Review Article

# Roles of mathematics in physics education: A systematic review

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Mathematics plays many roles in physics and physics education. While these roles have previously been extensively discussed in the physics education research community, no systematic picture of the multifaceted considerations has yet been formed. To gain a comprehensive overview of the previous studies on the topic, we conducted a systematic literature review on 122 journal articles published between 2000 and 2023 that examine the role of mathematics in physics and physics learning. In the reviewed articles, we employed qualitative content analysis, coded each article for its characteristics, and used network maps for visualization. We identified eight thematic article categories, highlighting the complex integration of mathematics in, for example, physical reasoning, problem solving, modeling, and experiments. Additionally, the review examines theoretical frameworks and contexts, revealing an overemphasis on problem solving in mechanics and a limited exploration of advanced physics topics like quantum mechanics. A detailed inductive analysis further identified six overarching roles that mathematics plays in physics and physics education: (i) supporting learning and achievement, (ii) enabling mathematical manipulations, (iii) guiding reasoning and sensemaking, (iv) facilitating experiments and modeling, (v) serving as a language, and (vi) providing a structural foundation for physics as a science. Based on the analysis, we discuss how the roles assigned to mathematics have been conceptualized in the reviewed articles and provide an overall picture of the types and features of the reviewed corpus. Our findings suggest opportunities for future research, including deeper explorations of underrepresented physics contexts and targeted investigations into specific roles of mathematics in teaching and learning.

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## I. INTRODUCTION

In physics education research, mathematics is recognized as serving multiple roles in both physics and physics learning. It is often regarded as, for example, a tool, language, or representation, see, e.g. [1,2]. In most cases, these roles are merely asserted as facts and remain unexplored; however, recent years have seen growing interest in offering a more nuanced description of the roles of mathematics in physics education.

In physics instruction, mathematics is sometimes associated with solving end-of-chapter problems. In this role, mathematics often serves as a calculation tool that is algorithmically applied in problem solving, e.g. [3]. Often, it is also assumed that the students have already learned the mathematical knowledge and skills in their mathematics class before entering the physics class. Consequently, many student difficulties in learning physics have been attributed to a lack of mathematical competencies and difficulty of transferring mathematical knowledge to the context of physics, cf. [4]. Among others, Karam *et al.* [5], however, argue against this stance and instead suggest that, fundamentally, many difficulties in transferring knowledge between the disciplines emerge from students' failure to recognize the interconnections between physics and mathematics.

The role of mathematics in physics and physics education is indeed far more complex than mathematics merely serving as prerequisite knowledge and skills for physics learning and problem solving. Mathematics and physics are deeply entangled, which stems from the long mutual history of the two disciplines. For example, Gingras [6] provides an overview of how the mathematization of physics has had long-term social, epistemological, and ontological effects on physics as a discipline in the past, and Brush [7] even argues that mathematics is often responsible for instigating scientific revolutions. To back up his argument, Brush introduces multiple examples of physical revolutions started by mathematics, ranging from

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the heliocentric theory by Copernicus to advances in quantum mechanics. As mentioned also by Brush, it was already Wigner who famously praised "the unreasonable effectiveness of mathematics" in the context of quantum theory [8]. However, this dynamic is not one-sided: physics, among other sciences, has also shaped mathematics. For example, Kjeldsen and Lützen [9] emphasize the two-way dynamics between physics and mathematics, discussing the history of the concept of function.

In the field of physics education research, many authors have contributed to analyzing the interconnections between physics education and mathematics, and multiple models exist to describe these entangled interactions. For example, Pietrocola [10] examines the various aspects of handling mathematics in physics, while Brahmia [11] models the relation of physical and mathematical understanding. Focusing on student modeling and reasoning activities, for example, Redish and Kuo [12] provide a diagrammatic presentation of a modeling and problem-solving process, which Uhden *et al.* [13] complement by incorporating aspects of Pietrocola's considerations into it. In discussing these two models, Palmgren and Rasa [14] give a layered characterization of the roles mathematics plays in physics and physics education. Meanwhile, in mathematics education research literature, the interconnections between mathematics and physics have been actively explored from the perspective of mathematics learning. On this side of the field, for example, the modeling cycle by Blum and Borromeo [15] exemplifies the complexity of the mathematical modeling process in detail. Moreover, there exists a vast body of mathematics education research examining, for example, mathematical problem-solving processes and the construction of mathematical models of physical systems. For instance, Wawro *et al.* [16] investigate how quantum mechanics students interpret eigenequations in different contexts, while Naranjo and Jones [17] study how students construct differential equations to model real-world systems, just to mention some recent contributions.

An extensive overview of the previous discussions on the roles mathematics plays in physics education has been provided by Pospiech [18] in the introductory chapter of the book *Mathematics in Physics Education* [19]. From an educational perspective, Pospiech separates three overarching aspects of the role mathematics has in physics, following Krey [20]: "it serves as a tool (pragmatic perspective), it acts as a language (communicative function), and it provides a logical and structural framework for describing, ordering and classifying physical processes and theories." On one hand, in Pospiech's overview, the roles of mathematics span "the range from mathematics as a technical tool to an inseparable mathematical-physical reasoning." Mathematics serves as a tool, for example, for technical calculations. At the same time, physics inherits deductive power from mathematics needed for shaping new concepts and theories. On the other hand, mathematics is seen to provide means for representing physical constructs. Beyond this representational function, however, mathematics also provides structural insights (see also Ref. [21]).

A lot of Pospiech's [18] discussion is primarily based on the idea of separating what they label technical and structural roles of mathematics in physics. Following Pietrocola [10], Pospiech defines the technical role as comprising "aspects or activities mainly related to numerical procedures" and the structural role as building "the skeleton of physical theories" and providing "valuable general theorems allowing to proceed into the unknown." As they stress, there is no sharp separation between the technical and structural aspects; instead, they lie at opposite ends of the same spectrum. The continuum between technical and structural roles of mathematics captures and generalizes both the above-mentioned continuums between technical use and reasoning, as well as representational and structural roles.

As is apparent from the above discussion, the relationship between physics and mathematics is highly complex and multifaceted. This being the case, it is not surprising that conveying this deep interaction of the disciplines to students is not an easy task. Pospiech [18] underscores the importance of adopting an integrated approach to teaching physics, emphasizing the need to develop students' understanding of the interplay between physics and mathematics from early stages of education.

While there has been a lot of interest in the roles of mathematics in physics and physics learning in the physics education research community, no systematic picture of this multifaceted discussion has yet been formed. Thus, the aim of this study is to form a comprehensive overview of physics education research on the topic published in recent years and to investigate how the role of mathematics has been framed in this literature. To this end, we conducted a systematic literature review of articles in physics education research and related fields, covering publications from 2000 to 2023.

The article has two main objectives reflected in two research questions. Regarding the aim of forming an overview of the studies on the topic, the first objective is to create an overall picture of the types and features of the reviewed articles. Here, we are especially interested in the theoretical and conceptual frameworks employed in the reviewed studies, as those might affect how the role of mathematics in physics is conceptualized—that is, what kind of roles are assigned to mathematics in physics. Furthermore, we examine other study characteristics: the type of study (empirical or theoretical), research strategy for empirical studies (qualitative, quantitative, or mixed), population, sample size, and the context within physics in which the study was conducted. Moreover, we consider the main themes and topics of the articles. The second objective is to form an overview of how the roles mathematics plays in physics and physics education are framed in the reviewed literature. The specific research questions are as follows:

RQ1. What are the characteristics of recent physics education research articles that examine the relation between mathematics and physics or physics learning?

- RQ1a. What are the generic characteristics of the studies (the type of study, research strategy for empirical studies, population, sample size, and the physics context)?
- RQ1b. What are the research themes examined in the studies?
- RQ1c. What are the theoretical frameworks employed in the studies?

RQ2. What roles of mathematics in physics and physics learning can be identified in these studies?

To address these research questions, we start by describing the study methods in Sec. II. Here we describe the steps of the literature review, the content analysis, and coding processes, as well as using network maps for visualization. In Sec. III, we present the results of the analyses, and in Sec. IV, we discuss their implications. Finally, we close the paper with conclusions in Sec. V.

## II. METHODS

To acquire a comprehensive overview of previously published studies and reveal the various viewpoints on the role mathematics plays in physics education presented in the literature, we conducted a systematic literature review [22]. The context of the reviewed articles and the steps of the review process are presented below.

To address the research questions, we employed qualitative content analysis [23,24] on our selected articles. In addition, in our answer to RQ1, we coded each article for characteristics based on our reading and used network maps as visualizations. The analysis methods are discussed in detail below.

### A. Identifying the relevant literature

To select relevant articles for review, we conducted a search of the ERIC database via ProQuest for peer-reviewed journal articles written in English and published between 2000 and 2023. As the search terms, we used "mathematics" AND "physics," "multiple representations" AND "physics," "equations" AND "physics," and "mathematical model" AND "physics." This search resulted in 3001 articles. We initially screened potential articles based on the title and removed duplicate entries. At this initial stage, our broad inclusion criterion was that the articles explicitly examined the relation between mathematics and physics. Our aim was to include all potentially relevant articles. After the initial screening, 264 articles remained.

Next, the articles were screened based on their abstracts, narrowing down to 181 articles. At this stage of screening, the selection criterion was whether the articles explicitly discussed the relation of mathematics and physics or examined the role of mathematical understanding in physics education. Approximately 80% of the abstracts were reviewed independently by two authors, and the inclusion criterion was discussed when there was a disagreement about inclusion between the authors.

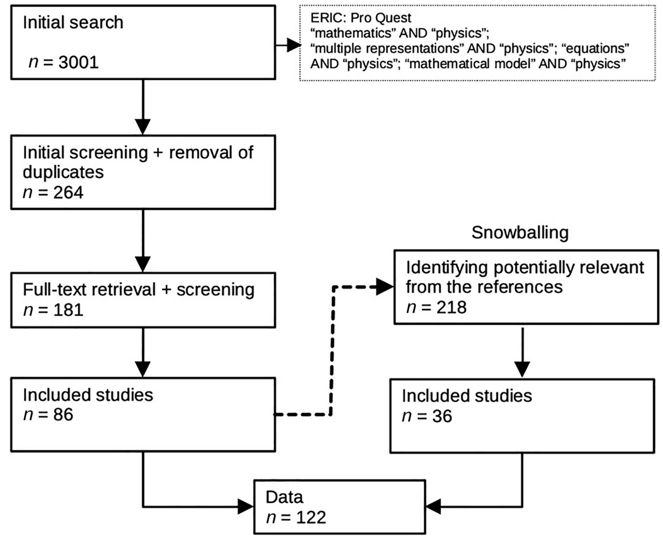


FIG. 1. The screening process and number of articles chosen in each stage.

Finally, the full texts of the 181 articles were retrieved and screened, resulting in 86 articles being included in the review. The above-mentioned criterion was also applied in reading the full texts, but it was refined by excluding studies that used physics, for example, solely as a context for mathematics problems or discussed physical modeling using spreadsheets. We also left out studies examining computer-based calculation methods in physics education and studies examining the ways in which physics can enhance mathematics learning. Finally, studies whose primary aim was to validate questionnaires were left out. Inclusion and exclusion of the articles were discussed between the two authors.

When reading the articles, we also identified potentially relevant articles from the reference lists (a method known as "snowballing"). The snowballing process was recursively repeated for all the reviewed articles, and in this way, we identified 218 potentially relevant journal articles based on their titles. After retrieving the full texts, 36 articles were selected for the review using the criteria described above. In total, 122 articles were selected for the analysis. The number of articles chosen in each stage is presented in Fig. 1.

### B. Content analysis and coding for characteristics

The analysis of the selected articles and their characteristics included two main phases that are described below.

#### 1. Identifying generic characteristics

First, to answer RQ1a, we coded the articles for a preagreed set of generic codes to characterize the types of studies included in the review [25]. Such codes are useful for describing the available studies in systematic reviews [22]. While there are standardized coding systems for generic coding for purposes of building a database (see, e.g. [25]), the set of codes and level of detail are often dependent on the aims of the researchers [22,25]. In line with our aim of forming an overview of the studies on the topic, we chose a set of codes that cover the type of study (empirical or theoretical), research strategy for empirical studies (qualitative, quantitative, or mixed), population (e.g., university or high school students, or teachers), sample size, and the physics context in which the study was conducted (e.g., mechanics or electricity and magnetism). Examining these aspects allows us to qualify our findings regarding the roles of mathematics in physics covered by RQ2. The codes were chosen following the suggestions of Gough *et al.* [22] and Bennett *et al.* [25].

#### 2. Thematic categorization and theoretical frameworks

To answer RQ1b, 1c, and 2, we employed inductive content analysis [23,24,26]. In inductive content analysis, the codes are derived from the data, in contrast to deductive content analysis, where the codes are operationalized prior to coding based on existing knowledge or theory [23]. Thus, inductive analysis potentially enables us to uncover novel viewpoints on the analysis and to examine aspects possibly missing in previous similar studies.

The content analysis process itself begins with choosing the unit of analysis. Regarding RQ1b, we chose the entire article as a unit of analysis, and for RQ1c, we chose a passage of text describing the theoretical framework used (depending on the article, this can vary from a sentence to a paragraph). For RQ2, we chose a sentence expressing one idea related to the research question. That is, a sentence expressing a role for mathematics was coded.

As for coding, several techniques exist [27,28]. The choice of the technique depends, for example, on the aims and research questions. For coding the research themes and frameworks, we used *descriptive coding*, which means assigning labels or short descriptive phrases to summarize passages of data [28]. For coding the roles, we used *in vivo* coding (to be described in detail later) [27]. Moreover, coding is often cyclical rather than linear in nature. In other words, codes are refined, combined, and categorized across multiple cycles, each of which may require different techniques [27].

To gain an overview of the themes examined in the research (RQ1b), we identified and coded the articles for research themes (such as the use of mathematics in physics problem solving). We then categorized the articles thematically. This process involved examining the focus of each study and grouping similar themes together to form a comprehensive overview of the types of research included in the review. Even though one article may include multiple themes, each article was categorized based on only one theme or topic, which was interpreted to be the main one. Categories and their boundaries were refined and established based on discussions between the two authors. Characteristics of the reviewed literature are discussed in Sec. III A.

To answer RQ1c, we coded the articles for the theoretical frameworks employed in the studies. Theoretical frameworks are here loosely understood as containing the assumptions and orientations of the research and as such provide a particular perspective for explaining and interpreting a phenomenon [29]. Theoretical frameworks guide the questions, methods, and interpretations of the findings. Therefore the frameworks are relevant to exploring how the role of mathematics in physics education has been studied. Some of the framework codes here represent somewhat established theoretical frameworks with their own theoretical assumptions (e.g., conceptual blending), and others merely reflect a certain point of view or general approach to analyzing the relationship of mathematics and physics. For instance, articles with the code "achievement" typically look at the correlations between scores in mathematics and physics achievement tests. The coding for frameworks was also discussed between the two authors. Frameworks are discussed in Sec. III B.

The characteristics of the reviewed articles were visualized via network maps. The process of network analysis is explained in the next section (Sec. II C). The visualizations and their implications are discussed in detail in Sec. III C.

#### 3. Identifying the roles of mathematics

To identify the roles of mathematics in physics education for answering RQ2, multiple coding cycles were performed. In the first coding cycle, as we aimed to capture the landscape and varied terminology of the rather diverse set of studies included in the analysis, we chose to use *in vivo* coding (also called literal coding, verbatim coding, inductive coding, etc.) [27]. While sometimes the analyst names the codes, during *in vivo* coding, the code names are derived using the language and terms appearing in the data. We performed the first coding cycle for articles from 2000 to 2019, and it resulted in 331 unique role codes. In the second coding cycle, we aimed at reducing the number of codes by merging similar codes into broader ones. This cycle proceeded iteratively through merging and revising codes. The second coding cycle resulted in 20 role codes. This analysis was carried out and discussed by two authors. The analysis process is schematically presented in Fig. 2.

The 20 role codes were subjected to the third and final coding cycle, aiming at creating the final categorization of the role codes for theorizing and identifying broader themes in the data. In this cycle, we identified the codes that we considered to belong to broader role categories and grouped them together. This grouping was discussed and

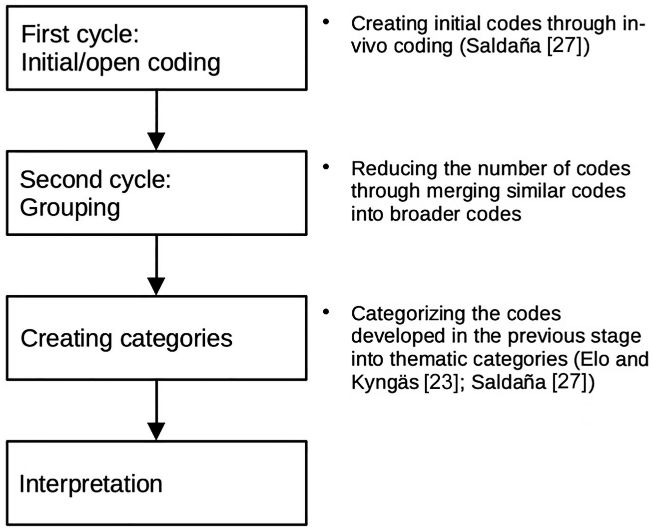


FIG. 2. Schematic description of the analysis process, cf. [23].

established between two authors. This final cycle resulted in a coding scheme of six role categories. After the initial analysis process, we expanded the dataset to include papers from 2020 to 2023. We coded these articles applying the final categorization scheme (the final six broader role categories). The role categorization is discussed in more detail in Sec. III D.

### C. Visualizing the characteristics via networks

In the analysis process, the reviewed articles were thematically categorized based on their topics and coded for their generic characteristics (e.g., physics context of the study and studied population) as well as the employed theoretical framework. In some cases, more than one code was assigned to an article. For instance, an article could be labeled with "mechanics" as well as "electricity and magnetism" to describe the physics contexts of the article. This was true also for the theoretical frameworks employed in the articles; an article might use, for example, both a problem-solving framework and a students' epistemologies framework. Thus, articles could show multiple connections to categories and codes.

We created network maps to visualize and analyze the relationships between articles and coding categories. A network generally consists of nodes that represent entities and links that represent relationships between these entities [30]. Our network maps of articles and coding categories are bipartite networks [31], meaning that there are two types of nodes, and links never connect to the same type of node. In our bipartite networks, coding categories represent one type of node, while articles represent another type of node. A link was established between an article A and a coding category C if C was assigned to A in the coding process.

Network maps are relevant in this case because they are apt for showing complex relationships in data. In creating networks, choices need to be made with regard to what counts as a node, what counts as a link, and what counts as node and link attributes [30]. We follow Bruun [30], who argues that these choices depend on the study at hand. In our case, we made the choice that having articles as one type of node would provide fruitful information for interpretations. As for the other type of node in our bipartite networks, our rule of thumb was that the category chosen should allow for multiple code assignments for each article. Furthermore, the category should give information about physics, the role of mathematics, and/or how the relationship was studied, since it is the aim of this study to characterize these. Thus, while population (here, e.g., undergraduate students) as a coding category could satisfy the first rule of thumb (for instance, by addressing multiple populations), it does not provide direct information about the physics, the role of mathematics, or how it was studied. The physics context of an article, however, would be a good candidate, since a given study could examine the role of mathematics in, for instance, mechanics and electromagnetism, and this does provide information that is directly relevant to our study. With these rules of thumb, we used bipartite networks to provide network maps to show overarching patterns between articles and coding categories. We also used co-occurrence networks to show direct relationships between coding categories. Such networks have been used, for instance, in Module Analysis for Multiple Choice Responses (MAMCR) to gauge relationships between responses to questionnaires [32,33]. Instead of questionnaire items, our focus was on relationships between coding categories. The procedures are described in detail in those studies [32,33], and we followed them to make backbone networks that show only the most prevalent connections [34]. These networks were then interpreted to characterize overall patterns between relevant coding categories.

To assess whether visual patterns in the network maps reflect meaningful structure rather than chance, we supplemented our qualitative inspection with two quantitative analyses. First, we used a Monte Carlo version of the chi-squared test [35] to check whether the distribution of thematic categories differs significantly across coding categories, taking into account the irregular margins in our data. Second, we used a node-level version of Bruun and Bearden's [36] segregation measure, which in our case uses Kullback-Leibler divergence to quantify how much categorical distributions around nodes deviate from a global distribution, comparing the observed value to a randomized null model to produce interpretable $Z$ scores (patterns are significant to the 0.05 level for $Z > 1.96$). This is appropriate for network data where nodes (articles, frameworks, and contexts in our study) vary in how many articles they are linked to. This method allowed us to assess, for each node, whether the distribution of connected article categories deviates from what would be expected by chance. Together, these methods provided both a global test

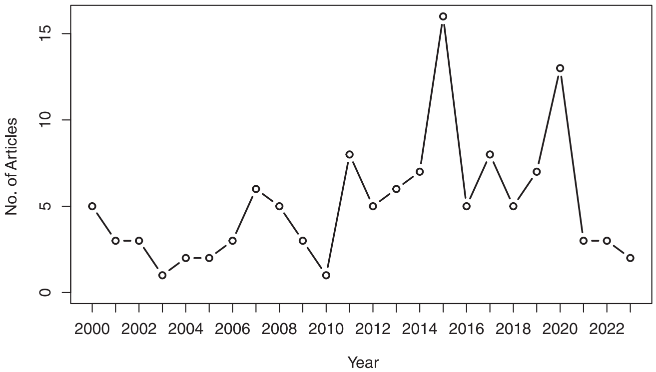


FIG. 3. The number of articles published per year. This subfield does not seem to follow the usual exponential growth in papers but appears more periodic.

of association and a more fine-grained view of which nodes structure the thematic landscape. Details of the procedures are available in [37].

## III. RESULTS

The analysis of the selected articles allowed us to provide an overview of the recent physics education research, examining the role of mathematics in physics and physics learning. Most importantly, we analyzed the roles mathematics has been assigned in the reviewed studies. The findings of the analysis are presented below.

### A. Description of the reviewed articles

The reviewed literature consists of 122 journal articles published in physics education research and the related fields in the years 2000–2023. The number of reviewed articles published per year is presented in Fig. 3. Unexpectedly, the number of articles does not follow the typical exponential-like growth but appears more periodic. In addition, the number of reviewed articles per journal is shown in Table I. As is apparent looking at these numbers, the most popular journals for discussions on the themes of interest have been *Physical Review Physics Education Research* (43 articles; 35%) and *Science & Education* (23; 19%). It should, however, be noted that *Physical Review Special Topics—Physics Education Research* was renamed *Physical Review Physics Education Research* in 2016, and here, we treat them as the same journal.

#### *1. Generic characteristics of the articles*

In the analysis process, the articles were coded for a preagreed set of generic codes to characterize the types of studies included in the review, e.g., based on the physics context in which the study was conducted. In all the reviewed articles, at least one of the primary contexts of the discussion belongs to the field of physics. For example, Silva [38] explores the historical development of electromagnetism using the viewpoint of mathematical analogies, and Karakok [39] focuses on undergraduate students' understanding of eigenvalues and eigenvectors in quantum mechanics. Some studies combine multiple fields. For example, Rodriguez *et al.* [40] discuss graphical reasoning in biology, calculus, and chemistry in addition to physics (mechanics). The most typical physics contexts of the reviewed studies are mechanics (in 51 articles; 42%) and electromagnetism (40; 33%). Also, quantum mechanics (18; 15%) and thermodynamics (14; 11%) are relatively frequently used as contexts.

TABLE I. The number of articles per journal. *Physical Review Special Topics—Physics Education Research was renamed Physical Review Physics Education Research in 2016. Here, we treat them as the same journal.

| Journal | No. of articles |
|---|---|
| Physical Review Physics Education Research* | 43 |
| Science & Education | 23 |
| American Journal of Physics | 8 |
| International Journal of Science and Mathematics Education | 4 |
| ZDM Mathematics Education, Research in Science Education, Physics Education, International Journal of Science Education (4 journals) | 3 |
| Science Education International, Science Education, International Journal of Mathematical Education in Science and Technology, European Journal of Engineering Education (4 journals) | 2 |
| Acta Didactica Norden, Canadian Journal of Science, Mathematics and Technology Education, Cognition and Instruction, Education Sciences, EURASIA Journal of Mathematics, Science and Technology Education, European Journal of Physics, European Journal of Science and Mathematics Education, For the Learning of Mathematics, International Education Studies, International Journal of Computers for Mathematical Learning, International Journal of Environmental & Science Education, Journal of Baltic Science Education, Journal of Chemical Education, Journal of Intelligence, Journal of Research in Science Teaching, Journal of Science Education and Technology, Journal of Technology and Science Education, Mathematics Education Research Journal, Nordic Studies in Science Education, Studies in Educational Evaluation, Studies in Science Education, The Asia-Pacific Education Researcher, The Journal of the Learning Sciences, Topics in Cognitive Science (25 journals) | 1 |

About a quarter of the reviewed articles (29; 24%) are theoretical studies, such as historical or philosophical analyses, task analyses, and document reviews (textbooks, curricula, etc.). Respectively, approximately three-quarters of the articles (93; 76%) are empirical studies utilizing student or expert data. When analyzing the articles and their research methodologies, 58 of the empirical articles were coded as qualitative studies (62%), 17 as quantitative (18%), and 18 as mixed (19%). This coding was based on the interpretation of the coder, as the analysis methods were not always clearly stated in the articles.

In the empirical studies, the data were collected from various populations: university students, students in lower educational levels, preservice and in-service teachers, university instructors and researchers, as well as other experts. University student data were collected from physics majors as well as other majors (e.g., mathematics and engineering). Overall, the grade levels of the student samples spanned from elementary to graduate. In 64 of the articles (69% of the empirical studies), data were collected from undergraduate students, sometimes combined with data from some other group. In the reviewed empirical studies, the sample sizes also vary from case studies with only a couple of participants to large studies with sample sizes larger than 500. However, almost half of the empirical studies have small samples with 5–50 participants (45; 48%).

### 2. Thematic categorization of the articles

From our reading and coding of the articles, we formed eight thematic categories that describe the main themes or topics of the articles. These categories were inductively produced during the data analysis process, and they provide a comprehensive overview of the themes of the studies included in the review. Each article was categorized based on only one theme or topic, which was interpreted to be the main one. However, it should be noted that an article may include and blend multiple themes, and the boundaries of the thematic categories are not clear-cut, with some categories overlapping. The categories are briefly described below and presented in Table II.

*Students' mathematical modeling of physical phenomena* is the smallest category, with only five articles, examining students' use of mathematical concepts and formalisms in modeling and inquiry contexts. More specifically, these articles investigate students' abilities to make connections between physical phenomena and mathematical models representing those in laboratory settings.

TABLE II. Thematic article categories, describing the topics of the articles, and the reviewed articles are assigned to the categories.

| Thematic category | Description | No. of articles | References |
|---|---|---|---|
| Category 1: Students' mathematical modeling of physical phenomena. | Studies that examine students' use of mathematical concepts or formalism in modeling and inquiry contexts. | 5 | [1,41–44] |
| Category 2: Role of mathematics in physics teaching. | Studies that examine how mathematical and physical concepts are blended in instruction and instructional materials. | 8 | [45–52] |
| Category 3: Mathematics as a representation for physics. | Studies that examine the features of mathematics as representations of physics concepts (e.g., how different notational systems may affect learning). | 9 | [40,53–60] |
| Category 4: Students' understanding/ interpretation of mathematical concepts in physics contexts. | Studies that examine students' understanding and/or use of mathematical concepts in physical contexts. | 10 | [39,61–69] |
| Category 5: Statistical analyses of relations between mathematics and physics achievement. | Studies that examine, e.g., correlations between mathematics and physics achievement measures. | 11 | [70–80] |
| Category 6: Conceptions about the relationship of mathematics and physics. | Studies that examine students' and teachers' conceptions of the relationship between mathematics and physics. | 15 | [2,3,81–93] |
| Category 7: Historical and philosophical accounts of the role of mathematics in physics. | Studies that examine the relation of mathematics and physics from a historical and/or philosophical point of view. | 21 | [7,13,21,38,94–110] |
| Category 8: Use of mathematics in physics problem solving. | Studies that examine students' use of mathematics in problem solving. | 43 | [12,111–152] |

The aim is to find ways to help students better perceive these connections.

*Role of mathematics in physics teaching* consists of eight articles, investigating how the role of mathematics appears in physics instruction, instructional materials, and, for example, physics and mathematics curricula. The articles in this category discuss how the role of mathematics should be framed in physics instruction and instructional materials to bring the connections between physics and mathematics into the forefront and present examples of teaching sequences targeting this aim.

*Mathematics as a representation for physics* features nine articles, the aim of which is to study how different representations may affect physics learning, for example, how the chosen mathematical notation system affects student understanding of physics concepts or reasoning in problem-solving situations. Another key focus of the studies in this category is students' understanding and fluency in using multiple mathematical representations in physics contexts.

*Students' understanding or interpretation of mathematical concepts in physics contexts* includes 10 articles that examine how students perceive and use mathematical constructs in physics. The main focus of these articles is on students' conceptual understanding of mathematical constructs in physics contexts, but also students' abilities to apply these constructs are investigated.

*Statistical analyses of relations between mathematics and physics achievement* is formed by 11 articles, examining correlations between mathematics and physics achievement measures. Correlations are searched for between physics or science achievement and learning gains, and, for example, mathematical preinstructional knowledge and flexibility. Some of the studies utilize large datasets, such as the Trends in International Mathematics and Science Study (TIMSS), Longitudinal Study of American Youth (LSAY), and National Matriculation Examination Data.

*Conceptions about the relationship of mathematics and physics* involves 15 articles that examine students' and experts' conceptions of the connections between mathematics and physics. The goal of these studies is to understand the epistemological views people hold on physics, mathematics, and their interconnections, and how these views evolve. Another focus of the studies is supporting the development of students' conceptions toward a more multifaceted and scientific direction during instruction.

*Historical and philosophical accounts of the role of mathematics in physics* contains 21 articles that examine the relation of mathematics and physics from a historical or philosophical point of view. Many of the studies in this category present historical accounts of the role mathematics plays in physics, for example, in the development of physical theories or scientific advancements. Others focus more on philosophical aspects of the interconnections of physics and mathematics and their effects on physics education. Notably, 15 (65%) of the articles have been published in *Science & Education*, while all the other articles in this category represent sporadic journals.

*Use of mathematics in physics problem solving* is the largest category with 43 articles. The studies of this category explore students' mathematical reasoning and sensemaking in physics problem-solving situations. These issues have been investigated from many points of view. Many of the studies in this category, for instance, apply frameworks stemming from Hammer's resources [153] in their exploration. About two-thirds (29; 67%) of the articles have been published in *Physical Review Special Topics—Physics Education Research* or *Physical Review Physics Education Research*.

### B. Theoretical frameworks employed in the articles

The theoretical frameworks employed in the articles are presented in Table III. While theoretical frameworks represent somewhat distinct perspectives for looking at a phenomenon, a study might employ more than one framework. Therefore, an article can have more than one framework code, and it was indeed rather typical. The biggest category was Other, which has frameworks that have equal to or fewer than three mentions. Frameworks included in this category are, for example, mental models, distributed cognition, formal analogies, and concept image. It is notable that despite mental models being rather ubiquitous in educational research, they did not appear as such in our data.

Regarding frameworks with four or more mentions, students' epistemologies (21; 17%), conceptions (17; 14%), and resources were three of the most frequent. These were followed by history of science and problem solving (13; 11%). Regarding the most frequent frameworks, students' epistemologies is a framework looking at students' views and understanding of knowledge, the role of mathematics, the nature of physics and, for example, the activities related to learning physics such as problem solving. For example, Bajracharya and Thompson [113] examine students' epistemological framing of physics problems, which is concerned with how students interpret a problem and its possible solution strategies. The conceptions framework is concerned with describing students' understanding of, for example, physics concepts. As an example, Planinic *et al.* [69] examine the understanding of the concept of a line graph slope in the contexts of mathematics and physics. The resources framework aims at building a "phenomenological model of high-level thinking" [115]. It is commonly used in examining students' problem-solving behavior and understanding the variety of context-specific cognitive elements students use during a problem-solving process. History of science is an example of a more overarching framework that looks at the history of science with the aim of deriving implications for teaching and learning. For example, Kanderakis [100] investigates the mathematization of physics through historical examples.

TABLE III. Frameworks employed in the reviewed articles.

| Framework | Description | No. of articles | References |
|---|---|---|---|
| Students' epistemologies | Examines students' understanding of the nature of physics, knowledge, role of mathematics etc. | 21 | [68,81,82,84–86,113,115,117,118,120,121,124,129,130,132,133,142,149–151] |
| Conceptions | Describes students' understanding of e.g., physics concepts. | 17 | [2,3,44,46,53,64,66–69,83,87–89,92,136,140] |
| Resources | A phenomenological model of thinking comprising connected chunk of diverse knowledge elements. | 14 | [12,84,113,115,117,122,124,131,133,142,149–152] |
| History of science | Examines historical development of physics to inform education. | 13 | [7,82,90,94,95,97,100,102–104,107–109] |
| Problem solving | Models the skills, strategies and processes of problem solving. | 13 | [99,112,116,118,125,126,134,135,137,141,143,144,147] |
| Achievement | Looks at outcomes of learning as measured by concept tests, and their correlations. | 9 | [70–72,74,75,77–80] |
| Modeling | Views models as central conceptual structures in physics. | 9 | [13,38,41,43,45,47,98,100,108] |
| Symbolic forms | Vocabulary of conceptual schemas associated with mathematical symbols. | 9 | [12,40,42,59,119,127,131,138,139] |
| External representations | Views e.g., mathematics symbols as external representations of physics phenomena or concepts. | 7 | [42,45,54,57,58,60,112] |
| ACER (Activation, construction, execution, and reflection) | Organizing structure for students' mathematics use when solving problems. | 6 | [117,141,148–151] |
| Conceptual blending | Models cognition as blending of "mental spaces" into a "blended space." | 6 | [56,63,123,127,145,146] |
| Constructivism | Suggests that learners construct knowledge in connection to their existing knowledge. | 4 | [21,60,106,111] |
| Other | Frameworks with $\leq 3$ mentions. | 37 | [1,2,12,39,41,47,56,57,59–62,65,67,73,76,77,87,94,96,97,104,105,110,114,118,120–122,125,128,129,132,133,135,143,144] |
| Unclear | | 9 | [48–52,55,91,93,101] |

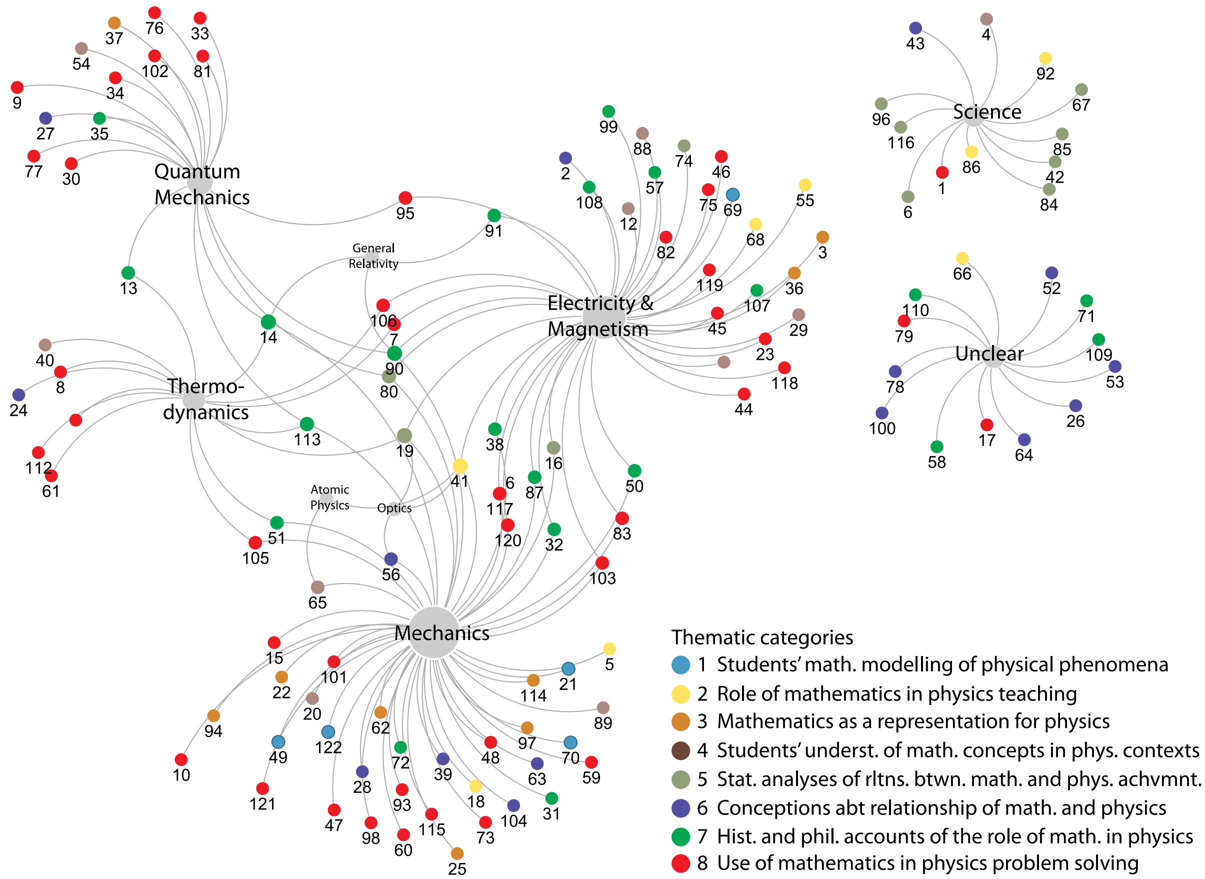


FIG. 4. A network map with context codes ($n = 9$), articles ($n = 122$), and links between articles and context codes ($n = 156$). Numbers refer to each article's number (see Ref. [37]). Size of nodes is proportional to the number of the node's connections. Colors indicate the thematic category assigned to each article.

As mentioned, the articles might include more than one framework code; for example, "students' epistemologies" shares more than one article with the code "resources." It turns out that such connections between frameworks reveal overall structural patterns between articles and frameworks. This aspect of the data is explored by the network analysis and is presented in the next section.

### C. Network overview of the articles

To analyze and visualize patterns in the data, we created three networks presented in Figs. 4–6. Links to interactive versions of each network are available in [37]. In presenting each network, we highlight what the reader can explore on the interactive versions. We have used the software Gephi for the visualization, and here, we have employed the Yifan-Hu layout algorithm. Yifan-Hu arranges nodes in a network graph to reveal overall patterns, with dense clusters forming naturally and intermediate nodes positioned as visible connectors between them.

Figure 4 shows the bipartite network of the context codes and articles included in our review. For the purposes of this figure, we represent articles as numbered and colored nodes. The number is an identifier, while the color represents the thematic article category. In the interactive version, the reader may explore article characteristics such as authors, title, journal, year, and codes that we have created and used to label the articles.

The first pattern we notice in Fig. 4 is that "mechanics" and "electricity and magnetism" are the most prevalent context codes, meaning that these two physics contexts have been studied most extensively with regard to the role of mathematics in physics. Second, most articles have been labeled with one context node, meaning that we could only identify one context as the focus of the article. About 25 articles are visible as connectors between context nodes; they have been identified as studying more than one context. Of the 21 articles belonging to thematic article category 7 (*historical and philosophical accounts of the role of mathematics in physics*), 10 are connectors, which could indicate a trend in historical and philosophical accounts to relate different contexts.

No articles connect science with other context nodes, which is indicative of a broader approach. This may be explained by the fact that most articles connected to science are in the thematic article category 5, which deals with

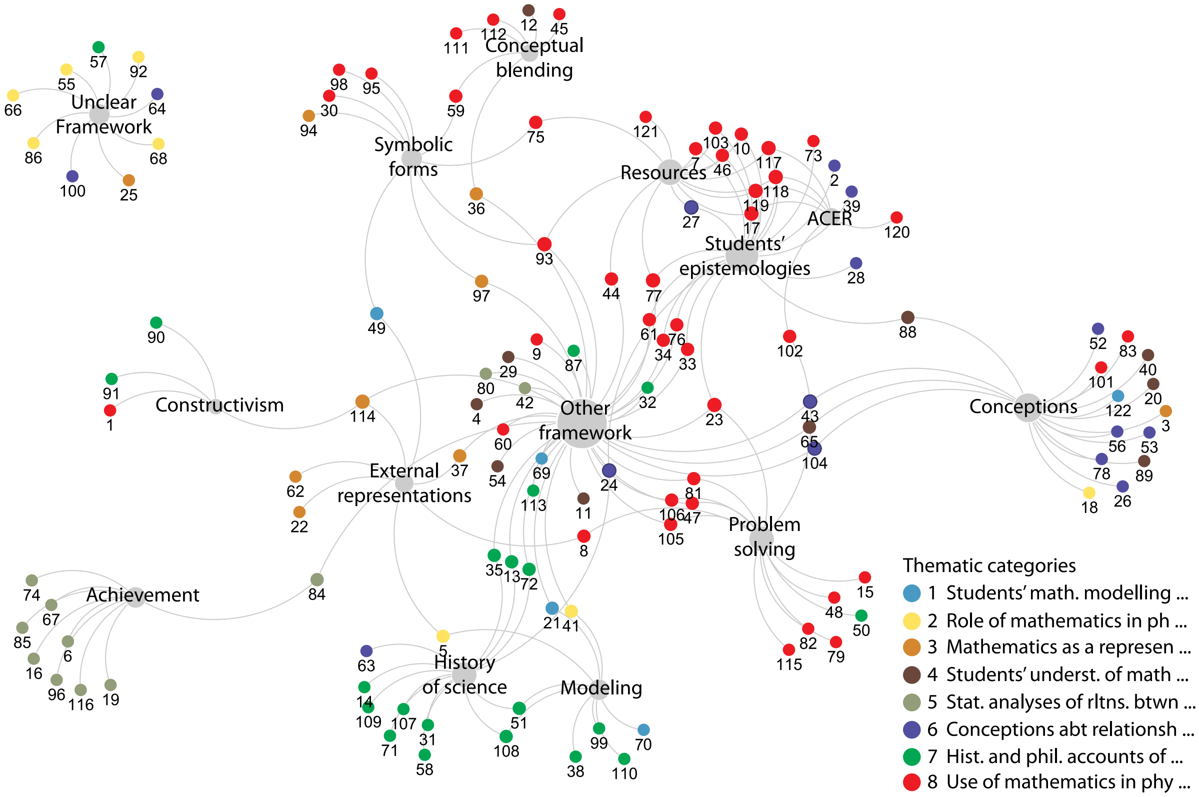


FIG. 5. A network map with framework codes ($n = 14$), articles ($n = 122$), and links between articles and framework codes ($n = 174$). Numbers refer to each article's number (see Ref. [37]). Size of nodes is proportional to the number of the node's connections. Colors indicate the thematic category assigned to each article.

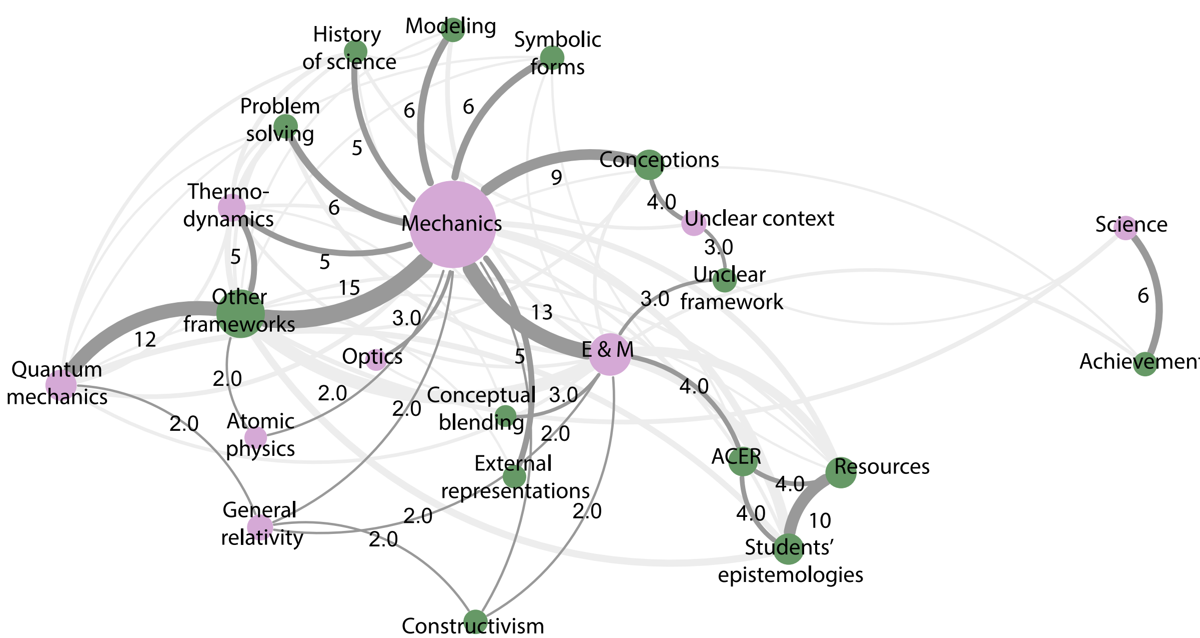


FIG. 6. Backbone network of a framework and context co-occurrence network. Nodes represent context (purple) and framework (green) codes. A link between two nodes represents the number of articles that have been assigned both codes. Thus, science and achievement have both been assigned to six articles. The light gray lines in the background show links with a weight larger than one that were part of the original co-occurrence network but had been filtered out when extracting the backbone. There are 23 nodes, 29 backbone links, and 40 nonbackbone links.

statistical analyses of achievement. Thus, physics may here be part of a broader set of disciplines that is analyzed.

To examine whether thematic article categories are related to specific physics contexts or theoretical frameworks, we combined visual inspection with quantitative analysis. Initially, Fig. 4 gave the impression that articles in each thematic category (as indicated by color) are evenly distributed across physics context nodes. This impression is supported by our statistical analysis: Only two of the nine context nodes exhibit significant divergence from expected distributions ($Z \geq 1.96$), suggesting that most contexts do not strongly structure the thematic focus of papers. However, science ($Z = 4.91$) and unclear context ($Z = 3.03$) exhibit significant overrepresentation of thematic categories (likely category 5 for science, and categories 6 and 7 for unclear context). These over-representations drive the overall Monte Carlo chi-squared test ($\chi^2 = 104.2$, d.o.f. $= 56$, $p \approx 1 \times 10^{-4}$), as well as the overall segregation ($Z = 3.27$, $p = 5 \times 10^{-4}$) toward a significant but modest association between thematic categories and physics contexts. Removing science and unclear context from this analysis also removes the significant overrepresentation ($\chi^2 = 47.74$, d.o.f. $= 42$, $p > 0.2$) and segregation patterns ($Z = 0.313$, $p > 0.3$).

In contrast, Fig. 5 suggests a clearer pattern linking thematic categories to theoretical frameworks, particularly for category 8 (*use of mathematics in physics problem solving*). This visual impression is supported quantitatively: both the Monte Carlo chi-squared test ($\chi^2 = 386.9$, d.o.f. $= 91$, $p < 2.2 \times 10^{-16}$) and the segregation ($Z = 15.45$, $p \approx 0$) indicate strong nonrandom structure. At the node level, 11 out of 14 framework codes show significant overrepresentation for thematic categories, with particularly high $Z$ scores for achievement ($Z = 13.6$), history of science ($Z = 9.8$), and modeling ($Z = 6.0$). These results indicate a strong association between thematic categories and theoretical frameworks. This structure seems especially pronounced for category 8, which is frequently linked to frameworks like problem solving, students' epistemologies, and resources.

For most framework nodes, there is a correspondence between the identified framework and the thematic article category. This was seen above for category 8, which has to do with problem solving, and for which problem solving was a prominent framework. The same pattern can be seen for thematic category 7, *historical and philosophical accounts of the role of mathematics in physics*. Many, but not all of the articles in this category are linked to history of science and/or modeling and the two framework nodes share a couple of articles. The trend of connecting to additional less used frameworks can also be observed here; three articles linked to history of science are also linked to other frameworks.

Articles in thematic category 5, *statistical analyses of relations between mathematics and physics achievement*, are primarily (9 out of 11) linked to the achievement framework. These studies are almost never combined with other theoretical frameworks, which raises the question of commensurability [154]: to what extent can the knowledge produced in achievement-focused, often observational studies be integrated with knowledge developed in frameworks that emphasize students' reasoning, representations, or epistemologies? This may point to deeper differences in epistemological commitments and methodological approaches across traditions. It is also interesting that articles in thematic category 2, *role of mathematics in physics teaching*, are to a large extent linked to an unclear framework. The explanation for this pattern does not seem immediately clear.

In contrast to most of the other frameworks, conceptions can be identified as a broad framework—one that is used to examine many different themes. This is evident in the variation of colors of the article nodes that surround the framework node. While, as could be expected, articles in categories 4 and 6 (both having to do with student conceptions) feature prominently, categories 1, 2, 3, and 8 are also used with this framework. Furthermore, while conceptions have been linked with other frameworks 3 times, it has only been coupled with one other prominent framework (students' epistemologies) once. Thus, one could again ask whether knowledge produced using a conceptions framework is really commensurable with knowledge produced with the other frameworks.

Figure 6 shows the backbone of a framework-context co-occurrence network. To create this backbone network, we first created a network consisting of all connections between framework and context nodes. In that network, nodes represented either framework or context codes. Two nodes were connected if they were applied to describe the same paper. The more frequently the same two codes had been applied to the same paper, the larger the weight of the connection between the two codes. For instance, general relativity and constructivism had been used to describe the papers by Quale [21,106]. No other papers had received the combination of general relativity and constructivism. Therefore, the weight of the link between general relativity and constructivism had the weight 2. Applying this procedure for all combinations of codes resulted in a dense network (23 nodes and 109 links). To create the backbone network, we retained only the link(s) with the largest weight(s) for each node. The result was a sparser network (23 nodes and 29 links) that highlights the strongest patterns of co-occurrence.

As an illustration, consider mechanics, which had a link to other frameworks with the highest observed weight ($w = 15$), so this was kept, while, from the perspective of the mechanics node, all other links were deleted. However, for symbolic forms, the link with the highest weight ($w = 6$) was to mechanics. Thus, this link was also kept, while from the perspective of the symbolic forms node, all other links were deleted. In case of ties, all links were kept.

For example, general relativity had four links, each with a weight of 2. Since they had the same weight, all four were retained.

One limitation of this approach is that some high-weight links may still be filtered out if they are not the strongest from a given node's perspective. For example, a link from resources to electricity and magnetism with weight 9 was excluded because resources had other, stronger links. To retain a sense of these important—but relatively weaker—connections, we visualized such filtered-out links with weight >1 as faint, backgrounded edges in Fig. 6.

Figure 6 provides an overview of the most prominent structures in our framework–context backbone network. For example, we see that mechanics is the most used context for a variety of frameworks. Also, we see that one of the prominent features of electricity and magnetism is that it is studied to a large extent in conjunction with mechanics. Another feature worth noting is that the node electricity and magnetism is linked to ACER (activation, construction, execution, and reflection), which is, in turn, connected to resources and students' epistemologies. These frameworks form a small cluster, and from the background links, we can see that they are often used in the context of both mechanics and electricity and magnetism. Notably, this cluster may reflect the work of a small group of authors, raising the question of whether such clustering stems from shared conceptual ground or from patterns of collaboration. However, we view this not as a limitation, but as an illustration of how intellectual structures often emerge through authoring practices. Frameworks typically gain traction and coherence through the sustained efforts of research communities, and the observed clustering likely reflects this co-construction of knowledge and scholarly identity.

We also observe that the node electricity and magnetism, like mechanics, is linked to several other frameworks—those that appear no more than 3 times in our dataset. Quantum mechanics is most often associated with these other frameworks as well. Of the background links removed from quantum mechanics, two had a weight of 4 (to mechanics) and 5 (to students' epistemologies), while all others had a weight of 3 or less. This suggests that quantum mechanics is also studied with a variety of frameworks but may represent a relatively newer or more exploratory research context. Finally, science and achievement form a small island of their own, with only weak background links connecting them to the rest of the corpus.

### D. Roles of mathematics in physics and physics education

In the analysis, 20 roles for mathematics in physics and physics learning were identified in the reviewed literature. These roles span from algorithmic use of mathematics in solving end-of-chapter problems to an epistemic tool in the investigation of the physical world around us. For theorizing and identifying broader themes in the data, these 20 roles were thematically categorized into six role categories. It should, however, be noted that the division between the roles and thematic role categories is not always clear-cut. Since the relationship between physics and mathematics is so interconnected, it is natural for some of the roles and categories to overlap.

The roles and their thematic categorization are presented in Table IV. The role categories are described in detail below.

#### *1. Mathematics and physics learning*

As this review is focused on physics and science education research literature, it is expected that assertions about the relationship between mathematics and *physics learning* are ubiquitous in the reviewed corpus. In many cases, mathematics is seen to provide foundations for learning physics: the learner needs to have good basic knowledge in mathematics upon which they may build when learning physics. This is especially relevant for physics problem solving, but also for building conceptual understanding, and it becomes more and more relevant when proceeding to more advanced physics topics. For example, Dini and Hammer [84] write that in quantum mechanics, conceptual "understanding ultimately anchors in mathematics." This also entails that instructors need to make sure that their students have strong enough skills and understanding of mathematics before proceeding with the physics curriculum. Importantly, the connections between physics and mathematics need to be actively made also during instruction, e.g. [48]. However, not only do physics learners benefit from the knowledge of mathematics. Professional physicists also need mathematics to be able to construct new physical models and theories, and thus expand our collective understanding of physics. For example, Martinez-Torregrosa *et al.* [103] highlight the role of differential calculus as a tool to progress in understanding scientific problems.

Because mathematics can be seen as crucial for physics learning and understanding, mathematical knowledge and skills can be expected to affect academic achievement in physics. However, when studied statistically, the evidence for the effect of the level of mathematics skills on physics achievement is mixed. Some studies in the review find little to no correlation between the initial mathematics skills and physics learning, see, e.g. [78], while others show stronger connections between achievements in the subjects, e.g. [76]. In fact, Torigoe and Gladding [140] suggest that student difficulties in physics are not due to a lack of mathematics skills but merely to confusion of symbolic meaning. Thus, it seems that making connections between physics and mathematics during instruction might be more important than students' preinstructional knowledge.

#### *2. Mathematical manipulations and calculations*

The role of mathematics as a calculation tool in physics is frequently discussed across the reviewed corpus.

TABLE IV. Final set of 20 role codes categorized into six role categories.

| Role category | Role | Description |
|---|---|---|
| Mathematics and physics learning | Mathematics affects learning and achievement. | Knowledge in mathematics affects physics learning and thus achievement in physics. |
| | Mathematics affects understanding. | Mathematics helps in constructing physical understanding. |
| Mathematical manipulations and calculations | Mathematics aids in predicting and retrodicting. | Mathematics is used to make inferences about the future or past of physical systems. In learning contexts, predictions are often solutions to exercise problems. |
| | Mathematics aids in problem solving. | Mathematics is a tool for problem solving. It enables handling complicated problems mathematically in a simplified form. |
| | Mathematics is a practical tool. | Mathematics is an external instrument that is applied in physics. |
| | Mathematics provides means for mathematical manipulations. | Mathematics is used as a tool to manipulate mathematically expressed information. This role refers especially to rote calculations, the so-called "plug and chug" way of using mathematics. |
| Mathematics as a reasoning guide | Mathematics aids in defining and formulating new concepts and theories. | Mathematics is used in developing physical concepts and theories. |
| | Mathematics helps to establish, generalize, unify, and extend the reach of theories. | Mathematics provides a structure for physical theories, upon which new knowledge is inserted. As this structure is uniform, it enables us to make generalizations and extend the reach of physical theories. |
| | Mathematics provides a derivation and proof mechanism. | Mathematics provides a way to derive new results from the known facts. |
| | Mathematics provides means for reasoning and sense making. | Mathematics serves as a reasoning guide; it gives us the ability to think about the physical world in a logical manner. It provides a framework that guides thought. |
| | Mathematics provides verified knowledge and rules. | As a logical structure, mathematics offers a way to lean on formerly established limits and rules. |
| Mathematics in experiments and modeling | Mathematics aids in modeling and application of models. | Mathematics enables building idealized and simplified models of complicated phenomena. |
| | Mathematics is a part of experiments. | Mathematics is used in planning and setting up experiments and interpreting their results. |
| Mathematics as a language | Mathematics contains and conveys information. | Mathematics is utilized to express and communicate information about the physical world. |
| | Mathematics makes/expresses meaning. | With mathematics, it is possible to express ideas and thoughts about the physical world. Using mathematically expressed information, e.g., by combining it, we may create new meanings. |
| | Mathematics provides means for representing and describing. | Mathematics enables us to describe and create unambiguous representations of the physical world to make physical systems, relations, and processes understandable. |

*(Table continued)*

TABLE IV. *(Continued)*

| Role category | Role | Description |
|---|---|---|
| Mathematics and physics as a science | Mathematics allows new knowledge and explanations. | Mathematics helps us to make sense of physical phenomena. In doing so, it leads to new understanding and enables us to create new explanations about the physical world. |
| | Mathematics drives/is required for the development of science. | Mathematics contributes to the development of physics, as it allows systematic study of physical phenomena and provides tools for exploring the physical world. |
| | Mathematics is a structural constituent. | Mathematics provides a structure for physical theories that serves as a basis or foundation for all physical knowledge. As such, mathematics has a huge influence over the form of physical theories. |
| | Mathematics is an epistemic tool. | While reality cannot be directly accessed, mathematics provides us with a way to examine it. |

Mathematics allows us to handle physical systems in a simplified and structured form, in which manipulating information mathematically becomes possible. Thanks to this, we may plan the steps to the desired solution and proceed toward it along a logical path. We can, for example, calculate new quantities from other quantities or find new symbolic or numerical solutions to posed problems. In turn, these solutions may be modified into a form that can be more readily interpreted. This way, mathematics allows us to make inferences about physical systems, cf. [148].

A large part of this category is the use of mathematics in physics problem solving. Undeniably, problem solving is an important aspect of physics expertise, and many studies have examined the phases of problem solving as well as student reasoning and sensemaking during the problem-solving process. For example, Redish and Kuo [12] present their modeling cycle adapted from mathematics education research to describe the reasoning process, and Dreyfus *et al.* [119] analyze mathematical sensemaking in undergraduate quantum mechanics. A recurring observation in the analyzed studies has been that most often students know the required mathematical techniques and procedures but, for some reason, fail to utilize them in the physics context, see, e.g. [140]. This difficulty affects the whole problem-solving process as students are not able to recognize the appropriate mathematical structures in representing the problem situations, finding the solution, and interpreting it.

Problem solving, in an ideal scenario, involves what Uhden *et al.* [13] call "structural" use of mathematics (i.e., using mathematics entangled with conceptual understanding in physics) alongside the rote mathematical manipulations (see also [10,18]). These manipulations are often regarded as "technical" use of mathematics or so-called plug and chug. This category includes both these aspects of mathematical problem solving in physics, but more emphasis on the structural use of mathematics in physics is given in the following category.

### 3. Mathematics as a reasoning guide

The aforementioned structural use of mathematics in physics (defined in Ref. [13]) is frequently emphasized in the analyzed articles. Here, mathematics is not seen as a mere calculation tool, but as an asset for structuring physical thought and "a reasoning instrument to think about the physical world" [48]. On one hand, mathematics provides a logical framework for reasoning and sense-making, and on the other hand, it sets limits for physical thought. This way mathematics provides a pathway along which physical thought can proceed. For example, Bing and Redish [115] describe the epistemic roles of mathematics for physicists, writing that mathematics "reflects physical relations, provides a calculation framework, forms a web of interconnected ideas, and provides a packaging system for encoding rules and previous results."

As mathematics provides a coherent structure for physical theories, it enables us to attach new information to the existing structure and logically organize the acquired knowledge. Because of this, physicists are able to extend the reach of theories, e.g., via generalizations and analogies [101,105]. Moreover, as a reasoning guide mathematics provides means for logical thought as well as a derivation and proof mechanism [109]. These features are crucial when mathematics is used in defining and developing new concepts and constructing new models and laws, see, e.g. [21,106].

For the physics learning context, reasoning aspects are discussed a lot, and acquiring these skills is often seen as an ultimate goal of physics instruction: students should learn to "think like a physicist," see, e.g. [142]. In order to reach this goal, students should be explicitly taught to analyze physical situations, identify the underlying mathematical structures, as well as blend physical and mathematical reasoning. For example, Kuo *et al.* [127] argue that what they call "opportunistically blending" of conceptual and formal mathematical reasoning in physics problem solving is an important aspect of problem-solving expertise. However, as Hull *et al.* [125] point out, the value of such blended reasoning is often not acknowledged in physics instruction, and for example problem-solving rubrics do not reward students for blended reasoning.

#### 4. Mathematics in experiments and modeling

Mathematics has an important role in physics experiments and modeling activities. For example, Angell *et al.* [45] write that physics "research is essentially about developing and improving models (usually formulated in mathematical language) describing phenomena." In the context of experiments, mathematics is involved in multiple steps of the process. First of all, mathematics enables us to develop laws and models of physical phenomena and draw predictions that can be tested by measurements, see, e.g. [92]. Second, mathematics affects planning of experiments and interpreting their results, e.g. [104]. Often the results are compared to existing mathematical models, while sometimes measurement data are used in building new mathematical models that requires standardization and generalization of the measurement results. However, mathematical modeling does not always start with experiments and experimental data; models may as well be theoretical or data models. Nevertheless, common for all modeling activities is that in constructing a mathematical model we use mathematics in abstracting, simplifying, idealizing and/or approximating physical situations.

A handful of analyzed articles study empirically students' use of mathematical concepts and formalisms in modeling and inquiry contexts, e.g. [1]. Many studies also bring up mathematical modeling of physical situations in problem-solving settings; the physical situation needs to be modeled in mathematical terms, i.e., "mathematized," to be able to analyze the situation mathematically, e.g. [130].

#### 5. Mathematics as a language

Mathematics is sometimes regarded as a language of physics or as a tool to represent physical phenomena. Mathematics acts as a language of physics in the sense that it gives a way to present information and communicate it with others. However, Redish and Kuo [12] argue that the mathematical language used in physics is not the same as the one taught by mathematicians, because physicists "not only use math in doing physics, we use physics in doing math." By this, they mean that when using mathematics in representing physical systems, physical meaning is encoded in the used mathematical symbols, while in pure mathematics, symbols often express abstract relationships.

In presenting physical information, we may use different external representations and symbolic languages, cf. [59,139]. Due to these representations, mathematics enables many ways we can process physical information. We can, for example, use mathematical models to describe and analyze physical situations or tabulated data to make sense of measurement results, and distinct mathematical representations differently support these processes. For example, Gire and Price [56,57] study how features of used mathematical representations affect undergraduate students' reasoning in electromagnetism and quantum mechanics. Another important aspect of mathematical representations is that they provide a way to communicate physical information. Especially in the context of research, mathematical representations are used in distributing information to the science community.

To become fluent in using the mathematical language of physics, students should be able to utilize multiple external representations, such as formulas, tables, and graphs, cf. [54,58]. To this end, it has been suggested that in physics instruction, students should be asked to explore different external representations, and using and coordinating multiple representations should be explicitly practiced [57]. Moreover, instructors should clearly explain their reasoning behind constructing and using certain representations of physical systems [48].

#### 6. Mathematics and physics as a science

Finally, in the reviewed corpus, mathematics is discussed many times in relation to physics as a field of science. For example, Redish and Kuo [12] state that "mathematics is a critical part of much scientific research," while Meli *et al.* [130] write that "mathematics is considered the spine of physics." According to Ceuppens *et al.* [54], mathematics has become "a primary driver of new physical knowledge" because of "the ever-growing complexity of physical knowledge and the need for a consistent framework and unified theories to describe, to explain, to predict, and to understand physical reality." Often in the analyzed articles, mathematics is seen as a necessity for the development of physics, science in general, technological innovations, or modern society, see, e.g. [76].

Looking deeper into this role of mathematics in the development of physics, it is seen that mathematics provides structural foundations for physics as a science. For example, Kanderakis [100] describes how mathematics historically became a structural constituent of new physics and the necessity for constructing physics' basic concepts. Put in another way, mathematics forms an underlying theoretical structure to which physical information is attached and thus affects its knowledge construction and the shape of physical theories. Because of this structure, mathematics allows systematic study of physical phenomena and guides knowledge exploration, cf. [100]. Through this process, mathematics may lead to new understanding and explanations about the physical world, cf. [96]. In addition, some authors emphasize the significance of mathematization of physics [13] and the ability to think about the world mathematically it provides us [101]. For example, de Ataíde and Greca [82] discuss the role of mathematics in modern physics of the 20th century, "in which an abstract construction modeled the real world in mathematical reasoning, in such a way that it was not even possible to think empirically without sophisticated mathematical symbols." In this regard, mathematics is treated as an epistemological tool that allows us to explore physical reality that cannot be directly accessed. In a sense, in this view, mathematics gives us the ability to see what would otherwise not be possible to see.

## IV. DISCUSSION

In this review article, we have presented a systematic overview of the studies published in 2000–2023 examining the role of mathematics in physics and physics learning. In addition, we have discussed how these roles have been conceptualized in the analyzed articles, and we have provided an overall picture of the types and features of the reviewed corpus. Ample studies on the topic have been published during recent years. The themes have been discussed from various points of view, and others have previously presented comprehensive overviews on the published literature, most notably Pospiech [18]. However, to our knowledge, ours is the first systematic review on the topic. Even though Pospiech's overview [18] presents carefully selected key contributions of the field, it is more thematic in nature. The aim of our study is to systematically summarize the previous discussion and draw an overall picture of the field.

### A. Characteristics of the articles

The articles included in the review were categorized into eight thematic categories, spanning from students' mathematical modeling of physical phenomena to statistical analyses of relations between mathematics and physics achievement, showing that the theme has been examined from many viewpoints. By far, the largest category features the studies examining the use of mathematics in physics problem-solving settings and includes one-third of all the reviewed articles.

Approximately a quarter of the analyzed articles are theoretical studies, e.g., historical analyses, and the rest are empirical studies. The majority of the empirical studies were found to use qualitative analysis methods. Most typically, data were collected from undergraduate students. The sample sizes in the empirical studies vary a lot as well, but typical samples were found to be relatively small. The most typical physics contexts used in the studies are mechanics and electromagnetism.

To summarize, the review data show some inclination toward empirical (qualitative) research among undergraduate students in courses on topics such as mechanics and electromagnetism. A lot of research has focused on physics problem solving. These findings are not unexpected as mechanics and electromagnetism are perhaps the most studied contexts within physics education in general, see, e.g. [155]. Moreover, there is a long tradition in physics education research on examining learning among undergraduate students, one of the reasons probably being the ease of implementing teaching interventions and collecting data on large lower-division courses.

### B. Theoretical frameworks

Coding of theoretical frameworks employed in the articles shows that the role of mathematics has been investigated through diverse theoretical lenses. That is, the role of mathematics is seen as complex and multifaceted—in contrast to seeing mathematics merely as a prerequisite skill or knowledge needed to learn physics. Nine articles included in this review approach the topic from the point of view of "achievement." Such studies look at correlations between different achievement measures or predictors thereof (see, e.g. [74]), and do not necessarily explore the nature of the relationship between mathematics and physics any further.

Many of the frameworks are aimed at conceptualizing the role of mathematics in reasoning and understanding. The most prominent framework, students' epistemologies, looks at how students approach problems, how they view learning, or physics, and, crucially for this review, how they assign physical meaning to mathematical symbols and equations (see, e.g. [84]). The external representations framework is also used to investigate how students understand mathematical representations, but the focus is perhaps more on analyzing the representations (such as graphs, tables, and diagrams) and how they convey information, as well as how well students understand the representations and relations between them (see, e.g. [45,58]).

While employing different concepts and points of view, all the above frameworks seek to conceptualize students' understanding, meaning making, and reasoning, particularly when it comes to using mathematics in physics contexts. On one hand, as Bollen *et al.* [63] point out, there are several frameworks that can be used to investigate how students use mathematics, and "these frameworks often are not mutually exclusive and may even strongly overlap." Some of the frameworks are more or less exclusive to investigating the role of mathematics in physics (such as symbolic forms), others are more general (such as conceptual blending or students' epistemologies) but applied to gain understanding of the role of mathematics in physics. These findings underscore the diversity of the theoretical underpinnings of the roles of mathematics in physics learning and align with the picture provided by Pospiech [18].

### C. Network analysis

The network analysis reveals several insights into the relationships between thematic article categories, employed frameworks, and used physics contexts in the reviewed articles. An examination of the physics contexts in which the studies were conducted reveals a noteworthy absence of discernible patterns between the article categories and particular contexts. This absence of clustering suggests that the identified themes of articles are not strongly tied to the physics context being studied. This observation underscores the diverse ways in which mathematics is integrated into physics education research, regardless of the specific thematic focus of the studies.

In contrast to the contexts, the network analysis shows strong connections between the thematic article categories and the employed theoretical frameworks. These patterns provide insights into how specific frameworks align with the themes of the studies and raise questions about the connections—or lack thereof—between different frameworks. For example, the problem-solving framework is strongly connected to the corresponding thematic category, featuring articles that explore the use of mathematics in problem-solving settings. This correspondence is expected given the thematic alignment between the framework and the focus of these studies [29].

Similarly, the history of science and modeling frameworks are linked to the thematic category that includes historical and philosophical accounts of the role of mathematics in physics. Interestingly, modeling as a framework does not connect significantly to any other thematic category, such as the problem-solving category, where one might expect a relationship, given the role of modeling in problem-solving strategies. This suggests that while modeling is a central theme in the philosophy of science, its application in empirical studies on physics problem solving remains limited. This raises a question of whether modeling as a framework is being underutilized in studies beyond historical and philosophical accounts.

Another notable observation on the connections between the article categories and frameworks is that the unclear framework is particularly prominent for the category consisting of articles examining the role of mathematics in physics teaching. This is puzzling, as one might expect established frameworks to guide studies on teaching, especially in the context of the physics-mathematics relationship. The prevalence of the unclear framework suggests a potential gap in the theoretical foundations used to examine teaching in this context, which could warrant further development of domain-specific frameworks.

In some cases, the lack of connections between the employed frameworks raises concerns about the commensurability of the knowledge produced using different frameworks. This is especially true for the achievement framework that is closely tied to the corresponding thematic article category, dealing with statistical analyses of the relationship between mathematics and physics achievement. These studies rarely incorporate other frameworks, and this pattern suggests that statistical studies on achievement may operate in an isolated epistemological domain, limiting their integration with broader theoretical discussions. Another such framework is conceptions that, while emerging as broad and versatile and appearing across several thematic article categories, has been linked with other prominent frameworks (e.g., student epistemologies) only once. These patterns suggest that while thematic article categories and frameworks are often aligned, some frameworks are more isolated or less utilized across thematic boundaries. This raises questions about the integration and application of frameworks in studies on the role of mathematics in physics, as well as potential opportunities for more interdisciplinary and theoretically cohesive research approaches.

The network analysis highlights the prevalence of mechanics, along with electricity and magnetism, as primary physics contexts explored in the reviewed studies. As mentioned before, this is not surprising given the long-standing tradition in physics education research of using these contexts. This, however, begs the question of whether the research should be extended to cover more diverse and advanced physics contexts. Especially in examining the roles of mathematics in physics and physics education, focusing on a more mathematically intensive area, such as quantum mechanics, could provide deeper, more comprehensive insights into the topic. However, it is worth noting that studying more advanced physics contexts has already gained increased interest in recent years, and the literature in this regard is expected to grow (see, e.g. [156,157]).

The backbone of the co-occurrence network for framework and context codes reveals the central role of mechanics. Mechanics appears as a highly connected node, linking to almost all other contexts and frameworks. As discussed in Sec. IV A, this extensive connectivity is not surprising, but likely has the consequence that mechanics serves as a foundational context for a variety of theoretical explorations in the field. The node electricity and magnetism exhibits the same pattern albeit with fewer connections, while the context of quantum mechanics is predominantly linked to “other frameworks,” which are less commonly used or emerging theoretical perspectives. The strong association between quantum mechanics and these varied frameworks suggests that quantum mechanics is a relatively new or evolving context of study within this research area.

Looking at the connections between contexts and frameworks, the backbone network (Fig. 6) shows that the unclear context and unclear framework nodes are connected, signifying instances where studies may lack a clearly defined context or theoretical foundation. This could point to areas in the research that require further clarification or development of more precise theoretical frameworks. Additionally, the pairing of the achievement framework and science as a context appears somewhat isolated from the rest of the network. The articles that were coded with codes "achievement" and "science" mostly belong to the thematic article category, focusing on statistical analyses of achievement in science education. This suggests that the studies in this category constitute a specialized area within the broader research landscape.

Overall, our network analysis underscores the importance of theoretical frameworks in shaping research themes and highlights areas where certain contexts, like mechanics, may be disproportionately emphasized. It also brings attention to the need for broader integration of contexts like quantum mechanics within various frameworks to foster a more balanced and comprehensive understanding of the role of mathematics in physics education.

### D. Roles of mathematics

In the analysis, 20 different roles mathematics has been given in the reviewed literature were identified. These roles range from so-called "plug and chug," i.e., utilizing mathematics in rote calculations, to use of mathematics in constructing physical understanding, highlighting the multitude of ways mathematics contributes to physics and physics learning. Some of the roles concern the interconnection of mathematics and physics as a field of science, e.g., mathematics driving the development of science, while others are more directly related to physics education, e.g., mathematics' effect on learning outcomes in physics. These roles were further grouped into six broad thematic categories.

The first role category considers mathematics and physics learning, and as the review focuses on physics and science education literature, it is not surprising that this theme is often discussed in the analyzed corpus. Indeed, mathematics is often regarded as a foundation for physics learning and understanding, or being a prerequisite for physics.

The second and third role categories focus on mathematical calculations and reasoning. Even though these roles are often considered inseparably in the literature, here we have loosely separated the use of mathematics as a practical instrument, especially in problem-solving settings, to the second category and the more "structural" use of mathematics to the third one. Viewing mathematics as a practical instrument in problem solving emphasizes its role as a tool that enables, for example, calculations and precise predictions. Hence, from this point of view, the important aspect of problem solving is whether students know or are able to utilize the correct mathematical procedures to solve a particular problem. When mathematics is used structurally, mathematical skills and knowledge are applied blended with conceptual understanding in physics, see Ref. [13]. For example, when knowledge and understanding of mathematics and physics are entangled in a problem-solving process, mathematics serves not merely as a calculation tool but also provides a reasoning guide. As mentioned above, these roles are often discussed alongside each other, and this is why the division between the categories is not always clear-cut.

The fourth role category discusses the role of mathematics in experiments and modeling, while the fifth role category considers mathematics as a language of physics. For example, mathematics has an important role in building coherent models and representations for physical phenomena and communicating physical information with others. Mathematics also plays a crucial role in carrying out experiments, starting from planning the experimental setup to interpreting the results and comparing them with the existing models. Importantly, different mathematical representations and models have their own advantages and limitations—both for learning physics and conducting research—and in physics instruction, students should learn how to navigate and utilize different external representations. To support this process, instructors should clearly explain the motivation for using certain representations and encourage students to practice using multiple representations, e.g. [48,57].

The final role category examines the various roles mathematics plays in physics at a more general level. Mathematics is often regarded as a necessity for the development of physics as a field of science and is sometimes considered an epistemological tool in the exploration of the physical world. Here, the historical aspect of the interplay is clearly recognizable; the deep entanglement of physics and mathematics is rooted in the long mutual history of the disciplines, and mathematics is seen to provide structural foundations for physics as a science.

The thematic categorization of the roles reveals the same kind of logic as Pospiech's overview [18] on the topic but provides a more systematic picture of the field. One major difference between Pospiech's and our work is that they do not bring up the role mathematics plays in physics experiments. However, the modeling aspects of experimental work are discussed in the contexts of physical-mathematical modeling and representations. On the other hand, a perspective that is emphasized by Pospiech but is not considered in our review is the effect of physics on the development of mathematics as a field of science. As an example of an instance when physics has inspired the development of mathematical elements or structures, Pospiech mentions that "the problem of the three-body system in gravitational physics boosted the development of the theory of nonlinear systems or the development of the theory of distributions."

Both Pospiech's [18] and our overviews highlight the fundamental entanglement of physics and mathematics, stemming from the mutual history of the disciplines, and the need for teaching this interconnection to students. As Pospiech stresses, understanding this interaction is of utmost importance, and they call for an integrated approach to teaching physics, emphasizing the need to develop students' understanding of the interplay between the disciplines. According to them, such an approach should be adopted already in the early stages of education as well as in the training of preservice teachers. However, to make the relationship between physics and mathematics a stronger part of physics curricula, it is first important to coherently conceptualize the identified roles. To this end, more work and discussions reflecting on the existing models are needed.

With the above-mentioned goal in mind, one possible viewpoint to make sense of the roles identified in our analysis is to look at internalized and externalized roles of mathematics in physics. In an externalized role, mathematics and physics are treated as separate domains that interact in certain contexts of use (e.g., problem solving or modeling). Physics as a *practical tool*, means for *conveying information* or *mathematical manipulations,* are perhaps more explicit cases of such an external role. In contrast, seeing mathematics as a *structural constituent*, *epistemic tool*, or being *part of experiments*, considers mathematics fundamentally intertwined with, for example, the structure, knowledge construction, and meaning making of physics, so that the two domains are, in principle, inseparable. Matthews [104] describes the historical origins of this development in his treatise on Galileo's pendulum experiments. Galileo used mathematics to guide the design of the experiments and "tried to have the real world mirror his ideal" (i.e., the mathematical description) [104]. Karam [48] also describes how physics involves using mathematical structures (e.g., functions or vectors) for representing the physical world (e.g., experiments and objects). This, in turn, allows for example precise analogies between different physical phenomena (e.g., central force fields in mechanics and electrostatics), and thereby enables constructing new models and expanding theories [48]. In many cases, however, the division between internalized and externalized roles is not clear-cut, and for example, *mathematics expressing physical meaning* could be seen as either one.

## V. CONCLUSIONS

It is almost a truism to say that mathematics is indispensable for physics and physics learning. While this is widely acknowledged, often the role of mathematics is understood merely as a prerequisite skill or a tool for learning physics. This view has been criticized and elaborated on by recent research. Here, we have provided a systematic overview of the studies published in 2000–2023 exploring the role of mathematics in physics education. Our results underscore the multiplicity of the roles mathematics has in physics and physics education. We identified 20 distinct roles assigned to mathematics in the physics education research literature, and these roles were further grouped into six broad thematic categories. Our analysis of the roles reveals the deep entanglement of physics and mathematics and highlights the need to explicitly teach this interconnection in the physics class.

In the corpus, we identified overemphasis of focusing on problem solving as well as using mechanics as the physics context to conduct studies in. These findings reveal the need for expanding the discussion on the theme outside of problem-solving research and further extending the discussion to more advanced physics contexts. Looking into the employed frameworks, we also discovered a lack of connections between some of them, raising concerns about the commensurability of the knowledge produced using different frameworks.

To make the relation between physics and mathematics a more integral part of physics curricula, more work is needed to conceptualize the identified roles coherently. As the topic of the role of mathematics in physics is so broad, there is certainly room for more focused investigations, precise analyses, and limited research scopes. For example, every role category would deserve its own in-depth review and analysis to probe the roles mathematics plays in physics and physics learning more deeply. However, this is outside of the scope of our study, as the aim of the review is to give an overall picture of the field.

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## DATA AVAILABILITY

The data that support the findings of this article are openly available [37].

[1] T. Mäntylä and A. Hämäläinen, Obtaining laws through quantifying experiments: Justifications of pre-service physics teachers in the case of electric current, voltage and resistance, Sci. Educ. **24**, 699 (2015).

[2] S. Turşucu, J. Spandaw, S. Flipse, and M. J. de Vries, Teachers' beliefs about improving transfer of algebraic skills from mathematics into physics in senior pre-university education, Int. J. Sci. Educ. **39**, 587 (2017).

[3] R. Karam and O. Krey, *Quod erat demonstrandum*: Understanding and explaining equations in physics teacher education, Sci. Educ. **24**, 661 (2015).

[4] I. Basson, Physics and mathematics as interrelated fields of thought development using acceleration as an example, Int. J. Math. Educ. Sci. Technol. **33**, 679 (2002).

[5] R. Karam, O. Uhden, and D. Höttecke, The "math as prerequisite" illusion: Historical considerations, and implications for physics teaching, in *Mathematics in Physics Education*, edited by G. Pospiech, M. Michelini, and B.-S. Eylon (Springer, Cham, 2019), pp. 37–52.

[6] Y. Gingras, What did mathematics do to physics?, Hist. Sci. **39**, 383 (2001).

[7] S. G. Brush, Mathematics as an instigator of scientific revolutions, Sci. Educ. **24**, 495 (2015).

[8] E. P. Wigner, The unreasonable effectiveness of mathematics in the sciences, Commun. Pure Appl. Math. **13**, 1 (1960).

[9] T. H. Kjeldsen and J. Lützen, Interactions between mathematics and physics: The history of the concept of function—Teaching with and about nature of mathematics, Sci. Educ. **24**, 543 (2015).

[10] M. Pietrocola, Mathematics as structural language of physical thought, in *Connecting Research in Physics Education with Teacher Education*, edited by M. Vicentini and E. Sassi, Volume 2 of ICPE Book (International Commission on Physics Education, 2008), https://www.per-central.org/items/detail.cfm?ID=9881.

[11] S. M. Brahmia, Mathematization in introductory physics, Ph.D. thesis, Rutgers University-Graduate School, New Brunswick, 2014.

[12] E. F. Redish and E. Kuo, Language of physics, language of math: Disciplinary culture and dynamic epistemology, Sci. Educ. **24**, 561 (2015).

[13] O. Uhden, R. Karam, M. Pietrocola, and G. Pospiech, Modelling mathematical reasoning in physics education, Sci. Educ. **21**, 485 (2012).

[14] E. Palmgren and T. Rasa, Modelling roles of mathematics in physics, Sci. Educ. **33**, 365 (2024).

[15] W. Blum and F. R. Borromeo, Mathematical modelling: Can it be taught and learnt?, J. Math. Model. Appl. **1**, 45 (2009), https://api.semanticscholar.org/CorpusID:56256990.

[16] M. Wawro, A. Pina, J. R. Thompson *et al.*, Student interpretations of eigenequations in linear algebra and quantum mechanics, Int. J. Res. Undergrad. Math. Educ. **11**, 314 (2025).

[17] O. A. Naranjo and S. R. Jones, How students construct sophisticated differential equations to model real-world contexts, Int. J. Sci. Math. Educ. **22**, 945 (2024).

[18] G. Pospiech, Framework of mathematization in physics from a teaching perspective, in *Mathematics in Physics Education*, edited by G. Pospiech, M. Michelini, and B.-S. Eylon (Springer, Cham, 2019), pp. 1–33.

[19] *Mathematics in Physics Education*, edited by G. Pospiech, M. Michelini, and B.-S. Eylon (Springer, Cham, 2019).

[20] O. Krey, Zur Rolle der Mathematik in der Physik: Wissenschaftstheoretische Aspekte und Vorstellungen Physiklernender, Ph.D. thesis, University of Potsdam, Logos, 2012.

[21] A. Quale, On the role of mathematics in physics: A constructivist epistemic perspective, Sci. Educ. **20**, 609 (2011).

[22] D. Gough, J. Thomas, and S. Oliver, *An Introduction to Systematic Reviews* (Sage, Thousand Oaks, CA, 2017).

[23] S. Elo and H. Kyngäs, The qualitative content analysis process, J. Adv. Nurs. **62**, 107 (2008).

[24] K. Krippendorf, *Content Analysis: An Introduction to Its Methodology* (Sage, Beverly Hills, CA, 1980).

[25] J. Bennett, F. Lubben, S. Hogarth, and B. Campbell, Systematic reviews of research in science education: Rigour or rigidity?, Int. J. Sci. Educ. **27**, 387 (2005).

[26] P. Mayring, Qualitative content analysis: Demarcation, varieties, developments, Forum **20** (2019).

[27] J. Saldaña, *The Coding Manual for Qualitative Researchers* (Sage, London, 2016).

[28] M. B. Miles, A. M. Huberman, and J. Saldaña, *Qualitative Data Analysis: Methods Sourcebook* (Sage, Los Angeles, CA, 2014).

[29] J. A. Luft, S. Jeong, R. Idsardi, and G. Gardner, Literature reviews, theoretical frameworks, and conceptual frameworks: An introduction for new biology education researchers, CBE Life Sci. Educ. **21**, rm33 (2022).

[30] J. Bruun. Networks as integrated in research methodologies in PER, in *presented at PER Conf. 2016, Sacramento, CA*, 10.1119/perc.2016.plenary.002.

[31] A. L. Barabási and M. Pósfai, *Network Science* (Cambridge University Press, Cambridge, England, 2016).

[32] E. Brewe, J. Bruun, and I. G. Bearden, Using module analysis for multiple choice responses: A new method applied to force concept inventory data, Phys. Rev. Phys. Educ. Res. **12**, 020131 (2016).

[33] C. Linder, J. Bruun, A. Pohl, and B. Priemer, Relationship between semiotic representations and student performance in the context of refraction, Phys. Rev. Phys. Educ. Res. **20**, 010103 (2024).

[34] N. J. Foti, J. M. Hughes, and D. N. Rockmore, Nonparametric sparsification of complex multiscale networks, PLoS One, **6**, e16431 (2011).

[35] A. Agresti, *Categorical Data Analysis*, 2nd ed. (John Wiley & Sons, Hoboken, NJ, 2002).

[36] J. Bruun and I. G. Bearden, Time development in the early history of social networks: Link stabilization, group dynamics, and segregation, PLoS One **9**, e112775 (2014).

[37] J. Bruun, jbruun/roleMathInPhys: Interactive Networks: Roles of Mathematics in Physics Education (v.1.0.0). Zenodo (2025), 10.5281/zenodo.16894515.

[38] C. C. Silva, The role of models and analogies in the electromagnetic theory: A historical case study, Sci. Educ. **16**, 835 (2007).

[39] G. Karakok, Making connections among representations of eigenvector: What sort of a beast is it?, ZDM Int. J. Math. Educ. **51**, 1141 (2019).

[40] J.-M. G. Rodriguez, K. Bain, and M. H. Towns, Graphical forms: The adaptation of Sherin's symbolic forms for the analysis of graphical reasoning across disciplines, Int. J. Sci. Math. Educ. **18**, 1547 (2020).

[41] D. J. Carrejo and J. Marshall, What is mathematical modelling? Exploring prospective teachers' use of

experiments to connect mathematics to the study of motion, Math. Educ. Res. J. **19**, 45 (2007).
[42] A. Izsak, Inscribing the winch: Mechanisms by which students develop knowledge structures for representing the physical world with algebra, J. Learn. Sci. **9**, 31 (2000).
[43] J. A. Marshall and D. J. Carrejo, Students' mathematical modeling of motion, J. Res. Sci. Teach. **45**, 153 (2008).
[44] J. Woolnough, How do students learn to apply their mathematical knowledge to interpret graphs in physics?, Res. Sci. Educ. **30**, 259 (2000).
[45] C. Angell, P. M. Kind, E. K. Henriksen, and Ø. Guttersrud, An empirical-mathematical modelling approach to upper secondary physics, Phys. Educ. **43**, 256 (2008).
[46] J. C. P. Campuzano, K. E. Matthews, and P. Adams, On the use of history of mathematics: an introduction to Galileo's study of free fall motion, Int. J. Math. Educ. Sci. Technol. **49**, 517 (2018).
[47] L. Hansson, Ö. Hansson, K. Juter, and A. Redfors, Reality–theoretical models–mathematics: A ternary perspective on physics lessons in upper-secondary school, Sci. Educ. **24**, 615 (2015).
[48] R. Karam, Framing the structural role of mathematics in physics lectures: A case study on electromagnetism, Phys. Rev. ST Phys. Educ. Res. **10**, 010119 (2014).
[49] I. Lyublinskaya and E. Petrova, Mathematics learning in physics classrooms of Russian schools: A changing landscape from the Soviet period to the present, ZDM Int. J. Math. Educ. **53**, 1485 (2021).
[50] C. A. Manogue, K. Browne, T. Dray, and B. Edwards, Why is Ampère's law so hard? A look at middle-division physics, Am. J. Phys. **74**, 344 (2006).
[51] T. Orton and T. Roper, Science and mathematics: A relationship in need of counselling?, Stud. Sci. Educ. **35**, 123 (2000).
[52] C. Radtka, Negotiating the boundaries between mathematics and physics: The case of late 1950s French textbooks for middle schools, Sci. Educ. **24**, 725 (2015).
[53] V. Albe, P. Venturini, and J. Lascours, Electromagnetic concepts in mathematical representation of physics, J. Sci. Educ. Technol. **10**, 197 (2001).
[54] S. Ceuppens, J. Deprez, W. Dehaene, and M. De Cock, Design and validation of a test for representational fluency of 9th grade students in physics and mathematics: The case of linear functions, Phys. Rev. Phys. Educ. Res. **14**, 020105 (2018).
[55] S. R. De Lozano and M. Cardenas, Some learning problems concerning the use of symbolic language in physics, Sci. Educ. **11**, 589 (2002).
[56] E. Gire and E. Price, Arrows as anchors: An analysis of the material features of electric field vector arrows, Phys. Rev. ST Phys. Educ. Res. **10**, 020112 (2014).
[57] E. Gire and E. Price, Structural features of algebraic quantum notations, Phys. Rev. ST Phys. Educ. Res. **11**, 020109 (2015).
[58] A. Lichtenberger, C. Wagner, S. I. Hofer, E. Stern, and A. Vaterlaus, Validation and structural analysis of the kinematics concept test, Phys. Rev. Phys. Educ. Res. **13**, 010115 (2017).
[59] B. L. Sherin, A comparison of programming languages and algebraic notation as expressive languages for physics, Int. J. Comput. Math. Learn. **6**, 1 (2001).
[60] J. Von Korff and N. S. Rebello, Teaching integration with layers and representations: A case study, Phys. Rev. ST Phys. Educ. Res. **8**, 010125 (2012).
[61] N. Amos and A. F. Heckler, Mediating relationship of differential products in understanding integration in introductory physics, Phys. Rev. Phys. Educ. Res. **14**, 010105 (2018).
[62] L. Bollen, P. van Kampen, and M. De Cock, Students' difficulties with vector calculus in electrodynamics, Phys. Rev. ST Phys. Educ. Res. **11**, 020129 (2015).
[63] L. Bollen, P. van Kampen, C. Baily, and M. De Cock, Qualitative investigation into students' use of divergence and curl in electromagnetism, Phys. Rev. Phys. Educ. Res. **12**, 020134 (2016).
[64] M. Carli, S. Lippiello, O. Pantano, M. Perona, and G. Tormen, Testing students ability to use derivatives, integrals, and vectors in a purely mathematical context and in a physical context, Phys. Rev. Phys. Educ. Res. **16**, 010111 (2020).
[65] L. Doughty, E. McLoughlin, and P. van Kampen, What integration cues, and what cues integration in intermediate electromagnetism, Am. J. Phys. **82**, 1093 (2014).
[66] L. C. Hadfield and C. E. Wieman, Student interpretations of equations related to the first law of thermodynamics, J. Chem. Educ. **87**, 750 (2010).
[67] R. López-Gay, J. Martínez Sáez, and J. Martínez Torregrosa, Obstacles to mathematization in physics: The case of the differential, Sci. Educ. **24**, 591 (2015).
[68] R. E. Pepper, S. V. Chasteen, S. J. Pollock, and K. K. Perkins, Observations on student difficulties with mathematics in upper-division electricity and magnetism, Phys. Rev. ST Phys. Educ. Res. **8**, 010111 (2012).
[69] M. Planinic, Z. Milin-Sipus, H. Katic, A. Susac, and L. Ivanjek, Comparison of student understanding of line graph slope in physics and mathematics, Int. J. Sci. Math. Educ. **10**, 1393 (2012).
[70] O. Badmus and L. Jita, Does further-mathematics performance predict student's achievement in physics? A correlational diagnosis in senior secondary schools, J. Technol. Sci. Educ. **13**, 381 (2023).
[71] E. W. Burkholder, G. Murillo-Gonzalez, and C. Wieman, Importance of math prerequisites for performance in introductory physics, Phys. Rev. Phys. Educ. Res. **17**, 010108 (2021).
[72] M. C. Capizzo, S. Nuzzo, and M. Zarcone, The impact of the pre-instructional cognitive profile on learning gain and final exam of physics courses: A case study, Eur. J. Eng. Educ. **31**, 717 (2006).
[73] P. Hästö, R. Palkki, D. Tuomela, and J. R. Star, Relationship between mathematical flexibility and success in national examinations, Eur. J. Sci. Math. Educ. **7**, 1 (2019).
[74] L. Ma and X. Ma, Estimating correlates of growth between mathematics and science achievement via a multivariate multilevel design with latent variables, Stud. Educ. Eval. **31**, 79 (2005).

[75] D. E. Meltzer, The relationship between mathematics preparation and conceptual learning gains in physics: A possible "hidden variable" in diagnostic pretest scores, Am. J. Phys. **70**, 1259 (2002).

[76] Y. Nakakoji and R. Wilson, First-year mathematics and its application to science: Evidence of transfer of learning to physics and engineering, Educ. Sci. **8**, 8 (2018).

[77] T. Nilsen, C. Angell, and L. S. Grønmo, Mathematical competencies and the role of mathematics in physics education: A trend analysis of TIMSS Advanced 1995 and 2008, Acta Didact. Norden **7** (2013).

[78] R. A. Olatoye, Effect of further mathematics on students' achievement in mathematics, biology, chemistry and physics, Int. J. Environ. Sci. Educ. **2**, 48 (2007), https://eric.ed.gov/?id=EJ901266.

[79] J. M. P. Sanchez and M. A. Ponce, Physics-mathematics associations: Evidence from TIMSS student achievements, Sci. Educ. Int. **31**, 229 (2020).

[80] J. Wang, Relationship between mathematics and science achievement at the 8th grade, Int. J. Sci. Math Educ. **5**, 1 (2005), https://eric.ed.gov/?id=ED490042.

[81] W. Al-Omari and R. Miqdadi, The epistemological perceptions of the relationship between physics and mathematics and its effect on problem-solving among pre-service teachers at Yarmouk University in Jordan, Int. Educ. Stud. **7**, 39 (2014).

[82] A. R. P. de Ataíde and I. M. Greca, Epistemic views of the relationship between physics and mathematics: Its influence on the approach of undergraduate students to problem solving, Sci. Educ. **22**, 1405 (2013).

[83] J. de Winter and J. Airey, Pre-service physics teachers' developing views on the role of mathematics in the teaching and learning of physics, Phys. Educ. **57**, 065007 (2022).

[84] V. Dini and D. Hammer, Case study of a successful learner's epistemological framings of quantum mechanics, Phys. Rev. Phys. Educ. Res. **13**, 010124 (2017).

[85] D. Domert, J. Airey, C. Linder, and R. L. Kung, An exploration of university physics students' epistemological mindsets towards the understanding of physics equations, Nordic Stud. Sci. Educ. **3**, 15 (2007).

[86] A. Gupta and A. Elby, Beyond epistemological deficits: Dynamic explanations of engineering students' difficulties with mathematical sense-making, Int. J. Sci. Educ. **33**, 2463 (2011).

[87] M. Holmberg (née González Sampayo) and J. Bernhard, University teachers' perspectives on the role of the Laplace transform in engineering education, Eur. J. Eng. Educ. **42**, 413 (2017).

[88] S. Kapucu, University students' conceptions of the relationship between mathematics and physics and the relationship between mathematics and physics learning, J. Balt. Sci. Educ. **13**, 622 (2014).

[89] S. Kapucu, M. F. Öçal, and M. Simsek, Evaluating high school students' conceptions of the relationship between mathematics and physics: Development of a questionnaire, Sci. Educ. Int. **27**, 253 (2016), https://eric.ed.gov/?id=EJ1104662.

[90] P.-H. Liu, A structuralist view for interpreting the role of mathematics in physics, ZDM Int. J. Math. Educ. **54**, 1657 (2022).

[91] P.-H. Liu and S.-Y. Liu, A cross-subject investigation of college students' epistemological beliefs of physics and mathematics, Asia-Pac. Educ. Res. **20**, 336 (2011), https://ejournals.ph/article.php?id=4123.

[92] P. Mulhall and R. Gunstone, Views about physics held by physics teachers with differing approaches to teaching physics, Res. Sci. Educ. **38**, 435 (2008).

[93] A. Sokolowski, Unpacking structural domain of mathematics to aid inquiry in physics: A pilot study, Phys. Educ. **56**, 015009 (2020).

[94] L. Branchetti, A. Cattabriga, and O. Levrini, Interplay between mathematics and physics to catch the nature of a scientific breakthrough: The case of the blackbody, Phys. Rev. Phys. Educ. Res. **15**, 020130 (2019).

[95] K. Forinash, W. Rumsey, and C. Lang, Galileo's mathematical language of nature, Sci. Educ. **9**, 449 (2000).

[96] I. Galili, Physics and mathematics as interwoven disciplines in science education, Sci. Educ. **27**, 7 (2018).

[97] Y. Gingras, The creative power of formal analogies in physics: The case of Albert Einstein, Sci. Educ. **24**, 529 (2015).

[98] I. M. Greca and M. A. Moreira, Mental, physical, and mathematical models in the teaching and learning of physics, Sci. Educ. **86**, 106 (2002).

[99] J. H. Jensen, M. Niss, and U. T. Jankvist, Problem solving in the borderland between mathematics and physics, Int. J. Math. Educ. Sci. Technol. **48**, 1 (2017).

[100] N. Kanderakis, The mathematics of high school physics, Sci. Educ. **25**, 837 (2016).

[101] F. B. Kneubil and M. R. Robilotta, Physics teaching: Mathematics as an epistemological tool, Sci. Educ. **24**, 645 (2015).

[102] H. Kragh, Mathematics and physics: The idea of a pre-established harmony, Sci. Educ. **24**, 515 (2015).

[103] J. Martínez-Torregrosa, R. López-Gay, and A. Gras-Martí, Mathematics in physics education: Scanning historical evolution of the differential to find a more appropriate model for teaching differential calculus in physics, Sci. Educ. **15**, 447 (2006).

[104] M. R. Matthews, Idealisation and Galileo's pendulum discoveries: Historical, philosophical and pedagogical considerations, Sci. Educ. **13**, 689 (2004).

[105] C. Pask, Mathematics and the science of analogies, Am. J. Phys. **71**, 526 (2003).

[106] A. Quale, On the role of mathematics in physics, Sci. Educ. **20**, 359 (2011).

[107] R. D. Tweney, Mathematical representations in science: A cognitive–historical case history, Top. Cognit. Sci. **1**, 758 (2009).

[108] R. D. Tweney, Representing the electromagnetic field: How Maxwell's mathematics empowered Faraday's field theory, Sci. Educ. **20**, 687 (2011).

[109] C. Tzanakis and Y. Thomaidis, Integrating the close historical development of mathematics and physics in mathematics education: Some methodological and epistemological remarks, For Learn. Math. **20**, 44 (2000), https://eric.ed.gov/?id=ej607175.

[110] G. K. Vemulapalli and H. C. Byerly, Carl Hempel's philosophy of science: How to avoid epistemic

discontinuity and pedagogical pitfalls, Sci. Educ. **13**, 85 (2004).
[111] A. H. Akatugba and J. Wallace, An integrative perspective on students' proportional reasoning in high school physics in a West African context, Int. J. Sci. Educ. **31**, 1473 (2009).
[112] R. R. Bajracharya, P. J. Emigh, and C. A. Manogue, Students' strategies for solving a multirepresentational partial derivative problem in thermodynamics, Phys. Rev. Phys. Educ. Res. **15**, 020124 (2019).
[113] R. R. Bajracharya and J. R. Thompson, Analytical derivation: An epistemic game for solving mathematically based physics problems, Phys. Rev. Phys. Educ. Res. **12**, 010124 (2016).
[114] T. J. Bing and E. F. Redish, Symbolic manipulators affect mathematical mindsets, Am. J. Phys. **76**, 418 (2008).
[115] T. J. Bing and E. F. Redish, Analyzing problem solving using math in physics: Epistemological framing via warrants, Phys. Rev. ST Phys. Educ. Res. **5**, 020108 (2009).
[116] E. Burkholder, L. Blackmon, and C. Wieman, Characterizing the mathematical problem-solving strategies of transitioning novice physics students, Phys. Rev. Phys. Educ. Res. **16**, 020134 (2020).
[117] M. D. Caballero, B. R. Wilcox, L. Doughty, and S. J. Pollock, Unpacking students' use of mathematics in upper-division physics: Where do we go from here?, Eur. J. Phys. **36**, 065004 (2015).
[118] D. N. Chari, H. D. Nguyen, D. A. Zollman, and E. C. Sayre, Student and instructor framing in upper-division physics, Am. J. Phys. **87**, 875 (2019).
[119] B. W. Dreyfus, A. Elby, A. Gupta, and E. R. Sohr, Mathematical sense-making in quantum mechanics: An initial peek, Phys. Rev. Phys. Educ. Res. **13**, 020141 (2017).
[120] J. D. Gifford and N. D. Finkelstein, Categorical framework for mathematical sense making in physics, Phys. Rev. Phys. Educ. Res. **16**, 020121 (2020).
[121] J. D. Gifford and N. D. Finkelstein, Applying a mathematical sense-making framework to student work and its potential for curriculum design, Phys. Rev. Phys. Educ. Res. **17**, 010138 (2021).
[122] D. Hu and N. S. Rebello, Understanding student use of differentials in physics integration problems, Phys. Rev. ST Phys. Educ. Res. **9**, 020108 (2013).
[123] D. Hu and N. S. Rebello, Using conceptual blending to describe how students use mathematical integrals in physics, Phys. Rev. ST Phys. Educ. Res. **9**, 020118 (2013).
[124] D. Hu and N. S. Rebello, Shifting college students' epistemological framing using hypothetical debate problems, Phys. Rev. ST Phys. Educ. Res. **10**, 010117 (2014).
[125] M. M. Hull, E. Kuo, A. Gupta, and A. Elby, Problem-solving rubrics revisited: Attending to the blending of informal conceptual and formal mathematical reasoning, Phys. Rev. ST Phys. Educ. Res. **9**, 010105 (2013).
[126] B. Ibrahim, L. Ding, A. F. Heckler, D. R. White, and R. Badeau, How students process equations in solving quantitative synthesis problems? Role of mathematical complexity in students' mathematical performance, Phys. Rev. Phys. Educ. Res. **13**, 020120 (2017).
[127] E. Kuo, M. M. Hull, A. Gupta, and A. Elby, How students blend conceptual and formal mathematical reasoning in solving physics problems, Sci. Educ. **97**, 32 (2013).
[128] E. Kuo, M. M. Hull, A. Elby, and A. Gupta, Assessing mathematical sensemaking in physics through calculation-concept crossover, Phys. Rev. Phys. Educ. Res. **16**, 020109 (2020).
[129] M. B. Kustusch, D. Roundy, T. Dray, and C. A. Manogue, Partial derivative games in thermodynamics: A cognitive task analysis, Phys. Rev. ST Phys. Educ. Res. **10**, 010101 (2014).
[130] K. Meli, K. Zacharos, and D. Koliopoulos, The integration of mathematics in physics problem solving: A case study of Greek upper secondary school students, Can. J. Sci. Math. Technol. Educ. **16**, 48 (2016).
[131] D. C. Meredith and K. A. Marrongelle, How students use mathematical resources in an electrostatics context, Am. J. Phys. **76**, 570 (2008).
[132] B. Modir, J. D. Thompson, and E. C. Sayre, Students' epistemological framing in quantum mechanics problem solving, Phys. Rev. Phys. Educ. Res. **13**, 020108 (2017).
[133] B. Modir, J. D. Thompson, and E. C. Sayre, Framing difficulties in quantum mechanics, Phys. Rev. Phys. Educ. Res. **15**, 020146 (2019).
[134] G. Murillo-Gonzalez and E. W. Burkholder, What decisions do experts make when doing back-of-the-envelope calculations?, Phys. Rev. Phys. Educ. Res. **18**, 010125 (2022).
[135] Y. Nakakoji and R. Wilson, Interdisciplinary learning in mathematics and science: Transfer of learning for 21st century problem solving at university, J. Intell. **8**, 32 (2020).
[136] D.-H. Nguyen and N. S. Rebello, Students' understanding and application of the area under the curve concept in physics problems, Phys. Rev. ST Phys. Educ. Res. **7**, 010112 (2011).
[137] D.-H. Nguyen and N. S. Rebello, Students' difficulties with integration in electricity, Phys. Rev. ST Phys. Educ. Res. **7**, 010113 (2011).
[138] Q. X. Ryan and B. P. Schermerhorn, Students' use of symbolic forms when constructing equations of boundary conditions, Phys. Rev. Phys. Educ. Res. **16**, 010122 (2020).
[139] B. L. Sherin, How students understand physics equations, Cognit. Instr. **19**, 479 (2001).
[140] E. T. Torigoe and G. E. Gladding, Connecting symbolic difficulties with failure in physics, Am. J. Phys. **79**, 133 (2011).
[141] T. Tu, C.-F. Li, Z.-Q. Zhou, and G.-C. Guo, Students' difficulties with partial differential equations in quantum mechanics, Phys. Rev. Phys. Educ. Res. **16**, 020163 (2020).
[142] J. Tuminaro and E. F. Redish, Elements of a cognitive model of physics problem solving: Epistemic games, Phys. Rev. ST Phys. Educ. Res. **3**, 020101 (2007).
[143] S. Turşucu, J. Spandaw, and M. J. de Vries, Search for symbol sense behavior: Students in upper secondary

education solving algebraic physics problems, Res. Sci. Educ. **50**, 2131 (2020).

[144] S. Turşucu, J. Spandaw, and M. J. de Vries, The effectiveness of activation of prior mathematical knowledge during problem-solving in physics, EURASIA J. Math. Sci. Technol. Educ. **16**, em1837 (2020).

[145] S. Van den Eynde, B. P. Schermerhorn, J. Deprez, M. Goedhart, J. R. Thompson, and M. De Cock, Dynamic conceptual blending analysis to model student reasoning processes while integrating mathematics and physics: A case study in the context of the heat equation, Phys. Rev. Phys. Educ. Res. **16**, 010114 (2020).

[146] S. Van den Eynde, M. Goedhart, J. Deprez, and M. De Cock, Role of graphs in blending physical and mathematical meaning of partial derivatives in the context of the heat equation, Int. J. Sci. Math. Educ. **21**, 25 (2023).

[147] L. N. Walsh, R. G. Howard, and B. Bowe, Phenomenographic study of students' problem solving approaches in physics, Phys. Rev. ST Phys. Educ. Res. **3**, 020108 (2007).

[148] B. R. Wilcox, M. D. Caballero, D. A. Rehn, and S. J. Pollock, Analytic framework for students' use of mathematics in upper-division physics, Phys. Rev. ST Phys. Educ. Res. **9**, 020119 (2013).

[149] B. R. Wilcox and G. Corsiglia, Cross-context look at upper-division student difficulties with integration, Phys. Rev. Phys. Educ. Res. **15**, 020136 (2019).

[150] B. R. Wilcox and S. J. Pollock, Upper-division student difficulties with the Dirac delta function, Phys. Rev. ST Phys. Educ. Res. **11**, 010108 (2015).

[151] B. R. Wilcox and S. J. Pollock, Upper-division student difficulties with separation of variables, Phys. Rev. ST Phys. Educ. Res. **11**, 020131 (2015).

[152] M. C. Wittmann and K. E. Black, Mathematical actions as procedural resources: An example from the separation of variables, Phys. Rev. ST Phys. Educ. Res. **11**, 020114 (2015).

[153] D. Hammer, Student resources for learning introductory physics, Am. J. Phys. **68**, S52 (2000).

[154] T. S. Kuhn, *The Structure of Scientific Revolutions* (University of Chicago Press, Chicago, 1997), Vol. 962.

[155] S. Kanim and X. C. Cid, Demographics of physics education research, Phys. Rev. Phys. Educ. Res. **16**, 020106 (2020).

[156] G. Corsiglia, B. P. Schermerhorn, H. Sadaghiani, A. Villaseñor, S. Pollock, and G. Passante, Exploring student ideas on change of basis in quantum mechanics, Phys. Rev. Phys. Educ. Res. **18**, 010144 (2022).

[157] B. P. Schermerhorn, H. Sadaghiani, A. E. Mansour, A. E. Mansour, S. Pollock, and G. Passante, Impact of problem context on students' concept definition of an expectation value, Phys. Rev. Phys. Educ. Res. **17**, 020141 (2021).