection coefficients in the two period model when $\beta_1^R \to 1$.

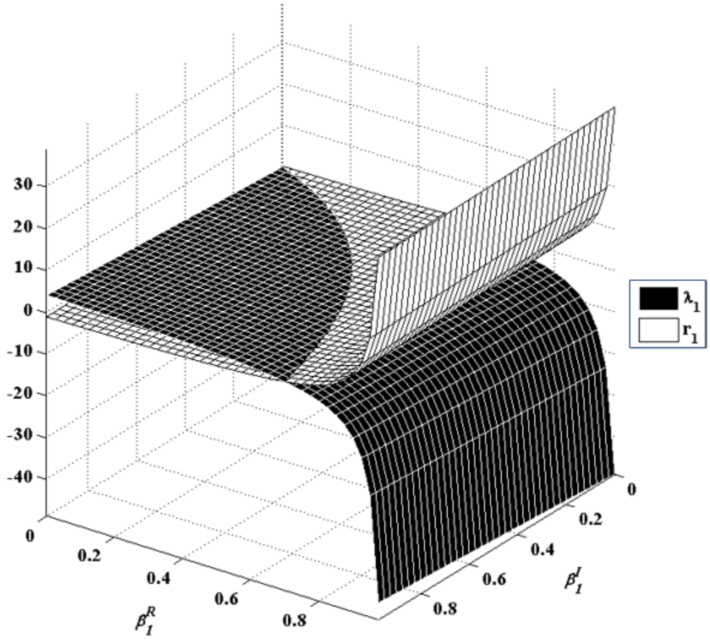


Figure 9: (**Projection coefficient surfaces**) This figure depicts the values taken by the projection coefficients λ_1 and r_1 when the pair (β_1^I, β_1^R) is allowed to vary on the plane $[0,1] \times [0,1)$.

The last two solution pairs (10. and 11.) in Figure 8 both satisfy all equilibrium conditions and one cannot discard neither of them based on, for instance, the second order condition as in Kyle (1985). Does this mean there are two equilibria in this case?

The answer is *no*. Looking at the initial moments implied by the two solutions sheds light on this. Denote by $\hat{\Sigma}_0^{(i)}$, $i = 1,2,3$, the implied initial solutions and recall that the fixed initial parameter values are: $\Sigma_0^{(1)} = \sigma_a^2 = 1$, $\Sigma_0^{(2)} = \sigma_v^2 = 1$, and $\Sigma_0^{(3)} = \rho\sigma_v\sigma_a = 0$. Initial moments implied by the two remaining solutions are given below.

$\hat{\Sigma}_0^{(1)}$	$\hat{\Sigma}_0^{(2)}$	$\hat{\Sigma}_0^{(3)}$
1.000	1.000	0.001
0.247	0.713	−0.294

Thus, recalling that $\rho_{av} \in [0,1]$ and therefore $\Sigma_0^{3)} \geq 0$, the latter solution pair (0.307, 0.319) can be discarded and (0.519, 0.542) remains as the unique period $n = 1$ solution pair. Alternatively, one could have discarded (0.307, 0.319) based on the fact that the implied moments are hardly the same as the fixed initial moments.

However, noting that this solution also violates the assumption regarding ρ_{av}, means that it cannot be considered an equilibrium in any case.

In the above example, a numerically unique equilibrium is obtained in a rather unambiguous fashion. Things are not always so straightforward. In fact, it would be surprising if they were. Returning for a while to real solutions to the β-equations at $n = 1$, Figure 10 highlights the two solutions which pass all the standard equilibrium conditions.

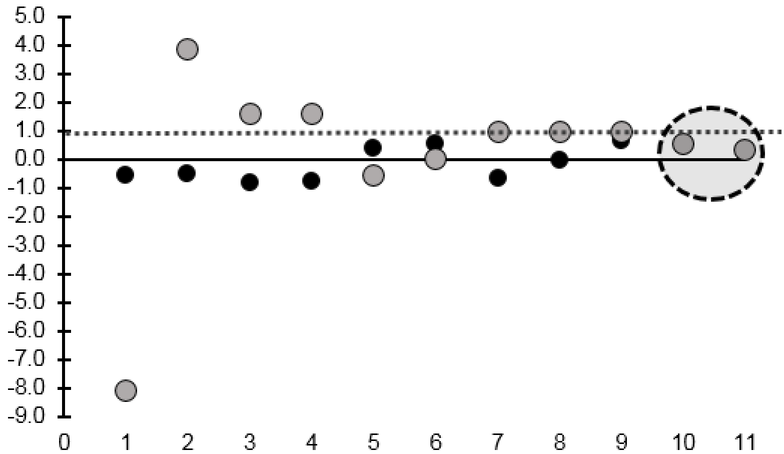


Figure 10: (**Real solutions to the β-equations revisited**) This figure presents the real β-solutions for the two period example with $\sigma_u^2 = 0.1$. Solutions passing the standard equilibrium conditions are circled. The dotted line represents the proviso $\beta_1^R \neq 1$.

The main take way from Figure 10 is that to "succeed" the $n = 1$ solution pairs (1) need to be relatively close to each other in terms of magnitude[53] and (2) they must also, in general, be of correct magnitude with respect to other model parameters, i.e., too small or large solutions do not pass the second order conditions.

Now, noting that the above example represents just one possible model parametrization, one might be skeptical with respect to *global uniqueness*, i.e., uniqueness over the whole range of feasible parametrizations. Indeed, one needs only to look at the case where (ceteris paribus) $\sigma_u^2 = 0.5$ to observe that the above example does not cover all possible cases. Indeed, in this new case, after trimming out all solutions that do not meet the equilibrium conditions, one is left with:

$$1.\ \left(\beta_1^I = 0.592,\ \beta_1^R = 0.588\right) \implies \hat{\mathbf{\Sigma}}_0 = \left(17.700,\ 2.383,\ 4.942\right)\ \&\ \rho_{av} = 0.761,$$

$$2.\ \left(\beta_1^I = 0.592,\ \beta_1^R = 0.459\right) \implies \hat{\mathbf{\Sigma}}_0 = \left(0.999,\ 1.001,\ 0.002\right)\ \&\ \rho_{av} = 0.000.$$

53 Note the almost perfect overlap.

Now, even though the first solution pair seems dubious, one cannot rule it out based on the same principles applied to the case where $\sigma_u^2 = 0.1$.[54] In fact, it could be interpreted as a market where the variance σ_a^2 and σ_v^2 are of utterly different magnitude compared to σ_u^2. This issue may be pondered from the point of view of *relative volume* (cf. Moallemi et al. 2012), and in the particular situation portrayed above, the assumed amount of noise trading is hardly enough to conceal the private information that motivates the trades of I and R respectively. Hence, almost all uncertainty is lifted after just one round of trading.

Theoretically, the situation depicted by the first solution pair must then be considered as possible. However, to be realistic, it is unlikely. Instead one would, for example, expect that R would seek to split her trading to various different markets so that at each individual market her trading target would not represent such a dominant part of the total trading volume. This is especially true in today's more fragmented market where a myriad of new exchanges and trading systems have risen and volume from old exchanges has drifted to these alternative venues (cf. Angel et al. 2015).

Therefore, from an economic viewpoint, only the second solution pair seems to imply a sensible equilibrium outcome which is, in addition, consistent with the initial parameter values $\sigma_a^2 = \sigma_v^2 = 1$. However, one could ask whether there is an alternative (economically meaningful) solution to the parametrization $\sigma_u^2 = 0.500$ and $\sigma_a^2 = 17.700$, $\sigma_v^2 = 2.383$, $\rho_{av}\sigma_a\sigma_v = 4.942$? Generally speaking, it is reasonable to expect that among all possible initial parametrizations there are ones that are more susceptible to equilibrium multiplicity. Is there a way to locate or to characterize these parametrizations based on the equilibrium conditions and equations? This question and other further inquiries into the equilibrium numerics are left for future research.

5.3 Related Applications of All-Solutions Methods

In the preceding numerical example, the focus was on the aspects of numerical uniqueness and the central goal was to verify numerically—at least for specific examples from the entire parameter space—that the equilibrium obtained was unique. An alternative use for the all-solutions methods, polynomial homotopy continuation and Gröbner bases, relates to verifying numerically whether or not there exists an equilibrium under some conditions.

The key idea here would be that algorithms, which focus on finding a single solution, might end up converging to a solution which does not support equilibrium existence. The advantage of finding all solutions lies in the fact that one can then examine the whole solution set at once and determine whether a feasible solution

[54] It is good to note that β_1^I has the same value in both cases and only the value β_1^R changes. This essentially means that, fixing β_1^I, the remaining equations have, at least, two distinct solutions.

belongs to that set.

Similar methods as introduced above can naturally be applied to, for example, the risk-averse extension of Choi et al. (2019) as illustrated in Hannula (2019). The solution method in the risk averse case is for the most part similar to the one utilized in the risk neutral case, although the coupled polynomial equations tend to be more complicated. However, the problem of potential equilibrium multiplicity due polynomial systems characterizing key model restrictions is by no means limited only to discrete-time dynamic models.

A case in point is Marinovic and Varas (2018) in which a blockholder (activist investor) has the ability to both trade strategically and to control firm value via costly effort. Their model features a continuous-time Gaussian linear quadratic model with CARA-preferences, and due to the emergence of coupled optimization problems between the blockholder and a set of market makers, combined with a feedback loop between stock price and firm productivity, multiple equilibria may arise. Indeed, like Choi et al. (2019), equilibrium multiplicity in Marinovic and Varas (2018) boils down to analyzing a system of polynomial equations. This observation supports the notion—and a major motivation for the present paper—that as strategic (trading) models become more complicated—in both discrete- and continuous-time—one is more likely to have to deal with potential equilibrium multiplicity.

In addition to the above discussed "partial equilibrium" examples, there are several other examples of using homotopy continuation methods and Gröbner bases in general equilibrium analysis. Recently, Amisano and Tristani (2009) and Amisano and Tristani (2011) utilize polynomial homotopy continuation in computing exact likelihoods for nonlinear dynamic stochastic general equilibrium (DSGE) models solved using second order approximations and Robatto (2019) takes advantage of Gröbner bases in dealing with multiple equilibria in a general equilibrium banking model. Other applications to general equilibrium models include Schmedders (1998) and Schmedders (1999).

6 CONCLUSION

This paper overviews selected methods from the field of (numerical) algebraic geometry, which has recently undergone rapid progress, and points out a few promising applications of these methods in finance. Examples selected are closely related to strategic trading and cover both theoretical and practical viewpoints. Explicit numerical examples—including many new findings—are given for optimal execution problems, mean-variance-skewness portfolio optimization, and determining equilibrium uniqueness in a dynamic strategic trading model.

The purpose of the examples is to both illustrate the applicability of the introduced methods and to provide some unique insights to widely recognized problems in the field of finance. First, in the case of optimal execution, it is illustrated how polynomial programming provides an effective way to optimize trading under non-linear price impact and trading-enhanced risk. Moreover, it is shown how one can parsimoniously formulate a problem of optimal execution of contingent claims as a polynomial optimization problem (POP).

Second, a portfolio selection example demonstrates that even if standard performance measures, such as the Sharpe ratio and asymmetric Sharpe ratio, cannot clearly differentiate between mean-variance (MV) optimal and mean-variance-skewness (MVS) optimal portfolios, these two portfolios can still have very different properties. In particular, it is documented that the portfolio turnover is consistently lower for the MVS portfolio. This result has practical implications for trading in the presence of transaction costs.

Third, a strategic trading example shows how one can numerically study equilibrium uniqueness in a complex dynamic model using all-solutions methods such as polynomial homotopy continuation and Gröbner bases. After providing an overview of the model under study, a detailed discussion pertaining to the simplest dynamic model, i.e., the two period model, is given. Discussion is supplemented with numerical examples, shedding light on various factors crucial for the existence of a unique (linear) equilibrium.

Finally, multiple possible venues for further research are identified and several other potential application areas are proposed. With the recent development in algebraic geometry combined with the advances in, for example, high-performance computing and semidefinite programming, many problems previously out of reach are now open for inquiry. Accordingly, additional research towards uncovering the full potential of algebraic geometry-based methods in the field of finance is certainly advisable.

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APPENDICES

A POLYNOMIAL OPTIMIZATION DETAILS

In Appendix A details pertaining to polynomial optimization are presented. The representation follows Henrion and Lasserre (2005) and Lasserre (2015).

Consider the following problem for $f \in \mathbb{R}[\mathbf{x}]$:

$$f^* := \inf_{\mathbf{x}} \Big\{ f(\mathbf{x}) : \mathbf{x} \in \mathbb{S} \Big\}, \tag{37}$$

where the feasible (semialgebraic) set is given by:

$$\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^n \ : \ g_j(\mathbf{x}) \geq 0, \quad j = 1, \ldots, m\}.$$

Moreover, suppose $\mathbb{S}$ is compact and a technical Archimedean condition holds.[55]

Fix $g_0 = 1$, $v_j := \left\lceil \mathrm{deg(g_j)}/2 \right\rceil$, $j = 0, \ldots, m$, $d_0 := \max\{\lceil \mathrm{deg(f)}/2 \rceil, \max_{j=1,\ldots,m} v_j\}$, and consider the following (convex) linear matrix inequality (LMI; see, for example, Boyd et al. 1994) relaxation of order d $(= d_0, d_0 + 1, \ldots)$:[56]

$$\begin{aligned} \rho_d = & \inf_{\mathbf{y}} \ L_{\mathbf{y}}(f) \\ \text{s.t.} \quad & \mathbf{M}_d(\mathbf{y}) \succeq \mathbf{0}, \\ & \mathbf{M}_{d-v_j}(g_j \mathbf{y}) \succeq \mathbf{0}, \ j = 1, \ldots, m \\ & y_{\mathbf{0}} = 1, \end{aligned}$$

where $L_{\mathbf{y}}$ is the Riesz linear functional:

Definition 12 (**Riesz linear functional**). *For a real sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}}) \subset \mathbb{R}$, denote by $L_{\mathbf{y}} : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ the Riesz linear functional given by:*

$$f \mapsto L_{\mathbf{y}}(f) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} f_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}},$$

$\mathbf{M}_d(\mathbf{y})$ is the (order d) positive semidefinite *moment matrix* defined as:

Definition 13 (**Moment matrix**). *Given a sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}})$, the positive definite moment matrix with $\boldsymbol{\alpha} \in \mathbb{N}^n_d$ used to label rows and columns is defined by:*[57]

$$\mathbf{M}_d(\mathbf{y})(\boldsymbol{\alpha}, \boldsymbol{\beta}) = L_{\mathbf{y}}(\mathbf{x}^{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\beta}}) = y_{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \quad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n_d,$$

and $\mathbf{M}_{d-v_j}(g_j \mathbf{y})$ is the (order $d - v_j$) positive semidefinite *localizing matrix* related to polynomial g_j, for all $j = 1, \ldots, m$. More specifically, the localizing matrix is defined as:

---

55 In short, the Archimedean condition requires that there exists $u \in Q(g) = Q(g_1, \ldots, g_m)$, for $(g_j)_{j=1}^m \subset \mathbb{R}[\mathbf{x}]$, such that the set $\{\mathbf{x} \in \mathbb{R}^n \ : \ u(\mathbf{x}) \geq 0\}$ is compact. Above, $Q(g)$ is the so-called quadratic module and it is defined by:

$$Q(g) = Q(g_1, \ldots, g_m) := \left\{ q_0 + \sum_{j=1}^m q_j g_j : (q_j)_{j=0}^m \subset \Sigma[\mathbf{x}] \right\},$$

where $\Sigma[\mathbf{x}]$ is the set of SOS polynomials (recall, Definition 3).

56 The ceiling function is given by: $\lceil z \rceil = \min\{k \in \mathbb{N} : \ k \geq z\}$.

57 Suppose $n = 2$ and $d = 1$, then $\mathbb{N}^2_1 = \{(0,0), (1,0), (0,1)\}$. Generally:

$$\mathbb{N}^n_d := \{\boldsymbol{\alpha} \in \mathbb{N}^n : \ |\boldsymbol{\alpha}| \leq d\},$$

where $|\boldsymbol{\alpha}| := \sum_{i=1,\ldots,n} \alpha_i$.

Definition 14 (**Localizing matrix**). *Given* $u \in \mathbb{R}[\mathbf{x}]$ *with coefficient vector* $\mathbf{u} = (u_{\boldsymbol{\gamma}})$*, the localizing matrix with respect to* $\mathbf{y}$ *and* u *is the matrix* $\mathbf{M}_d(u\mathbf{y})$ *given by:*

$$\mathbf{M}_d(u\mathbf{y})(\boldsymbol{\alpha},\boldsymbol{\beta}) = L_{\mathbf{y}}(u(\mathbf{x})\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}}) = \sum_{\boldsymbol{\gamma}\in\mathbb{N}^n} u_{\boldsymbol{\gamma}} y_{\boldsymbol{\gamma}+\boldsymbol{\alpha}+\boldsymbol{\beta}}, \quad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n_d,$$

where rows and columns of $\mathbf{M}_d(u\mathbf{y})$ *are indexed by* $\boldsymbol{\alpha} \in \mathbb{N}^n_d$*.*

The central idea in polynomial optimization, as implemented in GloptiPoly, is to solve a sequence $(\mathcal{R}_d)_{d=1,2,...}$ of LMI relaxations of the original problem (37). By doing this, under relatively mild assumptions, one obtains a sequence of optimal values $(\rho_d)_{d=1,2,...}$. Moreover, this sequence is nondecreasing and it can be shown (see, Lasserre 2008 for *asymptotic* convergence and Nie 2014 for *finite* convergence) to converge to the global optimal value f^*. For many relevant problems such as the MVS portfolio optimization discussed in Section 4, the convergence is immediate, i.e., relaxation order $d = 1$ is enough for obtaining the exact, numerically verified, global optimal solution(s).

One can also utilize special techniques to transform the original problem to simpler one to keep the required relaxation order as low as possible. This approach is demonstrated in Section 3.4. Generally, the main computational cost in polynomial optimization, using the above described approach, is due to solving the semidefinite optimization problem (SDP) $\mathcal{R}_d$. The size of this problem increases quickly with d which instead can pose a challenge for existing SDP solvers.

After solving, say, a relaxation $\mathcal{R}_d$ and obtaining an optimal solution $\mathbf{y}^*$, a special solution extraction algorithm is utilized to obtain the (list of) global minimizers. Description of this extraction algorithm is omitted here; for further details the reader is referred to Lasserre (2015, Ch. 6.1.2). Instead, the general algorithm for solving a semialgebraic constrained polynomial optimization problem is given below.

Algorithm 1: *Constrained Polynomial optimization*

Input: Objective polynomial $f(\mathbf{x})$, the (compact) semialgebraic feasible set
$\mathbb{K} := \{\mathbf{x} \in \mathbb{R}^n : \; g_j(\mathbf{x}) \geq 0, \; j = 1,...,m\}$, where $\deg(g_j) = 2v_j, \; 2v_j - 1$ (depending on parity),
$v := \max_j v_j, \; j = 1,...,m$, and the relaxation order $k \in \mathbb{N}$.

Output: Conditional on reaching the exact solution: the value $f^* = \inf_{\mathbf{x}\in\mathbb{K}} f(\mathbf{x})$ and the set of global minimizers. Otherwise: a lower bound ρ_k on f^*.

Solve the SDP given in (37), obtaining the optimal value ρ_d and, if the optimal solutions exists, also the optimal solution $\mathbf{y}^*$. If there is no optimal solution, the optimal value ρ_d provides a lower bound for f^*.

if $\operatorname{rank}(M_{d-v}(\mathbf{y}^*)) = \operatorname{rank}(M_d(\mathbf{y}^*))$ **then**
  $\rho_d = f^*$ and at least $\operatorname{rank}(M_d(\mathbf{y}^*))$ global minimizers can be extracted.
else if $\operatorname{rank}(M_{d-v}(\mathbf{y}^*)) \neq \operatorname{rank}(M_d(\mathbf{y}^*))$ and $d < k$ **then**
  Increase d by one and return to the first step.
else
  Rank condition does not hold and $d = k$. Stop and return ρ_k which provides a lower bound for f^*.
end

Suppose that $\mathbf{y}^*$ is an optimal solution of relaxation $\mathcal{R}_d$. Then, a *sufficient*, albeit not necessary, condition for global optimality is:

$$\operatorname{rank}(M_{d-v}(\mathbf{y}^*)) = \operatorname{rank}(M_d(\mathbf{y}^*)).$$

One may observe that this condition appears in Algorithm 1. From the perspective of numerics, one can use the *singular value decomposition* (SVD) to verify the global optimality condition (see, Lasserre 2015, Ch. 6.1.2).

One can also observe from Algorithm 1, how upon failure of the rank condition, one increases the relaxation order d and tries solving the new relaxation $\mathcal{R}_{d+1}$. In practice, however, increasing the relaxation order multiple times is often computationally impracticable as then the size of related SDP increases sharply. Even with the recent promising development in the field, solving a very large SDP can generate a notable bottleneck in the polynomial optimization procedure.

B ALGORITHM FOR PORTFOLIO REBALANCING

In Appendix B the simulation procedure for rolling (out-of-sample) portfolio optimization is described.[58] Additionally, the multivariate Gaussian mixture distribution is reviewed briefly.

Denote by $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ the specified preferences over the moments, i.e., the weights on expected returns, (co)variances, and (co)skewnesses respectively. Moreover, denote by T the sample size, the frequency of which can be determined freely, by N the number of rebalancing periods, and by $\mathbf{x}_0$ the initial weights in the portfolio prior to optimization.

Suppose, for instance, that a year's worth of trading days is used estimate the initial input moments and that $N = 12$, i.e., one year out-of-sample trading horizon with monthly rebalancing. Then, the size of the sample T with daily data is 2×252. Rolling estimation means that the first estimation window (or, the first training set) is 1–252, the second is (1+21)–(252+21), and so forth.

The portfolio optimization algorithm with rolling moment estimation is presented below.

Algorithm 2: *Rolling portfolio optimization via polynomial optimization*

Input: Sample size $T \in \mathbb{N}$, number of rebalancing periods N, initial weight vector $\mathbf{x}_0$, the level of transaction costs $\lambda \geq 0$, preferences $\boldsymbol{\phi}$ which specify the objective function (14), distribution specific parameters θ for simulation, and constraint (feasible) set $\mathcal{P}_{\mathbf{x}}$ for optimization.

Output: Realized return vector R_p, portfolio turnover TO, matrix $\mathbf{X}_p$ of stacked (optimal) portfolio weights over rebalancing periods $t_1, ..., t_N$, SR, and ASSR.

Draw a random sample of length T from a distribution specified by θ.

for ($l = 1$; $l = N$; $l = l + 1$)

> Evaluate the sample moments from the training data set for period t_l. Training data is updated on a rolling fashion for each t_l.
>
> Construct the objective function f using the in-sample moments, λ and $\boldsymbol{\phi}$. Construct the semialgebraic constraint set using $\mathcal{P}_{\mathbf{x}}$.
>
> Solve the polynomial optimization problem (14) and verify global optimality numerically.
>
> Save the optimal weight vector, calculate and save realized portfolio return and turnover.
>
> Determine and save the performance measures SR and ASSR.

endfor

Above the data is drawn from either a multivariate Gaussian or a multivariate Gaussian mixture distribution. In a finance context, mixture distributions are useful in, for example, modelling stereotypical empirical properties of financial return time series, such as negative skewness and positive excess kurtosis, and to capture the dynamics of potentially time-varying means and volatilities. Here, mixtures are used to ensure the simulated data has at least some of the empirical properties mentioned. To exemplify, two examples of simulated data together with a fitted normal distribution are given next.

[58] If the reader is wary about the use of simulated data, one can think of the procedure described in the spirit of the resampling procedure put forth by Jobson and Korkie (1981). See also Michaud and Michaud 2008, Ch. 6.

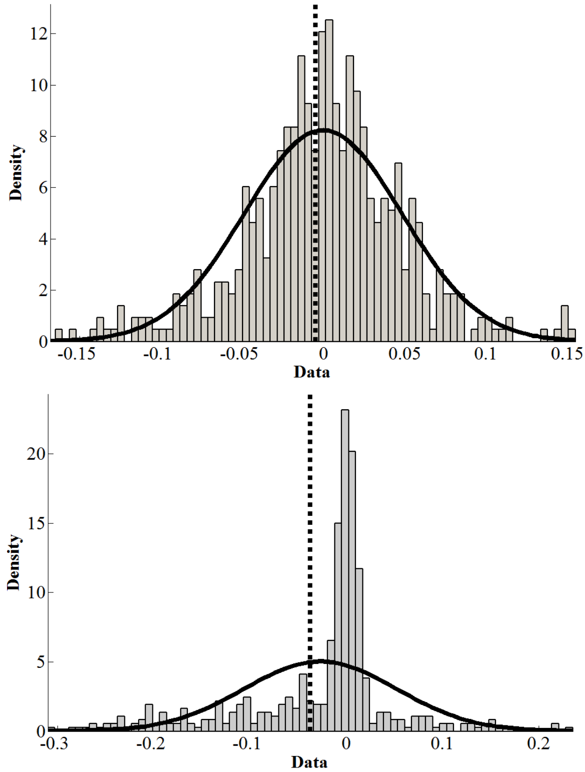

Figure 11: (**Examples of simulated negative skewed distributions**) In both figures the samples are drawn from a mixture distribution with the same initial parameters. The black curve represents a normal distribution fitted to the simulated sample and the dotted vertical line indicates the mean of the simulated distribution. While the sample on the top is closer to being normally distributed both samples feature negative skewness and excess kurtosis.

Formally, it is assumed that the mixtures utilized are finite and of the form:

$$\mathbf{y} \sim \int h(\cdot|\theta)H(d\theta),$$

where $\mathbf{y}$ is the data, h is a parametric density function, and the mixture measure H places probability mass m_g on each atom θ_g, where g belongs to some finite set, for instance, $\{1, ..., G\}$. Consequently, the finite mixture density is given by a convex combination of the component densities.

Gaussian distributions are parameterized by $\theta_g = (\mu_g, \sigma_g)$. While it would be possible to allow for slightly more generality using h_g instead of h as the component density, this generalization is not relevant here as the focus is solely on Gaussian mixtures. For more information about finite mixtures the reader is referred to Frühwirth-Schnatter (2006).

C HATTED PROCESSES AND INFORMATION SETS

Appendix C gives additional details about the strategic trading model. More specifically, in order to illustrate the equivalence $\boldsymbol{\sigma}(\tilde{\xi}, \hat{y}_1, ..., \hat{y}_n) = \boldsymbol{\sigma}(\tilde{\xi}, y_1, ..., y_n)$, with $\tilde{\xi} \in \{\tilde{a}, \tilde{v}\}$, consider the case from the point of view of R:

$$\boldsymbol{\sigma}(\tilde{a},\ y_1) = \boldsymbol{\sigma}(\tilde{a},\ \beta_1^I\tilde{v} + \theta_1^R + \tilde{u}_n) \overset{\theta_1^R \in \boldsymbol{\sigma}(\tilde{a})}{=} \boldsymbol{\sigma}(\tilde{a},\ \beta_1^I\tilde{v} + \tilde{u}_n) \text{ and}$$

$$\boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1) = \boldsymbol{\sigma}(\tilde{a},\ \beta_1^I\tilde{v} + \hat{\theta}_1^R + \tilde{u}_n) \overset{\hat{\theta}_1^R \in \boldsymbol{\sigma}(\tilde{a})}{=} \boldsymbol{\sigma}(\tilde{a},\ \beta_1^I\tilde{v} + \tilde{u}_n)$$

Via an induction argument it then follows that:

$$\begin{aligned}
\boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1, ..., \hat{y}_{n+1}) &\overset{\text{ind. hypo.}}{=} \boldsymbol{\sigma}(\tilde{a},\ y_1, ..., y_n, \hat{y}_{n+1}) \\
&= \boldsymbol{\sigma}(\tilde{a},\ y_1, ..., y_n, y_{n+1} + (\Delta\hat{\theta}_{n+1}^R - \Delta\theta_{n+1}^R) + (\Delta\hat{\theta}_{n+1}^I - \Delta\theta_{n+1}^I)) \\
&= \boldsymbol{\sigma}(\tilde{a},\ y_1, ..., y_n, y_{n+1}),
\end{aligned}$$

where the last equality follows since $\Delta\theta_{n+1}^R$ (similarly $\Delta\hat{\theta}_{n+1}^R$) is $\boldsymbol{\sigma}(\tilde{a},\ y_1, ..., y_n)$ measurable and $\Delta\hat{\theta}_{n+1}^I - \Delta\theta_{n+1}^I = \beta_{n+1}^I(p_n - \hat{p}_n) \in \boldsymbol{\sigma}(\tilde{a},\ y_1, ..., y_n)$. Now, defining $\hat{z}_n^R := \hat{y}_n - \mathbb{E}_n^R[\hat{y}_n]$, $1 \leq n \leq N$, it is observed that:

$$\hat{z}_n^R \sim \mathcal{N}(0, \mathbb{V}[\hat{z}_n^R]),\ \hat{z}_n^R \perp (\tilde{a}, y_1, ..., y_{n-1}) \text{ and}$$
$$\boldsymbol{\sigma}(\tilde{a}, y_1, ..., y_n) = \boldsymbol{\sigma}(\tilde{a}, \hat{y}_1, ..., \hat{y}_n) = \boldsymbol{\sigma}(\tilde{a}, \hat{z}_1^R, ..., \hat{z}_n^R),$$

where:

$$\mathbb{V}[\hat{z}_n^R] = \left(\beta_n^I\right)^2 \left(\Sigma_{n-1}^{(2)} - \frac{\left(\Sigma_{n-1}^{(3)}\right)^2}{\Sigma_{n-1}^{(1)}}\right) + \sigma_u^2.$$

More specifically, orthogonality facilitates Markovian dynamics, thus resolving the issue of expanding state history and allowing for a tractable treatment of the *forecast the forecast of others* problem. In addition, by verifying the above property for $\hat{y}_k$ one simultaneously verifies that the other hatted processes are also in $\boldsymbol{\sigma}(\tilde{\xi}, y_1, ..., y_n)$. This setting is aptly described in Foster and Viswanathan (1996) as the need for a specified trader i to "keep two sets of books". The first one documents the equilibrium path while the second one documents realized outcomes given possible non-equilibrium path actions by i.

Pertaining to the second book, deviations from the equilibrium path cause deviations in prices p_n as well as in θ_n^I, θ_n^R (since the orders of the strategic traders are not independent of each other), and q_n. To keep track of these deviations, the following quantities are paramount in determining the sufficient state variables:

$$\hat{p}_n - p_n, \quad \hat{\theta}_n^I - \theta_n^I, \quad \hat{\theta}_n^R - \theta_n^R, \quad \hat{q}_n - q_n.$$

It is immediately clear that all these deviations are equal to zero on the equilibrium path.

D ALGORITHM FOR DYNAMIC STRATEGIC EQUILIBRIUM

In Appendix D the search and verification algorithm for determining the dynamic linear Markovian BNE in the strategic trading model of Section 5 is described. In a nutshell, the algorithm is based on the principle of dynamic programming and proceeds via backward induction. The logic of Algorithm 3 follows closely the structure of the verification theorem in Choi et al. (2019).

Instead of utilizing a naive three dimensional search, involving initial guesses for all $\Sigma_{N-1}^{(i)}$, $i = 1,2,3$, a "diagonal" approach is used. Namely, since the quantities that truly matter are $\frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}$ and $\frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}$, only $\Sigma_{N-1}^{(3)}$ is guessed and then $\Sigma_{N-1}^{(1)}$ and $\Sigma_{N-1}^{(2)}$ are obtained as proportions of $\Sigma_{N-1}^{(3)}$. This approach helps to narrow down the search space.

Algorithm 3: *Linear equilibrium search and verification via backward induction*

Input: Number of periods N, termination parameter $\epsilon > 0$, variance of noise trading demand σ_u, variance of risky asset payoff σ_v^2, variance of (latent) trading demand σ_a^2, correlation ρ between $\tilde{v}$ and $\tilde{a}$ initial guess $\Sigma_{N-1}^{(3)}$.

Output: For all $n = N, N-1, ..., 1$, the trading parameters $(\beta_n^I, \beta_n^R, \alpha_n^R)$, the pricing parameters (λ_n, r_n), the moments $(\Sigma_{n-1}^{(i)})$, with $i = 1,2,3$, the value function coefficients for I, $(I_n^{(i,j)})_{1 \le i \le j \le 2}$, and the value function coefficients for R, $(L_n^{(i,j)})_{1 \le i \le j \le 3}$.

for ($n = N$; $n = 1$; $n = n-1$)

  if $n = N$ **then**

    Solve constants β_N^I and λ_N using (38) and (39) and fix $\beta_N^R := 1$, $\alpha_N^R := 0$, & $r_N := 0$.

    Using $(\beta_N^I, \beta_N^R, \alpha_N^R, \lambda_N, r_N)$ and $I_N^{(i,j)} = L_N^{(i,j)} = 0$ determine the value function coefficients $(L_{N-1}^{(i,j)})$, for $1 \le i \le j \le 3$, and $(I_{N-1}^{(i',j')})$ for $1 \le i' \le j' \le 2$.

  else

    Taking as inputs the constants from the previous period, solve backwards the system of moment condition (29)-(31).

    Plug the solved moments to equations (32) and (33) and solve λ_n and r_n as functions of β's and plug these solutions back to the expressions for the backward solved moments.

    Plug the solved moments and λ_n and r_n which at this point are all expressed as functions of only β_n^I and β_n^R to the β-equations (34) and (35) and solve the pair of equations using all-solutions methods (polynomial homotopy continuation or Gröbner bases).

    Extract the set real solution pairs from the set of all solutions and determine whether the set contains feasible solutions by using the obtained real solutions to solve other equilibrium constants $\lambda_n, r_n, \alpha_n^R$ as well as $(\Sigma_{n-1}^{(i)})$, $i = 1,2,3$, and to verify that the SOCs and other equilibrium constraints are satisfied.

    if *Equilibrium constraints are satisfied* **then**

      Determine the value function coefficients $(I_{n-1}^{(i,j)})_{1 \le i \le j \le 2}$ and $(L_{n-1}^{(i,j)})_{1 \le i \le j \le 3}$ and proceed to round $n-1$.

    else

      Break loop and adjust the initial guess $\Sigma_{N-1}^{(3)}$.

    end

  end

endfor

if $|\Sigma_0^{(1)} - \sigma_a^2| < \epsilon$ **and** $|\Sigma_0^{(2)} - \sigma_v^2| < \epsilon$ **and** $|\Sigma_0^{(3)} - \rho\sigma_v\sigma_a| < \epsilon$ **then**

  The algorithm terminates.

else

  Adjust initial guesses $\Sigma_{N-1}^{(i)}$, $i = 1,2,3$, and start the algorithm from the beginning.

end

E EXISTENCE AND UNIQUENESS IN FINAL PERIOD

This appendix studies the uniqueness of the pair (β_N^I, λ_N) when $\beta_N^I > 0$. This case can be justified as follows: In the numerical analysis, $\Sigma_n^{(3)}$ tends to be negative. Moreover, $\lambda_N > 0$—a condition stemming from the second order condition of I—means that $\beta_N^I + \Sigma_{N-1}^{(3/2)} > 0$ must hold. In other words, it must hold that $\beta_N^I > 0$.

Moving on, recall that β_N^I and λ_N are given by:

$$\beta_N^I = \frac{1}{2\lambda_N} - \Sigma_{N-1}^{(3/2)} \tag{38}$$

$$\lambda_N = \frac{\beta_N^I \Sigma_{N-1}^{(2)} + \Sigma_{N-1}^{(3)}}{(\beta_N^I)^2 \Sigma_{N-1}^{(2)} + \Sigma_{N-1}^{(1)} + 2\beta_N^I \Sigma_{N-1}^{(3)} + \sigma_u^2}. \tag{39}$$

Now, observe that, after fixing $\sigma_a > 0$ and $\sigma_v > 0$, it holds for the initial guesses:

$$\begin{aligned}
&\Sigma_{N-1}^{(1)} \in (0, \sigma_a^2),\\
&\Sigma_{N-1}^{(2)} \in (0, \sigma_v^2),\\
&(\Sigma_{N-1}^{(3)})^2 \leq \Sigma_{N-1}^{(1)} \Sigma_{N-1}^{(2)} < \sigma_a^2 \sigma_v^2.
\end{aligned}$$

From (38) and (39) one obtains the quadratic equation:

$$(\Sigma_{N-1}^{(2)})^2 (\beta_N^I)^2 + 2\Sigma_{N-1}^{(2)} \Sigma_{N-1}^{(3)} \beta_N^I - \left(\Sigma_{N-1}^{(2)} (\Sigma_{N-1}^{(1)} + \sigma_u^2) - 2(\Sigma_{N-1}^{(3)})^2\right). \tag{40}$$

Assume further that:

$$\sigma_u^2 > \Sigma_{N-1}^{(3)} \Sigma_{N-1}^{(3/2)} > 0.$$

This assumption rules out only non-feasible parts of the parameter space and poses essentially no restrictions on the numerical analysis.

Armed with the above assumption, it is observed that (40) exhibits only one sign change. Thus, by *Descartes' rule of signs*, at most one positive solution exists. This guarantees uniqueness if a solution indeed exists. For existence, note that the quadratic expression is negative if $\beta_N^I = 0$ and must unavoidably change sign for β_n^I large enough. Hence, for a large enough interval, a unique solution exists. As β_N^I uniquely determines λ_N the existence and uniqueness for the pair (β_N^I, λ_N) is now verified.

MIKA HANNULA

Risk Preferences, Dynamic Equilibrium, and Trading Targets

Preprint

II

Risk Preferences, Dynamic Equilibrium, and Trading Targets

Mika Hannula[*]

Abstract

This paper studies the effect of risk preferences on a dynamic financial market equilibrium with constrained and unconstrained, privately informed, strategic traders. The constrained trader (rebalancer) has a *strict* terminal trading target which dictates, ex ante, the risky asset position the trader *must* hold at the end of the day. The unconstrained (informed) trader is in possession of fundamental information and trades to maximize profits. It is shown that risk preferences play a pivotal role in determining the equilibrium interactions between the two strategic traders and the resulting intraday patterns in the market.

Keywords: Risk preferences, trading target, strategic trading, rebalancing, optimal execution, market microstructure

JEL Classification Numbers: G10, G11, G14, G23

1 INTRODUCTION

It is widely acknowledged that savvy traders employ dynamic trading strategies such as order splitting to increase profits by minimizing trading costs.[1] In fact, there has been a sharp rise in the utilization of optimized trade execution (algorithms) during the last two decades, and it is estimated that algorithms are behind roughly 80–90% of trades in the US. Major reasons for this are the development of electronic markets (e.g., NASDAQ) and improved access to order book information. Both can be utilized in the design of more effective execution strategies. As a natural result the user base—referred to as *strategic traders* in this paper—of these types of trading tools has become both wider and more heterogenous. The question then arises: What implications do the interactions between different types of strategic traders have on

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[1] See, for example, Hendershott et al. (2015).

the financial markets? This important question is studied recently in, for example, Choi et al. (2019).

In this paper, a market in which traders with different objectives and differential information engage in a dynamic (sequential) trading game is modeled. The main purpose of this modeling exercise is to study how introducing different risk preferences to this setting changes equilibrium outcomes. Formally speaking, the model constructed below is an extension of Choi et al. (2019) which is—in its own right—an extension of the seminal Kyle (1985) model and the more recent Foster and Viswanathan (1996).

The emphasis on risk preferences can be motivated as follows. First, recent evidence (cf. Cohn et al. 2015, Hanaoka et al. 2015, and Guiso et al. 2018) supports the notion that risk preferences are time varying and affected by changes in the financial environment. Thus, it is conceivable—for example—that even the traders generally regarded as risk neutral experience periods during which their behavior is better described by aversion to risk rather than plain neutrality towards risk.

Second, it is clear that traders representing financial institutions (the informed trader and the rebalancer in this paper) generally operate under a compensation contract. The space of different financial contracts is vast, but chiefly the goal of these contracts is to induce certain behavior deemed desirable by the principal of the contract. As a side product, these contracts can—on purpose or by accident—induce risk averse or even risk seeking behavior.[2]

Hence, by examining how shifts in risk preferences change the properties of the dynamic trading equilibrium, one can better understand how different initial conditions can affect the characteristics of financial markets. In addition to being of theoretical interest, this issue is also important for market participants with daily trading activities. For instance, certain trading strategies designed based on historical market data and related assumptions pertaining to the behavior of other traders might prove to be less efficient if the (large) traders' attitude towards risk has incurred a marked shift, which instead has resulted in an adjustment in their trading activities.

At the center of the model discussed in this paper there are two strategic traders. The *informed trader* trades to maximize expected utility given an endowment of private information regarding the payoff of a risky asset. The *rebalancer* maximizes expected utility of trading under a strict terminal position target in the risky asset. The realized size of the trading target is private information of the rebalancer. Private information in the model is long-lived and learned before any trading commences. In addition to the strategic traders, the market is populated by several competitive market makers and exogenously motivated noise traders.

Both strategic traders are assumed to have negative exponential, i.e., CARA, utility. CARA utility renders the decision problems of the traders independent of wealth

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2 Recent papers investigating these issues, among others, include Panageas and Westerfield (2009) and He and Xiong (2013).

considerations, and hence (cf. Schied et al. 2010) this particular utility form is well-suited for modelling institutional investors. Moreover, CARA utility, in the context of the present model, is a tractable instrument, which allows one to look at various risk preferences.

This paper offers various contributions to the literature. First, the main theoretical contribution is the construction of a financial market equilibrium such that the strategic traders are allowed to exhibit different risk preferences. Since the equilibrium is dynamic in nature, in addition to the optimal strategies of the traders, their value functions are also characterized. The approach taken also extends the differential information procedure developed in Foster and Viswanathan (1994) and Foster and Viswanathan (1996) to a setting with more general risk preferences. Some caveats brought on by adding CARA utility to the model are also discussed. Further, due to the inherent complexity of the full dynamic model, the basic intuition of the model is first detailed using a two period example.

The theoretical foundations are exploited to address the main purpose of the paper. Namely, to study the equilibrium implications of different risk preferences. Regarding these implications, it is shown that changes in the risk preferences of the informed trader generally induce larger equilibrium changes than changes in the risk preferences of the rebalancer. This result is intuitive as the presence of the (strict) trading constraint restraints the degree to which the equilibrium behavior of the rebalancer can change. The informed trader, however, is not subject to such constraints.

Nevertheless, there are exceptions to the rule. A case in point, where changes in the rebalancer's risk preferences are more important than changes in the informed trader's risk preferences, is aggregate order flow autocorrelation. This is to be expected as the main driver behind the autocorrelation dynamics is the presence of exogenous rebalancing needs. Hence, even small changes to the rebalancer's strategy carry over to changes in the autocorrelation dynamics.

Pertaining to the changes in the informed trader's risk preferences, it is shown that, in line with Holden and Subrahmanyam (1994), the intraday price impact becomes monotonically decreasing when the informed trader is more risk averse. This contrasts Choi et al. (2019) who document a non-monotonic (twisted) price impact pattern. Moreover, a similarly unique pattern—distinct from the risk neutral case—arises when the informed trader is allowed to be risk seeking, while maintaining the risk neutrality assumption for the rebalancer. This result verifies that even in the presence of a rebalancer, the trades of the informed trader mainly determine the price impact. This result is related to the fact that the price impact in the model, following Kyle (1985), is assumed to be information-based. Therefore, the trades of the most informed individuals are the most important components in the determination of the (endogenous) price impact parameter.

Another interesting intraday observation relates to the correlation between the trades of the rebalancer and the informed trader. It is noted by Choi et al. (2019) that negative correlation is mutually beneficial due to the liquidity provision effect. The presence of risk aversion seems to amplify liquidity provision. Conversely, if the informed trader is risk seeking and the rebalancer is risk neutral liquidity provision is repressed. This speaks on behalf of the potential detrimental effects of risk seeking on the financial market equilibrium.

Outside the main purpose of the model, studying the model solution brings forth some curious issues of independent interest. Because the model is essentially linear-quadratic and Gaussian, conditional moments and objective functions pose little problems as they can be dealt with via exact calculations. In other words, one does not have to rely on approximating the objective functions in solving the model numerically. However, the equilibrium conditions of the model generate a system of polynomial equations to which there are a large number of solutions. Moreover, as there is no general existence theory to which one could rely on, one faces two key questions: (1) does an equilibrium for some given parametrization exists? (2) is this equilibrium unique? To answer these questions one has to rely on numerical analysis. General answers, as one might expect, cannot be found using numerical methods. With the right tools, however, one can gain a better grasp of the inner workings of the model.

Consequently, this paper makes contributions pertaining to numerical procedures used in analyzing models of strategic trading. This is an eminently relevant topic as closed form solutions to these types of models are generally in short supply. Further, since Kyle (1985) the solution procedure has remained more or less the same although the models themselves have evolved rapidly. The proposed numerical methods are applicable to various strategic trading frameworks, naturally including the risk neutral and risk averse discrete time models, and extending to—for instance—continuous time models (cf. Marinovic and Varas 2018) as well as, via the use of polynomial approximations, models with even more general preferences and uncertainty specifications (cf. Bernardo and Judd 2000).

In the numerical equilibrium analysis conducted in the subsequent sections, the main emphasis is on the determination of equilibrium uniqueness. As noted earlier, due to the complicated nonlinear nature of the system of equations that describes the model equilibrium, it is very difficult to ascertain through analytical methods whether or not the model has a unique equilibrium. Hence, one must resort to the second best approach and study equilibrium uniqueness numerically.

Indeed, since the model solution boils down to a solution to a system of polynomial equations, one can take advantage of *all-solutions homotopy continuation methods* (see, e.g., Judd et al. 2012), in conjunction with the dynamic programming algorithm to obtain *all* solutions to a given equilibrium system of equations and then proceed by process of elimination to check whether the final solution set has cardinality one. Naturally, by this same process one can study the uniqueness of the equilibria presented in Choi et al. (2019), i.e., the risk neutral case (cf. Hannula 2019a). Details and issues related to the numerical approach are covered thoroughly in the forthcoming sections, the main observation being that the more intricate equilibrium dynamics in Choi et al. (2019) and its extensions raise new issues—in particular, issues not present, for instance, in the solution of the Kyle (1985) model—to be considered in the numerical implementation.

On top of the theoretical and computational contributions discussed above, this paper also contributes to the literature by expanding the understanding of, and expounding on the potential reasons behind, the patterns of trade observed in real trading data. From an industry perspective, modeling intraday dynamics is of interest since it serves the development of methods to predict trading volume. Improved forecasts of future trading volume can have a decided positive effect on the performance of various algorithmic (e.g., *Volume Weighted Average Price*, VWAP) and non-algorithmic trading strategies (see, e.g., Satish et al. 2014). Intraday trading dynamics may also have relevance in asset pricing applications (see, e.g., Huh 2014).

Furthermore, there is abundant empirical evidence supporting the economic significance of the trade implementation process (see, e.g., Anand et al. 2011), emphasizing the need for further inquiry on this area. Makarov and Schoar (2019) utilize the Kyle (1985) model in an empirical study of cryptocurrency trading, exemplifying the wide applicability of these types of models. Finally, using a novel data set of professional asset managers and utilizing the Foster and Viswanathan (1996) model to develop hypotheses, Di Mascio et al. (2017) provide empirical evidence on behalf of strategic trading behavior by informed traders. However, Di Mascio et al. (2017)—due to the nature of their theoretical framework—only consider informed trading and competition between similarly motivated informed traders, while ignoring other types of strategic traders potentially present in the market and the conceivably diverse risk preferences between different traders. Thus, there is still work to be done to gain a more holistic view of strategic trading and strategic traders. Unsurprisingly then the pressing need for models dealing with contemporaneous optimal execution of heterogenous orders is acknowledged in, for instance, Van Kervel and Menkveld (2019).

The rest of the paper is organized as follows. Section 2 reviews prior literature and discusses the positioning of the present paper. Section 3 introduces the model, the notation, and the equilibrium concepts used throughout the paper. In Section 4 a two period version of the model is discussed while Section 5 describes the full dynamic model. Section 6 concludes the paper. All proofs are allocated to Appendix A and the description of the numerical procedures is given in B.

2 PRIOR LITERATURE

This section gives a brief overview of the related literature. From a theoretical point of view, the two most relevant areas for the model at hand are: (1) the market microstructure-based approach to strategic trading (2) the literature on optimal execution strategies. The former is discussed first.

To this end, a natural starting point is the two seminal, most widely cited and upon further inspection closely linked classical papers regarding market microstructure theory and strategic trading, i.e., Kyle (1985) and Glosten and Milgrom (1985).[3] In fact, concerning the linkage between the aforementioned papers, it is shown in Back and Baruch (2004) that, under certain assumptions, the equilibria of the latter converge to the equilibrium of the former. Unsurprisingly, there is a voluminous literature dealing with possible extensions and empirical tests regarding these two important papers. Due to the fact that the model below is based on the first mentioned paper—as it is more suitable to studying dynamic trading strategies of individual traders—the literature covered here is for the most part limited to contributions that adopt a similar starting point.[4]

As for Kyle (1985), the framing of the paper features a simple financial market with one risky asset and a privately informed insider. The private information of the insider is long-lived, and the model has multiple trading periods so that the insider is induced to split his total demand to smaller child orders in order to minimize price impact. There is also a (number of) competitive market maker(s) who observes the aggregate order flow, which represents the orders submitted by the insider and a number of noise (liquidity) traders.

Both the insider and the market maker(s) are assumed risk neutral. Kyle (1985) studies the continuous time limit of his discrete time sequential model, while Back (1992) provides a formal description and a generalization of Kyle (1985) under the assumption of continuous trading. Risk aversion in the context of strategic informed trading in the framework proposed by Kyle (1985) is studied, in a static setting, by Subrahmanyam (1991) and, in a dynamic setting, by Holden and Subrahmanyam (1994). Baruch (2002) examines risk averse strategic trading in a continuous time environment.[5]

Holden and Subrahmanyam (1992) study a dynamic model with homogeneous long-lived private information with competing informed traders. They show that

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3 Market microstructure literature is surveyed in, for example, Madhavan (2000) and Biais et al. (2005).

4 Models building on Glosten and Milgrom (1985) usually assume that (informed) traders arrive sequentially and trade only once.

5 In addition to examining risk averse traders, Subrahmanyam (1991) also allows for risk averse market makers (see also Bernardo and Welch 2004). This is facilitated by the static setting utilized in the paper. In a dynamic context modeling risk averse market makers is much more subtle; see, for example, Çetin et al. (2016). Due to this issue, the question of market maker risk aversion is omitted in this paper.

competition results in rapid the revelation of private information when compared to a market with a monopolist informed trader. When the interval between auctions approaches zero, the speed of information revelation increases. The notion of (relatively) long-lived information plays an important role in models of strategic trading, since it forces the traders to utilize dynamically optimal decisions instead of one-shot optimal decisions. Back et al. (2000) examine the continuous time version of Holden and Subrahmanyam (1992). They find that there does not exist a linear equilibrium when the signals of the informed traders are perfectly correlated.

Foster and Viswanathan (1994) and Foster and Viswanathan (1996) extend the multiple informed traders setting by allowing heterogeneous information. Indeed, Foster and Viswanathan (1994) study a discrete time model with two informed traders, where one trader has access to more private information. Common information, available to both traders, is revealed quickly due to competition but the additional information available to the better-informed trader is only revealed during the last rounds of trading. Foster and Viswanathan (1996) extend the analysis to multiple heterogeneously informed traders and show that informed traders compete fiercely but also learn from each other. In fact, the competing traders seek to forecast the private information of the other traders. This brings about the *forecast the forecasts of others* problem (also present in this paper) to which Foster and Viswanathan (1996) provide an elegant solution. The authors show that the initial correlation between the signals observed by different traders has an important role in determining equilibrium profits and price informativeness.

Recently, Collin-Dufresne and Fos (2016) have extended the Kyle (1985) model to allow the noise trading process to exhibit stochastic volatility. Biais et al. (2015) and Foucault et al. (2016) study strategic trading when traders have differences in speed. Guo et al. (2015) study a continuous time model where an insider trades on both fundamental and non-fundamental (e.g., noise supply) information while Yang and Zhu (2020) study a two-period Kyle (1985) model with a *back-runner* trader who is allowed to observe a signal of the informed trader's previous order. In the Yang and Zhu (2020) setting a mixed strategy equilibrium may arise as the informed trader seeks to better hide his private information in the first round of trading. Mixed strategies are also encountered when an informed trader (insider) is required to disclose his trades (see, Huddart et al. 2001). The recent contributions exemplify the vibrant discussion revolving around models of strategic trading.

This paper is also related to the literature on optimal trade execution.[6] In fact, one could view the rebalancer in this paper as a trader pursuing optimal execution of an ex ante decided trade. The main difference between optimal execution literature and the present paper is that here trading strategies and price impact are jointly and endogenously determined in equilibrium, whereas it is common in the trade ex-

[6] See, for example, Bertsimas and Lo (1998), Almgren and Chriss (2001), Gatheral and Schied (2011), Predoiu et al. (2011), Bayraktar and Ludkovski (2011), and Robert et al. (2012).

ecution literature to work with an exogenously given price impact function. Even though this approach is efficient in facilitating tractability, there are many potential, empirically feasible, price impact functions to choose from and not necessarily any general guidelines to assist in providing the microfoundations to any given choice.

Furthermore, simplifying the notion of price impact to a fixed functional form usually ignores the learning aspect which plays an important role, e.g., in the model studied in this paper. Relatedly, papers focusing on optimal execution and the algorithmic side of the problem typically consider a single agent setting. Hence, the modeled trader operates in a vacuum without any interactions with other market participants.[7] This again marks a clear distinction to the model presented below in which the main focus is on intraday interactions between two strategic traders.

Finally, in the optimal execution literature one usually separates between *permanent* and *transient* price impact (see, e.g., Huberman and Stanzl 2005). In short, permanent price impact may be understood as the changes in the actual equilibrium price while transient price impact merely means deviations from the equilibrium price, which itself remains unchanged. Conversely, in this paper, following Kyle (1985) and its numerous extensions, the price impact is assumed to be *linear and permanent*.

From a modelling point of view, in addition to Choi et al. (2019), the papers closest to this paper are Seppi (1990), Foster and Viswanathan (1994), Foster and Viswanathan (1996), Moallemi et al. (2012), and Degryse et al. (2014).[8] Of the papers listed, only Moallemi et al. (2012) can be considered to directly deal with an optimal execution problem.

Relaxation of the risk neutrality assumption, endogenous learning in a dynamic framework, and the utilization of heterogenous long-lived private information separate the current model from previously suggested models. As a consequence, the problem of large number of state variables and the occurrence of high degree polynomials in the recursive equations describing the equilibrium encountered, for example, by Degryse et al. (2014) resurfaces in the present paper. Nevertheless, effective numerical methods are employed to deal with the problem.

These modeling complications mainly stem from dynamic learning and the forecasting the forecasts of others problem. Foster and Viswanathan (1994) model a similar type of learning problem with two traders having nested information sets while Foster and Viswanathan (1996) solve the problem of timewise increasing state history under a more general, non-nested, differential information structure. More specifically, they proceed under the assumption that the signals representing the information possessed by the traders are drawn from a joint normal distribution. The

7 Recently, new approaches deviating from the single agent setting have emerged. See, for instance, Brunnermeier (2005), Moallemi et al. (2012), Huang et al. (2015), and Schied and Zhang (2017).

8 See, Choi et al. (2019) for a discussion regarding the differences between their model and the model of Degryse et al. (2014).

approach developed in these two aforementioned papers to deal with the learning problem of the differentially informed traders is one of the backbones of Choi et al. (2019) and the model presented below.

In addition to dealing with differential information, this paper relates to, e.g., Subrahmanyam (1991) and Holden and Subrahmanyam (1994), two papers which also feature strategic risk averse traders. The first paper utilizes a static approach while the latter deals with a dynamic setting with multiple competing and homogeneously informed traders. These papers provide a useful albeit somewhat distant benchmark against which to contrast the results obtained in the proposed framework. Another important and related example—this time from the optimal execution literature—is Schied and Schöneborn (2009) who study the effect of risk aversion on optimal execution strategies.[9] The authors conclude that the coefficient of absolute risk aversion is a key parameter determining the optimal strategy of the trader. Empirically, risk aversion may help to provide a rational explanation for different types of liquidation strategies, e.g., aggressive in-the-money and passive in-the-money, utilized by real-world investors.[10]

Finally, this paper is also related to literature discussing computational methods in solving game-theoretic equilibrium models and finding all equilibria of a given game. Earlier examples of this literature are Kostreva and Kinard (1991), Schmedders (1998) and Schmedders (1999). More recently, homotopy methods are applied in Bajari et al. (2010), Besanko et al. (2010), Borkovsky et al. (2010), Herings and Peeters (2010), Judd et al. (2012), and Kubler et al. (2014). Especially in the three last papers, various insightful examples from economics are discussed.

3 MODEL

3.1 Overview

This section introduces the key components as well as the timeline of the model analyzed in this paper. In addition, the section covers the traders' objective functions, the equilibrium concept utilized, and a description of the endogenous learning process. For the sake of easy comparability, notation closely resembling the one used in Choi et al. (2019) is utilized.

ASSETS . There is one risky asset which has a terminal payoff $\tilde{v} \sim \mathcal{N}(0, \sigma_v^2)$ for

[9] See also, Schied et al. (2010).

[10] The notions *aggressive in-the-money* and *passive in-the-money* relate to the question whether optimal strategies are increasing or decreasing functions of revenues.

some constant $\sigma_v^2 > 0$. In addition to the risky asset, there is a riskless asset that pays zero interest rate. The realization of $\tilde{v}$ becomes public information after N rounds of trading, i.e., due to the discrete time setting, at the next possible instance $N+1$, where $N \in \mathbb{N}$.[11] Trading takes place at discrete time points $n \in \{1,2,...,N\}$ with evenly spaced time steps.

TRADERS. There are four types of traders in the model, two of which are assumed to be strategic. It is assumed that the market composition—i.e., which types of traders are present in the market—is common knowledge among all market participants.[12]

(1) A (fundamentally) *informed trader* who has access to private information about the terminal payoff of the risky asset prior to the start of the trading game, i.e., the informed trader privately observes the realization of $\tilde{v}$ at $n = 0$ and trades strategically to exploit *his* informational advantage. The letter I is used to refer to the informed trader.

(2) A constrained trader, hereinafter referred to as the *rebalancer*, who trades strategically to reach a terminal trading target, $\tilde{a} \sim \mathcal{N}(0, \sigma_a^2)$ with constant $\sigma_a^2 > 0$. $\tilde{a}$ is jointly normally distributed with $\tilde{v}$ such that $\mathbb{C}\text{orr}(\tilde{v}, \tilde{a}) = \rho \in [0,1]$. The realization of $\tilde{a}$ is privately learned by the rebalancer at $n = 0$. The trading target dictates the terminal position in the risky asset the rebalancer needs to acquire in the form of a *strict* constraint, i.e., the terminal position must be equal to the realization of $\tilde{a}$ exactly.[13] The rebalancer has two distinct trading motives: (1) to meet the terminal trading target (2) to exploit *her* potential informational advantage. The letter R is used to refer to the rebalancer.

(3) Non-strategic *liquidity* (or *noise*) *traders* whose aggregate demand at $n = 1,...,N$ is $\tilde{u}_n \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_u^2)$ for some $\sigma_u > 0$.[14] The liquidity trader demand is assumed to be price inelastic and independent of both $\tilde{v}$ and $\tilde{a}$.[15]

(4) A number of risk-neutral *market makers* who set prices competitively such that the resulting financial market can be considered *semi-strong form efficient*. The

[11] In this paper, $\mathbb{N} = \{1,2,...\}$, i.e., $\mathbb{N}$ refers the set of positive integers.

[12] It may be possible to relax this assumption but it would require an extensive modification of the present model. Uncertainty about the presence of an informed trader is examined, for example, in Li (2013) and Wang and Yang (2016).

[13] *Soft* constraints could be considered but with the cost of added complexity. A strict target can be interpreted to have an infinite penalty from deviation.

[14] One could just as well model the noise trader demand via independent and normally distributed increments Δu_n with variance $\Delta\sigma_u^2$. This variant is not used here as it exacerbates notational clutter.

[15] Utilizing non-strategic liquidity traders is a standard—see, for example, Grossman (1976), Grossman and Stiglitz (1980), Hellwig (1980), and Admati (1985), yet rather controversial—approach. In the present model, this modelling choice is related to the tractability of the model. Moreover, the results are likely to carry over to any equilibrium with endogenous liquidity trading, whenever the rebalancer, together with the informed trader, have an information advantage over the rest of the traders. For a model with strategic uninformed traders see, for example Spiegel and Subrahmanyam (1992) or Mendelson and Tunca (2004).

market makers observe the aggregate order flow at each $n = 1,...,N$ but not individual orders (trader anonymity). The letter M is used to refer to market makers.

INFORMATION STRUCTURE. All information is costless to those who observe it and (strategic) traders in the model are assumed perfectly attentive and rational in the sense that they do not ignore any relevant information. Private information, for those who are endowed with it, is learned at time $n = 0$, before trading commences, and no additional private information is received in later periods $n = 1,2,...,N$.[16] Furthermore, private information is available only to the trader type that observes it and information cannot be bought or sold.[17]

Order flow information is generated throughout the model at each $n = 1,...,N$. At a given period n, the information generated by the order flow process is denoted by $\boldsymbol{\sigma}(y_1,...,y_n) =: \boldsymbol{\sigma}_n$.[18] This effectively describes the information available to the market makers. The market makers utilize the order flow information to set prices and to update beliefs regarding the private information possessed by I and R. The (aggregate) order flow y_n is available only to the market makers before the price p_n—for period n—is determined. In other words, the traders move first by simultaneously submitting market orders, after which the market makers observe the sum of these orders and determine the price for the current period. The traders obviously know their own order but do not observe the orders of others.

However, since there is a one-to-one mapping from aggregate order flows to prices, all traders learn the realized aggregate order flow *after* observing p_n. Nonetheless, the traders do not have this information available upon placing orders for period n. Hence, at any $n = 1,...,N$, no one type of market participant is in possession of all the price relevant information.[19] More specifically, denoting I's and R's information sets at time n by $\boldsymbol{\sigma}(\tilde{v}, y_1,...,y_{n-1})$ and $\boldsymbol{\sigma}(\tilde{a}, y_1,...,y_{n-1})$ respectively, it can be seen that the information sets of the different market participants are not nested.[20]

MODEL TIMELINE. Figure 1 illustrates the model timeline for a generic N trading rounds model.

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[16] Strategic trading with information flow is examined recently in Sastry and Thompson (2019).

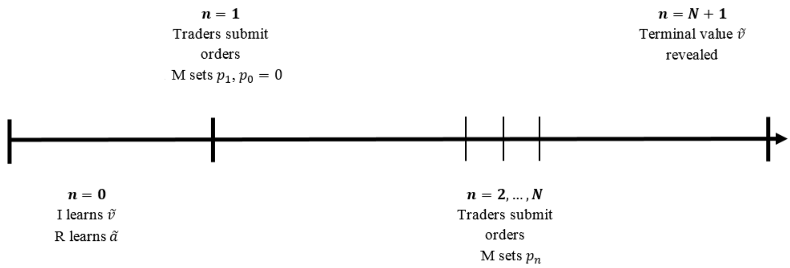
[17] Information sales is examined in, for instance, Cespa (2008) and García and Sangiorgi (2011).

[18] $\sigma(y_1,...,y_n)$ denotes the sigma-algebra generated by $\{y_1,...,y_n\}$. This, with slight abuse of notation, is the unique smallest sigma-algebra that contains all sets of type $\{y_i \leq \mu\}$ for $\mu \in \mathbb{R}$ and $i = 1,...,n$. Thus, to avoid confusion, it should be noted that $\mathbb{E}[\tilde{\zeta} \mid y_1,...,y_n] = \mathbb{E}[\tilde{\zeta} \mid \boldsymbol{\sigma}(y_1,...,y_n)] = \mathbb{E}[\tilde{\zeta} \mid \boldsymbol{\sigma}_n]$ with $\tilde{\xi} \in \{\tilde{v},\tilde{a}\}$.

[19] In a continuous time setting, under the usual right-continuous filtration, the invertibility of the pricing rule implies that an informed trader can infer the aggregate noise trader demand immediately at any time t. See, for instance, Back (1992). This does not happen in a discrete time setting, since the information sets of market participants remain constant in between periods. Due to this, the information sets of the informed traders cannot be considered as initial enlargements of the information set of the market makers, even though one could regard them as initial enlargements of the period n "public" information set $\boldsymbol{\sigma}(y_1,...,y_{n-1})$, i.e., $\boldsymbol{\sigma}(\tilde{\xi}, y_1,...,y_{n-1}) = \boldsymbol{\sigma}(y_1,...,y_{n-1}) \vee \boldsymbol{\sigma}(\tilde{\xi})$ for $\tilde{\xi} \in \{\tilde{v},\tilde{a}\}$.

[20] Generally speaking, nested information sets simplify the analysis. See, for example, Foster and Viswanathan (1994).

Figure 1: (**Model timeline**) This figure illustrates the model timeline. In the initial period ($n = 0$) the strategic traders receive their private information endowments, after which there are N sequential and identical trading rounds. At the terminal period ($n = N+1$) all remaining uncertainty regarding $\tilde{v}$ in the market is lifted.



Periods $n = 0$ (the initial period) and $n = N+1$ (the terminal period) are special and are assumed to involve no strategic actions. At the initial period, private information is observed, and no trading occurs, and at the terminal period, after the market has closed, the liquidation value of the risky asset is revealed. All trading rounds, $n = 1, \ldots, N$, are identical regarding the sequence of actions. The most natural interpretation for the model timeline is that it takes place during a single trading day.

3.2 Objective Functions

Let $\Delta\theta_n^I$ and $\Delta\theta_n^R$ denote respectively the informed trader's and the rebalancer's order at time n. Similarly, θ_n^I and θ_n^R represent the *total* risky asset position at time n. In other words, one obtains, for example, that $\sum_{n=1}^{N} \Delta\theta_n^R =: \theta_N^R = \tilde{a}$, where the last equality follows from the terminal constraint of the rebalancer. All initial positions are assumed to be zero, i.e., $\theta_0^I = \theta_0^R = 0$.

Utilizing the notation introduced above, the aggregate order flow can be written as $y_n = \Delta\theta_n^I + \Delta\theta_n^R + \tilde{u}_n$, with $y_0 = 0$. Due to the risk neutrality of the market makers and the assumption of perfect competition, the market makers earns zero expected profits and sets prices according to:[21]

$$p_n = \mathbb{E}\left[\tilde{v} \mid \boldsymbol{\sigma}_n^M\right] = \mathbb{E}_n^M\left[\tilde{v}\right], \text{ for all } n = 1, 2, \ldots, N, \tag{1}$$

where $\boldsymbol{\sigma}_n^M = \boldsymbol{\sigma}(y_1, \ldots, y_n)$ along with the initial condition $p_0 = 0$. As with the traders' orders, define $\Delta p_n := p_n - p_{n-1}$.

Given the pricing rule, the informed trader seeks to maximize:

$$\mathbb{E}\left[U^I\left(\sum_{n=1}^{N}(\tilde{v} - p_n)\Delta\theta_n^I\right) \mid \boldsymbol{\sigma}(\tilde{v})\right] = \mathbb{E}_0^I\left[U^I\left(\sum_{n=1}^{N}(\tilde{v} - p_n)\Delta\theta_n^I\right)\right], \tag{2}$$

[21] One may, as is usual in the literature, assume that the market makers are subject to Bertrand competition and consequently (1) holds.

where U^I is given by:

$$U^I(x) = \begin{cases} \frac{-e^{-A^I x}}{A^I}, & \text{if } A^I \neq 0 \\ x, & \text{if } A^I = 0, \end{cases} \tag{3}$$

and $\Delta\theta_n^I$ is $\boldsymbol{\sigma}(\tilde{v}, y_1, ..., y_{n-1}) =: \boldsymbol{\sigma}_n^I$ measurable for all $n = 1, 2, ..., N$. Note especially that $\boldsymbol{\sigma}_0^I := \boldsymbol{\sigma}(\tilde{v})$. The measurability constraint is the only explicit constraint imposed on I's (similarly R's) optimal orders (controls). Put differently, no additional ex ante constraints, e.g., functional form constraints, are introduced.

As for the rebalancer, it is easy to see that:

$$\begin{aligned} &\mathbb{E}\Big[U^R\Big((\tilde{a} - \theta_{N-1}^R)(\tilde{v} - p_N) + \Delta\theta_{N-1}^R(\tilde{v} - p_{N-1}) + ... + \Delta\theta_1^R(\tilde{v} - p_1)\Big) \,\Big|\, \boldsymbol{\sigma}(\tilde{a})\Big] \\ &= \mathbb{E}\Big[U^R\Big(\tilde{a}(\tilde{v} - p_N) - \theta_{N-1}^R(\tilde{v} - p_N) + \Delta\theta_{N-1}^R(\tilde{v} - p_{N-1}) + ... + \Delta\theta_1^R(\tilde{v} - p_1)\Big) \,\Big|\, \boldsymbol{\sigma}(\tilde{a})\Big] \\ &= \mathbb{E}\Bigg[U^R\Big(\tilde{a}\tilde{v} - \sum_{n=1}^{N}(\tilde{a} - \theta_{n-1}^R)\Delta p_n\Big) \,\Big|\, \boldsymbol{\sigma}(\tilde{a})\Bigg] = \mathbb{E}_0^R\Bigg[U^R\Big(\tilde{a}\tilde{v} - \sum_{n=1}^{N}(\tilde{a} - \theta_{n-1}^R)\Delta p_n\Big)\Bigg], \end{aligned} \tag{4}$$

where the last equality follows once it is noted that $p_N = \sum_{n=1}^N \Delta p_n$, with $p_0 = 0$, $\theta_M^R = \sum_{n=1}^M \Delta\theta_n^R$ for $M \leq N$, and U^R is given by:

$$U^R(x) = \begin{cases} \frac{-e^{-A^R x}}{A^R}, & \text{if } A^R \neq 0 \\ x, & \text{if } A^R = 0, \end{cases} \tag{5}$$

and where the measurability condition dictates that $\Delta\theta_n^R \in \boldsymbol{\sigma}(\tilde{a}, y_1, ..., y_{n-1}) =: \boldsymbol{\sigma}_n^R$ with $\boldsymbol{\sigma}_0^R := \boldsymbol{\sigma}(\tilde{a})$. It is important to note that in (4) the whole expression in the utility function—the term $\tilde{a}\tilde{v}$ minus the sum–refers to (maximization of) terminal wealth. The latter sum term simply captures the trading costs for R. So, omitting the term $\tilde{a}\tilde{v}$ from (4) one obtains a trading cost minimization objective:

$$\mathbb{E}_0^R\Bigg[U^R\Big(-\sum_{n=1}^{N}(\tilde{a} - \theta_{n-1}^R)\Delta p_n\Big)\Bigg]. \tag{6}$$

The reason why this is important is that the term $\tilde{a}\tilde{v}$, even though $\tilde{a}$ belongs the information set of R, is not stochastically independent of the rest of the expression and thus cannot be ignored as can be done (due to linearity) in the risk neutral case. Due to this, the resulting expressions obtained using the expected terminal wealth objective differ somewhat from the ones obtained using the trading cost objective. The most notable difference—originating from the need to deal with a complicated learning problem—is that one needs to introduce additional (quasi) state variables and additional conditional moments. This instead complicates the model further quite heavily in terms of both the dynamic programming equations and the numerical methods. For this reason the model is solved only for the cost minimization objective (6).[22]

3.3 Equilibrium Definition

The equilibrium concept utilized in this paper, following Kyle (1985) and the references therein, is *Bayesian Nash* equilibrium. The formal definition is:

Definition 1 (Bayesian Nash Equilibrium (BNE)). *A collection of functions* $(\theta_n^I, \theta_n^R, p_n)$ *is a BNE if:*

(i) p_n *satisfies (1) given* (θ_n^I, θ_n^R)*,*

(ii) θ_n^I *maximizes (2) (informed trader's utility) given* (θ_n^R, p_n)*,*

(iii) θ_n^R *maximizes (4) (rebalancer's utility) given* (θ_n^I, p_n)*.*

Similarly to Choi et al. (2019), the equilibria constructed under different risk preferences have a linear structure.[23] Namely, it is shown below that, for constants β_n^I, α_n^R, β_n^R, the optimal orders of I and R are given by:

$$\Delta\theta_n^I = \beta_n^I(\tilde{v} - p_{n-1}), \tag{7}$$

$$\begin{aligned} \Delta\theta_n^R &= \alpha_n^R q_{n-1} + \beta_n^R(\tilde{a} - \theta_{n-1}^R) \\ &= \beta_n^R(\tilde{a} - \theta_{n-1}^R - q_{n-1}) + (\alpha_n^R + \beta_n^R) q_{n-1}, \end{aligned} \tag{8}$$

where q_n denotes the market makers' expectation of the R's remaining demand for the risky asset, i.e., $q_n := \mathbb{E}_n^M[\tilde{a} - \theta_n^R]$. The terminal constraint of R imposes that $\beta_N^R = 1$ and $\alpha_N^R = 0$.

3.4 Learning and Beliefs

This section details the endogenous learning that takes place in the model and the beliefs of the market participants. Starting with the market makers, there are two key quantities:

$$p_n = \mathbb{E}_n^M[\tilde{v}], \tag{9}$$

[22] The impact of this choice on the trading strategies of I and R remain an open question. It is conjectured, however, that this impact will be negligible. Numerical tests show that the extra components in the wealth maximization formulation typically have very small values.

[23] The existence of a linear equilibrium in a Kyle-type CARA-Gaussian setting is to be expected. See, for example, Holden and Subrahmanyam (1994).

$$q_n = \mathbb{E}_n^M[\tilde{a} - \theta_n^R], \tag{10}$$

where equation (9) represents the market makers' period n risky asset valuation and equation (10) their belief about the remaining trading demand of R. These quantities, for all $n = 1,...,N$, adhere to the following dynamics:

$$\Delta p_n = \lambda_n\Big(y_n - \mathbb{E}_{n-1}^M[y_n]\Big) \tag{11}$$

$$\Delta q_n = r_n\Big(y_n - \mathbb{E}_{n-1}^M[y_n]\Big) - Q_n^R q_{n-1} \tag{12}$$

where $\{\lambda_n\}_{n=1,...,N}$ and $\{r_n\}_{n=1,...,N}$, are projection coefficients given by:

$$\lambda_n = \frac{\mathbb{C}\Big[\tilde{v} - \hat{p}_{n-1}, \hat{z}_n^M\Big]}{\mathbb{V}\Big[\hat{z}_n^M\Big]}, \tag{13}$$

$$r_n = \frac{\mathbb{C}\Big[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}, \hat{z}_n^M\Big]}{\mathbb{V}\Big[\hat{z}_n^M\Big]} \times \Big(1 - \beta_n^R\Big), \tag{14}$$

where $\mathbb{V}[\,\bullet\,]$ represents variance and $\mathbb{C}[\,\bullet\,]$ represents covariance and where $\hat{z}_n^M := \hat{y}_n - \mathbb{E}\Big[\hat{y}_n \;\Big|\; \boldsymbol{\sigma}(\hat{y}_1,...,\hat{y}_{n-1})\Big]$, while $Q_n^R := (\alpha_n^R + \beta_n^R)$ is a constant stemming from the orders of R. Moreover, using (7) and (8), it follows that:[24]

$$\mathbb{E}_{n-1}^M[y_n] = \underbrace{Q_n^R q_{n-1}}_{\text{Sunshine trading component}}.$$

Clearly, prices are unaffected by the *sunshine trading component* which can be readily seen from equations (11)-(12). As noted by Choi et al. (2019), the sunshine trading component in the optimal order of R is a unique feature that, among other things, distinguishes the strategy of R from that of I.

Moving on to the strategic traders, one should note that due to the presence of heterogenous (asymmetric) information, both R and I try to learn about the information possessed by the other. The relevant quantity for I is:

$$\begin{aligned}\mathbb{E}_n^I[\tilde{a} - \theta_{n-1}^R] &= q_{n-1} + \mathbb{E}_n^I[\tilde{a} - \theta_{n-1}^R - q_{n-1}] \\ &= q_{n-1} + \mathbb{E}[\tilde{a} - \theta_{n-1}^R - q_{n-1} | \boldsymbol{\sigma}(\tilde{v} - p_{n-1}]\end{aligned} \tag{15}$$

and for R:

$$\mathbb{E}_n^R[\tilde{v} - p_{n-1}] = \mathbb{E}[\tilde{v} - p_{n-1} | \boldsymbol{\sigma}(\tilde{a} - \theta_{n-1}^R - q_{n-1})]. \tag{16}$$

The quantity given in (15) is I's expectation of R's remaining trading demand and (16) gives R's view on the misvaluation of the risky asset. Related to the learning dynamics, the following (co)variances are heavily utilized:

$$\Sigma_n^{(1)} = \mathbb{V}\Big[\tilde{a} - \hat{\theta}_n^R - \hat{q}_n\Big], \tag{17}$$

[24] Note that the expectation is conditional on $\boldsymbol{\sigma}(y_1,...,y_{n-1})$.

$$\Sigma_n^{(2)} = \mathbb{V}\left[\tilde{v} - \hat{p}_n\right], \tag{18}$$

$$\Sigma_n^{(3)} = \mathbb{C}\left[\tilde{a} - \hat{\theta}_n^R - \hat{q}_n,\ \tilde{v} - \hat{p}_n\right]. \tag{19}$$

Since ratios of these conditional moments are often needed, let $\Sigma_n^{(3/1)} := \Sigma_n^{(3)}/\Sigma_n^{(1)}$ and $\Sigma_n^{(3/2)} := \Sigma_n^{(3)}/\Sigma_n^{(2)}$.

Therefore, in sum, both the market makers and strategic traders use all information available to refine their expectations. Further, the privately informed traders can utilize their information endowment to endogenously filter new information from observed prices in a superior fashion as compared to the market makers who are using only public information. This learning mechanism is the main channel driving the trading dynamics in the model.

One might wonder the significance of the "hatted" expressions, e.g., $\hat{\theta}_n^R$ above. To understand their usage, it is first important to recognize that heterogeneous information introduces an additional dimension of complexity to learning and belief formation. More specifically, in order to obtain a complete view regarding the equilibrium learning dynamics, one must consider the possibility that a trader deviates from the equilibrium path. In the present model, this sort of deviation is not directly observable to other (non-deviating) traders. Matters are, however, somewhat simplified by the fact (see, Foster and Viswanathan 1996) that since all order flows are observed with non-zero probability, there is no need to describe off-equilibrium-path beliefs.

To account for the deviations from the equilibrium path, suppose I, instead of applying the equilibrium strategy, has submitted arbitrary orders during a number of initial trading rounds and define hatted versions of y_n, $\Delta\theta_n^R$, and $\Delta\theta_n^I$ (as well as the other relevant quantities given above) such that $\hat{y}_n = \Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R + \tilde{u}_n$. These hatted processes describe what would have happened if I had followed the equilibrium strategy instead.[25] More specifically, the hatted processes capture what I believes (knows) that R can infer from past order flows subject to I utilizing his equilibrium strategy. A standard rational expectations argument dictates that the hatted processes must equal the unhatted ones in equilibrium, i.e., $\hat{y}_n = y_n$.

In order for the hatted processes to be useful, it is essential to verify that $\hat{y}_k \in \boldsymbol{\sigma}(\tilde{\xi}, y_1, ..., y_k)$ for all $1 \leq k \leq n$, where $\tilde{\xi} \in \{\tilde{a}, \tilde{v}\}$. That is to say, it should hold that $\boldsymbol{\sigma}(\tilde{\xi}, \hat{y}_1, ..., \hat{y}_n) = \boldsymbol{\sigma}(\tilde{\xi}, y_1, ..., y_n)$ which is indeed shown to be the case below. This property is important and useful as it guarantees the feasibility of certain orthogonality properties that are utilized, for instance, in determining the state variable dynamics for the traders R and I.

More specifically, orthogonality facilitates Markovian dynamics, thus resolving

[25] When solving the optimization problem of I, the same hatted processes are not used to determine the strategy of R as this would mean that the strategy chosen by R would be independent of the actual strategy of I. This contradicts the premise that the traders act strategically and utilize all the information available to them. See, Foster and Viswanathan (1996).

the issue expanding state history and allowing for a tractable treatment of the *forecast the forecast of others* problem arising in the model. In addition, by verifying the above property for $\hat{y}_k$ one simultaneously verifies that the other hatted processes are also in $\sigma(\tilde{\xi}, y_1, ..., y_n)$. This setting is aptly described in Foster and Viswanathan (1996) as the need for a specified trader i to "keep two sets of books". The first one documents the equilibrium path while the second one documents realized outcomes given possible non-equilibrium path actions by i.

Pertaining to the second book, deviations from the equilibrium path cause deviations in prices p_n as well as in θ^I_n, θ^R_n (since the orders of the strategic traders are not independent of each other), and q_n. To keep track of these deviations, the following quantities are paramount in determining the sufficient state variables:

$$\hat{p}_n - p_n, \quad \hat{\theta}^I_n - \theta^I_n, \quad \hat{\theta}^R_n - \theta^R_n, \quad \hat{q}_n - q_n.$$

All of the above play a role in solving for the traders' optimization problems (cf. Choi et al. 2019) Moreover, based on the above discussion it is immediately clear that all these deviations are equal to zero on the equilibrium path.

4 TWO-PERIOD EXAMPLE

4.1 Equilibrium Properties

The expressions stemming from the multiperiod dynamic optimization in the N period model can be relatively complicated, and it might be hard to grasp the intuition behind the lengthy formulas. Hence, illustrating the model mechanics in the simplest possible dynamic setting, namely the two period version of the model, seems well advised. The two period example also enables one to introduce many of the relevant details and concepts employed in the general multiperiod model as well as, through computational illustrations, to depict some *static* comparative statistics in the spirit of Subrahmanyam (1991). The focus in this section, for brevity, is only on risk aversion; possible implications of risk-seeking behavior are discussed in Section 5.3.[26]

Start with R whose problem does not involve optimization at $N = 2$ due to the condition $\theta^R_2 = \tilde{a}$ which directly implies that $\beta^R_2 = 1$ and $\alpha^R_2 = 0$ in (8). The only task left then is to determine the value function coefficients for period $n = 1$. Since R's

[26] Due to the focus on risk aversion the denominator coefficient A^K, $K \in \{I, R\}$, in (3) and (5) is omitted. Generally speaking, the inclusion of the risk aversion coefficient in the denominator ensures convexity in the risk-seeking case

objective is to minimize expected trading costs, plugging in the final period trading parameters α_2^R and β_2^R yields:

$$-\mathbb{E}_2^R\Big[e^{A^R\left(\tilde{a}-\theta_1^R\right)\lambda_2\left((\tilde{a}-\theta_1^R-q_1)+\beta_2^I(\tilde{v}-p_1)+\tilde{u}_2\right)}\Big]. \tag{20}$$

Denote, for all $n = 1,...,N$, by $\mathbf{Y}_n$ the state variable vector of R, i.e., $\mathbf{Y}_n = (Y_n^{(1)}\ Y_n^{(2)}\ Y_n^{(3)})^\top$, where[27]

$$Y_n^{(1)} = \tilde{a} - \theta_n^R, \tag{21}$$

$$\begin{aligned} Y_n^{(2)} &= (\hat{p}_n - p_n) + \mathbb{E}_n^R[\tilde{v} - \hat{p}_n] \\ &= (\hat{p}_n - p_n) + \Sigma_n^{(3/1)}(\tilde{a} - \hat{\theta}_n^R - \hat{q}_n), \end{aligned} \tag{22}$$

$$Y_n^{(3)} = q_n. \tag{23}$$

The state variables given in (21)-(23)—with $Y_0^{(1)} = \tilde{a}$, $Y_0^{(2)} = \frac{\sigma_v\rho}{\sigma_a}\tilde{a}$, and $Y_0^{(3)} = 0$—correspond to the state variables used in Choi et al. (2019). $Y_n^{(1)}$ keeps track of the remaining trading demand, $Y_n^{(2)}$ is a composite state variable related to the risky asset misvaluation, and $Y_n^{(3)}$ tracks the market makers' belief regarding latent trading demand. The invariability of the state variables with respect to changes in risk preferences is standard in CARA-Gaussian settings. However, one should recall from Section 3.2 that, in the present case, this invariability result hinges on how the objective function of R is set up. Similar issues are not present in most Kyle (1985) extensions.

Composite state variables, e.g., (22), are used to minimize the number of required state variables. Moreover, as noted earlier, all deviations are zero in equilibrium, i.e., the hatted processes concur with the realized values, and so one obtains the following identity:

$$Y_n^{(2)} = \Sigma_n^{(3/1)}\left(Y_n^{(1)} - Y_n^{(3)}\right). \tag{24}$$

Utilizing the state variables, one can write (20) as:

$$\begin{aligned} &-e^{A^R\lambda_2\left((Y_1^{(1)})^2+\beta_2^I(Y_1^{(1)}Y_1^{(2)})-(Y_1^{(1)}Y_1^{(3)})\right)} \\ &\times\mathbb{E}_1^R\Big[e^{A^R\lambda_2\beta_2^I\left(Y_1^{(1)}\left((\tilde{v}-\hat{p}_1)-\mathbb{E}_1^R[\tilde{v}-\hat{p}_1]\right)+Y_1^{(1)}\tilde{u}_2\right)}\Big]. \end{aligned} \tag{25}$$

It follows from (25) that the value function coefficients of R for $n = 1$ are given by:

$$L_1^{(2,0,0)} = -\frac{\lambda_2}{2}\left(2 + A^R\lambda_2\left(\left(\beta_2^I\right)^2\Sigma_1^{(R)} + \sigma_u^2\right)\right), \quad L_1^{(1,1,0)} = -\lambda_2\beta_2^I, \quad L_1^{(1,0,1)} = -\lambda_2,$$
$$L_1^{(0,2,0)} = 0, \quad L_1^{(0,1,1)} = 0, \quad L_1^{(0,0,2)} = 0,$$

[27] All vectors are assumed to be column vectors unless otherwise stated.

where[28]

$$\Sigma_1^{(R)} := \mathbb{V}\Big[(\tilde{v} - \hat{p}_1) - \mathbb{E}_1^R[\tilde{v} - \hat{p}_1]\Big]$$
$$= \Sigma_1^{(2)} - \frac{\left(\Sigma_1^{(3)}\right)^2}{\Sigma_1^{(1)}}.$$

For brevity, the value function coefficients are denoted by $L_n^{(\boldsymbol{\phi})}$, where $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3) \in \Phi := \left\{(i, j, k) \in \mathbb{N}^3 \ : \ 0 \le i, j, k \le 2, \ i + j + k = 2\right\}$.

Moving on to the informed trader, the first thing to note is that—unlike R—I has an actual optimization problem to solve in the final period. This problem is:

$$\max_{\Delta\theta_2^I} \ -\mathbb{E}_2^I\Big[e^{-A^I\left(\tilde{v} - \Delta p_2\right)\Delta\theta_2^I}\Big]. \tag{26}$$

Denote by $\mathbf{X}_n$ the state variable vector of I, i.e., $\mathbf{X}_n = (X_n^{(1)} \ X_n^{(2)})^\top$, where

$$X_n^{(1)} = \tilde{v} - p_n, \qquad X_0^{(1)} = \tilde{v}, \tag{27}$$
$$X_n^{(2)} = (\hat{\theta}_n^R - \theta_n^R) + (\hat{q}_n - q_n) + \Sigma_n^{(3/2)}(\tilde{v} - p_n), \quad X_0^{(2)} = \frac{\sigma_a \rho}{\sigma_v}\tilde{v}. \tag{28}$$

Again, the state variables given in (27)-(28) corresponds the state variables used in Choi et al. (2019). $X_n^{(1)}$ captures misvaluation and $X_n^{(2)}$ is a composite state variable, capturing expectations related to the latent trading demand of R. On the equilibrium path, it holds that:

$$X_n^{(2)} = \Sigma_n^{(3/2)} X_n^{(1)}. \tag{29}$$

The above identity is used in conjunction with the first order condition (FOC) to obtain an expression for β_2^I. The FOC for (26), after using (29), is:[29]

$$(\Delta\theta_2^I)^* = \underbrace{\left[\frac{1 - \lambda_2 \Sigma_1^{(3/2)}}{\lambda_2\left(2 - A^I \lambda_2 (\Sigma_1^{(I)} + \sigma_u^2)\right)}\right]}_{=:\beta_2^I} X_n^{(1)}, \tag{30}$$

where:[30]

$$\lambda_2 = \frac{\beta_2^I \Sigma_1^{(2)} + \Sigma_1^{(3)}}{\left(\beta_2^I\right)^2 \Sigma_1^{(2)} + \Sigma_1^{(1)} + 2\beta_2^I \Sigma_1^{(3)} + \sigma_u^2} \tag{31}$$

and

28 $\Sigma_n^{(R)}$ is obtained analogously.

29 Note that by setting $A^I = 0$ one recovers β_2^I in the risk neutral case.

30 Note especially that $\Sigma_n^{(I)} \neq \Sigma_n^{(1)}$.

$$\Sigma_1^{(I)} := \Sigma_1^{(1)} - \frac{\left(\Sigma_1^{(3)}\right)^2}{\Sigma_1^{(2)}}.$$

The second order condition (SOC) says:

$$-\lambda_2\Big(2 - A^I\lambda_2(\Sigma_1^{(I)} + \sigma_u^2)\Big) < 0.$$

The implications of the SOC on the sign of λ_2 (or, more generally, on λ_N) are ambiguous. However, if one allows for $\lambda_2 < 0$, the SOC is always true when $A^I > 0$ and $\Sigma_1^{(I)} + \sigma_u^2 > 0$.[31] This quickly leads to counter-intuitive equilibrium properties such as I trading against his private information. These issues are avoided when $\lambda_2 > 0$. For this reason, in what follows, the focus is on equilibria in which λ_N is positive. Lemma 1 guarantees that such a solution exists.

Lemma 1 (**Period** N **existence result**). *There exists a solution, satisfying* $\lambda_N > 0$*, to the bivariate period N system:*

$$\begin{cases} \dot{\Sigma}_{N-1}\lambda_N^2\beta_N^I + 2\lambda_N\beta_N^I + \Sigma_{N-1}^{(3/2)}\lambda_N - 1 & = 0, \\ \Sigma_{N-1}^{(2)}\lambda_N\left(\beta_N^I\right)^2 + 2\Sigma_{N-1}^{(3)}\lambda_N\beta_N^I + \bar{\Sigma}_{N-1}\lambda_N - \Sigma_{N-1}^{(2)}\beta_N^I - \Sigma_{N-1}^{(3)} & = 0, \end{cases} \tag{32}$$

where:

$$\begin{aligned} \dot{\Sigma}_{N-1} &:= -A^I\left(\Sigma_{N-1}^{(I)} + \sigma_u^2\right) \\ \bar{\Sigma}_{N-1} &:= \Sigma_{N-1}^{(1)} + \sigma_u^2. \end{aligned}$$

To understand the role of the equations in Lemma 1, note that by combining (30) with the expression for λ_2 given in (31) one can form a pair of *coupled* equations—referred to here as the period N system—to solve the two unknowns. These are in fact the only two unknowns in the last period—at least from the point of view of solving the model numerically. The reason for this is that, due to the restriction $\beta_2^R = 1$, one has $r_2 = 0$ (see, (14)). Further, to initiate the numerical procedure, one makes an educated guess for the period $n = N - 1$ (here, period $n = 1$) moments $(\Sigma_{N-1}^{(i)})_{i=1,2,3}$. Therefore, one can treat the moments appearing in (32) as constants.

It is worth pointing out that the above result likewise holds in the general case $N \geq 2$ and not just in the two period case. However, even with $\lambda_N > 0$, one cannot guarantee that β_N^I is positive, as the sign of β_N^I depends also on the values of the exogenous parameters. Similarly, standard comparative statistics yield ambiguous results when applied to the period N system. A case in point is the sensitivity of β_N^I, obtained via an application of the *Implicit Function Theorem*, to changes in the risk

[31] Alternatively, if one requires that $\lambda_n > 0$ and lets $A^I < 0$, then the second order condition is always true. Clearly, the case where $\lambda_n = 0$ is not relevant.

aversion parameter A^I:

$$\frac{\partial \beta_N^I}{\partial A^I} = \frac{-1}{|\mathbf{M}|}\left(-\dot{\Sigma}_{N-1}\beta_N^I \lambda_N^2\left(\left(\beta_N^I\right)^2 + 2\Sigma_{N-1}^{(3/2)}\beta_N^I + \bar{\Sigma}_{N-1}\left(\Sigma_{N-1}^{(2)}\right)^{-1}\right)\right) \lessgtr 0,$$

where

$$\mathbf{M} := \begin{bmatrix} \lambda_N\left(2 - A^I\dot{\Sigma}_{N-1}\lambda_N\right) & 2\beta_N^I\left(1 - A^I\dot{\Sigma}_{N-1}\lambda_N\right) + \Sigma_{N-1}^{(3/2)} \\ 2\lambda_N\left(\beta_N^I + \Sigma_{N-1}^{(3/2)}\right) - 1 & \beta_N^I\left(\beta_N^I + 2\Sigma_{N-1}^{(3/2)}\right) + \bar{\Sigma}_{N-1}(\Sigma_{N-1}^{(2)})^{-1} \end{bmatrix}.$$

The above is just one individual example but it plainly justifies the focus on numerical techniques in examining the equilibrium properties.[32]

To obtain the value function coefficients $I_1^{(\boldsymbol{\omega})}$, where $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \Omega$ and $\Omega := \left\{(i,j) \in \mathbb{N}^2 \ : \ 0 \le i, j \le 2, \ i + j = 2\right\}$, one plugs the general expression obtained from the FOC of I back into I's objective function and groups together the relevant terms. Then, after having determined the period $n = 2$ unknowns and the value function coefficients for both R and I, one can proceed to solving the period $n = 1$ problems.

The first period problems of R and I respectively are:

$$\max_{\Delta\theta_1^R} \ -\mathbb{E}_1^R\left[e^{-A^R\left(-\left(\tilde{a}-\theta_0^R\right)\Delta p_1 + \sum_{\boldsymbol{\phi}\in\Phi} L_1^{(\boldsymbol{\phi})}\mathbf{Y}_1^{\boldsymbol{\phi}}\right)}\right], \tag{33}$$

$$\max_{\Delta\theta_1^I} \ -\mathbb{E}_1^I\left[e^{-A^I\left(\left(\tilde{v}-\Delta p_1\right)\Delta\theta_1^I + \sum_{\boldsymbol{\omega}\in\Omega} I_1^{(\boldsymbol{\omega})}\mathbf{X}_1^{\boldsymbol{\omega}}\right)}\right], \tag{34}$$

where $\mathbf{Y}_1^{\boldsymbol{\phi}} = (Y_1^{(1)})^{\phi_1}(Y_1^{(2)})^{\phi_2}(Y_1^{(1)})^{\phi_3}$ and $\mathbf{X}_1^{\boldsymbol{\omega}} = (X_1^{(1)})^{\omega_1}(X_1^{(2)})^{\omega_2}$. The expressions in the exponent will be quadratic in the decision variables after evaluating the respective expectations. Since $x \mapsto e^x$ is increasing, it is sufficient to focus on these quadratic expressions.

In evaluating the expectations in (33) and (34) the dynamics of the state variables are in a central role. Namely, one wishes to understand how $\mathbf{Y}_n$ and $\mathbf{X}_n$ evolve as functions of past state variables and other model parameters. In turns out that, these dynamics are Markovian and hence the expectations can be evaluated efficiently. State variable dynamics are detailed in Section 5.

From the FOCs stemming from (33) and (34), in conjunction with the equilibrium path restrictions (24) and (29) and some algebraic manipulations, one obtains the pair of *coupled* polynomial equations for β_1^I and β_1^R. This pair of equations cannot be solved explicitly and thus one must resort to numerical methods in finding the solution(s).[33] The solutions obtained are crucial to the task of solving the entire model as one can use the solutions to (uniquely) determine all other equilibrium unknowns.

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32 The particular question of $\frac{\partial \beta_N^I}{\partial A^I}$, within the context of the two period model and for one example parametrization, is answered in Figure 2b.

33 Alternatively, one can skip the algebraic manipulations and resort to solving a higher dimensional system of polynomials. This, more direct, approach is described in Appendix B.2.

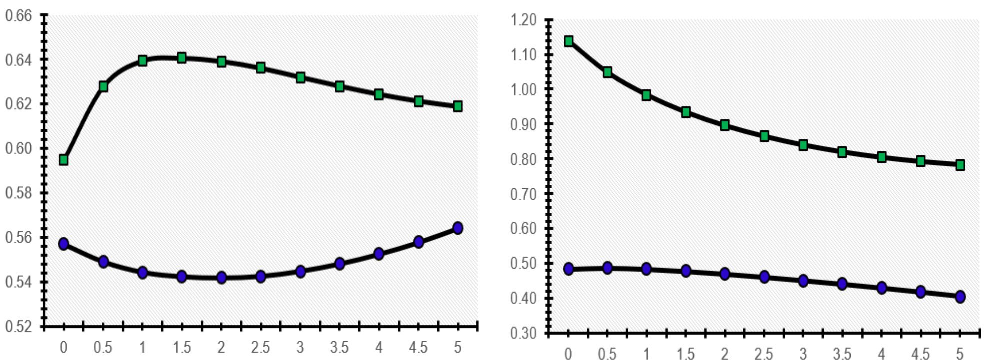
4.2 Numerical Results for Two Period Model

Solving the two period model numerically closely follows the steps in the above presentation and is reminiscent of the static analysis in Subrahmanyam (1991). Detailed discussion of numerical solution methods can be found in Appendix B.

The goal of the following selected numerical examples is to illustrate some of the consequences of risk aversion on dynamic trading in as simple a setting as possible. One could also study the consequences of risk-seeking behavior. However, as the intention is to provide a concise and introductory overview of the model mechanics, only risk aversion is examined here. A good starting point for the overview is the intensity of informed trading and price impact.

Figure 2 shows the evolution of β_n^I (green squares) and λ_n (blue circles), $n = 1, 2$, as a function of risk aversion of the strategic traders. In line with the risk neutral model, it is assumed for now that I and R have an identical coefficient of absolute risk aversion, i.e., $A^I = A^R$.

Figure 2: (**Informed trading intensity and price impact**) This figure depicts β_n^I (denoted by ▪) and λ_n (denoted by •) for $n = 1, 2$. Model parameter values used: $A^R = A^I$ (values on the x-axis), $\sigma_a^2 = \sigma_v^2 = 1$, $\sigma_u^2 = \frac{1}{N}$, and $\rho = 0$.

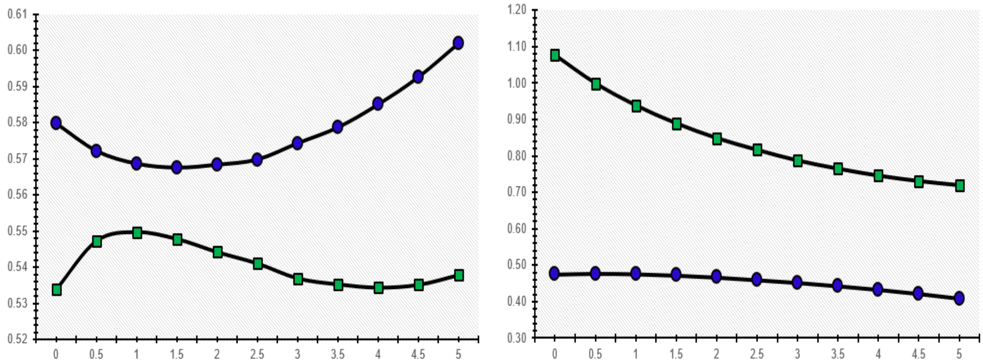


(a) β_1^I and λ_1 (b) β_2^I and λ_2

An interesting observation from Figure 2 is that the relationship between informed trading intensity and risk aversion—and thus the relationship between price impact and risk aversion—is *non-monotonic* in the first round. For example, the trading intensity of the informed trader β_1^I initially rises sharply for low risk aversion levels only to decrease again with higher levels of risk aversion, resulting in an inverted U-shape. Conversely, although in a less extreme fashion, λ_1 exhibits a U-shaped pattern to balance changes in the intensity of informed trading. Non-monotonicities are also discovered in Subrahmanyam (1991). It is also good to note that, in terms of magnitude, the trading intensity β_1^I dominates the price impact parameter λ_1.

Above, due to setting $\rho = 0$, information about the risky asset payoff is only conveyed through the trades of the informed trader. What happens if one allows for positive correlation, say $\rho = 1/3$? Figure 3 answers this question.

Figure 3: (**Informed trading intensity and price impact revisited**) This figure depicts β_n^I (denoted by ■) and λ_n (denoted by ●) for $n = 1, 2$. Model parameter values used: $A^R = A^I$ (values on the x-axis), $\sigma_a^2 = \sigma_v^2 = 1$, $\sigma_u^2 = \frac{1}{N}$, and $\rho = 1/3$.



(a) β_1^I and λ_1 (b) β_2^I and λ_2

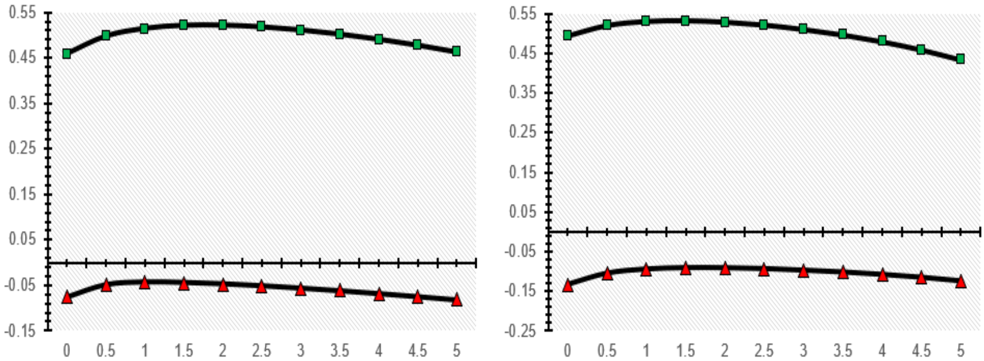
As seen from Figure 3, when $\rho = 1/3$ the results pertaining to β_1^I and λ_1 incur noticeable changes (compare to Figure 2a), while the results pertaining to β_2^I and λ_2 remain almost unchanged. On one hand, the reason for the change in the first period constants is due to the fact that shifting from $\rho = 0$ to $\rho = 1/3$ changes the behavior of R who now, in addition to having private information about $\tilde{a}$, also possesses private information about $\tilde{v}$. On the other hand, the reason for why β_2^I and λ_2 are only slightly affected stems from the fact that R's trade at $t = 2$ is fixed due to need to meet the hard trading target, and thus unaffected by the change in ρ.

Moving on to the rebalancer, β_1^R (green square) as well as α_1^R (red triangle) in Figure 4 can be seen to exhibit a mild inverted U-shape. This hints at the interconnectedness of the strategies of the two traders as well as the price impact, all of which are determined endogenously in equilibrium. Even though a positive ρ merely induces a level shift in the parameters α_1^R and β_1^R, it is enough to cause a clear change in β_1^I and λ_1. This "mere level shift" observation captures a wider trend present also in the $N > 2$ versions of the model. Namely, it demonstrates how changes in the initial conditions tend to have a smaller impact on the equilibrium behavior of R than on the equilibrium behavior of I. This can be seen as a reflection of the first order importance of the trading constraint $\tilde{a}$ in governing the behavior of R.

Generally speaking, allowing R to possess some private information about $\tilde{v}$ (i.e., $\rho > 0$) affects the values of the various equilibrium constants in different ways. In the rest of this paper, when this additional information channel is found to be substantial, results are reported for both $\rho = 0$ and $\rho = 1/3$.[34]

[34] Naturally, choosing $\rho = 1/3$ is somewhat arbitrary but, in any case, it provides an idea on how positive ρ may

Figure 4: (**Rebalancer trading intensities**) This figure depicts β_1^R (denoted by ■) and α_1^R (denoted by ▲) as functions of risk aversion. Model parameter values used: $A^R = A^I$ (values on the x-axis), $\sigma_a^2 = \sigma_v^2 = 1$, and $\sigma_u^2 = \frac{1}{N}$.

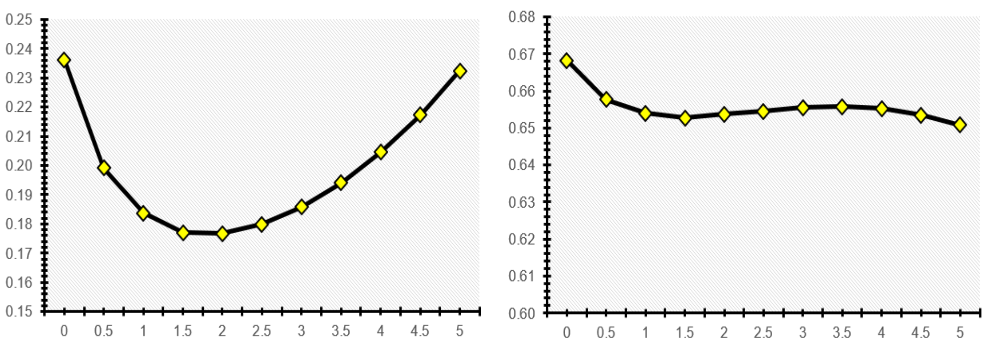


(a) $\rho = 0$ (b) $\rho = 1/3$

It is also evident from Figure 4 that $\alpha_1^R + \beta_1^R > 0$ which indicates, as noted in Choi et al. (2019), that the rebalancer tends to buy more when the market expects the rebalancer to have a large latent trading demand. Moreover, due to the few trading opportunities present in the two period version of the model, the difference between β_1^R and α_1^R is relatively big as the rebalancer relies more heavily on sunshine trading.

As a final demonstration, Figure 5 illustrates the behavior of the period $n = 1$ conditional variances as functions of risk aversion; recall that $\Sigma_n^{(1)}$ is the conditional variance of $\tilde{a} - \theta_n^R$ (unrealized, or latent, demand of R) and $\Sigma_n^{(2)}$ is the conditional variance of $\tilde{v}$.

Figure 5: (**Conditional variances in the first round**) This figure depicts $\Sigma_1^{(1)}$ (on the left) and $\Sigma_1^{(2)}$ (on the right) as functions of risk aversion. Model parameter values used: $A^R = A^I$ (values on the x-axis), $\sigma_a^2 = \sigma_v^2 = 1$, $\sigma_u^2 = \frac{1}{N}$, and $\rho = 0$.



(a) $\Sigma_1^{(1)}$ (b) $\Sigma_1^{(2)}$

affect the equilibrium results. Higher levels of ρ amplify the effect and lower levels weaken it. It is also possible that for some positive ρ's equilibrium fails to exist or that numerical instabilities occur (cf. Choi et al. 2019).

Unsurprisingly, both conditional variances reach their lowest levels when the trading intensities of both the informed trader and the rebalancer hit their peaks, i.e., information is revealed the fastest. However, where $\Sigma_1^{(1)}$ again follows an U-shaped pattern, $\Sigma_1^{(2)}$ interestingly exhibits an S-shape as seen in Figure 5b. This difference stems from the fact that solving $\Sigma_n^{(2)}$ backwards yields noticeably more complicated dynamics for $\Sigma_{n-1}^{(2)}$ as does solving $\Sigma_n^{(1)}$ backwards yield for $\Sigma_{n-1}^{(1)}$. The key piece of intuition here is then that while $\Sigma_1^{(1)}$ is a rather simple function of β_1^I and β_1^R, $\Sigma_1^{(2)}$ involves more complicated (nonlinear) expressions of these two trading intensities. Hence, the conditional variance of the risky asset payoff ends up having a unique pattern as compared to the conditional variance of the trading target.

In conclusion, the two period model numerics illustrate some interesting aspects of the model while at the same time raise some further questions. For instance, will the intraday dynamics and trading patterns presented in Choi et al. (2019) remain valid when one or both strategic traders are risk averse? Before looking into this question and others numerically, the next section presents the solution to the full N period model.

5 FULL MODEL

5.1 Equilibrium Derivation

In this section a verification theorem for the full dynamic equilibrium is constructed. Start with the set of constants $\{\lambda_n,\ r_n,\ \beta_n^I,\ \alpha_n^R,\ \beta_n^R\}_{n=1,\ldots,N}$ and processes:

$$\Delta\hat{\theta}_n^I = \beta_n^I(\tilde{v} - \hat{p}_{n-1}), \tag{35}$$

$$\Delta\hat{\theta}_n^R = \beta_n^R(\tilde{a} - \hat{\theta}_{n-1}^R) + \alpha_n^R \hat{q}_{n-1}, \tag{36}$$

$$\hat{y}_n = \Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R + \tilde{u}_n, \tag{37}$$

$$\Delta\hat{p}_n = \lambda_n(\hat{y}_n - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1}), \tag{38}$$

$$\Delta\hat{q}_n = r_n\hat{y}_n - (1 + r_n)(\alpha_n^R + \beta_n^R)\hat{q}_{n-1}, \tag{39}$$

for all $n = 1,\ldots,N$, with boundary conditions $\beta_N^R = 1,\ \alpha_N^R = 0,\ \hat{\theta}_0^I = 0,\ \hat{\theta}_0^R = 0,\ \hat{y}_0 = 0,\ \hat{p}_0 = 0,$ and $\hat{q}_0 = 0$. The first two boundary conditions stem from R's trading target while the next two are assumptions. These assumptions and the model primitives imply the final three boundary conditions.

Recall that the hatted processes depict the conjectures concerning other traders' beliefs about the equilibrium dynamics. This formulation captures the idea that an optimal strategy for a market participant, at any given round n, is optimal even if this particular participant has followed an arbitrary strategy in earlier rounds. Nevertheless, in equilibrium the beliefs must be correct, i.e., $\hat{\theta}^I_n = \theta^I_n$. In addition, off-equilibrium-path beliefs are not a concern since all orders flows are observed with a positive probability.

Endogenous learning on part of the strategic traders (R and I) is one of the key drivers behind the model dynamics. This feature, however, does bring about some challenges. Namely, even though the CARA-Gaussian setting enables one to utilize effectively standard filtering theory, the resulting expressions are complicated to the extent that the equilibrium cannot be represented in closed form. Moreover, without learning the equilibrium would have a much simpler form and, for instance, the forecast the forecasts of others problem would cease to be an issue.

Yet, it is imperative to study models that allow for endogenous learning so that one can better understand how this class of models differs—if it indeed differs—from models where traders have fixed beliefs. The justification is straightforward. On one hand, if the differences are found to be significant, then one has a chance to uncover why this is the case. On the other hand, if the differences are found to be small or negligible, then one obtains a good justification to focus on other important questions and to utilize simpler models as proxies for ones that are more complicated.

The following lemma gives the equilibrium properties of the projection coefficients and the equilibrium moments.[35]

Lemma 2 (**Equilibrium pricing constants and moments**). *Recall that* $\hat{p}_n := \mathbb{E}\left[\tilde{v} \mid \hat{y}_1,...,\hat{y}_n\right]$ *and* $\hat{q}_n := \mathbb{E}\left[\tilde{a} - \hat{\theta}^R_n \mid \hat{y}_1,...,\hat{y}_n\right]$. *Since the market makers are rational and take advantage of Bayesian updating it follows that, for all* $n = 1,...,N$*:*

$$\lambda_n = \frac{\beta^I_n \Sigma^{(2)}_{n-1} + \beta^R_n \Sigma^{(3)}_{n-1}}{(\beta^R_n)^2 \Sigma^{(1)}_{n-1} + (\beta^I_n)^2 \Sigma^{(2)}_{n-1} + 2\beta^I_n \beta^R_n \Sigma^{(3)}_{n-1} + \sigma^2_u}, \tag{40}$$

$$r_n = \frac{(1-\beta^R_n)(\beta^I_n \Sigma^{(3)}_{n-1} + \beta^R_n \Sigma^{(1)}_{n-1})}{(\beta^R_n)^2 \Sigma^{(1)}_{n-1} + (\beta^I_n)^2 \Sigma^{(2)}_{n-1} + 2\beta^I_n \beta^R_n \Sigma^{(3)}_{n-1} + \sigma^2_u}. \tag{41}$$

Moreover, the moment recursions are given by:

$$\Sigma^{(1)}_n = \left(1-\beta^R_n\right)\left((1-\beta^R_n - r_n\beta^R_n)\Sigma^{(1)}_{n-1} - r_n\beta^I_n\Sigma^{(3)}_{n-1}\right), \qquad \Sigma^{(1)}_0 = \sigma^2_a, \tag{42}$$

$$\Sigma^{(2)}_n = \left(1-\lambda_n\beta^I_n\right)\Sigma^{(2)}_{n-1} - \lambda_n\beta^R_n\Sigma^{(3)}_{n-1}, \qquad \Sigma^{(2)}_0 = \sigma^2_v, \tag{43}$$

$$\Sigma^{(3)}_n = \left(1-\beta^R_n\right)\left((1-\lambda_n\beta^I_n)\Sigma^{(3)}_{n-1} - \lambda_n\beta^R_n\Sigma^{(1)}_{n-1}\right), \qquad \Sigma^{(3)}_0 = \rho\sigma_a\sigma_v. \tag{44}$$

These coefficients are mainly related to the filtering problem of the market makers albeit the moments given in equations (42)-(44), or more specifically, the ratios of

[35] The moments are conditional on period n information after the order flow for the period has realized.

these moments, play a crucial role also in I's and R's filtering problems.[36] Note from the above that if $\beta_n^R = 1$, for some $n = 1,...,N-1$, then $\Sigma_n^{(1)} = \Sigma_n^{(3)} = r_n = 0$. This would essentially mean that R would choose to prematurely reach the target $\tilde{a}$, after which R would refrain from trading. This would be suboptimal in the present setting, as R has a profitable trading opportunity in each period, and only in the limit, after an infinite number of trading rounds, is R's private information completely revealed. For this reason, a constraint is set so that $\beta_n^R \neq 1$ for $n = 1,...,N-1$ (see also Choi et al. 2019).

Continuing with R's trading coefficients, denote again, for all $n = 1,...,N$, by $\mathbf{Y}_n$ the state variable vector of R, i.e., $\mathbf{Y}_n = (Y_n^{(1)}\ Y_n^{(2)}\ Y_n^{(3)})^\top$. In the following lemma the measurability conditions and the dynamics for R's state variables are described.

Lemma 3 (***R*****'s state variable dynamics and measurability**). *The dynamics of R's state variables are given by:*

$$\Delta Y_n^{(1)} = -\Delta\theta_n^R, \qquad Y_0^{(1)} = 0, \tag{45}$$

$$\Delta Y_n^{(2)} = -\lambda_n\Big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)} - (\alpha_n^R + \beta_n^R) Y_{n-1}^{(3)}\Big) - r_n \frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}} \hat{z}_n^R, \quad Y_0^{(2)} = \frac{\sigma_v \rho}{\sigma_a}\tilde{a}, \tag{46}$$

$$\Delta Y_n^{(3)} = r_n\Big(\Delta\theta_n^R + \beta_n^I Y_{n-1}^{(2)}\Big) - (1 + r_n)(\alpha_n^R + \beta_n^R) Y_{n-1}^{(3)} + r_n \hat{z}_n^R, \ Y_0^{(3)} = 0, \tag{47}$$

where

$$\hat{z}_n^R = \hat{y}_n - \Delta\hat{\theta}_n^R - \beta_n^I \frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}\Big(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}\Big), \ \ n = 1,2,...,N. \tag{48}$$

Moreover, for all $\mathbb{N} \ni n \leq N$*:*[37]

$$\begin{aligned} &\hat{z}_n^R \sim \mathcal{N}(0, \mathbb{V}[\hat{z}_n^R]), \\ &\boldsymbol{\sigma}(\tilde{a}, y_1, ..., y_n) = \boldsymbol{\sigma}(\tilde{a}, \hat{y}_1, ..., \hat{y}_n) = \boldsymbol{\sigma}(\tilde{a}, \hat{z}_1^R, ..., \hat{z}_n^R), \ \textit{and} \\ &\hat{z}_n^R \perp (\tilde{a}, y_1, ..., y_{n-1}). \end{aligned}$$

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36 The derivation of equations (42) - (44) is straightforward by first noting that:

$$\begin{aligned} \Sigma_n^{(1)} &= \mathbb{V}\Big[\tilde{a} - \hat{\theta}_n^R - \hat{q}_n\Big] = \mathbb{V}\Big[\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \Delta\hat{\theta}_n^R - \Delta\hat{q}_n\Big] \\ &= \mathbb{V}\Big[(1 - (1 + r_n)\beta_n^R)(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1}) - r_n\beta_n^I(\tilde{v} - \hat{p}_{n-1}) - r_n\Delta w_n\Big] \\ \Sigma_n^{(2)} &= \mathbb{V}\Big[\tilde{v} - \hat{p}_n\Big] = \mathbb{V}_n\Big[\tilde{v} - \hat{p}_{n-1} - \Delta\hat{p}_n\Big] \\ &= \mathbb{V}\Big[\tilde{v} - \hat{p}_{n-1} - \lambda_n(\Delta\hat{\theta}_n^I + \Delta\hat{\theta}_n^R + \Delta w_n - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1})\Big] \end{aligned}$$

and

$$\begin{aligned} \Sigma_n^{(3)} &= \mathbb{E}\Big[(\tilde{a} - \hat{\theta}_n^R - \hat{q}_n)(\tilde{v} - \hat{p}_n)\Big] \\ &= \mathbb{E}\Big[(\tilde{a} - \hat{\theta}_{n-1}^R - \hat{q}_{n-1} - \Delta\hat{\theta}_n^R - \Delta\hat{q}_n)(\tilde{v} - \hat{p}_{n-1} - \Delta\hat{p}_n)\Big], \end{aligned}$$

and utilizing equations (40) and (41).

37 $\mathbb{V}[\hat{z}_n^R] = \big(\beta_n^I\big)^2 \Sigma_{n-1}^{(R)} + \sigma_u^2$. The term $\Sigma_n^{(R)}$ is covered in Section 4.

It can be seen from Lemma 3 that R's state variable dynamics are identical to the risk neutral case. It is mainly due to this, i.e., the invariance of R's and I's state variables under CARA-utility, that the description of the equilibrium is almost identical to the risk neutral case. Similar invariance holds also between the risk neutral and risk averse Kyle (1985) model.

The following lemma concludes the description of R's problem and gives R's value function.

Lemma 4 (R**'s value function**). *For each $n = 1,...,N$, the value function of R is of the form:*

$$\max_{\substack{\Delta\theta_m^R \in \boldsymbol{\sigma}_m^R,\\ n+1\le m\le N-1}} -\mathbb{E}\left[\frac{-1}{A^R} e^{A^R\left(\sum_{m=n+1}^{N}(\tilde{a}-\theta_{m-1}^R)\Delta p_m\right)} \middle| \boldsymbol{\sigma}_{n+1}^R\right] = \Psi_n^R e^{-A^R\left(L_n^{(0)}+\sum_{\boldsymbol{\phi}\in\Phi} L_n^{(\boldsymbol{\phi})}\mathbf{Y}_n^{\boldsymbol{\phi}}\right)} \tag{49}$$

where Ψ_n^R, $L_n^{(0)}$, and $(L_n^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$ are constants independent of R's state variables, $\boldsymbol{\sigma}_{n+1}^R := \boldsymbol{\sigma}(\tilde{a}, y_1,...,y_n)$, and:

$$\begin{aligned}\boldsymbol{\phi} &=(\phi_1,\phi_2,\phi_3) \in \Phi := \left\{(i,j,k)\in\mathbb{N}^3 \ : \ 0\le i,j,k\le 2,\ i+j+k=2\right\},\\ \mathbf{Y}_n^{\boldsymbol{\phi}} &=\left(Y_n^{(1)}\right)^{\phi_1}\left(Y_n^{(2)}\right)^{\phi_2}\left(Y_n^{(3)}\right)^{\phi_3}.\end{aligned}$$

Moreover, the first and second order conditions of R's problem are given by:

$$(\Delta\theta_n^R)^* =\frac{\Lambda_n^{(1)}}{D_n^R}Y_{n-1}^{(1)} + \frac{\Lambda_n^{(2)}}{D_n^R}Y_{n-1}^{(2)} + \frac{\Lambda_n^{(3)}}{D_n^R}Y_{n-1}^{(3)}, \tag{50}$$

$$-D_n^R < 0. \tag{51}$$

Since the conjectured processes must correspond to the realized processes in equilibrium, it is possible to write $Y_n^{(2)} = \Sigma_n^{(3/1)}\left(Y_n^{(1)} - Y_n^{(3)}\right)$, where $\Sigma_n^{(3/1)} := \frac{\Sigma_n^{(3)}}{\Sigma_n^{(1)}}$, and hence:

$$(\Delta\theta_n^R)^* = \underbrace{\left[\frac{\Lambda_n^{(1)}}{D_n^R}+\Sigma_n^{(3/1)}\frac{\Lambda_n^{(2)}}{D_n^R}\right]}_{=:\beta_n^R}\left(Y_{n-1}^{(1)}\right)+\underbrace{\left[\frac{\Lambda_n^{(3)}}{D_n^R}-\Sigma_n^{(3/1)}\frac{\Lambda_n^{(2)}}{D_n^R}\right]}_{=:\alpha_n^R}\left(Y_{n-1}^{(3)}\right). \tag{52}$$

This expression corresponds to the one given in (8).

After having described R's problem, it is now time to move on to I. Denote again by $\mathbf{X}_n$ the state variable vector of I, i.e., $\mathbf{X}_n = (X_n^{(1)}\ X_n^{(2)})^\top$. Lemma 5 gives I's state variable dynamics and measurability conditions.

Lemma 5 (I**'s state variable dynamics and measurability**). *The dynamics of I's state variables are given by:*

$$\Delta X_n^{(1)} = -\lambda_n\left(\Delta\theta_n^I+\beta_n^R X_{n-1}^{(2)}\right) - \lambda_n\hat{z}_n^I, \qquad X_0^{(1)} = 0,$$

$$\Delta X_n^{(2)} = -r_n \Delta\theta_n^I - (1+r_n)\beta_n^R X_{n-1}^{(2)} - \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}\lambda_n \hat{z}_n^I,\ X_0^{(2)} = 0,$$

where

$$\hat{z}_n^I := \hat{y}_n - \Delta\hat{\theta}_n^I - (\alpha_n^R + \beta_n^R)\hat{q}_{n-1} - \beta_n^R \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}(\tilde{v} - \hat{p}_{n-1}),\ \ n = 1, ..., N.$$

Moreover, for all $\mathbb{N} \ni n \leq N$*:*[38]

$$\hat{z}_n^I \sim \mathcal{N}(0, \mathbb{V}[\hat{z}_n^I]),$$
$$\boldsymbol{\sigma}(\tilde{v}, y_1, ..., y_n) = \boldsymbol{\sigma}(\tilde{v}, \hat{y}_1, ..., \hat{y}_n) = \boldsymbol{\sigma}(\tilde{v}, \hat{z}_1^I, ..., \hat{z}_n^I),\ \textit{and}$$
$$\hat{z}_n^I \perp (\tilde{v}, y_1, ..., y_{n-1}).$$

The next lemma completes the description of I's problem by stating the form of I's value function and giving the corresponding optimality conditions.

Lemma 6 (**I's value function**). *For each* $n = 1, ..., N$*, the value function of* I *is of the form:*

$$\max_{\substack{\Delta\theta_m^I \in \boldsymbol{\sigma}_m^I,\\ n+1 \leq m \leq N-1}} \mathbb{E}\left[\frac{-1}{A^I} e^{-A^I\left(\sum_{m=n+1}^N (\tilde{v} - p_m)\Delta\theta_m^I\right)} \middle| \boldsymbol{\sigma}_{n+1}^I\right] = \Psi_n^I e^{-A^I\left(I_n^{(0)} + \sum_{\boldsymbol{\omega}\in\mathbb{N}_2^2} I_n^{(\boldsymbol{\omega})}\mathbf{X}_n^{\boldsymbol{\omega}}\right)}, \tag{53}$$

where Ψ_n^I*,* $I_n^{(0)}$*, and* $(I_n^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$ *are constants independent of* I*'s state variables,* $\boldsymbol{\sigma}_{n+1}^I := \boldsymbol{\sigma}(\tilde{v}, y_1, ..., y_n)$*, and:*

$$\boldsymbol{\omega} = (\omega_1, \omega_2) \in \Omega := \left\{(i,j) \in \mathbb{N}^2\ :\ 0 \leq i, j \leq 2,\ i + j = 2\right\},$$
$$\mathbf{X}_n^{\boldsymbol{\omega}} = \left(X_n^{(1)}\right)^{\omega_1}\left(X_n^{(2)}\right)^{\omega_2}.$$

Moreover, the first and second order conditions of I*'s problem are given by:*

$$(\Delta\theta_n^I)^* = \frac{\Gamma_n^{(1)}}{D_n^I} X_{n-1}^{(1)} + \frac{\Gamma_n^{(2)}}{D_n^I} X_{n-1}^{(2)}, \tag{54}$$
$$-D_n^I < 0. \tag{55}$$

In equilibrium it holds that $X_n^{(2)} = \Sigma_n^{(3/2)} X_n^{(1)}$, where $\Sigma_n^{(3/2)} := \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}}$. Thus, utilizing (54), it can be seen that I's optimal order has the form:

$$(\Delta\theta_n^I)^* = \underbrace{\left[\frac{\Gamma_n^{(1)}}{D_n^I} + \Sigma_n^{(3/2)}\frac{\Gamma_n^{(2)}}{D_n^I}\right]}_{=:\beta_n^I}\left(X_{n-1}^{(1)}\right), \tag{56}$$

[38] $\mathbb{V}[\hat{z}_n^I] = \left(\beta_n^R\right)^2 \Sigma_{n-1}^{(I)} + \sigma_u^2$. The term $\Sigma_n^{(I)}$ is covered in Section 4.

which corresponds to expression given in (7). It is worth emphasizing that the general form of I's optimal order remains unchanged from the risk neutral case. An analogous observation can be made when comparing the expression in (56) to I's optimal order in, e.g., Kyle (1985) or Holden and Subrahmanyam (1994). All in all, the notion that I's trades are proportional to the remaining information advantage has proven to be a robust feature of Kyle-type strategic trading equilibria.

Naturally, this does not mean that the equations that define, for example, β_n^I in equilibrium are invariant across different models. Nor does it mean that the values taken by the equilibrium constants are identical across models. This point becomes clear in Section 5.3. Before that, one needs to state the set of equilibrium conditions and to discuss some of the challenges these conditions pose in explicitly solving for the model equilibrium. With this aim in mind, the following *verification theorem* summarizes the sufficient conditions for the Bayesian Nash equilibrium.

Theorem 1 (**Verification theorem**). *The set of constants:*

$$\left\{\lambda_n,\ r_n,\ \beta_n^I,\ \alpha_n^R,\ \beta_n^R\right\}_{n=1,\ldots,N}$$

with $\alpha_N^R = 0$*,* $\beta_N^R = 1$*, and* $\beta_n^R \neq 1$*, for* $n < N$*, constitutes a linear Bayesian Nash equilibrium (BNE) if,* $\forall n = 1, \ldots, N$*, it holds that:*

(i) Pricing constants and moments satisfy the equations given in Lemma 4. Conditional variance terms are non-negative, $\Sigma_n^{(2)}$ *is non-increasing, and:*

$$\left(\Sigma_n^{(3)}\right)^2 \leq \Sigma_n^{(1)} \Sigma_n^{(2)}.$$

(ii) R's first and second order conditions given in Lemma 3, in conjunction with $Y_n^{(2)} = \Sigma_n^{(3/1)}\left(Y_n^{(1)} - Y_n^{(3)}\right)$*, are satisfied and R's value function corresponds to (49).*

(iii) I's first and second order conditions given in Lemma 6, in conjunction with $X_n^{(2)} = \Sigma_n^{(3/2)} X_n^{(1)}$*, are satisfied and I's value function corresponds to (53).*

Before proceeding, it is important to acknowledge that Theorem 1 is silent about equilibrium existence and uniqueness. The reason for this is the rather complicated nature of the equilibrium dynamics, preventing one to approach the existence and uniqueness questions with explicit methods. Exhaustive results pertaining to these equilibrium properties are not pursued in this paper.[39] However, it is possible to give some guidelines which can then be utilized, in conjunction with appropriate numerical methods, to study, for instance, *case-by-case* uniqueness of any equilibrium found. Details relating to this issue are covered in the next section.

[39] General equilibrium properties of the Kyle-model have proven to be quite resistant to thorough analysis. McLennan et al. (2017) study uniqueness of the single period version of Kyle (1985); to the best knowledge of the author, notable advances regarding the dynamic version of the model range from scarce to non-existent.

5.2 Solutions to Equilibrium System of Equations

The case-by-case qualification above refers to the issue that based on the uniqueness of a given equilibrium under some *initial parametrization*—say $\mathcal{S}_{\epsilon}^{N}$, consisting of the values for the exogenous parameters and the initial guesses—one cannot necessarily conclude the uniqueness of another equilibrium under an alternative parametrization $\check{\mathcal{S}}_{\epsilon}^{N}$.[40] While general results pertaining to this matter would obviously provide more widely applicable information, "pointwise" uniqueness is not without merit. Indeed, it takes only one instance of equilibrium multiplicity to show via a counter-example that general uniqueness does not hold. Moreover, one can focus on specific situations to conduct detailed analysis regarding why uniqueness may fail under certain conditions and persist in some other cases.

To pave way for the ensuing discussion, it is worthwhile to revisit the standard solution approach to the Kyle (1985) model in which, as in the present model, the equilibrium boils down to finding solutions to a system of difference equations. Two distinctive features are especially important:

1. First, the Kyle (1985) equilibrium system can be reduced to finding a root of a univariate cubic equation in λ_n. With appropriate parametrizations, this cubic polynomial features a unique feasible solution, which can be used to determine other equilibrium constants.

2. Second, to initiate the backward induction algorithm one needs an initial guess Σ_N. Via direct inspection, due to the relative simple structure in the Kyle (1985) model, one can verify that $\Sigma_N \propto \Sigma_0$, which can then be used to verify that the initial guess and the initial value Σ_0 are uniquely related.

The two features can be summarized schematically as:

$$\Sigma_N \to \left((\beta_N)^*, ..., (\beta_1)^*\right) \to \Sigma_0,$$

which is interpreted as follows: the initial guess Σ_N induces a (unique) path of β_n's—and as a by-product all other equilibrium constants required—and via backward iteration, this path eventually produces the initial value Σ_0.[41] If the initial value obtained corresponds to a preset value the algorithm terminates.

The present model differs from the Kyle (1985) model in various respects. First, instead of a single univariate polynomial—even after simplifications—one has to solve a polynomial system. Indeed, the equilibrium system features seven equations and seven unknowns: individual equations for the pricing constants λ_n (40) and r_n

[40] $\mathcal{S}_{\epsilon}^{N}$ and $\check{\mathcal{S}}_{\epsilon}^{N}$ have identical N and $\epsilon > 0$ (the termination parameter) but differ in terms of numerical values of other input constants and/or initial guesses. For more information about the initial parametrizations the reader is referred to see the inputs used in the algorithms of Appendix B.

[41] Note that one can also reduce the Kyle (1985) equilibrium system to a univariate polynomial in β_n.

(41) and the trading constants β_n^I (56) and β_n^R (52), as well as the equations for the conditional moments obtained from (42)-(44). While this seven-by-seven system can be directly solved via numerical methods, for instance, using the approach described in B.2, for talking about equilibrium multiplicity one is better off examining a lower dimensional system. To this end, it is described in B.3 how the equilibrium system of equations can be reduced to a bivariate system of coupled β-polynomials (or, β-equations). While the resulting system is far from simple, it enables one to describe the potential issues with equilibrium multiplicity in a rather straightforward manner.

Second, instead of one initial guess one is required to make *three*, i.e., instead of Σ_N one is required to use $\mathbf{\Sigma}_{N-1} = \left(\Sigma_{N-1}^{(1)}\ \Sigma_{N-1}^{(2)}\ \Sigma_{N-1}^{(3)}\right)^\top$.[42] Also, due to the more complex dynamics of these three conditional moments, it is not completely clear how a change in the initial guesses is transmitted to the initial values $\mathbf{\Sigma}_0 = \left(\Sigma_0^{(1)}\ \Sigma_0^{(2)}\ \Sigma_0^{(3)}\right)^\top$. Choi et al. (2019) make progress on this front by examining, similarly to Kyle (1985), how changes in the pricing error variance affect the initial values, but these special case results cannot be used to make general conclusions.

To understand what kind of conclusions could be made, some additional notation is required. Denote by $\boldsymbol{\beta}^*$ *a feasible path* (if it exists) :[43]

$$\left((\beta_N^I)^*,\ \left((\beta_{N-1}^I)^*,\ (\beta_{N-1}^R)^*\right), \ldots,\ \left((\beta_1^I)^*, (\beta_1^R)^*\right)\right),$$

i.e., a solution path—to the period N system as well as equilibrium solutions to the β-equations for all $n = N-1, \ldots, 1$—which meets all the required equilibrium conditions.[44] *A feasible equilibrium path* is a feasible path which implies initial values $\mathbf{\Sigma}_0$ consistent with some ex ante parametrization, i.e., the preset numerical values for σ_v, σ_a, and ρ. *A terminating path* is such that the path breaks, i.e., at some $1 \leq n \leq N$ there is no feasible solution and thus the path "terminates". Moreover, individual paths can diverge at some point to produce a number of new paths. This happens when there are multiple feasible solutions to the equilibrium system of equations for some period n. Figure 6 provides a simple illustration.

In Figure 6, the path on the left ($\boldsymbol{\beta}^*$) can be considered as the unique feasible path and the path of the right ($\boldsymbol{\beta}'$) is an example of a terminating path. To recap how the solution paths generate sequences of conditional moments, denote by $\mathbf{\Sigma}_n = \left(\Sigma_n^{(1)}\ \Sigma_n^{(2)}\ \Sigma_n^{(3)}\right)^\top$. Further, recall that β_n^I and β_n^R can be used to determine λ_n and r_n, and denote by $\mathbf{H}_n$ the matrix of coefficients in the period $n-1$ moment recursions (42)-(44). Now, one can simply write:

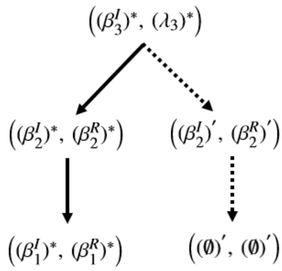
$$\mathbf{\Sigma}_n = \mathbf{H}_n \mathbf{\Sigma}_{n-1},$$

[42] The fact that in the present model one uses guesses for period $N-1$ instead of period N as in the Kyle-model is due to the differences in the model setups and inconsequential for the following discussion.

[43] More specifically, one could use $\boldsymbol{\beta}^*_{\mathcal{S}_\epsilon^N}$ to denote the solution path under model parametrization $\mathcal{S}_\epsilon^N$. For the sake of notational simplicity, $\boldsymbol{\beta}^*$ is used instead.

[44] An asterisk is used to separate the solution from the variable.

Figure 6: (**Solution paths**) This figure exemplifies solution path divergence. The path on the left is a feasible path while the path on the right terminates at $n = 1$.



where all the elements $h_n^{(i,j)} \in \mathbf{H}_n$ are known real numbers. Thus, conditional on $\det(\mathbf{H}_n) \neq 0$, one uniquely obtains the conditional moments for $n-1$.

Now, based on these notions, the question at hand is: what can one say about how the initial guesses and the induced solution paths relate to the initial values ($\mathbf{\Sigma}_0$) implied by them? After all, in the context of the Kyle-model this relationship was found to be very clear-cut.

A good starting place is the uniqueness of the feasible equilibrium path. Can one have multiple, two or more, feasible equilibrium paths induced by a single $\mathbf{\Sigma}_{N-1}$ such that these paths produce the same initial values $\mathbf{\Sigma}_0$? A schematic description of this situation is:

$$\begin{array}{ccccc} & & \boldsymbol{\beta}^* & & \\ & \nearrow & & \searrow & \\ \mathbf{\Sigma}_{N-1} & & \vdots & & \mathbf{\Sigma}_0 \\ & \searrow & & \nearrow & \\ & & \boldsymbol{\beta}' & & \end{array} \tag{S.1}$$

In terms of questions related to potential equilibrium multiplicity, the situation depicted in (S.1) is perhaps the most pressing. This is due to the fact that if multiple feasible equilibrium paths were found it would, in general, be rather difficult to discern which of them would be the most likely (dominating) equilibrium outcome, and therefore the equilibrium predictions of the model would be ambiguous at best.

Luckily numerical methods—especially the approach described in B.3—can be used, and are used, to verify that the feasible equilibrium paths found in Section 5.3 are *unique*. Namely, the paths found either feature no diverging paths or if diverging paths emerge, then all but one path are terminating or, at the very least, the initial values implied by diverging paths are inconsistent with the ex ante parametrization used. [45]

It turns out that the last qualification, i.e., that the initial values implied by diverging paths fail to converge to the values preset in the model parametrization, is important. Due to "slack" in the equilibrium conditions, one can find instances which feature multiple feasible paths, something that cannot occur in, for example, the Kyle (1985) model. This is exemplified in the context of the risk neutral version of the model in Hannula (2019a). Distance to initial values, as exemplified in B.2 and B.3, can then be used to consistently rule out these extraneous paths.

This observation that a single initial guess can produce multiple paths does, however, bring forth another question: is it possible that two (or more) distinct initial guesses produce two (or more) distinct feasible equilibrium paths, which imply the same initial values? A schematic description of this situation is:

$$\begin{array}{lll} \boldsymbol{\Sigma}_{N-1} \rightarrow \boldsymbol{\beta}^{*} & \searrow & \\ \vdots \qquad\quad \vdots & & \boldsymbol{\Sigma}_0 \\ \boldsymbol{\Sigma}'_{N-1} \rightarrow \boldsymbol{\beta}' & \nearrow & \end{array} \tag{S.2}$$

The above case relates to the second feature of the Kyle-model, i.e., that initial guesses are unique. However, in the context of the present model, where a single initial guess can produce multiple feasible paths, the uniqueness of the initial guesses need not hold.

Indeed, the possibility that multiple initial guesses produce the same initial values through different paths cannot be completely ruled out. Nevertheless, it can be argued that this is highly unlikely to occur with non-extreme parametrizations.[46] For one, this issue is not encountered in the numerical tests. Second, suppose, for instance, that the path $\boldsymbol{\beta}'$ features lower trading intensities (β_n^I) for I as compared to some other path. As a result R's trading intensity needs to be adjusted because otherwise the lower β_n^I's would result in a higher $\Sigma_0^{(2)}$. But all these changes also (nonlinearly) affect the paths for $\Sigma_n^{(1)}$ and $\Sigma_n^{(3)}$. Therefore, even if a parametrization and an initial guess with two (or more) feasible equilibrium paths exists, the possibility that they imply the same initial values can be deemed improbable.

Furthermore, in numerical runs, there are certain characteristics pertaining to the feasible equilibrium paths which seem to always hold.[47] One such characteristic is that β_n^I are monotonically increasing, moving from the first period towards the end of the trading horizon. One could thus use, if necessary, these characteristics as means

[45] If a feasible equilibrium path exists, all paths deviating from it are diverging paths. Diverging paths can be feasible or terminating. There can be feasible paths even if there is no feasible equilibrium path, but in this case, all paths can be considered as diverging with respect to each other.

[46] It is clear, when dealing with polynomials, that with extreme enough parametrizations one can produce very ill-behaved equations which instead can produce strange equilibrium outcomes.

[47] Actually, many properties are such that they hold in almost all extensions of Kyle (1985).

to rule out potential ill-behaved equilibria. Therefore, the situation (S.2) is not of primary concern in the numerical equilibrium analysis.

Based on the discussion about (S.2) one might still wonder about the possibility that two (or more) initial guesses produce, not different, but instead the same feasible equilibrium path. A schematic description of this situation is:

$$\begin{array}{ll} \boldsymbol{\Sigma}_{N-1} \searrow & \\ \vdots & \boldsymbol{\beta}^* \rightarrow \boldsymbol{\Sigma}_0 \\ \boldsymbol{\Sigma}'_{N-1} \nearrow & \end{array} \tag{S.3}$$

To rule out situation (S.3), one should note that from Lemma 1 it follows that changing the initial guess changes the period N system. More specifically, it changes the coefficients of the λ_N quartic. Unless the coefficients change in a way that can be factored out—which is generically not the case due to the complicated ways the coefficients depend on the initial guesses—these changes will change the solutions of the λ_N quartic, and thereby also β_N^I will be changed. Thus, the resulting paths cannot be the same.

Having obtained a better understanding of the equilibrium dynamics, the next task is to analyze these dynamics numerically. Lessons from above are good to keep in mind throughout the analysis. The path uniqueness of all equilibria presented is studied using the algorithm presented in B.3.

5.3 Numerical Results for Full Model

This section presents the numerical results for the full model. The main focus here is on the comparison between the risk neutral and risk averse cases. Namely, whether or not the equilibrium observations, i.e., trading and intraday patterns, from the risk neutral model carry over to the risk averse case, and if not, what changes and for what reason.

To keep track of the parameter values used, the next table reports the fixed parameter choices utilized in the rest of this section.
For conciseness, σ_u, σ_a, and σ_v are fixed to the values given in Table 1 throughout this section. Unreported numerical runs verify that the results obtained by varying these parameters are in line with results below. ρ is allowed to take two values: $\rho = 0$ pertains to the case where R has no (ex ante) private fundamental information, whereas $\rho = 1/3$ pertains to the case where R has some private fundamental information.[48] In line with the main focus of this section, both risk aversion parameters (A^I and A^R) are allowed to vary over the range $0, 1/2, ..., 5$.

[48] Over the range $[0, 1/3]$ for ρ, the results seem to be well-behaved.

Table 1: (**Parameter values**) This table present the fixed values for exogenous parameters utilized in the numerical analysis of the full model.

Parameter	Value
N	8
σ_u^2	$1/N$
σ_a^2	1
σ_v^2	1
ρ	$0, 1/3$
A^I	$0, 1/2, ..., 5$
A^R	$0, 1/2, ..., 5$

In addition to the case where both traders are risk averse, special attention is paid to the case where only R is risk averse. This is arguably a somewhat more interesting setting compared to the other special case, i.e., the case where I is risk averse and R is risk neutral.[49]

5.3.1 Trading coefficients and price impact

Since the trading intensities and price impact are determined endogenously in equilibrium, it is crucial for the big picture to understand how risk aversion affects each equilibrium constant individually. The first illustration in Figure 7 focuses on the equilibrium trading coefficients (intensities) β_n^I, β_n^R, and α_n^R.

It is evident from Figure 7 that high risk aversion has a greater impact on I's behavior in contrast to the behavior of R. Namely, high risk aversion raises the trading intensity of I in all periods, i.e., there is a clear upward translation in the intensity curve. Generally, this implies that the private information of I will be incorporated in the price faster as compared to the case where I is risk neutral. While this is a standard result for I (cf. Holden and Subrahmanyam 1994), the fact that risk aversion has little impact on R's trading curves—for instance, the case $A^I = 0$, $A^R = 5$ is omitted from Figure 7b as it is nearly indistinguishable from the risk neutral case—is somewhat surprising and not at all clear ex ante.

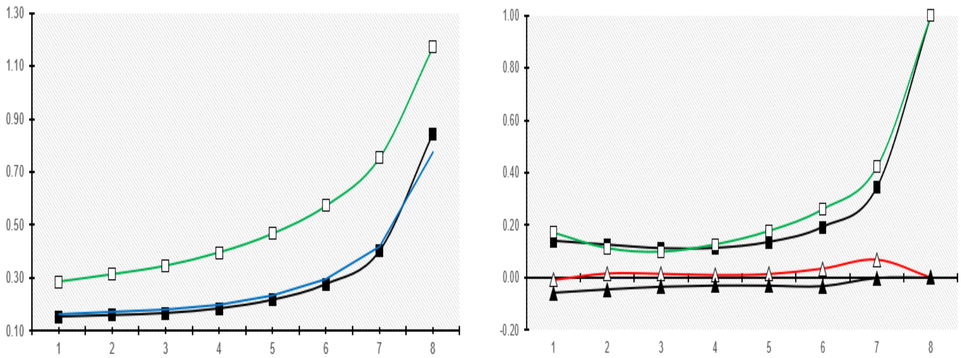
[49] For instance, think of I as representing a hedge fund and R as representing a mutual fund.

Figure 7: (**Trading intensities**) This figure provides a comparison between the risk neutral and risk averse trading intensities under the assumption that $\rho = 0$.

β_n^I: the **black** line represents $A^I = A^R = 0$, the blue line represents $A^I = 0,\ A^R = 5$, and the green line represents $A^I = A^R = 5$.

β_n^R: the **black** line represents $A^I = A^R = 0$ and the green line represents $A^I = A^R = 5$.

α_n^R: the bottom **black** line represents $A^I = A^R = 0$ and the red line represents $A^I = A^R = 5$.



(a) β_n^I (b) β_n^R and α_n^R

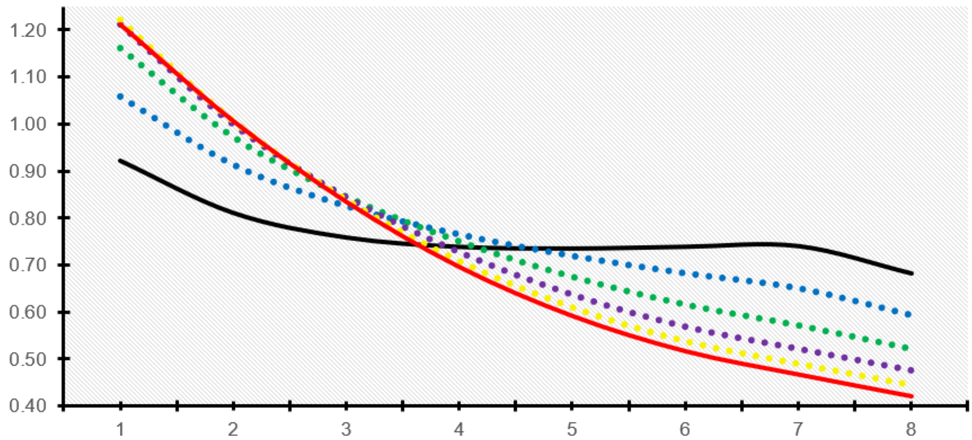
The explanation for this observation stems from the presence of the constraint $\tilde{a}$ which, more or less, stipulates the optimal execution path for R. I does not face a similar constraint and is thus able to adjust his trading more. One might pose the question whether the results pertaining to R could be an artefact of the modified objective of R, as discussed in Section 3.2. There is little reason to believe this would be the case because—as indicated earlier—the extra terms related to the wealth maximization objective seem to be generally very small.

Next, Figure 8 depicts the intraday patterns in the endogenous price impact parameter λ_n under various risk aversion levels. In interpreting Figure 8 it is important to take note of the dynamics of the trading intensity parameters discussed above.

In essence, λ_n captures the signal-to-noise ratio obtained from the order flow. In the risk neutral market with $\rho = 0$ the signal to noise ratio first decreases then remains (almost) fixed during the middle of the day, and then again decreases at the end to the trading horizon. The result is a distinctive twisted S-shape. In the risk averse case, however, this S-shape seems to vanish and the signal-to-noise ratio decreases monotonically even when $\rho = 0$.[50] This is in line with Holden and Subrahmanyam (1994) and stems from the fact that as I's private information is revealed at a faster pace, the noise component in the signal-to-noise ratio quickly starts to dominate.

[50] Similar monotonicity can be found when more private fundamental information is introduced to the market, i.e., $\rho > 0$. In this context, the monotonicity is due to a competition effect.

Figure 8: (**Price impact**) This figure presents the endogenous price impact parameter λ_n for symmetric $A^K = 0,1,2,3,4,5$, where $K \in \{I,R\}$. The risk neutral case in presented in **black**, $A^K = 1$ is presented in blue, $A^K = 2$ is presented in green, $A^K = 3$ is presented in violet, $A^K = 4$ is presented in yellow, and finally $A^K = 5$ is presented in red. Throughout it is assumed that $\rho = 0$.



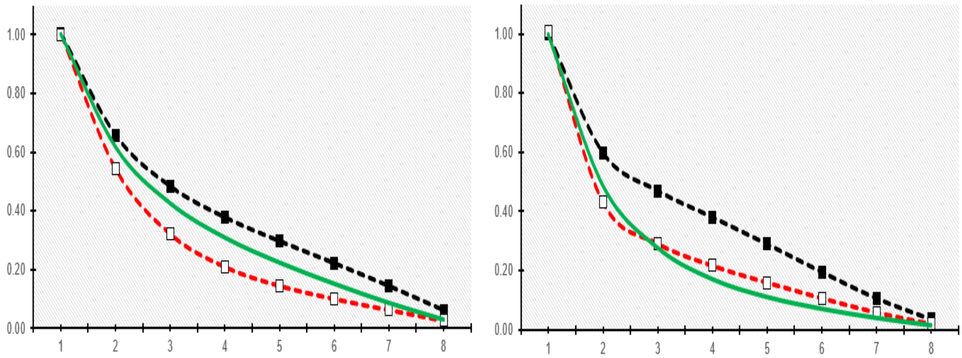
To wrap up, it should be pointed out that the results with a positive ρ and with $A^I = 0$ and $A^R > 0$ are similar to those of Choi et al. (2019), and therefore these results are not reported. As earlier, one can conclude that the apparent change in the behavior of the price impact parameter is driven by the change in the trading strategy of I. This is because—in the cases under study—a bulk of the assumed, price-relevant, private information is endowed to I, and for this reason it is the trading strategy of I which has a more prominent effect on the equilibrium price impact parameter. This result instead is driven by the fact that the price impact in the model is assumed to be purely information-based, i.e., trades which do not carry any price relevant information will have no price impact. In terms of robustness, it is conceivable that the above results continue to hold in a modified model with a more complicated price impact function—including, e.g., non-information-based components—as long as one maintains the assumption that information-based price impact is first-order and, more or less, dominates over the other price impact components.

5.3.2 Intraday patterns

The above results provide a good basis for proceeding to study intraday patterns. These patterns, in a large part, reflect the dissimilar trading motives of I and R. Looking back, the expectation is that I's risk aversion tends to cause more pronounced deviations from the risk neutral baseline. A good way to examine the validity of this intuition is to look the two key conditional moments $\Sigma_n^{(1)}$ and $\Sigma_n^{(2)}$.

Figure 9 shows how the variance of latent (unrealized) trading demand of R changes over the trading horizon. Choi et al. (2019) find in their numerical analysis that $\Sigma_n^{(1)}$ is monotonically decreasing and the same observation can be made from Figure 9.

Figure 9: (**Variance of latent trading demand**) This figure presents the evolution of latent trading demand variance $\Sigma_n^{(1)}$ over the trading horizon. The green line always represents the case $A^I = 0$ and $A^R = 5$, while the **black** and red on the left (right) represent the case $A^I = A^R = 0$ ($A^I = A^R = 5$).



(a) $\rho = 0$ (■) and $\rho = 1/3$ (□). (b) $\rho = 0$ (■) and $\rho = 1/3$ (□).

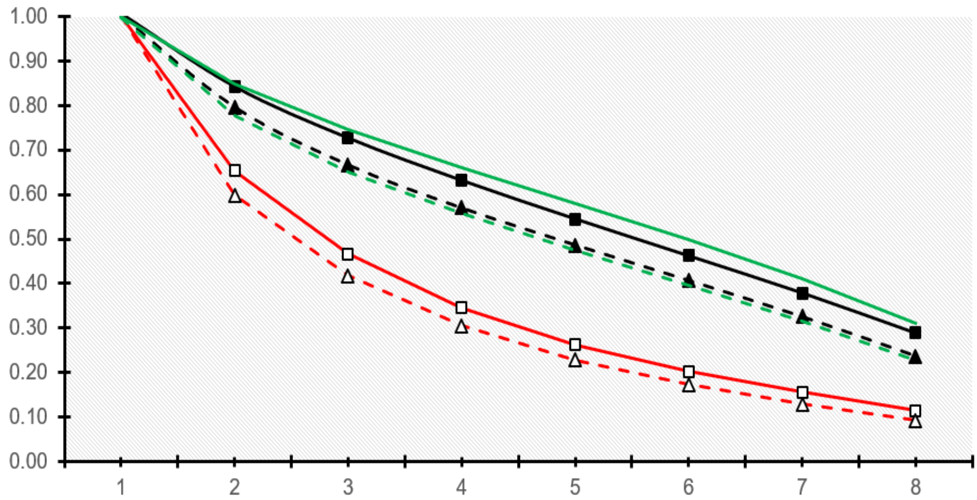
On top of verifying the monotonic evolution of $\Sigma_n^{(1)}$, Figure 9 also gives some additional indication on how risk aversion affects $\Sigma_n^{(1)}$. Evidently, the variance decreases faster in the risk averse case. However, somewhat surprisingly, $\Sigma_n^{(1)}$ decreases the fastest when R is risk averse and endowed with private information ($\rho = 1/3$) and I is risk neutral. This can be seen by looking at the green line in Figure 9b.

A natural explanation for this is that while a risk averse R slightly intensifies early trading, the same does not hold for the risk neutral I. Thus, R has a weaker camouflage and more information is revealed. Furthermore, as both traders now have fundamental information, the latent trading demand of R is more closely tied to informed trading. This issue also factors in to the speed with which $\Sigma_n^{(1)}$ decreases. Additional evidence on the matter can be gleaned by looking at the evolution of $\Sigma_n^{(2)}$.

For this purpose, Figure 10 depicts the intraday patterns for the pricing error variance $\Sigma_n^{(2)}$. It can be seen from the figure that—as one would expect—risk aversion of I leads to faster information revelation. Moreover, information revelation is slightly faster if R is also endowed with fundamental information. The latter observation reflects the competition effect and it is in line with, for example, Holden and Subrahmanyam (1992) and Foster and Viswanathan (1994).

Finally, and perhaps most interestingly, when I is risk neutral, R is risk averse, and $\rho = 0$, the pricing error variance is higher than in the full risk neutral case. This indicates that a risk averse R, through her rebalancing activities, provides additional camouflage for I over and above the camouflage provided by a risk neutral R. This effect is not, however, monotonically increasing in R's risk aversion. Instead, numerical tests imply that the additional camouflage is greatest when R is moderately risk

Figure 10: (**Pricing error variance**) This figure presents the evolution of pricing error variance $\Sigma_n^{(2)}$ over the trading horizon. The **black** (solid and dashed) lines represent the case $A^I = A^R = 0$ with $\rho = 0$ (■) and $\rho = 1/3$ (▲). The red (solid and dashed) lines represent the case $A^I = A^R = 5$ with $\rho = 0$ (□) and $\rho = 1/3$ (△). The solid (dashed) green line covers the case $A^I = 0$ and $A^R = 5$ with $\rho = 0$ ($\rho = 1/3$).



averse, i.e., $A^R \in (0, 2.5)$. Nonetheless, it does seem that a risk averse R generally provides more camouflage for I as compared to a risk neutral R and this effect can mostly be attributed to early trading rounds.

Moving on, Figure 11 shows the intraday correlations between I's and R's orders.[51] In the risk neutral case, Choi et al. (2019) show that these correlations tend to be negative—at least towards the end of the trading horizon. The authors conclude that negative correlation is beneficial for both traders due to the liquidity improving effect.

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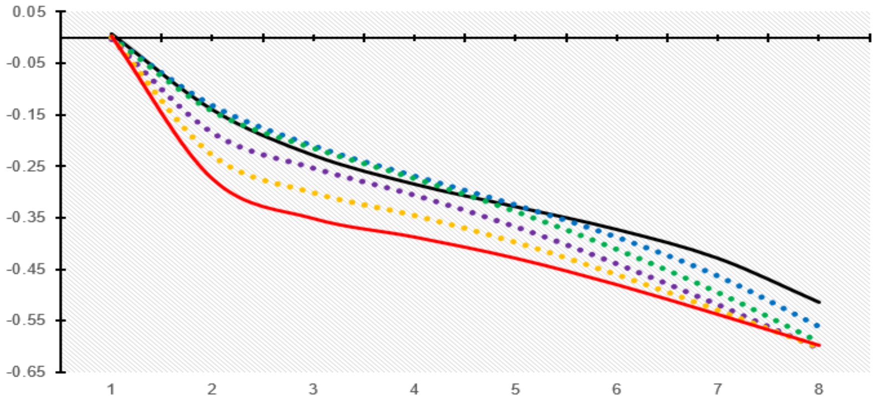
51 Here it is helpful to note that:

$$\mathbb{C}[\Delta\hat{\theta}_n^I, \Delta\hat{\theta}_n^R] = \beta_n^I \beta_n^R \Sigma_{n-1}^{(3)},$$

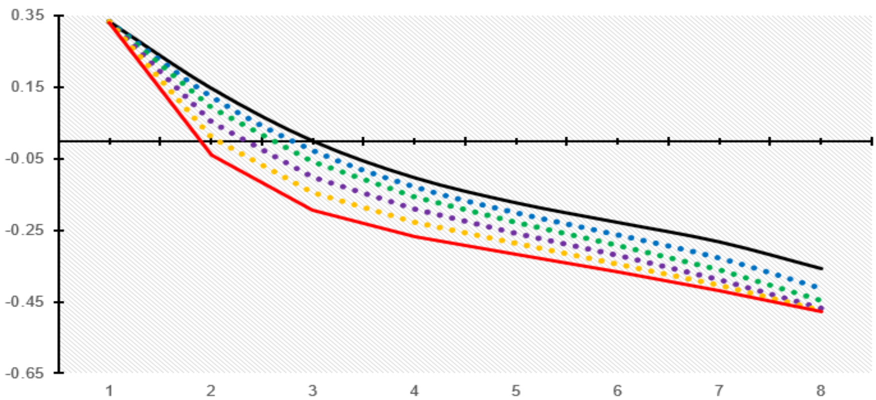
since it follows from iterated expectations that:

$$\begin{aligned}&\mathbb{E}[(\tilde{v} - \hat{p}_{n-1})\hat{q}_{n-1}]\\ =&\mathbb{E}\Big[\hat{q}_{n-1}\mathbb{E}[\tilde{v} - \hat{p}_{n-1}|\boldsymbol{\sigma}(\hat{y}_1, ..., \hat{y}_{n-1})]\Big] = 0.\end{aligned}$$

Figure 11: (**Order correlation**) This figure depicts the correlation between $\Delta\theta_n^I$ and $\Delta\theta_n^R$ for various (symmetric) risk aversion levels, i.e., $A^K = 0, 1, 2, 3, 4, 5$, where $K \in \{I, R\}$. The risk neutral case in presented in **black**, $A^K = 1$ is presented in blue, $A^K = 2$ is presented in green, $A^K = 3$ is presented in violet, $A^K = 4$ is presented in yellow, and finally $A^K = 5$ is presented in red.



(a) $\rho = 0$



(b) $\rho = 1/3$

One may observe from Figure 11 that for higher levels of risk aversion the correlation between the orders tends to become more negative. A similar observation pertains to the case where only R is risk averse. This implies that risk aversion tends to strengthen the mutually beneficial liquidity provision effect.

Furthermore, when R is uninformed ($\rho = 0$) the correlation is—for lower levels of risk aversion—initially close to or slightly higher than in the risk neutral case, decreasing faster towards the end of the trading period. When R is informed ($\rho = 1/3$), correlation between the orders of R and I decreases monotonically when the traders become more risk averse.

Continuing with the topic of correlations, the *unconditional* aggregate order flow autocorrelation between periods $n-1$ and n is given by:

$$R(y_{n-1}, y_n) = \frac{\mathbb{E}[y_{n-1}y_n]}{\sqrt{\mathbb{E}\left[y_{n-1}^2\right]\mathbb{E}\left[y_n^2\right]}}.$$

In order to calculate the autocorrelation coefficient numerically it is helpful to note first note that:

$$y_n = z_n^M + Q_n^R q_{n-1}, \quad n = 1, ..., N,$$

where Q_n^R is a constant and q_n can be further decomposed:

$$q_n = \sum_{k=1}^{n} \mu_{k,0}^{(n)} z_k^M, \quad \text{with constants}$$
$$\mu_{k,0}^{(n)} = (1 - Q_n^R) \cdots (1 - Q_{k+1}^R) r_k.$$

Then, using the above decompositions and taking advantage of the properties of z_n^M, the following formulas are obtained:

$$\mathbb{E}[y_n^2] = \sum_{k=1}^{n} \mu_{k,1}^{(n)} \mathbb{E}\left[(z_n^M)^2\right],$$
$$\mathbb{E}[y_{n-1}y_n] = \sum_{k=1}^{n-1} \mu_{k,2}^{(n)} \mathbb{E}\left[(z_n^M)^2\right],$$

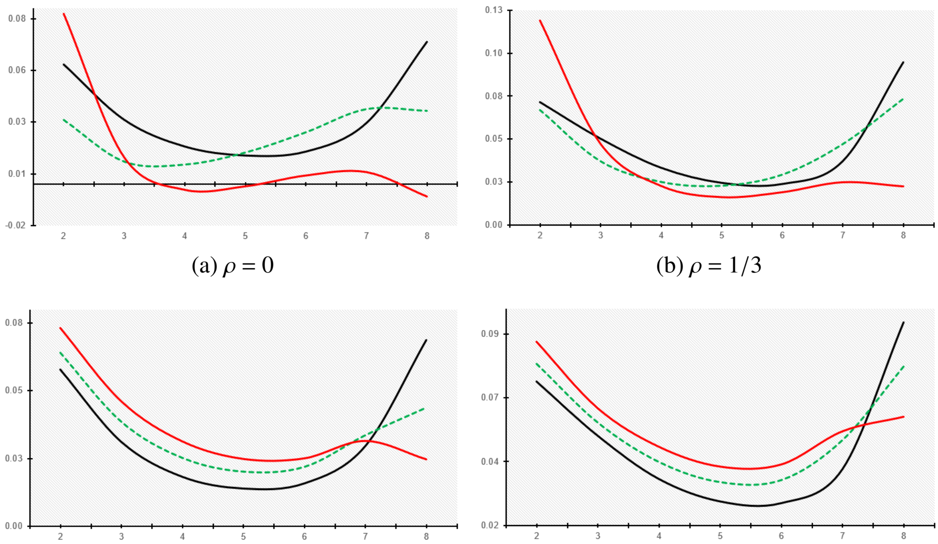
where $\mu_{k,1}^{(n)}$ and $\mu_{k,2}^{(n)}$ are new constants obtained via grouping the relevant terms in the intermediate equations. Using the above, the autocorrelation coefficient can be expressed in as:

$$R(y_{n-1}, y_n) = \frac{\sum_{k=1}^{n-1} \mu_{k,2}^{(n)} \mathbb{E}\left[(z_k^M)^2\right]}{\sqrt{\left(\sum_{k=1}^{n-1} \mu_{k,2}^{(n-1)} \mathbb{E}\left[(z_k^M)^2\right]\right)\left(\sum_{k=1}^{n} \mu_{k,1}^{(n)} \mathbb{E}\left[(z_k^M)^2\right]\right)}}.$$

The above expression is used in the numerical illustrations of Figure 12.

Figure 12 depicts the patterns of unconditional intraday autocorrelation in the aggregate order flow y_n. It is obvious from the figure that risk aversion has an apparent impact on the shape of the curve depicting intraday autocorrelation evolution. More specifically, risk aversion seems to level out the rise in autocorrelation observed during the last periods in conjunction with risk neutral trading. This result can be explained by noting that risk aversion induces both I and R to trade in a more front-loaded fashion and, as a direct consequence, I and R trade less during the final periods. The results in Figure 12 are in line with Dufour and Engle (2000) who document stronger autocorrelation during periods with higher trading activity.

Figure 12: (**Order flow autocorrelation**) This figure presents the (unconditional) aggregate order flow (y_n) autocorrelation patterns under various parametrizations. The **black** line is used to denote the risk neutral case, the red line corresponds to the case $A^I = A^R = 5$ ($A^I = 0$ and $A^R = 5$) in the top two figures (in the bottom two figures), and finally the green line dashed line is used for the case $A^I = A^R = 2.5$ ($A^I = 0$ and $A^R = 2.5$) in the top two figures (in the bottom two figures).



(a) $\rho = 0$ (b) $\rho = 1/3$

(c) $\rho = 0$ (d) $\rho = 1/3$

Furthermore, an interesting feature regarding the autocorrelation patterns is the fact that even in the case where I is risk neutral and only R is risk averse—Figures 12c and 12d—autocorrelation exhibits a clear deviation from the risk neutral baseline. This is in contrast to some the earlier results which seem to be mostly driven by I's risk aversion. A similar explanation as in the high risk aversion case ($A^I = A^R = 5$) pertains to this case. A risk averse R trades more during the initial periods and less towards the end.

Based on this observation one may conclude that while it is possible to use a risk neutral R as a proxy for a risk averse R in some instances, how good this proxy turns out to be depends greatly on what aspects of the market one is examining. For instance, if the goal is to study intraday patterns, one should pay close attention on how to model R's motives and preferences.

Finally, while the impact of risk aversion on the intraday autocorrelation patterns is plainly visible, the impact on the level of autocorrelation is, however, relatively small. For instance, in Figure 12a autocorrelation is close to zero for most of the trading horizon. This observation is in line with Choi et al. (2019).

5.3.3 Risk seeking informed trader

So far, the discussion has revolved around the outcomes stemming from the introduction of risk aversion. In this section the tables are turned and attention is directed towards the case where I is—instead of risk neutral or risk averse— (moderately) *risk seeking*, i.e., $A^I = -1$, while R is risk neutral.[52] The goal is to exemplify that even a conservative step to the "wild side"—i.e., a step from risk neutrality towards risk seeking— is (again) sufficient to generate decided changes in the equilibrium dynamics.

As noted above, the motive for studying the impact of changes in risk preferences stems from the fact that the risk preferences of actual traders are likely to be time-varying and affected by changes in the economic environment as well as, e.g., shifts in financial regulation. With respect to risk seeking behavior, especially when I (the informed trader or, e.g., a portfolio manager) is concerned, the main justification can be drawn from financial contracting (cf. Panageas and Westerfield 2009). It is plausible that, for example, convex compensation contracts, specifically towards the end of a (finite) contract horizon, can induce risk seeking behavior. Therefore, it is important to understand the impact of these alternative risk preferences on the dynamic trading equilibrium.

As with the risk averse case, a natural starting point is to examine the trading intensities β_n^I, β_n^R, and α_n^R. However, changes related to these are found to be far from dramatic and hence omitted. It suffices to note that, as intuition would predict, I's trading intensity moves to the opposite direction when I is risk seeking as compared to the case where I is risk averse.

More specifically, I's trading intensity falls, as with $A^I = -1$ he is more willing to look for better trading opportunities, instead of locking in profits early to protect himself from potential future price shocks as is the case when I is risk averse.[53] This leaves room for R to be more aggressive than in the risk averse setting, but, as above, changes in R's trading strategy, for the most part, have a weaker effect on the equilibrium dynamics. Changes in the trading intensities are useful in understanding the behavior of λ_n depicted in Figure 13.

Earlier it was observed that in a risk averse market the twist in the λ_n curve, clearly present in the risk neutral case, straightens out and price impact becomes monotonically decreasing over the trading horizon. It is evident that the same does not, by any means, hold when I is risk seeking. Indeed, it can be seen from Figure 13 that in the risk seeking market, with $\rho = 0$, the twist very much remains and, in addition, becomes rotated. Liquidity, the reciprocal of λ_n, is higher at the start of the trading horizon because I trades with less intensity, only to decrease towards the end

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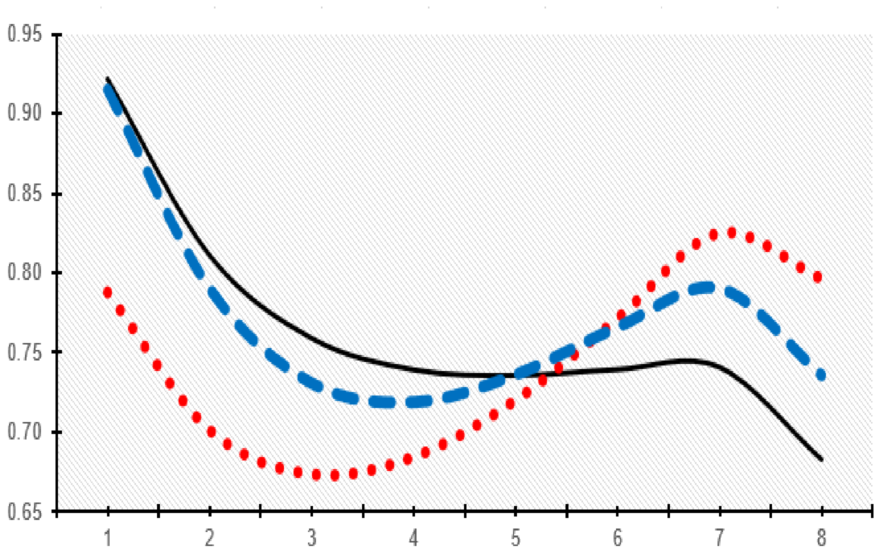
52 In unreported numerical experiments it is found that R's risk aversion does not materially change the results.

53 One outcome from this decreased intensity is that the pricing error variance $\Sigma_n^{(2)}$ tends to be higher than in other specifications (risk neutral baseline or risk averse models) when I is risk seeking.

Figure 13: (**Price impact revisited**) This figure presents the endogenous price impact parameter λ_n for the risk seeking case: $A^I = -1$ and $A^R = 0$. The solid **black** line represents the risk neutral baseline ($A^I = A^R = 0$) with $\rho = 0$, the dashed blue line represents the risk seeking case with $\rho = 1/3$, and the dotted red line represents the risk seeking case with $\rho = 0$.

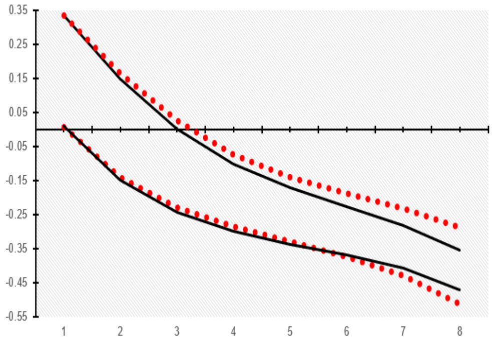


due to more pronounced informed trading.

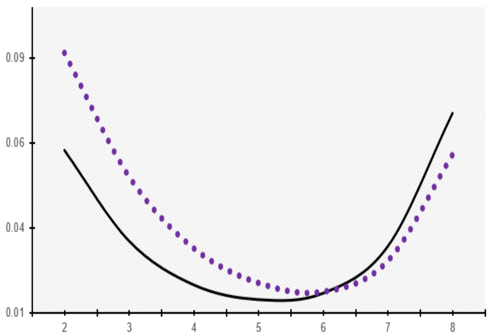
Endowing R with some initial fundamental information ($\rho = 1/3$) attenuates the rotation while still preserving the twisted shape. This is unsurprising: higher λ_n in this case only reflects the fact that the informed R picks up the slack left by the risk seeking I during the initial trading rounds.

Proceeding to the intraday observations, Figure 14 shows the patterns in trade correlation and autocorrelation. First, regarding the unconditional trade correlation, one can see from 14a that deviations from the risk neutral baseline stemming from risk seeking behavior are small. This is a reflection of the fact that the trading strategies of I and R are coupled and mutually beneficial liquidity provision extends also to the risk seeking case.

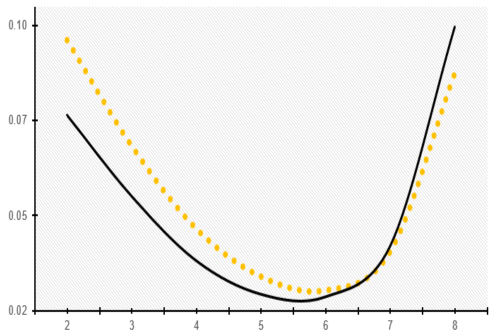
Figure 14: (**Intraday patterns in the risk seeking case**) This figure depicts the correlation between $\Delta\theta_n^I$ and $\Delta\theta_n^R$ (cf. Figure 11) as well as the aggregate order flow (y_n) autocorrelation (cf. Figure 12) for both $\rho = 0$ (the bottom two lines in the first subfigure) and $\rho = 1/3$ (the top two lines in the first subfigure) and for both the risk neutral $A^I = A^R = 0$ (solid **black** lines) and the risk seeking case $A^I = -1$ and $A^R = 0$ (dotted lines).



(a) $\mathbb{C}\text{orr}\left(\Delta\theta_n^I, \Delta\theta_n^R\right)$



(b) Autocorrelation, $\rho = 0$



(c) Autocorrelation, $\rho = 1/3$

Second, Figures 14b and 14c illustrate that risk seeking has a greater impact on order flow autocorrelation than it has on the correlation between the trades of I and R. Although the U-shape is preserved, the autocorrelation magnitude is clearly elevated in the risk seeking case, especially in the case where R initially has no fundamental information. Autocorrelation is mainly driven by the rebalancing needs of R. Therefore, higher levels of autocorrelation in the risk seeking case simply echo the fact that R has more room to trade and to be aggressive in the beginning of the trading horizon when the risk seeking I trades with lower intensity.

6 CONCLUSION

This paper studied a model of strategic trading under asymmetric information based on Choi et al. (2019), featuring two strategic traders—the informed trader and the rebalancer—with distinct trading motives and risk preferences. One of the traders (the rebalancer) operates under a strict terminal position constraint dictating the amount of risky asset the trader needs to hold when trading commences. The size of this position constraint is private information. The other trader (the informed one) is unconstrained and trades to take advantage of an endowment of private fundamental information. The main theoretical contribution of the paper was to analyze the dynamic equilibrium of a model where these two traders, under various risk preferences, are allowed to interact with each other. Due to the intractability of the equilibrium system of equations, computational methods were advanced to effectively study the equilibrium numerically.

In the actual numerical analysis, special attention was paid on the impact of risk preferences on equilibrium constants and on intraday properties emerging in the studied financial market. On this front, it was found that risk preferences indeed have a marked impact on the equilibrium properties but that this impact is not evenly distributed between the two traders. More specifically, the results implied that changes in the risk preferences of the informed trader tend to have greater equilibrium implications than changes in the risk preferences of the rebalancer.

Changes in the risk preferences of the informed trader had a particularly notable effect on price impact and on the symbiotic liquidity provision between the two traders. Changes in the risk preferences of the rebalancer had most weight in shifting the order flow autocorrelation patterns. The results can be utilized, for example, in justifying the risk neutrality assumption in versions of the model, where research questions are such that changes in risk preferences do not have first order effects.

Moreover, as the model is susceptible to equilibrium multiplicity, emphasis was also put on studying—instead of a single feasible equilibrium solution—the entire solution set of the equilibrium system of equations. It was shown that the potential equilibrium multiplicity in the model has many dimensions, which must be considered separately. Efficient solution methods tend to involve a two tier approach where costly parameter value search is conducted using efficient local methods and uniqueness examination is conducted using, computationally more expensive, global solution methods.

There are a number of ways to extend the analysis further. Different forms of trading constraints (e.g., a soft constraint with quadratic penalties) combined with new ways to model the private information of the rebalancer (e.g., short-lived private information independent of the trading target) would be fruitful avenues for further research. Another extension is to move into a continuous time framework.

On one hand, a continuous time approach circumvents the complexities involved in dealing with the rigid system of equations that is characteristic for the discrete model, and one could thus obtain a simpler description of the model equilibrium. On the other hand, the continuous time framework requires dealing with a number of non-trivial technical issues not present in the discrete time model. Finally, one could consider modifying the (endogenous) price impact dynamics from permanent to transient (cf. Hannula 2019b). This extension would provide valuable information with regards to the robustness of the equilibrium properties to changes in the way trades are assumed to impact prices.

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APPENDICES

A PROOFS

Appendix A contains the proofs from the main text.

A.1 Proofs from Section 4

Proof of Lemma 1. Assume that $A^I > 0$ and note first that:

$$\dot{\Sigma}_{N-1} := -\underbrace{A^I\left(\Sigma^I_{N-1} + \sigma^2_u\right)}_{>0} < 0,$$
$$\bar{\Sigma}_{N-1} := \Sigma^{(1)}_{N-1} + \sigma^2_u > 0,$$
$$\Sigma^{(2)}_{N-1} > 0 \text{ and } \Sigma^{(3)}_{N-1} \in \mathbb{R}$$

The system in question can be simplified to a quartic (univariate) polynomial in λ_N. The coefficients of the quartic are:

$$\begin{aligned}
\lambda^4_N &: & (\dot{\Sigma}_{N-1})^2\bar{\Sigma}_{N-1} &> 0,\\
\lambda^3_N &: & -\dot{\Sigma}_{N-1}\Sigma^{(3)}_{N-1}\left(2\Sigma^{(3/2)}_{N-1} + \dot{\Sigma}_{N-1}\right) + 4\dot{\Sigma}_{N-1}\bar{\Sigma}_{N-1} &\gtrless 0,\\
\lambda^2_N &: & \dot{\Sigma}_{N-1}\Sigma^{(3)}_{N-1} + \Sigma^{(3)}_{N-1}\Sigma^{(3/2)}_{N-1} - 2\Sigma^{(3)}_{N-1}\left(\dot{\Sigma}_{N-1} + 2\Sigma^{(3/2)}_{N-1}\right) + 4\bar{\Sigma}_{N-1} &\gtrless 0,\\
\lambda^1_N &: & -\dot{\Sigma}_{N-1}\Sigma^{(2)}_{N-1} &> 0,\\
\lambda^0_N &: & -\Sigma^{(2)}_{N-1} &< 0.
\end{aligned}$$

By looking at the possible permutations, one can observe that the quartic exhibits either one or three signs changes. A similar observation can be made in the case where $A^I < 0$. The only difference there is that $\dot{\Sigma}_{N-1} > 0$ and the coefficient of λ^1_N is now negative. Finally, setting $A^I = 0$ reduces the problem further and the desired result is obtained from a quadratic equation. For additional details the reader is referred to Hannula (2019a).

Therefore, by *Descartes' rule*, the number of positive roots is either exactly one or three, i.e., a positive root, either unique or not, always exists. This concludes the proof. ∎

A.2 Proofs from Section 5

Proof of Lemma 2. Omitted. See Choi et al. (2019) for details. ∎

Proof of Lemma 3. Begin with the state variable dynamics. Since the dynamics of $Y^{(1)}_n$ are trivial, one can move straight to $Y^{(2)}_n$:

$$\begin{aligned}
\Delta Y^{(2)}_n &= (\Delta\hat{p}_n - \Delta p_n) + \mathbb{E}[\tilde{v} - \hat{p}_n \mid \boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1, ..., \hat{y}_n)] - \mathbb{E}[\tilde{v} - \hat{p}_{n-1} \mid \boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1, ..., \hat{y}_{n-1})]\\
&= (\Delta\hat{p}_n - \Delta p_n) - \mathbb{E}[\Delta\hat{p}_n \mid \boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1, ..., \hat{y}_n)] + \mathbb{E}[\tilde{v} - \hat{p}_{n-1} \mid \boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1, ..., \hat{y}_n)] - \mathbb{E}[\tilde{v} - \hat{p}_{n-1} \mid \boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1, ..., \hat{y}_{n-1})]\\
&= -\Delta p_n + \frac{\mathbb{E}[(\tilde{v} - \hat{p}_n)\hat{z}^R_n]}{\mathbb{V}[\hat{z}^R_n]}\hat{z}^R_n
\end{aligned}$$

$$= -\lambda_n\Big(\Delta\theta^R_n + \beta^I_n Y^{(2)}_{n-1} - Q^R_n Y^{(3)}_{n-1}\Big) + \left[\frac{\mathbb{E}[(\tilde{v}-\hat{p}_n)\hat{z}^R_n]}{\mathbb{V}[\hat{z}^R_n]} - \lambda_n\right]\hat{z}^R_n$$

$$= -\lambda_n\Big(\Delta\theta^R_n + \beta^I_n Y^{(2)}_{n-1} - Q^R_n Y^{(3)}_{n-1}\Big) - r_n\frac{\Sigma^{(3)}_n}{\Sigma^{(1)}_n}\hat{z}^R_n,$$

where the *third* equality follows from the recursive property of projections and since $\Delta\hat{p}_n \in \boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1,\dots,\hat{y}_n)$, the *fourth* equality follows from equations (35) - (39), and the *fifth* equality is obtained using Lemma 2. The dynamics of $Y^{(3)}_n$ are derived similarly.

As for the second part, start by fixing an arbitrary $m \in \{1,\dots,n-1\}$. The claim $\hat{z}^R_n \perp (\tilde{a}, y_1,\dots,y_{n-1})$ follows from the joint normality of the random variables and from the fact that $\mathbb{E}[\hat{z}^R_n\hat{y}_m] = 0$.

Regarding the equivalence of the different sigma-algebras note that:

$$\boldsymbol{\sigma}(\tilde{a},\ y_1) = \boldsymbol{\sigma}(\tilde{a},\ \beta^I_1\tilde{v} + \theta^R_1 + \tilde{u}_1) \overset{\theta^R_1\in\boldsymbol{\sigma}(\tilde{a})}{=} \boldsymbol{\sigma}(\tilde{a},\ \beta^I_1\tilde{v} + \tilde{u}_1) \text{ and}$$

$$\boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1) = \boldsymbol{\sigma}(\tilde{a},\ \beta^I_1\tilde{v} + \hat{\theta}^R_1 + \tilde{u}_1) \overset{\hat{\theta}^R_1\in\boldsymbol{\sigma}(\tilde{a})}{=} \boldsymbol{\sigma}(\tilde{a},\ \beta^I_1\tilde{v} + \tilde{u}_1)$$

Via an induction argument it then follows that:

$$\begin{aligned}\boldsymbol{\sigma}(\tilde{a},\ \hat{y}_1,\dots,\hat{y}_{n+1}) &\overset{\text{ind. hypo.}}{=} \boldsymbol{\sigma}(\tilde{a},\ y_1,\dots,y_n,\hat{y}_{n+1})\\ &= \boldsymbol{\sigma}\Big(\tilde{a},\ y_1,\dots,y_n,y_{n+1} + (\Delta\hat{\theta}^R_{n+1} - \Delta\theta^R_{n+1}) + (\Delta\hat{\theta}^I_{n+1} - \Delta\theta^I_{n+1})\Big)\\ &= \boldsymbol{\sigma}(\tilde{a},\ y_1,\dots,y_n,y_{n+1}),\end{aligned}$$

where the last equality follows since $\Delta\theta^R_{n+1}$ (similarly $\Delta\hat{\theta}^R_{n+1}$) is $\boldsymbol{\sigma}(\tilde{a},\ y_1,\dots,y_n)$ measurable and $\Delta\hat{\theta}^I_{n+1} - \Delta\theta^I_{n+1} = \beta^I_{n+1}(p_n - \hat{p}_n) \in \boldsymbol{\sigma}(\tilde{a},\ y_1,\dots,y_n)$ which proves the second claim in Lemma 3. ∎

Proof of Lemma 4. Recall the notation $\mathbb{E}[\,\bullet \mid \sigma(\tilde{a},y_1,\dots,y_{n-1})] =: \mathbb{E}^R_n[\,\bullet\,]$, for $n \in \{0,1,\dots,N\}$, and denote R's state variable vector by $\mathbf{Y}_n = (Y^{(1)}_n\ Y^{(2)}_n\ Y^{(3)}_n)^\top$. Using the principle of optimality and backward induction, one obtains, for period n, the Bellman equation:

$$\max_{\Delta\theta^R_n} \frac{-1}{A^R}\mathbb{E}^R_n\left[\exp\left\{-A^R\Big(-(\tilde{a}-\theta^R_{n-1})\Delta p_n + \sum_{\boldsymbol{\phi}\in\Phi} L^{(\boldsymbol{\phi})}_n \mathbf{Y}^{\boldsymbol{\phi}}_n\Big)\right\}\right], \tag{A.1}$$

where $\Delta\theta^R_n$ is required to be $\boldsymbol{\sigma}^R_n$-measurable, $\boldsymbol{\phi} = (\phi_1,\ \phi_2,\ \phi_3)$, $\mathbf{Y}^{\boldsymbol{\phi}}_n = (Y^{(1)}_n)^{\phi_1}(Y^{(2)}_n)^{\phi_2}(Y^{(3)}_n)^{\phi_3}$, and

$$\Phi := \Big\{(2,0,0),\ (1,1,0),\ (1,0,1),\ (0,2,0),\ (0,1,1),\ (0,0,2)\Big\}.$$

Moving forward, the term outside the expectation operator is omitted. The goal now is to evaluate the expectation (A.1). Using Lemma 3 one can verify that:

$$\begin{aligned}&\mathbb{E}^R_n\left[\exp\left\{-A^R\Big(-(\tilde{a}-\theta^R_{n-1})\Delta p_n + \sum_{\boldsymbol{\phi}\in\Phi} L^{(\boldsymbol{\phi})}_n \mathbf{Y}^{\boldsymbol{\phi}}_n\Big)\right\}\right]\\ &= \exp\left\{-A^R\Big(F^R_n(\mathbf{Y}_n) + d^{(1)}_n\Delta\theta^R_n + d^{(2)}_n(\Delta\theta^R_n)^2\Big)\right\}\\ &\quad\times\mathbb{E}^R_n\left[\exp\left\{-A^R\Big((s^{(0)}_n + s^{(1)}_n\Delta\theta^R_n)\hat{z}^R_n + s^{(2)}_n(\hat{z}^R_n)^2\Big)\right\}\right],\end{aligned} \tag{A.2}$$

where $F^R_n : \mathbb{R}^3 \to \mathbb{R}$ is a quadratic function of the state variables of R, and where the constants $d^{(i)}_n$, $i = 1,2$, as well as $s^{(j)}_n$, $j = 0,1,2$, are independent of $\Delta\theta^R_n$. Now, noting that $\hat{z}^R_n \sim \mathcal{N}(0,\mathbb{V}[\hat{z}^R_n])$, the last line in (A.2) is evaluated as:

$$\mathbb{E}^R_n\left[\exp\left\{-A^R\Big((s^{(0)}_n + s^{(1)}_n\Delta\theta^R_n)\hat{z}^R_n + s^{(2)}_n(\hat{z}^R_n)^2\Big)\right\}\right]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \mathbb{V}[\hat{z}_n^R]}} \exp\left\{\left(\hat{s}_n^{(0)} + \hat{s}_n^{(1)}\Delta\theta_n^R\right)\hat{z}_n^R + \hat{s}_n^{(2)}(\hat{z}_n^R)^2\right\} \times \exp\left\{\frac{-(\hat{z}_n^R)^2}{2\mathbb{V}[\hat{z}_n^R]}\right\} d\hat{z}_n^R$$

$$= \underbrace{\frac{1}{\sqrt{1+2A^R s_n^{(0)}\mathbb{V}[\hat{z}_n^R]}}}_{\text{constant absorbed by } \Psi_n^R} \exp\left\{-A^R\left(\frac{-A^R\mathbb{V}[\hat{z}_n^R]\left(s_n^{(0)} + s_n^{(1)}\Delta\theta_n^R\right)^2}{2\left(1+2A^R s_n^{(2)}\mathbb{V}[\hat{z}_n^R]\right)}\right)\right\}, \tag{A.3}$$

where $\hat{s}_n^{(i)} = -A^R s_n^{(i)}$, for $i = 0, 1, 2$, and where it is required that:

$$\hat{s}_n^{(2)} - \frac{1}{2\mathbb{V}[\hat{z}_n^R]} < 0. \tag{A.4}$$

Condition (A.4) ensures the above expectation is finite.

Combining (A.2) and (A.3) one can verify that the resulting expression inside the exponential function is quadratic in $\Delta\theta_n^R$ and thus it suffices to check the usual optimality conditions:

$$\frac{\partial}{\partial\Delta\theta_n^R}\left(F_n^R(\mathbf{Y}_n) + W_n^R\left(s_n^{(0)}\right)^2 + \left[d_n^{(1)} + 2W_n^R s_n^{(0)} s_n^{(1)}\right]\Delta\theta_n^R + \left[d_n^{(2)} + W_n^R\left(s_n^{(1)}\right)^2\right]\left(\Delta\theta_n^R\right)^2\right) = 0,$$
$$\frac{\partial^2}{\partial\Delta(\theta_n^R)^2}\left(F_n^R(\mathbf{Y}_n) + W_n^R\left(s_n^{(0)}\right)^2 + \left[d_n^{(1)} + 2W_n^R s_n^{(0)} s_n^{(1)}\right]\Delta\theta_n^R + \left[d_n^{(2)} + W_n^R\left(s_n^{(1)}\right)^2\right]\left(\Delta\theta_n^R\right)^2\right) < 0,$$

whence, after some algebra, one obtains:

$$\begin{aligned}(\Delta\theta_n^R)^* &= \underbrace{\frac{\Lambda_n^{(1)}}{D_n^R}}_{=:\bar{\Lambda}_n^{(1)}} Y_{n-1}^{(1)} + \underbrace{\frac{\Lambda_n^{(2)}}{D_n^R}}_{=:\bar{\Lambda}_n^{(2)}} Y_{n-1}^{(2)} + \underbrace{\frac{\Lambda_n^{(3)}}{D_n^R}}_{=:\bar{\Lambda}_n^{(3)}} Y_{n-1}^{(3)} \\ &= \mathbf{\Lambda}_n \cdot \mathbf{Y}_{n-1},\end{aligned} \tag{A.5}$$

and the second order condition $-D_n^R < 0$, where:

$$D_n^R = -2\Bigg(\underbrace{\left(L_n^{(2,0,0)} + \lambda_n L_n^{(1,1,0)} - r_n L_n^{(1,0,1)} + \lambda_n^2 L_n^{(0,2,0)} - \lambda_n r_n L_n^{(0,1,1)} + r_n^2 L_n^{(0,0,2)}\right)}_{=:\ d_n^{(2)};\ \text{see also, Choi et al. (2019)}} + W_n^R\left(s_n^{(1)}\right)^2\Bigg),$$

$$W_n^R = \frac{-A^R\mathbb{V}[\hat{z}_n^R]}{2\left(1+2A^R s_n^{(2)}\mathbb{V}[\hat{z}_n^R]\right)}.$$

This verifies (50) and (51). Furthermore, from the above presentation it is straightforward to see that, by setting $A^R = 0$, one immediately recovers the risk neutral solution presented in Choi et al. (2019). In other words, the model nests the risk neutral version of the model.

Now, it is easy to check that $d_n^{(1)}$ and $s_n^{(0)}$ from (A.2) are linear in the state variables:

$$\begin{aligned} d_n^{(1)} &= M_n^{(1)} Y_{n-1}^{(1)} + M_n^{(2)} Y_{n-1}^{(2)} + M_n^{(3)} Y_{n-1}^{(3)}, \\ s_n^{(0)} &= m_n^{(1)} Y_{n-1}^{(1)} + m_n^{(2)} Y_{n-1}^{(2)} + m_n^{(3)} Y_{n-1}^{(3)}. \end{aligned}$$

Moreover, $d_n^{(2)}$ and $s_n^{(1)}$ are independent of the state variables. Therefore, plugging $(\Delta\theta_n^R)^* = \mathbf{\Lambda}_n \cdot \mathbf{Y}_{n-1}$ from (A.5) back into:

$$F_n^R(\mathbf{Y}_n) + W_n^R\left(s_n^{(0)}\right)^2 + \left[d_n^{(1)} + 2W_n^R s_n^{(0)} s_n^{(1)}\right]\Delta\theta_n^R + \left[d_n^{(2)} + W_n^R\left(s_n^{(1)}\right)^2\right]\left(\Delta\theta_n^R\right)^2,$$

yields the expression inside the exponent function on the right hand side of (49). This concludes the proof. ■

Proof of Lemma 5. Omitted. See, the proof of Lemma 3. ■

Proof of Lemma 6. The proof is similar to the proof of Lemma 4. ■

Proof of Theorem 1. The proof of the verification theorem follows from Lemmas 2 - 6. ■

B NUMERICAL ANALYSIS

Appendix B introduces the numerical methods.

B.1 General Discussion

In this appendix, the procedures to determine an equilibrium numerically are described. This is necessary as, due to lack of analytical solutions, most models of strategic trading in finance are forced to utilize numerical methods in illustrating various equilibrium properties. The standard numerical approach, more or less applicable to the present model, is detailed in Kyle (1985) and it consists of a combination of an "educated" initial guess and a backward induction based solution procedure, involving simple numerical (univariate) root-finding. The approach terminates when one finds an initial guess, which produces a set of starting values matching those set by the modeler ex ante. This approach is well-suited for solving many important extensions of the standard Kyle-model such as versions with risk averse informed traders, multiple informed traders, and so forth. [54]

In the model at hand there are two additional complications, not present in Kyle (1985) and its simpler extensions. **(1)** Instead of a single initial guess, one is required to provide three initial guesses $\mathbf{\Sigma}_{N-1} = \left(\Sigma^{(1)}_{N-1}\ \Sigma^{(2)}_{N-1}\ \Sigma^{(3)}_{N-1}\right)^{\top}$, all of which should be, in some shape or form, mutually consistent (recall the discussion in Section 5.2), and **(2)** instead of simple univariate root finding problem one is faced with the task of solving a system of (simultaneous) polynomial equations. As the implications of these complications on equilibrium existence and multiplicity are already dealt with in the main text, the focus here is instead on the implications on applicable solution methods and the solution procedure.

Two approaches for determining the (set of) numerical solution(s) to the model presented in this paper are discussed in detail. The first one emphasizes speed and is useful due to the fact that finding the correct initial guess usually boils down to a three dimensional grid search with bisection—a procedure which can easily take a large chunk of computing time.[55] The second one emphasizes uniqueness of the solution path and is best used in conjunction with the first approach, as the second approach is orders of magnitude slower than the first one. Using the combination of these two approaches could be deemed as a sort of *hybrid approach*, benefiting from the best of both worlds—the first approach is used to narrow down feasible initial guesses and the second approach is used to examine the number of solutions which meet the equilibrium conditions.

[54] However, due to the dynamic nature of the problem, solutions from later rounds are used as inputs in solving model equations in earlier rounds. This feedback leads to the possibility of *dynamic (propagating) error* which may cause (numerical) instabilities. To be more specific, in each round, after having solved the key equilibrium constants, one solves the value function coefficients using the constants solved earlier. These value function coefficients are then (again) plugged into their respective objective functions to be used in the subsequent optimization problem. Increasing N, the number of trading periods, one increases the number of these backwards substitutions where the solutions from the previous round are directly used to solve the next period problem.

[55] Typically, in economics and finance, one utilizes this sort of search in a situation where, for instance, one has to find a *discrete approximation* of a complicated objective function over some predetermined subset of $\mathbb{R}^n$. In this type of situation, determining the "optimal grid" is usually a rule-based problem, whereas in the present model one initially faces a substantial amount of uncertainty pertaining to how to best set up the initial grid.

The key steps in the numerical procedures are:

(1) The starting step, similarly to Choi et al. (2019), involves guessing the period $N-1$ values of the conditional moments $(\Sigma_{N-1}^{(k)})_{k=1,2,3}$, as well as fixing the values for other model constants. Due to the trading constraint, the trading parameters of the rebalancer in period N are fixed as $\beta_N^R = 1$ and $\alpha_N^R = 0$. Thus, one only needs to solve for β_N^I and λ_N as depicted in Lemma 1. Finally, before moving to period $N-1$, one needs determine the value function coefficients $(I_{N-1}^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$ and $(L_{N-1}^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$ for both I and R respectively. These value function coefficients depend only on parameters which are known at N and can thus be treated as constants in the period $N-1$ problem.

(2) In rounds $n = N-1, N-2, ..., 1$ the algorithm proceeds by backward induction, taking as inputs the conditional moments and value function coefficients solved in the previous round and proceeding to determine, under the equilibrium path restrictions, the trading parameters $\{\beta_n^I, \beta_n^R, \alpha_n^R\}$ pricing parameters $\{\lambda_n, r_n\}$, and the conditional moments $\boldsymbol{\Sigma}_{n-1} = \left(\Sigma_{n-1}^{(1)}\ \Sigma_{n-1}^{(2)}\ \Sigma_{n-1}^{(3)}\right)^\top$. Initially, the aforementioned parameters are solved using local methods, ignoring the fact that multiple solutions may exist for the system of nonlinear equations. Finally, the value function coefficients $(I_{n-1}^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$ and $(L_{n-1}^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$ are updated. The algorithm returns to step (1) until the initial guesses produce moments that concur with the fixed starting values $(\Sigma_0^{(k)})_{k=1,2,3}$.

(3) For a previously determined set of equilibrium starting values, denoted by $\mathcal{S}_{\epsilon,EQ}^N$, the equilibrium path is recalculated, this time taking into account all solutions to the equilibrium system of equations.[56] The goal is to explicitly rule out all solutions which do not adhere to the equilibrium conditions and by doing this to verify whether or not there are, for a given parametrization, multiple equilibria.[57] It is evident that the solution found earlier, the de facto starting point of step (3), should survive this process of elimination and thus at least one equilibrium remains at the end.

B.2 Description of Basic Nonlinear Algorithm

Algorithm 1 describes the solution method for the dynamic Markovian equilibrium based on solving the complete nonlinear equilibrium system of equations. For reference, define the so-called *equilibrium system of equations* to constitute of equations (40) and (41), which define the pricing parameters, λ_n and r_n, equations (42)-(44), which define the conditional moments $\Sigma_n^{(i)}$, $i = 1, 2, 3$, and expressions (56) and (52), which give β_n^I as well as β_n^R.

56 For the sake of efficiency, $\mathcal{S}_{\epsilon,EQ}^N$—or, more specifically, the tuple of initial guesses related to $\mathcal{S}_{\epsilon,EQ}^N$—is initially determined using an algorithm (Algorithm 1) which, at each round, finds only one solution to the set of equilibrium equations. Finding $\mathcal{S}_{\epsilon,EQ}^N$ usually requires one to go through a large grid of initial guesses and consequently, to speed up this process, a fast algorithm is crucial. Once the iterative part is over and $\mathcal{S}_{\epsilon,EQ}^N$ is obtained, one can use a more specialized algorithm (Algorithm 2) to study equilibrium uniqueness.

57 Complex solutions may be removed immediately, while real solutions are ruled out by establishing that they are not part of any feasible equilibrium path.

Algorithm 1: *Linear equilibrium search and verification using Newton's method*

Input: Number of periods N, termination parameter $\epsilon > 0$, risk aversion coefficients A^I and A^R, variance of noise trading demand σ_u^2, variance of risky asset payoff σ_v^2, variance of (latent) trading demand σ_a^2, correlation ρ between $\tilde{v}$ and $\tilde{a}$, and initial guesses for $\Sigma_{N-1}^{(1)}$, $\Sigma_{N-1}^{(2)}$, and $\Sigma_{N-1}^{(3)}$.

Output: For all $n = N, N-1, ..., 1$, the trading parameters $(\beta_n^I,\ \beta_n^R,\ \alpha_n^R)$, pricing parameters $(\lambda_n,\ r_n)$, moments $\Sigma_{n-1}^{(i)}$, with $i = 1,2,3$, the value function coefficients for I, $(I_n^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$, and value function coefficients for R, $(L_n^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$.

for ($n = N$; $n = 1$; $n = n-1$)
- **if** $n = N$ **then**
  - Fix $\beta_N^R := 1$, $\alpha_N^R := 0$, & $r_N := 0$ and solve constants β_N^I and λ_N using (30) and (31)[a].
  - Using $(\beta_N^I,\ \beta_N^R,\ \alpha_N^R,\ \lambda_N,\ r_N)$, and the fact that $I_N^{(\boldsymbol{\omega})} = 0$, $\forall \boldsymbol{\omega} \in \Omega$ and $L_N^{(\boldsymbol{\phi})} = 0$, $\forall \boldsymbol{\phi} \in \Phi$, determine the value function coefficients $(I_{N-1}^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$ and $(L_{N-1}^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$.
- **else**
  - Plug in the known constants (i.e., value function coefficients and current period conditional moments) to the equilibrium system of equations and solve the resulting system using Newton's method.
  - Using the solved equilibrium parameters, determine α_n^R via (52).
  - Check the second order conditions $D_n^R > 0$ and $D_n^I > 0$.
  - **if** *Equilibrium constraints are satisfied*[b] **then**
    - Determine the value function coefficients $(I_{n-1}^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$ and $(L_{n-1}^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$ and proceed to round $n-1$.
  - **else**
    - Adjust initial guesses $\Sigma_{N-1}^{(i)}$, $i = 1,2,3$, and start the algorithm from the beginning.
  - **end**
- **end**

endfor

if $|\Sigma_0^{(1)} - \sigma_a^2| < \epsilon$ **and** $|\Sigma_0^{(2)} - \sigma_v^2| < \epsilon$ **and** $|\Sigma_0^{(3)} - \rho\sigma_v\sigma_a| < \epsilon$ **then**
- The algorithm terminates.

else
- Adjust initial guesses $\Sigma_{N-1}^{(i)}$, $i = 1,2,3$, and start the algorithm from the beginning.

end

[a] Note that these equations need to be modified to the general N period case in an obvious way.
[b] Equilibrium constraints are taken to mean that each equilibrium constant is obtained from the equilibrium system of equations and the second order conditions as well as the natural conditions (positivity etc.) for the conditional moments are satisfied.

Finding the appropriate initial guesses for the conditional moments over the three dimensional search space is nontrivial and, due to the complex interconnections between the model parameters, the equilibrium system of equations gives little information concerning the correct values of these initial guesses. For this reason, one needs to have an efficient way to search for—basically a three dimensional grid search—a fitting tuple of initial guesses. Algorithm 1 is used for this purpose. More specifically, in each period the nonlinear equilibrium system is solved with the fast converging (local) Newton's method.[58]

[58] The fast convergence of the Newton's method relies on providing the method a starting point relatively close to the eventual root of the target system.

B.3 Description of Homotopy Algorithm

The search and verification algorithm for the dynamic Markovian equilibrium using *polynomial homotopy continuation* is described in Algorithm 2. It assumed that the equilibrium system of equations is manipulated to yield two coupled polynomials in β_n^I and β_n^R instead of the seven equation nonlinear system used in Algorithm 1. The mechanics of this manipulation operation are briefly presented below.

Let $\mathcal{F}_n^I$ denote the polynomial related to (56) and $\mathcal{F}_n^R$ denote the polynomial related to (52). To see that solving the equilibrium constants can be reduced to solving the system $\mathcal{F}_n^I = \mathcal{F}_n^R = 0$ start by noting that solving (42)-(44) backwards yields:

$$\Sigma_{n-1}^{(1)} = \frac{(1-\lambda_n\beta_n^I)\Sigma_n^{(1)} + r_n\beta_n^I\Sigma_n^{(3)}}{(\beta_n^R-1)(\beta_n^R(1+r_n)+\lambda_n\beta_n^I(1-\beta_n^R)-1)}, \tag{B.6}$$

$$\Sigma_{n-1}^{(2)} = \frac{(\lambda_n\beta_n^R)^2\Sigma_n^{(1)} + B_n^{(2)}\Sigma_n^{(2)} + B_n^{(3)}\Sigma_n^{(3)}}{(1-\lambda_n\beta_n^I)(\beta_n^R-1)(\beta_n^R(1+r_n)+\lambda_n\beta_n^I(1-\beta_n^R)-1)}, \text{ where} \tag{B.7}$$

$$B_n^{(2)} := 1+\beta_n^I\lambda_n\Big(\beta_n^R(2-\beta_n^R)-1\Big)+\beta_n^R\Big(\beta_n^R(1+r_n)-(2+r_n)\Big)$$

$$B_n^{(3)} := \beta_n^R\lambda_n\Big(1-\beta_n^R(1+r_n)\Big)$$

$$\Sigma_{n-1}^{(3)} = \frac{(1-\beta_n^R(1+r_n))\Sigma_n^{(3)} + \lambda_n\beta_n^R\Sigma_n^{(1)}}{(\beta_n^R-1)(\beta_n^R(1+r_n)+\lambda_n\beta_n^I(1-\beta_n^R)-1)} \tag{B.8}$$

Plugging the above expressions to equations (40) and (41), solving for λ_n and r_n, and simplifying yields λ_n and r_n as equations of β_n^I and β_n^R only. These expressions can then be substituted back into (B.6)-(B.8) which in turn yields $\Sigma_{n-1}^{(1)}$, $\Sigma_{n-1}^{(2)}$, and $\Sigma_{n-1}^{(3)}$ expressed in terms of β_n^I and β_n^R only.

The final step is to plug in the new expressions for λ_n, r_n, $\Sigma_{n-1}^{(1)}$, $\Sigma_{n-1}^{(2)}$, and $\Sigma_{n-1}^{(3)}$ to:

$$\begin{cases} \beta_n^R D_n^R - \Lambda_n^{(1)} - \Sigma_n^{(3/1)}\Lambda_n^{(2)} = & 0, \\ \beta_n^I D_n^I - \Gamma_n^{(1)} - \Sigma_n^{(3/2)}\Gamma_n^{(2)} = & 0, \end{cases}$$

obtained from the FOCs of R and I respectively.[59] Simplification and algebraic manipulation of the resulting expressions then gives the two (coupled) polynomials $\mathcal{F}^I$ and $\mathcal{F}^R$. The equations obtained in this manner are referred to as the *(coupled) β-equations*.

Due to the fact that λ_n, r_n, $\Sigma_{n-1}^{(1)}$, $\Sigma_{n-1}^{(2)}$, and $\Sigma_{n-1}^{(3)}$ are at this point expressed as *functions* of β_n^I and β_n^R only, it follows that solving $\mathcal{F}_n^I = \mathcal{F}_n^R = 0$ is sufficient for obtaining the values of all of the remaining equilibrium constants. For a given solution pair, the values of the other equilibrium parameters are thus uniquely determined.

59 Unfortunately these equations are too complicated to be presented explicitly.

Algorithm 2: *Linear equilibrium search and verification using PHC*

Input: Number of periods N, termination parameter $\epsilon > 0$, risk aversion coefficients A^I and A^R, variance of noise trading demand σ_u^2, variance of risky asset payoff σ_v^2, variance of (latent) trading demand σ_a^2, correlation ρ between $\tilde{v}$ and $\tilde{a}$, and initial guesses $\Sigma_{N-1}^{(1)}$, $\Sigma_{N-1}^{(2)}$, and $\Sigma_{N-1}^{(3)}$.

Output: For all $n = N, N-1, ..., 1$, the trading parameters $(\beta_n^I, \beta_n^R, \alpha_n^R)$, pricing parameters (λ_n, r_n), moments $(\Sigma_{n-1}^{(i)})$, with $i = 1,2,3$, the value function coefficients for I, $(I_n^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$, and value function coefficients for R, $(L_n^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$.

for ($n = N$; $n = 1$; $n = n-1$ **)**

  if $n = N$ **then**

    Fix $\beta_N^R := 1$, $\alpha_N^R := 0$, & $r_N := 0$ and solve for the constants β_N^I and λ_N using (30) and (31)[a] .
    Using $(\beta_N^I, \beta_N^R, \alpha_N^R, \lambda_N, r_N)$, and the fact that $I_N^{(\boldsymbol{\omega})} = 0$, $\forall \boldsymbol{\omega} \in \Omega$ and $L_N^{(\boldsymbol{\phi})} = 0$, $\forall \boldsymbol{\phi} \in \Phi$, determine the value function coefficients $(I_{N-1}^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$ and $(L_{N-1}^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$.

  else

    Plug in the constants from the previous round and extract the set of real solution pairs from the set of all ($\mathbb{C}^2$) solutions to the coupled β-equations.

    Determine whether the set of real solutions contains feasible solutions by solving for the other equilibrium constants $\lambda_n, r_n, \alpha_n^R$, and $(\Sigma_{n-1}^{(i)})$, $i = 1,2,3$, and verifying that the SOCs and other equilibrium conditions hold.

    If multiple feasible solution pairs are found, all pairs are tested one by one.

    if *Equilibrium constraints are satisfied* **then**

      Determine the value function coefficients $(I_{n-1}^{(\boldsymbol{\omega})})_{\boldsymbol{\omega}\in\Omega}$ and $(L_{n-1}^{(\boldsymbol{\phi})})_{\boldsymbol{\phi}\in\Phi}$ and proceed to round $n-1$.

    else

      Break the loop and adjust the initial guesses $\Sigma_{N-1}^{(i)}$, $i = 1,2,3$.

    end

  end

endfor

if $|\Sigma_0^{(1)} - \sigma_a^2| < \epsilon$ **and** $|\Sigma_0^{(2)} - \sigma_v^2| < \epsilon$ **and** $|\Sigma_0^{(3)} - \rho\sigma_v\sigma_a| < \epsilon$ **then**

  The algorithm terminates.

else

  Adjust initial guesses $\Sigma_{N-1}^{(i)}$, $i = 1,2,3$, and start the algorithm from the beginning.

end

[a] Note that these equations need to be modified to the general N period case in an obvious way.

MIKA HANNULA

Order Execution Game with Transient Price Impact and Asymmetric Information

Preprint

III

Order Execution Game with Transient Price Impact and Asymmetric Information

Mika Hannula*

Abstract

In this paper an order execution game between a large *trader* and an adversary *arbitrageur* under transient price impact and asymmetric information is studied. The model extends previous work by Moallemi et al. (2012) and shows that shifting from a permanent (linear) price impact to transient (linear) price impact markedly changes the optimal trading strategies of both the trader and the arbitrageur. Furthermore, the impact of transaction fees on the equilibrium strategies and expected profits is examined and it is shown that small transaction fees may be beneficial for a large trader utilizing suboptimal trading strategies. The paper concludes with an overview of numerous directions for future research, illustrating the wealth of remaining open questions.

Keywords: Strategic trading, optimal execution, transient price impact, asymmetric information, transaction fees

JEL Classification Numbers: G14, G23, G24

1 INTRODUCTION

Optimal execution problems have recently gained much attention in the finance literature and industry. While there are a number of reasons behind this development, the advent of new trading technologies—floor trading has given way to electronic trading—and introduction of new regulation (e.g., *MiFID II* & *Reg. NMS* best execution rules) are among the chief culprits for the ever increasing use of trading algorithms and reinforced attention towards trade execution quality (cf. Anand et al. 2011). Increased adoption has sparked the interest of researches, which instead has fueled innovation and development in this growing and evolving area of finance.

In short, an optimal execution problem asks how to optimally buy or sell a given

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number of some target security while taking into account the fact that trading, in contrast to what some well-known models of financial markets would have you believe, tends to be costly. First, trades tend to have an impact on the price of the security traded, i.e., liquidity is limited. In fact, one of the key questions in how to set up an optimal execution problem is how the price impact of trades is modeled, and most strategic trading models, i.e., models where traders acknowledge that their actions are costly and influence prices, focus solely on the effects of price impact. There are, however, other costs such as various trading (transaction) fees, adding to the total execution costs.[1] Further, a trader $\mathcal{T}$, while executing a given trade, may face a situation where there are some adversary traders present in the market, trying to profit from the (predictable) trading program of $\mathcal{T}$.[2]

Returning to the issue of price impact, in classical models, such as Kyle (1985) and its many extensions, price impact is assumed to be *linear* and *permanent*. Linear permanent price impact is theoretically well-founded in that it rules out quasi-arbitrage (see, Huberman and Stanzl 2004). Many, more practice oriented models, such as Bertsimas and Lo (1998), Almgren and Chriss (1999), Almgren and Chriss (2001), and Huberman and Stanzl (2005) distinguish between (typically linear) *permanent* price impact and *temporary* price impact, where the latter may be nonlinear and affects only current trades.[3]

More recently, a view that price impact should in fact be modelled as a *transient* effect has gained foothold. This view is taken, for instance, in the influential paper by Obizhaeva and Wang (2013), where the authors focus on modeling the dynamics of a *limit order book* (LOB), arriving at a tractable optimal execution model which features transient price impact. A large literature follows in the footsteps of Obizhaeva and Wang (2013).[4] The model developed in this paper also embraces the transient price impact view. However, different from most earlier models, the two additional sources of trading costs, namely, transaction fees and the existence of potential adversary traders is also recognized.

The first is achieved by augmenting the objective functions of the model agents with quadratic transaction fees. As for the second source of additional trading costs, a distinguishing feature of all of the theoretical papers discussing optimal execution listed above (excl. Kyle 1985) is that they deal with a "single-player in a vacuum" setting, i.e., there are no strategic interactions. This is naturally a notable shortcom-

[1] Generally speaking, trading fees could be categorized as being *visible* costs while, for example, price impact costs may be considered as invisible (cf. Treynor 1994). Explicit (transaction fees) and implicit (price impact) costs is another commonly used categorization.

[2] The case of LTCM is (by now) a classic example. See, Brunnermeier (2005).

[3] Obviously, in order to be able to separate permanent from temporary, one must assume that there is a long enough time between distinct trades.

[4] Working paper versions of Obizhaeva and Wang (2013) date back to at least 2005 and many extensions to the model were developed and even published prior to the publication of the original paper. For instance, Alfonsi et al. (2010) replace the block-shaped LOB of Obizhaeva and Wang (2013) with a more general shape function and derive optimal strategies. Similarly, Predoiu et al. (2011) study a generalization of Obizhaeva and Wang (2013).

ing as real markets thrive on dynamic interactions between different market participants. To reconcile this issue, the model developed below features, in addition to a large trader seeking to execute optimally an exogenously given trade, an adversary arbitrageur whose goal is solely to maximize trading profits.

More precisely, the framework studied involves two players, one of who is exogenously required to execute a given trade. This exogenously constrained player is henceforth referred to as trader $\mathcal{T}$. The other player, henceforth referred to as arbitrageur $\mathcal{A}$, only seeks to profit from the trading needs of $\mathcal{T}$. Asymmetric information is introduced in the form that the size of the initial position of $\mathcal{T}$ is observed by $\mathcal{T}$ only, and $\mathcal{A}$ must infer the size of the evolving asset position from the (partially observed) trades of $\mathcal{T}$.

The papers closest to the present model are Moallemi et al. (2012), on which the transient price impact order execution game featured in this paper is built upon, and Choi et al. (2019), who study a related extension of Kyle (1985) featuring non-nested asymmetric information and linear permanent price impact. Other closely related papers include Yang and Zhu (2020), who also study an extension of Kyle (1985) where "back-runners" can observe past orders of informed investors who, in turn, are forced to resort to using randomized strategies, as well as Strehle (2017) and Schied and Zhang (2018), who study an order execution game under transient price impact and symmetric information.[5]

The present model differs from Moallemi et al. (2012) in that instead of permanent linear price impact, the model is formulated under transient linear price impact. As is shown below this change yields non-trivial changes to the resulting optimal trading strategies. Moreover, Moallemi et al. (2012) assume that there are no additional trading fees, whereas quadratic fees are included in the present model.

Similarly to Moallemi et al. (2012), Choi et al. (2019) and Yang and Zhu (2020) utilize permanent linear price impact.[6] However, in these Kyle-based models price impact is endogenous. In the model below the price impact, as noted earlier, is transient and *exogenous*.[7] Moreover, Choi et al. (2019) conduct their analysis under non-nested information sets (two-sided asymmetric information), while the present model, together with Moallemi et al. (2012), features nested information sets (one-sided asymmetric information). Yang and Zhu (2020) instead focus on a two period model and assume that there are back-runners in the market who obtain signals pertaining to the past trades of a fundamental investor which forces the fundamental investor to adopt a mixed trading strategy and, as such, the motivation behind their

[5] See also, Schied and Zhang (2017) and Huang et al. (2019).

[6] Linear permanent price impact is best suited for models with private fundamental information which is slowly incorporated into prices. While there is private information in the present model, it is not considered to be fundamental, i.e., information directly tied to the economic state of a company whose securities are traded or information related to the state of the surrounding economy.

[7] Endogenous transient price impact would essentially require one to model explicitly the dynamics of the underlying limit order book. This task is beyond the scope of this paper.

paper is quite different from the present paper. Choi et al. (2019) and Yang and Zhu (2020) also ignore costs other than those stemming from price impact.

Strehle (2017) and Schied and Zhang (2018) feature transient price impact and quadratic transaction costs but the models analyzed in these papers are formulated under the assumption of symmetric (perfect) information. In reality traders are not equally informed and a more appropriate setting features asymmetrically informed traders as in Moallemi et al. (2012), Choi et al. (2019), and the present paper.[8]

In sum, the objective of this paper is to push the boundaries of existing models and to study a more multifaceted order execution game with transient price impact, trading fees, as well as strategic interactions and asymmetric information. On top of this main objective, the main contributions to the existing literature are:

1. Analysis and comparison of the equilibrium strategies between (linear) permanent and (linear) transient price impact order execution game.

2. Insights into additional equilibrium properties and equilibrium existence in this new framework.

3. Numerical analysis of expected execution costs and factors affecting these costs under various trading strategies.

4. Analysis of the impact of introducing quadratic trading fees to the above-described framework.

5. A comprehensive look at the assumptions behind most current models supplemented with thoughts on potential modifications.

The rest of this paper is organized as follows. Section 2 introduces the model. Section 3 presents the model analysis. Section 4 provides the results from the numerical analysis. Section 5 discusses some specific model details and potential extensions. Section 6 concludes the paper. All proofs are allocated to Appendix A.[9]

2 MODEL

2.1 Preliminaries

The trade execution game involves two risk neutral players and takes place in discrete time, i.e., $t = 0, ..., T, T + 1$. The first player is referred to as the *trader* ($\mathcal{T}$, she) and

[8] Predatory trading models such as Brunnermeier (2005), Carlin et al. (2007), and Schied and Schöneborn (2009a) also feature perfect information.

[9] Appendices B and C provide additional details pertaining to the numerical methods and equilibrium analysis.

the second player as the *arbitrageur* ($\mathcal{A}$, he). One may interpret $\mathcal{T}$, as a (large) trader who is subject to a privately observed exogenous shock, resulting in a need to acquire/liquidate a number of shares in a risky asset.[10] $\mathcal{A}$ in turn can be interpreted as an opportunistic trader, seeking to profit from short-term price movements. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is taken as given and all random variables featured in the model are defined on this space.

As in Moallemi et al. (2012), both $\mathcal{T}$ and $\mathcal{A}$ are subject to boundary conditions (i.e., *hard* trading constraints). Namely, letting x_t (y_t) denote the target stock position of $\mathcal{T}$ ($\mathcal{A}$) at time t, it is required that $x_T = 0$ and $y_{T+1} = 0$. The assumption that $\mathcal{A}$ has one extra period to close his position again reflects the notion that $\mathcal{A}$ has more freedom in pursuing profit seeking trading strategies. Trade sizes at time t of $\mathcal{T}$ and $\mathcal{A}$ respectively are denoted by u_t and v_t so that the target stock position has dynamics:

$$x_t = x_{t-1} + u_t,$$
$$y_t = y_{t-1} + v_t.$$

The game features asymmetric information in the sense that the initial stock position of $\mathcal{T}$, denoted by $\tilde{x}_0$, is assumed to be private information of $\mathcal{T}$ while $\mathcal{A}$ begins the game with a (Gaussian) prior:[11]

$$\tilde{x}_0 \sim N(\mu_0, \sigma_0^2), \qquad \phi_0 \triangleq (\mu_0, \sigma_0). \tag{1}$$

The realization x_0 of $\tilde{x}_0$, observed exclusively by $\mathcal{T}$, is thought being a direct consequence of the aforementioned exogenous shock. $\mathcal{A}$'s initial position is denoted by y_0 and it is assumed that y_0 is common knowledge; in what follows it is generally assumed that $y_0 = 0$ which is consistent with the assumption that $\mathcal{A}$ is free to pursue profits from short horizon strategies and prefers to end up with zero inventory. Due to the common knowledge assumption, it follows that the information set of $\mathcal{T}$ nests the information set of $\mathcal{A}$ and hence there will be no *forecasting the forecasts of others problem* as in, for instance, Foster and Viswanathan (1996).

All period t trades are assumed to be executed at price p_t, where, for $t = 1, ..., T+1$, and $p_0 = 0$ fixed, p_t has the following dynamics:

$$\Delta p_t = \gamma q_t - (1-\alpha) s_{t-1}, \tag{2}$$

where[12]

$$q_t = u_t + v_t + e_t, \text{ with } \tilde{e}_t \overset{i.i.d.}{\sim} N(0, \sigma_e^2) \text{ and}$$
$$s_t = \sum_{\tau=0,\dots,t} \alpha^{t-\tau} \gamma q_\tau \text{ with } \alpha \in (0,1] \text{ and } s_0 = 0;$$

---

[10] The exogenous shock could be, for example, arrival of new fundamental information or a change in market conditions that plunges $\mathcal{T}$ into a state of distress.

[11] The private information assumption is interpreted as $\mathcal{T}$ observing at $t = 0$ the realization $x_0 \in \mathbb{R}$ of $\tilde{x}_0$.

[12] Note that q_t, the realization of $\tilde{q}_t$, is used to denote the total order flow for period t. After period t trading, q_t becomes public knowledge due to the fact that the total order flow is inferable from price changes.

In other words, trades impact prices and price impact is assumed to be *transient* (whenever $0 < \alpha < 1$) with factor α and an exogenous intensity γ. As noted by Obizhaeva and Wang (2013) this form of price impact is consistent with the notion of replenishing liquidity backed up by empirical literature (see, for example, Biais et al. 1995, Degryse et al. 2005, and Large 2007).

Above, $\tilde{e}_t$'s represent periodic random (noise) trades which, from a modeling perspective, enable $\mathcal{T}$ to hide part of her private information from $\mathcal{A}$ over an extended period of time. This is a standard assumption in finance literature, especially in the context of strategic trading models.

A small digress to further discuss the repercussions of the transient price impact assumption is in order. Classical market microstructure literature (cf. Kyle 1985) assume *permanent linear* price impact leading to price dynamics of the form:

$$\Delta p_t = \lambda_t q_t,$$

where λ_t is allowed to be endogenous. Generally, λ_t is derived using Gaussian filtering, in which case it has a clear interpretation as being the result of market makers' efforts to filter price relevant information from aggregate order flows. As a result, prices are semi-strong form efficient in that they reflect, within the context of a given model, all public information. This interpretation for the permanent price impact intensity λ does not extend to the transient intensity γ, or more generally γ_t, which should be thought of as stemming from the features of an unmodeled underlying limit order book and not from the filtering efforts of a set of market markers. Hence, γ is taken to be an exogenous constant and one can interpret price impact as being liquidity-driven.

Moreover, looking at the present model, the notion that prices would be governed by a linear permanent price impact and would reflect all public information would essentially mean that $\mathcal{A}$ could not trade profitably on any price relevant information obtained from aggregate order flows as this information would already be reflected in the prices. In addition, trades that do not contain price relevant information would have no price impact. However, under transient price impact, and because price impact is not assumed to be purely information-driven, $\mathcal{A}$ has a clear incentive to pursue trading strategies which are designed to reap profits from short-term price movements. Section 5 discusses incorporating the Kyle-lambda (λ) in to the present model.

2.2 Optimal Execution Problem

Given the dynamics for the stock positions, price, and information sets ($\mathcal{F}_t^{\mathcal{T}}, \mathcal{F}_t^{\mathcal{A}}$)—which are described properly in (11) and (12)—the objective of both $\mathcal{T}$ and $\mathcal{A}$, for each $t = 0, ..., T-1, T$, is to maximize expected profits given by:

$$\Pi_t^{\mathcal{T}} = \mathbb{E}\left[\sum_{\tau=t}^{T-1} \Delta p_{\tau+1} x_\tau - \theta_c \Delta x_\tau^2 \mid \mathcal{F}_t^{\mathcal{T}}\right], \tag{3}$$

$$\Pi_t^{\mathcal{A}} = \mathbb{E}\left[\sum_{\tau=t}^{T} \Delta p_{\tau+1} y_\tau - \theta_c \Delta y_\tau^2 \mid \mathcal{F}_t^{\mathcal{A}}\right], \tag{4}$$

over admissible, $\mathbf{u} = (u_1, ..., u_T) \in \mathcal{U}$ and $\mathbf{v} = (v_1, ..., v_{T+1}) \in \mathcal{V}$, where u_t (resp. v_t) is required to be $\mathcal{F}_t^{\mathcal{T}}$-adapted ($\mathcal{F}_t^{\mathcal{A}}$-adapted) and the sets $\mathcal{U}$ and $\mathcal{V}$ refer to all admissible controls for $\mathcal{T}$ and $\mathcal{A}$ respectively. To understand the objectives in terms of controls one should recall that, for example, $\Delta x_\tau = u_\tau$.

In addition, $\theta_c \geq 0$ is the common transaction fee parameter and Δ is used to denote differences, i.e., $\Delta w_\tau = w_\tau - w_{\tau-1}$. Naturally, setting $\theta_c = 0$ corresponds to the situation where transaction fees are ignored.

2.3 Equilibrium Concept

The relevant state variables for the model as described above are:[13]

$x_t -$ describes the evolution of $\mathcal{T}$'s position
$y_t -$ describes the evolution of $\mathcal{A}$'s position
$\mu_t -$ describes the evolution of $\mathcal{A}$'s beliefs
$s_t -$ describes the fading impact of past trades.

As in Moallemi et al. (2012) and Choi et al. (2019), the trading strategies of the players, for all $t = 1, ..., T+1$ are conjectured to be Gaussianity preserving, linear functions of the state variables, i.e.,

$$u_t = a_{x,t} x_{t-1} + a_{y,t} y_{t-1} + a_{\mu,t} \mu_{t-1} + a_{s,t} s_{t-1}, \tag{5}$$

$$v_t = b_{y,t} y_{t-1} + b_{\mu,t} \mu_{t-1} + b_{s,t} s_{t-1}, \tag{6}$$

which in turn implies that the value functions of $\mathcal{T}$ and $\mathcal{A}$ will be quadratic in the state variables. More specifically, let $\zeta_t^{\mathcal{T}} \in \mathbb{R}^4$ and $\zeta_t^{\mathcal{A}} \in \mathbb{R}^3$ represent the state variable vectors of $\mathcal{T}$ and $\mathcal{A}$ respectively as follows:

$$\zeta_t^{\mathcal{T}} = (x_t, y_t, \mu_t, s_t), \tag{7}$$

$$\zeta_t^{\mathcal{A}} = (y_t, \mu_t, s_t). \tag{8}$$

[13] Recall that $\mathcal{T}$ can determine μ_t the same way $\mathcal{A}$ can due to the nested nature of the information sets of the two traders. Thus, one should keep in mind that μ_t always refers to the conditional expectation restricted to $\mathcal{A}$'s information set.

Then the value functions take the forms:[14]

$$U_t = \mathbf{c}_t \cdot \mathrm{vech}(\boldsymbol{\zeta}_t^{\mathcal{T}} \otimes \boldsymbol{\zeta}_t^{\mathcal{T}})^\top, \tag{9}$$

$$V_t = \mathbf{d}_t \cdot \mathrm{vech}(\boldsymbol{\zeta}_t^{\mathcal{A}} \otimes \boldsymbol{\zeta}_t^{\mathcal{A}})^\top, \tag{10}$$

where the constants $\mathbf{c}_t \triangleq (c_{xx,t}, c_{xy,t}, c_{x\mu,t}, c_{xs,t}, c_{yy,t}, c_{y\mu,t}, c_{ys,t}, c_{\mu\mu,t}, c_{\mu s,t}, c_{ss,t})$ represent the value function coefficients of $\mathcal{T}$ and in a similar manner the constants $\mathbf{d}_t \triangleq (d_{yy,t}, d_{y\mu,t}, d_{ys,t}, d_{\mu\mu,t}, d_{\mu s,t}, d_{ss,t})$ represent the value function coefficients of $\mathcal{A}$.

The goal is to find a (weak) Perfect Bayesian equilibrium in pure strategies according to the following definition.[15]

Definition 1 (Perfect Bayesian Equilibrium (PBE)). *A perfect Bayesian equilibrium of the model is a system of beliefs $\phi \triangleq (\mu, \sigma)$ and a pair of strategies (u^*, v^*) such that:*

(1) Taking ϕ_0 as given, ϕ_t, for each $t = 1, ..., T$, is obtained via (14) and (15), consistently with Bayes' rule.

(2) $\mathcal{A}$'s strategy $\mathbf{v}^* \in \mathcal{V}$ is sequentially rational, i.e., for each $t = 1, ..., T+1$, it maximizes (4) given $\{\phi_t\}$ and the conjectured strategy $\hat{\mathbf{u}}$.

(3) $\mathcal{T}$'s strategy $\mathbf{u}^* \in \mathcal{U}$, such that $\mathbf{u}^* = \hat{\mathbf{u}}$, is sequentially rational, i.e., for each $t = 1, ..., T$, it maximizes (3) given $\mathbf{v}^* \in \mathcal{V}$ and $\{\phi_t\}$.

Absent from the above definition is the fact that the state variables $\{x_t, y_t, \mu_y, s_t\}$ will adhere to Markovian dynamics, which enables the use of standard dynamic optimization tools. In spite of this, the equilibrium of the model cannot be expressed in closed form and numerical methods are instead utilized to obtain model solutions and to evaluate model properties. The numerical algorithm is adapted from Moallemi et al. (2012) and described in Appendix B.

Another point in Definition 1 that deserves more attention is the belief system. Indeed, the beliefs in the definition are referred to only in a simplified fashion, depicting them as they were separate from the explicit sources of uncertainty brought on by the underlying probability structure. This issue is remedied in the following section.

[14] The term vech(·) refers *half-vectorization*. This specific linear transformation is utilized since, for example, the *outer product* $\boldsymbol{\zeta}_t^{\mathcal{A}} \otimes \boldsymbol{\zeta}_t^{\mathcal{A}}$ yields a symmetric square matrix, i.e.,

$$\boldsymbol{\zeta}_t^{\mathcal{A}} \otimes \boldsymbol{\zeta}_t^{\mathcal{A}} = \begin{pmatrix} y_t^2 & y_t\mu_t & y_t s_t \\ y_t\mu_t & \mu_t^2 & \mu_t s_t \\ y_t s_t & \mu_t s_t & s_t^2 \end{pmatrix}.$$

[15] For simplicity, beliefs at zero-measure information sets are not restricted, hence the qualifier *weak*. This is not consequential for the results obtained below. The pure strategy restriction is taken to be implicit in the two sets of feasible strategies $\mathcal{U}$ and $\mathcal{V}$.

2.4 Information and Filtering

Even though $\mathcal{A}$ cannot directly observe the realization of $\tilde{x}_0$, he can still improve his view on the target stock position of $\mathcal{T}$ over time. Due to the fact that the underlying uncertainty takes the form of Gaussian random variables, linear filtering techniques can be used to obtain $\mathcal{A}$'s beliefs in a manner consistent with Bayesian updating. To see this, note that each price change involves, for $\mathcal{A}$, an uncertain component:[16]

$$\begin{aligned}\tilde{z}_t \triangleq \Delta\tilde{p}_t - \mathbb{E}_t^{\mathcal{A}}[\Delta\tilde{p}_t] &= \gamma(\tilde{u}_t + \tilde{e}_t) - \gamma(\hat{a}_{y,t}y_{t-1} + \hat{a}_{\mu,t}\mu_{t-1} + \hat{a}_{s,t}s_{t-1}) \\ &= \gamma(\hat{a}_{x,t}\tilde{x}_{t-1} + \tilde{e}_t),\end{aligned}$$

where $\mathbb{E}_t^{\mathcal{A}}[\cdot] \equiv \mathbb{E}[\cdot \mid \mathcal{F}_t^{\mathcal{A}}]$ is the conditional expectation of $\mathcal{A}$ based on his period t information specified below, and where the first term after the second equality, $\gamma(\tilde{u}_t + \tilde{e}_t)$, is clearly Gaussian and the term after that is a known constant. Call these terms *innovations* and denote $\mathbf{z}^t = (z_1, ..., z_t)$.

The hatted versions of $\mathcal{T}$'s trading parameters refer to the fact that $\mathcal{A}$ cannot directly observe the trades of $\mathcal{T}$ and thus cannot verify the strategy $\mathcal{T}$ is using. Instead, $\mathcal{A}$ makes inferences based on his own actions and the conjectured strategy of $\mathcal{T}$. As it is possible that the conjectured strategy differs from the actual strategy used by $\mathcal{T}$, hatted versions, i.e., $\hat{a}_{x,t}$, $\hat{a}_{y,t}$, $\hat{a}_{\mu,t}$, and $\hat{a}_{s,t}$ are used to designate the conjectured trading parameters. A reasonable follow-up question is whether one should also determine hatted versions of $\mathcal{A}$'s trading parameters, i.e., $\hat{b}_{i,t}$, $i \in \{y, \mu, s\}$. To answer this, it is wise to have a look at the information sets of $\mathcal{T}$ and $\mathcal{A}$.

First, it is important to emphasize that the innovation z_t only becomes available for $\mathcal{A}$ *after* period t trading. For this reason, the information set of $\mathcal{A}$ just *prior* to period t trading is given by:

$$\mathcal{F}_t^{\mathcal{A}} \triangleq \boldsymbol{\sigma}(y_{t-1}, z_1, ..., z_{t-1}), \tag{11}$$

where $\boldsymbol{\sigma}(S)$ refers to the σ-algebra generated by the a S. Similarly, the information set of $\mathcal{T}$ just *prior* to period t trading is given by:

$$\mathcal{F}_t^{\mathcal{T}} \triangleq \boldsymbol{\sigma}(x_{t-1}, y_{t-1}, z_1, ..., z_{t-1}). \tag{12}$$

Now, to solve the problem pertaining to $\hat{b}_{i,t}$'s, the decisive question is: Does it hold that $v_t \in \mathcal{F}_t^{\mathcal{T}}$? To answer this, first note that $\mathcal{T}$, in addition to her private information, has access to the same observations as $\mathcal{A}$. Moreover, since y_0 is common knowledge, one observes that μ_{t-1} and s_{t-1} indeed belong to $\mathcal{F}_t^{\mathcal{T}}$. Furthermore, it holds that:

$$v_1 = b_{y,1}y_0 + b_{\mu,1}\mu_0 + b_{s,1}s_0 \in \mathcal{F}_1^{\mathcal{T}},$$

16 Due to the Gaussian noise assumption all order flows occur with nonzero probability and hence $\mathcal{A}$ cannot directly infer the actions of $\mathcal{T}$. Due to this $\mathcal{T}$'s private information is not immediately revealed. The equilibrium of the order execution game will feature rational expectations in the sense that in equilibrium $\mathcal{A}$' s beliefs and the actual strategy utilized by $\mathcal{T}$ concur.

where $\mathcal{F}_1^{\mathcal{T}} = \boldsymbol{\sigma}(x_0, y_0)$. Suppose now (inductively) that $v_{t-1} \in \mathcal{F}_{t-1}^{\mathcal{T}}$ holds for $t-1$. This implies that $y_{t-1} = y_{t-2} + v_{t-1} \in \mathcal{F}_t^{\mathcal{T}}$, and since it is already confirmed above that μ_{t-1} and s_{t-1} are in $\mathcal{F}_t^{\mathcal{T}}$, one obtains that $v_t \in \mathcal{F}_t^{\mathcal{T}}$. Therefore, $\mathcal{T}$ cannot be misled and defining $\hat{b}_{i,t}$'s would be redundant.

Moving on, by Bayes' theorem one obtains that:

$$\mathbb{P}[\tilde{x}_t \mid \mathbf{z}^t] = \frac{\mathbb{P}[\tilde{x}_t, \tilde{z}_t \mid \mathbf{z}^{t-1}]}{\mathbb{P}[\tilde{z}_t \mid \mathbf{z}^{t-1}]} \propto \underbrace{\mathbb{P}[\tilde{z}_t \mid x_t]}_{\text{Current likelihood}} \times \underbrace{\mathbb{P}[\tilde{x}_t \mid \mathbf{z}^{t-1}]}_{\text{Last best estimate}}.$$

It is well established in Bayesian statistics that a Gaussian prior, e.g., $\mathbb{P}[\tilde{x}_0 \mid \mathbf{z}^0] = \mathbb{P}_0[\tilde{x}_0]$ (see, (1)), multiplied by a Gaussian likelihood yields a posterior (filter) distribution that will also be Gaussian as long as, in each period, the Gaussian random variables are mapped through affine (Gaussianity preserving) functions. Consequently, the filter distribution can be characterized through its first two conditional moments, i.e., conditional mean and conditional variance. Proposition 1 characterizes these moments.

Proposition 1 (**Filtering Formulas**). *Suppose that $\mathcal{A}$'s prior on $\mathcal{T}$'s initial position $\tilde{x}_0$ is $\tilde{x}_0 \sim N(\mu_0, \sigma_0^2)$, $\tilde{e}_t \overset{i.i.d.}{\sim} N(0, \sigma_e^2)$ is independent of $\tilde{x}_0$, and that Gaussianity is preserved in rounds $t' = 1, \ldots, t$. Then, it holds that:*

$$\mathbb{E}\Big[\tilde{x}_t \mid y_t, \Delta p_1, \ldots, \Delta p_t\Big] = \mathbb{E}\Big[\tilde{x}_t \mid y_t, \mathbf{z}^t\Big]. \tag{13}$$

Moreover, $\mathcal{A}$'s conditional distribution for $\tilde{x}_t$ is characterized by:

$$\begin{aligned}
\mu_t \triangleq& \mathbb{E}\Big[\tilde{x}_t \mid \mathcal{F}_{t+1}^{\mathcal{A}}\Big] = \mathbb{E}\Big[\tilde{x}_t \mid y_t, \mathbf{z}^t\Big] \\
=& \hat{a}_{y,t} y_{t-1} + \hat{a}_{s,t} s_{t-1} + (1 + \hat{a}_{\mu,t} + \hat{a}_{x,t})\mu_{t-1} + (1 + \hat{a}_{x,t}) g_t\Big(z_t - \mathbb{E}\Big[\tilde{z}_t \mid \mathbf{z}^{t-1}\Big]\Big) \\
=& \hat{a}_{y,t} y_{t-1} + \hat{a}_{s,t} s_{t-1} + \Big(\hat{a}_{\mu,t} + \theta_t / (\hat{a}_{x,t} \rho_{t-1}^2)\Big)\mu_{t-1} + \hat{a}_{x,t}\theta_t x_{t-1} + \theta_t e_t,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
g_t \triangleq& \frac{\mathbb{COV}[\tilde{x}_t, \tilde{z}_t \mid \mathbf{z}^{t-1}]}{\mathbb{V}[\tilde{z}_t \mid \mathbf{z}^{t-1}]} = \frac{a_{x,t}}{\gamma(\rho_{t-1}^{-2} + \hat{a}_{x,t}^2)}, \\
\theta_t \triangleq& \gamma(1 + \hat{a}_{x,t}) g_t = \frac{\hat{a}_{x,t}(1 + \hat{a}_{x,t})}{\Big(\rho_{t-1}^{-2} + \hat{a}_{x,t}^2\Big)},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_t^2 \triangleq& \mathbb{V}\Big[\tilde{x}_t \mid \mathcal{F}_{t+1}^{\mathcal{A}}\Big] = \mathbb{V}\Big[\tilde{x}_t \mid y_t, \mathbf{z}^t\Big] = \mathbb{V}\Big[\tilde{x}_{t-1} + \tilde{u}_t \mid y_t, \mathbf{z}^t\Big] \\
=& (1 + \hat{a}_{x,t})^2 \frac{\sigma_\epsilon^2 \sigma_{t-1}^2}{\Big(\sigma_\epsilon^2 + \sigma_{t-1}^2 \hat{a}_{x,t}^2\Big)}.
\end{aligned} \tag{15}$$

It is easily seen from (15) that the scaled variance (see, Moallemi et al. 2012) $\rho_t = \sigma_t/\sigma_\epsilon$ satisfies the relation:

$$\rho_t^2 = \frac{(1+\hat{a}_{x,t})^2}{\left(\rho_{t-1}^{-2}+\hat{a}_{x,t}^2\right)}. \tag{16}$$

The dynamics of ρ_t have an important role to play in the numerical determination of the model equilibrium. This point becomes evident from the algorithm presented in Appendix B.

3 MODEL ANALYSIS

3.1 Illustrative Example

In this section, to gain a better grasp of some of the model features, a three period version ($t = 1,2,3$) of the model is examined.[17] This example is suitable for demonstrating certain aspects that play an important role in the inner workings of the model, yet simple enough that the main intuition is not buried under lengthy and complicated equations. One of the key observations from below is that the due to the state variable treatment of transient price impact, the model mechanics remain close to the permanent price impact version of the model.

First, it is good to see at how the prices look in a fully explicit form. Using (2) one obtains:

$$\begin{aligned} p_0 &= 0, \\ p_1 &= \Delta p_1 = \gamma q_1 \\ p_2 &= \gamma\left(q_2 + \alpha q_1\right) \\ p_3 &= \gamma\left(q_3 + \alpha q_2 + \alpha^2 q_3\right), \end{aligned}$$

where $q_t = u_t + v_t + e_t$, for $t = 1,2,3$. Moving forward, it is assumed—without loss of generality[18]—that $\gamma = 1$. This assumption is made mainly for simplicity. From the explicit prices, it is easy to see that, for example:

$$\Delta p_2 = q_2 - (1-\alpha)\underbrace{q_1}_{\triangleq s_1},$$

[17] Time $t = 0$ is special in that no trading takes place.
[18] Because all other parameters can be scaled appropriately.

whence it is straightforward to obtain the general relation given in (2) and where the state variable s_t captures the (fading) impact of past order flows.

At this point it is useful to consider what does the optimization problem of $\mathcal{T}$ look when it is assumed that there are no adversary traders, i.e., no $\mathcal{A}$. Noting that $\mathcal{T}$ is required to complete her trade by $t = 2$, one obtains:[19]

$$\begin{aligned} &\mathbb{E}\Big[\Delta p_1 x_0 + \Delta p_2 x_1\Big] \\ =&\mathbf{u} \underbrace{\begin{pmatrix} -(2-\alpha) & -1/2 \\ -1/2 & -1 \end{pmatrix}}_{\text{negative definite for } \alpha\in(0,1]} \mathbf{u}^\top, \end{aligned}$$

which is maximized over strategies:

$$\Big\{\mathbf{u} = (u_1, u_2) \text{ s.t. } u_1 + u_2 = -x_0\Big\}.$$

The unique solution to the above problem, which is extendable to a general $T \in \mathbb{N}$, is the optimal strategy in the class of deterministic strategies for the case where $\mathcal{T}$ faces no adversary traders. It is easy to see that this strategy is also optimal in the class of adapted strategies since there is no useful information for $\mathcal{T}$ to glean from the price process in this case. Arguments along these lines leads one to see that, for example, the dynamic programming formulation in Obizhaeva and Wang (2013) can be substituted with a deterministic (convex minimization) problem; see, for instance, Fruth et al. (2014) or Lin and Fahim (2017).

However, with both $\mathcal{T}$ and $\mathcal{A}$ present, the market features strategic interactions as well as asymmetric information, and in this case both players learn new information from changes in the market price. In particular, $\mathcal{A}$ updates his beliefs (infers new information) about the position x_t of $\mathcal{T}$ based on past price changes as described in Proposition 1. Deterministic strategies overlook this learning aspect and hence, on average, underperform adaptive strategies. This observation is summarized in the first model feature.[20]

MODEL FEATURE 1: *Deterministic strategies will be suboptimal in equilibrium due to strategic interactions and asymmetric information.*

Utilizing backward induction, one sees that since $\mathcal{T}$ must finish her program at $t = 2$, it must be that $u_2^* = -x_1$. Similarly, since $\mathcal{A}$ must finish his program at $t = 3$, it follows that $v_3^* = -y_2$. Plugging v_3^* back into the objective function of $\mathcal{A}$ yields:

$$\begin{aligned} &\mathbb{E}\Big[(-y_2 + \tilde{e}_3)y_2 - (1-\alpha)y_2 s_2 \mid \mathcal{F}_3^{\mathcal{A}}\Big] \\ =&\underbrace{-1}_{\triangleq d_{yy,2}} y_2^2 + \underbrace{(\alpha - 1)}_{\triangleq d_{ys,2}} y_2 s_2, \end{aligned} \tag{17}$$

[19] Eigenvalues are given by: $L_i = \frac{1}{2}\Big(\pm\sqrt{\alpha^2 - 2\alpha + 2} + (\alpha - 3)\Big)$, for $i = 1, 2$.

[20] The relation of deterministic and adaptive strategies is revisited in Section 4.2.

whence, using the value function coefficients from above, $u_2^* = -x_1$, and the dynamics of s_t from (2), one arrives at the $t = 2$ objective of $\mathcal{A}$:

$$\mathbb{E}\Big[\big(v_2 - x_1 + \tilde{e}_2 - (1-\alpha)s_1\big)y_1 \\ - (y_1 + v_2)\big((y_1 + v_2) - (\alpha - 1)(\alpha s_1 + v_2 - x_1 + \tilde{e}_2)\big) \mid \mathcal{F}_2^{\mathcal{A}}\Big],$$

which is maximized over feasible v_2. The above objective is immediately recognized as a quadratic and it turns that a similar observation can be made for each trading period and for the objectives of both $\mathcal{T}$ and $\mathcal{A}$. As a result, it is, at each period, enough to verify the standard first and second order optimality conditions.

MODEL FEATURE 2: *The objective functions of $\mathcal{T}$ and $\mathcal{A}$ are quadratic under transient price impact.*

Continuing the backward analysis and plugging $u_2^* = -x_1$ back into the objective of $\mathcal{T}$ and assuming $y_0 = 0$, one finds that the first period objective is:

$$\mathbb{E}\Big[(u_1 + b_{\mu,1}\mu_0 + \tilde{e}_1)x_0 + c_{xx,1}(x_0 + u_1)^2 \\ + c_{xy,1}(x_0 + u_1)(y_0 + b_{\mu,1}\mu_0) + c_{x\mu,1}(x_0 + u_1)(\theta_1 u_1) \\ + c_{xs,1}(x_0 + u_1)(u_1 + b_{\mu,1}\mu_0 + \tilde{e}_1) \mid \mathcal{F}_1^{\mathcal{T}}\Big].$$

The first order condition with respect to u_1 is:

$$-2(c_{xx,1} + \theta_1 c_{x\mu,1} + c_{xs,1})u_1 = (1 + 2c_{xx,1})x_0 + b_{\mu,1}(c_{xy,1} + c_{xs,1})\mu_0,$$

whence, solving u_1 and equating appropriately, one obtains:

$$a_{x,1} = \frac{-(1 - c_{xx,1})}{2(c_{xx,1} + \theta_1 c_{x\mu,1} + c_{xs,1})},$$

which, recalling that:

$$\theta_1 = \frac{a_{x,1}(1 + a_{x,1})}{\left(\rho_0^{-2} + a_{x,1}\right)},$$

simplifies to a polynomial equation in $a_{x,1}$. Having obtained the roots of this polynomial, one can proceed to solving $a_{\mu,1}$ from the first order condition. Indeed, determining the coefficients of the a_x-polynomial and solving it is a key step in each round $t = 1, ..., T-1$, as a_x is used to obtain the values for all of the remaining constants a_y, a_μ, and a_s.

The last thing to note is that the model features transient price impact as long as $\alpha \in (0,1)$. Setting $\alpha = 0$ would essentially mean that there is only temporary price impact; this case is not of interest here. A question that remains is what happens when $\alpha = 1$? To make an educated guess, one should see what happens to (17):

the coefficient $d_{ys,2}$ goes to zero. In fact, it is easy to verify that $\alpha = 1$ means that all model parameters featuring the subscript s will go to zero and what remains is summarized in the final model feature.

MODEL FEATURE 3: *Linear permanent price impact model can be recovered by setting* $\alpha = 1$.

3.2 Dynamic Equilibrium

Suppose the current period is t. Using the approach put forth in Section 3.1, one can verify that $a_{x,t}$ from the first order condition of $\mathcal{T}$ satisfies the following cubic equation:

$$\sum_{k=0,1,2,3} f_{k,t} a_{x,t}^k = 0, \tag{18}$$

where $f_{k,t} \in \mathbb{R}$ are (reasonably complicated) constants obtained again from the first order condition. These constants—whose numerical values change moving from one period to another—depend only on $\mathbf{c}_t$—obtained from the previous round via value function recursions—and ρ_{t-1}.

Having solved for $a_{x,t}$, one can solve ρ_{t-1} using (16). After this one proceeds to successively solving three pairs of linear equations in $(a_{y,t}, b_{y,t})$, $(a_{\mu,t}, b_{\mu,t})$ and $(a_{s,t}, b_{s,t})$, also obtained from the first order conditions of $\mathcal{A}$ and $\mathcal{T}$. Finally, to check the feasibility of the solutions obtained, one must verify that the second order conditions:

$$D_t^{\mathcal{T}} \triangleq 2\Big(c_{xx,t} + \gamma c_{xs,t} + \gamma^2 c_{ss,t} + \theta_t (c_{x\mu,t} + \gamma c_{\mu s,t} + \theta_t c_{\mu\mu,t}) - \tau_c\Big) < 0, \tag{19}$$

$$D_t^{\mathcal{A}} \triangleq 2\Big(d_{yy,t} + \gamma d_{ys,t} + \gamma^2 d_{ss,t} - \tau_c\Big) < 0, \tag{20}$$

hold in the current period. The final step, for each $t = 2, ..., T+1$, is to update the value function coefficients backwards for period $t-1$, at which point one starts from the beginning by again solving the polynomial for $a_{x,t-1}$. Following this backward procedure, one is able to determine the set of equilibrium (if one exists) constants from $t = T+1$ to $t = 1$. As a final note, one should recall that periods $T+1$ and T are special (see, Section 3.1) due to the limited strategic interactions and hence finding the roots of the a_x-polynomial is only required for periods $t = T-1, ..., 1$.

Having traced back the above procedure until the first period, one requires a way to corroborate that the solution path found indeed constitutes an equilibrium. To this end, the following theorem states the conditions required for a given candidate equilibrium to be a PBE for the order execution game. In other words, Theorem 1 is a verification theorem.[21]

[21] Bayesian belief evolution is embedded in the state variable dynamics.

Theorem 1 (**Verification Theorem**). *The collection:*

$$\Big\langle \{a_{x,t}, a_{y,t}, a_{\mu,t}, a_{s,t}\},\ \{b_{y,t}, b_{\mu,t}, b_{s,t}\},\ \rho_t \Big\rangle, \quad t = 1, ..., T+1,$$

constitutes a PBE in pure (linear) strategies (5) and (6) of the order execution game if:

(I) $a_{x,t}$ *solves (18), for* $t = 1, ..., T-1$.

(II) The parameters pairs $(a_{y,t}, b_{y,t})$, $(a_{\mu,t}, b_{\mu,t})$ *and* $(a_{s,t}, b_{s,t})$, *for* $t = 1, ..., T-1$, *are obtained as the unique solutions to the linear systems given in (a.2).*

(III) The second order conditions (19) and (20) are satisfied.

(IV) The scaled variance satisfies:

$$\rho_t^2 = \frac{(1 + a_{x,t})^2}{\left(\rho_{t-1}^{-2} + a_{x,t}^2\right)}.$$

The proof of Theorem 1 is provided in Appendix A.

The limitation of Theorem 1 is that while it can be used to recognize whether a given candidate is a PBE equilibrium solution, it does not itself guarantee that, for a given set of initial values, an equilibrium actually exists. Moreover, Theorem 1 remains silent about equilibrium uniqueness. This issue deserves some attention.

Facing a similar situation, Kyle et al. (2011) analyze a quintic polynomial to establish equilibrium existence and uniqueness in a strategic trading model, and one might wonder why their approach could not be applied here also. The reason is the dynamic nature of the present model as compared to the static model developed in Kyle et al. (2011). To understand the difference, start by fixing T while supposing that the initial values $(x_0, y_0, \mu_0, \sigma_0, \sigma_\epsilon, \gamma, \alpha, \theta_c) \in \mathcal{I} \subset \mathbb{R}^8$, where $\mathcal{I}$ is a compact set, are not fixed, and note that now the polynomial (18), at a given period t, is in fact a *univariate interval polynomial*:[22]

$$[f](a_{x,t}) \triangleq \left\{ \sum_{k=0,1,2,3} f_{k,t} a_{x,t}^k \ :\ f_{k,t} \in I_{k,t} \subset \mathbb{R} \right\}, \tag{21}$$

where the constants $(f_{k,t})$ and the corresponding intervals $(I_{k,t})$ are dependent on the initial values and, for example, constants from earlier rounds.

The question then boils down to whether or not this interval polynomial, for $t = 1, ..., T-1$, has a unique solution or many solutions on an appropriate interval $I_a \subset \mathbb{R}$.

---

[22] Essentially, an interval polynomial is a polynomial whose coefficients are considered to be intervals instead of constants. The coefficients of (21) vary with ρ_t and $\mathcal{T}$'s value function coefficients and hence the polynomial coefficients effectively take values from intervals.

Existence of a unique solution would imply existence of a unique solution to the order execution game, or more precisely, to the restriction of the order execution game to the initial value set $\mathcal{I}$. Similarly, multiple solutions or no solution would have straightforward implications. The reason for why looking at the a_x-polynomial is key, is that through a_x all other equilibrium parameter values can be determined in a unique way. Appendix C provides a further illustration of the proposed approach in an example.

Nevertheless, a detailed analysis of (21), while certainly worth looking at, is beyond the scope of this paper.[23] Instead, equilibria in the next section are solved using numerical methods without a theoretical existence proof. This can be justified as follows (cf. Judd 1998): if an equilibrium does not exist, the numerical methods will with all likelihood fail. Otherwise, proceeding to look at numerical solutions is in the current setting exactly what one would do even if one could find a formal existence proof.

4 NUMERICAL RESULTS

4.1 Values for Exogenous Parameters

Prior to moving to the actual numerical analysis, it is helpful to discuss briefly the values chosen for the exogenous parameters in the model. Following Moallemi et al. (2012), three regimes describing the trading volumes in the market are designated:[24]

Low relative volume regime (LRVR):	$\rho_0 = 1$,
Moderate relative volume regime (MRVR):	$\rho_0 = 10$,
High relative volume regime (HRVR):	$\rho_0 = 100$.

Regarding naming conventions, it should be noted that, for example, a MRVR market with permanent (transient) price impact is sometimes referred to as the MRVR-perm. (MRVR-trans.) market. Similar naming convention is utilized for the LRVR and HRVR markets.

Numerical illustrations below mainly involve liquidation problems, i.e., $x_0 > 0$.[25] The initial position of $\mathcal{A}$ is assumed to be zero, i.e., $y_0 = 0$. This choice is made for convenience and for the sake of uncluttered presentation. Finally, the effect of

[23] Example applications of interval methods are given by Stradi and Haven (2005) and Stradi-Granados and Haven (2010).

[24] Changes in ρ_0 are obtained by adjusting σ_0 while holding σ_e unchanged.

[25] The case with $x_0 < 0$ can be handled analogously.

transaction fees is studied in Section 4.4. Until then, for simplicity and comparability, it is assumed that trades are free of transaction fees, i.e., $\theta_c = 0$.

4.2 Liquidation without Adversary

This section discusses the liquidation problem without an adversary, i.e., without $\mathcal{A}$. Since most existing optimal execution models are formulated in a single trader setting without strategic interactions, the results in this section enable one to compare optimal strategies with earlier models.

4.2.1 Deterministic and Adaptive Components

As noted earlier, the optimal execution strategy—when both $\mathcal{T}$ and $\mathcal{A}$ are present—features both adaptive and deterministic components. This contrasts prominent single trader models such as Bertsimas and Lo (1998), Almgren and Chriss (2001), and Obizhaeva and Wang (2013). One might thus expect that in the special case of solo $\mathcal{T}$ the optimal execution strategy would reduce to a deterministic equal-weight (EQ) strategy (cf. Moallemi et al. 2012).[26] Interestingly, this would be the case when $\alpha = 1$, but with transient price impact the optimal strategy will feature an adaptive component even without $\mathcal{A}$.

To see this, note that, without $\mathcal{A}$, $\mathcal{T}$'s linear strategy takes the form:

$$u_t = a_{x,t}x_{t-1} + a_{s,t}s_{t-1},$$

with simplified s-dynamics:

$$s_t = \alpha s_{t-1} + \gamma(u_t + e_t).$$

Hence, one obtains a forward recursive formula:

$$\begin{aligned}
u_1 &= a_{x,1}x_0, \\
u_2 &= a_{x,2}(1 + a_{x,1})x_0 + a_{s,2}\gamma(u_1 + e_1), \\
&\vdots \\
u_t &= a_{x,t}\Big(x_0 + u_1 + \cdots + u_{t-1}\Big) + \gamma a_{s,t}\Big(\alpha^{t-2}(u_1 + e_1) + \cdots \\
&\qquad + \alpha^1(u_{t-2} + e_{t-2}) + \alpha^0(u_{t-1} + e_{t-1})\Big),
\end{aligned}$$

whence it is easy to see that, in addition to the deterministic component, which is specified as the coefficient in front of x_0, there is an adaptive component which makes

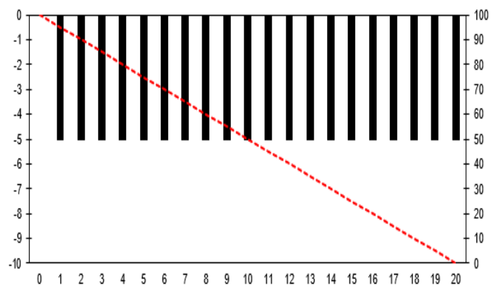
[26] The equal-weight (liquidation) strategy, for $t = 1, \ldots, T$, is characterized by:

$$u_t^{EQ} = \frac{-1}{T - t + 1}x_{t-1}.$$

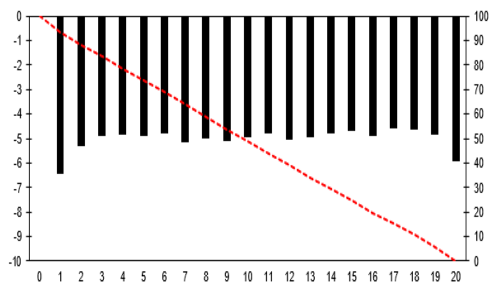
use of the noise demand at each $t = 1, \ldots, T-1$. The inclusion of adaptive components stems from the state-variable approach. More specifically, adaptive components arise since for $\alpha \in (0,1)$, the coefficient $a_{s,t}$ is nonzero, and $\mathcal{T}$ adjusts her trading program to take into account the weakening impact from past order flow shocks.

Figure 1 shows examples of execution programs of solo $\mathcal{T}$ in the MRV-regime. From the figure one can, for example, see how changes in the sizes of trades start to twist the evolution of x_t (depicted by the red line) from a straight line (Figure 1a) towards a mild S-shape (Figure 1c). Moreover, an eagle-eyed reader might notice some "dissymmetries" in Figures 1b and 1c. These stem from the fact that the trading sequences shown feature adaptive components as opposed to the fully deterministic trading sequence shown in 1a.

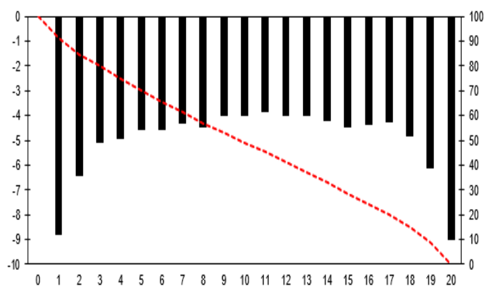
Figure 1: This figure shows the solo $\mathcal{T}$ strategy under permanent ($\alpha = 1$) and transient price impact ($\alpha = e^{-1}$, $2e^{-1}$). The first (left) vertical axis is trade size, the horizontal axis is time, and the second (right) vertical axis is the remaining asset position. The red line depicts the evolution of x_t.



(a) Permanent price impact, MRVR



(b) Transient price impact $\alpha = e^{-1}$, MRVR

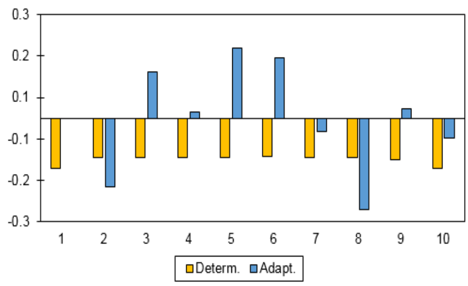


(c) Transient price impact $\alpha = 2e^{-1}$, MRVR

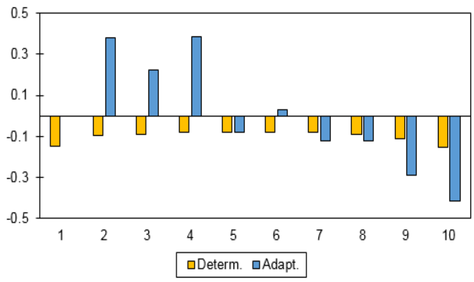
The impact of adaptive trades is most visible in Figure 1c, where the trading pattern has noticeably more distinct features compared to the other two examples. It is good to note also that the adaptive components vanish at the boundaries, i.e., when price impact becomes almost temporary (α approaches 0) or when price impact becomes purely permanent ($\alpha = 1$). This is a standard result for the permanent price impact case. With almost temporary price impact, $\mathcal{T}$ loses the ability to utilize the information in past order flows and a deterministic strategy is again optimal.

To further study the role of adaptive trading under transient price impact, Figure 2 separates the deterministic and adaptive components under different market regimes and α-levels. Looking at the figure it is obvious that, for example, the discernible pattern in Figure 1c derives from deterministic components of $\mathcal{T}$'s strategy.

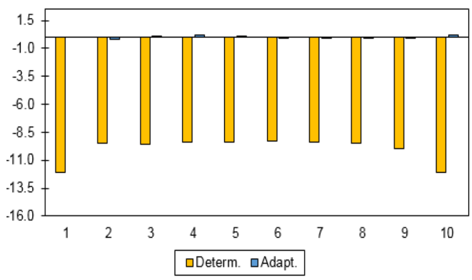
Figure 2: This figure demonstrates explicitly the deterministic and adaptive components of the solo $\mathcal{T}$'s equilibrium strategy under different market conditions.



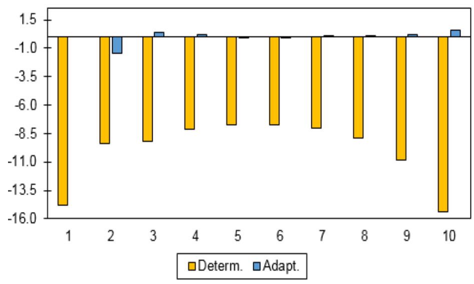
(a) LRVR, $\alpha = e^{-1}$, $x_0 = 1$



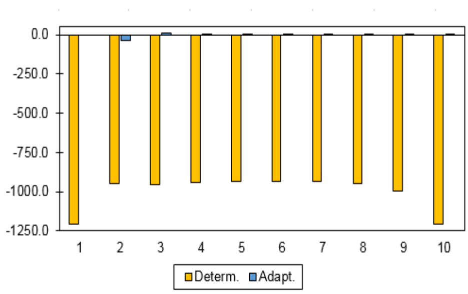
(b) LRVR, $\alpha = 2e^{-1}$, $x_0 = 1$



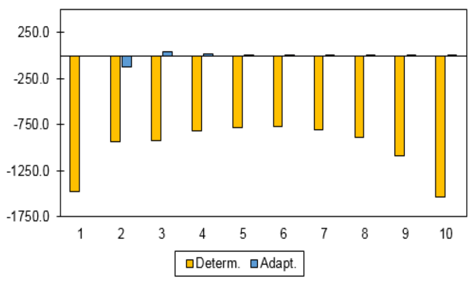
(c) MRVR, $\alpha = e^{-1}$, $x_0 = 100$



(d) MRVR, $\alpha = 2e^{-1}$, $x_0 = 100$



(e) HRVR, $\alpha = e^{-1}$, $x_0 = 100^2$



(f) HRVR, $\alpha = 2e^{-1}$, $x_0 = 100^2$

It is observed from Figure 2 that the market regime and the size $\mathcal{T}$'s trading target are crucial in determining how much adaptive trading occurs in equilibrium. Evidently, when $\mathcal{T}$'s trading accounts for most of the trading volume, adaptive components are small with respect to the deterministic components. It should also be noted that as $\mathcal{T}$ faces no adversary traders, motives to utilize adaptive trading are diminished to a single objective: taking advantage of random profit-making opportunities. These opportunities are understandably scarce when outside trading volume is low. Furthermore, for example, Choi et al. (2019) document that adaptive components can be relatively small even in a setting featuring strategic interactions. The question of adaptive versus deterministic components is revisited in Section 4.3 when $\mathcal{T}$ must design an optimal trading strategy taking into account the presence of (potentially adversary) player $\mathcal{A}$.

4.2.2 Comparison to Other Transient Price Impact Models

The purpose of this section is to provide a quick comparison of the present model to some earlier transient price impact models. This comparison helps one to better position the current paper with respect to existing models. As mentioned before, the comparison is done using the solo $\mathcal{T}$ version—more specifically, using the deterministic components of the solo $\mathcal{T}$ strategy—as the earlier models considered do not feature adaptive trading and because adaptive components in the present model are small for all but the LRVR market. In essence, the restriction to deterministic strategies simply means that any potential adaptive components featured in the strategies are omitted. The trades obtained in this fashion are essentially *expected trades* in the context of the current model.

Figure 3 depicts the optimal trading sequence under four different price impact specifications in the MRVR market. As seen from the figure, all four optimal sequences have highly similar features.

Figure 3a illustrates the optimal strategy from Obizhaeva and Wang (2013), featuring two larger trades at the beginning and end as well as constant trades in between these two trades. Figure 3b shows a variant of the exponential resilience, namely the Gaussian resilience, with:

$$G(t) = e^{-t^2},$$

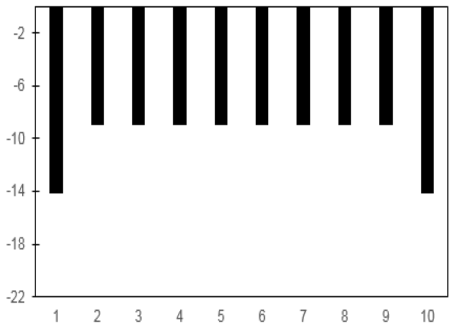
while Figure 3c features a power law resilience function:

$$G(t) = (1+t)^{-k} \text{ with } k = e^{-1}.$$

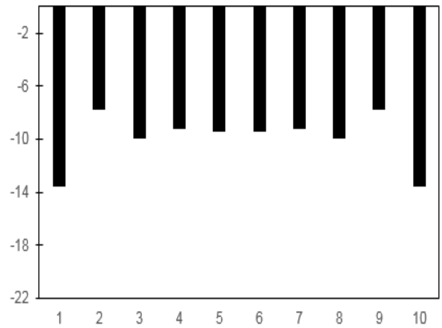
Finally, Figure 3d represents the (deterministic) trading sequence under the present model with α resilience, where, for comparability, $\alpha = e^{-1}$.

It is quickly observed that α resilience shares the shape of the power law trading sequence but with individual trade sizes closer to the exponential and Gaussian resilience strategies. Overall, one may conclude that the present model is able to capture the same features as the more explicit Obizhaeva and Wang (2013) model and its various extensions. Moreover, α resilience, as depicted above, facilitates a

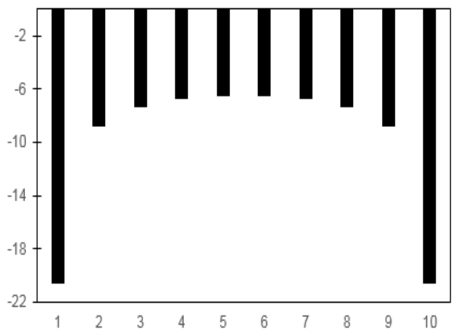
Figure 3: This figure presents optimal trading sequences in the MRVR market under different price impact specifications. Parameter values: $T = 10$, $x_0 = 100$, $\alpha = e^{-1}$, $\gamma = 1$, $\rho_0 = 10$, and finally the power law exponent $k = e^{-1}$.



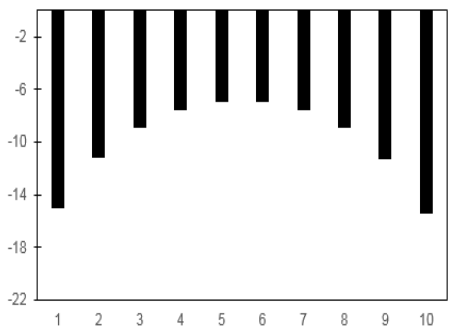
(a) Exponential resilience



(b) Gaussian resilience



(c) Power law resilience



(d) α resilience

clear-cut state-space representation for the model which is essential from the point of view of efficient (economic) analysis and interpretation. Therefore, the rest of the paper focuses exclusively on the α resilience case.

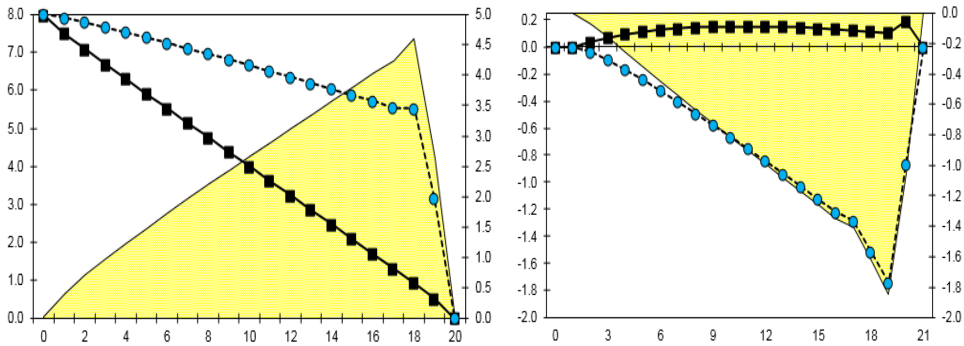
4.3 Liquidation Facing Adversary

Having looked at the solo $\mathcal{T}$ case, it is time to consider the model version with strategic interactions. To be more specific, this section provides a comparison of the equilibrium trading strategies under permanent and transient price impact when both $\mathcal{T}$ and $\mathcal{A}$ are assumed to be present in the market.

4.3.1 Evolution of Traders' Positions

Looking at how the traders' average positions evolve over several simulated runs reveals the marked change caused by the move from permanent to transient price impact. This is highlighted in Figure 4.

Figure 4: This figure portrays the average evolution of x_t and y_t when $x_0 = 8$, $\mu_0 = 0$, and $\sigma_0 = 10$. The line with the blue circles depicts the average x_t (left) or y_t (right) in the permanent price impact ($\alpha = 1$) market while the line with the black squares depicts he average x_t (left) or y_t (right) in the transient price impact ($\alpha = e^{-1}$) market. The yellow area represents the difference $z_t^{\text{perm.}} - z_t^{\text{trans.}}$, where $z_t = x_t, y_t$ and $t = 1, ..., 21$.



(a) Average evolution of x_t (b) Average evolution of y_t

As shown in Figure 4 moving the parameter α from 1 to e^{-1} turns around $\mathcal{A}$'s trading approach almost completely. When price impact is permanent $\mathcal{A}$'s strategy is, in a nutshell, to start trading in the same direction as $\mathcal{T}$ (Figure 4b). Since x_0 is assumed to be positive and $y_0 = 0$, this means that $\mathcal{A}$ accumulates a short position, expecting the price to fall further in the future due to the permanent nature of price impact. Therefore, $\mathcal{A}$'s strategy relies on the view that the short position can eventually be closed on the cheap.[27] In this case, $\mathcal{T}$'s strategy closely resembles the minimum revelation (MR) strategy (cf. Moallemi et al. 2012).[28]

When price impact is transient, however, similar logic does not apply, or—at least—is not optimal. In fact, to maximize profits under transient price impact, $\mathcal{A}$ instead provides liquidity to $\mathcal{T}$ while at the same time acquiring information about x_t. One should also note that $\mathcal{A}$ trades noticeably more conservatively under transient price impact as compared to the permanent price impact case (Figure 4b). As a result $\mathcal{T}$ liquidates at a faster pace (Figure 4a), and consequently $\mathcal{A}$ seeks make profits

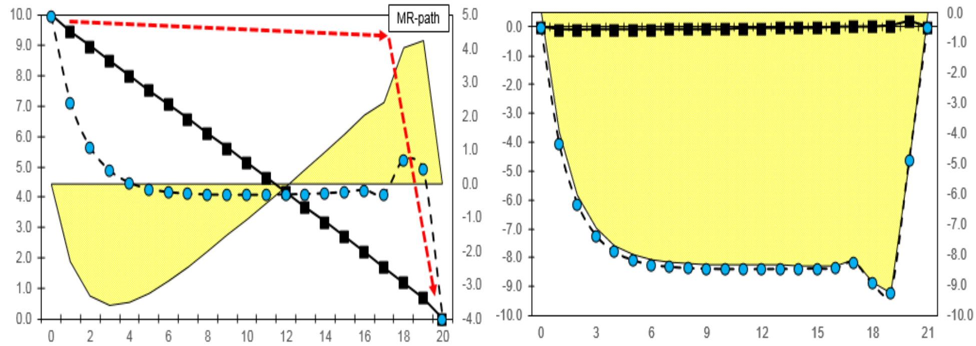
27 Naturally, from the perspective of $\mathcal{A}$ at $t = 0$, $\tilde{x}_0$ is a Gaussian random variable with mean μ_0 and variance σ_0^2.

28 The MR-strategy is obtained by setting $u_t = 0$, for $t = 1, ..., T-2$, $u_{T-1} = -0.5x_{T-2}$, and $u_T = -x_{T-1}$. The minimum revelation name stems from the fact that by using this strategy $\mathcal{T}$ ensures that $\mathcal{A}$ cannot obtain any useful information about $\mathcal{T}$'s position before trading ceases. In a sense, one can interpret the choice of submitting nonzero trades only at the last two rounds as an optimal trading horizon choice for $\mathcal{T}$ under certain market conditions; for additional discussion the reader is referred to Easley et al. (2015).

during the intermediary trading-phase by acting as a liquidity provider. In the final trading-phase, both traders share the common goal of closing their positions.

Evolution of average positions can also be illustrated by examining, instead of a fixed x_0, what happens when x_0 is drawn randomly from $N(\mu_0, \sigma_0^2)$. This is exemplified in Figure 5.

Figure 5: This figure portrays the average evolution of x_t and y_t when $\tilde{x}_0 \sim N(\mu_0, \sigma_0^2)$, $\mu_0 = 10$, and $\sigma_0 = 10$. The line with the blue circles depicts the average x_t (left) or y_t (right) in the permanent price impact ($\alpha = 1$) market while the line with the black squares depicts he average x_t (left) or y_t (right) in the transient price impact ($\alpha = e^{-1}$) market. The yellow area represents the difference $z_t^{\text{perm.}} - z_t^{\text{trans.}}$, where $z_t = x_t, y_t$ and $t = 1, ..., 21$. The red dashed line illustrates the minimum revelation path.



(a) Average evolution of x_t

(b) Average evolution of y_t

Distinctions between the permanent and transient price impact case are again clearly visible and it is verified that the average evolution of x_t in the MRVR-trans. market with strategic interactions is similar to the position evolution of the equal-weight strategy. As above, the intuition is that with transient price impact the ability of $\mathcal{A}$ to profitably front-run $\mathcal{T}$ is decreased and hence $\mathcal{T}$ can focus more on minimizing the impact of individual trades. In conclusion, the observation from Figure 4 carries over even when x_0 is drawn randomly and $\mu_0 > 0$.

The results for the MRVR-perm. market in Figure 5, however, seem quite different. First, it should be noted that in this case, $\mathcal{A}$ knows that $\mu_0 = 10$ which explains the more aggressive short position during the first periods (Figure 5b). Second, the evolution of x_t under permanent price impact (Figure 5a) illustrates the various motivations affecting $\mathcal{T}$'s trading. Initially, when the quality of $\mathcal{A}$'s information is low, it is optimal to liquidate of decent chunk of the initial position. This is again related to the fact that $\mathcal{A}$'s prior features a positive μ_0. During the intermediate periods camouflage motives dominate and in the final periods $\mathcal{T}$ must liquidate the remaining position.

The "buy jump" (Figure 5a) just prior to the last two trades is driven by a cost minimization objective. $\mathcal{T}$ knows that in the final periods $\mathcal{A}$ is forced to start buying to close his short position. This pushes the price of the risky asset up and hence $\mathcal{T}$ is able to, on average, sell her remaining shares at a higher price. It makes sense to have a bit more—just enough so that the equilibrium does not break down—to sell to take advantage of this.

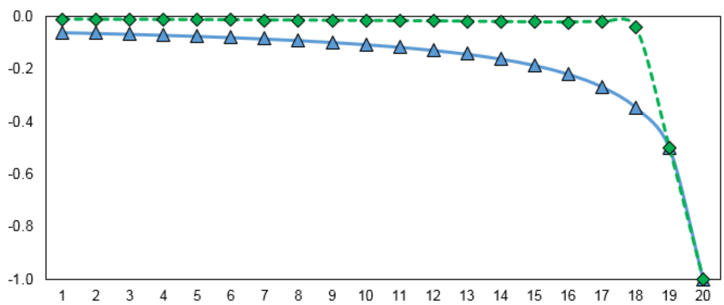
In sum, while $\mathcal{T}$'s strategy is very stable in the MRVR-trans. market, in the MRVR-perm. market the implications of having $\mu_0 > 0$ are more pronounced. Comparing the x_t-path in the MRVR-perm. market (Figure 5a) to the MR-path (dashed red line) highlights this difference.[29]

4.3.2 Trading Coefficients

At this point it is reasonable to wonder what (in a mechanical sense) causes the differences observed in Figure 4a and Figure 4b. Due to the fact that the equilibrium strategies considered have a similar functional form in both permanent and transient price impact cases, all changes in the trading sequences are transmitted through changes in the equilibrium trading parameters ($a_{x,t}$, $a_{y,t}$, $a_{\mu,t}$, $a_{s,t}$) and ($b_{y,t}$, $b_{\mu,t}$, $b_{s,t}$). Thus, it is worthwhile to examine how these parameters change with the price impact specification.

Figure 6 initiates this examination by focusing on $a_{x,t}$ in the stereotypical MVRV market.[30]

Figure 6: This figure depicts the evolution of the trading parameter $a_{x,t}$ in the MRVR market under permanent (green) and transient (blue) price impact ($\alpha = e^{-1}$).



The green line in Figure 6 shows that, in the MRVR market with permanent price impact, $\mathcal{T}$ avoids trading directly towards the target until the end of trading horizon. This is again akin to the minimum revelation strategy. At the same time, it is observed that the same does not hold when price impact is transient. Instead, with $\alpha = e^{-1}$, one sees that $\mathcal{T}$ proceeds with the liquidation program, using an—in absolute terms—increasing trading intensity towards the terminal target position. Evidently, this increasing intensity merges with the "almost MR-strategy" from the

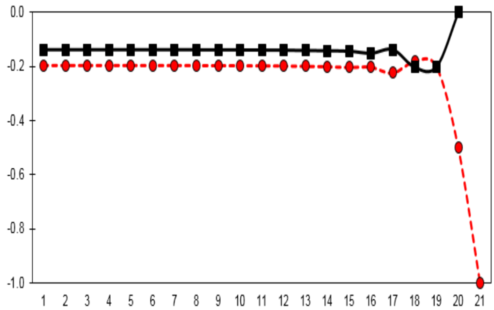
[29] In general, the MR-strategy is mostly utilized in the HRVR-perm. market, but the "almost MR-strategy" is also possible in the MRVR-perm. market as shown in Figure 4a.

[30] Stereotypical in the sense that the results are similar in the LRVR and HRVR cases.

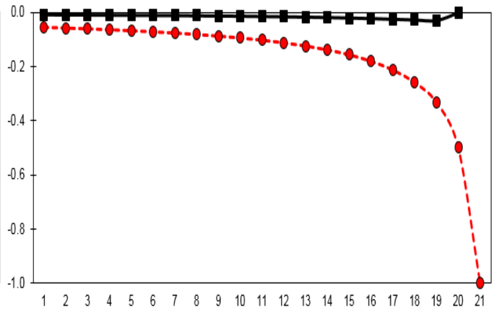
permanent price impact market only during the final two periods in $\mathcal{T}$'s execution horizon. Clearly, the increased intensity explains in part the faster and more straightforward execution pace discovered earlier.

Next, Figure 7 focuses on $a_{y,t}$ and $b_{y,t}$.

Figure 7: This figure shows the evolution $a_{y,t}$ (black) and $b_{y,t}$ (red) in the MRVR market under permanent and transient price impact.



(a) Permanent price impact



(b) Transient price impact ($\alpha = e^{-1}$)

First thing to note is that $\mathcal{T}$'s intensity towards y_t (i.e., $a_{y,t}$) is negative in the permanent price impact case, meaning that, for example, when $\mathcal{A}$ starts to accumulate a short position, $a_{y,t} < 0$ slows down the liquidation program of $\mathcal{T}$. Further, $a_{y,t}$ remains nearly constant throughout the execution horizon of $\mathcal{T}$ under both permanent and transient price impact, exhibiting notable variation only at the end. This variation stems from $\mathcal{T}$'s need to close down her execution program and the goal to benefit from the fact that the same need holds true for $\mathcal{A}$ also.

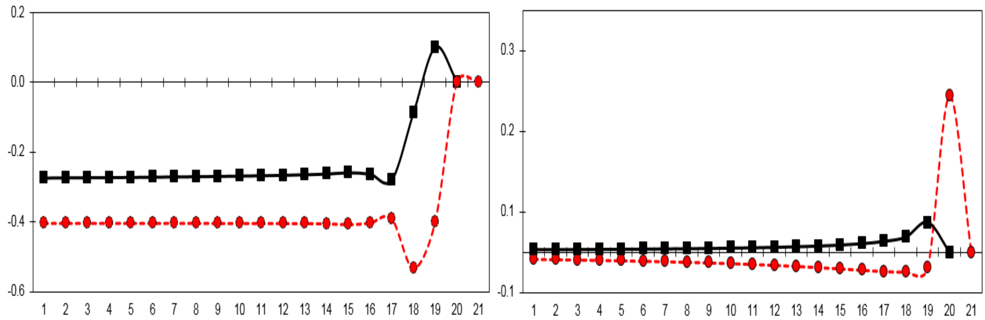
Interestingly, as observed from Figure 7b, in the transient price impact case $a_{y,t}$ is close to zero instead of clearly negative. As $\mathcal{A}$ does not engage in a straightforward adversary trading, $\mathcal{T}$ has less need to slow down her execution speed for protection purposes. $\mathcal{A}$'s coefficient $b_{y,t}$ remains negative, yet distinctly smaller in absolute terms. Negativity of $b_{y,t}$ reflects the fact $\mathcal{A}$ also aims the end the day with zero inventory. Hence, it is not surprising that the path $b_{y,t}$ resembles closely the path of $a_{x,t}$.

Next to be discussed are $a_{\mu,t}$ and $b_{\mu,t}$. For this purpose, Figure 8 contrasts the permanent and transient price impact equilibrium parameters in the MRVR market. At this point it is good to recap how $\mathcal{T}$'s MR-strategy under permanent price impact works in terms of the trading coefficients. As seen from Figures 7 and 8, the coefficients $a_{y,t}$ and $a_{\mu,t}$ differ from zero in the MRVR-perm. market. Their impact on the evolution of x_t is, however, negligible due to the fact that when $\mathcal{T}$ trades slowly the state variables y_t and μ_t remain close to zero.[31] Thus, the evolution of

[31] See, Figure 10.

x_t is dominantly driven by $a_{x,t}$ which is also close to zero during all but the last two rounds (see, Figure 6). This explains the evolution of x_t under permanent price impact demonstrated in Figure 4.

Figure 8: This figure shows the evolution $a_{\mu,t}$ (black) and $b_{\mu,t}$ (red) in the MRVR market under permanent and transient price impact.



(a) Permanent price impact

(b) Transient price impact ($\alpha = e^{-1}$)

As seen from Figure 8 the dynamics of $a_{\mu,t}$ and $b_{\mu,t}$ are clearly affected by the change in the form of price impact. Starting with Figure 8a it is observed that both $\mathcal{T}$ and $\mathcal{A}$ have, until the very final periods, a negative weight (trading intensity) on the conditional expected value of $\mathcal{T}$'s position. In particular, initially ($y_0 = s_0 = 0$) $\mathcal{A}$ trades—provided that μ_t has the correct sign—against $\mathcal{T}$'s expected position. Thus, as in Figure 4b, $\mathcal{A}$ trades to same direction as $\mathcal{T}$ and thus acquires a short position. This makes sense in the permanent price impact case when $\mathcal{A}$ believes he is facing a liquidating $\mathcal{T}$. In this case $\mathcal{A}$ wishes to push price as low as possible so that he can close the short position cheaply during the final trading periods.

What about Figure 8b then? Not only is the intensity $b_{\mu,t}$ much smaller in absolute terms, but $a_{\mu,t}$ also takes positive instead of negative values. This change reflects the fact that with transient price impact it is not optimal for $\mathcal{A}$ to (slowly) build an opposing position to $\mathcal{T}$, relying on profitable trading opportunities and the ability to close down the position at later rounds. Instead $\mathcal{A}$ is forced, consistently with Figure 4b, to glean potential profits from liquidity provision.

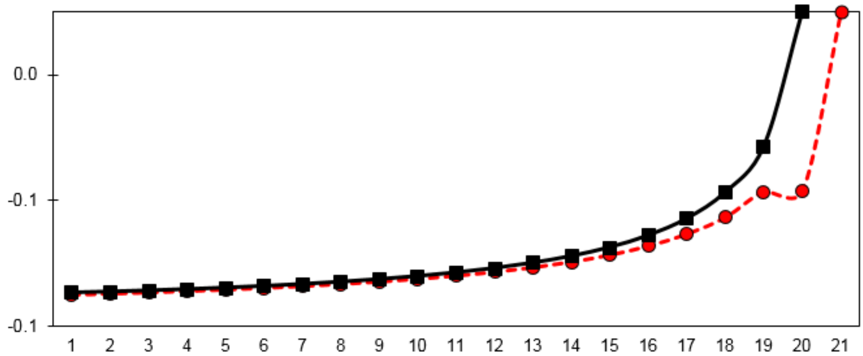
$\mathcal{T}$'s optimal correction to change in $\mathcal{A}$'s trading is to accelerate the liquidation process with more trading towards the goal. As a result, the intensity on μ_t is adjusted to a small positive value and the trading intensities (in absolute terms) $a_{x,t}$ and $a_{s,t}$ are adjusted upwards accordingly.[32] To exemplify, Figure 9 shows the paths for $a_{s,t}$ and $b_{s,t}$ for two different levels of α in the MRVR market.

In interpreting the implications of Figure 9 on the evolution of y_t, it is first good

32 Note that when $\alpha = 1$, $a_{s,t} = b_{s,t} = 0$, for all $t = 1, ..., T+1$.

to note that $b_{s,t} < 0$ implies that—when s_t's are negative, which is typical when $\mathcal{T}$ is a liquidating trader—$b_{s,t}$ drives y_t towards a long position instead of a short one. When this effect dominates one obtains a result such as the one in Figure 4b. Evidently, whether or not this dominance occurs depends also on the other trading parameters. The intensity $b_{y,t} < 0$ slows down this accumulation effect while the significance of $b_{\mu,t}$ depends, for the most part, on μ_0. All in all, the end result tends to be conservative trading by $\mathcal{A}$ as illustrated in Figure 5b.

Figure 9: This figure shows the evolution $a_{s,t}$ (black) and $b_{s,t}$ (red) in the MRVR market under permanent and transient price impact.



The conservative strategy makes sense from the point of view of expected price evolution. Indeed, if instead $\mathcal{A}$ would have, for instance, acquired a substantial short position, he would face the risk of rising prices stemming from both the attenuating impact of past order flows and exogenous demand from other market participants. This risk is mitigated by choosing a more conservative strategy, which consequently—in the case of transient price impact—also turns out to have higher expected profits.

In essence, under transient price impact, $\mathcal{A}$ seeks to maintain his total asset position near neutral in order to be able to flexibly take advantage of any emerging profitable trading opportunities. A crucial aspect of this approach is that $\mathcal{A}$'s position cannot swell too much, otherwise the immediate price impact from reversing the position could wipe out all potential profits. This aspect is clearly visible in Figures 4b and 5b in that $\mathcal{A}$'s position remains (on average) closer to zero in the transient price impact case.

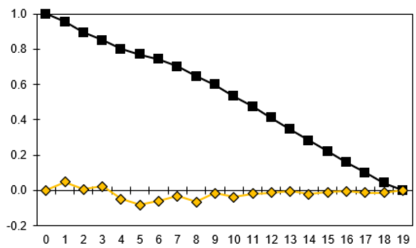
4.3.3 Belief Evolution

Looking deeper at the reasons and rationale as to why $\mathcal{A}$ transforms from a predator to a conservative trader, one key reason can be found from the evolution of $\mathcal{A}$'s beliefs. These beliefs inherently carry uncertainty, which diminishes as more information is learned. This is also very much the case in everyday financial markets.

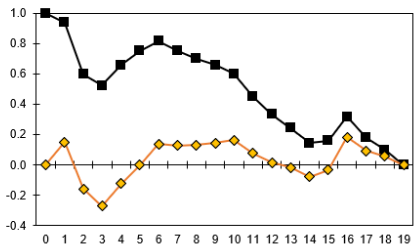
To exemplify this point, a short digression to tie the discussion at hand to real world phenomena is in order here. For this purpose, consider the recent empirical analysis by Van Kervel and Menkveld (2019). Looking at high frequency trading around (large) institutional orders, the authors observe, among other things, that: (1) institutional investors trade judiciously on their information (2) *it takes a significant amount of time* for high frequency traders to detect and target these aforementioned institutional orders. [33]

The latter observation can be illustrated in the context of the present framework. For this purpose, Figure 10 illustrates the belief evolution of $\mathcal{A}$ over a number of trading rounds.

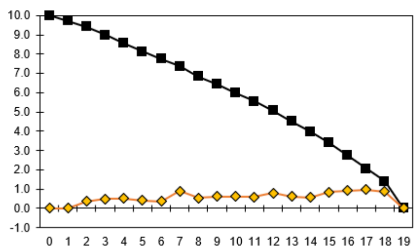
Figure 10: This figure illustrates path realizations of the belief evolution (orange) and the evolution of x_t (black) under different market conditions.



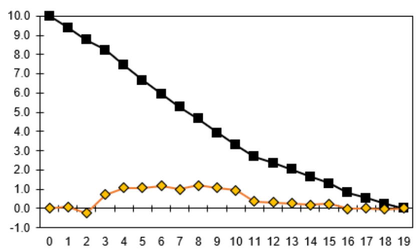
(a) LRVR-perm.



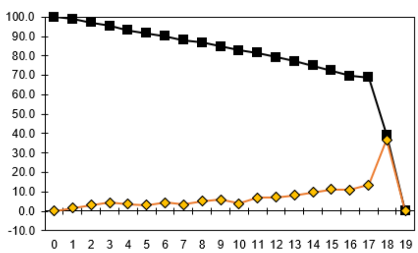
(b) LRVR-trans. ($\alpha = e^{-1}$)



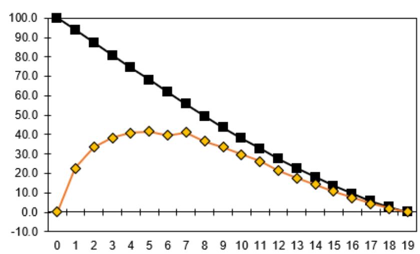
(c) MRVR-perm.



(d) MRVR-trans. ($\alpha = e^{-1}$)



(e) HRVR-perm.



(f) HRVR-trans. ($\alpha = e^{-1}$)

[33] See also Korajczyk and Murphy (2018) and Hirschey (2018). Moreover, it is worth pointing out that assuming $\mathcal{A}$ is unsure about both whether $\mathcal{T}$ is present in the market and the realization of $\tilde{x}_0$ leads to a considerably more involved model. By interpreting the realization $x_0 = 0$ as $\mathcal{T}$ not being present in the market in the current model, it is noted that the above analysis does not specialize to this probability zero event.

From Figure 10 it can be seen how the beliefs of $\mathcal{A}$ converge to the actual position of $\mathcal{T}$ only during the final trading periods. Trading on noisy beliefs thus involves considerable risk, and—taking the point of view of a real trader—one could easily see some constraints imposed on $\mathcal{A}$'s trading activities. For example, constraints preventing $\mathcal{A}$ from trading when his information quality is questionable. Then, as observed by Van Kervel and Menkveld (2019), it would take an extended period of time for $\mathcal{A}$ to react to the trading activities of $\mathcal{T}$.[34] This is especially relevant when price impact is transient and future prices are less dependent on current trades.

Indeed, with transient price impact, the shorting strategy for $\mathcal{A}$ is clearly less efficient as $\mathcal{A}$ must start to acquire the short position during the very first rounds with low quality information, facing the problem that impact from past trades has already mostly faded when $\mathcal{A}$'s information quality has finally improved. Hence, relying on the looming big payoff during the final trading rounds is largely wishful thinking and it is optimal for $\mathcal{A}$ to instead opt for a conservative trading strategy, which is much less dependent on the belief evolution process.

4.3.4 Strategic Interactions and Size of Adaptive Component

The purpose of this section is to quickly revisit the question of deterministic versus adaptive component in the trading strategy of $\mathcal{T}$.[35] It is generally believed (cf. Schied et al. 2010 and Aldridge 2013) that different types—e.g., adaptive and static (deterministic)—of optimal execution programs perform well in different market environments. This raises the question of how important is adaptive trading when traders face strategic interactions under transient price impact? Table 1 seeks to provide insights to the matter.

Table 1: This table reports the adaptive-deterministic ratio $\mathcal{R}_{u,t} \triangleq u_t^{\text{adapt}}/u_t^{\text{determ}}$, for $t = 1, ..., 10$, obtained from 10^6 Monte Carlo runs under three different relative volume regimes and for both linear and transient price impact ($\alpha = e^{-1}$) markets.

Market	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
LRVR-perm.	0.00	0.87	0.72	0.58	0.52	0.53	0.62	0.86	1.16	1.56
LRVR-trans.	0.00	4.49	4.51	4.49	4.45	4.35	4.18	3.96	3.63	4.04
MRVR-perm.	0.00	9.13	7.88	6.04	4.54	3.52	2.79	3.19	1.46	0.32
MRVR-trans.	0.00	0.80	0.73	0.68	0.63	0.60	0.57	0.54	0.50	0.73
HRVR-perm.	0.00	65.81	56.99	43.78	31.70	23.09	17.17	15.89	0.07	0.01
HRVR-trans.	0.00	0.12	0.11	0.10	0.09	0.08	0.07	0.07	0.06	0.17

Table 1 provides yet another clear illustration of the consequences of moving from permanent to transient price impact. First thing to note is that under permanent

[34] Section 5 discusses the case where, on top of asymmetric information about $\tilde{x}_0$, $\mathcal{A}$ is uncertain about whether $\mathcal{T}$ is actually present in the market.

[35] Examining the same question for $\mathcal{A}$ is less relevant as $\mathcal{A}$'s strategy, by its very nature, is based on adaptive learning and taking advantage of arising profitable trading opportunities.

price impact adaptive trading tends to attenuate significantly in the middle of the day, while with transient price impact the fraction of adaptive trades tends to remain rather stable throughout the day. The more stable trading under transient price impact reflects the notion that periodic adaptive trades are in a sense "cheaper" since the impact of these trades fades quickly.

A perhaps more striking finding is that adaptive trading is by and far dominant in the HRVR when price impact is permanent, while with transient price impact adaptive trading is most utilized in the LRVR market. While this stark difference may feel puzzling at first, it can be explained based on the earlier discussion. In the HRVR market with permanent price impact $\mathcal{T}$ is desperately trying to camouflage her position. This involves aggressive adaptive trading during periods when $\mathcal{A}$'s information is still relatively noisy. In fact, essentially all of $\mathcal{T}$'s trading in periods $t = 1, ..., 8$ is adaptive. Hence, as the deterministic component is small, the total trade size during these periods is small (close to zero). Small deterministic trades also explain why the adaptive-deterministic ratios are an order of magnitude larger in the HRVR-perm. market than in other market parametrizations. Conversely, during periods $t = 9$ and $t = 10$ there are essentially no adaptive trading but plenty of deterministic trading. An avid reader may notice that the reason for this is that the HRV-regime and permanent price impact forces $\mathcal{T}$ to utilize the MR-strategy, which is characterized by two large, deterministic trades in the final two periods.

With transient price impact, however, it is optimal for $\mathcal{T}$—in the MRVR-trans. and HRVR-trans. market—to trade in a more balanced manner. This is because in these markets information revelation is less important relative to optimally utilizing the transient nature of price impact via effective trade splitting. In contrast, in the LRVR-trans. market, it is optimal for $\mathcal{T}$ to engage in speculative trading and to actively mislead $\mathcal{A}$. These goals manifest in the form of large adaptive components. Key facilitators here are the fact that $\mathcal{T}$ believes that her private information is well camouflaged by the high inflow of exogenous (noise) trades and the observation that due to $\alpha < 1$, the impact of past trades fades rapidly.

4.4 Expected Execution Costs

This section illustrates expected execution costs under various market parametrizations and strategies. The impact of quadratic transaction fees on expected execution costs is also discussed.[36] Finally, to better understand how additional transaction fees affect the equilibrium outcomes, the question of adaptive-deterministic ratios in $\mathcal{T}$'s trading strategy is revisited.

[36] Empirical approaches to execution cost assessment are given in, for instance, Bessembinder (2003) and Davies and Kim (2009).

4.4.1 Expected Costs of Optimal and Suboptimal Strategies

The following proposition summarizes the expected execution costs for the different strategies introduced.[37]

Proposition 2 (Ex Ante Expected Execution Costs and Profits). *The scaled expected execution costs for $\mathcal{T}$ and scaled expected trading profits for $\mathcal{A}$ are given, respectively, by:*

$$\begin{aligned}
\sigma_0^{-2}\Pi_0^{\mathcal{T}} &= c_{xx,0} + \frac{c_{0,0}}{\rho_0^2},\\
\sigma_0^{-2}\Pi_0^{\mathcal{A}} &= \frac{d_{0,0}}{\rho_0^2},\\
\sigma_0^{-2}\Pi_{MR,0}^{\mathcal{T}} &= -\frac{\gamma}{4}\left(\alpha + 2\left(1 + \frac{\theta_c}{\gamma}\right)\right),\\
\sigma_0^{-2}\Pi_{MR,0}^{\mathcal{A}} &= 0,
\end{aligned}$$

where the value function coefficients are obtained from recursions:

$$\begin{aligned}
c_{xx,t-1} &= \gamma a_{x,t} + \left(1 + a_{x,t}\right)\left((1 + a_{x,t})c_{xx,t} + a_{x,t}\theta_t c_{x\mu,t} + \gamma a_{x,t} c_{xs,t}\right)\\
&\quad + a_{x,t}^2\left(\theta_t^2 c_{\mu\mu,t} + \gamma^2 c_{ss,t} + \gamma\theta_t c_{\mu s,t} - \theta_c\right)\\
c_{0,t-1} &= c_{0,t} + \sigma_e^2\left(\theta_t^2 c_{\mu\mu,t} + \gamma^2 c_{ss,t} + \gamma\theta_t c_{\mu s,t}\right),\\
d_{0,t-1} &= d_{0,t} + \sigma_e^2\left(1 + \rho_{t-1} a_{x,t}^2\right)\left(\theta_t^2 d_{\mu\mu,t} + \gamma^2 d_{ss,t} + \gamma\theta_t d_{\mu s,t}\right).
\end{aligned}$$

Ex ante expected costs and profits from $\Pi_{EQ,0}^{\mathcal{T}}$ and $\Pi_{EQ,0}^{\mathcal{A}}$ are obtained as special cases of $\Pi_0^{\mathcal{T}}$ and $\Pi_0^{\mathcal{A}}$.

It should also be noted that, while for the most part not explicitly visible, the potential existence of transaction fees is transmitted to the ex ante expected costs and profits via the relevant value function coefficients. Table 2 illustrates expected costs and profits numerically.

There are a few to things to note in Table 2. First, utilizing her optimal trading strategy, $\mathcal{T}$ is considerably better off in terms of expected execution costs when price impact is transient. Second, both the MR-strategy and the EQ-strategy perform better from the viewpoint of $\mathcal{T}$ under transient price impact. These results are a further elaboration on how the form of price impact changes the nature of interactions between $\mathcal{T}$ and $\mathcal{A}$. In term of expected execution costs, $\mathcal{T}$'s optimal strategy dominates both the MR- and EQ-strategies in all markets.

[37] $\Pi_{EQ,0}^{\mathcal{A}}$ captures $\mathcal{A}$'s expected profits when $\mathcal{T}$ uses the EQ-strategy and $\mathcal{A}$ trades optimally.

Table 2: This table gives expected execution costs ($\mathcal{T}$) and trading profits ($\mathcal{A}$) under three different relative volume regimes and for both linear and transient price impact markets. $\Pi_0^{\mathcal{T}}$ (resp. $\Pi_0^{\mathcal{A}}$) refers to the optimal strategy of $\mathcal{T}$ ($\mathcal{A}$), $\Pi_{MR,0}^{\mathcal{T}}$ (resp. $\Pi_{MR,0}^{\mathcal{A}}$) refers to the minimum revelation (MR) strategy, and $\Pi_{EQ,0}^{\mathcal{T}}$ (resp. $\Pi_{EQ,0}^{\mathcal{A}}$) refers to the equal-weighting (EQ) strategy. Throughout, it is assumed that $T = 10$.

Market	$\Pi_0^{\mathcal{T}}$; $\Pi_0^{\mathcal{A}}$	$\Pi_{MR,0}^{\mathcal{T}}$; $\Pi_{MR,0}^{\mathcal{A}}$	$\Pi_{EQ,0}^{\mathcal{T}}$; $\Pi_{EQ,0}^{\mathcal{A}}$
LRVR-perm.	-0.56; 0.00	-0.75 ; 0.00	-0.57 ; 0.01
LRVR-trans.	-0.06 ; 0.10	-0.59 ; 0.00	-0.14 ; 0.19
MRVR-perm.	-0.75 ; 0.02	-0.75 ; 0.00	-1.09 ; 0.25
MRVR-trans.	-0.14 ; 0.00	-0.59 ; 0.00	-0.15 ; 0.00
HRVR-perm.	-0.75 ; 0.00	-0.75 ; 0.00	-1.33 ; 0.37
HRVR-trans.	-0.14 ; 0.00	-0.59 ; 0.00	-0.14 ; 0.00

Finally, for $\mathcal{A}$, under permanent price impact and when both traders utilize their optimal strategies, the expected trading profits are highest in the MRVR market where $\mathcal{T}$ represents a lions' share of the trading volume but not quite so much that invoking the MR-strategy would be optimal. In stark contrast, under transient price impact, the best case for both traders is the LRVR market where adaptive (symbiotic) trading is most beneficial. More specifically, in the LRVR-trans. market, as noted earlier, the outside (noise) trading volume is large enough to facilitate more aggressive speculative trading.

However, the best situation for $\mathcal{A}$, among all possibilities considered, is the case where $\mathcal{T}$ utilizes the deterministic EQ-strategy. Indeed, one can observe from Table 2 that $\Pi_{EQ,0}^{\mathcal{A}} \geq \Pi_0^{\mathcal{A}}$ for all markets. This is in large part due to the fact that the deterministic strategy of $\mathcal{T}$ enables $\mathcal{A}$ to effectively predict the trading need of $\mathcal{T}$ towards the end of the trading horizon T.

4.4.2 Impact of Transaction Fees

The question posed in this section is how does imposing an across the board transaction fee on trades impact the expected execution costs and profits. This question has clear-cut implications on liquidity and market volume as increased (total) costs can be linked to decreased volume and liquidity and vice versa. Moreover, it is found in Schied and Zhang (2018) that introducing transaction fees may "stabilize" equilibrium strategies and lead to overall lower execution costs. This welfare implication is important from, for instance, a policy making point of view and hence addressed below. Table 3 displays the numerical results.

It is obvious from Table 3 that $\mathcal{T}$—utilizing her optimal strategy, adjusted to take into account the existence of transaction fees—is not better-off after the introduction of transaction fees. As a matter of fact, $\mathcal{T}$ ($\mathcal{A}$) is consistently worse-off (better-off) in the MRVR-perm. market with nonzero transaction fees. In the LRVR and HRVR market, when both traders utilize optimal strategies, the introduction of transaction fees increases expected execution costs and decreases expected profits.

Table 3: This table gives expected execution costs ($\mathcal{T}$) and trading profits ($\mathcal{A}$) under three different relative volume regimes and for both linear and transient price impact markets. $\Pi_0^{\mathcal{T}}$ (resp. $\Pi_0^{\mathcal{A}}$) refers to the optimal strategy of $\mathcal{T}$ ($\mathcal{A}$), $\Pi_{MR,0}^{\mathcal{T}}$ (resp. $\Pi_{MR,0}^{\mathcal{A}}$) refers to the minimum revelation (MR) strategy, and $\Pi_{EQ,0}^{\mathcal{T}}$ (resp. $\Pi_{EQ,0}^{\mathcal{A}}$) refers to the equal-trade (EQ) strategy. θ_c is used to denote the level of the additional transaction costs. Cases where the introduction of transaction fees has *decreased* the expected execution costs of $\mathcal{T}$ are boxed. Cases where the introduction of transaction fees has improved the expected profits of $\mathcal{A}$ are presented in **bold**. Throughout, it is assumed that $T = 10$.

Panel **A**: $\theta_c = 3^{-3}$

Market	$\Pi_0^{\mathcal{T}}$; $\Pi_0^{\mathcal{A}}$	$\Pi_{MR,0}^{\mathcal{T}}$; $\Pi_{MR,0}^{\mathcal{A}}$	$\Pi_{EQ,0}^{\mathcal{T}}$; $\Pi_{EQ,0}^{\mathcal{A}}$
LRVR-perm.	-0.57 ; 0.00	-0.77 ; 0.00	-0.58 ; 0.01
LRVR-trans.	-0.07 ; 0.09	-0.61 ; 0.00	-0.15 ; 0.19
MRVR-perm.	**-0.76** ; **0.03**	-0.77 ; 0.00	-1.05 ; 0.24
MRVR-trans.	-0.15 ; 0.00	-0.61 ; 0.00	-0.15 ; 0.00
HRVR-perm.	-0.77 ; 0.00	-0.77 ; 0.00	-1.28 ; 0.34
HRVR-trans.	-0.15 ; 0.00	-0.61 ; 0.00	-0.15 ; 0.00

Panel **B**: $\theta_c = 3^{-2}$

Market	$\Pi_0^{\mathcal{T}}$; $\Pi_0^{\mathcal{A}}$	$\Pi_{MR,0}^{\mathcal{T}}$; $\Pi_{MR,0}^{\mathcal{A}}$	$\Pi_{EQ,0}^{\mathcal{T}}$; $\Pi_{EQ,0}^{\mathcal{A}}$
LRVR-perm.	-0.57 ; 0.00	-0.81 ; 0.00	-0.58 ; 0.01
LRVR-trans.	-0.08 ; 0.09	-0.65 ; 0.00	-0.15 ; 0.18
MRVR-perm.	**-0.80** ; **0.05**	-0.81 ; 0.00	-1.00 ; 0.21
MRVR-trans.	-0.15 ; 0.00	-0.65 ; 0.00	-0.15 ; 0.00
HRVR-perm.	-0.81 ; 0.00	-0.81 ; 0.00	-1.20 ; 0.30
HRVR-trans.	-0.15 ; 0.00	-0.65 ; 0.00	-0.16 ; 0.00

Panel **C**: $\theta_c = 3^{-1}$

Market	$\Pi_0^{\mathcal{T}}$; $\Pi_0^{\mathcal{A}}$	$\Pi_{MR,0}^{\mathcal{T}}$; $\Pi_{MR,0}^{\mathcal{A}}$	$\Pi_{EQ,0}^{\mathcal{T}}$; $\Pi_{EQ,0}^{\mathcal{A}}$
LRVR-perm.	-0.59 ; 0.00	-0.92 ; 0.00	-0.59 ; 0.01
LRVR-trans.	-0.11 ; 0.08	-0.76 ; 0.00	-0.18 ; 0.15
MRVR-perm.	**-0.83** ; **0.08**	-0.92 ; 0.00	-0.90 ; 0.15
MRVR-trans.	-0.18 ; 0.00	-0.76 ; 0.00	-0.18 ; 0.00
HRVR-perm.	-0.92 ; 0.00	-0.92 ; 0.00	-1.05 ; 0.22
HRVR-trans.	-0.18 ; 0.00	-0.76 ; 0.00	-0.18 ; 0.00

However, looking at the column stating the results for the EQ-strategy, an opposite remark can be made. $\mathcal{T}$ is in fact better-off (lower expected costs) in some cases using the EQ-strategy in a market with nonzero transaction fees as compared to a market with zero transaction fees, while $\mathcal{A}$ is generally worse-off (lower expected profits). This result on quadratic transaction fees is reminiscent of the one in Schied and Zhang (2018) and highlights some important issues.

First, in Schied and Zhang (2018), where nonzero fees have a positive impact, trading strategies are deterministic. Similarly, the EQ-strategy of $\mathcal{T}$ is deterministic, leading one to conjecture that the positive impact of nonzero fees may only apply to market settings where deterministic strategies are pursued. Generally speaking, it is, on one hand, reasonable to purport that a trader utilizing a suboptimal trading strategy benefits from the extra "protection" provided by transaction fees which limit, for example, predatory trading. On the other hand, it is equally reasonable to purport that nonzero fees constrain the utilization optimal adaptive strategies and thus the positive impact turns to negative when adaptive strategies are studied. In sum, in considering how to limit, for instance, predatory trading one should take into account the sophistication of the trading strategies utilized by the (average) market participants in order to avoid making the market uniformly less attractive for all traders.

Another point of view from which the impact of transaction fees can be examined further is how these fees affect the relative proportions of adaptive and deterministic components. This point of view can be considered as an additional robustness check to corroborate the above discussion.

Table 4 exhibits the numerical results and it is quickly observed from the table, though a comparison with the results in Table 1, that introducing nonzero transaction fees do not fundamentally change the relationship between adaptive and deterministic components when price impact is transient. This indicates that $\mathcal{T}$'s optimal strategy under transaction fees remains close to the optimal strategy without fees. This is unsurprising as price impact is the main source of execution costs and given the straightforward way in which quadratic transaction fees affect the strategies of $\mathcal{T}$ and $\mathcal{A}$.

Table 4: This table reports the adaptive-deterministic ratio $\mathcal{R}_{u,t} \triangleq u_t^{\text{adapt}}/u_t^{\text{determ}}$, obtained from 10^6 Monte Carlo runs under three different relative volume regimes and for both linear and transient price impact markets. θ_c is used to denote the level of the additional transaction costs. Throughout, it is assumed that $T = 10$.

Panel **A**: $\theta_c = 3^{-3}$

Market	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
LRVR-perm.	0.00	0.81	0.67	0.53	0.48	0.51	0.61	0.81	0.88	1.44
LRVR-trans.	0.00	4.31	4.33	4.34	4.27	4.15	4.04	3.81	3.45	3.82
MRVR-perm.	0.00	8.32	7.14	5.62	4.27	3.37	2.66	3.86	1.46	0.36
MRVR-trans.	0.00	0.76	0.71	0.67	0.62	0.59	0.57	0.53	0.51	0.70
HRVR-perm.	0.00	61.37	53.82	41.66	30.75	22.64	16.60	19.31	0.08	0.01
HRVR-trans.	0.00	0.12	0.11	0.10	0.09	0.09	0.08	0.07	0.07	0.16

Panel **B**: $\theta_c = 3^{-2}$

Market	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
LRVR-perm.	0.00	0.68	0.57	0.45	0.42	0.47	0.57	0.69	0.55	1.21
LRVR-trans.	0.00	4.14	4.16	4.07	4.07	3.94	3.82	3.61	3.30	3.62
MRVR-perm.	0.00	6.87	5.98	4.74	3.70	3.02	2.38	5.17	1.47	0.45
MRVR-trans.	0.00	0.71	0.67	0.61	0.58	0.56	0.53	0.49	0.48	0.63
HRVR-perm.	0.00	53.52	47.78	37.68	28.55	21.59	15.40	25.06	0.08	0.02
HRVR-trans.	0.00	0.10	0.09	0.09	0.08	0.07	0.06	0.06	0.06	0.14

Panel **C**: $\theta_c = 3^{-1}$

Market	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
LRVR-perm.	0.00	0.46	0.37	0.29	0.30	0.37	0.44	0.42	0.19	0.80
LRVR-trans.	0.00	3.65	3.66	3.63	3.56	3.46	3.32	3.13	2.86	3.15
MRVR-perm.	0.00	4.51	4.02	3.24	2.56	1.89	2.27	1.60	0.50	0.81
MRVR-trans.	0.00	0.59	0.56	0.52	0.49	0.47	0.46	0.43	0.41	0.52
HRVR-perm.	0.00	36.97	33.89	28.05	22.52	17.18	11.58	30.76	0.11	0.02
HRVR-trans.	0.00	0.09	0.08	0.08	0.07	0.06	0.06	0.05	0.06	0.10

There are, however, differences between the transient price impact and permanent price impact cases. In particular, when looking at the MRVR-perm. market one can observe a clear drop in the adaptive-deterministic ratio. This observation directly supports the view that additional transaction fees can constrain the utilization of optimal adaptive strategies and helps to explain the MRVR-perm. results in Table 3. Indeed, one can observe from Table 4 that the adaptive-permanent composition is in fact generally more affected when price impact is permanent. One reason for this is that $\mathcal{T}$'s strategy is more direct and to the point in the transient price impact case. This combined with the fact that nonzero transaction fees hinder $\mathcal{A}$ from taking advantage of $\mathcal{T}$'s forthright approach amounts to transaction fees having a limited impact on $\mathcal{R}_{u,t}$ under transient price impact.

5 DISCUSSION

This section provides a discussion of the significance of various model assumptions and how these assumptions could be altered, extended, or relaxed entirely. Directions for future research are set forth. A reasonable starting point is the way information and price impact are modeled in the paper.

Information, Price Impact, and Transaction Fees

First, it should be noted that the price impact in the model is interpreted to be *liquidity-driven* (or, *flow-driven*). This is in contrast to, e.g., Kyle (1985) where price impact is *information-driven*; should the market makers in the Kyle-model believe (with probability one) that none of the traders in the market possess any fundamental information, then orders from these traders would not have a price impact. Hence, as it is assumed that the trading needs of $\mathcal{T}$ are not driven by fundamental information, the traders in the present model would not induce information-driven price impact. For this reason the model presented does not feature an additional information-based component.[38]

It is, however, worth pointing out that a potential extension is to augment the information structure of the present model and to consider a model where at least some of the traders are endowed with fundamental information and where the price dynamics take the form:

$$\Delta p_t = \lambda_t q_t + \gamma q_t - (1-\alpha)s_{t-1},$$

where λ_t is the endogenous Kyle-lambda.[39] This hybrid model would feature both linear permanent price impact and transient price impact and optimal execution strategies in this setting are presently an open question. In general, models featuring combinations of different types of price impact functions are scarce. Park and Van Roy (2015) is one recent example.

Second, in the present model it is assumed that asymmetric information is one-sided, i.e., $\mathcal{A}$ is uncertain about $\mathcal{T}$'s initial position while $\mathcal{A}$'s initial position is common knowledge. Relaxing this assumption would require one to take into account the issues, which ensue from the use of non-nested information sets, and this would add an additional layer of complexity to the model. Moreover, since the zero initial inventory assumption for the arbitrageur seems reasonable from an economics perspective, the two-sided asymmetric information extension is not developed further

38 One needs to be somewhat careful in specifying the form of different price impact functions to avoid potential (quasi-)arbitrage (cf. Huberman and Stanzl 2004) or transaction-triggered price manipulation (cf. Alfonsi et al. 2012).

39 Naturally, one could allow γ to be time-varying as well.

here. For the interested reader, Choi et al. (2019) is an example of a closely related model with two-sided asymmetric information.

Another extension related to information structure is robust trading in a setting where $\mathcal{T}$ is ambiguity averse and faces uncertainty pertaining to the prior beliefs of $\mathcal{A}$. For instance, $\mathcal{A}$ could either be totally oblivious with regards to the trading needs of $\mathcal{T}$ or, alternatively, $\mathcal{A}$ could have been extensively studying $\mathcal{T}$'s financial position before trading commences and thus have a very accurate idea about the initial position $\tilde{x}_0$.[40]

Finally, pertaining to different specifications for transaction fees, Schied and Zhang (2018) illustrate that, under certain restrictions, quadratic transaction fees can be replaced with proportional ones without changing equilibrium outcomes. Ultimately, the choice of the functional form depends on what aspect of total costs one wishes capture. Quadratic costs are typically linked to *slippage costs*[41] while (linear) proportional costs may be seen as representing, e.g., *transaction taxes*.

Resilience, Limit Order Book, and Queuing

Continuing with topics related to price impact modeling, the model discussed assumes a recovery parameter α to capture the key aspects of transient price impact, abstracting over the micro-foundations of this assumption. Whilst this approach is sufficient and justifiable for the goals of this paper, one could pursue a more detailed procedure. For example, one could superimpose a model of a limit order book, as in Obizhaeva and Wang (2013), on top of the current model and specify from that a (at least semi-endogenous) resilience function to be used instead of the parameter α.

However, delving deeper into modeling LOBs, one must acknowledge the existence of typical price-time priority rules which would represent a deviation from the assumption utilized above that all period t trades are executed simultaneously and at the same price. Taking priority rules into account may provide new insights and, at the same time, complicate the model in various ways, depending on the approach taken.

The simplest way to consider time priority in the present model is to assume that one trader—the logical choice is $\mathcal{A}$—has a natural speed advantage and thus is, at each period, able to basically front-run the other trader. Alternatively, one could assume that the priority at each period is determined via a (possibly biased) coin-flip. The first mentioned approach clearly makes matters worse for $\mathcal{T}$ in the form of increased execution costs. Furthermore, the minimum revelation strategy of $\mathcal{T}$ becomes less usable and an optimal strategy drifts away from larger, more revealing, trades. The second approach has a more ambiguous impact on profits and execution costs.

40 Ambiguity-averse strategic trading is studied in, for example, Vitale (2018) and Schied (2013).

41 Slippage costs could arise from, for instance, temporary price impact which causes a difference between the expected price of a trade and the price at which the trade is eventually executed.

Schied and Zhang (2018) illustrate the emergence of distinctive ("hot potato") oscillations in a perfect information, "coin-flip priority", transient price impact execution game. A reasonable question to ask is whether or not this could happen in the present model as well. The answer is a tentative *no* and the justification for the answer is as follows.

Suppose that, at each period, there is an exogenous probability $\xi \in (0,1)$ that $\mathcal{T}$'s period t order is executed at price:

$$p_t^* = p_{t-1} + \gamma(u_t + \epsilon_t) - (1-\alpha)s_{t-1},$$

and $\mathcal{A}$'s order is executed, immediately after, at price:

$$p_t^{**} = p_{t-1} + \gamma(u_t + v_t + \epsilon_t) - (1-\alpha)s_{t-1}.$$

In other words, if $\mathcal{T}$ gets to trade first at period t, she is able to avoid the additional trading cost stemming from v_t, while $\mathcal{A}$ incurs the price impact of both his own and $\mathcal{T}$'s trade. Naturally, with probability $1-\xi$ the roles are reversed.

It is easy to see that the effect of this change in the model is transmitted through changes in the value function coefficients of $\mathcal{T}$ and $\mathcal{A}$, leaving the state variable dynamics unchanged. Hence, the end result will be, depending on what values certain parameters take, either amplification or attenuation of the effects at play in the analysis above, instead of the rise of completely new trading strategies. Furthermore, it should be noted that an important prerequisite for synchronous oscillatory trading is precise knowledge about the position of the opposing trader—in the present asymmetric information setting $\mathcal{A}$ does not possess this knowledge.

Multiple Traders

The model discussed features two prominent players: $\mathcal{T}$ and $\mathcal{A}$. A natural extension would be to consider an extension where there are multiple arbitrageurs and a single $\mathcal{T}$. A straightforward way to achieve this is to assume there are $N > 1$ arbitrageurs with identical prior beliefs, initial positions, and strategies (cf. Holden and Subrahmanyam 1992).

An extension featuring multiple heterogenous arbitrageurs is trickier since in this case the interactions between the arbitrageurs become more complicated even if there is no information asymmetry among the arbitrageurs. Complicated interactions are likely to yield inconclusive inferences. Alternatively, one could devise the multi-player model in a *mean field game* framework (cf. Huang et al. 2019) to avoid some of the problems related to games with a finite number of heterogenous players.

Continuous Time Limit and Endogenous Trading Horizons

The question of existence of the continuous time limit of a discrete model, or alternatively, determining the discrete time model, assuming one exists, of which a given continuous time model is a limit of is an important topic with practical relevance (cf. Duffie et al. 1992). The former can be achieved in the context of the present model by examining equilibrium outcomes when the time between trades approaches zero. The version attained using this approach can be coined as the (continuous time) limit model.

As illustrated, for example, by Huberman and Stanzl (2005), Caldentey and Stacchetti (2010) and Schied et al. (2017) equivalence of the continuous time limit model and a pure continuous time model—i.e., a model developed directly using the principles of continuous time stochastic processes—is not guaranteed nor is the existence of continuous time equilibria or the continuous time limit for that matter. More specifically, Caldentey and Stacchetti (2010) show, utilizing a Kyle-type model, that the limit equilibrium obtained by letting the time between trades to go to zero does not constitute an equilibrium in the pure continuous time model. A proper treatment of the continuous time limit is beyond the scope of this paper and is left for further research.

Another important timing related assumption is that the model has a fixed terminal date after which trading ceases. As an alternative Moallemi et al. (2012) suggest studying an infinite horizon model with suitable objective functions such that, up to a point, faster liquidation is preferred to slower.[42] This approach is certainly worth looking at as, in addition to endogenizing the trading horizon, it allows one to effectively supplement the model with time-varying or regime-switching dynamics in the spirit of Becherer et al. (2018) and Siu et al. (2019).

Soft Targets with Inventory Costs

Above it is assumed that $\mathcal{T}$ ($\mathcal{A}$) must end up with $x_T = 0$ ($y_{T+1} = 0$), i.e., the traders are subject to *hard* trading constraints. Consider instead a situation where $\mathcal{T}$ is allowed to choose x_T freely, but incurs a penalty of, say, $c(x_T) > 0$, whenever $|x_T| \neq 0$.[43] How does shifting from a hard constraint to a soft one change $\mathcal{T}$'s trading?

Simple terminal penalty, as depicted above, has a rather predictable impact. Indeed, one can think of the present model as one with an infinite terminal penalty and therefore it may be conjectured that there exists a threshold terminal penalty, above which $\mathcal{T}$ finds it optimal not to deviate from the target and below which deviations become profitable when the relevant performance criteria is total execution costs. Naturally, if one lets the terminal penalty to go to zero then, subject to a profit maximization objective, $\mathcal{T}$ does not trade at all thus securing zero execution costs.

Running inventory penalty, however, could change the equilibrium dynamics in a fundamental way. A case in point is Cartea et al. (2017) who illustrate that, in

[42] Endogenous horizons in strategic trading models are previously featured in, for example, Holden and Subrahmanyam (1996), Easley et al. (2015), and Brunovský et al. (2018).

[43] This example focuses solely on $\mathcal{T}$ but nothing prevents one to relax assumptions about $\mathcal{A}$'s terminal position as well.

a market-making context, running inventory penalties can be linked to ambiguity aversion. Simply put, in this case $\mathcal{T}$ incurs periodic costs from holding inventory and thus is encouraged to trade faster. Under running inventory costs, optimal strategies may feature *pace changes*: initially the trader wishes to shed inventory quickly to reduce costs after which a slower camouflage strategy is adopted to delay revelation of the remaining private information.

Utility Functions and Gaussian Noise

The model involves assumptions pertaining to utility functions and return distributions. A simple risk averse extension with negative exponential utility of the present model is feasible and potentially valuable. Indeed, risk aversion has been shown to yield important insights in conjunction to optimal execution problems (cf. Schied and Schöneborn 2009b).

Relaxing the assumption of Gaussian uncertainty is more complicated as this assumption is linked to the "shape" of the resulting equilibrium. While numerical methods could be used to solve the filtering equations and to solve the resulting optimization problems, one should be wary of not ignoring the technical issues stemming from moving beyond Gaussian uncertainty. For example, one may have to consider an alternative equilibrium concept to the PBE used above due to an inability to impose Bayes' rule.

Nevertheless, a model with more general distributional assumptions is without a doubt an interesting modeling challenge. Even more ambitiously, one could consider a model with general objectives as well as general return and noise distributions in the spirit of Bernardo and Judd (2000). This type of extension would be a very comprehensive exercise in the efficiency and flexibility of modern computational methods.

Uncertainty about Presence of $\mathcal{A}$

Uncertainty about the presence of $\mathcal{A}$ brings about a new, novel component by changing the nature of the asymmetric information in the model.[44] While $\mathcal{A}$, if present, is still in the dark about the initial position x_0, $\mathcal{T}$ also now faces incomplete information pertaining to whether $\mathcal{A}$ is actually present in the market. So, in a similar fashion as $\mathcal{A}$ tries to learn the stock position of $\mathcal{T}$ from price changes, $\mathcal{T}$ now tries to infer whether there are, hidden in the aggregate order flow, adversary trades of $\mathcal{A}$.[45]

This type of setting with two-sided asymmetric information results in a model noticeably more involved than the one discussed in this paper. Nonetheless, the main

[44] Uncertainty about the presence of $\mathcal{T}$ could also be considered. This angle to the problem is, however, less interesting from an economic point of view. Further, one can think of uncertainties pertaining to $\mathcal{T}$ to be embedded in the prior beliefs about $\tilde{x}_0$.

[45] Models of strategic trading featuring uncertainty about the presence of certain traders are studied in, for example, Chakraborty and Yilmaz (2004) and Back et al. (2017).

intuition is rather straightforward. From the viewpoint of $\mathcal{T}$, protecting against $\mathcal{A}$ is costly if $\mathcal{A}$ is actually not present and not protecting is costly if $\mathcal{A}$ is present. From the viewpoint of $\mathcal{A}$, having $\mathcal{T}$ believe that she can essentially operate adversary-free—along the lines of *the greatest trick the devil ever pulled*—is the key to effectively learning $\mathcal{T}$'s position at which point it will be too late for $\mathcal{T}$ to shift into a camouflage mode. Thus, $\mathcal{A}$'s optimal strategy in this environment is likely to involve a slow learning phase at the beginning and a faster profit phase towards the end of the trading horizon.

A proper analysis of this case is left for future research. One can, however, look at the present model as a certain special case of a more general model, featuring a form of uncertainty about the presence of $\mathcal{A}$. To see this, suppose $\mathcal{Q}^{\mathbf{u}} \triangleq \{\mathbb{Q}_1, ..., \mathbb{Q}_k\}$, for some $k \in \mathbb{N}$, is a set of appropriate subjective measures, consistent with a control $\mathbf{u}$, and assume that $\mathbb{Q}_1$ assigns probability 1 to the event that $\mathcal{A}$ is present while other measures in the set assign a probability between 0 and 1 to the same event. Then, the present model can be viewed through the lens of robust optimization such that $\mathcal{T}$ solves:

$$\max_{\mathbf{u} \in \mathcal{U}} \left\{ \min_{\mathbb{Q} \in \mathcal{Q}^{\mathbf{u}}} \mathbb{E}^{\mathbb{Q}} \left[\sum_{\tau=t}^{T-1} \Delta p_{\tau+1} x_\tau \mid \mathcal{F}_t^{\mathcal{T}} \right] \right\},$$

where $\mathcal{U}$ is the set of admissible controls (cf. Iyengar 2005). Interpreting the above as a game of its own, it is noted that for any control $\mathbf{u}$, "nature" chooses $\mathbb{Q} \in \mathcal{Q}$ such that the objective is minimized. In this case, under the working assumptions, $\mathcal{T}$ always solves the problem as if $\mathcal{A}$ is present with probability one. Therefore, the solution in this special case concurs with the solution of the model without ambiguity.

6 CONCLUSION

While optimal execution under transient price impact has been studied extensively, the existing models typically feature a single trader without strategic interactions in a complete information setting. This setting, for the most part, is not the most realistic one, and—in light of the present paper—features modeling choices, such as ignoring strategic interactions and asymmetric information, not strictly necessary for establishing a parsimonious optimal execution model. Indeed, complementing existing literature, the model depicted above illustrates an order execution game, featuring coupled strategic trading of a large liquidating trader and a (predatory) arbitrageur under transient price impact, asymmetric information, and quadratic transaction fees.

The main theoretical contribution presented in the paper is the description of a linear equilibrium of the order execution game together with an appropriate verification theorem and the analysis of the properties of this linear equilibrium. Furthermore, equilibrium existence and uniqueness are discussed in detail. For example, a way to proceed with the problem of verifying equilibrium existence and uniqueness in the discrete time dynamic programming formulation is put forth. This approach is based on an application of interval polynomial theory and could prove to be an interesting area for further research. Furthermore, to emphasize the links to earlier attempts to model optimal execution problems with transient price impact, the present model is compared to other existing models and it is illustrated that, despite different foundations, these models share key common features. For instance, it is verified that without an adversary player, the deterministic part of the liquidating trader's strategy corresponds to deterministic optimal strategies derived in earlier models. Finally, an extensive discussion pertaining to the model assumptions and extensions is provided together with an overview of possible venues for future research.

Theoretical results are complemented with an extensive numerical analysis. A key observation from this analysis is that shifting from permanent to transient price impact markedly changes the equilibrium trading strategies of both the liquidating trader and the arbitrageur. Namely, under transient price impact the arbitrageur trades conservatively, focusing mostly on liquidity provision. As a consequence, the liquidating trader adopts a more straightforward strategy and trades, for the most part, as if there were no arbitrageur. Moreover, the dynamic programming treatment of the order execution game combined with the nature of uncertainty in the game yields adaptive, instead of deterministic, equilibrium trading strategies.

In additional numerical examinations, focusing on expected execution costs, it is discovered that both optimal and suboptimal trading strategies for the liquidating trader perform better under transient rather than permanent price impact. This is mainly because transient price impact changes the nature of interactions between the liquidating trader and the arbitrageur. Furthermore, it is shown that small transaction fees may improve the performance of suboptimal strategies of the liquidating trader, especially if price impact is permanent. The intuition of this result circles back to the fact that transaction fees may alleviate the problem of adversary trading and thus in fact reduce the execution costs faced by the liquidating trader.

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APPENDICES

A PROOFS

Appendix A contains the proofs from Sections 2 and 3. Prior to moving to the proofs, a short refresher is in order.

The order execution game features four states variables: x_t, y_t, μ_t, and s_t. All of these state variables evolve from period to period with the trades of $\mathcal{T}$ and $\mathcal{A}$. Recall first, that $x_t = x_{t-1} + u_t$ from which it is obtained that

$$x_t = (1 + a_{x,t})x_{t-1} + a_{y,t}y_{t-1} + a_{\mu,t}\mu_{t-1} + a_{s,t}s_{t-1}. \tag{a.1}$$

Similarly, from $y_t = y_{t-1} + v_t$ it follows that:

$$y_t = (1 + b_{y,t})y_{t-1} + b_{\mu,t}\mu_{t-1} + b_{s,t}s_{t-1}.$$

Regarding s_t, one obtains:

$$\begin{aligned} s_t &= \gamma\left(\sum_{\tau=0,\ldots,t} \alpha^{t-\tau} q_\tau\right) \\ &= \alpha s_{t-1} + \gamma q_t. \end{aligned}$$

The dynamics of μ_t are given in (14) and verified in the subsequent proof.

Proof of Proposition 1. Equality (13) follows the fact that the innovations (z_t) are obtained from the observations (Δp_t) via *invertible* linear causal operations (cf. Anderson and Moore 1979). Hence, the $\boldsymbol{\sigma}$-algebras generated by the two are indeed equivalent. Further, the first equalities in both (14) and (15) follow from (11).

Due to the Gaussianity of the random variables involved, the conditional mean (14) is obtained using (a.1) and noting that g_t is essentially a linear regression slope coefficient. Similarly, the conditional variance σ_t is obtained using (a.1) and:

$$\sigma_t^2 \triangleq \mathbb{V}\left[\tilde{x}_t \mid y_t, \mathbf{z}^t\right] = \left(1 - \left(\mathbb{CORR}[\tilde{x}_t, \tilde{z}_t \mid y_t, \mathbf{z}^{t-1}]\right)^2\right) \times \mathbb{V}\left[\tilde{x}_t \mid \mathbf{z}^{t-1}\right],$$

where $\mathbb{CORR}[\,\cdot\,]$ refers to correlation and where it is recalled that $\mathbf{z}^t = (z_1, ..., z_t)$. ■

Due to the fact that the state variables have stochastic (Markovian) dynamics, one faces the question of conditional moments of the state variables. This question is addressed in Lemma A.1.

Lemma A.1 (State variable moments). *The conditional moments for the state variables μ_t and s_t under linear strategies are:*

$$\begin{aligned}
\mathbb{E}[\tilde{\mu}_t \mid \mathcal{F}_t^{\mathcal{T}}] &= a_{y,t} y_{t-1} + a_{s,t} s_{t-1} + \left(a_{\mu,t} + \theta_t/(\hat{a}_{x,t}\rho_{t-1}^2)\right)\mu_{t-1} + \theta_t\left(u_t - a_{y,t} y_{t-1} - a_{\mu,t}\mu_{t-1}\right),\\
\mathbb{E}[\tilde{\mu}_t \mid \mathcal{F}_t^{\mathcal{A}}] &= \hat{a}_{y,t} y_{t-1} + \hat{a}_{s,t} s_{t-1} + \left(1 + \hat{a}_{\mu,t} + \hat{a}_{x,t}\right)\mu_{t-1},\\
\mathbb{V}[\tilde{\mu}_t \mid \mathcal{F}_t^{\mathcal{T}}] &= \gamma^{-2}\theta_t^2\sigma_\epsilon^2,\\
\mathbb{V}[\tilde{\mu}_t \mid \mathcal{F}_t^{\mathcal{A}}] &= \gamma^{-2}\theta_t^2\sigma_\epsilon^2\left(1 + (\hat{a}_{x,t}\rho_{t-1})^2\right),\\
\mathbb{E}[\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{T}}] &= \alpha s_{t-1} + \gamma\left(u_t + b_{y,t} y_{t-1} + b_{\mu,t}\mu_{t-1} + b_{s,t} s_{t-1}\right),\\
\mathbb{E}[\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{A}}] &= \alpha s_{t-1} + \gamma\left(v_t + \hat{a}_{y,t} y_{t-1} + (\hat{a}_{x,t} + \hat{a}_{\mu,t})\mu_{t-1} + \hat{a}_{s,t} s_{t-1}\right),\\
\mathbb{V}[\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{T}}] &= \gamma^2\sigma_\epsilon^2,\\
\mathbb{V}[\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{A}}] &= \gamma^2\sigma_\epsilon^2\left(1 + (\hat{a}_{x,t}\rho_{t-1})^2\right),\\
\mathbb{E}[\tilde{\mu}_t\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{T}}] &= \mathbb{E}[\tilde{\mu}_t \mid \mathcal{F}_t^{\mathcal{T}}] \times \mathbb{E}[\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{T}}] + \gamma\theta_t\sigma_\epsilon^2,\\
\mathbb{E}[\tilde{\mu}_t\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{A}}] &= \mathbb{E}[\tilde{\mu}_t \mid \mathcal{F}_t^{\mathcal{A}}] \times \mathbb{E}[\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{A}}] + \gamma\theta_t\sigma_\epsilon^2.
\end{aligned}$$

Proof of Lemma A.1. The first two conditional expectations are obtained via the results from Proposition 1, while the following two conditional variances are obtained using the properties of variance as well as the scaling relation $\rho_t = \gamma\sigma_t/\sigma_\epsilon$. Identical approach can be utilized for the next four conditional moments. The last two relations are obtained by using the law of total expectation and recalling that $\tilde{e}_t$, for $t = 1, ..., T+1$, is an i.i.d., zero mean Gaussian random variable independent of the other random variables in the model. ■

In the proof of Theorem 1, the Lemma A.1 has a key role in verifying the quadratic nature of the periodic optimization problems. For example, using the above conditional moments, for $K \in \{\mathcal{T}, \mathcal{A}\}$, one immediately obtains:

$$\mathbb{E}[(\tilde{s}_t)^2 \mid \mathcal{F}_t^{\mathcal{K}}] = \mathbb{V}[\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{K}}] + \left(\mathbb{E}[\tilde{s}_t \mid \mathcal{F}_t^{\mathcal{K}}]\right)^2,$$

which can in turn be used to evaluate the value function terms related to $c_{ss,t}$ and $d_{ss,t}$ respectively.

***Proof of Theorem** 1*. The first step is to set up the optimization problem for period t. Starting with $\mathcal{T}$, using (7), one obtains:

$$\begin{aligned}&\max_{u_t} \mathbb{E}\Big[\Big(\gamma(u_t + v_t + \tilde{\epsilon}_t) - (1-\alpha)s_{t-1}\Big)x_{t-1} + U_t \mid \mathcal{F}_t^{\mathcal{T}}\Big]\\ =&\max_{u_t} \mathbb{E}\Big[\Big(\gamma(u_t + v_t + \tilde{\epsilon}_t) - (1-\alpha)s_{t-1}\Big)x_{t-1} + \mathbf{c}_t \cdot \mathrm{vech}(\boldsymbol{\zeta}_t^{\mathcal{T}} \otimes \boldsymbol{\zeta}_t^{\mathcal{T}})^\top \mid \mathcal{F}_t^{\mathcal{T}}\Big],\end{aligned}$$

where $\boldsymbol{\zeta}_t^{\mathcal{T}}$ is given in (7) and where it is recalled that:

$$\mathbf{c}_t \triangleq (c_{xx,t}, c_{xy,t}, c_{x\mu,t}, c_{xs,t}, c_{yy,t}, c_{y\mu,t}, c_{ys,t}, c_{\mu\mu,t}, c_{\mu s,t}, c_{ss,t}).$$

The quadratic nature of the problem can be verified by fixing v_t to (6) and using the above described state variable dynamics and Lemma A.1. Based on Proposition 1 and Lemma A.1 the system of beliefs is consistent with Bayes' rule. Therefore, due to the quadratic nature of the problem, sequential rationality is ensured via the first and second order optimality conditions.

The first order condition for $\mathcal{T}$, obtained from the above is:

$$\begin{aligned}&-2\Big(c_{xx,t} + \gamma c_{xs,t} + \gamma^2 c_{ss,t} + a_{x,t}\theta_t(c_{x\mu,t} + \gamma c_{\mu s,t} + c_{\mu\mu,t}a_{x,t}\theta_t)\Big)u_t\\ &= \Big(\gamma + 2c_{xx,t} + \gamma x_{xs,t} + a_{x,t}\theta_t c_{x\mu,t}\Big)x_{t-1} + A_y y_{t-1} + A_\mu \mu_{t-1} + A_s s_{t-1},\end{aligned}$$

from which, by matching coefficients, one can recover the a_x-polynomial which is clearly independent of $a_{y,t}$, $a_{\mu,t}$, and $a_{s,t}$. The scaled variance ρ_{t-1} does, however, appear in the polynomial, for instance, through θ_t (cf. Proposition 1). Thus, one must utilize (16) to simplify the first order condition to a standard polynomial in a_x and hence the recursive equation for ρ_{t-1} is an integral part of the equilibrium construction.

The lengthy terms in front of $a_{y,t}$, $a_{\mu,t}$, and $a_{s,t}$ are suppressed (A_y, A_μ, and A_s) to keep the presentation concise. These terms can be recovered from the first order condition in a straightforward manner. The second order condition (19) is immediate.

To proceed, the first order condition for $\mathcal{A}$ is presented next. The optimization problem of $\mathcal{A}$ is:

$$\begin{aligned}&\max_{v_t} \mathbb{E}\Big[\Big(\gamma(u_t + v_t + \tilde{\epsilon}_t) - (1-\alpha)s_{t-1}\Big)y_{t-1} + V_t \mid \mathcal{F}_t^{\mathcal{A}}\Big]\\ \Leftrightarrow&\max_{v_t} \mathbb{E}\Big[\Big(\gamma(u_t + v_t + \tilde{\epsilon}_t) - (1-\alpha)s_{t-1}\Big)y_{t-1} + \mathbf{d}_t \cdot \mathrm{vech}(\boldsymbol{\zeta}_t^{\mathcal{A}} \otimes \boldsymbol{\zeta}_t^{\mathcal{A}})^\top \mid \mathcal{F}_t^{\mathcal{A}}\Big],\end{aligned}$$

where $\boldsymbol{\zeta}_t^{\mathcal{A}}$ is given in (8) and where it is recalled that:

$$\mathbf{d}_t \triangleq (d_{yy,t}, d_{y\mu,t}, d_{ys,t}, d_{\mu\mu,t}, d_{\mu s,t}, d_{ss,t}).$$

Again, the quadratic nature of the problem can be verified by fixing u_t to (5) and using the state variable dynamics given in Lemma A.1. Sequential rationality follows as above.

The first order condition for $\mathcal{A}$ is:[46]

$$-2\left(d_{yy,t} + \gamma d_{ys,t} + \gamma^2 d_{ss,t}\right)v_t = B_y y_{t-1} + B_\mu \mu_{t-1} + B_s s_{t-1}.$$

The second order condition (20) is immediate.

Similarly to the a_x-polynomial, one obtains from the above first order conditions for $\mathcal{T}$ and $\mathcal{A}$ the following set of linear systems, for $i \in \{y, \mu, s\}$ and $A_i^{(1)} B_i^{(1)} \neq 1$:

$$\begin{cases} b_{i,t} = & \frac{B_i^{(1)} A_i^{(0)} + B_i^{(0)}}{\left(1 - A_i^{(1)} B_i^{(1)}\right)}, \\ a_{i,t} = & A_i^{(0)} + A_i^{(1)} b_{i,t}, \end{cases} \tag{a.2}$$

where one utilizes the fact that (A_i, B_i) parameters have decompositions:

$$\begin{aligned} A_i =& A_i^{(0)} + A_i^{(1)} b_{i,t} \text{ and} \\ B_i =& B_i^{(0)} + B_i^{(1)} a_{i,t}. \end{aligned}$$

Thus, for instance, (A_y, B_y) only depend on $a_{x,t}$—which is obtained from the a_x-polynomial and thus known at this point—and $(a_{y,t}, b_{y,t})$. Consequently, the equilibrium linear systems are square—two equations and two unknowns—and therefore any solution pair obtained will be unique. ■

***Proof of Proposition* 2.** Omitted. ■

B NUMERICAL APPROACH

Appendix B gives the details of the numerical algorithm used to solve for a PBE in the model.

The final step in each period $t = T + 1, T, ..., 2$, moving backwards in time, is to update the value function coefficients to be used in the preceding period. Value

[46] Note that setting $\alpha = 1$ effectively shuts down the s-channel, i.e., all coefficients with the s subscript will be equal to zero. Then, setting $\gamma = \lambda$, one can recover the Moallemi et al. (2012) model.

function coefficients are updated via (lengthy) linear recursions, which are omitted here for the sake of brevity.

ALGORITHM: *Iterative equilibrium search*

Input: Trading horizon for $\mathcal{T}$, denoted by T, prior parameters for $\mathcal{A}$, denoted by μ_0 and σ_0. Model constants: y_0, σ_ϵ, γ, α, and θ_c. Iteration control: Set *counter* := 0, fix m, referring to the maximum number of iterations permitted, κ referring to strategy update step-size, and a stopping threshold $\epsilon > 0$.
Output: For all $t = T+1, T, ..., 1$, the equilibrium trading and value function coefficients:

$$\mathbf{a}_t^* = (a_{x,t},\ a_{y,t},\ a_{\mu,t},\ a_{s,t})^\top,\ \mathbf{A}^* = \begin{pmatrix} (\mathbf{a}_{T+1}^*)^\top \\ \vdots \\ (\mathbf{a}_1^*)^\top \end{pmatrix} \in \mathbb{R}^{T+1,4}$$

****

$$\mathbf{b}_t^* = (b_{y,t},\ b_{\mu,t},\ b_{s,t})^\top, \qquad \mathbf{B}^* = \begin{pmatrix} (\mathbf{b}_{T+1}^*)^\top \\ \vdots \\ (\mathbf{b}_1^*)^\top \end{pmatrix} \in \mathbb{R}^{T+1,3}$$

and two sequences of scaled variances: $\hat{\boldsymbol{\rho}} = (\hat{\rho}_0,\ \hat{\rho}_1,\ ...,\hat{\rho}_T)$, referring to initial scaled variances, and $\check{\boldsymbol{\rho}} = (\check{\rho}_0,\ \check{\rho}_1,\ ...,\check{\rho}_T)$, referring to closing scaled variances.

Fix $\hat{\rho}_0$ and set $\hat{\mathbf{a}} := \mathbf{a}^{\text{eq}}$, *heureka* := 0.[a]

---

while *(heureka* = 0 **and** *counter < m)* **do**
- Compute $\hat{\boldsymbol{\rho}}$ using $\hat{\mathbf{a}}_t$'s, for $t = 1, ..., T$, and $\hat{\rho}_0$ and, using the traders' boundary conditions, assign values for $\mathbf{a}_{t'}$, $\mathbf{b}_{t'}$, for $t' = T+1, T$, as well as $\mathbf{c}_{T-1}$ and $\mathbf{d}_{T-1}$.[b]
- **for** ($t = T-1, T-2, ..., 1$)
  - Solve the a_x-polynomial to obtain $a_{x,t}$.
  - Using $a_{x,t}$, find ρ_{t-1} via (16).[c]
  - Obtain the rest of the constants in $\mathbf{a}_t$ and $\mathbf{b}_t$ using (a.2).
  - Check the second order conditions (19) and (20).
  - Collect $\mathbf{a}_t$ and $\mathbf{b}_t$ to $\mathbf{A}$ and $\mathbf{B}$ respectively.
  - Find $\mathbf{c}_t$ and $\mathbf{d}_t$ using $\mathbf{a}_t$ and $\mathbf{b}_t$ and the value function coefficients from the present round.
- **endfor**
- Compute $\check{\boldsymbol{\rho}}$ using $a_{x,t}$'s solved in the **for**-loop.
- **if** $\|\hat{\rho} - \check{\rho}\| < \epsilon$ **then**
  - HEUREKA! I.e, set *heureka* := 1. The algorithm terminates.
  - Return $\mathbf{A}$ and $\mathbf{B}$ from the last iteration.
- **else**
  - Update guess: $\hat{\mathbf{a}} := \kappa\hat{\mathbf{a}} + (1-\kappa)\text{col}_1(\mathbf{A})$.
- **end**

end

---

[a] $\mathbf{a}^{\text{eq}}$ refers to the Bertsimas and Lo (1998) strategy in which, for all t, $a_{x,t} = (t-(T+1))^{-1}$, and $a_{y,t} = a_{\mu,t} = a_{s,t} = 0$.

[b] Due to the special nature of the last two periods, these coefficients are determined easily. See Section 3.1. Note also that $\mathbf{a}_{t'}$, $\mathbf{b}_{t'}$ (without hats) refer to the equilibrium candidate values for the trading coefficients as opposed to $\hat{\mathbf{a}}_t$ which is used to refer to the conjectured strategy.

[c] ρ_t is an intermediary scaled variance not to be confused with $\hat{\boldsymbol{\rho}}$, the elements of which are all known at beginning of the for-loop or $\check{\boldsymbol{\rho}}$, the elements of which are determined outside the for loop.

C EQUILIBRIUM ANALYSIS VIA INTERVAL POLYNOMIALS

In this appendix a short example on using univariate interval polynomials in analyzing equilibrium uniqueness and existence is discussed. For this purpose, suppose that $T+1=4$. It is easy to see that equilibrium solutions at periods $t=4,3$ are unique. This observation stems from the fact that there are only limited strategic interactions in the last two rounds since the options of both $\mathcal{T}$ and $\mathcal{A}$ are constrained by their respective trading targets.

What about round $T-1=2$? One can verify that the cubic polynomial for $a_{x,2}$ takes the form:

$$f(a_{x,2}) \triangleq 2\Big(-1-(1-\alpha)W_1+W_2\Big)a_{x,2}^3+\Big(-5-5(1-\alpha)W_1+3W_2\Big)a_{x,2}^2 \\ +\Big(-4-4(1-\alpha)W_1+W_2\Big)a_{x,2}+\Big(-1-(1-\alpha)W_1\Big)=0, \tag{22}$$

where

$$W_1 \triangleq \frac{\alpha}{2(2-\alpha)}+1, \\ W_2 \triangleq \frac{(1-\alpha)\hat{\rho}_T}{2(2-\alpha)}.$$

Via direct substitution it is easy to check that $a_{x,2}=-1/2$ and $a_{x,2}=-1$ solve (22) for $\alpha \in (0,1]$. Of these roots, noting that $a_{x,3}=-1$, the latter is not consistent with $\mathcal{T}$'s equilibrium strategy.[47]

There is still one root not accounted for. To proceed, fix $\alpha=1$.[48] Then, (22) simplifies to:

$$-2a_{x,2}^3-5a_{x,2}^2-4a_{x,2}-1=0, \tag{23}$$

whence one obtains $\Delta_f=0$, where Δ_f is the cubic discriminant, indicating that (23) has a multiple root. In this case the multiple root is $a_{x,2}=-1$ which, as noted earlier, is incompatible with the equilibrium strategy. It follows that $a_{x,2}=-1/2$ is the unique solution to (23).

How crucial is to assumption $\alpha=1$? By examining (22) with $\alpha \in (0,1)$, one observes that $a_{x,2}=-1/2$ generally continues to be the unique (feasible) solution with the other two solutions lying somewhere in $(-1-\epsilon,-1]$, for small $\epsilon>0$.

Now, only period $t=1$ remains. It is crucial to note that by simply looking at the solutions of the $t=1$ a_x-polynomial f for fixed coefficient values would not be

[47] To see why this is the case, one should consider what happens to the scaled variance (16) if $a_{x,t}=-1$, for some $t<T$.

[48] With this assumption, one can delay introducing interval polynomials to period $t=1$.

sufficient in determining the equilibrium path uniqueness over a larger parameter space. The reason for this is intuitive. Changes in the a_x-polynomial coefficients can change the real solution set for this polynomial and thus, while feasible solutions exist for some part of the parameter space, this might not hold for the whole space.

However, starting with a fixed parametrization for which an appropriate solution does exist, one should expect that—at least around a small enough neighborhood—this existence continues to hold, i.e., that the polynomial is well-behaved. The question then becomes: Where are the limits of the parameter space where existence is compromised and do the limits belong to the interior of the so-called economically sensible part of the parameter space? This is just an example question to which interval methods can be applied.[49]

The above discussion implies that:

$$(\alpha, \gamma) \in I_\alpha \times I_\gamma \subset \mathbb{R}^2,$$

where $I_\alpha \triangleq (0,1]$ and $I_\gamma \subset \mathbb{R}_{>0}$ are appropriate intervals for α and γ respectively. Therefore, the period $t = 1$ a_x-polynomial can be treated as an interval polynomial:

$$[f](a_{x,1}) \triangleq \left\{ \sum_{k=0}^{3} f_k a_{x,1}^k \;:\; f_k \in [f_l^k, f_u^k] \subset \mathbb{R} \right\},$$

with piecewise polynomial *upper* (f_u) and *lower* (f_l) *bound functions*:

$$f_u(a_x) \triangleq \begin{cases} f_u^+(a_x) = \sum_{k=0}^{3} f_u^k a_{x,1}^k & x \geq 0, \\ f_u^-(a_x) = f_l^3 a_{x,1}^3 + f_u^2 a_{x,1}^2 + f_l^1 a_{x,1} + f_u^0 & x < 0, \end{cases}$$

$$f_l(a_x) \triangleq \begin{cases} f_l^+(a_x) = \sum_{k=0}^{3} f_l^k a_{x,1}^k & x \geq 0, \\ f_l^-(a_x) = f_u^3 a_{x,1}^3 + f_l^2 a_{x,1}^2 + f_u^1 a_{x,1} + f_l^0 & x < 0. \end{cases}$$

One can now utilize *Theorem 12.* from Zhang and Deng (2013) and the related algorithm to determine the number of interval zeros of $[f](a_x)$. Should one obtain that this number is zero, then the implication is that no equilibrium (of the desired form) exists for the given parametrization. Alternatively, if the interval polynomial has a single, i.e., a unique, solution on $(-1,0)$ then each stage problem ($t = 4,3,2,1$) in the dynamic program has a unique solution. In other words, the solution path is unique. Preliminary numerical tests support this view. Naturally, one could also adopt the above described approach to a general $T(> 4)$ in which case interval methods need to applied multiple times.

Although the interval approach is compelling and relatively simple to use, there are some caveats related to this approach in the present context. First, it is implicitly assumed above that $a_{x,1}$ is restricted to $(-1,0)$. While this holds in the numerical tests

[49] Interval arithmetic is implemented in the INTLAB toolbox (cf. Rump 1999) for MATLAB®. For the purposes of the analysis described here, one does not require numerical interval arithmetic.

and is consistent with the equilibrium characterization, it is possible that there are cases in which this assumption is violated. At the same time, extending the interval to which the solutions are assumed to belong, increases the likelihood of "spurious solutions", which are indeed interval zeros, but which cannot be considered to part of a sensible equilibrium solution. Second, deriving exact bounds for coefficient intervals $[f_l^k, f_u^k]$, needed to make sharp conclusions about the equilibrium properties, is complicated.

Finally, related to the last point, a large T implies that interval methods need to be utilized multiple times and this considerably increases the computational effort required to solve the problem. For example, one must take into account a multitude of different coefficient intervals. Nonetheless, these caveats are surpassed by the benefits and insights gained from better understanding of equilibrium properties, especially in cases where these properties are central to answering the questions posed.

UNIVERSITY OF TURKU

Painosalama Oy, Turku, Finland 2020

ISBN 978-951-29-8055-0 (PRINT)
ISBN 978-951-29-8056-7 (PDF)
ISSN 2343-3159 (Painettu/Print)
ISSN 2343-3167 (Verkkojulkaisu/Online)