On Regular and New Types of Codes for Location-Domination

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Abstract

Identifying codes and locating-dominating codes have been designed for locating irregularities in sensor networks. In both cases, we can locate only one irregularity and cannot even detect multiple ones. To overcome this issue, self-identifying codes have been introduced which can locate one irregularity and detect multiple ones. In this paper, we define two new classes of locating-dominating codes which have similar properties. These new locating-dominating codes as well as the regular ones are then more closely studied in the rook's graphs and binary Hamming spaces.

In the rook's graphs, we present optimal codes, i.e., codes with the smallest possible cardinalities, for regular location-domination as well as for the two new classes. In the binary Hamming spaces, we present lower bounds and constructions for the new classes of codes; in some cases, the constructions are optimal. Moreover, one of the obtained lower bounds improves the bound of Honkala *et al.* (2004) on codes for locating multiple irregularities.

Besides studying the new classes of codes, we also present record-breaking constructions for regular locating-dominating codes. In particular, we present a locating-dominating code in the binary Hamming space of length 11 with 320 vertices improving the earlier bound of 352; the best known lower bound for such code is 309 by Honkala *et al.* (2004).

Keywords: Locating-dominating set; locating-dominating code; rook's graph; Hamming space; sensor network

1 Introduction

Sensor networks are systems designed for environmental monitoring. Various location detection systems such as fire alarm and surveillance systems can be viewed as examples of sensor networks. For location detection, a sensor can be placed in any location of the network. The sensor monitors its neighbourhood (including the location of the sensor itself) and reports possible irregularities such as a fire or an intruder in the neighbouring locations. Based on the reports of the sensors, a central controller attempts to determine the location of a possible irregularity in the network. Usually, the aim is to minimize the number of sensors in the network. More explanation regarding location detection in sensor networks can be found in [4, 12, 16].

A sensor network can be modeled as a simple and undirected graph $G = (V(G), E(G)) = (V, E)$ as follows: the set of vertices V of the graph represents the locations of the network and the edge set E of the graph represents the connections between the locations. In other words, a sensor can be placed in each vertex of the graph and the sensor placed in the vertex u monitors u itself and the vertices neighbouring u. Besides being simple and undirected, we assume that the graphs in this paper are connected and have order of at least two. In what follows, we present some basic terminology and notation regarding graphs. The *open neighbourhood* of $u \in V$ consists of the

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vertices adjacent to u and it is denoted by $N(u)$. The *closed neighbourhood* of u is defined as $N[u] = \{u\} \cup N(u)$. A nonempty subset C of V is called a *code* and the elements of the code are called *codewords*. In this paper, the code C represents the set of locations where the sensors have been placed on. For the set of sensors monitoring a vertex $u \in V$, we use the following notation:

$$I(u) = N[u] \cap C.$$

In order to emphasize the graph G and/or the code C, we sometimes write $I(u) = I(C; u) = I(G, C; u)$. We call $I(u)$ the *identifying set* (or the *I-set*) of u. The notation of identifying set can also be generalized for a subset U of V as follows:

$$I(U) = \bigcup_{u \in U} I(C; u).$$

Here we also use the notation $I(U) = I(C; U) = I(G, C; U)$.

As stated above, a sensor $u \in V$ reports that an irregularity has been detected if there is (at least) one in the closed neighbourhood $N[u]$. In what follows, we divide into two different situations depending on the capability of a sensor to distinguish whether the irregularity has been spotted in the location of the sensor itself or in its (open) neighbourhood. More precisely, we have the following two cases:

(i) In the first case, we assume that a sensor $u \in V$ reports 1 if there is an irregularity in $N[u]$, and otherwise it reports 0.

(ii) In the second case, we assume that a sensor $u \in V$ reports 2 if there is an irregularity in u, it reports 1 if there is one in $N(u)$ (and none in u itself), and otherwise it reports 0.

Assume first that the sensors work as in (i). Notice then that if the sensors in the code C are located in such places that $I(C; u)$ is nonempty and unique for all $u \in V$, then an irregularity in the network can be located by comparing $I(C; u)$ to identifying sets of other vertices. This leads to the following definition of *identifying codes*, which were first introduced by Karpovsky *et al.* in [11]. For various papers regarding identification and related problems, we refer to the online bibliography [13].

Definition 1. A code $C \subseteq V$ is *identifying* in G if for all distinct $u, v \in V$ we have $I(C; u) \neq \emptyset$ and

$$I(C; u) \neq I(C; v).$$

An identifying code C in a finite graph G with the smallest cardinality is called *optimal* and the number of codewords in an optimal identifying code is denoted by $\gamma^{ID}(G)$.

Let C be an identifying code in G. By the definition, the identifying code C works correctly if there is simultaneously at most one irregularity in the network. However, using the identifying code C, we cannot locate or even detect more than one irregularity in the network. Indeed, for example, consider the graph G in Figure 1 and the code $C = \{a, b, c\}$ in the graph. Clearly, C is an identifying code in G. However, all the sensors a, b and c are alarming if there is a single irregularity in b, or multiple ones in d, e and f. Hence, no distinction can be made between these two cases. Thus, we might determine a false location and more disturbingly not even notice that something is wrong. To overcome this problem, in [7], self-identifying codes, which are able to locate one irregularity and detect multiple ones, were introduced. (Notice that in the original paper self-identifying codes are called 1^+-identifying.) The formal definition of self-identifying codes is given as follows.

Definition 2. A code $C \subseteq V$ is called *self-identifying* in G if the code C is identifying in G and for all $u \in V$ and $U \subseteq V$ such that $|U| \geq 2$ we have

$$I(C; u) \neq I(C; U).$$

A self-identifying code C in a finite graph G with the smallest cardinality is called *optimal* and the number of codewords in an optimal self-identifying code is denoted by $\gamma^{SID}(G)$.

In addition to [7], self-identifying codes have also been previously discussed in [9, 10]. Separately in these papers, two useful characterizations have been presented for self-identifying codes.

Theorem 3 ([7, 9, 10])**.** *Let C be a code in G. Then the following statements are equivalent:*

(i) The code C is self-identifying in G.

(ii) For all distinct $u, v \in V$, we have $I(C; u) \setminus I(C; v) \neq \emptyset$.

(iii) For all $u \in V$, we have $I(C; u) \neq \emptyset$ and

$$\bigcap_{c \in I(C;u)} N[c] = \{u\}.$$

As stated earlier, self-identifying codes can locate one irregularity and detect multiple ones. Besides that, the characterization (iii) of the previous theorem also gives another useful property for self-identifying codes. Namely, the location of an irregularity can be determined without comparison to other identifying sets, since for all $u \in V$ the neighbourhoods of the codewords in $I(u)$ intersect uniquely in u.

So far, we have discussed the case where it is assumed that each sensor outputs 1 or 0 depending on whether there is an irregularity in the neighbourhood or not. In what follows, we now focus on the case (ii) where a sensor can also distinguish if the irregularity is on the location of the sensor itself. Then notice that if the sensors in the code C are located in such places that $I(C; u)$ is nonempty and unique for all $u \in V \setminus C$, then an irregularity in the network can be located by comparing $I(C; u)$ to identifying sets of other non-codewords. Indeed, we do not have to worry about vertices in C as an irregularity in such locations is immediately determined by a sensor outputting 2. This leads to the following definition of *locating-dominating codes*, which were first introduced by Slater in [15, 17, 18].

Definition 4. A code $C \subseteq V$ is *locating-dominating* in G if for all distinct $u, v \in V \setminus C$ we have $I(C; u) \neq \emptyset$ and

$$I(C; u) \neq I(C; v).$$

A locating-dominating code C in a finite graph G with the smallest cardinality is called *optimal* and the number of codewords in an optimal locating-dominating code is denoted by $\gamma^{LD}(G)$.

Comparing the definitions of identifying and locating-dominating codes, we immediately notice their apparent similarities; in the case of identification we require that the identifying sets $I(u)$ are unique for all vertices and in the case of location-domination the same is required for non-codewords. Therefore, as self-identifying codes are a natural specialization of regular identifying codes, it is obvious to consider if something similar could be done for locating-dominating codes. Indeed, the characterizations of Theorem 3 gives two natural ways to define new types of locating-dominating codes with similar kind of beneficial properties as self-identifying codes have over regular identifying codes. The definitions of these codes are given as follows.

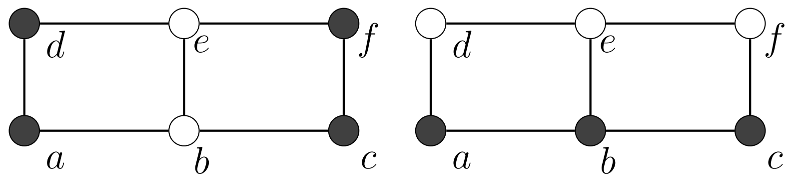
Definition 5. A code $C \subseteq V$ is *self-locating-dominating* in G if for all $u \in V \setminus C$ we have $I(C; u) \neq \emptyset$ and

$$\bigcap_{c \in I(C;u)} N[c] = \{u\}.$$

A self-locating-dominating code C in a finite graph G with the smallest cardinality is called *optimal* and the number of codewords in an optimal self-locating-dominating code is denoted by $\gamma^{SLD}(G)$.

Definition 6. A code $C \subseteq V$ is *solid-locating-dominating* in G if for all distinct $u, v \in V \setminus C$ we have

$$I(C; u) \setminus I(C; v) \neq \emptyset.$$



(a) Self-locating-dominating code in G (b) Solid-locating-dominating code in G

Figure 1: Optimal self-locating-dominating and solid-locating-dominating codes in G

A code $C \subseteq V$ is dominating if $I(C;u) \neq \emptyset$ for all $u \in V$. Since G is a connected graph on at least two vertices, a solid-locating-dominating code is also dominating. A solid-locating-dominating code C in a finite graph G with the smallest cardinality is called *optimal* and the number of codewords in an optimal solid-locating-dominating code is denoted by $\gamma^{DLD}(G)$.

The previous definitions are illustrated in the following example. In particular, we show that the given definitions are different. Compare this observation to self-identifying codes for which the analogous requirements are just other characterizations for the codes.

Example 7. Let G be the graph illustrated in Figure 1. Let C be a self-locating-dominating code in G. Observe first that if $a \notin C$, then $I(C;a) \subseteq \{b,d\}$ and we have

$$\{a,e\} \subseteq \bigcap_{c \in I(C;a)} N[c].$$

This implies a contradiction and therefore the vertex a belongs to C. An analogous argument also holds for the vertices c, d and f. Hence, we have $\{a,c,d,f\} \subseteq C$. Moreover, the code $C_1 = \{a,c,d,f\}$, which is illustrated in Figure 1(a), is self-locating-dominating in G since for the non-codewords b and e we have $I(C_1;b) = \{a,c\}$ and $N[a] \cap N[c] = \{b\}$, and $I(C_1;e) = \{d,f\}$ and $N[d] \cap N[f] = \{e\}$. Hence, C_1 is an optimal self-locating-dominating code in G and we have $\gamma^{SLD}(G) = 4$.

Let us then consider the code $C_2 = \{a,b,c\}$, which is illustrated in Figure 1(b). Now we have $I(C_2;d) = \{a\}$, $I(C_2;e) = \{b\}$ and $I(C_2;f) = \{c\}$. Therefore, it is easy to see that C_2 is a solid-locating-dominating code in G. Moreover, it can be shown that there are no solid-locating-dominating codes in G with smaller number of codewords. Thus, C_2 is an optimal solid-locating-dominating code in G and we have $\gamma^{DLD}(G) = 3$.

In the previous example, we showed that the definitions of self-locating-dominating and solid-locating-dominating codes are different. In the following theorem, we present new characterizations for self-locating-dominating and solid-locating-dominating codes. Comparing these characterizations to the original definitions of the codes, the differences of the codes become apparent.

Theorem 8. *Let $G = (V,E)$ be a connected graph on at least two vertices:*

(i) A code $C \subseteq V$ is self-locating-dominating if and only if for all distinct $u \in V \setminus C$ and $v \in V$ we have

$$I(C;u) \setminus I(C;v) \neq \emptyset.$$

(ii) A code $C \subseteq V$ is solid-locating-dominating if and only if for all $u \in V \setminus C$ we have $I(C;u) \neq \emptyset$ and

$$\left(\bigcap_{c \in I(C;u)} N[c] \right) \setminus C = \{u\}.$$

Proof. (i) Assume first to the contrary that there exist $u \in V \setminus C$ and $v \in V$ such that $I(C;u) \setminus I(C;v) = \emptyset$. This implies that $I(C;u) \subseteq I(C;v)$ and we have a contradiction as

$$\{u,v\} \subseteq \bigcap_{c \in I(C;u)} N[c].$$

On the other hand, if there exists $u \in V \setminus C$ such that

$$\{u,v\} \subseteq \bigcap_{c \in I(C;u)} N[c]$$

for some $v \in V$, then $I(C;u) \setminus I(C;v) = \emptyset$ (a contradiction).

(ii) Assume first to the contrary that there exist $u, v \in V \setminus C$ such that $I(C;u) \setminus I(C;v) = \emptyset$. This implies that $I(C;u) \subseteq I(C;v)$ and we have a contradiction as

$$\{u,v\} \subseteq \left(\bigcap_{c \in I(C;u)} N[c] \right) \setminus C.$$

On the other hand, if there exists $u \in V \setminus C$ such that

$$\{u,v\} \subseteq \left(\bigcap_{c \in I(C;u)} N[c] \right) \setminus C$$

for some $v \in V \setminus C$, then $I(C;u) \setminus I(C;v) = \emptyset$ (a contradiction). □

The previous theorem immediately gives the following corollary.

Corollary 9. *If C is a self-locating-dominating code in G, then C is also solid-locating-dominating in G. Furthermore, we have $\gamma^{DLD}(G) \leq \gamma^{SLD}(G)$.*

As discussed earlier, self-identifying codes have benefits over regular identifying codes; they detect more than one irregularity and locate one irregularity without comparison to other identifying sets. Next we study the same properties concerning self-locating-dominating and solid-locating-dominating codes:

- Let us begin by considering the ability to locate an irregularity without comparison to other identifying sets. For self-locating-dominating codes, this property immediately follows from the definition. Analogously, the property is obtained for solid-locating-dominating codes by Theorem 8(ii).
- Consider then the ability to detect more than one irregularity. Let first C be a self-locating-dominating code in G. If more than one sensor is reporting 2, then we immediately detect that there are multiple irregularities. Hence, we may assume that there is at most one sensor reporting 2. Let U be the set of sensors reporting 1 (U can be empty). Consider then the intersection
$$X = \bigcap_{c \in U} N[c].$$
Here we assume that $X = V$ if the set U of sensors reporting 1 is empty. Now, by the definition of self-locating-dominating codes (as at most one sensor is reporting 2), there are multiple irregularities if and only if the intersection X is empty, or a sensor reporting 2 does not belong to X. Indeed, if $X = \emptyset$ or a sensor reporting 2 does not belong to X, then there are clearly multiple irregularities. On the other hand, if there is an irregularity in a location u with a sensor and at least one without a sensor, then $X = \emptyset$ or $u \notin X$, and if there is no irregularity in a location with a sensor and at least two without a sensor, then $X = \emptyset$. Thus, self-locating-dominating codes can detect multiple irregularities.

On the other hand, solid-locating-dominating codes do not always detect multiple irregularities. For a counterexample, consider the graph G and the solid-locating-dominating code C_2 of Example 7. If the vertex b is reporting 2 and the vertices a and c are reporting 1, then there might be a single irregularity in b or multiple irregularities in b, d and f. However, if it is assumed that the irregularities occur only in the locations without a sensor, then we can detect multiple irregularities using similar arguments as in the case of self-locating-dominating codes.

In the paper our main focus is on the new types of locating-dominating codes. However, we also present some results for regular locating-dominating codes. In Section 2, we consider the different types of locating-dominating codes in the Cartesian product of two complete graphs, which is also called the rook's graph. In particular, we obtain optimal codes for regular location-domination, self-location-domination and solid-location-domination. In Section 3, we consider similar problems in the binary Hamming space (or hypercube) $\mathbb{F}^n$, where n is a positive integer. In particular, we present an infinite family of optimal self-locating-dominating codes and construct regular locating-dominating codes with the smallest known cardinalities; especially proving that $309 \leq \gamma^{LD}(\mathbb{F}^{11}) \leq 320$. Moreover, our bound in Theorem 20 on solid-locating-dominating codes implies an improvement on Honkala *et al.* bound in [8], see Remark 21.

2 Location-domination in the rook's graphs

In this section, we consider the different locating-dominating codes in the Cartesian product of two complete graphs. The Cartesian product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is $G_1 \square G_2 = (V_1 \times V_2, E)$ where $(x, y)(x', y') \in E$ if and only if $x = x'$ and $yy' \in E_2$, or $y = y'$ and $xx' \in E_1$. If K_n and K_m are two complete graphs of order n and m, respectively, then $K_n \square K_m$ is known as rook's graph and can be viewed as a chess board with m rows and n columns. The closed neighbourhood of a vertex is determined by the movement of a rook in chess. We denote $V(K_n) = \{x_1, \ldots, x_n\}$, $V(K_m) = \{y_1, \ldots, y_m\}$ and the kth *row* by $R_k = \{(x_i, y_k) \mid i = 1, \ldots, n, 1 \leq k \leq m\}$ (resp. the hth *column* by $P_h = \{(x_h, y_i) \mid i = 1, \ldots, m, 1 \leq k \leq n\}$).

In our considerations, the columns are ordered from left to right and the rows from bottom to top. However, we will occasionally permute the order of rows and/or columns. If C is a locating-dominating, self-locating-dominating or solid-locating-dominating code, it will also be such code in the graph gained through these permutations since all neighbourhoods remain the same after these permutations. Previously, in [5], [6], [9] and [10], optimal codes have been respectively found for identification and self-identification in the rook's graphs. These results are combined in the following theorem.

Theorem 10 ([5, 6, 9])**.** *Let $G = K_n \square K_m$ be a rook's graph with $m \geq n \geq 2$. We have the following formulas for the sizes of optimal identifying and self-identifying codes in $K_n \square K_m$:*

$$\gamma^{ID}(K_n \square K_m) = \begin{cases} m + \left\lfloor \frac{n}{2} \right\rfloor, & m \leq \frac{3n}{2} \\ 2m - n, & m \geq \frac{3n}{2} \end{cases} \quad \text{and}$$

$$\gamma^{SID}(K_n \square K_m) = 2m, \ m \geq n.$$

In what follows, we are going to find optimal locating-dominating, self-locating-dominating and solid-locating-dominating codes in the rook's graphs. For this purpose, we first introduce the following helpful lemma.

Lemma 11. *For $v = (x_i, y_j) \in V(K_n \square K_m)$ the following statements hold:*

(i) If $|I(v)| = 2$ and vertices in $I(v)$ are not on the same row or column, then $|\bigcap_{c \in I(v)} N[c]| = 2$.

(ii) If $|I(v)| \geq 2$ and vertices in $I(v)$ are on the same row, then $\bigcap_{c \in I(v)} N[c] = R_j$.

(iii) If $|I(v)| \geq 2$ and vertices in $I(v)$ are on the same column, then $\bigcap_{c \in I(v)} N[c] = P_i$.

(iv) If $|I(v)| = 3$ and all vertices in $I(v)$ are not on the same row or column, then $\bigcap_{c\in I(v)} N[c] = v$.

Proof. Let $v = (x_i, y_j)$.

(i) If we have $I(v) = \{(x_i, y_{j'}), (x_{i'}, y_j)\}$, when $i \neq i'$ and $j \neq j'$, then $\bigcap_{c\in I(v)} N[c] = \{(x_i, y_j), (x_{i'}, y_{j'})\}$.

(ii) If we have $\{(x_{i'}, y_j), (x_{i''}, y_j)\} \subseteq I(v)$, when $i' \neq i''$, then $\bigcap_{c\in I(v)} N[c] = R_j$.

(iii) If we have $\{(x_i, y_{j'}), (x_i, y_{j''})\} \subseteq I(v)$, when $j' \neq j''$, then $\bigcap_{c\in I(v)} N[c] = P_i$.

(iv) Without loss of generality, we may assume that there are two codewords in the same column as v and one in the same row. Hence we have $I(v) = \{(x_i, y_{j'}), (x_i, y_{j''}), (x_{i'}, y_j)\}$ for some i, j with $i \neq i'$ and $j' \neq j''$. This implies that $\bigcap_{c\in I(v)} N[c] = \{(x_i, y_j)\}$.

□

Let us first consider self-locating-dominating codes. We will give optimal cardinality of such codes in the next theorem.

Theorem 12. *Let $G = K_n \square K_m$ be a rook's graph with $m \geq n \geq 1$. We have*

$$\gamma^{SLD}(G) = \begin{cases} m, & m \geq 2n, \text{ or } n = 1, \\ 2n, & 2n > m > n \geq 2, \\ 2n-1, & m = n > 2, \\ 4, & n = m = 2. \end{cases}$$

Proof. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$, $V(K_m) = \{y_1, y_2, \ldots, y_m\}$ and $C \subseteq V(G)$ be an optimal self-locating-dominating code in G.

- **Fact 1**: Lemma 11 gives that for each non-codeword $v \in V(G)$ we have $|I(C; v)| \geq 3$ and we know that there are no rows or columns empty of codewords.

Let us first consider the case where $\mathbf{m \geq 2n}$. The fact 1 tells us that there has to be at least one codeword on each row. Hence, we get $\gamma^{SLD}(G) \geq m$. The condition $m \geq 2n$ includes most of the cases under $n = 1$ and the case $n = m = 1$ is clear.

By selecting as our code

$$C_1 = \{(x_i, y_j) \in V(G) \mid i - j \equiv 0 \mod n\}$$

we get $|C_1| = m$. Since $|C_1| = m \geq 2n$, there is exactly one codeword on each row and, as there are n vertices on each row, there are at least two vertices on each column. Therefore, we have at least three vertices which are not in the same row or column in each I-set of a non-codeword. Now we get from Lemma 11 that C_1 is a self-locating-dominating code.

Let us now consider the case where $\mathbf{2n > m > n}$. If we had $|C| \leq 2n - 1 \leq 2m - 3$, then there would be a column P_i and at least two rows R_j and $R_{j'}$ with only one codeword (or an empty row or column). Now at least one of the vertices (x_i, y_j) and $(x_i, y_{j'})$ is not a codeword. We can assume that $(x_i, y_j) \notin C$, now we have $|I(x_i, y_j)| = 2$ and based on the fact 1 C cannot be a self-locating-dominating code. Thus, we have $\gamma^{SLD}(G) \geq 2n$.

If we choose

$$C_2 = \{(x_i, y_j) \in V(G) \mid j - i = 0 \text{ or } j - i = m - n\},$$

we get $|C_2| = 2n$ and there are two vertices on each column and at least one vertex on each row. Therefore, we have at least three vertices which are not in the same row or column in each I-set of a non-codeword. Thus, based on Lemma 11, C_2 is a self-locating-dominating code. In Figure 2 code C_2 is illustrated for $K_5 \square K_7$.

Let us consider the case where $\mathbf{m = n > 2}$. If we had $|C| \leq 2n - 2 = 2m - 2$, then there would be at least two columns and rows with only one codeword (or an empty row or column). Hence, we can again choose a non-codeword v with $|I(v)| = 2$ and fact 1 tells that C cannot be a self-locating-dominating code. Now we have $\gamma^{SLD}(G) \geq 2n - 1$.

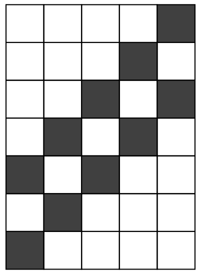


Figure 2: Optimal self-locating-dominating code for $K_5 \square K_7$.

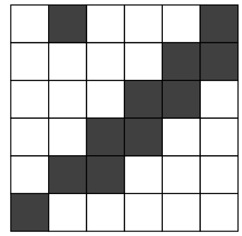


Figure 3: Optimal self-locating-dominating code for $K_6 \square K_6$.

If we choose

$$C_3 = \{(x_i, y_j) \in V(G) \mid i - j = 0,\ i - j = 1 \text{ or } (i, j) = (2, n)\} \setminus \{(x_2, y_1)\},$$

we have two vertices on each row and column except for R_1 and P_1. But since $R_1 \cap P_1 = (x_1, y_1) \in C_3$ each intersection of a row and column with a single codeword belongs to code C_3. Thus, for each non-codeword $v \in V(G)$, we have at least three vertices in $I(v)$ and they are not all in the same row or column. Now Lemma 11 says that C_3 is a self-locating-dominating code. We also have $|C_3| = 2n - 1$. In Figure 3 code C_3 is illustrated for $K_6 \square K_6$.

Let us finally consider the case $\mathbf{m = n = 2}$. If we have only three codewords in C, then the I-set of the non-codeword contains only two codewords and the intersection of their neighbourhoods contains two words. On the other hand, the whole graph only contains four vertices so we have $\gamma^{SLD}(K_2 \square K_2) = 4$. □

We will see in the next theorem that optimal solid-locating-dominating codes are mostly of the same size as optimal self-locating-dominating codes. However, this is only a superficial similarity. It will be seen in the proof that the structures of solid-locating-dominating codes vary more and there are more of them. For example, the codes in the Figures 4 and 5 are optimal solid-locating-dominating codes for $K_5 \square K_6$ and $K_5 \square K_5$. However, they are not self-locating-dominating codes.

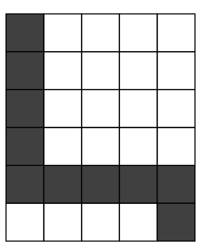


Figure 4: Optimal solid-locating-dominating code for $K_5 \square K_6$.

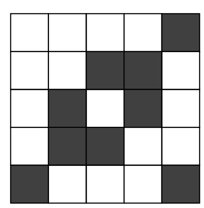


Figure 5: Optimal solid-locating-dominating code for $K_5 \square K_5$.

Theorem 13. *Let $G = K_n \square K_m$ be a rook's graph with $m \geq n \geq 1$. We have*

$$\gamma^{DLD}(G) = \begin{cases} m, & m \geq 2n \geq 4 \text{ or } n = 2, \\ 2n, & 2n > m > n > 2, \\ 2n - 1, & m = n > 2, \\ m - 1, & m > n = 1. \end{cases}$$

Proof. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$, $V(K_m) = \{y_1, y_2, \ldots, y_m\}$ and $C \subseteq V(G)$ be an optimal solid-locating-dominating code.

If there is a row R_i such that $R_i \cap C = \emptyset$ and there are no vertices with empty I-sets, then $P_j \subseteq \bigcap_{c \in I(x_j,y_i)} N[c]$ for each j. Thus, $C = V(G) \setminus R_i$.

- **Fact 2**: The code C, $|C| < m(n-1)$, is not a solid-locating-dominating code if there is a row or a column without any codewords.

Let us first consider the case where $\mathbf{m \geq 2n \geq 4}$. Fact 2 says that there has to be at least one codeword on each row so we have $\gamma^{DLD}(G) \geq m$. We also have $\gamma^{DLD}(G) \leq \gamma^{SLD}(G) = m$ by Theorem 12. Hence, $\gamma^{DLD}(G) = m$.

Let us next consider the case where $\mathbf{2n > m > n > 2}$. Theorem 12 gives an upper bound $\gamma^{DLD}(n) \leq \gamma^{SLD}(n) = 2n$. Let $|C| \leq 2n-1 \leq 2m-3$. Without loss of generality we may assume that the rows with a single codeword are consecutive rows and numbered as the first ones in the notation R_i and the same is true for the columns P_i. Denote the rows and columns which contain exactly one codeword as follows:

$$\mathcal{K} = \{R_1, \ldots, R_K\} \text{ and } \mathcal{T} = \{P_1, \ldots, P_T\}.$$

Since there are no empty rows or columns by fact 2 and $|C| \leq 2n-1 \leq 2m-3$, we have $K \geq 3$ and $T \geq 1$. Hence, $K + T \geq 4$.

Let us denote codewords on column P_i as (x_i, y_{s_i}), where $1 \leq i \leq T$ and codewords on row R_j as (x_{h_j}, y_j), where $1 \leq j \leq K$. Let us further denote

$$S = \{(x_i, y_j) \mid 1 \leq i \leq T, 1 \leq j \leq K\}.$$

If we have a codeword, say $(x_t, y_{s_t}) \in S$, then at least one of the vertices (x_1, y_{s_t}) or (x_2, y_{s_t}) is a non-codeword, say (x_1, y_{s_t}). Now $I(x_1, y_{s_t}) = \{(x_t, y_{s_t}), (x_1, y_{s_1})\}$ and $s_1 \neq s_t$. Furthermore, $\bigcap_{c \in I(x_1,y_{s_t})} N[c] = \{(x_t, y_{s_1}), (x_1, y_{s_t})\}$. Since column P_t has only one codeword neither of vertices $(x_t, y_{s_1}), (x_1, y_{s_t})$ belongs to C and C is not a solid-locating-dominating code. Thus, no vertex in S can be a codeword.

For each vertex $(x_i, y_j) \in S$ we have $I(x_i, y_j) = \{(x_{h_j}, y_j), (x_i, y_{s_i})\}$. Hence, in order for C to be a solid-locating-dominating code, vertex (x_{h_j}, y_{s_i}) has to be a codeword if $(x_i, y_j) \in S$. We can assume that codewords (x_{h_j}, y_j) in $\mathcal{K}$ are located in Y different columns and codewords (x_i, y_{s_i}) in $\mathcal{T}$ are located in Z different rows. Now each of the YZ vertices (x_{h_j}, y_{s_i}) has to be a codeword.

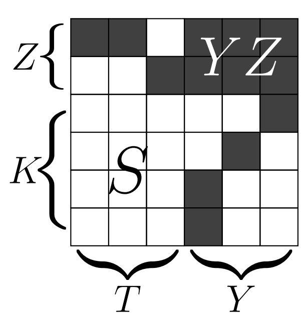


Figure 6: Part of a solid-locating-dominating code in a rook's graph with $Y = 3, Z = 2$, $T = 3, K = 4$.

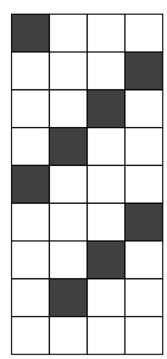


Figure 7: Optimal locating-dominating code for $K_4 \square K_9$.

Let a be a positive integer. Observe that if we have more than two codewords in a row, say $2 + a$ codewords, then there are a rows with exactly one codeword since we have $|C| \leq 2n-1$. The same is also true for columns. Hence we have at least $3 + T + YZ - 2Z$ rows with one codeword due to rows with multiple codewords since we have $T + YZ$ codewords on Z rows. Similarly we see that we have at least $1+K+YZ-2Y$ columns with one codeword due to columns with multiple codewords.

Thus, we get the following inequality which implies a contradiction:

$$K+T \geq 4+(K+YZ-2Y)+(T+YZ-2Z) = 4+K+T+2(YZ-Y-Z) \geq K+T+2.$$

The latter inequality is due to the fact that $YZ-Y-Z \geq -1$, when $Y,Z \geq 1$. Therefore, we have $|C| \geq 2n$.

Let us then consider the case $\mathbf{m}=\mathbf{n}>\mathbf{2}$. If $|C| \leq 2n-2 = 2m-2$, then $K \geq 2$, $T \geq 2$, $K+T \geq 4$ and $Y,Z \geq 1$. Now as in the previous case, there are no codewords in S, there is a $Y \times Z$ rectangle filled with codewords and we gain the same contradiction with similar reasoning. Hence the same arguments also apply here and we have $|C| \geq 2n-1$. Since $\gamma^{DLD}(G) \leq \gamma^{SLD}(G) = 2n-1$, we have $\gamma^{DLD}(G) = 2n-1$.

As the next case we consider $\mathbf{n}=\mathbf{2}$. If $|C|<m$, then there is a row without a codeword. On the other hand we can choose $C=P_1$ as our code. Thus $|C|=m$.

Finally as the last case we consider $\mathbf{m}>\mathbf{n}=\mathbf{1}$. If $|C|<m-1$, then there are two non-codewords with I-set equal to C and so C is not a solid-locating-dominating code. On the other hand if $C=V(G)\setminus\{v\}$, then $I(v)=C$ and $I(v)$ is unique as the only I-set of non-codeword. □

Finally, in the following theorem, we construct optimal locating-dominating codes in rook's graphs.

Theorem 14. *Let $G=K_n \square K_m$ be a rook's graph with $m \geq n \geq 1$. We have*

$$\gamma^{LD}(G) = \begin{cases} m-1, & m \geq 2n, \\ \lceil \frac{2n+2m}{3} \rceil - 1, & n \leq m \leq 2n-1. \end{cases}$$

Proof. Let $V(K_n)=\{x_1,x_2,\dots,x_n\}$, $V(K_m)=\{y_1,y_2,\dots,y_m\}$, $\gamma=\gamma^{LD}(K_n \square K_m)$ and $C \subseteq V(G)$ be an optimal locating-dominating code.

We first observe that if there are two rows R_i and R_j without codewords, then vertices (x_1,y_i) and (x_1,y_j) have the same I-set. The case for two columns without codewords is similar. If we have a row R_j and a column P_i without codewords, then $I(x_i,y_j)=\emptyset$.

- **Fact 3**: The total number of rows and columns without codewords in G is less than two.

Let us consider the case $\mathbf{m} \geq \mathbf{2n}$. If we had $|C| \leq m-2$, we would have at least two rows empty of codewords. Hence we have $\gamma > m-2$. We can choose

$$C_1 = \{(x_i,y_j) \mid i-j \equiv 0 \mod n, j \geq 2\}.$$

The code C_1 is illustrated in Figure 7 for values $n=4$ and $m=9$. Each non-codeword, which is not located in R_1 or P_1, has at least two codewords on the same column and one codeword on the same row. Hence, they are uniquely determined by Lemma 11. If the vertex is located in R_1, then $I(x_i,y_1)=\{(x_i,y_{i+kn}) \mid i+kn \in [2,m], k \in \mathbb{Z}\}$, which is a unique I-set since all other I-sets contain a codeword from two different columns. For each vertex (x_1,y_j), $j>1$, we have (x_1,y_{n+1}) in its I-set and some codeword from the row R_j. Hence vertices on P_1 have unique I-set. Note that column P_1 has to be considered only when $m=2n$.

Let us then consider the case $\mathbf{n} \leq \mathbf{m} < \mathbf{2n}$. Let

- s_p denote the number of columns with exactly one codeword,
- s_{p0} denote the number of columns without codewords,
- s_r denote the number of rows with exactly one codeword and
- s_{r0} denote the number of rows without codewords.

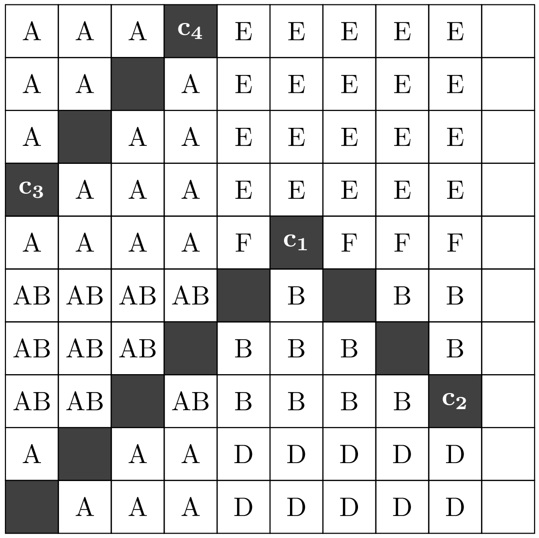


Figure 8: Optimal locating-dominating code for $K_{10}\square K_{10}$.

We can assume that $s_{p0} + s_{r0} \le 1$. If we had $s_{p0} + s_{r0} \ge 2$, then we would not have a locating-dominating code by the fact 3.

By counting the size of the code column by column, we get

$$\gamma \ge 0 \cdot s_{p0} + s_p + 2(n - s_p - s_{p0})$$

which gives us

$$s_p \ge 2n - \gamma - 2s_{p0}. \tag{1}$$

When we count the size of the code row by row, we get similarly

$$s_r \ge 2m - \gamma - 2s_{r0}. \tag{2}$$

If we have two codewords c_1 and c_2 with I-sets $I(c_i) = \{c_i\}$, $1 \le i \le 2$, then C is not a locating-dominating code. Let $c_1 = (x_{i_1}, y_{j_1})$ and $c_2 = (x_{i_2}, y_{j_2})$. Now $I(x_{i_2}, y_{j_1}) = I(x_{i_1}, y_{j_2}) = \{c_1, c_2\}$. Hence our s_r rows with exactly one codeword and s_p columns with exactly one codeword share at most one codeword. Now we get from inequalities (1) and (2)

$$\gamma \ge s_p + s_r - 1 \ge 2n + 2m - 2\gamma - 2(s_{p0} + s_{r0}) - 1,$$

$$\gamma \ge \left\lceil \frac{2(m+n)}{3} \right\rceil - 1.$$

We can choose

$$C_2 = A_1 \cup A_2 \cup A_3, \text{ where}$$

$$A_1 = \left\{ (x_i, y_j) \mid i = j \le \left\lfloor \frac{m+n}{3} \right\rfloor \right\}, A_2 = \left\{ (x_i, y_j) \mid j - i = \left\lfloor \frac{m+n}{3} \right\rfloor \right\} \text{ and}$$

$$A_3 = \left\{ (x_i, y_j) \mid i + j = 2\left\lfloor \frac{m+n}{3} \right\rfloor \text{ and } \left\lfloor \frac{m+n}{3} \right\rfloor + 1 \le i \le n-1 \right\}$$

(and if $n = 2$, $m = 3$ we can choose $C_2 = P_1$). Now we have $|C_2| = \lfloor \frac{m+n}{3} \rfloor + (n - 1 - \lfloor \frac{m+n}{3} \rfloor) + (m - \lfloor \frac{m+n}{3} \rfloor) = \lceil \frac{2m+2n}{3} \rceil - 1$. In Figure 8, we have an optimal locating-dominating code for $K_{10}\square K_{10}$. The labelling of areas in what will follow corresponds to that of the figure. Codeword c_1 has coordinates $\left(x_{\lfloor \frac{m+n}{3} \rfloor}, y_{\lfloor \frac{m+n}{3} \rfloor}\right)$, c_2 has coordinates $\left(x_{n-1}, y_{2\lfloor \frac{m+n}{3} \rfloor + 1 - n}\right)$, c_3 has coordinates

$\left(x_1, y_{\lfloor \frac{m+n}{3} \rfloor+1}\right)$ and c_4 has coordinates $\left(x_{m-\lfloor \frac{m+n}{3} \rfloor}, y_m\right)$. The I-set of a vertex on P_n is the set of codewords on the same row as it is. All other I-sets have also vertices from different rows. Hence the vertices on P_n have unique I-set. The non-codewords on columns P_i, $1 \leq i \leq m - \lfloor \frac{m+n}{3} \rfloor$ have at least three codewords in their I-sets of which two are on the same column and at least one on the same row. Thus by Lemma 11 they have unique I-set. Let us denote the set of these vertices by A. The vertices in set B on rows R_j, $2\lfloor \frac{m+n}{3} \rfloor - n + 1 \leq j \leq \lfloor \frac{m+n}{3} \rfloor - 1$, have at least three codewords in their I-sets if they are not on column P_n. Out of these three codewords two are on the same row and at least one on the same column. Hence by Lemma 11 they have unique I-set.

The vertices D on (x_i, y_j), $m - \lfloor \frac{m+n}{3} \rfloor + 1 \leq i \leq n - 1$, $1 \leq j \leq 2\lfloor \frac{m+n}{3} \rfloor - n \leq m - \lfloor \frac{m+n}{3} \rfloor$, have codeword (x_j, y_j) in their I-set and one codeword from a different row. Thus by Lemma 11 there is only one other vertex which has these codewords in its I-set. However, the other vertex would have to be in A and vertices in A have unique I-set. Vertices E; (x_i, y_j), $m - \lfloor \frac{m+n}{3} \rfloor + 1 \leq i \leq n - 1$, $\lfloor \frac{m+n}{3} \rfloor + 1 \leq j \leq m$, have two vertices in their I-set. One of which is from columns P_i, $1 \leq i \leq m - \lfloor \frac{m+n}{3} \rfloor$ and the other one is from a different row. Hence by Lemma 11 the only other vertex that could have the same I-set is in A but vertices in A have unique I-set.

Finally the vertices F (x_i, y_j), $m - \lfloor \frac{m+n}{3} \rfloor + 1 \leq i \leq n-1$, $j = \lfloor \frac{m+n}{3} \rfloor$, have two vertices in their I-sets. Codeword $(\lfloor \frac{m+n}{3} \rfloor, \lfloor \frac{m+n}{3} \rfloor)$ and a codeword from a different row R_l with $2\lfloor \frac{m+n}{3} \rfloor - n + 1 \leq l \leq \lfloor \frac{m+n}{3} \rfloor - 1$. The only other vertex that could share this I-set is located on rows R_l, but such vertices had at least three vertices in their I-sets. $\square$

In conclusion, by the previous theorems, we determine the cardinalities of optimal locating-dominating, self-locating-dominating and solid-locating-dominating codes in all graphs $K_m \square K_n$.

3 Location-domination in the binary Hamming spaces

In this section, we consider self-locating-dominating and solid locating-dominating codes in binary Hamming spaces of length n. A binary Hamming space of length n is a graph with the vertex set $\mathbb{F}^n = \{0,1\}^n$, and two vertices have an edge between them if they differ in exactly one coordinate. Vertices of $\mathbb{F}^n$ are called *words*. The distance $d(x, y)$ is the number of coordinates where words x and y differ. We define $\mathbf{0}$ and $\mathbf{1}$ as the all zero word and respectively all one word. We define e_i as the almost all zero word which has a 1 at i'th coordinate. The *weight* $w(x)$ of a word $x \in \mathbb{F}^n$ is the number of coordinates equal to 1, i.e., $w(x) = d(x, \mathbf{0})$. When we speak about the *cover* of a word x, we mean the cardinality $|I(x)|$. The sizes of optimal self-locating-dominating and solid-locating-dominating codes in $\mathbb{F}^n$ are denoted by $\gamma^{SLD}(\mathbb{F}^n) = \gamma^{SLD}(n)$ and $\gamma^{DLD}(\mathbb{F}^n) = \gamma^{DLD}(n)$, respectively.

In what follows, we first concentrate on the case of self-locating-dominating codes. In particular, we present an infinite family of optimal self-locating-dominating codes in binary Hamming spaces. This result is based on the observation that a code C is self-locating-dominating in $\mathbb{F}^n$ if and only if for each $x \in \mathbb{F}^n \setminus C$ we have $|I(C;x)| \geq 3$ (see Theorem 16). An analogous result for self-identifying codes has been presented in [7]: a code C is self-identifying in $\mathbb{F}^n$ if and only if for each $x \in \mathbb{F}^n$ we have $|I(C;x)| \geq 3$.

The following well-known observation is useful in the following proofs of the paper.

Observation 15. Let $a, b \in \mathbb{F}^n$. We have

$$\big|N[a] \cap N[b]\big| = \begin{cases} 0, & \text{if } d(a,b) \geq 3, \\ 2, & \text{if } d(a,b) = 2, \\ 2, & \text{if } d(a,b) = 1, \\ n+1, & \text{if } a = b. \end{cases}$$

Theorem 16. *A code C is a self-locating-dominating code in $\mathbb{F}^n$ if and only if for each non-codeword w we have $|I(w)| \geq 3$.*

Proof. By Observation 15 we have that if two non-codewords w and w' have at least three common neighbours, then they are the same word. On the other hand if w has only two codewords in its I-set, then there is another word which has those same codewords in its I-set. □

Now we get the lower bound for self-locating-dominating codes.

Theorem 17. *Let $n \geq 3$. We have*

$$\gamma^{SLD}(n) \geq \left\lceil \frac{3 \cdot 2^n}{n+3} \right\rceil.$$

Proof. Let C be a self-locating-dominating code in $\mathbb{F}^n$. Theorem 16 says that each non-codeword w has at least three codewords in $I(w)$. We also have $|I(c)| \geq 1$ for all $c \in C$. Thus by double counting pairs (c, x) such that $c \in C$, $x \in \mathbb{F}^n$ and $d(c, x) \leq 1$, we get $(n+1)|C| \geq |C| + 3|\mathbb{F}^n \setminus C| = 3 \cdot 2^n - 2|C|$. This gives us

$$|C| \geq \left\lceil \frac{3 \cdot 2^n}{n+3} \right\rceil.$$

□

In the proof of the following theorem, we are going to need some basics of linear codes. For more details, the interested reader is referred to [14]. The binary Hamming space $\mathbb{F}^n$ is a vector space under the normal addition of vectors and multiplication with scalars. We call code C *linear* if it is a subspace of $\mathbb{F}^n$. If C is a linear code, then we call H its *parity-check matrix* if $Hx^T = \mathbf{0}$ if and only if $x \in C$. If $Hy^T = d$, then we get a codeword of C by finding columns of H which form d as their sum and adding e_i to y if i'th column is in the sum. We denote the *covering radius* of C by $R(C) = \max_{x \in \mathbb{F}^n} \min_{c \in C} d(x, c)$. Hence, we have $R(C) = 1$ if each word has a codeword in its closed neighbourhood.

Theorem 18. *Let n and k be positive integers such that $n = 3(2^k - 1)$. We have*

$$\gamma^{SLD}(n) = 2^{3(2^k-1)-k}.$$

Proof. Theorem 17 gives us the lower bound $|C| \geq \frac{3 \cdot 2^n}{n+3} = 2^{3(2^k-1)-k}$.

Let C be a linear code such that its $k \times n$ parity-check matrix H, where $k \in \mathbb{Z}_+$ and $n = 3 \cdot (2^k - 1)$, contains each non-zero column of $\mathbb{F}^k$ three times and no zero columns. We now have $R(C) = 1$ since each non-zero word is a column of H. Furthermore, each non-codeword is covered by three codewords since there are three copies of each non-zero column. The cardinality of the code is $|C| = 2^{n-k} = 2^{3(2^k-1)-k}$ and it is a self-locating-dominating code by Theorem 16. □

Let $C \subseteq \mathbb{F}^n$ and $D \subseteq \mathbb{F}^m$ be codes. The *direct sum* of C and D is defined as $C \oplus D = \{(x, y) \mid x \in C, y \in D\}$. In the following theorem, it is shown that new self-locating-dominating codes can be formed from known ones using a direct sum.

Theorem 19. *If $C \subseteq \mathbb{F}^n$ is a self-locating-dominating code in $\mathbb{F}^n$, then $D = C \oplus \mathbb{F}$ is also a self-locating-dominating code in $\mathbb{F}^{n+1}$.*

Proof. Let $(a, x) \in \mathbb{F}^{n+1}$ where $a \in \mathbb{F}^n, x \in \mathbb{F}$ and $a \notin C$. We have $I(D; (a, x)) = \{(c, x) \mid c \in I(C; a)\}$. Since $|I(C; a)| \geq 3$, also $|I(D; (a, x))| \geq 3$. Therefore, D is a self-locating-dominating code. □

Let us then concentrate on solid-locating-dominating codes. We will first give a lower bound such that its ratio to $2\frac{2^n}{n+1}$ approaches 1 as n tends to infinity. After that we will give an infinite sequence of solid-locating-dominating codes with the same limit. When we compare the sizes of optimal self-locating-dominating and solid-locating-dominating codes we see from Theorems 17 and 18 that optimal solid-locating-dominating codes are essentially smaller. In the following theorem, we first give a lower bound for solid-locating-dominating codes.

Theorem 20. *Let n be an integer such that $n \geq 5$. We have*

$$\gamma^{DLD}(n) \geq \left\lceil \left(1 + \frac{n-1}{n^2+n+2}\right) \frac{2^{n+1}}{n+1} \right\rceil .$$

Proof. Let $C \subseteq \mathbb{F}^n$ be a solid-locating-dominating code. Let us define sets

$$C_i = \{c \in C \mid |I(c)| = i\} \text{ and } N_i = \{v \notin C \mid |I(v)| = i\}.$$

Let us also denote $C_{i^+} = \bigcup_{j=i}^{n+1} C_j$ and $N_{i^+} = \bigcup_{j=i}^{n} N_j$.

Each of the $|C|$ codewords covers $n+1$ words. Hence together they give total of $(n+1)|C|$ cover. On the other hand, if each of the $|\mathbb{F}^n|$ words is covered on average by at least $2 + \frac{2n-2}{n^2+n+2}$ codewords, then we have an inequality $(n+1)|C| \geq (2 + \frac{2n-2}{n^2+n+2})|\mathbb{F}^n|$ which gives the desired result when we solve $|C|$.

If $w \in N_1$ and $I(w) = \{c\}$, then $c \in C_n$. Otherwise, there would be a non-codeword v such that $I(w) \setminus I(v) = \emptyset$ which would mean that C is not a solid-locating-dominating code. If $w \in N_2$ and $I(w) = \{c_1, c_2\}$, then $N(c_1) \cap N(c_2) = \{w, w'\}$. We have $I(w) \subseteq I(w')$ and this implies that $w' \in C$ as in Figure 9.

When $n \geq 5$, we have $2 + \frac{2n-2}{n^2+n+2} \leq \frac{9}{4}$ and we will be moving covers from words to words according to following three rules:

R1. If $x \in N_i$, $3 \leq i \leq n$, we will be moving $\frac{1}{4}$ cover from it to each codeword in $N(x)$.

R2. If $c \in C_{3^+}$ and $N(c) \cap N_1 = \emptyset$, then we move $\frac{2n-2}{n^2+n+2}$ cover from c to each codeword in $I(c)$ (including c itself) and each word which is neighbour to two words in $I(c) \setminus \{c\}$ as in Figure 10.

R3. If $c \in C_n$ and $N(c) \cap N_1 = \{x\}$, then we first move one cover from c to x and then we move $\frac{n-3}{2+\binom{n-1}{2}}$ cover from c to x, c and each word which is neighbour to two words in $I(c) \setminus \{c\}$ as in Figure 11.

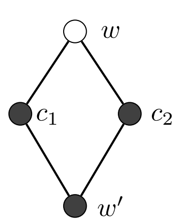


Figure 9: If $w \in N_2$, then $w' \in C$.

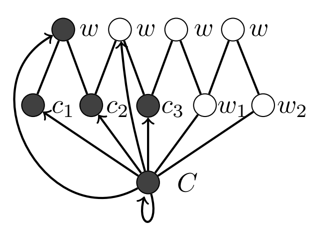


Figure 10: Rule R2: c gives cover to words pointed by arrows.

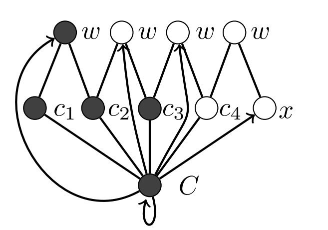


Figure 11: Rule R3: c gives cover to words pointed by arrows.

We immediately notice that we never move cover away from a word with two different rules. We will next go through all types of words in the following order: N_{3^+}, C_1, $c \in C_{3^+}$ with $N(c) \cap N_1 = \emptyset$, N_1, $c \in C_n$ for which $N(c) \cap N_1 \neq \emptyset$, N_2 and finally C_2.

Let us first consider words $x \in N_i$, $3 \leq i \leq n$. If we move $\frac{1}{4}$ cover from x to i codewords in its I-set, then x has $i - \frac{i}{4} \geq \frac{9}{4}$ cover left.

If codeword c is in C_1, then each word $x \in N(c)$ is in N_{3^+}. Since if $x \in N_1$, then $c \in C_n$ and if $x \in N_2$, then c has a codeword neighbour. Hence c has n neighbours which are in N_{3^+} and each of them gives $\frac{1}{4}$ cover to c by R1. Thus c has at least $\frac{9}{4}$ cover since $n \geq 5$.

Let us next consider a codeword $c \in C_{n+1-K}$ such that either $K = 1$ and $N(c) \cap N_1 = \emptyset$ or $K \in \{i \mid 0 \leq i \leq n-2, i \neq 1\}$. We move $\frac{2n-2}{n^2+n+2}$ cover from c to $n + 1 - K + \binom{n-K}{2}$ words. After

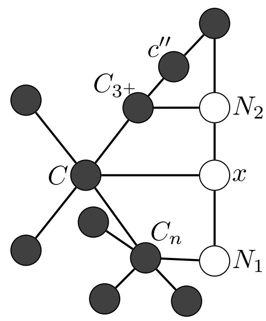


Figure 12: A solid-locating-dominating code when $x \in N_1$. Darker vertices are in code C.

that c has

$$\begin{aligned}
&n+1-K-\left(n+1-K+\binom{n-K}{2}\right)\frac{2n-2}{n^2+n+2}+\frac{2n-2}{n^2+n+2}\\
\overset{(*)}{\geq}&n+1-K-\left(n+1-K+\binom{n-K}{2}\right)\frac{n-1-K}{n+1-K+\binom{n-K}{2}}+\frac{2n-2}{n^2+n+2}\\
=&2+\frac{2n-2}{n^2+n+2}
\end{aligned} \tag{3}$$

cover. We get the inequality $(*)$ from the fact that $f(x)=\frac{x-1}{x+1+\binom{x}{2}}$ is decreasing when $x \geq 3$ and $f(2)=f(5)$. Hence $f(n-K)=\frac{n-1-K}{n+1-K+\binom{n-K}{2}}$ gets its minimum value when $K=0$ at $f(n)=\frac{2n-2}{n^2+n+2}$ since $n \geq 5$.

Let $x \in N_1$ and $\{c\}=I(x)$. Hence $c \in C_n$ and x has $1+1+\frac{n-3}{2+\binom{n-1}{2}}$ cover by R3. We have

$$\frac{n-3}{2+\binom{n-1}{2}} \geq \frac{2n-2}{n^2+n+2}, \text{ when } n \geq 5.$$

Hence x has enough cover.

When $c \in C_n$ and $x \in N_1 \cap N(c)$, we move $\frac{n-3}{2+\binom{n-1}{2}}$ cover from c to x, c and $\binom{n-1}{2}$ words neighbouring exactly two of codewords in $N(c)$ and 1 more cover to x in the rule R3. Now c has $n-1-\left(2+\binom{n-1}{2}\right)\frac{n-3}{2+\binom{n-1}{2}}+\frac{n-3}{2+\binom{n-1}{2}}=2+\frac{n-3}{2+\binom{n-1}{2}}$ cover left which is enough.

Let $x \in N_2$ and $I(x)=\{c_1,c_2\}$. Hence $N(c_1)\cap N(c_2)=\{x,c\}$. Since C is a solid-locating-dominating code, we have that $c \in C$. Thus $|I(c)| \geq 3$ and c gives to x either $\frac{2n-2}{n^2+n+1}$ cover by R2 or $\frac{n-3}{2+\binom{n-1}{2}}$ cover if $|I(c)|=n$ and $N(c)\cap N_1 \neq \emptyset$ by R3. Hence x has at least $2+\frac{2n-2}{n^2+n+2}$ cover.

Let $c \in C_2$ and $I(c)=\{c,c'\}$. If $c' \in C_2$, then c has $n-1$ non-codewords in N_{3+} in its neighbourhood. These words are in N_{3+}, since they clearly cannot be in N_1 and if $I(w)=\{c,c''\}$ for some $w \in N_2$, then c'' and c have a common codeword in their I-set but this is impossible, since $I(c)=I(c')=\{c,c'\}$ and $c' \neq c''$. Now each non-codeword in $N(c)$ gives c at least $\frac{1}{4}$ cover by R1. If $n \geq 5$, then it will have at least three cover. If $c' \in C_{3+}$ and $N(c')\cap N_1=\emptyset$, then c' gives $\frac{2n-2}{n^2+n+1}$ cover to c by R2 and c has enough cover. If $c' \in C_n$ and $N(c')\cap N_1=\{x\}$, then our situation is as in Figure 12 and c and x have a common non-codeword neighbour v which is in N_{3+} since if $v \in N_1$, then $c \in C_n$ and if $v \in N_2$, then $\bigcap_{y\in I(v)} N[y]=\{v,c''\}$, so $c'' \in C$ and $c'' \neq c'$ so $\{c,c',c''\}\subseteq I(c)$ which is a contradiction. Since c has a neighbour in N_{3+}, it has at least $\frac{9}{4}$ cover by R1.

Now we have considered every word and each of them has at least $2+\frac{2n-2}{n^2+n+2}$ cover. □

In the following remark, we briefly compare the previously obtained lower bound to one for locating-dominating codes locating multiple irregularities.

Remark 21. In this paper, we have mainly studied locating-dominating codes which can locate one and detect multiple irregularities. Previously, in [8], so called $(1, \leq \ell)$-locating-dominating codes of type B ($(1, \leq \ell)$-LDB codes for short), which can locate up to ℓ irregularities, have been studied. In [8, Theorem 5], the lower bound $\left\lceil \frac{2^{n+1}}{n+1} \right\rceil$ for $(1, \leq 2)$-LDB codes has been achieved. Since it can be shown that every $(1, \leq 2)$-LDB code is also a solid-locating-dominating code, our lower bound in Theorem 20 improves the lower bound for $(1, \leq 2)$-LDB codes in Hamming spaces.

When $n \geq 5$, the lower bound in Theorem 20 is attained by choosing as codewords all codewords and their neighbours of a code with covering radius two and minimum distance five. Unfortunately, codes like this are only known when $n = 5$ [2, Theorem 11.2.2]. Using this code, the following theorem is obtained.

Theorem 22. *We have* $\gamma^{DLD}(5) = 12$.

Proof. We have $\gamma^{DLD}(5) \geq \lceil (2 + \frac{10-2}{5^2+5+2}) \cdot \frac{2^5}{5+1} \rceil = 12$ by Theorem 20. We can choose as our code C all words of weight $0, 1, 4$ or 5 since then each non-codeword is covered by two codewords and the intersection of 1-balls of these two codewords contains a codeword ($\mathbf{1}$ or $\mathbf{0}$). □

In general, we can construct solid-locating-dominating codes from codes with covering radius two.

Theorem 23. *If* $D \subseteq \mathbb{F}^n$ *is a code with* $R(D) = 2$*, then the code*

$$C = \{c \in \mathbb{F}^n \mid c \in N[d], d \in D\}$$

is solid-locating-dominating.

Proof. Since D has covering radius of two, each non-codeword w is covered by at least two codewords $x, y \in C$. These codewords on the other hand have a common codeword d, the one which is also in D. Now $\{w, d\} \subseteq \bigcap_{c \in I(w)} N[c]$ so C is a solid-locating-dominating code. □

In [2, Theorem 4.5.8], Struik has constructed an infinite sequence of codes with covering radius two such that we can build on top of it such a sequence of solid-locating-dominating codes that they converge to our lower bound. We denote the cardinality of a ball with radius 2 in $\mathbb{F}^n$ with $V(n, 2)$.

Theorem 24. *There exists a sequence of solid-locating-dominating codes* $(C_n)_{n=1}^{\infty}$ *such that*

$$\lim_{n \to \infty} \frac{|C_n|}{2\frac{2^n}{n+1}} = 1.$$

Proof. Struik has constructed a sequence of codes $(D_n)_{n=1}^{\infty}$ with covering radius two such that $\frac{|D_n| V(n,2)}{2^n} \xrightarrow{n \to \infty} 1$. If we choose $C_n = \{x \in \mathbb{F}^n \mid a \in D_n, x \in N[a]\}$, this is a solid-locating-dominating code by Theorem 23. We have $|C_n| \leq (n+1)|D_n|$. Hence

$$\frac{|C_n|}{2\frac{2^n}{n+1}} \leq \frac{(n+1)|D_n|}{2\frac{2^n}{n+1}} = \frac{|D_n| V(n,2)}{2^n} + \frac{|D_n|(\frac{n}{2} - \frac{1}{2})}{2^n} \xrightarrow{n \to \infty} 1.$$

On the other hand we have from Theorem 20 $\frac{|C_n|}{2\frac{2^n}{n+1}} \geq \frac{(1 + \frac{n-1}{n^2+n+2}) \frac{2^{n+1}}{n+1}}{2\frac{2^n}{n+1}} \xrightarrow{n \to \infty} 1$, which proofs the theorem. □

Using direct sum we can construct new solid-locating-dominating codes from existing ones in a similar fashion as with self-locating-dominating codes.

Theorem 25. *If* $C \subseteq \mathbb{F}^n$ *is a solid-locating-dominating code, then code* $D = C \oplus \mathbb{F}$ *is also solid-locating-dominating.*

Proof. Let $(a, x) \in \mathbb{F}^{n+1}$ where $a \in \mathbb{F}^n, x \in \mathbb{F}$ and $a \notin C$. We have $I(D; (a, x)) = \{(y, x) \mid y \in I(C; a)\}$. If $|I(C; a)| \geq 3$, also $|I(D; (a, x))| \geq 3$. If $I(C; a) = \{c_1, c_2\}$, then there is a codeword $c_3 \in N(c_1) \cap N(c_2)$. Now we have $I(D; (a, x)) = \{(c_1, x), (c_2, x)\}$ and $(c_3, x) \in N(c_1, x) \cap N(c_2, x)$. If $I(C; a) = \{c\}$, then $|I(C; c)| = n$ and $|I(D; (c, x))| = n + 1$. Since $I(D; (a, x)) = \{(c, x)\}$, D is a solid-locating-dominating code. □

For small lengths n the sizes of optimal self-locating-dominating and solid-locating-dominating codes in $\mathbb{F}^n$ are presented in Table 1. The lower bounds of $\gamma^{DLD}(n)$ for $n \leq 4$ as well as $\gamma^{SLD}(1)$, $\gamma^{SLD}(2)$ and $\gamma^{SLD}(5)$ are achieved with computer search. The lower bounds of $\gamma^{SLD}(3)$ and $\gamma^{SLD}(4)$ are due to the fact $\gamma^{SLD}(n) \geq \gamma^{DLD}(n)$. The rest of the lower bounds are due to Theorems 17 and 20. The upper bound of $\gamma^{SLD}(1)$ comes from the size $|\mathbb{F}| = 2$ and the upper bound of $\gamma^{DLD}(1)$ is gained with the code $C = \{0\}$. The upper bounds of $\gamma^{SLD}(2)$, $\gamma^{SLD}(4)$, $\gamma^{SLD}(5)$ are from Theorem 19 and the upper bounds of $\gamma^{DLD}(2)$, $\gamma^{DLD}(3)$ and $\gamma^{DLD}(4)$ are from Theorem 25. We get $\gamma^{DLD}(5)$ from Theorem 22, $\gamma^{SLD}(3)$ is gained with the code $C = \{x \mid w(x) = 0 \text{ or } w(x) = 2\}$, the upper bound for $\gamma^{SLD}(6)$ with the code $C = \{x \in \mathbb{F}^6 \mid x \in A, w(x) = 1, w(x) = 4 \text{ or } w(x) = 6\}$, where $A = \{(1,1,1,0,0,0), (1,0,0,1,1,0), (1,1,0,0,0,1), (0,1,0,1,1,0), (0,0,1,1,0,1), (0,0,1,0,1,1)\}$ and the upper bound for $\gamma^{DLD}(6)$ with the code $C = \{x \in \mathbb{F}^6 \mid w(x) = 0, 1, 4 \text{ or } 6\}$.

n	$\gamma^{SLD}(n)$	$\gamma^{DLD}(n)$
1	2	1
2	4	2
3	4	4
4	8	8
5	16	12
6	$[22, 28]$	$[21, 23]$

Table 1: Optimal self-locating-dominating and solid-locating-dominating codes in binary Hamming spaces of short lengths.

Above, we have discussed self-locating-dominating and solid-locating-dominating codes in binary Hamming spaces. In what follows, we briefly consider regular locating-dominating codes. In particular, for certain lengths, we provide locating-dominating codes with the smallest known cardinalities. Previously, locating-dominating codes in $\mathbb{F}^n$ have been considered, for example, in [3, 8]. For future considerations, we first define the mapping $\pi : \mathbb{F}^n \to \mathbb{F}$ as follows:

$$\pi(u) = \begin{cases} 0, & \text{if } w(u) \text{ is even;} \\ 1, & \text{if } w(u) \text{ is odd.} \end{cases}$$

In the following theorem we introduce a novel approach for constructing new locating-dominating codes based on known (suitable) identifying codes.

Theorem 26. *Let C be an identifying code in $\mathbb{F}^n$ such that $|I(C; u)| \geq 2$ for all $u \in \mathbb{F}^n \setminus C$. Then*

$$D = \{(\pi(u), u, u + c) \mid u \in \mathbb{F}^n, c \in C\}$$

is a locating-dominating code in $\mathbb{F}^{2n+1}$.

Proof. Let a be an element of $\mathbb{F}$, u and v be words of $\mathbb{F}^n$ and $x = (a, u, u + v)$ be a word of $\mathbb{F}^{2n+1}$. Assume further that $I(C; v) = \{c_1, c_2, \ldots, c_k\}$ for some positive integer k. Then we have the following observations:

- If $a = \pi(u)$, then we have $I(D; x) = \{(a, u, u + c_1), (a, u, u + c_2), \ldots, (a, u, u + c_k)\}$.
- Assume then that $a \neq \pi(u)$. If v is not a codeword of C, then we have $I(D; x) = \{(a, u+v+c_1, u+v), (a, u+v+c_2, u+v), \ldots, (a, u+v+c_k, u+v)\}$. Indeed, the word $(a, u+v+c_i, u+v)$

belongs to $I(D;x)$ since its distance from x is equal to 1 and $(a, u+v+c_i, u+v) = (a, u+v+c_i, (u+v+c_i)+c_i) \in D$. If v is a codeword of C, say $v = c_1$, then we similarly have $I(D;x) = \{(a+1, u, u+v), (a, u+v+c_2, u+v), \ldots, (a, u+v+c_k, u+v)\}$.

Assume then that x is not a codeword of D. By the previous observation, we first obtain that $I(D;x) \neq \emptyset$ as $I(C;v) \neq \emptyset$. Furthermore, if $|I(D;x)| \geq 3$, then the identifying set of x immediately identifies the word x by Observation 15. Hence, we may assume that $|I(D;x)| \leq 2$. In what follows, we first suppose that $|I(D;x)| = 2$ implying $|I(C;v)| = 2$.

Assume first that $a = \pi(u)$. Then, by the previous observation, we have $I(D;x) = \{(a, u, u+c_1), (a, u, u+c_2)\}$. Assume then (to the contrary) that there exists $y \in \mathbb{F}^{2n+1} \setminus D$ such that $x \neq y$ and $I(D;x) = I(D;y)$. Since $(a, u, u+v+c_1+c_2)$ is the unique word in the set $(N[(a, u, u+c_1)] \cap N[(a, u, u+c_2)]) \setminus \{x\}$, we obtain that $y = (a, u, u+v+c_1+c_2)$. Therefore, as $I(D;x) = I(D;y)$, we have $I(C; v+c_1+c_2) = I(C;v)$. However, this is a contradiction since $v+c_1+c_2 \neq v$ and C is an identifying code in $\mathbb{F}^n$.

Assume then that $a \neq \pi(u)$. If v is not a codeword of C, then $I(D;x) = \{(a, u+v+c_1, u+v), (a, u+v+c_2, u+v)\}$ by the previous observation. Assume now (to the contrary) that there exists $y \in \mathbb{F}^{2n+1} \setminus D$ such that $x \neq y$ and $I(D;x) = I(D;y)$. Then we obtain that $y = (a, u+c_1+c_2, u+v)$ since $(a, u+c_1+c_2, u+v)$ is the unique word in the set $(N[(a, u+v+c_1, u+v)] \cap N[(a, u+v+c_2, u+v)]) \setminus \{x\}$. Denoting $u' = u+c_1+c_2$ and $v' = v+c_1+c_2$, we have $\pi(u) = \pi(u')$, $y = (a, u', u'+v')$ and $I(D;y) = \{(a, u'+v'+c_1, u'+v'), (a, u'+v'+c_2, u'+v')\}$. Therefore, as $I(D;x) = I(D;y)$, it follows that $I(C;v) = I(C;v')$ (a contradiction). Hence, we may assume that v is a codeword of C, say $v = c_1$. Then we have $I(D;x) = \{(a+1, u, u+v), (a, u+v+c_2, u+v)\}$. Now we obtain that $y = (a+1, u+v+c_2, u+v)$ since it is the unique word in the set $(N[(a+1, u, u+v)] \cap N[(a, u+v+c_2, u+v)]) \setminus \{x\}$. Denoting $a' = a+1$, $u' = u+v+c_2$, $v' = c_2$ and $c_2' = c_1$, we have $y = (a', u', u'+v')$ and $I(D;y) = \{(a'+1, u', u'+v'), (a', u'+v'+c_2', u'+v')\}$. Therefore, as $I(D;x) = I(D;y)$, it follows that $I(C;v) = \{v, c_2\} = \{v', c_2'\} = I(C;v')$ (a contradiction).

Finally, we assume that $|I(D;x)| = 1$. This implies that $|I(C;v)| = 1$. Hence, as $|I(C;u)| \geq 2$ for all $u \in \mathbb{F}^n \setminus C$, we know that $v \in C$. Then we may assume that $a \neq \pi(u)$ as otherwise $x = (a, u, u+v)$ belongs to D. Now, by the previous observation, we have $I(D;x) = \{(a+1, u, u+v)\}$. Assume to the contrary that there exists $y = (a', u', u'+v') \in \mathbb{F}^{2n+1} \setminus D$ such that $I(D;x) = I(D;y)$. As above, we obtain that $v' \in C$ and $I(D;y) = \{(a'+1, u', u'+v')\}$. Therefore, as $I(D;x) = I(D;y)$, we have $a' = a$, $u' = u$, $v' = v$ and $x = y$ (a contradiction). Thus, in conclusion, we have shown that D is a locating-dominating code in $\mathbb{F}^{2n+1}$. □

The best known upper bounds on $\gamma^{LD}(\mathbb{F}^n)$ for $1 \leq n \leq 10$ have been presented in [3, Table 3]. For lengths $n > 10$, the smallest known locating-dominating codes are actually identifying codes. (Recall that by the definitions any identifying code is also locating-dominating.) The currently best known upper bounds on $\gamma^{ID}(\mathbb{F}^n)$ can be found in [1]. In the following corollary, we present locating-dominating codes in $\mathbb{F}^n$ with the smallest known cardinalities for the lengths $n = 11$ and $n = 17$. These constructions significantly improve on the known upper bounds $\gamma^{LD}(\mathbb{F}^{11}) \leq \gamma^{ID}(\mathbb{F}^{11}) \leq 352$ and $\gamma^{LD}(\mathbb{F}^{17}) \leq \gamma^{ID}(\mathbb{F}^{17}) \leq 18558$.

Corollary 27. *We have* $\gamma^{LD}(\mathbb{F}^{11}) \leq 320$ *and* $\gamma^{LD}(\mathbb{F}^{17}) \leq 16384$.

Proof. Let C_1 be a code in $\mathbb{F}^5$ formed by the the words of weight 1 and 4. In [11], it has been shown that C_1 is an identifying code in $\mathbb{F}^5$ (with 10 codewords). Moreover, it is straightforward to verify that for all $u \in \mathbb{F}^5 \setminus C_1$ we have $|I(C_1;u)| \geq 2$. Indeed, we have $|I(\mathbf{0})| = 5$ and each word of weight two is covered by exactly two codewords of weight one. By symmetry, analogous observations also hold for the words of weight three and the word $\mathbf{1}$. Therefore, by Theorem 26, the code

$$D_1 = \{(\pi(u), u, u+c) \mid u \in \mathbb{F}^5, c \in C_1\}$$

is locating-dominating in $\mathbb{F}^{11}$. Thus, we have $\gamma^{LD}(\mathbb{F}^{11}) \leq |D_1| = 2^5 \cdot |C_1| = 320$.

Let C_2 be a code in $\mathbb{F}^8$ formed by the binary representations of length 8 of the following integers: 3, 6, 8, 13, 18, 21, 27, 28, 32, 39, 41, 46, 49, 52, 58, 63, 65, 68, 74, 79, 80, 87, 89, 94, 98, 101, 107, 108, 115, 118, 120, 125, 129, 132, 138, 143, 144, 151, 153, 158, 162, 165, 171, 172, 179, 182, 184, 189, 195, 198, 200, 205, 210, 213, 219, 220, 224, 231, 233, 238, 241, 244, 250, 255. It is straightforward to verify that C_2 has 64 codewords, C_2 is an identifying code in $\mathbb{F}^8$ and for all $u \in \mathbb{F}^8 \setminus C_2$ we have $|I(C_2; u)| \geq 2$. Therefore, by Theorem 26, the code

$$D_2 = \{(\pi(u), u, u + c) \mid u \in \mathbb{F}^8, c \in C_2\}$$

is locating-dominating in $\mathbb{F}^{17}$. Thus, we have $\gamma^{LD}(\mathbb{F}^{17}) \leq |D_2| = 2^8 \cdot |C_2| = 16384$. □

With the help of the following theorem, which has been shown in [8, Theorem 7], we can construct new improved locating-dominating codes from codes obtained in Corollary 27.

Theorem 28 ([8])**.** *If $C \subseteq \mathbb{F}^n$ is a locating-dominating code, then $C \oplus \mathbb{F}$ is also a locating-dominating code.*

The smallest currently known upper bounds for locating-dominating codes of lengths $n = 12$ and $n = 18$ are 684 and 35604 respectively [1].

Corollary 29. *We have $\gamma^{LD}(\mathbb{F}^{12}) \leq 640$ and $\gamma^{LD}(\mathbb{F}^{18}) \leq 32768$.*

Proof. The upper bounds follow immediately by applying Theorem 28 on codes obtained in Corollary 27. □

In [8, Theorem 15], a lower bound for $\gamma^{LD}(\mathbb{F}^n)$, which is currently the best known, has been presented. Applying the lower bound on the lengths $n = 11$, $n = 12$, $n = 17$ and $n = 18$, we obtain that $\gamma^{LD}(\mathbb{F}^{11}) \geq 309$, $\gamma^{LD}(\mathbb{F}^{12}) \geq 576$, $\gamma^{LD}(\mathbb{F}^{17}) \geq 13676$ and $\gamma^{LD}(\mathbb{F}^{18}) \geq 26006$. Thus, comparing the lower bounds to the constructions of the previous corollaries, we can state the gap between the new upper bound and existing lower bound is significantly smaller than the gap between the previous upper and lower bounds.

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