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ORIGINAL ARTICLE

A revised condition for harmonic analysis in generalized Orlicz spaces on unbounded domains

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Abstract

Conditions for harmonic analysis in generalized Orlicz spaces have been studied over the past decade. One approach involves the generalized inverse of so-called weak Φ-functions. It featured prominently in the monograph *Orlicz Spaces and Generalized Orlicz Spaces* [P. Harjulehto and P. Hästö, Lecture Notes in Mathematics, vol. 2236, Springer, Cham, 2019]. While generally successful, the inverse function formulation of the decay condition (A2) in the monograph contains a flaw, which we explain and correct in this note. We also present some new results related to the conditions, including a more general result for the density of smooth functions.

KEYWORDS

density of smooth functions, double phase, generalized Orlicz space, Musielak–Orlicz spaces, nonstandard growth, variable exponent

1 | INTRODUCTION

Generalized Orlicz spaces, or Musielak–Orlicz spaces, are useful in the analysis of a variety of models, including the variable exponent and double-phase cases that have been fashionable lately. Harmonic analysis in this context has been studied over the past 10 years. In the 2019 monograph [7], conditions for harmonic analysis in generalized Orlicz spaces inspired by [1, 5, 8, 13] were developed and refined. One idea was to formulate the assumptions in terms of the generalized inverse of so-called weak Φ-functions that are not necessarily bijections (cf. [2, 4, 6, 9–11, 14] for variants and developments of the conditions). The rationale for this is to obtain better behavior with respect to taking limits and other modifications. This approach has proven its power in numerous papers, but it turns out that the inverse function formulation of the decay condition (A2) for unbounded domains in [7] contains a flaw, which we explain and correct in this note. Additionally, related new results are contained in Section 3. In particular, we show in Theorem 3.2 that (A2) implies (A0).

1.1 | The mistake in the book

We start with two versions of the condition (A2) from the monograph [7]. For every $\sigma > 0$, there should exist $\beta \in (0, 1]$ and $h \in L^1(\Omega) \cap L^\infty(\Omega)$ such that the following hold for a.e. $x, y \in \Omega$:

$$\beta\varphi^{-1}(x,\tau) \leqslant \varphi^{-1}(y,\tau) \quad \text{when } \tau \in [h(x) + h(y), \sigma] \tag{1.1}$$

or

$$\varphi(x,\beta t) \leqslant \varphi(y,t) + h(x) + h(y) \quad \text{when } \varphi(y,t) \in [0,\sigma]. \tag{1.2}$$

In these (and later) conditions, we assume without loss of generality that $h \geqslant 0$ (since if $h < 0$ then Equation (1.1) cannot hold as φ^{-1} is defined only for non-negative values, and Equation (1.2) cannot hold when $t = 0$.) Condition (1.2) is called (A2) in [8] whereas Equation (1.1) carries that name in [7]. The idea was that this does not lead to confusion since the conditions were supposed to be equivalent. Indeed, in [7, Lemma 4.2.5] they are claimed to be equivalent, but this is not the case as the next example shows.

Example 1.3. Let $B(0,1) \subseteq \mathbb{R}^n$ be the unit ball centered at the origin and $\Omega := B(0,1) \setminus \{0\}$. We define $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ by

$$\varphi(x,t) = \frac{t^2}{|x|}.$$

Since Ω is bounded, $h := \sigma\chi_\Omega \in L^1(\Omega)$. Thus, $[h(x)+h(y),\sigma]$ is empty for all $x,y \in \Omega$, and so Equation (1.1) holds. It remains to show that φ does not satisfy (1.2).

Suppose that the inequality from Equation (1.2) holds for some $\sigma > 0$, $\beta \in (0,1]$ and $h \in L^1(\Omega) \cap L^\infty(\Omega)$. Fix a non-exceptional point $y \in \Omega$ and $t \in (0,\infty)$ such that $\varphi(y,t) \in [0,\sigma]$, the inequality (1.2) holds for a.e $x \in \Omega$ and $|h(y)| \leqslant \|h\|_\infty$. Then

$$\frac{\beta^2 t^2}{|x|} \leqslant \frac{t^2}{|y|} + 2\|h\|_\infty$$

for almost every $x \in \Omega$. When $x \to 0$, this gives a contradiction. Thus, Equation (1.2) does not hold.

The problem with the proof of [7, Lemma 4.2.5], which claimed that Equations (1.1) and (1.2) are equivalent, is that the set $[h(x)+h(y),\sigma]$ can be empty and hence Equation (1.1) cannot necessary be applied for τ' in the proof, as we explain next. Here is an excerpt from the proof of [7, Lemma 4.2.5] with the problematic part highlighted.

> Assume Equation (1.1) and denote $\tau := \varphi^{-1}(x,t)$. By [7, Lemma 2.3.3], $\varphi(x,\tau) = t$, and it follows from Equation (1.1) that
>
> $$\beta_2\tau \leqslant \varphi^{-1}(y,\varphi(x,\tau))$$
>
> for almost every $x,y \in \Omega$ whenever $\varphi(x,\tau) \in [h(x)+h(y),\sigma]$. Then we apply $\varphi(y,\cdot)$ to both sides and use [7, Lemma 2.3.3] to obtain that
>
> $$\varphi(y,\beta_2\tau) \leqslant \varphi(x,\tau)$$
>
> for the same range. If, on the other hand, $\varphi(x,\tau) \in [0,h(x)+h(y))$, then we can find $\tau' > \tau$ such that $\varphi(x,\tau') = h(x)+h(y)$ since $\varphi \in \Phi_s(\Omega)$. As $\varphi(x,\cdot)$ is increasing, we obtain by **the previous case applied for τ' that**
>
> $$\varphi(y,\beta_2\tau) \leqslant \varphi(y,\beta_2\tau') \leqslant \varphi(x,\tau') = h(x)+h(y).$$

In the next section, we solve the problem by introducing a corrected (A2)-condition and re-prove some lemmas. First, we recall some notation.

TABLE 1 Correspondence between results in the book and this note.

Book	This note	Change
Definition 4.2.1	Definition 2.1(A2)	Definition modified
Lemma 4.2.2	Lemma 2.2	Old proof works
Lemma 4.2.3	Lemma 2.3	Stronger assumption and new proof
Lemma 4.2.4	Lemma 2.4	Auxiliary lemma and old proof
Lemma 4.2.5	Lemma 2.5	New proof

1.2 | Notation and terminology

For functions $\varphi, \psi : \Omega \times [0,\infty) \to [0,\infty]$, we write $\varphi \approx \psi$ or $\varphi \simeq \psi$ if there exists $c \geqslant 1$ such that

$$\tfrac{1}{c}\varphi(x,t) \leqslant \psi(x,t) \leqslant c\varphi(x,t) \quad \text{or} \quad \varphi(x,\tfrac{t}{c}) \leqslant \psi(x,t) \leqslant \varphi(x,ct),$$

respectively, for a.e. $x \in \Omega$ and every $t \in [0,\infty)$. A function $f : [0,\infty) \to [0,\infty]$ is *almost increasing* (more precisely, a-almost increasing) if there exists $a \geqslant 1$ such that $f(s) \leqslant af(t)$ for all $s \leqslant t$. *Almost decreasing* is defined analogously. By *increasing* we mean that the inequality holds for $a = 1$ (some call this non-decreasing), similarly for *decreasing*.

Definition 1.4. We say that $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ is a *weak Φ-function*, and write $\varphi \in \Phi_{\mathrm{w}}(\Omega)$, if the following conditions hold for a.e. $x \in \Omega$:

- $\varphi(\cdot, |f|)$ is measurable for every measurable $f : \Omega \to \mathbb{R}$.
- $t \mapsto \varphi(x,t)$ is increasing.
- $\varphi(x,0) = \lim_{t\to 0^+} \varphi(x,t) = 0$ and $\lim_{t\to\infty} \varphi(x,t) = \infty$.
- $t \mapsto \frac{\varphi(x,t)}{t}$ is a-almost increasing on $(0,\infty)$ with constant $a \geqslant 1$ independent of x.

If $\varphi(x,\cdot)$ is additionally a convex and in $C([0,\infty);[0,\infty])$ for a.e. $x \in \Omega$, then φ is a *strong Φ-function* and we write $\varphi \in \Phi_{\mathrm{s}}(\Omega)$.

The *left-inverse* of $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ is defined as

$$\varphi^{-1}(x,\tau) := \inf\{t \geqslant 0 \mid \varphi(x,t) \geqslant \tau\}$$

and the *conjugate* is defined as

$$\varphi^*(x,t) := \sup\{st - \varphi(x,s) \mid s \geqslant 0\}.$$

2 | THE CORRECTED DECAY CONDITION (A2)

We introduce a new, slightly modified (A2)-condition, that solves the problem explained in the previous section. This yields changes to Lemmas 4.2.2–4.2.5 in the book [7] (cf. Table 1). No modification is needed in the book after Lemma 4.2.5 since (A2) is mostly used as Equation (1.2) via Lemma 4.2.5, and in other places changing to the new (A2) is trivial. Also the extra assumption in Lemma 2.3 compared to [7, Lemma 4.2.3] does not matter since when Lemma 4.2.3 is used in the book also (A0) and (A1) are assumed.

Apart from the elegance of defining all conditions, (A0), (A1), and (A2), in terms of inverse functions, a motivation is that some features are easier to handle in this formulation, in particular invariance and conjugate functions. Since the invariance of Equation (1.2) under conjugation was shown in [7, Lemma 4.2.4] using Equation (1.1), that proof breaks down. Therefore, we need to establish φ^* satisfies Equation (1.2) if φ does using our new (A2).

Our corrected (A2)-condition with the inverse φ^{-1} is included in the following definition. Compared to Equation (1.1), it is more directly related to Equation (1.2), but not as elegant.

Definition 2.1. For $\varphi : \Omega \times [0,\infty) \to [0,\infty]$ and $p, q > 0$ we define some conditions.

$(\mathrm{aInc})_p$ There exists $a_p \geqslant 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^p}$ is a_p-almost increasing in $(0,\infty)$ for a.e. $x \in \Omega$.

$(\mathrm{aDec})_q$ There exists $a_q \geqslant 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^q}$ is a_q-almost decreasing in $(0,\infty)$ for a.e. $x \in \Omega$.

(A0) There exists $\beta \in (0,1]$ such that $\beta \leqslant \varphi^{-1}(x,1) \leqslant \frac{1}{\beta}$ for a.e. $x \in \Omega$.

(A1) There exists $\beta \in (0,1]$ such that, for a.e. $x, y \in \Omega \cap B$,

$$\beta\varphi^{-1}(x,\tau) \leqslant \varphi^{-1}(y,\tau) \quad \text{when} \quad \tau \in \left[1, \frac{1}{|B|}\right].$$

(A2) For every $\sigma > 0$ there exist $\beta \in (0,1]$ and $h \in L^1(\Omega) \cap L^\infty(\Omega)$, $h \geqslant 0$, such that, for a.e. $x, y \in \Omega$,

$$\beta\varphi^{-1}(x,\tau) \leqslant \varphi^{-1}(y, \tau + h(x) + h(y)) \quad \text{when} \quad \tau \in [0,\sigma].$$

Since $\varphi \simeq \psi$ if and only if $\varphi^{-1} \approx \psi^{-1}$ by Theorem 2.3.6 of [7], we see that φ satisfies (A2) if and only if ψ does. This proves the next result.

Lemma 2.2 (A2). *Is invariant under equivalence of weak Φ-functions.*

Like Equation (1.1), the new (A2)-condition only concerns the behavior of φ at infinity, as the following result shows. The next lemma contains an extra assumption, (A0), compared to the book. The fact that Equation (1.1) holds in all bounded domains even when φ is not locally bounded (cf. Example 1.3) suggests that the old condition is not reasonable.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded. If $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfies (A0), then it satisfies (A2).*

Proof. Let $\sigma > 0$. Since φ satisfies $(\mathrm{aInc})_1$, φ^{-1} satisfies $(\mathrm{aDec})_1$ by Proposition 2.3.7 of [7]. If $\tau \in [0,1]$, then $\varphi^{-1}(x,\tau) \leqslant \varphi^{-1}(x,1) \leqslant \frac{1}{\beta}$ by (A0). If $\tau \in (1,\sigma]$, then by $(\mathrm{aDec})_1$ and (A0),

$$\varphi^{-1}(x,\tau) \leqslant a\tau\varphi^{-1}(x,1) \leqslant a\sigma\frac{1}{\beta}.$$

Choose $h := \chi_\Omega$. Since h and Ω are bounded, $h \in L^1(\Omega) \cap L^\infty(\Omega)$. Furthermore,

$$\varphi^{-1}(y, \tau + h(x) + h(y)) \geqslant \varphi^{-1}(y,1) \geqslant \beta$$

by (A0). Combining the inequalities, we obtain the (A2)-inequality

$$\frac{\beta^2}{\max\{1, a\sigma\}}\varphi^{-1}(x,\tau) \leqslant \varphi^{-1}(y, \tau + h(x) + h(y)) \quad \text{for every } \tau \in [0,\sigma].$$

□

The inverse is also useful when dealing with the conjugate function. The following proof is as in [7], once we use Theorem 3.2(3) proved.

Lemma 2.4. *If $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfies (A2), then so does φ^*.*

Proof. Let us write $\tilde\tau := \max\{\tau, h(x) + h(y)\}$ and use Theorem 3.2(3). Thus, it suffices to assume $\beta\varphi^{-1}(x,\tilde\tau) \leqslant \varphi^{-1}(y,\tilde\tau)$ and show the same inequality for φ^*. We multiply the inequality by $(\varphi^*)^{-1}(x,\tilde\tau)(\varphi^*)^{-1}(y,\tilde\tau)$ and use $\frac{1}{c}\tilde\tau \leqslant$

$(\varphi^*)^{-1}(x,\tilde\tau)\varphi^{-1}(x,\tilde\tau) \leqslant c\tilde\tau$ [7, Theorem 2.4.8] to obtain that

$$\frac{\beta}{c^2}\tilde\tau(\varphi^*)^{-1}(y,\tilde\tau) \leqslant \tilde\tau(\varphi^*)^{-1}(x,\tilde\tau).$$

For $\tilde\tau > 0$, we divide the previous inequality by $\tilde\tau$ to get (A2) of φ^*; for $\tilde\tau = 0$, (A2) is trivial, since $(\varphi^*)^{-1}(x,0) = 0$. □

Next, we prove the equivalence of (A2) and Equation (1.2). Note that the equivalence is an easy exercise if φ is a strictly increasing bijection, but here we consider a more general case.

Lemma 2.5. *The weak Φ-function $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfies (A2) if and only if it satisfies Equation (1.2).*

Proof. We first show that Equation (1.2) is invariant under equivalence of Φ-functions. We assume that φ satisfies Equation (1.2) and take an equivalent function $\psi \in \Phi_{\mathrm{w}}(\Omega)$ such that $\varphi \simeq \psi$ with constant $L \geqslant 1$. By Equation (1.2) of φ,

$$\psi(x, \tfrac{\beta}{L}t) \leqslant \varphi(x,\beta t) \leqslant \varphi(y,t) + h(x) + h(y) \leqslant \psi(y, Lt) + h(x) + h(y),$$

for $\varphi(y,t) \in [0,\sigma]$. Denote $t' := Lt$. If $\psi(y,t') \in [0,\sigma]$, then $\varphi(y,t) \in [0,\sigma]$ and the previous inequality gives $\psi(x, \frac{\beta}{L^2}t') \leqslant \psi(y,t') + h(x) + h(y)$, which is Equation (1.2) for ψ. By Lemma 2.2, (A2) is invariant under equivalence of weak Φ-functions.

By Theorem 2.5.10 of [7], every weak Φ-function is equivalent to a strong Φ-function. Since both claims are invariant under equivalence, and every Φ-function is equivalent to a strong Φ-function, it suffices to show the claim for $\varphi \in \Phi_{\mathrm{s}}(\Omega)$. For a strong Φ-function, the inverse is better behaved since φ is a surjection onto $[0,\infty)$. Specifically, $\varphi(x,\varphi^{-1}(x,\tau)) = \tau$ for every $\tau \geqslant 0$ and $\varphi^{-1}(x,\varphi(x,t)) = t$ provided $\varphi(x,t) \in (0,\infty)$ [7, Lemma 2.3.3 and Corollary 2.3.4].

Assume first that (A2) holds and $\varphi(y,t) \in [0,\sigma]$. Let $\tau := \varphi(y,t)$. If $\tau \in (0,\sigma]$, then

$$\begin{aligned}
\beta\varphi^{-1}(y,\tau) \leqslant \varphi^{-1}(x,\tau + h(x) + h(y)) \quad &\Leftrightarrow \quad \beta t \leqslant \varphi^{-1}(x,\varphi(y,t) + h(x) + h(y)) \\
&\Rightarrow \quad \varphi(x,\beta t) \leqslant \varphi\big(x,\varphi^{-1}(x,\varphi(y,t) + h(x) + h(y))\big) \\
&\Leftrightarrow \quad \varphi(x,\beta t) \leqslant \varphi(y,t) + h(x) + h(y)
\end{aligned}$$

and so Equation (1.2) holds. When $\tau = 0$, we denote $t_0 := \max\{s \geqslant 0 : \varphi(y,s) = 0\}$, so that $t \in [0,t_0]$. Let $t_i \searrow t_0$. Then, the previous argument applies to $\tau_i := \varphi(y,t_i) \in (0,\sigma]$ and yields

$$\varphi(x,\beta t) \leqslant \varphi(x,\beta t_i) \leqslant \varphi(y,t_i) + h(x) + h(y) \to h(x) + h(y),$$

where we used that φ is continuous. Also, in this case Equation (1.2) holds.

Assume conversely that Equation (1.2) holds and $\tau \in [0,\sigma]$. Let $t := \varphi^{-1}(y,\tau)$ so that $\varphi(y,t) = \tau$. If $\varphi(x,\beta t) > 0$, then

$$\begin{aligned}
\varphi(x,\beta t) \leqslant \varphi(y,t) + h(x) + h(y) \quad &\Rightarrow \quad \varphi^{-1}(x,\varphi(x,\beta t)) \leqslant \varphi^{-1}(x,\tau + h(x) + h(y)) \\
&\Leftrightarrow \quad \beta\varphi^{-1}(y,\tau) \leqslant \varphi^{-1}(x,\tau + h(x) + h(y))
\end{aligned}$$

If $\varphi(x,\beta t) = 0$ and $\tau = \varphi(y,t) > 0$, then

$$\beta\varphi^{-1}(y,\tau) = \beta t \leqslant \max\{s \geqslant 0 \mid \varphi(x,s) = 0\} \leqslant \varphi^{-1}(x,\tau).$$

Finally, if $\tau = 0$, then (A2) automatically holds since the left-hand side in the (A2)-inequality equals zero. In all cases, (A2) follows from Equation (1.2). □

2.1 OTHER RESULTS

Next, we collect conditions that are equivalent with (A2). The starting point of this paper was the observation that Equations (1.1) and (1.2) are not equivalent. However, the following theorem shows that this can be remedied by some additional assumptions. We define one more auxiliary condition: for every $\sigma > 0$, there exist $\beta \in (0,1]$ and $h \in L^1(\Omega) \cap L^\infty(\Omega)$ such that, for a.e. $x, y \in \Omega$,

$$\beta\varphi^{-1}(x, \max\{\tau, h(x)+h(y)\}) \leqslant \varphi^{-1}(y, \max\{\tau, h(x)+h(y)\}) \quad \text{when} \quad \tau \in [0,\sigma]. \tag{3.1}$$

Note also that Lemma 2.4, which used the next theorem, is not involved in the next proof.

Theorem 3.2. *Let $\varphi \in \Phi_{\mathrm{w}}(\Omega)$. The following are equivalent:*

(1) φ satisfies (A2).
(2) φ satisfies Equation (1.2).
(3) φ satisfies Equation (3.1).
(4) φ satisfies Equation (1.1) for h with $\|h\|_\infty \leqslant \frac{\sigma}{2}$.
(5) φ satisfies Equation (1.1) and (A0).

Proof. From Lemma 2.5, we have equivalence between (1) and (2). The implication from (4) to (2) follows from the proof of the [7, Lemma 4.2.5], since the additional assumption $\|h\|_\infty \leqslant \frac{\sigma}{2}$ ensures that $[h(x)+h(y), \sigma]$ is non-empty for a.e. $x, y \in \Omega$. We conclude the proof by showing that (1) ⇒ (3) ⇒ (4) and (4) ⇔ (5).

First, we show that (1) implies (3). Let $\tilde\tau := \max\{\tau, h(x)+h(y)\}$. Since φ satisfies $(\mathrm{aInc})_1$, φ^{-1} satisfies $(\mathrm{aDec})_1$ by Proposition 2.3.7 of [7]. Hence

$$\varphi^{-1}(y, \tau + h(x) + h(y)) \leqslant \varphi^{-1}(y, 2\tilde\tau) \leqslant 2a\varphi^{-1}(y,\tilde\tau) \leqslant 2a\varphi^{-1}(y, \tau + h(x)+h(y)).$$

Thus, (A2) is equivalent with

$$\beta\varphi^{-1}(x,\tau) \leqslant \varphi^{-1}(y,\tilde\tau)$$

for almost every $x, y \in \Omega$ and every $\tau \in [0,\sigma]$. To obtain (3), we need to also show that $\beta\varphi^{-1}(x, h(x)+h(y)) \leqslant \varphi^{-1}(y,\tilde\tau)$ with possibly different $\beta > 0$. If $h(x)+h(y) \in (0,\sigma]$, then we can use the previous inequality with $\tau = h(x)+h(y)$ and obtain

$$\beta\varphi^{-1}(x, h(x)+h(y)) \leqslant \varphi^{-1}(y, h(x)+h(y)) \leqslant \varphi^{-1}(y,\tilde\tau).$$

If $h(x)+h(y) \in (\sigma,\infty)$, then we use $(\mathrm{aDec})_1$ for φ^{-1}, and (A2) with $\tau = \sigma$ to conclude

$$\beta\varphi^{-1}(x, h(x)+h(y)) \leqslant \tfrac{a}{\sigma}(h(x)+h(y))\,\beta\varphi^{-1}(x,\sigma) \leqslant \tfrac{2a}{\sigma}\|h\|_\infty\,\varphi^{-1}(y,\sigma).$$

Since $\sigma \leqslant h(x)+h(y) \leqslant \tilde\tau$, (3) follows.

Next, we show that (3) implies (4). For $\sigma > 0$, let $\beta \in (0,1]$ and $h \in L^1(\Omega)\cap L^\infty(\Omega)$, $h \geqslant 0$, be from Equation (3.1). If $\|h\|_\infty \leqslant \frac{\sigma}{2}$, there is nothing to prove, so we consider $\|h\|_\infty > \frac{\sigma}{2}$. We denote $\tilde h := \frac{\sigma}{2\|h\|_\infty}h$ and show that Equation (1.1) holds with function $\tilde h$ and some constant $\tilde\beta$, which establishes (4). Let $\tau \in [\tilde h(x)+\tilde h(y), \sigma]$ so that $h(x)+h(y) \leqslant \frac{2\|h\|_\infty}{\sigma}\tau$. If $\tau \geqslant h(x)+h(y)$, then (4) follows from Equation (3.1). When $\tau < h(x)+h(y)$, we use that φ^{-1} is increasing, the inequality (3.1) and $(\mathrm{aDec})_1$ of φ^{-1} to obtain that

$$\beta\varphi^{-1}(x,\tau) \leqslant \beta\varphi^{-1}(x, h(x)+h(y)) \leqslant \varphi^{-1}(y, h(x)+h(y)) \leqslant \frac{2a\|h\|_\infty}{\sigma}\varphi^{-1}(y,\tau).$$

Thus, (4) holds with $\tilde\beta := \frac{\beta\sigma}{2a\|h\|_\infty}$.

It remains to show that (4) and (5) are equivalent. Assume first that (4) holds. Then, Equation (1.1) holds so we need only establish (A0). We use Equation (1.1) with $\sigma = 1$ and $\|h\|_\infty \leqslant \frac{1}{2}$ for $\tau = 1$ and find that

$$\beta\varphi^{-1}(x,1) \leqslant \varphi^{-1}(y,1)$$

for a.e. $x, y \in \Omega$. Thus

$$\beta \operatorname*{ess\,sup}_{y\in\Omega} \varphi^{-1}(y,1) \leqslant \varphi^{-1}(x,1) \leqslant \frac{1}{\beta} \operatorname*{ess\,inf}_{y\in\Omega} \varphi^{-1}(y,1)$$

for a.e $x \in \Omega$. Since $\varphi^{-1}(x,1) \in (0,\infty)$, this implies that

$$0 < \operatorname*{ess\,inf}_{y\in\Omega} \varphi^{-1}(y,1) \leqslant \operatorname*{ess\,sup}_{y\in\Omega} \varphi^{-1}(y,1) < \infty.$$

Hence, (A0) holds with constant

$$\tilde\beta = \beta \min\left\{ \operatorname*{ess\,sup}_{y\in\Omega} \varphi^{-1}(y,1), \left(\operatorname*{ess\,inf}_{y\in\Omega} \varphi^{-1}(y,1)\right)^{-1} \right\}.$$

Assume finally that (5) holds, let β and h be from Equation (1.1) and define $\tilde h := \min\{h, \frac{\sigma}{2}\}$. We show that Equation (1.1) holds with function $\tilde h$ (and different β), which gives (4). If $h(x), h(y) \leqslant \frac{\sigma}{2}$, then $\tilde h(x) = h(x)$ and $\tilde h(y) = h(y)$ and there is nothing to show. Otherwise $h(x) + h(y) \geqslant \frac{\sigma}{2}$, so it suffices to prove the inequality from Equation (1.1) for all $\tau \in [\frac{\sigma}{2}, \sigma]$. For this we use (aDec)$_1$ and (A0):

$$\varphi^{-1}(x,\sigma) \leqslant \max\{1, a\sigma\}\varphi^{-1}(x,1) \leqslant \frac{\max\{1, a\sigma\}}{\beta^2}\varphi^{-1}(y,1) \leqslant \frac{\max\{1, a\sigma\}\max\{1, \frac{2a}{\sigma}\}}{\beta^2}\varphi^{-1}(y, \tfrac{\sigma}{2}).$$

Thus, Equation (1.1) holds with function $\tilde h$ and constant $\tilde\beta := \frac{\beta^2}{\max\{1,a\sigma\}\max\{1,2a/\sigma\}}$. □

In Lemma 2.3, we showed that (A0) implies (A2) in bounded sets and in the previous proposition we saw that (A2) implies (A0) in any set. The next result shows that (A1) implies both of them.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded. If $\varphi \in \Phi_{\mathrm{w}}(\Omega)$ satisfies (A1), then it satisfies (A0) and (A2).*

Proof. Let φ satisfy (A1). By Lemma 2.3, it suffices to show (A0). Since $\lim_{t\to 0^+} \varphi(x,t) = 0$ (by $\varphi \in \Phi_{\mathrm{w}}(\Omega)$), we obtain from the definition of inverse that $\varphi^{-1}(x,1) > 0$ for a.e. $x \in \Omega$. Since Ω is precompact, we can cover it by finitely many balls $B_1, \dots, B_m$ of measure equal to 1. In each ball, we choose $x_i \in \Omega \cap B_i$ outside the exceptional set so (A1) implies that $\beta\varphi^{-1}(x_i,1) \leqslant \varphi^{-1}(y,1) \leqslant \frac{1}{\beta}\varphi^{-1}(x_i,1)$ for almost every $y \in B_i \cap \Omega$. Since there are only finitely many balls, we obtain that

$$0 < \beta \min_{i\in\{1,\dots,m\}} \varphi^{-1}(x_i,1) \leqslant \varphi^{-1}(y,1) \leqslant \frac{1}{\beta} \max_{i\in\{1,\dots,m\}} \varphi^{-1}(x_i,1) < \infty$$

for almost every $y \in \Omega$, and hence (A0) holds. □

The previous result was inspired by Kamińska and Żyluk's recent study [12] of density of smooth functions in Sobolev spaces (see also [3]), where they show that (A0) and (A2) are not needed for density, in contrast to earlier results like Theorem 6.4.4 from [7]. Using Proposition 3.3, we can easily obtain the theorem without assuming (A0) and (A2). Similar modifications can be done for the other results in [7, Section 6.4] and other results which are local in character.

Proposition 3.4. *Let $\varphi \in \Phi_{\mathrm{w}}(\mathbb{R}^n)$ satisfy (A1) and (aDec)$_q$ for some $q > 1$. Then, $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,\varphi}(\mathbb{R}^n)$.*

Proof. Let $u \in W^{1,\varphi}(\mathbb{R}^n)$ and $\varepsilon \in (0,1)$. By [7, Lemma 6.4.1], we may assume that u has compact support in $\mathbb{R}^n$. Denote $\Omega := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \operatorname{supp} u) < 1\}$. Then, Ω is bounded and by Proposition 3.3, φ satisfies (A0), (A1), and (A2) in Ω. Let σ_ε be a standard mollifier. Then, $u * \sigma_\varepsilon$ belongs to $C_0^\infty(\Omega)$ and

$$\nabla(u * \sigma_\varepsilon) - \nabla u = (\nabla u) * \sigma_\varepsilon - \nabla u.$$

Thus, the claim follows since $\|f * \sigma_\varepsilon - f\|_{L^\varphi(\Omega)} \to 0$ as $\varepsilon \to 0^+$ for every $f \in L^\varphi(\Omega)$ by [7, Theorem 4.4.7] and since the norm $\|u\|_{L^\varphi(\Omega)} + \|\nabla u\|_{L^\varphi(\Omega)}$ is equivalent with $\|u\|_{W^{1,\varphi}(\Omega)}$ by [7, Lemma 6.1.5] (see [7, Definitions 3.2.1 and 6.1.2] for the definition of the norms). □

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