The Quarterly Journal of Mathematics
Quart. J. Math. 00 (2023), 1–22; doi:https://doi.org/10.1093/qmath/haad017

ON THE BALOG–RUZSA THEOREM IN SHORT INTERVALS

by YU-CHEN SUN†

(*Department of Mathematics and Statistics, University of Turku, Turku, 20014, Finland*)

[Received 29 June 2022]

Abstract

In this paper we give a short interval version of the Balog–Ruzsa theorem concerning bounds for the L_1 norm of the exponential sum over r-free numbers. In particular, when $r=2$, for $H \geq N^{\frac{9}{17}+\epsilon}$, we have the lower bound result

$$\int_{\mathbb{T}} \left| \sum_{|n-N|<H} \mu^2(n) e(n\alpha) \right| d\alpha \gg H^{\frac{1}{3}},$$

and for $H \geq N^{\frac{18}{29}+\epsilon}$, we have the upper bound result

$$\int_{\mathbb{T}} \left| \sum_{|n-N|<H} \mu^2(n) e(n\alpha) \right| d\alpha \ll H^{\frac{1}{3}}.$$

As an application, we show that the L_1 norm of the exponential sum $\sum_{|n-N|<H} \mu(n)e(n\alpha)$ has the lower bound $\gg H^{\frac{1}{6}}$ when $H \geq N^{\frac{9}{17}+\epsilon}$.

1. Introduction

For an integer $r \geq 2$, we say an integer n is r-free if it has no factor $d>1$ which is an r-th power. In 1998, Brüdern, Granville, Perelli, Vaughan and Wooley [3] studied bounds of the L_1 norm for the exponential sum over r-free numbers and gave the first non-trivial bounds. In 2001, Balog and Ruzsa [2] improved on the bounds and gave the best possible bound for the L_1 norm of the exponential sum over r-free numbers.

Let $r \geq 2$ be fixed and a_n be the characteristic function of the r-free integers, that is

$$a_n := \begin{cases} 1, & \text{if } n \text{ is } r\text{-free}, \\ 0, & \text{otherwise}. \end{cases} \tag{1.1}$$

Balog and Ruzsa [2] proved that

†Corresponding author. E-mail: yuchensun93@163.com

THEOREM 1.1 (Balog–Ruzsa). *Let $N \geq 2$. Then*

$$N^{\frac{1}{r+1}} \ll \int_{\mathbb{T}} \left| \sum_{n=1}^{N} a_n e(n\alpha) \right| d\alpha \ll N^{\frac{1}{r+1}}.$$

One can get the following short interval version of Theorem 1.1 immediately by just following Balog–Ruzsa's original arguments (we will provide the proof in this paper).

THEOREM 1.2 *Let $\epsilon > 0$ and $N \geq H \geq N^{\frac{r+1}{2r}+\epsilon}$. Then*

$$H^{\frac{1}{r+1}} \ll \int_{\mathbb{T}} \left| \sum_{|n-N|<H} a_n e(n\alpha) \right| d\alpha \ll H^{\frac{1}{r+1}}.$$

The most interesting case is $r = 2$, since it is related to the square-free numbers and the Möbius function $\mu(n)$. In this case, Theorem 1.2 works for interval length $H = N^{3/4+\epsilon}$.

For the lower bound, we can improve the exponent from $3/4$ to $9/17$ and get that

THEOREM 1.3 *Let $\epsilon > 0$ and $N \geq H \geq N^{\frac{9}{17}+\epsilon}$. Then*

$$\int_{\mathbb{T}} \left| \sum_{|n-N|<H} \mu^2(n) e(n\alpha) \right| d\alpha \gg H^{\frac{1}{3}}. \tag{1.2}$$

In the upper bound case, we can improve the exponent $3/4$ to $18/29$ and get that

THEOREM 1.4 *Let $\epsilon > 0$ and $N \geq H \geq N^{\frac{18}{29}+\epsilon}$. Then*

$$\int_{\mathbb{T}} \left| \sum_{|n-N|\leq H} \mu^2(n) e(n\alpha) \right| d\alpha \ll H^{\frac{1}{3}}.$$

REMARK. The exponents $\frac{9}{17} = 0.529\ldots$ in Theorems 1.3 and $\frac{18}{29} = 0.620\ldots$ in Theorem 1.4 should be compared with the exponent $\frac{3}{4} = 0.75$ in Theorem 1.2.

As a corollary of Theorem 1.3, one can obtain a lower bound for the L_1 of the short interval exponential sum of the Möbius function.

THEOREM 1.5 *Let $\epsilon > 0$ and $N \geq H \geq N^{\frac{9}{17}+\epsilon}$. Then*

$$\int_{\mathbb{T}} \left| \sum_{|n-N|<H} \mu(n) e(n\alpha) \right| d\alpha \gg H^{\frac{1}{6}}.$$

Balog and Ruzsa [2] obtained the above for $H = N$ and Theorem 1.2 would yield it for $H \geq N^{3/4+\epsilon}$. We will show how Theorem 1.3 implies Theorem 1.5 in the end of this section.

REMARK. Let us also remark the early work of Balog and Ruzsa [1], where they gave a weaker lower bound $N^{1/8}/\log N$ for both square-free number case and Möbius case. Their work can be easily extended to the following short interval result.

THEOREM 1.6 *Let $\epsilon > 0$ and $N \geq H \geq N^{\epsilon}$. If the interval $[N, N+H]$ contains $\gg H$ square-free numbers, then*

$$\int_{\mathbb{T}} \left| \sum_{N<n\leq N+H} b_n e(n\alpha) \right| d\alpha \gg \frac{H^{\frac{1}{8}}}{\log N},$$

where b_n denotes the Möbius function or the indicator function of square-free numbers.

The assumption in Theorem 1.6 has been proved for much shorter intervals than those in Theorem 1.3, for example see [4] which proves that one can take any $H \gg N^{1/5}\log N$.

Balog–Ruzsa's argument utilizes the Fejér kernel, which has many good properties (see [9]) and has many applications in number theory (see [8]).

Let $e(x) := e^{2\pi i x}$. The Fejér kernel $F(\alpha)$ is the Cesàro mean of the Dirichlet kernel $D_N(\alpha) = \sum_{|n|\leq N} e(n\alpha)$ and is defined by

$$F(\alpha) := \sum_{|n|\leq N} \left(1 - \frac{|n|}{N}\right) e(n\alpha) = \frac{\sin^2(\pi N\alpha)}{N\sin^2(\pi\alpha)} \leq \min\left\{N, \frac{1}{N\|\alpha\|^2}\right\}. \tag{1.3}$$

For the short interval case, we need to introduce a short interval version of the Fejér kernel, which is motivated by the perspective of the Fourier analysis. For convenience we first introduce some notions in Fourier analysis.

Let $A \subset \mathbb{Z}$ be finite. For $f : A \to \mathbb{C}$, we define the Fourier transform $\hat{f} : \mathbb{T} \to \mathbb{C}$ of f by

$$\hat{f}(\alpha) = \sum_{n\in A} f(n)e(n\alpha).$$

If $f, g : A \to \mathbb{C}$, we define the convolution of f and g to be

$$f * g(n) = \sum_{x+y=n} f(x)g(y),$$

A basic property involving Fourier transform and convolution is that

$$\widehat{f * g}(\alpha) = \hat{f}(\alpha)\hat{g}(\alpha).$$

Similarly, we define the convolution of the Fourier transform as

$$\hat{f} * \hat{g}(\alpha) = \int_{\mathbb{T}} \hat{f}(\alpha - \beta)\hat{g}(\beta)d\beta = \widehat{fg}(\alpha). \tag{1.4}$$

Now the Fejér kernel $F(\alpha)$ can be described through the 'Fourier language'.

$$F(\alpha) = \frac{1}{N}\widehat{\mathbf{1}_{[-\frac{N}{2},\frac{N}{2})} * \mathbf{1}_{(-\frac{N}{2},\frac{N}{2}]}} = \frac{1}{N}\hat{\mathbf{1}}_{[-\frac{N}{2},\frac{N}{2})}\hat{\mathbf{1}}_{[-\frac{N}{2},\frac{N}{2})}$$

$$= \frac{1}{N}\left|\sum_{n\in(-\frac{N}{2},\frac{N}{2}]} e(n\alpha)\right|^2 = \frac{\sin^2(\pi N\alpha)}{N\sin^2(\pi\alpha)} \ll \min\left\{N, \frac{1}{N\|\alpha\|^2}\right\}. \tag{1.5}$$

Let us introduce some properties of the difference of two Fejér kernels, which will be used in Section 5 to prove the upper bound case of Theorem 1.2 and Theorem 1.4. By the difference of the two Fejér kernels, we mean

$$\frac{1}{K}\left(\widehat{\mathbf{1}_{(-\frac{N+K}{2},\frac{N+K}{2}]} * \mathbf{1}_{[-\frac{N+K}{2},\frac{N+K}{2})}} - \widehat{\mathbf{1}_{(-\frac{N}{2},\frac{N}{2}]} * \mathbf{1}_{[-\frac{N}{2},\frac{N}{2})}}\right). \tag{1.6}$$

From (1.3) and (1.5) one can easily get

$$\frac{1}{K}\left(\widehat{\mathbf{1}_{(-\frac{N+K}{2},\frac{N+K}{2}]} * \mathbf{1}_{[-\frac{N+K}{2},\frac{N+K}{2})}} - \widehat{\mathbf{1}_{(-\frac{N}{2},\frac{N}{2}]} * \mathbf{1}_{[-\frac{N}{2},\frac{N}{2})}}\right)$$

$$= \sum_{|n|\le N+K} \min\left\{1, \frac{N+K-|n|}{K}\right\} e(n\alpha) = \frac{\sin^2(\pi(N+K)\alpha) - \sin^2(\pi K\alpha)}{K\sin^2(\pi\alpha)}$$

$$= \frac{\sin(\pi(2N+K)\alpha)\sin(\pi K\alpha)}{K\sin^2(\pi\alpha)} \ll \min\left\{N+K, \frac{1}{\|\alpha\|}, \frac{1}{K\|\alpha\|^2}\right\}. \tag{1.7}$$

Now we define our short interval version of the Fejér kernel $F_H(\alpha)$, For convenience we can assume that both N and H are even integers. We write

$$F_H(\alpha) = \frac{1}{H}\widehat{\mathbf{1}_{(\frac{N-H}{2},\frac{N+H}{2}]} * \mathbf{1}_{[\frac{N-H}{2},\frac{N+H}{2})}}(\alpha). \tag{1.8}$$

Similar to (1.5), our $F_H(\alpha)$ also has two explicit formulas and an upper bound. Namely,

$$F_H(\alpha) = e(N\alpha)\sum_{|h|\le H}\left(1 - \frac{h}{H}\right)e(h\alpha) = \sum_{|n-N|\le H}\left(1 - \frac{|n-N|}{H}\right)e(n\alpha) \tag{1.9}$$

and then

$$F_H(\alpha) = \frac{1}{H}e(N\alpha)\frac{\sin^2(\pi H\alpha)}{\sin^2(\pi\alpha)} \tag{1.10}$$

$$\ll \min\left\{H, \frac{1}{H\|\alpha\|^2}\right\}. \tag{1.11}$$

Remark. Using this F_H we can reserve almost all properties from Fejér kernel, but from (1.10) we can see that this F_H is not always real, which might be the only property it loses from Fejér kernel.

Recall the definition of a_n from (1.1) and let $g_1(\alpha)=\sum_{|n-N|\leq H}a_ne(n\alpha)$. Before proving our results, we should notice that $\|g_1\|_{L_1}$ and $\|F_H*g_1\|_{L_1}$ are comparable. Namely, we have

$$\|F_H*g_1\|_{L_1}\leq\int_{\mathbb{T}}\int_{\mathbb{T}}|F_H(\alpha-\beta)||g_1(\beta)|d\beta d\alpha\leq\|g_1\|_{L_1}\|F_H\|_{L_1}\ll\|g_1\|_{L_1}. \tag{1.12}$$

The last inequality immediately follows from (1.10). Note that $a_n=\mu^2(n)$ when $r=2$ in (1.1). Thus, for Theorem 1.3 and the lower bound case of Theorem 1.4, it suffices to find a lower bound for $\|F_H*g_1\|_{L_1}$. The above idea can also be used to deduce that Theorem 1.5 implies Theorem 1.3.

Proof that Theorem 1.3 implies Theorem 1.5. Define

$$\mu_H(n):=\mu(n)\mathbf{1}_{|n-N|<H}.$$

Then we have

$$\widehat{\mu_H}(\alpha)=\sum_{|n-N|<H}\mu(n)e(n\alpha),$$

and by (1.4) we have

$$\widehat{\mu_H^2}(\alpha)=\widehat{\mu_H}*\widehat{\mu_H}(\alpha).$$

Similar to (1.12), we have

$$\|\widehat{\mu_H^2}\|_{L_1}=\|\widehat{\mu_H}*\widehat{\mu_H}\|_{L_1}\ll\|\widehat{\mu_H}\|_{L_1}^2.$$

Now by Theorem 1.3, $\|\widehat{\mu_H^2}\|_{L_1}\gg H^{1/3}$, and the claim follows. □

The rest of the paper is organized as follows. In Section 2, we collect some auxiliary lemmas concerning the Riemann zeta function, Dirichlet polynomials and van der Corput bounds. In Section 3, we will prove a key lemma, which improves Lemma 1 of [**2**], using an analytic approach and van der Corput bounds. Then in Sections 4 and 5, we will follow Balog's and Ruzsa's arguments to prove our lower bound and upper bound results.

2. Some auxiliary lemmata

LEMMA 2.1 (Perron's formula). *For $\Re(s)>1$, let*

$$f(s)=\sum_{n=1}^{\infty}a_nn^{-s},$$

where $a_n=O(\phi(n))$, $\phi(n)$ is non-decreasing. Let $\alpha>0$ and as $\sigma\to1^+$,

$$\sum_{n=1}^{\infty}a_nn^{-\sigma}=O\left(\frac{1}{(\sigma-1)^{\alpha}}\right).$$

Then if $c > 1$ and x is not an integer, we have

$$\sum_{n\le x} a_n = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s)\frac{x^s}{s}ds + O\left(\frac{x^c}{T(c-1)^\alpha}\right) + O\left(\frac{\phi(2x)x\log x}{T}\right) + O\left(\frac{\phi(N)x}{T\|x\|}\right).$$

Proof. See [10, Lemma 3.12]. □

LEMMA 2.2 *Let $\epsilon > 0$. Suppose that $\frac{1}{2} \le \sigma \le 1+\varepsilon$ and $t \ge 1$. Then*

$$\zeta(\sigma+it) = O(t^{\frac{1}{3}(1-\sigma)+\epsilon}).$$

Proof. See [10, Chapter 5, in particular, (5.12), Theorem 5.5 and the convexity of $\mu(\sigma)$.] □

LEMMA 2.3 *For $T \ge 2$, we have*

$$\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \ll T\log T.$$

Proof. See [10, Theorem 7.2(A)]. □

LEMMA 2.4 *For $T, N \ge 2$ and any complex number a_n we have*

$$\int_0^T \left|\sum_{0\le n\le N} a_n n^{it}\right|^2 dt = (T+O(N))\sum_{0\le n\le N}|a_n|^2.$$

Proof. See [6, Theorem 9.1]. □

LEMMA 2.5 (van der Corput bounds). *Suppose that (p, q) is an exponent pair defined by* [6, (8.57) *and* (8.58)], *f(x) behaves like a monomial so that*

$$|f^{(j)}(x)| \asymp_j FM^{-j}$$

for every $j \ge 0$, any $x \in [M,2M]$ and some $F \ge M$. We define $\psi(x) := \{x\} - \frac{1}{2}$ and let $I = [a,b] \subset [M,2M]$. Then

$$\sum_{m\in I}\psi(f(m)) \ll F^{\frac{p}{p+1}}M^{\frac{1+2q}{2(p+1)}+\epsilon}$$

for any $\epsilon > 0$.

Proof. By [5, Theorem A.6], there exist coefficients $\beta(r) \ll |r|^{-1}$ such that

$$\psi(x) \leq R^{-1} + \sum_{1 \leq |r| \leq R} \beta(r) e(rx).$$

Thus,

$$\sum_{m \in I} \psi(f(m)) \ll MR^{-1} + \sum_{1 \leq r \leq R} \frac{1}{r} \left| \sum_{m \in I} e(rf(m)) \right|.$$

By [6, (8.58)], we obtain that

$$\begin{aligned}\sum_{m \in I} \psi(f(m)) &\ll MR^{-1} + \sum_{1 \leq r \leq R} \frac{1}{r} (rFM^{-1})^p M^{q+\frac{1}{2}+\epsilon'} \\ &\ll MR^{-1} + F^p M^{q-p+\frac{1}{2}+\epsilon'} R^p.\end{aligned}$$

We choose R so that $R^{p+1} = M^{\frac{1}{2}+p-q-\epsilon'} F^{-p}$, and the claim follows. □

Remark. We will only need the lemma for the exponent pair $(p, q) = \left(\frac{2}{7}, \frac{1}{14}\right)$.

3. Key lemmas

We recall that $a_n = \sum_{d^r | n} \mu(d)$. Let $z > y \geq 1$ be any real number. For $n \neq 0$ we define $c_n = c_n(y, z)$ as a middle part of the above sum, namely

$$c_n = \sum_{\substack{d^r | n \\ y < d \leq z}} \mu(d). \tag{3.1}$$

Lemma 3.1 *For any $1 \leq K < N$ and $1 \leq y < z$ we have*

$$\sum_{N-K < n \leq N} |c_n|^2 \ll Ky^{1-r} + N^{1/r+\epsilon}. \tag{3.2}$$

Furthermore, for $r = 2$ and $y \leq K^{1/2-\epsilon}$, we have

$$\sum_{N-K < n \leq N} |c_n|^2 \ll Ky^{-1} + N^{\frac{12}{29}+\epsilon} y^{-\frac{10}{29}}. \tag{3.3}$$

Proof. In fact (3.2) is [**2**, Lemma 1], so we just need to prove (3.3). For $r=2$, we have

$$\begin{aligned}\sum_{N-K<n\le N}|c_n|^2 &\le \sum_{N-K<n\le N}\Big(\sum_{\substack{d^2|n\\ y<d\le z}}1\Big)^2=\sum_{N-K<n\le N}\sum_{\substack{d_1^2,d_2^2|n\\ y<d_i\le z}}1\\ &\ll \sum_{N-K<n\le N}\sum_{h^2|n}\sum_{\substack{(hd_1')^2,(hd_2')^2|n\\ (d_1',d_2')=1\\ y<hd_i'\le z}}1\\ &\ll \sum_{N-K<n\le N}\sum_{\substack{n=h^2d_1^2d_2^2a\\ y<hd_i\le z}}1. \qquad (3.4)\end{aligned}$$

So we only need to show that (3.4) is $\ll Ky^{-1}+N^{\frac{12}{29}+\epsilon}y^{-\frac{10}{29}}$. We first split d_i,h into dyadic ranges, so we consider the sum over $d_i\sim D_i$ and $h\sim H$ with $y\ll HD_i\ll z$ and then add those sums later.

Before proving the upper bound for (3.4), let us explain how we use two different techniques depending on the size of D_i and H. Without loss of generality, we can assume that $D_2\gg D_1$ and also notice that $D_1H\gg y$. If D_2 is large, we can use an analytic approach to give a good upper bound (see (3.7)). If D_2 is small, we can use the hyperbolic trick and the van der Corput method.

For convenience we can assume that neither N nor $N-K$ is an integer (for example using $\lfloor N\rfloor+\frac12$ and $\lfloor N-K\rfloor+\frac12$ to replace N and $N-K$). By Lemma 2.1, with $T=T_0\asymp N$,

$$\begin{aligned}&\sum_{\substack{N-K<h^2d_1^2d_2^2a\le N\\ d_i\sim D_i,h\sim H}}1\\ &=\frac{1}{2\pi i}\int_{1+\varepsilon-iT_0}^{1+\varepsilon+iT_0}\frac{N^s-(N-K)^s}{s}\zeta(s)P(2s)ds+O(N^\epsilon),\end{aligned}$$

where

$$P(s)=\Big(\sum_{d_1\sim D_1}\frac{1}{d_1^s}\Big)\Big(\sum_{d_2\sim D_2}\frac{1}{d_2^s}\Big)\Big(\sum_{h\sim H}\frac{1}{h^s}\Big).$$

We move the line of integration to $\Re(s)=\frac12$. The residue of

$$\frac{N^s-(N-K)^s}{s}\zeta(s)P(2s)$$

at $s=1$ is $O(K(HD_1D_2)^{-1})$.

By Lemma 2.2,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\frac12+iT_0}^{1+\varepsilon+iT_0}\frac{N^s-(N-K)^s}{s}\zeta(s)P(2s)ds\\ &\ll \max_{\frac12\le\sigma\le1+\varepsilon}\frac{N^\sigma}{T_0}T_0^{\frac13(1-\sigma)+\varepsilon}(HD_1D_2)^{1-2\sigma}\ll N^\epsilon.\end{aligned}$$

Similarly,

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-iT_0}^{1+\varepsilon-iT_0}\frac{N^s-(N-K)^s}{s}\zeta(s)P(2s)ds \ll N^{\epsilon}.$$

Hence the remaining task is to estimate the integral when $\Re(s)=\frac{1}{2}$. We have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\frac{1}{2}-iT_0}^{\frac{1}{2}+iT_0}\frac{N^s-(N-K)^s}{s}\zeta(s)P(2s)ds\\
=&\frac{1}{2\pi i}\int_{-T_0}^{T_0}\frac{N^{\frac{1}{2}+it}-(N-K)^{\frac{1}{2}+it}}{\frac{1}{2}+it}\zeta\left(\frac{1}{2}+it\right)P(1+2it)dt\\
\ll&\int_{-T_0}^{T_0}\min\left\{KN^{-\frac{1}{2}},\frac{N^{\frac{1}{2}}}{|t|}\right\}\left|\zeta\left(\frac{1}{2}+it\right)P(1+2it)\right|dt\\
\ll&N^{\frac{1}{2}}\left(\int_{|t|\le N/K}\frac{K}{N}\left|\zeta\left(\frac{1}{2}+it\right)P(1+2it)\right|dt+\int_{N\ge|t|>N/K}\frac{1}{|t|}\left|\zeta\left(\frac{1}{2}+it\right)P(1+2it)\right|dt\right). \qquad (3.5)
\end{aligned}$$

Using a dyadic trick to deal with the second term in the bracket, we have

$$\begin{aligned}
&\int_{|t|>N/K}\frac{1}{|t|}\left|\zeta\left(\frac{1}{2}+it\right)P(1+2it)\right|dt \ll \int_{N\gg|t|>N/K}\frac{1}{|t|}\left|\zeta\left(\frac{1}{2}+it\right)P(1+2it)\right|\frac{1}{|t|}\int_{|t|/2}^{|t|}1dTdt\\
\ll&\int_{N\gg T\gg N/K}\frac{1}{T^2}\int_T^{2T}\left|\zeta\left(\frac{1}{2}+it\right)P(1+2it)\right|dtdT.
\end{aligned}$$

Hence (3.5) is at most

$$\ll N^{\frac{1}{2}}\log N\sup_{N/K\ll T\ll N}\frac{1}{T}\int_{|t|\le T}\left|\zeta\left(\frac{1}{2}+it\right)P(1+2it)\right|dt. \qquad (3.6)$$

By Cauchy–Schwarz inequality and Lemmas 2.3 and 2.4, we deduce that (3.6) is

$$\begin{aligned}
\ll&N^{\frac{1}{2}}\log N\sup_{N/K\ll T\ll N}\frac{1}{T}\left(\int_{|t|\le T}\left|\zeta(\frac{1}{2}+it)\right|^2dt\right)^{\frac{1}{2}}\left(\int_{|t|\le T}|P(1+2it)|^2dt\right)^{\frac{1}{2}}\\
\ll&N^{\frac{1}{2}}\log N\sup_{N/K\ll T\ll N}\frac{1}{T}(T\log T)^{\frac{1}{2}}((T+O(D_1D_2H))^{\frac{1}{2}}\left(\sum_{D_1D_2H\ll n\ll D_1D_2H}\frac{\tau_3^2(n)}{n^2}\right)^{\frac{1}{2}}\\
\ll&N^{\frac{1}{2}}(D_1D_2H)^{-\frac{1}{2}}\log^{C_2}N+K^{\frac{1}{2}}\log^{C_1}N, \qquad (3.7)
\end{aligned}$$

where $\tau_3(n)=\sum_{n=n_1n_2n_3}1$ and $C_1,C_2>0$ are absolute constants. Hence, we have

$$\begin{aligned}&\sum_{\substack{N-K<h^2d_1^2d_2^2a\le N\\ d_i\sim D_i,h\sim H}}1\\ &\ll K(HD_1D_2)^{-1}+K^{\frac12}\log^{C_1}N+N^{\frac12}(D_1D_2H)^{-\frac12}+N^{\epsilon}\log^{C_2}N+N^{\epsilon}\\ &\ll K(HD_1D_2)^{-1}+Ky^{-1-\epsilon}+N^{\frac12}y^{-\frac12}D_2^{-\frac12}\log^{C_2}N+N^{\epsilon},\end{aligned}\tag{3.8}$$

since $y\le K^{\frac12-\epsilon}$ and $D_iH\gg y$.

On the other hand, (3.4) is no more than

$$\sum_{d_1,d_2}\sum_{N-K<n\le N}\sum_{\substack{n=h^2d_1^2d_2^2a\\ y<hd_i\le z}}1\le\sum_{d_1,d_2}\sum_{\substack{\frac{N-K}{d_1^2d_2^2}<h^2a\le\frac{N}{d_1^2d_2^2}\\ \frac{y}{\min\{d_1,d_2\}}<h\le\frac{z}{\max\{d_1,d_2\}}}}1.\tag{3.9}$$

Without loss of generality, we can assume that $d_1\le d_2$, and then by the hyperbolic trick, the inner sum on the right-hand side of (3.9) is at most

$$\begin{aligned}&\sum_{\frac{y}{d_1}\le h\le(\frac{N}{d_1^2d_2^2})^{\frac13}}\ \sum_{\frac{N-K}{h^2d_1^2d_2^2}\le a\le\frac{N}{h^2d_1^2d_2^2}}1+\sum_{a\le(\frac{N}{d_1^2d_2^2})^{\frac13}}\ \sum_{(\frac{N-K}{ad_1^2d_2^2})^{\frac12}\le h\le(\frac{N}{ad_1^2d_2^2})^{\frac12}}1\\ &:=\Sigma_1+\Sigma_2.\end{aligned}$$

We use the standard van der Corput method to handle Σ_1 and Σ_2, respectively. Recalling the definition of $\psi(x)=\{x\}-\frac12$ in Lemma 2.5, we have

$$\begin{aligned}\Sigma_1&=\sum_{\frac{y}{d_1}\le h\le(\frac{N}{d_1^2d_2^2})^{\frac13}}\frac{K}{h^2d_1^2d_2^2}+\psi\left(\frac{N-K}{h^2d_1^2d_2^2}\right)-\psi\left(\frac{N}{h^2d_1^2d_2^2}\right)\\ &\ll Ky^{-1}\frac{1}{d_1d_2^2}+O\left(\left|\sum_{\frac{y}{d_1}\le h\le(\frac{N}{d_1^2d_2^2})^{\frac13}}\psi\left(\frac{N}{h^2d_1^2d_2^2}\right)\right|+\left|\sum_{\frac{y}{d_1}\le h\le(\frac{N}{d_1^2d_2^2})^{\frac13}}\psi\left(\frac{N-K}{h^2d_1^2d_2^2}\right)\right|\right).\end{aligned}\tag{3.10}$$

Let (p,q) be an exponent pair (see Lemma 2.5) satisfying $1+2q-4p\ge0$. Using dyadic trick and Lemma 2.5 with $F=\frac{N}{M^2d_1^2d_2^2}$, we have

$$\begin{aligned}\sum_{\frac{y}{d_1}\le h\le(\frac{N}{d_1^2d_2^2})^{\frac13}}\psi\left(\frac{N}{h^2d_1^2d_2^2}\right)&\ll\log N\max_{\frac{y}{d_1}\le M\le M_1\le2M\le2(\frac{N}{d_1^2d_2^2})^{\frac13}}\left|\sum_{h\in(M,M_1]}\psi\left(\frac{N}{h^2d_1^2d_2^2}\right)\right|\\ &\ll\log N\max_{\frac{y}{d_1}\le M\le M_1\le2M\le2(\frac{N}{d_1^2d_2^2})^{\frac13}}\left(\frac{N}{d_1^2d_2^2}\right)^{\frac{p}{p+1}+\epsilon}M^{\frac{1+2q-4p}{2(p+1)}}\ll\left(\frac{N}{d_1^2d_2^2}\right)^{\frac{1+2q+2p}{6(p+1)}+\epsilon}.\end{aligned}$$

Thus

$$\Sigma_1 \ll Ky^{-1}\frac{1}{d_1d_2^2} + \left(\frac{N}{d_1^2d_2^2}\right)^{\frac{2(p+q)+1}{6(p+1)}+\epsilon}. \tag{3.11}$$

Similarly, for all exponent pairs (p,q), we have

$$\Sigma_2 = \sum_{a\le(\frac{N}{d_1^2d_2^2})^{\frac13}} \left(\frac{N}{ad_1^2d_2^2}\right)^{\frac12} - \left(\frac{N-K}{ad_1^2d_2^2}\right)^{\frac12} + \psi\left(\left(\frac{N-K}{ad_1^2d_2^2}\right)^{\frac12}\right) - \psi\left(\left(\frac{N}{ad_1^2d_2^2}\right)^{\frac12}\right)$$

$$\ll KN^{-\frac13}\frac{1}{(d_1d_2)^{\frac43}} + O\left(\left|\sum_{a\le(\frac{N}{d_1^2d_2^2})^{\frac13}} \psi\left(\left(\frac{N}{ad_1^2d_2^2}\right)^{\frac12}\right)\right| + \left|\sum_{a\le(\frac{N}{d_1^2d_2^2})^{\frac13}} \psi\left(\left(\frac{N-K}{ad_1^2d_2^2}\right)^{\frac12}\right)\right|\right). \tag{3.12}$$

As above, we split a to dyadic ranges $a\sim A$ and use Lemma 2.5 with $M=A$ and $F=(\frac{N}{Ad_1^2d_2^2})^{1/2}$. We obtain

$$\Sigma_2 \ll KN^{-\frac13}\frac{1}{(d_1d_2)^{\frac43}} + \left(\frac{N}{d_1^2d_2^2}\right)^{\frac{2(p+q)+1}{6(p+1)}+\epsilon}. \tag{3.13}$$

Let $\sum^\flat$ represent the sum over powers of two and let D be a parameter to be chosen later. Hence by (3.8), (3.9), (3.11) and (3.13) we get the following upper bound for (3.4)

$$\sum_{\substack{N-K\le h^2d_1^2d_2^2a\le N\\ y\le hd_i\le z}} 1 \ll \sum^\flat_{D_2\le D}\sum_{\substack{N-K\le h^2d_1^2d_2^2a\le N\\ y\le hd_i\le z\\ d_2\sim D_2\\ d_1\le d_2}} 1 + \sum^\flat_{D_2>D}\sum_{\substack{N-K\le h^2d_1^2d_2^2a\le N\\ y\le hd_i\le z\\ d_2\sim D_2\\ d_1\le d_2}} 1$$

$$\ll \sum^\flat_{D_2\le D}\sum_{\substack{d_2\sim D_2\\ d_1\le d_2}}\sum_{\substack{\frac{N-K}{d_1^2d_2^2}\le h^2a\le\frac{N}{d_1^2d_2^2}\\ \frac{y}{d_1}\le h\le\frac{z}{d_2}}} 1 + \sum^\flat_{\substack{D_2\gg D\\ D_1\ll D_2\\ y\ll HD_i\ll z}}\left(K(HD_1D_2)^{-1} + Ky^{-1-\epsilon} + N^{\frac12+\epsilon}y^{-\frac12}D_2^{-\frac12} + N^\epsilon\right)$$

$$\ll Ky^{-1} + Ky^{-1-\epsilon}\log^2 N + N^{\frac{2(p+q)+1}{6(p+1)}+\epsilon}D^{2-\frac{4p+4q+2}{3(p+1)}} + N^{\frac12+\epsilon}y^{-\frac12}D^{-\frac12} + \sum^\flat_{\substack{D_2\gg D\\ D_1\ll D_2\\ y\ll HD_i\ll z}} K(HD_1D_2)^{-1}. \tag{3.14}$$

We first calculate $\sum^\flat_{\substack{D_2\gg D\\ D_1\ll D_2\\ y\ll HD_i\ll z}} K(HD_1D_2)^{-1}$ in (3.14), writing $D_1=2^{n_1}, D_2=2^{n_2}$ and $H=2^{h_0}$, which is no more than

$$K\sum_{n_2\ge0}\sum_{n_1\le n_2+O(1)}\sum_{h_0\ge\log_2(\frac{y}{2^{n_1}})-O(1)}(2^{n_1}2^{n_2}2^{h_0})^{-1}$$

$$\ll Ky^{-1}\sum_{n_2\geq 0}2^{-n_2}\sum_{n_1\leq n_2+O(1)}1\ll Ky^{-1}. \tag{3.15}$$

Then we choose the exponent pair $(p,q)=(\frac{2}{7},\frac{1}{14})$, see [6, (8.16)–(8.17)], and $D=N^{\frac{5}{29}+\epsilon}y^{-\frac{9}{29}}$, which yields that

$$N^{\frac{2(p+q)+1}{6(p+1)}+\epsilon}D^{2-\frac{4p+4q+2}{3(p+1)}}+N^{1/2+\epsilon}y^{-1/2}D^{-\frac{1}{2}}\ll N^{\frac{12}{29}+\epsilon}y^{-\frac{10}{29}}. \tag{3.16}$$

Hence, the claim follows from (3.4), (3.14), (3.15) and (3.16). □

We can easily get the following lemma using Lemma 3.1.

LEMMA 3.2 *Let $N>K\geq 1$, $y=K^{\frac{1}{r+1}}$ and $z=N^{\frac{1}{r}}$. Then*

$$\sum_{N-K<n\leq N}|c_n(y,z)|^2\ll K^{\frac{2}{r+1}}$$

provided that

$$K\gg\begin{cases}N^{\frac{9}{17}+\epsilon} & \text{if } r=2,\\ N^{\frac{r+1}{2r}+\epsilon} & \text{if } r\geq 3.\end{cases}$$

REMARK. The proof of Lemma 3.1 is motivated by the methods used in dealing with the Dirichlet divisor problem, see for example [7]. In the above proof, we combine analytic methods with hyperbolic trick and van der Corput bounds. If we only use the analytic approach, we can also get a non-trivial result but the length K must be longer than $N^{\frac{3}{5}+\epsilon}$ in Lemma 3.2, which is weaker than our result. On the other hand, if we just use the hyperbolic trick and van der Corput bound, we cannot get any non-trivial improvement. The exponent $\frac{9}{17}$ is determined by the exponent pair $(p,q)=(\frac{2}{7},\frac{1}{14})$. One could use a stronger exponent pair to obtain a similar result for slightly smaller K, but the improvement would not be very significant.

4. Proof of the lower bound case of Theorem 1.2 and Theorem 1.3

In this section we will prove the lower bound $\|F_H * g_1\|_{L_1}\gg H^{\frac{1}{r+1}}$ which, by the argument in the end of Section 1, implies Theorem 1.3 and Theorem 1.5. We first define a 'q-analog' of F_H as

$$F_H^q(\alpha):=\frac{1}{q}\sum_{a=1}^{q}F_H\left(\alpha-\frac{a}{q}\right)=\sum_{\substack{|n-N|\leq H\\ n\equiv 0 \pmod q}}\left(1-\frac{|n-N|}{H}\right)e(n\alpha).$$

Let $y = H^{\frac{1}{r+1}}$ and $z = N^{\frac{1}{r}}$. By (1.4) and (1.9), we have

$$\begin{aligned}F_H * g_1(\alpha) &= \sum_{|n-N|\le H}\left(1-\frac{|n-N|}{H}\right)a_n e(n\alpha)\\ &= \sum_{|n-N|\le H}\left(1-\frac{|n-N|}{H}\right)c_n(1,y)e(n\alpha)\\ &+ \sum_{|n-N|\le H}\left(1-\frac{|n-N|}{H}\right)c_n(y,z)e(n\alpha)\\ &= \sum_{d_0\le y}\mu(d_0)F_H^{d_0^r}(\alpha) + \sum_{|n-N|\le H}\left(1-\frac{|n-N|}{H}\right)c_n(y,z)e(n\alpha). \end{aligned}\tag{4.1}$$

For the second term, we will use Lemma 3.1 to handle it. Now we follow Balog–Ruzsa's argument to deal with the first term. We denote the first term of (4.1) by $g_2(\alpha)$ and transform it to

$$\begin{aligned}g_2(\alpha) &= \sum_{d_0\le y}\frac{\mu(d_0)}{d_0^r}\sum_{a_0=1}^{d_0^r}F_H\left(\alpha-\frac{a_0}{d_0^r}\right)\\ &= \sum_{d_0\le y}\sum_{m|d_0}\sum_{\substack{a_0=1\\ m^r|a_0\\ (a_0/m^r,\,d_0^r/m^r)\,r\text{-free}}}^{d_0^r}\frac{\mu(d_0)}{d_0^r}F_H\left(\alpha-\frac{a_0}{d_0^r}\right).\end{aligned}$$

Writing $d_0 = md$ and $a_0 = m^r a$ and noticing that $\mu(d_0) = \mu(m)\mu(d)\mathbf{1}_{(m,d)=1}$, we obtain

$$\begin{aligned}g_2(\alpha) &= \sum_{d\le y}\frac{\mu(d)}{d^r}\sum_{\substack{m\le y/d\\ (m,d)=1}}\frac{\mu(m)}{m^r}\sum_{\substack{a=1\\ (a,d^r)\,r\text{-free}}}^{d^r}F_H\left(\alpha-\frac{a}{d^r}\right)\\ &= \sum_{d\le y}\mu(d)b_d G_d(\alpha),\end{aligned}\tag{4.2}$$

where

$$b_d = \sum_{\substack{m\le y/d\\ (m,d)=1}}\frac{\mu(m)}{m^r}$$

and

$$G_d(\alpha) = \frac{1}{d^r}\sum_{\substack{a=1\\ (a,d^r)\,r\text{-free}}}^{d^r}F_H\left(\alpha-\frac{a}{d^r}\right).\tag{4.3}$$

It is clear that

$$\frac{1}{3} \leq 1 - (\frac{\pi^2}{6} - 1) \leq 1 - \sum_{d=2}^{\infty} d^{-r} \leq b_d \leq 1 + \sum_{d=2}^{\infty} d^{-2} \leq \frac{\pi^2}{6} \leq \frac{5}{3}, \tag{4.4}$$

and by (1.10) we have

$$|F_H(\beta)| \geq \frac{H}{2}, \quad \text{whenever} \quad \|\beta\| \leq \frac{1}{2H}. \tag{4.5}$$

To show that $|F_H|$ is large, we must define a set with positive density in $\mathbb{T}$, in which $|F_H|$ is large. For any $d \leq y$, we define the set

$$\mathcal{X}_d = \bigcup_{\substack{1 \leq a \leq d^r \\ (a,d^r)\,r\text{-free}}} \left[\frac{a}{d^r} - \frac{1}{2H}, \frac{a}{d^r} + \frac{1}{2H}\right].$$

For any fixed $d \leq y$ and two distinct $1 \leq a_1, a_2 \leq d^r$ the distance between $\frac{a_1}{d^r}$ and $\frac{a_2}{d^r}$ is at least

$$\frac{1}{d^r} \geq \frac{1}{y^r} = \frac{1}{H^{\frac{r}{r+1}}}. \tag{4.6}$$

Thus we have

$$\left[\frac{a_1}{d^r} - \frac{1}{2H}, \frac{a_1}{d^r} + \frac{1}{2H}\right] \bigcap \left[\frac{a_2}{d^r} - \frac{1}{2H}, \frac{a_2}{d^r} + \frac{1}{2H}\right] = \varnothing. \tag{4.7}$$

In Balog–Ruzsa's argument, their $F(\alpha)$ is non-negative, so they can just pick up one large term $F(\alpha - \frac{a}{d^r})$ in the sum of $G_d(\alpha)$ directly as the lower bound of $G_d(\alpha)$. However, since $F_H(\alpha)$ is not always non-negative, we must be more careful.

LEMMA 4.1 *Let $1 \leq d \leq y$. For any $\alpha \in \mathcal{X}_d$, we have that*

$$|G_d(\alpha)| \geq \frac{1}{2}(1 + o(1))\frac{H}{d^r}.$$

Proof. For any $\alpha \in \mathcal{X}_d$, by (4.7) there is a unique a such that $\alpha \in [\frac{a}{d^r} - \frac{1}{2H}, \frac{a}{d^r} + \frac{1}{2H}]$. Thus by (4.5), (4.3), (1.11) and (4.6), we have, for any $d \leq y$ and $\alpha \in \mathcal{X}_d$,

$$\begin{aligned} |G_d(\alpha)| &\geq \frac{H}{2d^r} - \frac{1}{d^r} \sum_{\substack{1 \leq a_1 \leq d^r \\ (a_1,d^r)\,r\text{-free} \\ a_1 \neq a}} \left|F_H\left(\alpha - \frac{a_1}{d^r}\right)\right| = \frac{H}{2d^r} - \frac{1}{d^r} \sum_{\substack{1 \leq a_1 \leq d^r \\ (a_1,d^r)\,r\text{-free} \\ a_1 \neq a}} \frac{1}{H\|\alpha - \frac{a_1}{d^r}\|^2} \\ &\geq \frac{H}{2d^r} - \frac{1}{d^r} \sum_{\substack{1 \leq a_1 \leq d^r \\ (a_1,d^r)\,r\text{-free} \\ a_1 \neq a}} \frac{1}{H(\|\frac{a-a_1}{d^r}\| - \frac{1}{2H})^2} \geq \frac{H}{2d^r} - \frac{1}{d^r} \sum_{k=1}^{d^r/2} \frac{2}{H(\frac{k}{d^r} - \frac{1}{2H})^2} \\ &\geq \frac{H - O\left(H^{\frac{2r}{r+1} - 1}\right)}{2d^r} \geq \frac{1}{2}(1 + o(1))\frac{H}{d^r}. \end{aligned} \tag{4.8}$$

□

REMARK. To deal with $\sum_{\substack{1\le a_1\le d^r\\(a_1,d^r)\,r\text{-free}\\a_1\ne a}} |F(\alpha-\frac{a_1}{d^r})|$, one might consider to use the Cauchy–Schwarz inequality and then use the large sieve inequality. However, this method may fail.

Since the sets $\mathcal{X}_d$ are not necessarily disjoint for different $d\le y$, we cannot directly calculate the size of $\bigcup_d \mathcal{X}_d$. In order to overcome this obstacle, we define pairwise disjoint $\mathcal{Y}_d$ which have the similar size as $\mathcal{X}_d$. We define

$$\mathcal{Y}_d := \left\{ \alpha \in \mathcal{X}_d : \sum_{\substack{d'\le y\\ d'\ne d}} |G_{d'}(\alpha)| \le \frac{H}{20d^r} \right\}. \tag{4.9}$$

LEMMA 4.2 *The sets $\mathcal{Y}_d$ defined in (4.9), for $d\le y$, are pairwise disjoint.*

Proof. Suppose that $1\le d_1<d_2\le y$ such that $\alpha\in\mathcal{Y}_{d_1}\cap\mathcal{Y}_{d_2}\ne\varnothing$. By Lemma 4.1 and (4.9), we have

$$|G_{d_1}(\alpha)| \ge (1+o(1))\frac{H}{2d_1^r} \quad \text{and} \quad |G_{d_1}(\alpha)| \le \frac{H}{20d_2^r},$$

which contradict each other. □

In case $\alpha\in\mathcal{Y}_d$, (4.2), (4.8) and (4.4) imply that

$$|g_2(\alpha)| \ge b_d|G_d(\alpha)| - \sum_{\substack{d'\le y\\ d'\ne d}} b_{d'}|G_{d'}(\alpha)| \gg \frac{H}{d^r}. \tag{4.10}$$

Now we estimate the sizes of $\mathcal{X}_d$ and $\mathcal{Y}_d$. First, it is easy to see that

$$|\mathcal{X}_d| = \frac{1}{H}\sum_{\substack{a=1\\(a,d^r)\,r\text{-free}}}^{d^r} 1 = \frac{d^r}{H}\prod_{p|d}\left(1-\frac{1}{p^r}\right),$$

and therefore

$$\frac{d^r}{H} \ll |\mathcal{X}_d| \le \frac{d^r}{H}. \tag{4.11}$$

Next, we estimate the size of $\mathcal{Z}_d = \mathcal{X}_d\setminus\mathcal{Y}_d$. If $\alpha\in\mathcal{Z}_d$ then

$$|G_d(\alpha)| \ge (1+o(1))\frac{H}{2d^r} \quad \text{and} \quad \sum_{\substack{d'\le y\\ d'\ne d}} |G_{d'}(\alpha)| \ge \frac{H}{20d^r}.$$

This immediately implies that

$$\int_{\mathcal{Z}_d} |G_d(\alpha)| \sum_{\substack{d' \le y \\ d' \neq d}} |G_{d'}(\alpha)| d\alpha \ge |\mathcal{Z}_d| \frac{H^2}{40d^{2r}}. \tag{4.12}$$

Now we follow the argument of Balog–Ruzsa to prove a 'quasi-orthogonality' property of $|G_d(\alpha)|$. Namely,

LEMMA 4.3 (Quasi-orthogonality). *For any* $1 \le d_1 < d_2 \le y$, *we have*

$$\int_{\mathbb{T}} |G_{d_1}(\alpha)||G_{d_2}(\alpha)| d\alpha \ll 1.$$

Proof. First by (1.11) we observe that

$$\|F_H\|_{L_1} = \int_{\mathbb{T}} |F_H(\alpha)| d\alpha \ll 1.$$

By (1.11), for any $\alpha \in \mathbb{T}$, we have

$$\int_{\mathbb{T}} |F_H(\beta)||F_H(\alpha+\beta)| d\beta \tag{4.13}$$

$$\begin{aligned}
&\le \int_{\mathbb{T}} (|F_H(\beta)| + |F_H(\alpha+\beta)|) \min\{|F_H(\beta)|, |F_H(\alpha+\beta)|\} d\beta \\
&\le \int_{\mathbb{T}} (|F_H(\beta)| + |F_H(\alpha+\beta)|) \min\left\{H, \frac{1}{H\|\beta\|^2}, \frac{1}{H\|\alpha+\beta\|^2}\right\} d\beta \\
&\ll \int_{\mathbb{T}} (|F_H(\beta)| + |F_H(\alpha+\beta)|) \min\left\{H, \frac{1}{H(\|\beta\| + \|\beta+\alpha\|)^2}\right\} d\beta \\
&\le \int_{\mathbb{T}} (|F_H(\beta)| + |F_H(\alpha+\beta)|) \min\left\{H, \frac{1}{H\|\alpha\|^2}\right\} d\beta \\
&\le \min\left\{H, \frac{1}{H\|\alpha\|^2}\right\}.
\end{aligned} \tag{4.14}$$

By (4.3) and (4.14) we have

$$\int_{\mathbb{T}} |G_{d_1}(\alpha)||G_{d_2}(\alpha)|d\alpha$$
$$\le \frac{1}{d_1^r d_2^r} \sum_{\substack{a_1=1\\(a_1,d_1^r)\, r\text{-free}}}^{d_1^r} \sum_{\substack{a_2=1\\(a_2,d_2^r)\, r\text{-free}}}^{d_2^r} \int_{\mathbb{T}} \left|F_H(\alpha-\frac{a_1}{d_1^r})F_H(\alpha-\frac{a_2}{d_2^r})\right|d\alpha$$
$$\ll \frac{1}{d_1^r d_2^r} \sum_{\substack{a_1=1\\(a_1,d_1^r)\, r\text{-free}}}^{d_1^r} \sum_{\substack{a_2=1\\(a_2,d_2^r)\, r\text{-free}}}^{d_2^r} \min\left\{H, \frac{1}{H\|\frac{a_1}{d_1^r}-\frac{a_2}{d_2^r}\|^2}\right\}. \tag{4.15}$$

We can write

$$\left\|\frac{a_1}{d_1^r}-\frac{a_2}{d_2^r}\right\| = \frac{|m|}{[d_1^r,d_2^r]},$$

where m is the member of the residue class

$$a_1\frac{d_2^r}{(d_1^r,d_2^r)} - a_2\frac{d_1^r}{(d_1^r,d_2^r)} \pmod{[d_1^r,d_2^r]}$$

with least absolute value. Note that $m=0$ does not appear because $d_1 \neq d_2$ and $(a_1,d_1^r),(a_2,d_2^r)$ are r-free. Given a non-zero $|m| \le [d_1^r,d_2^r]/2$ the above holds when

$$a_1\frac{d_2^r}{(d_1^r,d_2^r)} \equiv m \pmod{\frac{d_1^r}{(d_1^r,d_2^r)}},$$

which happens for exactly (d_1^r,d_2^r) choices of $a_1 \pmod{d_1^r}$. When m and a_1 are fixed, a_2 is uniquely determined. In (4.15), we get that

$$\int_{\mathbb{T}} |G_{d_1}(\alpha)||G_{d_2}(\alpha)|d\alpha \ll \frac{(d_1^r,d_2^r)}{d_1^r d_2^r} \sum_{1\le m\le [d_1^r,d_2^r]/2} \min\left\{H, \frac{[d_1^r,d_2^r]^2}{Hm^2}\right\}$$
$$\ll \frac{H}{[d_1^r,d_2^r]} \sum_{1\le m\le [d_1^r,d_2^r]/H} 1 + \frac{[d_1^r,d_2^r]}{H} \sum_{m>[d_1^r,d_2^r]/H} \frac{1}{m^2} \ll 1.$$

□

By (4.12) and Lemma 4.3, we have

$$|\mathcal{Z}_d| \ll \frac{d^{2r}y}{H^2} \tag{4.16}$$

Let $\epsilon>0$ be a small fixed constant. By (4.11) and (4.16) we have

$$|\mathcal{Y}_d| = |\mathcal{X}_d| - |\mathcal{Z}_d| \gg \frac{d^r}{H} \gg |\mathcal{X}_d|, \quad \text{for every} \quad d \le \epsilon y. \tag{4.17}$$

Let

$$\mathcal{Y} = \bigcup_{d \le \epsilon y} \mathcal{Y}_d.$$

We have

$$|\mathcal{Y}| \le \sum_{d \le \epsilon y} |\mathcal{X}_d| \le \frac{(\epsilon y)^{r+1}}{H} = \epsilon^{r+1}. \tag{4.18}$$

By (4.1), Cauchy–Schwarz inequality and Parseval's identity, we have

$$\|F * g_1\|_{L_1} = \int_{\mathbb{T}} |F * g_1(\alpha)| d\alpha \ge \int_{\mathcal{Y}} |F * g_1(\alpha)| d\alpha$$

$$\ge \int_{\mathcal{Y}} |g_2(\alpha)| d\alpha - \left(|\mathcal{Y}| \sum_{|n-N| \le H} |c_n(y,z)|^2 \right)^{\frac{1}{2}}. \tag{4.19}$$

By (4.10), (4.17) and Lemma 4.2,

$$\int_{\mathcal{Y}} |g_2(\alpha)| d\alpha \gg \sum_{d \le \epsilon y} \frac{H}{d^r} |\mathcal{Y}_d| \gg \epsilon y \gg \epsilon H^{\frac{1}{r+1}},$$

and by Lemma 3.2 and (4.18)

$$\left(|\mathcal{Y}| \sum_{|n-N| \le H} |c_n(y,z)|^2 \right)^{\frac{1}{2}} \ll \epsilon^{\frac{r+1}{2}} H^{\frac{1}{r+1}},$$

when H satisfies the condition of K in Lemma 3.2. Combining these with (4.19) implies, when $\epsilon > 0$ is small enough, $\|F_H * g_1\|_{L_1} \gg H^{\frac{1}{r+1}}$. As explained in the beginning of this section, this completes the proof of Theorem 1.3 and the proof of the lower bound case of Theorem 1.2.

5. Proof of the upper bound case of Theorem 1.2 and Theorem 1.4

In this section, we will prove the upper for the L_1 norm of the exponential sum over r-free numbers. This proof does not involve F_H. Instead we involve the difference of two Fejér kernels, see (1.6), and the argument is essentially same as Balog–Ruzsa's.

Let $D = H^{\frac{1}{r+1}}$, and let S be such that $2^{S-1} < D \leq 2^S$. Note that

$$S \leq \frac{1}{r+1}\log_2 H + 1.$$

Recall (3.1) and decompose a_n as follows:

$$a_n = \sum_{s=1}^{S} c_n(D2^{-s}, D2^{-(s-1)}) + c_n(D, N^{\frac{1}{r}}).$$

We have

$$\int_{\mathbb{T}} \left| \sum_{|n-N|\leq H} a_n e(n\alpha) \right| d\alpha$$

$$\leq \sum_{s=1}^{S} \int_{\mathbb{T}} \left| \sum_{|n-N|\leq H} c_n(D2^{-s}, D2^{-(s-1)}) e(n\alpha) \right| d\alpha + \int_{\mathbb{T}} \left| \sum_{|n-N|\leq H} c_n(D, N^{\frac{1}{r}}) e(n\alpha) \right| d\alpha. \tag{5.1}$$

The second term can be estimated as in the proof of the lower bound case: The Cauchy–Schwarz inequality, Parseval's identity and Lemma 3.2 imply that

$$\int_{\mathbb{T}} \left| \sum_{|n-N|\leq H} c_n(D, N^{1/r}) e(n\alpha) \right| d\alpha \ll H^{1/(r+1)}. \tag{5.2}$$

The remaining task is to estimate the first term. Let $K = H2^{-rs}$ and rewrite

$$\begin{aligned}
&\sum_{|n-N|\leq H} c_n(D2^{-s}, D2^{-(s-1)}) e(n\alpha) = \sum_{|n|\leq H} c_{n+N}(D2^{-s}, D2^{-(s-1)}) e((n+N)\alpha) \\
&= \sum_{|n|\leq H+K} \min\left\{1, \frac{H+K-|n|}{K}\right\} c_{N+n}(D2^{-s}, D2^{-(s-1)}) e((n+N)\alpha) \\
&- \sum_{H<|n|\leq H+K} \frac{H+K-|n|}{K} c_{n+N}(D2^{-s}, D2^{-(s-1)}) e((n+N)\alpha) \\
&=: \Sigma_3(s) + \Sigma_4(s).
\end{aligned}$$

Before estimating each term, let us explain the motivation of this decomposition. The coefficients in $\Sigma_3(s)$ come from the difference of two Fejér kernels. By involving this difference of Fejér kernels we can make the coefficient 'smooth', which is useful in the later integral. However, the payoff is that we should add the 'tails'–$\Sigma_4(s)$ from H to $H+K$. Fortunately, we can control the 'tails' using Lemma 3.1.

Let us estimate $\int_{\mathbb{T}} \Sigma_4(s)$ first. In fact by the Cauchy–Schwarz inequality and Parseval's identity

$$\int_{\mathbb{T}} |\Sigma_4(s)| d\alpha \leq \left(\sum_{H<|n|\leq H+K} |c_{n+N}(D2^{-s}, D2^{-(s-1)})|^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{N+H<n\leq N+H+K} + \sum_{N-H-K\leq n<N-H} |c_n(D2^{-s}, D2^{-(s-1)})|^2 \right)^{\frac{1}{2}}. \tag{5.3}$$

For $r=2$, we have

$$D2^{-s} = H^{\frac{1}{3}} 2^{-s} \leq H^{\frac{1}{2}-\epsilon} 2^{-s} \leq (H2^{-2s})^{\frac{1}{2}-\epsilon} = K^{\frac{1}{2}-\epsilon},$$

which implies that (5.3) satisfies the condition of Lemma 3.1. By Lemma 3.1, (5.3) is

$$\ll \begin{cases} (K(D2^{-s})^{1-r} + N^{\frac{12}{29}+\epsilon})^{\frac{1}{2}} & r=2, \\ (K(D2^{-s})^{1-r} + N^{\frac{1}{r}+\epsilon})^{\frac{1}{2}} & \text{otherwise}, \end{cases}$$

$$\ll \begin{cases} H^{\frac{1}{3}} 2^{-\frac{s}{2}} + N^{\frac{6}{29}+\epsilon} & r=2, \\ H^{\frac{1}{r+1}} 2^{-\frac{s}{2}} + N^{\frac{1}{2r}+\epsilon} & \text{otherwise}. \end{cases}$$

Thus when H is as in Theorems 1.2 and 1.4, we have

$$\sum_{s=1}^{S} \int_{\mathbb{T}} |\Sigma_4(s)| d\alpha \ll H^{\frac{1}{r+1}}. \tag{5.4}$$

The rest argument for estimating the contribution of the $\Sigma_3(s)$ part is same as Balog–Ruzsa's (2.11), but to make paper self-contained, we write down it.

LEMMA 5.1 *For any $H \geq 0, K \geq 1, 1 \leq d \leq H+K$ and M, we have*

$$\sum_{\substack{|n|\leq H+K \\ n\equiv M \pmod d}} min\left\{1, \frac{H+K-|n|}{K}\right\} e(n\alpha) \ll min\left\{\frac{H+K}{d}, \frac{1}{\|d\alpha\|}, \frac{d}{K\|d\alpha\|^2}\right\}.$$

Proof. See [2, (2.5)]. □

Recall the definition of c_n in (3.1). By the above lemma, we have

$$\begin{aligned}
&\int_{\mathbb{T}} |\Sigma_3(s)| d\alpha \\
&= \int_{\mathbb{T}} \left| \sum_{|n| \le H+K} \min\left\{1, \frac{H+K-|n|}{K}\right\} \sum_{\substack{d^r | (n+N) \\ D2^{-s} < d \le D2^{-(s-1)}}} \mu(d) e((n+N)\alpha) \right| d\alpha \\
&\ll \sum_{D2^{-s} < d \le D2^{-(s-1)}} \int_{\mathbb{T}} \left| \sum_{\substack{|n| \le H+K \\ n \equiv -N \pmod{d^r}}} \min\left\{1, \frac{H+K-|n|}{K}\right\} e(n\alpha) \right| d\alpha \\
&\ll \sum_{D2^{-s} < d \le D2^{-(s-1)}} \int_{\mathbb{T}} \min\left\{ \frac{H+K}{d^r}, \frac{1}{\|d^r\alpha\|}, \frac{d^r}{K\|d^r\alpha\|^2} \right\} d\alpha \\
&\ll \sum_{D2^{-s} < d \le D2^{-(s-1)}} \int_{\mathbb{T}} \min\left\{ \frac{H}{d^r}, \frac{1}{\|\alpha\|}, \frac{d^r}{K\|\alpha\|^2} \right\} d\alpha \\
&\ll \sum_{D2^{-s} < d \le D2^{-(s-1)}} \left(\int_0^{\frac{d^r}{H}} \frac{H}{d^r} d\alpha + \int_{\frac{d^r}{H}}^{\frac{d^r}{K}} \frac{1}{\alpha} d\alpha + \int_{\frac{d^r}{K}}^{\frac{1}{2}} \frac{d^r}{K\alpha^2} d\alpha \right) \\
&\ll \sum_{D2^{-s} < d \le D2^{-(s-1)}} \left(1 + \log \frac{H}{K} \right) \ll s2^{-s} D.
\end{aligned}$$

So it is easy to see that

$$\sum_{s=1}^{S} \int_{\mathbb{T}} |\Sigma_3(s)| d\alpha \ll H^{\frac{1}{r+1}}. \tag{5.5}$$

Hence, the upper bound case of Theorem 1.2 and Theorem 1.4 follows from (5.1), (5.2), (5.5) and (5.4).

Acknowledgements

The author is grateful to his supervisor Kaisa Matomäki for many useful discussions, reading the paper carefully and giving a lot of helpful comments and to Hao Pan and Joni Teräväinen for drawing his attention to the van der Corput methods. The author wishes to thank the referee for comments. During the work Y.-C.S. was supported by Edufi funding.

References

1. A. Balog and I. Z. Ruzsa, A new lower bound for the L1 mean of the exponential sum with the Möbius function, *Bull. London Math. Soc.* **31** no. 4 (1999), 415–418. 10.1112/S0024609398005414.

2. A. Balog and I. Z. Ruzsa, On the exponential sum over r-free integers, *Acta Math. Hungar.* **90** no. 3 (2001), 219–230. 10.1023/A:1006754512257.
3. J. Brüdern, A. Granville, A. Perelli, R. C. Vaughan and T. D. Wooley, On the exponential sum over k-free numbers, *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* **356** no. 1738 (1998), 739–761. 10.1098/rsta.1998.0183.
4. M. Filaseta and O. Trifonov, On gaps between squarefree numbers. II, *J. London Math. Soc.* **45** no. 2 (1992), 215–221. 10.1112/jlms/s2-45.2.215.
5. S. W. Graham and G. Kolesnik, van der Corput's method of exponential sums, *London Mathematical Society Lecture Note Series*, Vol. 126, Cambridge University Press, Cambridge, 1991.
6. H. Iwaniec and E. Kowalski, *Analytic Number theory*, American Mathematical Society, Providence, RI, 2004, p. 53.
7. C. Jia and A. Sankaranarayanan, The mean square of the divisor function, *Acta Arith.* **164** no. 2 (2014), 181–208. 10.4064/aa164-2-7.
8. H. L. Montgomery, Topics in multiplicative number theory, *Lecture Notes in Mathematics*, Vol. 227, Springer-Verlag, Berlin-New York, 1971.
9. E. M. Stein and R. Shakarchi, Fourier analysis, *An Introduction, Princeton Lectures in Analysis*, Vol. 1, Princeton University Press, Princeton, NJ, 2003.
10. E. C. Titchmarsh, *The Theory of The Riemann Zeta-Function*, (eds. D. R. Heath-Brown), 2nd edn. The Clarendon Press, Oxford University Press, New York, 1986.