



Inequalities for a new hyperbolic type metric

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Abstract

We study a new hyperbolic type metric recently introduced by Song and Wang. We present formulas for it in the upper half-space and the unit ball domains and find its sharp inequalities with the hyperbolic metric and the triangular ratio metric. We also improve existing ball inclusion results and give bounds for the distortion of this new metric under conformal and quasiregular mappings.

Keywords Hyperbolic geometry · Hyperbolic metric · Hyperbolic type metrics · Triangular ratio metric

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1 Introduction

Due to its conformal invariance, the hyperbolic metric is a useful tool for studying different types of mappings in geometric function theory but this metric is not properly defined in higher dimensions outside the most simple domains of the upper half-space and the unit ball. For this reason, researchers have introduced several hyperbolic type metrics which share some of the properties of the hyperbolic metric but have a very easy definition [2, 5, 6, 10, 12]. The key feature is that the metrics measure the distance between two points by taking into account the position of the points with respect to each other and the boundary of their domain.

Typically, the hyperbolic metrics are defined by using a quotient where the numerator is the Euclidean distance of the two points and the denominator depends on their distance from the boundary. For instance, the distance ratio metric introduced by Gehring and Osgood [2] uses the minimum of the Euclidean distances from the two points to the boundary as the denominator. However, this definition leads to one of the differences from the hyperbolic metric: If we rotate the point further away from the boundary around the other point so that the center of the rotation stays the point closer to the boundary, the value of the distance ratio metric does not change at all. Given this type of a transformation would affect the hyperbolic distance between the points, there is an interest

in the hyperbolic type metrics that actually consider how the both points are positioned with respect to the boundary.

Recently, the following new hyperbolic type metric was introduced by Song and Wang [11]: In a domain $G \subsetneq \mathbb{R}^n$ whose boundary is denoted by ∂G , define the function $\tilde{c}_G : G \times G \rightarrow [0, \infty)$ as

$$\tilde{c}_G(x, y) = \frac{|x - y|}{\inf_{z \in \partial G} \max\{|x - z|, |y - z|\}}, \quad x, y \in G. \quad (1.1)$$

Given the combination of infimum and maximum in the denominator, it is clear that this function is affected by the distance from the both points x and y to a same boundary point z . By [11, Cor. 1], this function \tilde{c}_G is also truly a metric.

In this article, we continue the study of this metric. We give formulas for computing the value of this new metric in the upper half-space and the unit disk. We find the best possible constants for the inequalities between this new metric and both the hyperbolic metric and the triangular ratio metric. We use these inequalities to formulate the ball inclusion results, which improve the ones found in [11]. We also present the results about the distortion of the metric \tilde{c} under conformal and quasiregular mappings.

2 Preliminaries

For a point $x \in \mathbb{R}^n$, denote its j th coordinate by x_j so that $x = (x_1, \dots, x_n)$. The upper half-space can now be defined as

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$\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$. Denote also the unit vectors of \mathbb{R}^n by e_1, \dots, e_n . Denote the Euclidean open ball with a center $x \in \mathbb{R}^n$ and a radius $r > 0$ by $B^n(x, r)$ and let $S^{n-1}(x, r)$ be its boundary sphere. Use the simplified notation \mathbb{B}^n for the unit ball $B^n(0, 1)$. Denote the line segment between $x, y \in \mathbb{R}^n$ by $[x, y]$. For two distinct points $x, y \in \mathbb{R}^n \setminus \{0\}$, let $\angle XOY$ be the magnitude of the angle formed between the vectors from the origin to the points x and y so that $\angle XOY \leq \pi$.

By denoting the hyperbolic sine, cosine and tangent by sh , ch , and th , respectively, we can define the hyperbolic metric as [5, (4.8), p. 52 & (4.14), p. 55]

$$\begin{aligned} \text{ch}\rho_{\mathbb{H}^n}(x, y) &= 1 + \frac{|x - y|^2}{2x_n y_n}, \quad x, y \in \mathbb{H}^n, \\ \text{sh}^2 \frac{\rho_{\mathbb{B}^n}(x, y)}{2} &= \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}, \quad x, y \in \mathbb{B}^n. \end{aligned}$$

From this, we can solve that

$$\begin{aligned} \text{th} \frac{\rho_{\mathbb{H}^n}(x, y)}{2} &= \frac{|x - y|}{\sqrt{|x - y|^2 + 4x_n y_n}}, \\ \text{th} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} &= \frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}}. \end{aligned} \tag{2.1}$$

For a domain $G \subsetneq \mathbb{R}^n$, define then the distance ratio metric $j_G : G \times G \rightarrow [0, \infty)$ as [1, p. 685],

$$j_G(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d_G(x), d_G(y)\}} \right) \tag{2.2}$$

and the j^* -metric $j_G^* : G \times G \rightarrow [0, 1]$, as [4, 2.2, p. 1123 & Lemma 2.1, p. 1124]

$$j_G^*(x, y) = \text{th} \frac{j_G(x, y)}{2} = \frac{|x - y|}{|x - y| + 2 \min\{d_G(x), d_G(y)\}} \tag{2.3}$$

The triangular ratio metric introduced by Hästö [6] and studied in [7–9] is defined as $s_G : G \times G \rightarrow [0, 1]$, [1, (1.1), p. 683]

$$s_G(x, y) = \frac{|x - y|}{\inf_{z \in \partial G} (|x - z| + |z - y|)}.$$

For a hyperbolic type metric $d \in \{\tilde{c}, s, j, \rho\}$, define its balls in a domain $G \subsetneq \mathbb{R}^n$ so that $B_d(x, r) = \{y \in G : d_G(x, y) < r\}$ where $x \in \mathbb{R}^n$ and $r > 0$.

Theorem 2.4 ([4], Lemma 2.1, p. 1124; Lemma 2.2, p. 1125; Lemma 2.8 & Thm 2.9(1), p. 1129) *For all $x, y \in G \subsetneq \mathbb{R}^n$, the inequality $j_G^*(x, y) \leq s_G(x, y) \leq 2j_G^*(x, y)$ holds and, if G is convex, then $s_G(x, y) \leq \sqrt{2}j_G^*(x, y)$.*

3 Results

Lemma 3.1 *For all points x, y in any domain $G \subsetneq \mathbb{R}^n$, the inequality $s_G(x, y) \leq \tilde{c}_G(x, y) \leq 2s_G(x, y)$ holds and has the best constants regardless of the exact choice of $G \subsetneq \mathbb{R}^n$ and $x \in G$.*

Proof Clearly,

$$\begin{aligned} (1/2) \cdot \inf_{z \in \partial G} (|x - z| + |z - y|) &\leq 1/2 \cdot \\ &\inf_{z \in \partial G} (\max\{|x - z|, |y - z|\} + \max\{|x - z|, |y - z|\}) \\ &= \inf_{z \in \partial G} \max\{|x - z|, |y - z|\} \\ &\leq \inf_{z \in \partial G} (\max\{|x - z|, |y - z|\} + \min\{|x - z|, |y - z|\}) \\ &= \inf_{z \in \partial G} (|x - z| + |z - y|), \end{aligned}$$

so the inequality follows. Let us then prove that there cannot be better constants for any choice of G . For $x \in G, u \in S^{n-1}(x, d_G(x)) \cap \partial G$, and $y = u + k(x - u)$ with $0 < k < 1$,

$$\frac{\tilde{c}_G(x, y)}{s_G(x, y)} = \frac{|x - u| + |u - y|}{|x - u|} = 1 + k,$$

which approaches 1 when $k \rightarrow 0^+$ and 2 when $k \rightarrow 1^-$. \square

Corollary 3.2 *For any point x in a domain $G \subsetneq \mathbb{R}^n$, the ball inclusion*

$$B_s(x, R_0) \subseteq B_{\tilde{c}}(x, r) \subseteq B_s(x, R_1)$$

holds for all $0 < r < 1$ if and only if $R_0 \leq r/2$ and $R_1 \geq r$.

Proof By Lemma 3.1, we have

$$\begin{aligned} B_s(x, r/2) &= \{y \in G : s_G(x, y) < r/2\} \subseteq \{y \in G : \\ &\tilde{c}_G(x, y) < r\} = B_{\tilde{c}}(x, r) \subseteq B_s(x, r) \end{aligned}$$

and, since Lemma 3.1 has the best possible constants for any choices of $G \subsetneq \mathbb{R}^n$ and $x \in G$, the radii $r/2$ and r are the best possible ones here for an unspecified value of $0 < r < 1$. \square

Remark 3.3 Corollary 3.2 improves the earlier inclusion result [11, Cor. 2].

Lemma 3.4 *For all points x, y in any domain $G \subsetneq \mathbb{R}^n$, the inequality $j_G^*(x, y) \leq \tilde{c}_G(x, y) \leq 4j_G^*(x, y)$ holds. The inequality $j_G^*(x, y) \leq \tilde{c}_G(x, y)$ has always the best constant regardless of the exact choice of $G \subsetneq \mathbb{R}^n$ and $x \in G$, and the inequality $\tilde{c}_G(x, y) \leq 4j_G^*(x, y)$ has the best constant in the case $G = \mathbb{R}^n \setminus \{0\}$. If G is convex, then $\tilde{c}_G(x, y) \leq 2\sqrt{2}j_G^*(x, y)$.*

Proof The inequalities follow from Theorem 2.4 and Lemma 3.1. If $x \in G$, $u \in S^{n-1}(x, d_G(x)) \cap \partial G$, and $y = u + k(x - u)$ with $k \rightarrow 0^+$,

$$\frac{\tilde{c}_G(x, y)}{s_G(x, y)} = \frac{|x - u| + |u - y|}{|x - u|} = 1 + k \rightarrow 1.$$

For $x = e_1$ and $y = -e_1$,

$$\frac{\tilde{c}_{\mathbb{R}^n \setminus \{0\}}(x, y)}{J_{\mathbb{R}^n \setminus \{0\}}^*(x, y)} = \frac{2 + 2}{1} = 4.$$

□

Theorem 3.5 For all $x, y \in \mathbb{H}^n$,

$$\tilde{c}_{\mathbb{H}^n}(x, y) = \begin{cases} \frac{|x - y|}{\max\{x_n, y_n\}} & \text{if } |x - y| \leq \sqrt{(x_n - y_n)^2 + |x_n^2 - y_n^2|}, \\ 2\sqrt{\frac{|x - y|^2 - (x_n - y_n)^2}{|x - y|^2 + 4x_n y_n}} & \text{otherwise.} \end{cases}$$

Proof Denote $x' = (x_1, \dots, x_{n-1}, 0)$ and $y' = (y_1, \dots, y_{n-1}, 0)$. In the special case $x' = y'$, the infimum $\inf_{z \in \partial \mathbb{H}^n} \max\{|x - z|, |y - z|\}$ is trivially found by choosing $z = x' = y'$ and we have $\tilde{c}_{\mathbb{H}^n}(x, y) = |x - y| / \max\{x_n, y_n\}$. In this case, we have

$$|x - y| = |x_n - y_n| \leq \sqrt{(x_n - y_n)^2 + |x_n^2 - y_n^2|},$$

so the condition in the cases in the theorem is fulfilled.

Suppose then $x' \neq y'$. For any $u \in \partial \mathbb{H}^n \setminus [x', y']$,

$$\max\{|x - v|, |y - v|\} < \max\{|x - u|, |y - u|\}$$

where v is the closest point to u on the line segment $[x', y']$. It follows from this that the infimum $\inf_{z \in \partial \mathbb{H}^n} \max\{|x - z|, |y - z|\}$ is obtained when $z \in [x', y']$ or, equivalently, $z = x' + k(y' - x')$ for some $0 \leq k \leq 1$.

If $\max\{x_n, y_n\} \geq \min\{|x - y'|, |y - x'|\}$ or, equivalently,

$$\begin{aligned} \max\{x_n, y_n\} &\geq \sqrt{|x' - y'|^2 + \min\{x_n, y_n\}^2} \Leftrightarrow \\ |x - y| &\leq \sqrt{(x_n - y_n)^2 + |x_n^2 - y_n^2|}, \end{aligned} \tag{3.6}$$

then the infimum $\inf_{z \in \partial \mathbb{H}^n} \max\{|x - z|, |y - z|\}$ is found at either x' or y' and we have $\tilde{c}_{\mathbb{H}^n}(x, y) = |x - y| / \max\{x_n, y_n\}$.

Suppose then the inequality (3.6) does not hold. Now, there is a value of $0 < k < 1$ such that $|x - z| = |y - z|$ with $z = x' + k(y' - x')$. Since $|x - z|$ is increasing with respect to k and $|y - z|$ is decreasing with respect to k , the

infimum $\inf_{z \in \partial \mathbb{H}^n} \max\{|x - z|, |y - z|\}$ is found at the point where this equality holds. We can solve that

$$\begin{aligned} |x - z| = |y - z| &\Leftrightarrow \sqrt{k^2|x' - y'|^2 + x_n^2} \\ &= \sqrt{(1 - k)^2|x' - y'|^2 + y_n^2} \\ \Leftrightarrow k &= \frac{|x' - y'|^2 - x_n^2 + y_n^2}{2|x' - y'|^2} \\ \Leftrightarrow |x - z| &= \frac{\sqrt{(|x' - y'|^2 + x_n^2 + y_n^2)^2 - 4x_n^2 y_n^2}}{2|x' - y'|} \\ &= \frac{|x - y|}{2} \sqrt{\frac{|x - y|^2 + 4x_n y_n}{|x - y|^2 - (x_n - y_n)^2}} \\ \Rightarrow \tilde{c}_{\mathbb{H}^n}(x, y) &= 2\sqrt{\frac{|x - y|^2 - (x_n - y_n)^2}{|x - y|^2 + 4x_n y_n}}. \end{aligned}$$

The theorem follows. □

Corollary 3.7 For all $x, y \in \mathbb{H}^n$, the inequality

$$\text{th} \frac{\rho_{\mathbb{H}^n}(x, y)}{2} \leq \tilde{c}_{\mathbb{H}^n}(x, y) \leq 2 \text{th} \frac{\rho_{\mathbb{H}^n}(x, y)}{2}$$

holds and has the best possible constants.

Proof Follows directly from Lemma 3.1 because $s_{\mathbb{H}^n}(x, y) = \text{th}(\rho_{\mathbb{H}^n}(x, y)/2)$ by [5, p. 460]. □

Theorem 3.8 For all $x, y \in \mathbb{B}^n$,

$$\tilde{c}_{\mathbb{B}^n}(x, y) = \begin{cases} \frac{|x - y|}{1 - \min\{|x|, |y|\}} & \text{if } \cos(\mu) \geq \frac{||x|^2 - |y|^2| + 2 \min\{|x|, |y|\}}{2 \max\{|x|, |y|\}}, \\ \frac{|x - y|}{\sqrt{1 + |x|^2 - 2|x| \cos(k)}} & \text{otherwise,} \end{cases}$$

where μ is the magnitude of the angle $\angle XOY$ and $k \in (0, \mu)$ is chosen so that $|x| \cos(k) - |y| \cos(\mu - k) = (|x|^2 - |y|^2)/2$.

Proof In the special case where $x/|x| = y/|y|$, the infimum

$$\inf_{z \in S^{n-1}(0,1)} \max\{|x - z|, |y - z|\}$$

is obtained when $z = x/|x| = y/|y|$ and it is equal to $1 - \min\{|x|, |y|\}$. Suppose then $x/|x| \neq y/|y|$. Let C be such an arc that belongs to an origin-centered circle, has $x/|x|$ and $y/|y|$ as its end points, and has a center angle with magnitude $\mu = \angle XOY \in (0, \pi]$. For all $u \in S^{n-1}(0, 1) \setminus C$,

$$\max\{|x - v|, |y - v|\} < \max\{|x - u|, |y - v|\}$$

where v is the closest point to v on C . Consequently,

$$\begin{aligned} & \inf_{z \in S^{n-1}(0,1)} \max\{|x-z|, |y-z|\} = \inf_{z \in C} \max\{|x-z|, |y-z|\} \\ & = \inf_{k \in [0, \mu]} \max \left\{ \sqrt{1 + |x|^2 - 2|x| \cos(k)}, \right. \\ & \left. \sqrt{1 + |y|^2 - 2|y| \cos(\mu - k)} \right\}. \end{aligned} \tag{3.9}$$

Suppose first that $1 - \min\{|x|, |y|\}$ is greater than or equal to $\min\{|x-z|, |y-z|\}$ for all points $z \in C$. Since $\sup_{z \in C} |x-z| = |x-y/|y||$, this is the case when

$$1 - \min\{|x|, |y|\} \geq \min\{|x-y/|y||, |y-x/|x||\}. \tag{3.10}$$

We need to have either

$$1 - \min\{|x|, |y|\} \geq \sqrt{1 + \min\{|x|, |y|\}^2 - 2 \min\{|x|, |y|\} \cos(\mu)} \Leftrightarrow \cos(\mu) \geq 1$$

or

$$\begin{aligned} & 1 - \min\{|x|, |y|\} \geq \sqrt{1 + \max\{|x|, |y|\}^2 - 2 \max\{|x|, |y|\} \cos(\mu)} \\ & \Leftrightarrow \cos(\mu) \geq \frac{||x|^2 - |y|^2| + 2 \min\{|x|, |y|\}}{2 \max\{|x|, |y|\}}. \end{aligned} \tag{3.11}$$

The inequality (3.11) holds for $\cos(\mu) = 1$, so the inequality (3.10) holds if and only if the inequality (3.11) holds. Since

$$1 - \min\{|x|, |y|\} \geq \sup_{z \in C} \min\{|x-z|, |y-z|\},$$

we have

$$\inf_{z \in C} \max\{|x-z|, |y-z|\} = 1 - \min\{|x|, |y|\}.$$

While we have $\cos(\mu) < 1$ as $x/|x| \neq y/|y|$, the infimum was the same in the case $x/|x| = y/|y|$ and $\cos(\mu) = 1$ fulfills (3.11), so first part of the theorem follows.

Suppose then $1 - \min\{|x|, |y|\} < \sup_{z \in C} \min\{|x-z|, |y-z|\}$. We have

$$\begin{aligned} 1 - |x| &< \sqrt{1 + |y|^2 - 2|y| \cos(\mu)}, \\ 1 - |y| &< \sqrt{1 + |x|^2 - 2|x| \cos(\mu)}, \end{aligned} \tag{3.12}$$

so the infimum (3.9) is obtained with such $k \in (0, \pi)$ that

$$\begin{aligned} & \sqrt{1 + |x|^2 - 2|x| \cos(k)} = \sqrt{1 + |y|^2 - 2|y| \cos(\mu - k)} \\ & \Leftrightarrow |x| \cos(k) - |y| \cos(\mu - k) = (|x|^2 - |y|^2)/2. \end{aligned}$$

The theorem follows. □

Theorem 3.13 For all $x, y \in \mathbb{H}^n$, the inequality

$$\text{th} \frac{\rho_{\mathbb{B}^n}(x, y)}{2} \leq \tilde{c}_{\mathbb{B}^n}(x, y) \leq 2 \text{th} \frac{\rho_{\mathbb{B}^n}(x, y)}{2}$$

holds and has the best possible constants.

Proof If the inequality (3.11) holds, then by Theorem 3.8, we have

$$\begin{aligned} & \text{th}(\rho_{\mathbb{B}^n}(x, y)/2) \leq \tilde{c}_{\mathbb{B}^n}(x, y) \\ & \Leftrightarrow 1 - \min\{|x|, |y|\} \leq \sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)} \\ & = \sqrt{1 + |x|^2|y|^2 - 2|x||y| \cos(\mu)} \\ & \Leftrightarrow -2(1 - \max\{|x|, |y|\} \cos(\mu)) \\ & + \min\{|x|, |y|\}(1 - \max\{|x|, |y|\}^2) \leq 0 \\ & \Leftrightarrow (1 - \max\{|x|, |y|\})(-2 + \min\{|x|, |y|\} + |x||y|) < 0. \end{aligned}$$

Consider then the case where the inequality (3.11) does not hold. The infimum (3.9) is now obtained neither $k = 0$ nor $k = \mu$. We need to show that

$$\begin{aligned} & \text{th}(\rho_{\mathbb{B}^n}(x, y)/2) \leq \tilde{c}_{\mathbb{B}^n}(x, y) \\ & \Leftrightarrow \inf_{k \in (0, \mu)} \max \left\{ \sqrt{1 + |x|^2 - 2|x| \cos(k)}, \right. \\ & \left. \sqrt{1 + |y|^2 - 2|y| \cos(\mu - k)} \right\} \\ & \leq \sqrt{|x-y|^2 + (1-|x|^2)(1-|y|^2)} \\ & \Leftrightarrow \inf_{k \in (0, \mu)} \max \left\{ |x|^2 - 2|x| \cos(k), |y|^2 - 2|y| \cos(\mu - k) \right\} \\ & \leq |x|^2|y|^2 - 2|x||y| \cos(\mu) \end{aligned}$$

The function $f_0(k) = |x|^2 - 2|x| \cos(k)$ is increasing with respect to k on $(0, \mu)$ and $f_1(k) = |y|^2 - 2|y| \cos(\mu - k)$ is decreasing. Due to the inequalities (3.12), we know that these functions intersect for some $k \in (0, \mu)$ and this point of intersection is where the infimum above is obtained. Thus, to prove that this intersection point is less than or equal to $|x|^2|y|^2 - 2|x||y| \cos(\mu)$, we need to show that either $f_0(k)$ exceeds $|x|^2|y|^2 - 2|x||y| \cos(\mu)$ for a greater value of k than $f_1(k)$ drops under $|x|^2|y|^2 - 2|x||y| \cos(\mu)$ or one of the functions f_1, f_0 is below $|x|^2|y|^2 - 2|x||y| \cos(\mu)$ for all $0 < k < \mu$. Since we can solve

$$\begin{aligned} & f_0(k) \leq |x|^2|y|^2 - 2|x||y| \cos(\mu) \Leftrightarrow \\ & k \leq \text{acos} \left(\frac{|x| - |x||y|^2 + 2|y| \cos(\mu)}{2} \right) \end{aligned}$$

and

$$f_1(k) \leq |x|^2|y|^2 - 2|x||y| \cos(\mu) \Leftrightarrow k \geq \mu - \operatorname{acos}\left(\frac{|y| - |x|^2|y| + 2|x| \cos(\mu)}{2}\right),$$

we have to show that, for

$$u_0 = |x| - |x||y|^2 + 2|y| \cos(\mu), \\ u_1 = |y| - |x|^2|y| + 2|x| \cos(\mu),$$

we have either

$$\mu - \operatorname{acos}(u_1/2) \leq \operatorname{acos}(u_0/2), \\ u_0 \geq 2 \cos(\mu), \text{ or } u_1 \geq 2 \cos(\mu).$$

Consequently, we need to prove that the left-most inequality above holds when $\cos(\mu) \leq \min\{u_0, u_1\}/2$ or, equivalently, when

$$\cos(\mu) \leq \frac{\min\{|x|, |y|\} + |x||y|}{2}.$$

Given $\mu \leq \pi$, the inequality

$$\mu - \operatorname{acos}(u_1/2) \leq \operatorname{acos}(u_0/2)$$

holds trivially when

$$\operatorname{acos}(u_0/2) + \operatorname{acos}(u_1/2) \geq \pi \Leftrightarrow \\ \operatorname{acos}(u_0/2) \geq \pi - \operatorname{acos}(u_1/2) = \operatorname{acos}(-u_1/2) \\ \Leftrightarrow u_0/2 \leq -u_1/2 \\ \Leftrightarrow (|x| + |y|)(1 - |x||y| + 2 \cos(\mu)) \leq 0 \\ \Leftrightarrow \cos(\mu) \leq -(1 - |x||y|)/2.$$

Suppose then $\cos(\mu) > -(1 - |x||y|)/2$. We have now

$$\mu - \operatorname{acos}(u_1/2) \leq \operatorname{acos}(u_0/2) \Leftrightarrow \\ \cos(\operatorname{acos}(u_0/2) + \operatorname{acos}(u_1/2)) \leq \cos(\mu) \\ \Leftrightarrow u_0u_1 - \sqrt{4 - u_0^2}\sqrt{4 - u_1^2} - 4 \cos(\mu) \leq 0.$$

To prove the inequality above, it is enough to prove it holds when its left-side is at maximum. Since $u_0u_1 - \sqrt{4 - u_0^2}\sqrt{4 - u_1^2}$ is clearly increasing with respect to both u_0 and u_1 and

$$u_0 < 1 - |y|^2 + 2|y| \cos(\mu) < \max\{1, 2 \cos(\mu)\}, \\ u_1 < 1 - |x|^2 + 2|x| \cos(\mu) < \max\{1, 2 \cos(\mu)\},$$

we have

$$u_0u_1 - \sqrt{4 - u_0^2}\sqrt{4 - u_1^2} - 4 \cos(\mu) \\ < \max\{2 - 4 \cos(\mu), 8 \cos^2(\mu) - 4 \cos(\mu) - 4\} \\ = (1 + 2 \cos(\mu)) \max\{-2, -(1 - \cos(\mu))\} \leq 0 \\ \Leftrightarrow \cos(\mu) \geq -1/2$$

Since we assumed that $\cos(\mu) > -(1 - |x||y|)/2$ and $-(1 - |x||y|)/2 \geq -1/2$, the inequality above holds within our bounds.

We have now proven that $\operatorname{th}(\rho_{\mathbb{B}^n}(x, y)/2) \leq \tilde{c}_{\mathbb{B}^n}(x, y)$ holds, regardless of whether (3.11) holds or not. The inequality $\tilde{c}_{\mathbb{B}^n}(x, y) \leq 2\operatorname{th}(\rho_{\mathbb{B}^n}(x, y)/2)$ follows from Lemma 3.1 because $s_{\mathbb{B}^n}(x, y) \leq \operatorname{th}(\rho_{\mathbb{B}^n}(x, y)/2)$ by [5, p. 460]. We can also show that our theorem has the best constants: For $x = ue_1$ and $y = ve_1$ with $0 < u < v < 1$,

$$\frac{\tilde{c}_{\mathbb{B}^n}(x, y)}{\operatorname{th}(\rho_{\mathbb{B}^n}(x, y)/2)} = \frac{\sqrt{1 - 2uv + u^2v^2}}{1 - u} = \frac{1 - uv}{1 - u},$$

and, if $v \rightarrow u^+$, this quotient becomes $1 + u$, which approaches 0 when $u \rightarrow 0^+$ and 1 when $u \rightarrow 1^-$. \square

Corollary 3.14 For all $0 < r < 1$ and $x \in G \in \{\mathbb{H}^n, \mathbb{B}^n\}$, the ball inclusion

$$B_\rho(x, R_0) \subseteq B_{\tilde{c}}(x, r) \subseteq B_\rho(x, R_1)$$

holds if and only if $R_0 \leq \log\left(\frac{2+r}{2-r}\right)$ and $R_1 \geq \log\left(\frac{1+r}{1-r}\right)$.

Proof By Corollary 3.7 and Theorem 3.13,

$$B_\rho\left(x, \log\left(\frac{2+r}{2-r}\right)\right) = \left\{y \in G : \operatorname{th}\frac{\rho_G(x, y)}{2} \leq r/2\right\} \\ \subseteq B_{\tilde{c}}(x, r) \\ \subseteq \left\{y \in G : \operatorname{th}\frac{\rho_G(x, y)}{2} \leq r\right\} = B_\rho\left(x, \log\left(\frac{1+r}{1-r}\right)\right)$$

for $G \in \{\mathbb{H}^n, \mathbb{B}^n\}$ and, since Corollary 3.7, and Theorem 3.13 have the best possible constants in their inequalities, we also have the best bounds for R_0 and R_1 here. \square

Remark 3.15 Corollary 3.14 improves the earlier inclusion result [11, Thm 5].

Corollary 3.16 If $f : G \rightarrow f(G)$ is a conformal mapping between domains $G, f(G) \in \{\mathbb{H}^n, \mathbb{B}^n\}$,

$$\tilde{c}_{f(G)}(f(x), f(y)) \leq 2 \tilde{c}_G(x, y).$$

Proof By Corollary 3.7 and Theorem 3.13 and the conformal invariance of the hyperbolic metric,

$$\begin{aligned} \tilde{c}_{f(G)}(f(x), f(y)) &\leq 2 \operatorname{th} \frac{\rho_{f(G)}(f(x), f(y))}{2} \\ &= 2 \operatorname{th} \frac{\rho_G(x, y)}{2} \leq 2 \tilde{c}_G(x, y). \end{aligned}$$

□

Corollary 3.17 *If $f : G \rightarrow f(G)$ is a non-constant quasiregular mapping between domains $G, f(G) \in \{\mathbb{H}^n, \mathbb{B}^n\}$,*

$$\tilde{c}_{f(G)}(f(x), f(y)) \leq 2 \lambda_n^{1-\alpha} (\tilde{c}_G(x, y))^\alpha,$$

where $\alpha = K_I(f)^{1/(1-n)}$ with the inner dilatation $K_I(f) \leq K$ of the mapping f , and λ_n is a constant defined in [5, (9.6), p. 158] that fulfills $\lambda_2 = 4$ and $\lambda_n < 2e^{n-1}$ for $n \geq 2$.

Proof Follows by Corollary 3.7, Theorem 3.13, and [5, Thm 16.2(1), p. 300]. □

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Declarations

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Consent for publication I am the sole author and consent for publication.

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References

1. Chen, J., Hariri, P., Klén, R., Vuorinen, M.: Lipschitz conditions, triangular ratio metric, and quasiconformal maps. *Ann. Acad. Sci. Fenn. Math.* **40**, 683–709 (2015)
2. Gehring, F.W., Osgood, B.G.: Uniform domains and the quasi-hyperbolic metric. *J. Anal. Math.* **36**, 50–74 (1979)
3. Gehring, F.W., Palka, B.P.: Quasiconformally homogeneous domains. *J. Anal. Math.* **30**, 172–199 (1976)
4. Hariri, P., Vuorinen, M., Zhang, X.: Inequalities and bi-Lipschitz conditions for triangular ratio metric. *Rocky Mt. J. Math.* **47**(4), 1121–1148 (2017)
5. Hariri, P., Klén, R., Vuorinen, M.: Conformally Invariant Metrics and Quasiconformal Mappings. Springer, New York (2020)
6. Hästö, P.: A new weighted metric, the relative metric I. *J. Math. Anal. Appl.* **274**, 38–58 (2002)
7. Rainio, O.: Intrinsic quasi-metrics. *Bull. Malays. Math. Sci. Soc.* **44**(5), 2873–2891 (2021)
8. Rainio, O.: Intrinsic metrics under conformal and quasiregular mappings. *Publ. Math. Debrecen* **101**(1–2), 189–215 (2022)
9. Rainio, O., Vuorinen, M.: Triangular ratio metric in the unit disk. *Complex Var. Elliptic Equ.* **67**(6), 1299–1325 (2022)
10. Rainio, O., Vuorinen, M.: Introducing a new intrinsic metric. *Result. Math.* **77**, 2 (2022). <https://doi.org/10.1007/s00025-021-01592-2>
11. Song, X., Wang, G.: A new metric associated with the domain boundary. *Comput. Methods Funct. Theory* (2024). <https://doi.org/10.1007/s40315-024-00545-4>
12. Zhou, Q., Zheng, Z., Ponnusamy, S., Guan, T.: Dovgoshey-Hariri-Vuorinen’s metric and Gromov hyperbolicity. *Arch. Math.* (2024). <https://doi.org/10.1007/s00013-024-02021-w>