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Super domination: Graph classes, products and enumeration

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ABSTRACT

The dominating set problem (DSP) is one of the most famous problems in combinatorial optimization. It is defined as follows. For a given graph $G = (V, E)$, a dominating set of G is a subset $S \subseteq V$ such that every vertex in $V \setminus S$ is adjacent to at least one vertex in S. Furthermore, the DSP is the problem of finding a minimum-size dominating set and the corresponding minimum size, the domination number of G.

In this, work we investigate a variant of the DSP, the super dominating set problem (SDSP), which has attracted much attention during the last years. A dominating set S is called a super dominating set of G, if for every vertex $u \in \overline{S} = V \setminus S$, there exists a $v \in S$ such that $N(v) \cap \overline{S} = N(v) \setminus S = \{u\}$. Analogously, the SDSP is to find a minimum-size super dominating set, and the corresponding minimum size, the super domination number of G. The decision variants of both the DSP and the SDSP have been shown to be $\mathcal{NP}$-hard.

In this paper, we present tight bounds for the super domination number of the neighbourhood corona product, r-clique sum, and the Hajós sum of two graphs. Additionally, we present infinite families of graphs attaining our bounds. Finally, we give the exact number of minimum size super dominating sets for some graph classes. In particular, the number of super dominating sets for cycles has quite surprising properties as it varies between values of the set $\{4, n, 2n, \frac{5n^2-10n}{8}\}$ based on $n \mod 4$.

1. Introduction

Notations and definitions. Let $G = (V, E)$ be a finite, undirected, simple graph with vertex set V and edge set E, i.e., it has no loops and no multiple edges. For each vertex $v \in V$, the set $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ refers to the *open neighbourhood* of v in G and the set $N[v] = N_G[v] = N_G(v) \cup \{v\}$ refers to the *closed neighbourhood* of v in G. If the graph G is clear from the context, we will omit the corresponding index G. The *degree of* v is the cardinality of $N(v)$. Throughout this paper, for a set $S \subseteq V(G)$, the expression $\overline{S}$ always stands for $V(G) \setminus S$.

A set $S \subseteq V$ is called a *dominating set* if every vertex in $\overline{S} = V \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the cardinality of a minimum size dominating set in G. For a detailed treatment of domination theory, we refer the reader to [13,14].

A dominating set S of G is called a *super dominating set* of G, if for every vertex $u \in \overline{S}$, there exists a $v \in S$ such that $N(v) \cap \overline{S} = N(v) \setminus S = \{u\}$. In this case, we say that v super dominates u. The *super domination number* G is the cardinality

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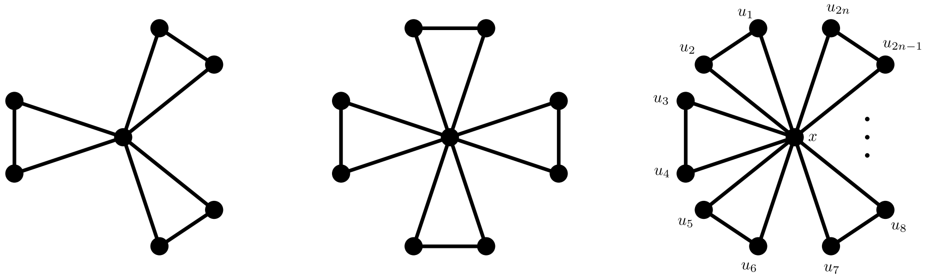

Fig. 1. Friendship graphs F_3, F_4 and F_n, respectively.

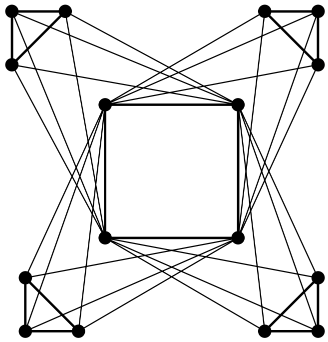

Fig. 2. $C_4 \star K_3$.

of a minimum size super dominating set of G, denoted by $\gamma_{sp}(G)$ [17]. We refer the reader to [1,5–7,15,16,22] for more details on super dominating sets of a graph. Some applications can be found in [17].

In this work, we often call vertices "black" (abbreviated by "B") if they are contained in a given super dominating set and "white" (abbreviated "W") if they are *not* contained in it. Finally, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and for $s \in \mathbb{N}$, let $[s] := \{1, 2, \ldots, s\}$.

Motivating example. In the following, we give an example for better intuitive understanding of super dominating sets and to demonstrate what kinds of applications super domination might offer. Previously, in [17], the authors gave examples related to worker–manager relationships and student–professor relationships.

Consider a setup in which the vertices of an undirected graph represent locations and edges the adjacencies or connections between these locations. Let S be a super dominating set in this underlying graph. Consider devices of types A and B which are in adjacent locations and where a device of type B measures or monitors an adjacent device of type A. Moreover, assume that an additional device of type A, adjacent to a device of type B, causes interference or disturbance with the measurements. Hence, we aim to ensure that each device of type B is adjacent to only one device of type A. If we now place devices of type A to locations corresponding to $\overline{S}$ and for each device of type A in location $s \in \overline{S}$, we place a device of type B to some adjacent location $s' \in S$ which super dominates s, then the problem of maximizing the possible number of devices of type A is the same as maximizing $|\overline{S}|$ or minimizing $|S|$.

Graph classes. We will consider the following well-known graph classes: *path graph*, *cycle graph*, *star graph*, *complete graph*, *complete bipartite graph*. A *friendship graph* F_n is a collection of n triangles, where all triangles have one vertex, the central vertex, in common (see Fig. 1).

Given two simple graphs G_1 and G_2, the *corona product* of G_1 and G_2, denoted by $G_1 \circ G_2$ is the graph arising from the disjoint union of G_1 with $|V(G_1)|$ copies of G_2, by adding edges between the ith vertex of G_1 and all vertices of the ith copy of G_2 [12]. The *neighbourhood corona product* of G_1 and G_2, denoted by $G_1 \star G_2$, is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the neighbours of the ith vertex of G_1 to every vertex in the ith copy of G_2 [10]. Thus, this graph has $|V(G_1)| \cdot (|V(G_2)| + 1)$ vertices. Fig. 2 shows $C_4 \star K_3$, where C_n is the cycle of order n and K_n is the complete graph of order n. For more results on the neighbourhood corona product of two graphs, we refer the reader to [2,8,18].

Let G_1 and G_2 be two graphs and $r \in \mathbb{N}_0$ with $r \leq \min\{\omega(G_1), \omega(G_2)\}$, where $\omega(G)$ is the clique number of G. Choose a clique K_r from each G_i, $i = 1, 2$, and form a new graph G from the union of G_1 and G_2 by identifying the two chosen r-cliques in an arbitrary manner. The graph G is called *r-clique sum* of G_1 and G_2 and denoted by $G_1 \cup_{K_r} G_2$, [21]. If $r = 0$, then $G_1 \cup_{K_0} G_2$ is just its disjoint union. $G_1 \cup_{K_i} G_2$ for $i = 1, 2$, is called vertex and edge gluing, respectively. Notice that there are sometimes several ways to create the r-clique sum of two graphs (see Fig. 3). We refer the reader for some results on the r-clique sum of two graphs to [8] (where r-clique sum is called r-gluing). Note that in some sources r-clique sum

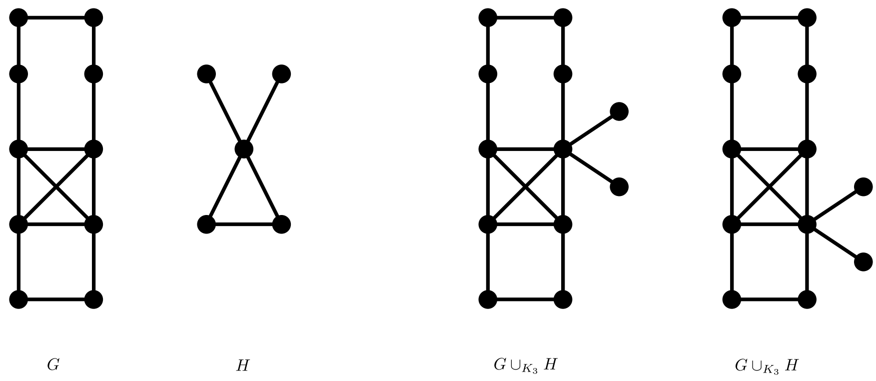

Fig. 3. Graphs G, H and all non-isomorphic graphs $G \cup_{K_3} H$, respectively.

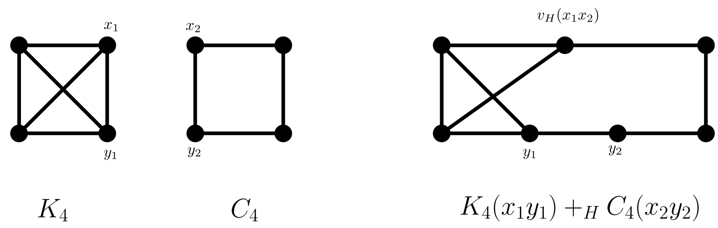

Fig. 4. Hajós construction of K_4 and C_4.

has a slightly different definition, namely sometimes, unlike this paper, some or all edges in the cliques are removed at the end of creating the clique sum [19].

Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets, an edge $x_1y_1 \in E_1$, and an edge $x_2y_2 \in E_2$, the *Hajós sum* $G = G_1(x_1y_1) +_H G_2(x_2y_2)$ is the graph obtained as follows: begin with $G = (V_1 \cup V_2, E_1 \cup E_2)$; then in G delete the edges x_1y_1 and x_2y_2, identify the vertices x_1 and x_2 as $v_H(x_1x_2)$, and add the edge y_1y_2 [11]. Fig. 4 shows the Hajós sum of K_4 and C_4 with respect to x_1y_1 and x_2y_2.

Previous work. Super dominating sets have been studied in multiple papers since the inception of the concept in 2015 [17]. In particular, the following tight lower and upper bounds are known for the super domination number.

Theorem 1.1 ([17]). *Let G be a graph of order n without isolated vertices. Then,*

$$1 \le \gamma(G) \le \frac{n}{2} \le \gamma_{sp}(G) \le n-1.$$

Besides general bounds, the super domination number is known exactly for many graph classes, some stated in the following theorem.

Theorem 1.2 ([17]). *Let $n \in \mathbb{N}$.*

(a) For the path graph P_n it holds that $\gamma_{sp}(P_n) = \lceil \frac{n}{2} \rceil$.
(b) For the cycle graph C_n it holds that

$$\gamma_{sp}(C_n) = \begin{cases} \lceil \frac{n+1}{2} \rceil & \text{if } n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil & \text{otherwise.} \end{cases}$$

(c) For the complete graph K_n, where $n \ge 2$, it holds that $\gamma_{sp}(K_n) = n-1$.
(d) For the complete bipartite graph $K_{n,m}$, where $\min\{n, m\} \ge 2$, it holds that $\gamma_{sp}(K_{n,m}) = n+m-2$.
(e) For the star graph $K_{1,n}$ it holds that $\gamma_{sp}(K_{1,n}) = n$.

Later we will refer to the following known results from [6,7].

Theorem 1.3 ([7]). *For the friendship graph F_n it holds that $\gamma_{sp}(F_n) = n+1$.*

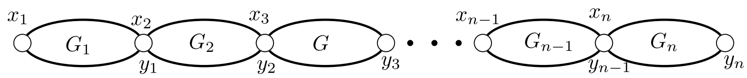

Fig. 5. Chain of n graphs $G_1, G_2, \ldots, G_n$.

Proposition 1.4 ([6]). *Let G be a disconnected graph with components G_1 and G_2. Then*

$$\gamma_{sp}(G) = \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

Theorem 1.5 ([6]). *Let $G_1, G_2, \ldots, G_n$ be a finite sequence of pairwise disjoint connected graphs and let $x_i, y_i \in V(G_i)$. Let $C(G_1, G_2, \ldots, G_n)$ be the chain of graphs $\{G_i\}_{i=1}^n$ with respect to the vertices $\{x_i, y_i\}_{i=1}^k$ obtained by identifying the vertex y_i with the vertex x_{i+1} for $i \in [n-1]$ (see Fig. 5). Then, for $n = 2$, we have:*

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 1 \leq \gamma_{sp}C(G_1, G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

Furthermore, these bounds are tight.

Besides being studied in exact graph classes, super domination has also been studied for different graph products. In particular, Dettlaff et al. [5] have studied the super domination number of lexicographic products and joins and also shown that determining the super domination number of a graph is $\mathcal{NP}$-hard. Klein et al. [15] have studied Cartesian products and (usual) corona products.

Our results. In this paper, we continue the study of the super domination number of a graph, started in [6,8]. First in Section 2, we present a key lemma which will be used throughout this paper. In Section 3, we find the exact value of the super domination number of the neighbourhood corona product of two graphs. In Section 4, we present tight lower and upper bounds on the r-clique sum of two graphs and provide infinite families of graphs attaining these bounds. We study the super domination number of the Hajós sum of two graphs and find tight upper and lower bounds for it, together with infinite families of examples attaining the bounds, in Section 5. In Section 6, we count exactly the number of minimum size super dominating sets of some graph classes. Finally, in Section 7, we give some conclusions and suggestions for future work.

2. Key lemma

We introduce a technical key lemma for analysing super dominating sets. It will be needed in most of the following results.

Lemma 2.1. *Let S be a super dominating set in a graph G.*

(a) Then there is a super dominating set S' with same cardinality with $\overline{S} \subseteq S'$ and $\overline{S'} \subseteq S$.
Furthermore, there is a bijective function $f : \overline{S'} \to \overline{S}$ so that $f(a) = b$ holds if and only if a super dominates b for the super dominating set S and b super dominates a for the super dominating set S'.
(b) Let $D = S \cap S'$. Then $V(G)$ can be partitioned as $V(G) = \overline{S} \,\dot\cup\, \overline{S'} \,\dot\cup\, D$, where it holds that $S = \overline{S'} \,\dot\cup\, D$, $S' = \overline{S} \,\dot\cup\, D$.
(c) Let S have cardinality $|V(G)|/2$. Then $\overline{S}$ is a super dominating set with the same cardinality and it holds that $\overline{S} = S'$ and $\overline{S'} = S$.
Furthermore, each vertex in S super dominates exactly one vertex in $\overline{S}$ and vice versa, i.e., the function $f : S \to S'$ from (a) is uniquely determined.

The following observation gives an interesting interpretation of (a).

Observation 2.2. *Lemma 2.1(a) states that in the induced graph $G[\overline{S} \cup \overline{S'}]$ the set of edges between the vertex sets $\overline{S}$ and $\overline{S'}$ forms a perfect matching.*

Proof (*Lemma 2.1*).
(a) Let $S \subseteq V(G)$ be a super dominating set in G. Then for each $b \in \overline{S}$ there exists a vertex $a \in S$ such that $N(a) \cap \overline{S} = \{b\}$. We construct the new super dominating set S' by replacing each $a \in S$ by the corresponding $b \in \overline{S}$. Denote $A \subseteq S$ as the set of all vertices a removed from S during this process. Let $f : A \to \overline{S}$ be the corresponding function with $f(a) = b$ for each $a \in A$ and $b \in \overline{S}$ super dominated by a. By construction, $A = \overline{S'}$, $|S| = |S'|$ and $|\overline{S}| = |\overline{S'}|$ hold. As each vertex $a \in \overline{S'}$ can super dominate only one vertex $b \in \overline{S}$, f is well defined. By construction, for each b we have only one a, i.e., f is also injective. Because of $|\overline{S'}| = |\overline{S}|$, it is also bijective.

Claim. S' is a super dominating set in G.

Proof (Claim): Let $f(a) = b$ with $a \in \overline{S'}$, $b \in \overline{S}$. Assume that b does not super dominate a. Then b is also adjacent to another $a' \in \overline{S'}$. Thus, there is a $b' \in \overline{S}$ with $f(a') = b'$. This would mean that a' super dominates b', but it is also adjacent to b, which is a contradiction. Thus, b super dominates a, and the claim follows.

(b) This follows easily from (a).

(c) Here $D = S \cap S' = \emptyset$ holds. From (b) it follows that $S = \overline{S'}$ and $S' = \overline{S}$. Since each vertex can super dominate at most one other vertex, each vertex in S (and each vertex in S') super dominates exactly one vertex. Thus, f is uniquely determined. □

3. Super domination number of neighbourhood corona product of two graphs

In this section, we study the super domination number of the neighbourhood corona product of two graphs. Let G and H be two graphs of orders $n, m \in \mathbb{N}$, respectively. It is clear that

$$\gamma_{sp}(G \star H) \leq \gamma_{sp}(G) + nm, \tag{1}$$

since if we consider all vertices of all copies of H in our super dominating set, then we only need to find a super dominating set for G and the claim follows by the definition of a super dominating set.

Later (in Corollary 3.4) we will show that, under some conditions, it holds that $\gamma_{sp}(G\star H) = n(\gamma_{sp}(H)+1)$. The following proposition shows that this is an improvement over the trivial upper bound from (1) for $m \geq 2$.

Proposition 3.1. *Let G and H be two connected graphs of orders n and $m \neq 1$, respectively. Then*

$$n(\gamma_{sp}(H) + 1) < \gamma_{sp}(G) + nm.$$

Proof. By Theorem 1.1, we know that $\gamma_{sp}(H) \leq m - 1$. So we have $n(\gamma_{sp}(H) + 1) = n\gamma_{sp}(H) + n \leq nm$. Hence, we have $n(\gamma_{sp}(H) + 1) < nm + \gamma_{sp}(G)$ and therefore, we have the result. □

Next we present a tight upper bound for $\gamma_{sp}(G \star H)$.

Theorem 3.2. *Let G and H be two connected graphs of orders $n, m \in \mathbb{N}$. Then*

$$\gamma_{sp}(G \star H) \leq n(\gamma_{sp}(H) + 1).$$

Proof. Let S_H be a super dominating set for H. We create a set S for $G \star H$ by placing all vertices of G in S. Next, for each copy of H, we place all vertices corresponding to S_H in S. In the following, we show that S is a super dominating set for $G \star H$.

Let $u \in \overline{S}$. Then $u \in V(H')$ for some copy of H. Let $v \in V(H')$ super dominate u in H'. If v does not super dominate u in $G\star H$, then v has another neighbour in $\overline{S}\cap(V(G\star H)\setminus V(H'))$. However, this is not possible, since $V(G) \subseteq S$ and there are no edges between different copies of H. Thus, S is a super dominating set for $G\star H$ with cardinality $n\gamma_{sp}(H)+n = n(\gamma_{sp}(H)+1)$ and the assertion follows. □

In the following theorem, we show that the upper bound in Theorem 3.2 is actually the super domination number of $G \star H$, when G and $H \neq K_1$ are connected graphs and $\gamma_{sp}(H) < m - 1$ or it holds that $H = K_m$.

Theorem 3.3. *Let G and H be two connected graphs of orders n and $m \neq 1$, respectively, where it holds that $\gamma_{sp}(H) < m - 1$ or it holds that $H = K_m$. Then*

$$n(\gamma_{sp}(H) + 1) \leq \gamma_{sp}(G \star H).$$

Proof. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$. Let S be a super dominating set for $G \star H$. Let H_w be the copy of H corresponding to $w \in V(G)$. Then it holds that $V(G \star H) = V(G) \cup \bigcup_{w\in V(G)} V(H_w)$.

Claim 1. *In each copy of H, the set S has at least $\gamma_{sp}(H)$ vertices.*

Proof (Claim 1): For a given $w \in V(G)$, consider the copy H_w of H. Assume that $r := |S \cap V(H_w)| < \gamma_{sp}(H)$. As H is connected, it follows by Theorem 1.1 that $r < \gamma_{sp}(H) \leq m - 1$. Thus, this copy of H has at least two vertices which are contained in $\overline{S}$.

As $S \cap V(H_w)$ is *not* a super dominating set of H_w, there must exist a vertex $u \in \overline{S} \cap V(H_w)$, such that there does *not* exist a vertex $v \in S \cap V(H_w)$ for which $N_{H_w}(v) \cap \overline{S} = \{u\}$. On the other hand, for this $u \in \overline{S} \cap V(H_w)$, there exists a vertex $v \in (V(G \star H) \setminus V(H_w)) \cap S$ such that $N_{G\star H}(v) \cap \overline{S} = \{u\}$.

As $v \in V(G \star H) \setminus V(H_w)$ holds, we have two possibilities. Either $v \in V(G)$ or $v \in V(H_x)$ for some $x \in V(G) \setminus \{w\}$.

Firstly, $v \in V(G)$ cannot hold, since then all vertices in $V(H_w) \setminus \{u\}$ would lie in S, because v is adjacent to all of them. Thus, $r = m - 1$ holds, a contradiction.

Secondly, $v \in V(H_x)$ cannot hold, as there are no adjacent vertices between different copies of H. Thus, we have a contradiction again. Claim 1 follows.

In the following, for a given $w \in V(G)$, we define a block of vertices $B_w(G \star H) = \{w\} \cup V(H_w)$ (or shortly B_w). The blocks B_w clearly partition the vertex set $V(G \star H)$.

For the given super dominating set S, we define a block as *over-satisfied*, if it has more than $\gamma_{sp}(H) + 1$ vertices (note that for $m = 1$, i.e., $H = K_1$, a block can never be over-satisfied), as *satisfied*, if it has exactly $\gamma_{sp}(H) + 1$ vertices, and as *under-satisfied*, if it has less than $\gamma_{sp}(H) + 1$ vertices. Note that by Claim 1, an under-satisfied block B_w has always exactly $\gamma_{sp}(H)$ vertices in S and that w lies in $\overline{S}$.

Claim 2. *Let $w \in V(G)$ and let B_w be under-satisfied. Let $x \in V(G) \setminus \{w\}$ be chosen so that a vertex in B_x super dominates $w \in V(G)$ (and $\{w, x\} \in E(G)$). Then B_x is an over-satisfied block.*

Proof (Claim 2): As we have mentioned above, in an under-satisfied block B_w there are exactly $\gamma_{sp}(H)$ vertices in S and $w \in \overline{S}$. Thus, at least one vertex $y \in V(H_w)$ lies in $\overline{S}$.

First, assume that $x \in \overline{S}$. On the one hand, each vertex $z \in V(H_w)$ adjacent to y is also adjacent to x, i.e., z cannot super dominate y. On the other hand, each vertex $z \notin V(H_w)$ adjacent to y is also adjacent to w, i.e., z cannot super dominate y either. In summary, y cannot be super dominated by another vertex of S. Thus, by contradiction $x \in S$ follows.

We continue by dividing the proof into two cases.

Case 1: $\gamma_{sp}(H) < m - 1$.

Assume that there are $b, c \in V(H_x) \cap \overline{S}$. Each vertex $z \in V(H_x)$, adjacent to b or c, is also adjacent to w, and if $z \notin V(H_x)$ is adjacent to b, then it is also adjacent to c and vice versa. So neither b nor c can be super dominated by another vertex. Thus, there are at least $\gamma_{sp}(H) + 1$ vertices in $V(H_x) \cap S$, and B_x is over-satisfied because x is in S.

Case 2: $H = K_m$.

Assume that there is a $b \in V(H_x) \cap \overline{S}$. By the choice of $x \in V(G)$, one vertex of B_x super dominates w. This cannot be x, as x is also adjacent to y. On the other hand, as $H = K_m$, each other vertex in B_x super dominating w is also adjacent to b, a contradiction. Thus, there does not exist such b, and B_x is over-satisfied.

Claim 2 follows.

Claim 3. *No two under-satisfied blocks B_w and $B_{w'}$ can be assigned to the same over-satisfied block B_x.*

Proof (Claim 3): Assume that this does not hold. If there is a $d \in B_x$ super dominating $w \in V(G)$, then it super dominates also $w' \in V(G)$ and vice versa. On the other hand, no vertex can super dominate two vertices. Thus, we have a contradiction, and Claim 3 follows.

By Claim 1, for each under-satisfied block B_w, where $w \in V(G)$, we have exactly $\gamma_{sp}(H)$ vertices in S. By Claims 2 and 3, we have a corresponding over-satisfied block with at least $\gamma_{sp}(H) + 2$ vertices in S.

In total, we have at least $\gamma_{sp}(H) + 1$ vertices in S for each block. Thus, each super dominating set has cardinality of at least $n(\gamma_{sp}(H) + 1)$ in $G \star H$. □

By using Theorems 3.2 and 3.3, we have the following result which gives us the exact value of the super domination number of $G \star H$.

Corollary 3.4. *Let G and H be two connected graphs of orders n and $m \neq 1$, respectively, with $\gamma_{sp}(H) < m - 1$ or $H = K_m$. Then*

$$\gamma_{sp}(G \star H) = n(\gamma_{sp}(H) + 1).$$

In the following example, we show that "$\gamma_{sp}(H) < |V(H)| - 1$ or $H = K_m$" is a necessary condition for Theorem 3.3 and Corollary 3.4.

Example 3.5. Consider the graph G from Fig. 6. One can easily check that $\gamma_{sp}(G) = 3$ and the set of black vertices in $P_3 \star G$ is a super dominating set for G. We have $\gamma_{sp}(P_3 \star G) = 11 < 12 = 3 \cdot (\gamma_{sp}(G) + 1)$.

Remark 3.6. The class of graphs with n vertices with super domination number $n - 1$ was determined in [5]. They give a set of rules which can be used to construct these graphs. Furthermore, although the authors in [5] do not mention it, this class is exactly the class of threshold graphs without the empty graphs (which have super domination number n). The class of *threshold graphs* [3] is the class of graphs which can be constructed by repeating the two following two operations: Adding an isolated vertex and adding a vertex with is adjacent to each other vertex (at this stage). In [9] the class of threshold graphs was characterized as the class of graphs in which no graph contains as an induced subgraph the path P_4, the cycle C_4 or two disconnected paths P_2, all having 4 vertices.

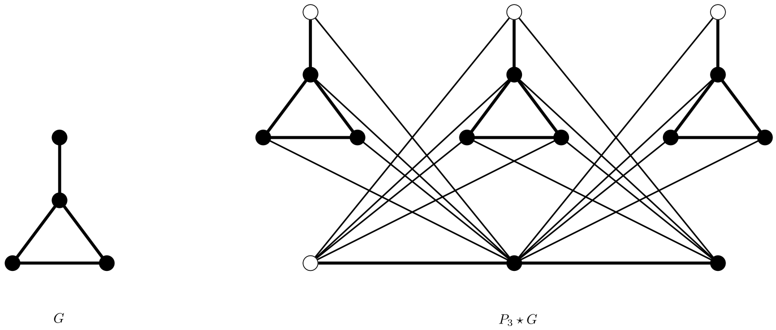

Fig. 6. Graphs G and $P_3 \star G$, respectively.

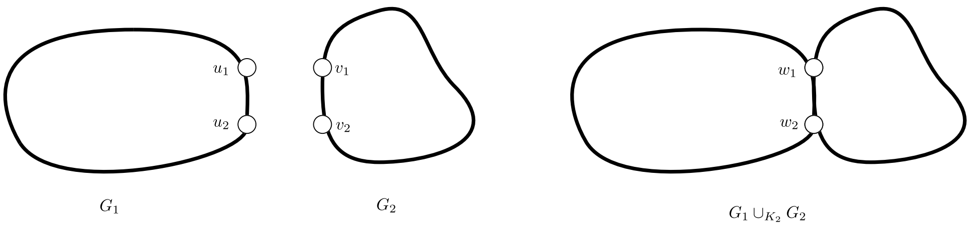

Fig. 7. Graphs G_1, G_2 and $G_1 \cup_{K_2} G_2$, respectively.

Interestingly, the value of Corollary 3.4 is equal to the value attained for the (usual) corona product of two graphs in [15, Theorem 10].

We end this section by determining the super domination number of the neighbourhood corona product of some specific graphs. These results follow directly from Theorems 1.2 (a)–(d) and 1.3 and Corollary 3.4.

Example 3.7. Let $n, m \in \mathbb{N}$.

(a) $\gamma_{sp}(C_n \star P_{2m}) = n(m+1)$.
(b) $\gamma_{sp}(P_{2n} \star C_{4m}) = 4nm + 2n$.
(c) $\gamma_{sp}(C_{2n} \star K_m) = 2nm$ for $m \geq 2$.
(d) $\gamma_{sp}(C_n \star K_{2,3}) = 4n$.
(e) $\gamma_{sp}(P_n \star F_n) = n^2 + 2n$.

4. Super domination number of the r-clique sum of two graphs

In this section, we give exact upper and lower bounds for the super domination number of the r-clique sum of two graphs. In Theorem 4.1, we consider the r-clique sum of two graphs. It is a generalization of the cases $r = 0$ (Proposition 1.4) and $r = 1$ (Theorem 1.5) to general $r \in \mathbb{N}_0$.

Theorem 4.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with clique number at least $r \in \mathbb{N}_0$. Then it holds:*

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - r \leq \gamma_{sp}(G_1 \cup_{K_r} G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

Proof. Let $G = G_1 \cup_{K_r} G_2 = (V, E)$. The case $r = 0$ holds by Proposition 1.4 since for every two graphs G_1 and G_2, $G_1 \cup_{K_0} G_2$ is their disjoint union, and then it follows:

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 0 \leq \gamma_{sp}(G_1 \cup_{K_0} G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

The case $r = 1$ holds by Theorem 1.5, since $G_1 \cup_{K_1} G_2$ is the chain of two graphs and then it follows:

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 1 \leq \gamma_{sp}(G_1 \cup_{K_1} G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

Hence, we assume from now on that $r \geq 2$. Let S_1 and S_2 be two minimum size super dominating sets for G_1 and G_2, respectively. Let the vertex sets $V' = \{v'_i \mid i \in [r]\}$ and $V'' = \{v''_i \mid i \in [r]\}$ form r-cliques in G_1 and G_2, respectively. Furthermore, let us create a vertex w_i by identifying the vertices v'_i and v''_i for each $i \in [r]$ (see Fig. 7 for the case $r = 2$)

and denote $W = w_i \mid i \in [r]$. Observe that for the graph $G = (V, E)$ it holds that $V = (V_1 \cup V_2 \cup W) \setminus (V' \cup V'')$. We divide the proof into two parts, namely the lower and the upper bound.

Lower bound $\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - r \leq \gamma_{sp}(G_1 \cup_{K_r} G_2)$. We use Lemma 2.1 to show that we can assume two cases for the minimum size super dominating set S:

- $W \subseteq S$,
- $|W \cap S| = r - 1$ and one vertex of W, say w_1, super dominates the missing vertex, say w_r.

To show this, assume that $|S \cap W| \leq r - 1$ and, without loss of generality, $w_r \notin S$. By Lemma 2.1, there is another minimum size super dominating set S', the corresponding set $D = S \cap S'$ and a bijective function $f : \overline{S'} \to \overline{S}$ with the mentioned characteristics. If no vertex of $W \cap S$ super dominates another vertex, then $W \cap S \subseteq D$ follows, and we can replace S by S', where $W \subseteq S'$ holds. On the other hand, if $W \nsubseteq S$ and one vertex of $W \cap S$ super dominates another vertex, then $|W \cap S| = r - 1$ and, without loss of generality, w_1 super dominates the missing vertex w_r. This finishes the proof of this assumption. We split our considerations into two cases.

(i) $W \subseteq S$.
Let

$$S_1 = (S \cap V_1) \cup V'$$

and

$$S_2 = (S \cap V_2) \cup V''.$$

Let $j \in [2]$. As $V_j \setminus S_j \subseteq V \setminus S$, if $v \in V_j \setminus S_j$ is super dominated by some vertex u in S, then it is super dominated by a vertex corresponding to u in S_j. Thus, S_1 is a super dominating set for G_1, and S_2 is a super dominating set for G_2. As we replace r vertices from S by $2r$ other vertices to reach $S_1 \cup S_2$, it follows that $|S_1| + |S_2| - r \leq \gamma_{sp}(G_1 \cup_{K_r} G_2)$.

(ii) $|W \cap S| = r - 1$.
Let

$$S_1 = (S \cap V_1) \cup (V' \setminus \{v'_r\})$$

and

$$S_2 = (S \cap V_2) \cup (V'' \setminus \{v''_r\}).$$

The vertex v'_r is super dominated by v'_1, and v''_r is super dominated by v''_1. Let $j \in [2]$. As $V_1 \setminus \left(S_1 \cup \{v'_r\}\right) \subseteq V \setminus S$ and $V_2 \setminus \left(S_2 \cup \{v''_r\}\right) \subseteq V \setminus S$, if any other vertex $v \in V_j \setminus S_j$ is super dominated by some vertex u in S, then it is super dominated by a vertex corresponding to u in S_j.
As we replace $r-1$ vertices from S by $2r-2$ other vertices to reach $S_1 \cup S_2$, it follows that $|S_1| + |S_2| - r \leq \gamma_{sp}(G_1 \cup_{K_r} G_2)$.

Upper bound $\gamma_{sp}(G_1 \cup_{K_r} G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$.
As in the proof of the lower bound, by Lemma 2.1 we can assume two cases for the minimum size super dominating set S_1. Note that this leads later to three cases in total.

- $V' \subseteq S_1$,
- $|V' \cap S_1| = r - 1$ and one vertex of V', say v'_1, super dominates the missing vertex, say v'_r.

(By symmetry, this holds analogously for S_2 and V''.)

(i) $V' \subseteq S_1$ and $V'' \subseteq S_2$.
Let $i \in [r]$. If we have $|N(v'_i) \setminus S_1| = 1$, then we denote $\{x_i\} = N(v'_i) \setminus S_1$. If $|N(v'_i) \setminus S_1| \neq 1$ but $\left|N(v''_i) \setminus S_2\right| = 1$, then we denote $\{x_i\} = N(v''_i) \setminus S_2$. Denote $X = \{x_i \mid i \in [r],\ \ x_i \text{ exists}\}$. (Notice that the vertices x_i are included in X only if they exist.)
Let

$$S = (S_1 \cup S_2 \cup W \cup X) \setminus (V' \cup V'').$$

Clearly, if a vertex $u \in V \setminus S$ was super dominated by a vertex v in $S_1 \setminus V'$ or in $S_2 \setminus V''$, then it is now super dominated by $v \in S \setminus W$. Let us then assume that the vertex $u \in V \setminus S$ was super dominated by a vertex v in $S_1 \cap V'$ or in $S_2 \cap V''$. Since $u \notin X$, we have $v \in S_2 \cap V''$. We may assume that $v = v''_h$ for some $h \in [r]$. Thus, $x_h \in X \subseteq S$ and $N_G(w_h) \setminus S = \{u\}$. Thus, S is a super dominating set of G.
As we replace $2r$ vertices from $S_1 \cup S_2$ by at most $2r$ other vertices to reach S, it follows that $|S| \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$, as claimed.

(ii) Either $|S_1 \cap V'| = r - 1$ or $|S_2 \cap V''| = r - 1$.
Notice that the two cases are essentially identical and we may assume, without loss of generality, that $|S_2 \cap V''| = r - 1$.

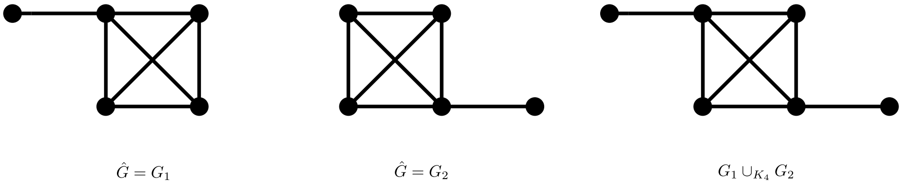

Fig. 8. $G_1 = G_2$ and $G_1 \cup_{K_4} G_2$, respectively.

As we have mentioned above, we can assume that v_1'' super dominates v_r'' in S_2. Let us have $x_i \in N(v_i') \setminus S_1$ for each $2 \leq i \leq r$ (if such vertex exists). Define $X = \{x_i \mid 2 \leq i \leq r, \quad x_i \text{ exists}\}$.
Let

$$S = (S_1 \cup S_2 \cup W \cup X) \setminus (V' \cup V'').$$

Again, if a vertex $u \in V \setminus S$ was super dominated by a vertex v in $S_1 \setminus V'$ or in $S_2 \setminus V''$, then it is now super dominated by $v \in S \setminus W$. If a vertex $u \in V \setminus S$ was super dominated by a vertex v in V' or in V'', then $v = v_1'$ since $u \notin W \cup X$ and the vertices in V'' could super dominate only the vertex v_r''. Since v_1'' super dominates v_r'', the vertex w_1 super dominates u. Thus, S is a super dominating set of G.
As we replace $2r - 1$ vertices from $S_1 \cup S_2$ by at most $2r - 1$ other vertices to reach S, it follows that $|S| \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$.

(iii) $|S_1 \cap V'| = r - 1$ and $|S_2 \cap V''| = r - 1$.
We can assume, without loss of generality, that v_1' super dominates v_r' in S_1 and v_h'' super dominates v_p'' in S_2, where $1 \leq h < p \leq r$.
Let

$$S = (S_1 \cup S_2 \cup W) \setminus (V' \cup V'').$$

In this case, no vertex in V' or in V'' super dominates any vertex in $V_1 \setminus \left(S_1 \cup V'\right)$ or $V_2 \setminus \left(S_2 \cup V''\right)$, respectively. Furthermore, $V \setminus S \subseteq (V_1 \setminus S_1) \cup (V_2 \setminus S_2)$ holds. Thus, if $u \in V \setminus S$, then $u \in (V_1 \setminus S_1) \cup (V_2 \setminus S_2)$ and u is super dominated by a vertex in S_1 or in S_2. That same vertex is in S and super dominates u in S. Thus, S is a super dominating set of G.
As we replace $2r - 2$ vertices from $S_1 \cup S_2$ by $r \geq 2$ other vertices to reach S, it follows that $|S| \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$.

This finishes the proof. □

In the following remark, we show that the lower bound in Theorem 4.1 is tight.

Remark 4.2. Let $r \in \mathbb{N} \setminus \{1\}$ and let an $(r + 1)$-vertex graph $\hat{G} = G_1 = G_2$ be formed from the complete graph K_r by attaching a single leaf to each vertex in K_r. Let G be the graph with two leaves attached to each vertex, obtained by the r-clique sum $G_1 \cup_{K_r} G_2$ (see Fig. 8). Then one can easily check that $\gamma_{sp}(G) = r = r + r - r = \gamma_{sp}(\hat{G}) + \gamma_{sp}(\hat{G}) - r$. So, the lower bound in Theorem 4.1 is tight.

We finish this section by showing that also the upper bound in Theorem 4.1 is tight.

Remark 4.3. Let $r \in \mathbb{N}$ and let an $3r$-vertex graph $\hat{G} = G_1 = G_2$ be formed from K_r by attaching two leaves to each vertex in K_r. Let G be the graph with four leaves attached to each vertex, obtained by the r-clique sum $G = G_1 \cup_{K_r} G_2$ (see Fig. 9). Then one can easily check that $\gamma_{sp}(G_1 \cup_{K_r} G_2) = 4r = 2r + 2r = \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$. So, the upper bound in Theorem 4.1 is tight.

5. Super domination number of the Hajós sum of graphs

In this section, we consider the Hajós sum of two graphs. We start with a tight lower bound for the super domination number of the Hajós sum of two graphs.

Theorem 5.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets, $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. Then for the Hajós sum*

$$G = G_1(x_1y_1) +_H G_2(x_2y_2),$$

it holds:

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 2 \leq \gamma_{sp}(G) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

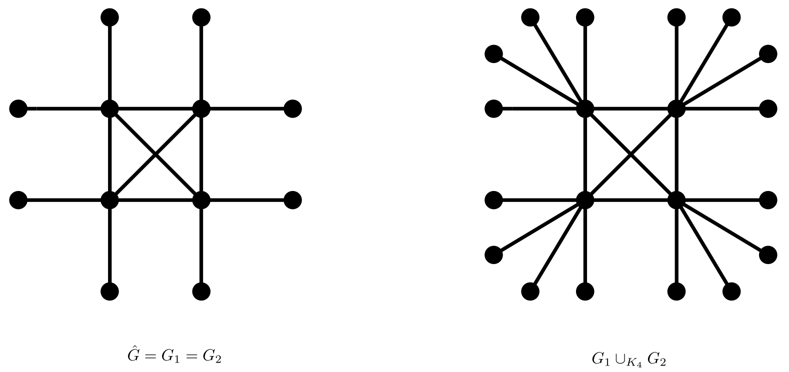

Fig. 9. $G_1 = G_2$ and $G_1 \cup_{K_4} G_2$, respectively.

Proof. Suppose that we have formed G. Let $v := v_H(x_1x_2)$ be the vertex identifying the vertices x_1 and x_2. Again we divide the proof into two parts, namely the lower and the upper bound.

Lower bound $\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 2 \leq \gamma_{sp}(G)$.

Let S be a super dominating set for G. With Lemma 2.1, we may assume that $v \in S$. Moreover, the cases $v \in S, y_1 \in S, y_2 \in V \setminus S$, and $v \in S, y_1 \in V \setminus S, y_2 \in S$ are symmetrical and can be studied together. Hence, we have the following three cases.

(i) $v \in S, y_1 \in V \setminus S, y_2 \in V \setminus S$.
There might exist one vertex in $V \setminus S$ which is super dominated by v. If such vertex exists, denote it by v'. Then it holds that $N(v) \setminus S = \{v'\}$. We may assume, without loss of generality, that $v' \in V_1$. Observe that $v' \neq y_1$ holds. Let

$$S_1 = (S \cup \{x_1, v'\}) \setminus (V_2 \cup \{v\})$$

(or $S_1 = (S \cup \{x_1\}) \setminus (V_2 \cup \{v\})$ if v' does not exist) and

$$S_2 = (S \cup \{x_2\}) \setminus (V_1 \cup \{v\}).$$

Let $j \in [2]$. We have $V_j \setminus S_j \subseteq V \setminus S$. Let $u \in V_j \setminus S_j$. If u is super dominated in G by some vertex $w \in S$, then $w \neq v$, since $u \neq v'$, and thus $w \in S_j$. Hence, S_1 is a super dominating set for G_1 and S_2 is a super dominating set for G_2. Thus,

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) \leq \gamma_{sp}(G) + 2.$$

(ii) $v \in S, y_1 \in S, y_2 \in V \setminus S$.
Let

$$S_1 = (S \cup \{x_1\}) \setminus (V_2 \cup \{v\}), \quad S_2 = (S \cup \{x_2, y_2\}) \setminus (V_1 \cup \{v\}).$$

In comparison to Case (i), y_2 is added to S_2, as it could occur that y_1 super dominates y_2 in S, but y_2 is not super dominated in S_2. Observe that a vertex v', which was super dominated by v in S, as in Case (i), does not have to be added. First, let $v' \in V_1$. Then because of $y_1 \in S$, x_1 super dominates v' in S_1. Second, let $v' \in V_2$. Then because of $y_2 \in S_2$, x_2 super dominates v' in S_2.
So S_1 is a super dominating set for G_1 and S_2 is a super dominating set for G_2. Thus,

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) \leq \gamma_{sp}(G) + 2.$$

(iii) $v \in S, y_1 \in S, y_2 \in S$.
Let

$$S_1 = (S \cup \{x_1\}) \setminus (V_2 \cup \{v\}), \quad S_2 = (S \cup \{x_2\}) \setminus (V_1 \cup \{v\}).$$

Observe that a vertex v', which was super dominated by v in S, as in Case (i), does not have to be added. Because $y_1 \in S, y_2 \in S$, x_1 super dominates v' in S_1 or x_2 super dominates v' in S_2.
So S_1 is a super dominating set for G_1 and S_2 is a super dominating set for G_2. Thus,

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) \leq \gamma_{sp}(G) + 1.$$

Upper bound $\gamma_{sp}(G) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$.

Let S_1 and S_2 be super dominating sets for G_1 and G_2, respectively. Let $j \in [2]$. Let $f_j : \overline{S'_j} \to \overline{S_j}$ be the bijective function introduced in Lemma 2.1 where S'_j is a super dominating set in G_j. In particular, we can now assume that $y_1 \in S_1$ and $y_2 \in S_2$. Moreover, we may assume that if $x_1 \in V_1 \setminus S_1$, then it is super dominated by y_1. Indeed, if x_1 is not super dominated by y_1, then y_1 does not super dominate any vertex in V_1 and so $y_1 \in D$. Thus, $y_1, x_1 \in S'_1$ and we could consider S'_1 instead of S_1. This holds analogously for x_2. We have the following three cases:

(i) $x_1 \in S_1$, $x_2 \in S_2$.
There might be vertices $t_1 \in V_1 \setminus S_1$ and $t_2 \in V_2 \setminus S_2$ such that t_1 is super dominated by x_1, and t_2 is super dominated by x_2. By the definition of a super dominating set, if t_1 exists, then all neighbours of x_1 are in S_1 except t_1, and the same is true for x_2. Without loss of generality, if exactly one exists, we assume it is t_1. Let

$$S = (S_1 \cup S_2 \cup \{t_1, v\}) \setminus \{x_1, x_2\}.$$

(or $(S_1 \cup S_2 \cup \{v\}) \setminus \{x_1, x_2\}$, if t_1 does not exist). The set S is super dominating in G since each vertex in $V \setminus S$ is super dominated by the same vertex in $S_1 \cup S_2$ as before except possibly t_2 which is super dominated by the vertex v. Thus,

$$\gamma_{sp}(G) \le \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

(ii) $x_1 \in V_1 \setminus S_1$, $x_2 \in S_2$.
By assumption, y_1 now super dominates x_1 and thus, $f_1(y_1) = x_1$. Thus, by Lemma 2.1, for the super dominating set S'_1 of G_1 it holds that $x_1 \in S'_1$, $y_1 \in V_1 \setminus S'_1$ and $N(x_1) \setminus S'_1 = \{y_1\}$, i.e., x_1 super dominates y_1. Let

$$S = (S'_1 \cup S_2 \cup \{y_1, v\}) \setminus \{x_1, x_2\}.$$

The set S is super dominating in G since each vertex in $V \setminus S$ is super dominated by the same vertex in $S'_1 \cup S_2$ as before with the (possible) exception that the vertex which was super dominated by x_2 is now super dominated by v. Thus,

$$\gamma_{sp}(G) \le \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

(iii) $x_1 \in V_1 \setminus S_1$, $x_2 \in V_2 \setminus S_2$.
Let $j \in [2]$. As in Case (ii) for x_1, by assumption y_j now super dominates x_j. Thus, in S'_1, x_1 super dominates y_1. Let us now consider the set

$$S = (S'_1 \cup S_2 \cup \{v\}) \setminus \{x_1\}.$$

The set S is super dominating in G since y_1 is super dominated by y_2, v does not super dominate any vertex and all other vertices in $V \setminus S$ are super dominated by the same vertices in $S'_1 \cup S_2$ as before. Thus,

$$\gamma_{sp}(G) \le \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

This finishes the proof. □

We finish this section by showing that both the lower and upper bound of Theorem 5.1 are tight.

Remark 5.2. Consider $G_1 = C_{4p+2}$ and $G_2 = C_3$. Then one can easily check that $G_1(x_1y_1) +_H G_2(x_2y_2) = C_{4p+4}$, for any two edges x_1y_1 and x_2y_2 from G_1 and G_2, respectively. By Theorem 1.2(b), we have $\gamma_{sp}(G_1(x_1y_1) +_H G_2(x_2y_2)) = 2p+2$, $\gamma_{sp}(G_1) = 2p+2$ and $\gamma_{sp}(G_2) = 2$. Thus, the lower bound of Theorem 5.1 is tight.

Remark 5.3. Consider the cycles $G_1 = C_{4p}$ and $G_2 = C_3$. Then, we have $G_1(x_1y_1) +_H G_2(x_2y_2) = C_{4p+2}$, for any two edges x_1y_1 and x_2y_2 from G_1 and G_2, respectively. By Theorem 1.2(b), we have $\gamma_{sp}(G_1(x_1y_1) +_H G_2(x_2y_2)) = 2p+2$, $\gamma_{sp}(G_1) = 2p$ and $\gamma_{sp}(G_2) = 2$. Thus, the upper bound of Theorem 5.1 is tight.

6. The number of minimum size super dominating sets of some graphs

In this section, we initiate the study of the number of minimum size super dominating sets of a graph. Similar research has been conducted, for example for the domination number in multiple papers, see for example, [4]. Let $\mathcal{N}_{sp}(G)$ be the family of super dominating sets of a graph G with cardinality $\gamma_{sp}(G)$ and let $N_{sp}(G) = |\mathcal{N}_{sp}(G)|$. By Theorem 1.1 and Lemma 2.1, for every non-empty graph G we have $N_{sp}(G) \ge 2$. In the following, we consider some special graph classes and compute their N_{sp} values. Following Theorem 1.2, by an easy argument, we have the following result for N_{sp} of the complete graph, the complete bipartite graph and the star graph.

Theorem 6.1.

(a) If K_n is the complete graph, then $N_{sp}(K_n) = n$.
(b) If $K_{n,m}$ is the complete bipartite graph, then $N_{sp}(K_{n,m}) = nm$, where $\min\{n, m\} \ge 2$.

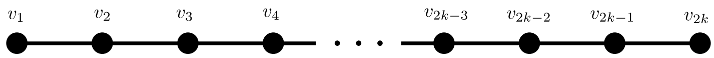

Fig. 10. Path graph of order $2k \geq 2$.

(c) If $K_{1,n}$ is the star graph, then $N_{sp}(K_{1,n}) = n + 1$.

Proof.

(a) By Theorem 1.2(c), $\gamma_{sp}(K_n) = n - 1$ holds. Thus, $|\overline{S}| = 1$ follows. Clearly, any single vertex of K_n can be chosen as $\overline{S}$. As K_n has exactly n vertices, it follows that $N_{sp}(K_n) = n$.

(b) Let $\min\{n, m\} \geq 2$. By Theorem 1.2(d), $\gamma_{sp}(K_{n,m}) = n + m - 2$ holds. Thus, $|\overline{S}| = 2$ follows. Clearly, these two vertices of $\overline{S}$ have to be chosen from two different sides of the bipartition. It follows that $N_{sp}(K_{n,m}) = nm$.

(c) By Theorem 1.2(e), $\gamma_{sp}(K_{1,n}) = n$ holds. Thus, $|\overline{S}| = 1$ follows. Clearly, any single vertex of $K_{1,n}$ can be chosen as $\overline{S}$. As $K_{1,n}$ has exactly $n + 1$ vertices, it follows that $N_{sp}(K_{1,n}) = n + 1$. □

In the following, we compute N_{sp} of the friendship graph.

Theorem 6.2. *Let F_n be the friendship graph of order n. Then*

$$N_{sp}(F_n) = 2^n.$$

Proof. Let x be the central vertex of the graph and u_i for $i \in [2n]$ be the degree two vertices, where $\{u_{2t-1}, u_{2t}\}$ for $t \in [n]$ are adjacent. By Theorem 1.3, we know that $\gamma_{sp}(F_n) = n + 1$. Now consider Fig. 1. For any dominating set S of the friendship graph with cardinality less than $2n$, if we do not have $\{x, u_{2t-1}\} \subseteq S$ or $\{x, u_{2t}\} \subseteq S$, where $t \in [n]$, then it is clear that S is not a super dominating set. So we need x in our super dominating set. Among u_{2t-1} and u_{2t}, where $t \in [n]$, we choose one of them. So we have 2^n super dominating sets of size $n + 1$, and we have the result. □

Now we consider the path graph and compute $N_{sp}(P_n)$.

Theorem 6.3. *Let P_n be the path graph of order $n \in \mathbb{N} \setminus \{1\}$. Then*

$$N_{sp}(P_n) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ \frac{3}{2}(n-1), & \text{if } n \text{ is odd.} \end{cases}$$

Also we have $N_{sp}(P_1) = 1$.

Proof. The case $n = 1$ is clear. Thus, in the following we consider $n \geq 2$. By Theorem 1.2(a), we have $\gamma_{sp}(P_n) = \lceil \frac{n}{2} \rceil$. We have two cases based on the parity of n:

(a) n even.

Let $n = 2k$ with $k \in \mathbb{N}$. Let $V = \{v_1, v_2, \ldots, v_{2k}\}$ be the vertex set of P_{2k} (see Fig. 10), and S be a super dominating set of P_{2k} with $|S| = k$. Since $|S| = n/2$, in the partition of Lemma 2.1(b), we have $D = \emptyset$ and $V(G) = S \dot{\cup} S'$, where S and S' have the same cardinality $n/2$. By Lemma 2.1(a), each vertex in S is adjacent to exactly one vertex in S' and vice versa. Thus, if we choose $v_1 \in S$, then $v_2 \in S'$, $v_3 \in S'$, $v_4 \in S$ and so on. After we know whether $v_1 \in S$, all other vertices have their set decided. The case with $v_1 \in S'$ is analogous; we can just swap S and S'. Thus, we have $N_{sp}(P_{2k}) = 2$.

(b) n odd.

Let $n = 2k + 1$ with $k \in \mathbb{N}_0$. It is easy to see that $N_{sp}(P_1) = 1$, $N_{sp}(P_3) = 3$. Now we consider $n \geq 5$ and thus $k \geq 2$. Let $V = \{v_1, v_2, \ldots, v_{2k+1}\}$ be the vertex set of P_{2k+1}.

By Theorem 1.2(a), we have $\gamma_{sp}(P_{2k+1}) = k + 1$. Let S be a super dominating set of cardinality $k + 1$ and S' be another super dominating set of the same cardinality with $\overline{S} \subseteq S'$ and $\overline{S'} \subseteq S$. Moreover, by Lemma 2.1, let $f : \overline{S'} \to \overline{S}$ be a bijective function for which $f(a) = b$ if and only if a super dominates b for S and b super dominates a for S'. Observe that $|\overline{S}| = |\overline{S'}| = k$. Again, by Lemma 2.1(b), $|D| = |S \cap S'| = 1$ holds. Let us denote by w the single vertex in $S \cap S'$. Since w is not necessary for super dominating any other vertices, $S \setminus \{w\}$ is a super dominating set for the induced subgraph P' of $V(P_{2k+1}) \setminus \{w\}$. Notice that P' consists of either two paths or a single even-length path (when w is the start or end vertex of the original path). We have $|S \setminus \{w\}| = k$. Thus, each path in P' has even-length since odd paths have more than half of their vertices in any super dominating set. Hence, $w = v_i$ where i is odd. Thus, we have $k + 1$ possible choices for w. Furthermore, each of these choices for w yields a super dominating set of the smallest cardinality.

When $w = v_1$, we have $P' = P_{2k}$. Since $N_{sp}(P_{2k}) = 2$, based on the proof of n even, we may choose in this case S in two different ways based on whether $v_2 \in S$. Moreover, if we choose S in one way, then S' is the other super dominating set of the smallest cardinality in P'. Thus, $w = v_1$ contributes two super dominating sets. Furthermore, the choice $w = v_{2k+1}$ is symmetrical.

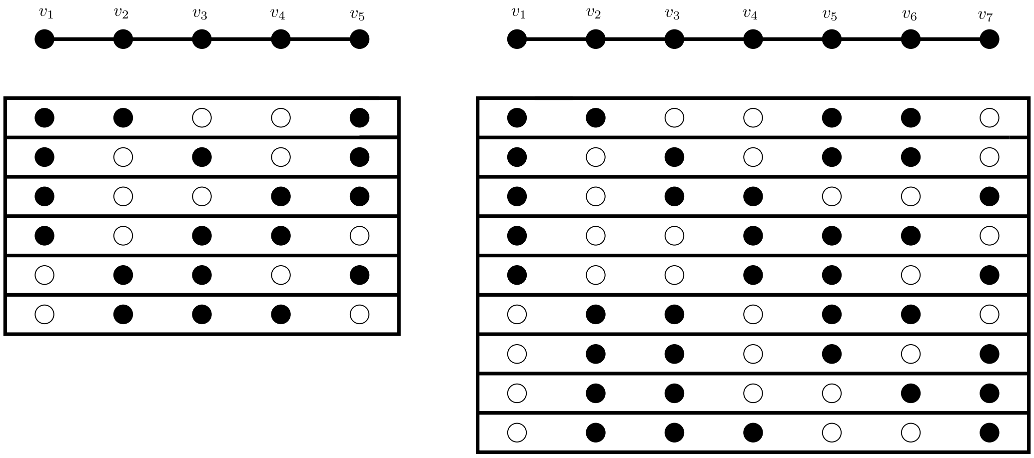

Fig. 11. Minimum size super dominating sets of P_5 and P_7, respectively.

Let us now consider the case $w = v_{2i+1}$ where $i \in [k-1]$. Now, P' consists of two even paths of lengths $2i$ and $2k-2i$. Hence, for each choice of w there are four different smallest super dominating sets (two for both even paths). In summary, we can classify the super dominating sets in eight classes, where by definition, the vertices w are always black.

SW(0): $w = v_1$, v_2 is white (i.e., does not lie in S),

SB(0): $w = v_1$, v_2 is black (i.e., lies in S),

EW(k): $w = v_{2k+1}$, v_{2k} is white,

EB(k): $w = v_{2k+1}$, v_{2k} is black,

MBB(i) $w = v_{2i+1}$, v_{2i} is black, v_{2i+2} is black, where $i \in [k-1]$,

MWW(i): $w = v_{2i+1}$, v_{2i} is white, v_{2i+2} is white, where $i \in [k-1]$,

MBW(i): $w = v_{2i+1}$, v_{2i} is black, v_{2i+2} is white, where $i \in [k-1]$,

MWB(i): $w = v_{2i+1}$, v_{2i} is white, v_{2i+2} is black, where $i \in [k-1]$.

This leads to

$$2 \cdot 2 + 4 \cdot (k-1) = 4k$$

super dominating sets. However, some super dominating sets appear in multiple classes. Below we give a formal explanation on the number these classes overlap.

Recall that the eight cases above are formed by first choosing the set $D = \{w\}$. Thus, a super dominating set S is counted in multiple classes if and only if there are multiple ways to choose the set D (and the super dominating set S') for S. Moreover, since $|D| = 1$, we can choose the set D in multiple ways if and only if there is a white vertex which is super dominated by two different black vertices. Since the maximum degree of a vertex is 2, we may choose the set D in these cases in exactly two ways, say D and D', where the corresponding vertices w and w' have distance 2 in the path. Hence, in each such case, a super dominating set is included in two classes with consecutive indices.

Finally, if we look into the eight classes above, we notice that in the classes SB(0), EB(k), MBB(i) and MWW(i), the vertex w does not super dominate any vertex. Thus, we do not have any white vertex, which is dominated twice in those classes. On the other hand, we have a white vertex that is super dominated by two black vertices in the classes SW(0), EW(k), MBW(i), and MWB(i). This twice super dominated white vertex is v_2, v_{2k}, v_{2i+2} and v_{2i}, respectively.

We illustrate these overlapping super dominating sets in Fig. 11 for P_5 and P_7. For P_5, in row IV, SW(0) and vertex $w = v_1$ overlap with MWB(1) and vertex $w = v_3$, and in row V, EW(2) and vertex $w = v_5$ overlap with MBW(1) and vertex $w = v_3$. For P_7, in row III, SW(0) and vertex $w = v_1$ overlap with MWB(1) and vertex $w = v_3$, in row V, EW(3) and vertex $w = v_7$ overlap with MBW(2) and vertex $w = v_5$, and in row VI, MBW(1) and vertex $w = v_3$ overlap with MWB(2) and vertex $w = v_5$.

That is, we are counting the super dominating sets twice in $1 + 1 + (k-1) + (k-1) = 2k$ classes. Hence, $N_{sp}(P_{2k+1}) = 4k - (2k/2) = 3k$ as claimed. □

In Theorem 6.5, we determine the value $N_{sp}(C_n)$ for each n. Interestingly, the value varies quite a lot based on n (mod 4). We will utilize necklace combinatorics for the proof of Case (c). See [20] for an algorithm to calculate the value $N_n(q_1, q_2, \ldots, q_k)$ in the following definition as well as some connections between necklaces and combinatorics on words.

Definition 6.4. Let $k, n, q_1, q_2, \ldots, q_k \in \mathbb{N}$ with $n = \sum_{i=1}^{k} q_i$.

(a) A $(q_1, q_2, \ldots, q_k)$-*necklace* of length n consists of a total of n beads where we have q_i beads of type i. The beads are placed into a cycle (necklace).
(b) $N_n(q_1, q_2, \ldots, q_k)$ is defined as the number of different $(q_1, q_2, \ldots, q_k)$-necklaces of length n, when the n rotations of a necklace are considered as the same necklace.

The following relation is easy to see.

$$N_{k+1}(k, 1) = 1 \text{ for } k \in \mathbb{N}. \tag{2}$$

Furthermore, we have

$$N_{2k+2}(2k, 2) = k + 1 \text{ for } k \in \mathbb{N} \text{ and } N_{2k+3}(2k + 1, 2) = k + 1 \text{ for } k \in \mathbb{N}_0. \tag{3}$$

Indeed, in Eq. (3), let us say that we have two red beads in the necklace and the other beads are blue. Since the rotations of the necklaces are not counted multiple times, each necklace is uniquely determined by the smallest number of blue beads between the two red beads.

Theorem 6.5. *Let C_n be the cycle graph of order $n \in \mathbb{N} \setminus \{1, 2\}$. Then*

$$N_{sp}(C_n) = \begin{cases} 4 & \text{if } n \equiv 0 \pmod 4, \\ 2n & \text{if } n \equiv 1 \pmod 4, \\ \frac{5n^2-10n}{8} & \text{if } n \equiv 2 \pmod 4, \\ n & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $n \in \mathbb{N} \setminus \{1, 2\}$. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n and S be a minimum size super dominating set for that. We consider the following cases:

(a) $n \equiv 0 \pmod 4$.

Let $n = 4k$ with $k \in \mathbb{N}$. By Theorem 1.2(b), we have $\gamma_{sp}(C_{4k}) = 2k$. Let S be a super dominating set of cardinality $2k$ in C_n. By Lemma 2.1, there is another super dominating set $\overline{S}$ with same cardinality as S, $D = \emptyset$, and there is a bijective function $f : \overline{S'} \to \overline{S}$. Thus, we cannot have 3 consecutive vertices in S and for each $v \in S = \overline{S'}$, there exists a unique $u \in \overline{S} = S'$ such that $N(v) \cap \overline{S} = \{u\}$ and for each $v \in S' = \overline{S}$ a unique $u \in \overline{S'} = S$ such that $N(v) \cap \overline{S'} = \{u\}$. Hence, the following four sets are the only super dominating sets in C_{4k}:

$$S_1 = \{v_1, v_2, v_5, v_6, \ldots, v_{4k-3}, v_{4k-2}\},$$
$$S_2 = \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\},$$
$$S_3 = \{v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}\},$$
$$S_4 = \{v_1, v_4, v_5, v_8, \ldots, v_{4k-3}, v_{4k}\}.$$

Hence, $N_{sp}(C_{4k}) = 4$.

(b) $n \equiv 1 \pmod 4$.

Let $n = 4k + 1$ with $k \in \mathbb{N}$. By Theorem 1.2(b), we have $\gamma_{sp}(C_{4k+1}) = 2k + 1$. Let S be a super dominating set of size $2k + 1$ in C_n. First, notice that $|S| = |\overline{S}| + 1$. Thus, if we consider Lemma 2.1 and the set D, we have $|D| = 1$. Without loss of generality, $v_{4k+1} \in D$ holds. If we now contract the edge $v_{4k}v_{4k+1}$, then we get a cycle C_{4k} in which $S \setminus \{v_{4k+1}\}$ is a super dominating set of cardinality $2k$. If we now recall the structure of a minimum size super dominating set for the cycle C_{4k}, then we notice that there are two possibilities for $v_{4k+1} \in D$, namely either $N(v_{4k+1}) \subseteq S$ or $N(v_{4k+1}) \subseteq \overline{S}$. Hence, v_{4k+1} is either one of 3 consecutive vertices in S or a single vertex in S surrounded by vertices in $\overline{S}$. Moreover, in both cases, we may rotate the super dominating set around the cycle in n different ways. Thus, $N_{sp}(C_{4k+1}) = 2n$ as claimed.

(c) $n \equiv 2 \pmod 4$.

Let $n = 4k + 2 = 8q + 4p + 2$ where $q \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $p \in \{0, 1\}$. By Theorem 1.2(b), we have $\gamma_{sp}(C_{4k+2}) = 2k + 2$. Since choosing any four vertices from C_6 gives us a super dominating set, $\binom{6}{4} = 15 = (5 \cdot 6^2 - 10 \cdot 6)/8$ is the number of super dominating sets, and it shows that the formula holds for $n = 6$. So, let now $n \geq 10$.

In the following, we write about k *consecutive vertices* in a minimum size super dominating set S implying that these k vertices are not contained in a subset of $k + 1$ consecutive vertices in S. We claim that the following cases cannot occur for vertices in S:

- At least 5 consecutive vertices,
- twice 4 consecutive vertices,
- once 4 consecutive vertices and once 1 consecutive vertex,
- once 4 consecutive vertices and once 3 consecutive vertices,
- three times 3 consecutive vertices,
- twice 3 consecutive vertices and once 1 consecutive vertex,
- once 3 consecutive vertices and twice 1 consecutive vertex,

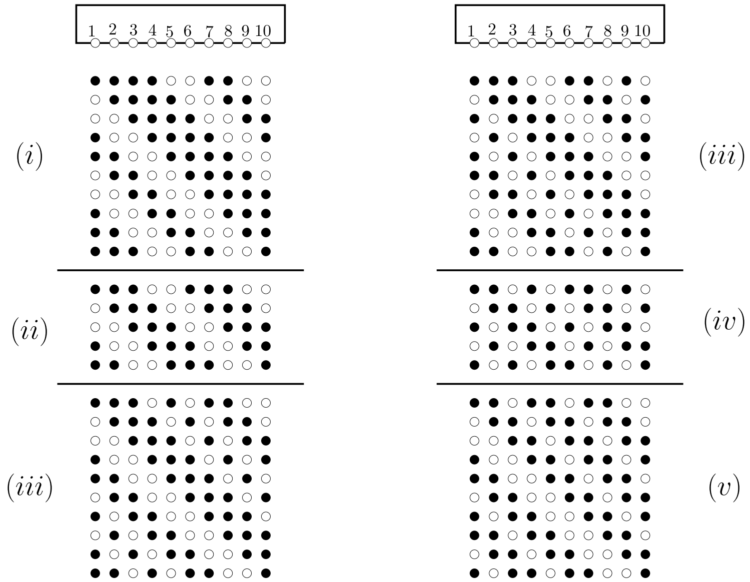

Fig. 12. All super dominating sets of C_{10}.

- three times 1 consecutive vertex.

The non-existence of all these cases can be shown in the same way. By Lemma 2.1, we have $|D| = 2$. On the other hand, if one of these cases occurred, then we would have at least three black vertices surrounded either by two black vertices or by two white vertices. As these vertices do not super dominate other vertices, it follows that $|D| \geq 3$ leading to a contradiction.

As we are in the case $n \equiv 2 \pmod 4$, it remains to consider the following cases for vertices in S, where we do not list the occurrences of 2 consecutive vertices.

(i) Exactly once 4 consecutive vertices,
(ii) exactly twice 3 consecutive vertices,
(iii) exactly once 3 consecutive vertices and exactly once 1 consecutive vertex,
(iv) exactly twice 1 consecutive vertex,
(v) none of that, i.e., only 2 consecutive vertices.

We have illustrated the minimum size super dominating sets for C_{10} in Fig. 12.

As mentioned, we will utilize necklaces (see Definition 6.4). We transform some sequences of vertices into beads according to the following list. After that, we consider the resulting cycle as a necklace.

- **B-bead** (black bead): B,
- **DB-bead** (double black bead): BB,
- **P-bead** (pair bead): WBBW,
- **T-bead** (triple bead): WBBBW,
- **Q-bead** (quadruple bead): WBBBBW,
- **DP-bead** (double pair bead): WBBWBBW.

We distinguish between the five cases (i)–(v) mentioned above.

(i) The 4 consecutive vertices form a Q-bead. If we convert the cycle together with the super dominating set into a necklace consisting of $2q-1+p$ P-beads and a single Q-bead, then by Eq. (2) we attain $N_{2q+p}(2q-1+p, 1) = 1$ different necklaces. Now, this single necklace provides n different super dominating sets with rotations.
(ii) We attain a necklace consisting of $2q-2+p$ P-beads and two T-beads. We have $N_{2q+p}(2q-2+p, 2) = q$ by Eq. (3). Furthermore, we can again rotate these n times. However, if $p = 0$ then the necklace with $q-1$ P-beads between the two T-beads yields only $n/2$ different super dominating sets. Thus, in this case we have $qn-(1-p)n/2$ different super dominating sets.
(iii) We attain a necklace consisting of $2q-1+p$ P-beads, one T-bead and one B-bead. We have $N_{2q+p+1}(2q-1+p, 1, 1) = 2q+p$. Indeed, the number of P-beads h which we can have between the B-bead and the T-bead is $0 \leq h \leq 2q-1+p$. Moreover, n rotations give n different super dominating sets for each of these necklaces. Thus, we have $n(2q+p)$ different super dominating sets in this case.

(iv) We attain a necklace consisting of $2q+p$ P-beads and two B-beads. We have $N_{2q+p+2}(2q+p,2)=q+1$ by Eq. (3). However, we do not consider the case where we have two adjacent B-beads since in that case we would actually have 2 consecutive black vertices and we would be considering Case (v). As in Case (ii), if we have $p=0$, then the necklace which has q P-beads between the two B-beads gives only $n/2$ different super dominating sets with rotations. In all other cases, we get n different super dominating sets. Thus, we have $qn-(1-p)n/2$ different super dominating sets in this Case (iv).

(v) In this case, we have two vertices $u_1, u_2 \in \overline{S}$ such that they are both super dominated by two different vertices.
Let us first consider the case where u_1 and u_2 have distance 3. We attain a necklace consisting of $2q+p$ P-beads and one DB-bead. By Eq. (2), we have $N_{2q+p}(2q+p,1)=1$. We can rotate this case n times and thus, we have n different super dominating sets.
Let us then assume that u_1 and u_2 have distance of at least 7. Here the white vertex in the middle corresponds to u_1 or u_2. In this case, we attain a necklace consisting of $2q-3+p$ DB-beads and two DP-beads. We have $N_{2q-1+p}(2q-3+p,2)=q-1+p$ by Eq. (3). Notice that when $p=1$ the necklace in which we have $q-1$ DB-beads between the two DP-beads gives us only $n/2$ different super dominating sets. All other necklaces give n different super dominating sets. Thus, we have $n(q-1)+pn/2$ different super dominating sets in this Case (v).

By summing all these different cases together, we get

$$\begin{aligned} & N_{sp}(C_n) \\ =\ & n+qn-(1-p)n/2+n(2q+p)+qn-(1-p)n/2+n+n(q-1)+pn/2 \\ =\ & 5qn+(5/2)pn \\ =\ & \frac{5n^2-10n}{8} \end{aligned}$$

super dominating sets, as claimed.

(d) $n \equiv 3 \pmod 4$.

Let $n=4k+3$ with $k \in \mathbb{N}_0$. It is easy to see that $N_{sp}(C_3)=3$. Now let $n \geq 7$. By Theorem 1.2(b), $\gamma_{sp}(C_{4k+3})=2k+2$. So we need to choose $2k+2$ vertices in a proper way to have a super dominating set. First, we show that it is not possible to have 3 consecutive vertices in a minimum size super dominating set S. Suppose that we have 3 consecutive vertices v, v', v'' and contract one of the two corresponding edges (vv' or $v'v''$). Then we have a super dominating set of size $2k+1$ in C_{4k+2}, a contradiction. Second, with the same contraction technique, we notice that it is not possible among 3 consecutive vertices for the middle one to be in S and for the two others to be in $\overline{S}$. Hence, we can only have 2 consecutive vertices in S.

Since $\gamma_{sp}(C_{4k+3})=2k+2$, we have $k+1$ sets of size two with consecutive vertices in S. Moreover, we have $|\overline{S}|=2k+1$. Thus, we have one vertex $v \in \overline{S}$ with $N(v) \subseteq S$ and k sets of size two with consecutive vertices in $\overline{S}$. Assume first that $v=v_1$. Now we can construct n different super dominating sets by rotating them around the cycle. Thus, $N_{sp}(C_{4k+3})=n$.

Therefore, we have the result. □

7. Conclusions and future work

In this paper, we obtained tight results on the super domination number of graphs, particularly on the neighbourhood corona product, r-clique sum and Hajós sum of two graphs. For each of these, we presented tight lower and upper bounds together with constructions attaining these upper bounds. Moreover, we gave the exact number of minimum size super dominating sets of some graph classes such as paths and cycles. Finally, we provide the following suggestions for future research.

(a) The formula $\gamma_{sp}(G \star H)=n(\gamma_{sp}(H)+1)$ of Corollary 3.4 does not cover the case $H=K_1$. Thus, it would be interesting to compute $\gamma_{sp}(G \star K_1)$. By Eq. (1), $\gamma_{sp}(G \star K_1) \leq \gamma_{sp}(G)+n$ is a (trivial) upper bound, but it is tight only for some graphs G.
(b) We found a connection for the super domination number between the usual corona product of two graphs [15] and the neighbourhood corona product of two graphs. Does a generalization exist for the corona products that preserve the result of Corollary 3.4?
(c) Give general upper and lower bounds for the number of minimum size super dominating sets with help of $\gamma_{sp}(G)$ and possibly some other graph parameters.
(d) Count the number of (minimum size) super dominating sets in trees.

Data availability

No data was used for the research described in the article.

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