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STRUCTURAL APPROACH FOR CREDIT RISK MODELING

An empirical analysis on European corporate bond prices

Master's Thesis
in Accounting and Finance

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NOTATIONS

The notations used in this thesis are summarized below. Although the list might not be all-inclusive it covers the most important notations used. The notations used in a given formula are also presented along with the formula in question when they are not obvious.

δ	Asset payout ratio
$\mathbb{E}$	Expected value
μ	Drift of the instrument in question
Φ	Cumulative normal distribution function
σ_E	Equity volatility
σ_V	Asset volatility
B	Face value of the firm's debt
D	Value of the firm's debt
E	Value of the firm's equity
I	Indicator function
K	Default barrier
q	Dividend yield
r_t	Risk-free interest rate at time t
T	Maturity
V	Value of the firm's assets

1 INTRODUCTION

1.1 Background

The interest in credit risk modeling has increased substantially during the last decade or so. This can be mainly accounted for two reasons. First, the Capital Accord 2006, known as Basel II, gives large banks an option to use their internal models to assess their capital requirement instead of the standardized model (Laajimi 2012, 54). The standardized approach, which previously had been the only choice, forced banks to use simple predefined rules for the calculation of risk-weights used for the determination of the regulatory capital. The use of banks' own models might improve the credit risk assessment compared to a simple rule of thumb, but if the models are applied wrong, the results might be rather unwanted. The use of banks' internal models continues under Basel III, which is currently under implementation.

Second, the development and increased usage of new off-balance-sheet derivatives and securitization of loans have made the developing of credit risk assessment methods crucial (Laajimi 2012, 54). The financial crisis in 2007–2008 showed the effects of incorrect evaluation of credit risk, when the sudden collapse of collateralized debt obligation (CDO) market shook the global banking system (Castagnolo & Ferro 2014, 52). Future prevention of these kinds of disasters calls for more accurate credit risk models and also better understanding of these models among the financial sector.

Although a given model might not perfectly describe the complexity of reality, modeling provides useful guidance, for example in detecting pricing errors in the market. A model can point out a feasible trading strategy to utilize a possible pricing error in the market and provide a prediction on the profit of this strategy. In risk management, on the other hand, models can be used to determine sensitivities of securities with respect to changes in factors that affect their prices. (Veronesi 2010, 531.)

One way to model credit risk, that is the risk of default of the obligor, is the approach pioneered by Merton (1974), which uses the principles of the option pricing framework of Black & Scholes (1973). This approach to risky debt has spawned huge amounts of theoretical literature (Eom, Helwege & Huang 2004, 499). Credit risk models based on Merton's approach are known as structural models, since the default risk is tied to the firm's value process and its capital structure (Lee, Chen Lee 2009, 630). One structural model that has widely been applied in both research and practice is, for example the KMV model used by Moody's credit rating agency.

The idea behind Merton's credit risk model is that the equity of the firm can be viewed as a call option on the firm's assets. At the maturity of the debt the bondholders are paid only if the value of the firm's assets exceeds the face value of the debt. In that case the value of the equity is the difference between the value of the firm's assets and the face value of its debt. If the asset value is below the face value of the debt, bondholders take over the firm and the shareholders receive nothing. This payoff structure equals that of a call option and can be priced based on Black-Scholes-Merton option pricing framework. Since the value of the firm's debt is the asset value less the value of equity, the Black-Scholes-Merton framework can be used also for the valuation of debt.

An alternative approach to credit risk is the so called reduced-form approach. Reduced-form approach uses a hazard process or a default intensity that reflects the probability of default and is not related to any firm-specific variable but is specified exongenously. Usually, the default intensity is described by a stochastic differential equation in which the parameters are determined by direct calibration to market data (Ballestra & Pacelli 2014, 262). The default intensity is therefore the same for all bonds in the relevant credit risk class (Jarrow & Turnbull 1995, 76). Agency-provided credit ratings are often utilized in the determination of the intensities.

The structural approach is attractive on theoretical grounds, as it links the valuation of debt to the financial condition of the firm. Another supporting factor is that corporate yields are found to be linked to stock market returns and macroeconomic business cycle indicators. (Anderson & Sundaresan 2000, 256.) Structural models are also able to deal with yields and default probabilities at the same time (Qi & Liu 2010, 1700). Moreover an individual default probability can be derived for each firm. The credit ratings are discontinuous so in ratings-based approaches the default probability is the same for all the firms in the same credit class. This is obviously unrealistic, since the default probabilities are continuous (Crouhy & Galai 2000, 84-85).

However, there are unrealistic assumptions and shortcomings in the original Merton (1974) model as well. One unrealistic assumption is that the default can only occur at the maturity of the debt. In response to this, alternative models have been developed that adopt the original form of the default process suggested by Merton (1974), but allow for the default to happen at anytime between the issuance and maturity. (Lipton & Rennie 2011, 41-42.)

The Merton (1974) model also excludes the effect of a stochastic interest rates and cannot be used for the valuation of coupon bearing bonds (Li & Wong 2008, 752). In addition, since the asset value is modelled as a continuous-time process, the default cannot occur as a surprise (Ballestra & Pacelli 2014, 261). Extensions and refinements to the original Merton (1974) model have been made to overcome these deficiencies as well (Li & Wong 2008, 752).

Both structural and reduced-form models are widely applied in both literature and practice. They both have their own advantages and as a generalization it can be said that structural models are successful where reduced-form models fail and vice versa (Ballestra & Pacelli 2014, 262). This study focuses on the structural models due to the theoretical attractiveness of the approach.

Every credit risk model claims to be able to theoretically capture a certain market phenomena, but it is important to test these models and to have empirical evidence with actual data. There have been many studies concerning the structural models' ability to predict bond prices and the usual observation has been that the models underestimate the yields (Li & Wong 2008, 752). As the option pricing theory has been a huge success in pricing various financial instruments, the poor performance of the structural models in corporate debt pricing may be quite surprising (Schaefer & Strebulaev 2008, 1).

On the other hand, the results of the study by Huang & Huang (2012) for example indicate that credit risk accounts only for a small fraction of the credit spreads, that is the return over the benchmark risk-free rate, of corporate bonds. The problem may therefore well be that the corporate bond prices are influenced by factors that are not included in the models (Schaefer & Strebulaev 2008, 2). Since an extensively used application of structural models is the prediction of default, an important question is whether or not these missing factors are related to credit risk.

The variation of corporate bond prices may be linked, for example to fluctuations of market liquidity. This component would not be captured by structural models, but in the same time the models could capture the credit risk component of bond prices and returns. The non-credit related component of the bond prices is unrelated to the firm's asset value. Therefore the bond's hedge ratio, that is the sensitivity of the bond return with respect to the return on the firm's equity, is equal to the hedge ratio of the credit related component. So by focusing on the sensitivities of the bond prices instead of the prices themselves, one could achieve better results. (Schaefer & Strebulaev 2008, 1-2.)

One problem concerning the empirical testing of the structural models is that the asset values are not directly observable. The asset value and its volatility are key parameters in all of the structural models and without them these model cannot be estimated (Duan & Fulop (2009), 288). Luckily there are methods developed for estimating these parameters. These methods include the proxy, volatility restriction and maximum likelihood approaches. So the question is whether the choice of the estimation method has a significant effect on the results. The maximum likelihood estimation for the structural framework suggested by Duan (1994) and empirically tested by Li & Wong (2008) seems to provide bond prices closer to those observed in the market, but more research is needed.

1.2 Research questions

The aim of this thesis is to test the validity of the structural approach in measuring credit risk and pricing credit risky bonds. The methods used by other researchers in the field of structural models are used to reach a better consensus of the models' empirical validity. By using data from a different market and time period it is possible to examine, if results of previous studies are universal and not data-specific. The goal is also to analyze which factors affect the performance of the models. This is pursued in two ways.

First the accuracy of the Merton (1974) model in pricing corporate bonds is tested and it is also tried, if a more sophisticated structural model and parameter estimation techniques significantly improve the results. Two structural models, namely the Merton (1974) and Longstaff & Schwartz (1995) models, are implemented using different parameter estimation methods. The prices and credit spreads[1] based on the models are then compared to the ones observed on the market at that same point in time.

Second the ability of the original Merton (1974) to capture the credit risk of corporate bonds is examined by focusing on the hedge ratios based on the model. Even if the structural models were in fact unable to accurately price corporate bonds like many studies have suggested, they could still be able to quantify the credit risk component of the bond prices. Hence by examining the hedge ratios, one could have better understanding of the models' usefulness in credit risk assessment.

The research concerning structural form models' ability to value corporate debt is quite dispersed. The structural models, especially the original Merton (1974) model, have been often seen to perform quite weakly in empirical tests and bond prices produced by the models tend to not match those observed in the market. (Lipton & Rennie 2011, 40-42.)

[1] a term yield spread is used here equivalently

When the model does not perform appropriately it is usually concluded that there exists some risk factors that the model is not taking into account. The traditional view is that default and recovery risks are the main components of corporate credit spreads. However in some studies it is detected that only a small part of credit spreads are accounted for these factors. Other possible factors that may affect the credit spreads include taxes, jumps, liquidity, market risk and interest rate factors (Delianedis & Geske 2001, 26). These are factors that are not considered by the original Merton (1974) model to affect prices.

On the other hand structural models are constantly improved by researchers and there have been some promising results. For example the maximum likelihood parameter estimation technique for the calibration of these models seems to improve the performance significantly (Duan & Fulop 2009 and Li & Wong (2008)). Also including the effect of personal taxes into the valuation process by Qi & Liu (2010) seems to improve the performance of the models even further. Same applies for the inclusion of stochastic volatility and jumps into the underlying asset value process as done by Bu & Liao (2014). All in all many studies suggest that there remains to be variables such as liquidity that affect prices and structural models are not able to capture.

Since many studies have pointed out that there are variables more or less affecting bond prices that are not taken into account by the structural models, the aim of this thesis is not to find a model that 100% accurately prices corporate bonds. What is tried to accomplish rather is a better understanding of the models' ability to capture the credit risk contained in the corporate bonds and how should the model parameters be estimated in order to enhance this ability. Interest lies also in the question of how large part of corporate bond yield spreads is caused by credit risk and how much is due to the other factors. There are not many studies that have done this with data from the period after the financial crisis when both the yield spreads and credit risk have been high.

The most important application of the structural models is in the credit risk assessment. They are not expected to do so well in the bond valuation. However a model's ability to produce accurate prices or at least a consistent approximation reflects how well the model's assumptions are in line with reality. The same features of the models and model parameters are used in both valuation and credit risk assessment. Thus, it is important to test which models and which estimation techniques produce the most accurate and consistent price estimations. Valuation and hedge ratios are therefore used here as a proxy to evaluate the models' ability to measure credit risk.

In some studies the results produced by different models differ quite substantially. However in other studies different models have been seen to produce very similar results. The original Merton (1974) model is often identified being the weakest performer, but some studies indicate that the simple Merton (1974) model can actually outperform a more complex Longstaff & Schwartz (1995) model. A consensus especially about the most accurate model is also yet to be achieved. Of the parameter estimation techniques the maximum likelihood estimation has been considered superior by many researchers.

There is however a problem concerning the studying most accurate model or parameter estimation technique in pricing of bonds. If there is supposed to be some variables not captured by the models and a resulting model error, the technique that produces the most accurate prices is not necessarily the one that produce the parameters closest to the real ones. The error in price caused by for example liquidity could merely be canceled out by the effect of false parameters. This should be taken into account when analyzing the pricing results and it should be carefully analyzed whether the more accurate price is due to more realistic parameters or is it a case of two wrongs making a right.

These considerations could be made by controlling certain variables. For example testing of whether a model error is caused by illiquidity of bonds or by the models inability of measuring asset volatility could be made by controlling the asset volatility. If error is caused by the estimation of asset volatility, large estimation errors should be associated with bonds with higher volatility. On the other hand more volatile bonds should be more liquid. Therefore, if liquidity is causing the errors, higher volatility should be associated with lower estimation errors. (Barsotti & Viva 2015, 99.) Therefore one of the main interests of this study is in the main causes of the possible empirical failure of the models.

When considering credit risk analysis, one has to also take into account what method is the most accurate, but also what is the cost of implementing the methods in consideration. Maximum likelihood estimation makes the models more complex and their usage more time consuming. Since the number of firms to be analyzed is often in thousands, the difficulty of the implementation becomes relevant (Li & Wong 2008). Some models have taken the structural approach into quite complex levels as the model suggested by Bu & Liao (2014), where the asset value is allowed to have jumps and the asset volatility is considered stochastic.

Allowing more general form for the model may produce more accurate results, but this comes with the cost of difficulties in implementation. This is why a key focus in this thesis lies also on the question, whether the models to be tested can feasibly be implemented and used in practice or whether their advantage lies merely on gaining better understanding of credit risk as a theoretical concept.

Finally, since the structural framework is such attractive from the theoretical perspective, much of the attention in this thesis is also focused on the risk theory of mathematical models. This especially the case in the section 2, where the theoretical foundations of the models are analyzed.

1.3 Methods, limitations and structure

The empirical part of the thesis was conducted by obtaining data needed for implementation of the structural models from Datastream and Bloomberg terminal. Data about coupon rates, maturities, market capitalizations, total liabilities and bond prices among others are needed for the implementation. Therefore the study had to be limited to corporate debtors that have issued equity and provide regular financial statements. The selection of bonds is based on the previous studies on the subject to make this thesis comparable to past research.

The bond price data is used in the bond pricing tests of this thesis in the form of panel data, meaning that the focus is in certain points in time. Eom et al. (2004) use price data on the last trading day of December from the years 1986-1997. By using the year-end prices the price observations can be matched up with year-end financial data (Eom et al. 2004, 503). The same method is also used by other researchers like Li & Wong (2008) and it is also applied in this thesis.

In the tests for hedge ratios the basic methods of time series regression analysis are used. Several regressions are performed for bond and equity returns to compare the applicability of the theoretical hedge ratios implied by the Merton (1974) model in practice.

Bonds nominated in euro currency are used in the tests and they are chosen based on the following criteria. As in Eom et al. (2004), nonfinancial firms are left outside of the empirical study. This is to secure that the leverage ratios are comparable across the firms. Financial firms usually use leverage ratios around 90% where as only the most trustworthy non-financial firms have leverage ratios this high (Eom et al. 2004, 503). There are, however, studies that use both financial and non-financial firms. For example Lyden & Saraniti (2000) implement structural models this way. In this thesis the approach used by Eom et al. (2004) is, however, considered more reasonable.

Bonds with maturities less than one year are not considered. This is for the reason that they are highly unlikely to trade. The structural modeling framework is based firms with simple capital structures. Using firms with complex capital structures therefore raises doubts about whether the pricing errors are due to model weaknesses or to the fact that the models are not trying to price complex capital structures (Eom et al. 2004, 503). However, unlike in the previous studies the number of bonds issued by the firm is not limited here as the number of bonds in the sample would otherwise be unbearably low. The effect of the number of bonds issued to the bond pricing results is separately tested.

In the previous studies the structural models are usually tested via both simulation and empirical testing. For example Ericsson & Reneby (2005) perform this type of simulation. They form four different scenarios of illustrative issuers based on financial and business risk. This way firms from different risk classes are covered. To describe business risk they set the percentage equity volatility as 40% for high business risk firms and as 20% for low business risk firms. For financial risk, two different debt-ratios are assigned. They use high values for risk parameters since these are expected to be harder to price and therefore performance of the models is more informative. (Ericsson & Reneby 2005, 717-718.)

In this thesis simulation is used as a part of the hedge ratio tests. In the approach of Schaefer & Strebulaev (2008) for the hedge ratio tests the model dependent parameter estimation techniques like maximum likelihood estimation are not appropriate since the pricing formula is assumed to possibly contain an error due to missing variables. Via simulation large sample sizes can be generated by relying on some assumptions about reality, therefore ruling out the possibility of statistical anomalies. Hedge ratios are, however, also tested with real data.

The empirical study of this thesis compares different implementation methods for structural models and considers how the choice of implementation method affects the models' ability to predict bond prices. Models' ability to predict corporate defaults is left outside of this study since sufficient amount of data on corporate defaults is hard to obtain.

The limited availability of data is a considerable issue also for the test that are performed in this thesis. The number of bonds found from Datastream fulfilling the selection criteria is significantly lower than that of the referenced studies. The time series used in the hedge ratio tests is also shorter on average due to the lack of longer time series for the European corporate bonds in Datastream. The quality of bond price data also has to be considered when interpreting the results of this thesis. The liquidity of the bonds in sample is in some occasions questionable. Price observations, where the price had not moved for number of consecutive months, were eliminated from the sample, but the liquidity of the remaining sample might still be questionable.

Rest of the thesis is structured in the following way. In section 2 the important theoretical concepts related to bonds and mathematical finance. The section 2 also deals with the theoretical background concerning the structural credit risk models and different parameter estimation techniques. In the section 3 different applications as well as the strengths and weaknesses of the models are considered. Prior research on the structural models is also reviewed. In the section 4 the data and the methods for the selection of the bonds are described along with specific methods for model implementation. In the section 5 the results of the bond pricing tests are analyzed. In the section 6 the analysis of results is continued with the hedge ratio tests. Finally the section 7 summarizes the thesis and presents the conclusions based on the results.

2 THEORETICAL BACKGROUND

2.1 Basic concepts

2.1.1 Bond types, yield spreads and risk-free interest rate

The simplest and theoretically most convenient bond type is a zero-coupon bond. This is simply an instrument that pays one predetermined cash flow in the maturity. The return for this instrument is of course the difference between the price of the instrument and the cash flow received in the maturity. Since the zero-coupon bond involves only one cash flow, the theoretical pricing considerations can be focused on the determination of the required rate of return for the bond to cover for different risks involved in the instrument (e.g. credit risk). The original Merton (1974) model, for example, focuses only in the pricing of zero-coupon bonds.

In practice, however, the market for zero-coupons is very limited and most of the actively traded bonds pay some sort of coupon. The simplest coupon bond is a fixed-coupon bond, which pays some fixed amount on predetermined dates plus the principal amount at the maturity. Since there are payments on multiple points in time it is not as theoretically convenient instrument as a zero-coupon bond. Fixed coupon bond can be, however, seen as a collection of zero-coupon bonds, which often makes the analysis easier. The price of a fixed-coupon bond can be expressed with prices of zero-coupon bonds $p(t, T_i)$ with maturity T_i and face value of one as follows:

$$P = \sum_{i=1}^{n} c_i \times p(t, T_i) \; + B \times p(t, T_n), \tag{1}$$

where B is the face value of the fixed-coupon bond and c_i is the coupon payment at time i. (Björk 2009, 482.)

As mentioned the original Merton (1974) model does not deal with fixed-coupon bonds, but since a fixed-coupon bond can be seen as a collection of zero-coupon bonds as described above, an extension of the original Merton (1974) model has been presented by researchers in order to implement the model with fixed-coupon bonds. In the empirical tests of this thesis all the bonds have a fixed coupon and therefore the extension of the original Merton (1974) model is used in the implementation. The more complex models such as Longstaff & Schwartz (1995) model are able to price fixed-coupon bonds in their original form. Longstaff & Schwartz (1995) model is also able to price floating rate bonds i.e. bonds, where the coupon payments vary based on some benchmark interest rate. For the simplicity and comparison purposes, however, only fixed-coupon bonds are dealt with in this thesis.

The prices of the zero-coupon bonds in the equation (1) depend on risk-free interest rate for the corresponding maturity as well as the risk characteristics of the bond in question. The modeling of one of the most important risks involved, namely the credit risk, is the main focus in this thesis. Since the size of the coupon payments and face value as well as the maturity characteristics of the bond affect the bond price as well, the comparison of different bonds is often done in terms other than prices.

The main factor for the comparison of different bonds in this thesis is the yield spread of the bond. Bond's yield-to-maturity (from now on just yield) is the internal rate of return of the bond. It is the discount rate for which the discounted cash flows promised by the bond match the price. For a standard coupon-bearing bond paying annual coupons, with n payments and face value of 1, yield y is defined as the solution for the equation:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + ... + \frac{1+C_n}{(1+y)^n}$$

$$\Leftrightarrow P = \sum_{i=1}^{n-1} \frac{C_i}{(1+y)^i} + \frac{1+C_n}{(1+y)^n}$$

or with continuous compounding and arbitrary coupon frequency:

$$P = \sum_{i=1}^{n-1} e^{-yt_i} C_i + e^{-yt_n}(1+C_n).$$

In the above equations P is the bond price and C_i for $i = 1, ..., n$ are the coupon payments promised by the bond. Bond's yield spread is the difference between bond's yield and yield of risk-free instrument promising same cash flows. Yield spread is therefore calculated by subtracting an appropriate benchmark risk-free interest rate from the bond's yield.

The reason why yield spreads are used instead of yields to measure the models' performance is that yield spread is the part of the price that the models considered in this thesis try to explain. The models use risk-free interest rate to discount expected payoffs under risk-neutral measure as is explained in the section 2.1.3. The fair value of the bond is therefore obviously smaller than if the payments were certain and every model will produce yield spreads larger than 0. Thus every model will be able to explain the risk-free part of the yield and the real focus should be on how much of the risk premium they can explain. This is why the models' performance is mainly measured with errors in yield spreads. (Eom et al. 2004, 510.)

The risk-free interest rate strongly affects the price of the bond as it is used as the discount rate in pricing. The risk-free interest rate also affects the bond's yield spread because it is used as the benchmark rate. The selection of the proxy for the risk-free rate is therefore important. Since this thesis focuses on corporate bonds issued in the Eurozone countries, the yields of German government bonds are used as a risk-free benchmark. Germany has for a long time had the strongest economy in the Eurozone, which is why the bonds issued by the German government are often regarded as the closest to a risk-free instrument in the Eurozone. For example instruments constructing of bonds issued by various All the Eurozone governments might not be considered as risk-free since there are many Eurozone countries that have fairly uncertain economic conditions as the sovereign debt crisis has shown.

The risk-free rate should be determined for the whole maturity spectrum as the risk-free rate is not constant across the maturities. For example the annualized rate for one year German government bond is very unlikely to be the same as the annualized rate for five year German government bond. The relationship between the maturities and interest rates is known as the term structure of interest rates or as the yield curve. The shortest maturity for German government bonds, however, is one year. Therefore another proxy is needed for the short term interest rates.

Historically one could have used the lending rate between large banks in the respective market that is the LIBOR (London interbank offered rate), which is interbank lending rate determined for ten different currencies. However, after the financial crisis the lending even between the largest banks cannot necessarily be considered riskless.

More appropriate proxy for risk-free rate is therefore overnight indexed swap (OIS) rate. The difference between these two is that the OIS rate is a swap rate and therefore includes only the coupon payments and not the principal payments whereas LIBOR includes the principal as well as the coupons. Therefore the risk of the counterparty bank defaulting on the principal is eliminated in the OIS rate and only the risk of defaulting on the swap's coupon payments is involved. This is why Eurozone OIS curve is used as the proxy for the short term risk-free rates in this thesis. (Hull & White 2012, 15-16.)

Although the German government bond rates and Eurozone OIS rates include both the lower and upper bound for the maturities of the bonds analyzed in this thesis, they do not include rates for every given maturity. The coupon and principal payments of the bonds, however, are dispersed across different time horizons and an exactly corresponding risk-free German bond or OIS rate rarely exists. The various dots formed by the proxy interest rate curve need to be therefore somehow connected. The most straightforward method would be do this by linear interpolation.

More sophisticated models include Nelson-Siegel and Nelson-Siegel-Svensson models former of which is used in this thesis. These models assume that a few specific factors account for the shape of the yield curve. In the Diebold & Rudebusch (2013) interpretation of Nelson-Siegel model the three factors of the model are interpreted as the level, the slope and the curvature of the yield curve. The usage of factors is convenient because in the world of finance in many cases there appears to be only a small number of systematic factors affecting the pricing of assets. The use of factors is also a useful way to compress the valuable information included in the price data. (Diebold & Rudebusch 2013, 24, 33-36.) The implementation of the Nelson-Siegel model is described more thoroughly in the section 4.2.1.

The Nelson-Siegel model assumes the term structure of interest rates to be constant over time. However, in practice the risk-free interest rate evolve through time meaning that they are stochastic in nature. The Longstaff & Schwartz (1995) model implemented in this thesis utilizes the Vasicek (1977) model for modeling the risk-free interest rate and therefore takes into account the stochastic nature of the risk-free interest rates. The Vasicek (1977) model assumes a certain parametric form for the stochastic process (i.e. the random evolution in time) of the interest rates. For the purposes of the Vasicek (1977) model and more importantly in order to understand the structural models in general the concepts described in the next two sections are needed.

2.1.2 Stochastic processes

As can be seen, for example from the equation (1), the pricing of fixed income securities is moderately easy when one assumes that the payoffs and interest rates are deterministic. For example a price of a simple coupon-bearing bond can be derived by merely discounting the predetermined coupon payments and the face value with an appropriate interest rate or in other words by multiplying with the price of an appropriate zero-coupon bond. Things, however, become more complicated when randomness is included into the calculations.

Since this thesis focuses on the pricing of corporate bonds with credit risk, it is assumed that the instruments in question have payoffs that are not deterministic, but uncertain. In the Merton (1974) model framework a stochastic process for the value of the firm is assumed and the payoffs for equity and debt are seen to be contingent on this value process. In addition some models used for bond pricing in this thesis also allow the interest rates to be stochastic. Therefore some clarification for the concept of stochastic processes is in place.

Randomness can be added into an otherwise deterministic time dependent process by adding a random component in the form of a Wiener process denoted from here on by W_t. A Wiener process is a stochastic process which has increments that are normally distributed and independent of each other. In financial theory the Wiener processes are often used to model the stochastic behavior of various instruments. The stock prices, for example, are often modeled by using geometric Brownian motion which is a process combining a deterministic proportional drift term and random deviations from that drift in the form of a Wiener process.

The resulting process Y is also a stochastic process and it can be described by a following stochastic differential equation:

$$dY_t = \mu(Y_t, t)dt + \sigma(Y_t, t)dW_t, \tag{2}$$

where $\mu(Y_t, t)$ and $\sigma(Y_t, t)$ are functions that can depend on the current state of the process and/or on time. The term $\mu(Y_t, t)dt$ is the drift term of the stochastic process and it represents the expected change of the process between $t + dt$. The term $\sigma(Y_t, t)dW_t$ is the diffusion term of the stochastic process and it is the unpredictable part of the process. (Veronesi 2010, 510-513; Lipton & Rennie 2011, 21-22.)

The rules of calculus that link together the dynamics of an underlying stochastic process and the securities contingent on it, are provided by an identity called Itô's lemma. The general version of Itô's lemma states that if the price of a security P_t depends on both time and a process like Y_t (i.e. $P_t = F(Y_t, t)$). Then, the capital gain process is

$$dP_t = \left[F_t + \mu(Y_t, t)F_y + \frac{1}{2}\sigma(Y_t, t)^2 F_{yy}\right] dt + \sigma(Y_t, t)F_y dW_t. \tag{3}$$

(Veronesi 2010, 515-520.) In the above equation the subscripts of F denote the partial derivatives of F. The drift and the volatility of P_t have taken the form $[F_t + \mu(Y_t, t)F_y + \frac{1}{2}\sigma(Y_t, t)^2 F_{yy}]$ and $\sigma(Y_t, t)F_y$, respectively. This is a key result that forms the basis for both the Black & Scholes (1973) option pricing framework and the Merton (1974) model. In the next sections the above concepts are utilised in the pricing of contingent claims, which is the general idea behind all the structural models.

2.1.3 Arbitrage theory and risk-neutral pricing

The groundbreaking innovation of the Merton (1974) model is viewing the debt of the firm as a contingent claim depending on the firm's asset value process. This is the fundamental idea behind all the structural models. An often applied method for valuing contingent claims is the use of no arbitrage principle and risk-neutral pricing.

In well functioning financial markets there are no arbitrage opportunities, because otherwise arbitrage profits could be made until the prices adjusted and arbitrage opportunity seized to exist. This approach in valuation is useful because it provides constraints that need to be satisfied for arbitrage not to exist. These constraints can then be used to form arbitrage-free prices of the securities. (Veronesi 2010, 531.) This is also the idea behind the famous Black-Scholes-Merton partial differential equation (PDE) and the similar equation for the Merton (1974) model.

This pricing approach utilises two important results known as the two Fundamental Theorems of Mathematical Finance. The first fundamental theorem states that:

Theorem 1. *The model is arbitrage free if and only if there exists a risk-neutral martingale probability measure Q that is equivalent to the probability measure P.*

The objective probability measure denoted as P describes the probabilities as they are in "real" world. The probability measure Q is called the equivalent martingale measure for the specific market model, the price process of the risk-free asset Y_0 and the time interval $[0, T]$. Here T can be interpreted as the maturity of the asset to be priced. With all the information up to time T (i.e. on $\mathcal{F}_T$), the price processes are equivalent under P and Q. In addition under Q all the assets are expected to have returns equivalent to the returns on the risk-free asset. This is why Q is often called the risk-neutral measure.

The risk-neutral measure Q can be seen to represent the probabilities in the world, where all the investors are risk-neutral. The reason why the concept of risk-neutral measure is needed in the contingent claim pricing is that by definition the price of a contingent claim is dependent on the price of the underlying asset and therefore it must be in line with the price of the underlying security for arbitrage not to exist. Therefore the preferences of the investors regarding risk do not affect the price of a contingent claim, but are reflected in the price of the underlying. This is often used rhetoric to describe the essence of the risk-neutral pricing (see e.g. Hull 2011).

What makes the above result so useful is that it suffices to know the risk-neutral dynamics of an instrument to derive prices. With this knowledge one only has to discount the expected payoffs under risk-neutral measure with risk-free rate. By using real-world probabilities different instruments would have different drifts and therefore choosing a discount factor for a portfolio would be impossible. One cannot either use any other constant drift for all the assets since the risk-free instrument would always by definition have drift equal to risk-free rate despite the probability measure.

The implications of the Theorem 1 are two-folded. The first part implies that by using the risk-neutral dynamics in deriving the prices of securities the absence of arbitrage is guaranteed. The second part of the theorem states that an arbitrage-free market implies the existence of Radon-Nikodym derivative which transforms the objective probanilities under P into the risk-neutral probabilities under Q. Therefore on an arbitrage-free market the risk-neutral dynamics of the securities always exist and can be used for pricing. (Björk 2009, 137-154.) For the thorough proof of the Theorem 1 one may refer to, for example, Björk (2009).

The second fundamental theorem states that:

Theorem 2. *Assuming the absence of arbitrage, the market model is complete if and only if the martingale measure Q is unique.*

The uniqueness of Q means it cannot be chosen otherwise still fulfilling the criteria for a martingale measure. The completeness means that every payoff structure can be reached and that the arbitrage-free price of a given security is unique. Therefore, when there is a sufficient number of financial instruments one can form a one unique and "correct" price for all the financial instruments. This is a sufficiently realistic assumption for real life markets, where number of different instruments is certainly vast. (Björk 2009, 155-156.) For the proof of the second fundamental theorem one must yet again refer to for example Björk (2009).

Based on these fundamental results the Black-Scholes-Merton PDE can be derived by applying the Girsanov theorem. According to the Girsanov theorem a Q-Wiener process, i.e. a Wiener process under measure Q, can be expressed as:

$$W_t^Q = W_t^P - \int_0^t \varphi_s ds, \tag{4}$$

where W_t^Q and W_t^P are multidimensional Wiener processes under Q and P respectively and φ is an adapted column vector process with corresponding dimensions i.e. the so-called Girsanov's kernel. As in the Black & Scholes (1973) model the price of the underlying claim is assumed to follow geometric Brownian motion, described in the equation (2), its price dynamics can be written by plugging in the equation (4) as follows:

$$dY_t = \{\mu + \sigma\varphi_t\}Y_t dt + \sigma Y_t dW_t^Q$$

(Björk 2009, 164-168.)

From the first fundamental theorem it is obtained that the drift factor must equal the short rate r. The Girsanov kernel is therefore defined by solving the equation:

$$\mu + \sigma\varphi_t = r \Leftrightarrow \varphi_t = -\frac{\mu - r}{\sigma}$$

Therefore in the Black & Scholes (1973) framework the Girsanov kernel is a deterministic constant. The solution has also an important economic interpretation as it corresponds negative of the well known constant, namely the Sharpe ratio. The P-Wiener process in the Black & Scholes (1973) setup can therefore be obtained by adding the market price of risk to the risk-neutral Q-Wiener process. (Björk 2009, 164-168.)

By using the Itô's lemma, as stated in the equation (3) and the Theorem 1 the price for a simple claim of the form $X = f(Y_T)$ can be written as a function F that solves the famous Black-Scholes-Merton equation:

$$\begin{aligned} F_t + rF_yY + \frac{1}{2}F_{yy}\sigma^2Y^2 - rF = 0 \\ F(T, y) = f(Y_T) \end{aligned} \tag{5}$$

The first equation sets the expected change of the price process of X as equal to the risk-free rate. The drift in the equation (3) has been replaced by r as a result of applying the Girsanov theorem. The second equation sets the price of the contingent claim at time T as equal to a specific function of the price of the underlying at time T. In case of a vanilla stock option, for example, this would be equal to the intrinsic value of the option at the maturity. (Björk 2009, 137-154.)

The solution to the PDE can be found by using the Feynman-Kač formula:

$$\Pi(t; X) = \mathbb{E}^Q\left[e^{-\int_t^T r(s)ds} X \middle| \mathcal{F}_t\right], \tag{6}$$

where $\mathbb{E}^Q$ denotes the expectation under measure Q and r is allowed to be stochastic. This way the price $\Pi(t; X)$ for any claim X can be computed as an expected future value under risk-neutral measure Q discounted with the risk-free rate. When the market is complete this price must also be unique. (Björk 2009, 178-179.) For example the Black-Scholes-Merton formula for option pricing can be derived by solving the corresponding expectation.

In the next section these results are applied to the derivation of the Moody's Investors Service (2011) model for the pricing of credit risky debt, an application which is in the center of this thesis.

2.2 The general framework of structural models

2.2.1 The original Merton model

The basic idea behind structural credit risk models is based on Merton (1974). Since then the model has been extended by other researchers, but the fundamental theory has remained the same, namely the use of Black-Scholes-Merton option pricing principles.

Merton was the first to apply option pricing framework in corporate bond valuation which has turned out to be one of the most influential ideas in the debt pricing theory. The basic idea is simple, but yet powerful. It is assumed that the dynamics for the value of the firm (V) through time can be described by a following stochastic differential equation:

$$dV = (\alpha V - \delta)dt + \sigma V dW_t,$$

where α is the expected rate of return on the firm per unit time, δ is the payout of the firm (e.g. dividends or interests), σ is the volatility of the return on the firm and W_t is a standard Wiener process. (Merton 1974, 450-456.)

By applying Itô's lemma as in Black-Scholes-Merton framework one can write the dynamics of a security Y that depends on both the value of the firm and time (i.e. $Y = F(V, t)$) as follows:

$$dY = \left[\frac{1}{2}\sigma^2 V^2 F_{vv} + (\alpha V - \delta)F_v + F_t\right] dt + \sigma V F_v W_t. \tag{7}$$

As demonstrated in the section 2.1.3, any security whose value can be written in the form $F(V,t)$, must satisfy the following partial differential equation:

$$\frac{1}{2}\sigma^2 V^2 F_{vv} + (rV - \delta)F_v + F_t + \delta_y - rF = 0,$$

where r is the risk-free rate and δ_y is the payout per unit time of the security Y. Here it can be seen that the price of the security Y does not depend on the expected rate of return on the firm nor on the risk preferences of the investor. This is a crucial result of the risk-neutral pricing of contingent claims. (Merton 1974, 450-452.)

In the Merton (1974) model for corporate debt valuation, the necessary boundary conditions are obtained by assuming that a firm has two classes of claims: a single homogeneous class of debt and the residual claim, equity. The firm is supposed to have a zero-coupon type debt and if the payment is not made at the maturity, the bondholders would takeover and the shareholders would receive nothing. For simplification it is also first assumed that no coupons, dividend payments or share repurchases can be made although this condition can be quite easily relaxed as in Black & Scholes (1973) model.

Since a firm acts in the best interest of its shareholders, the payment is made if and only if the firm's asset value is greater than the face value of the debt. Therefore at the maturity the value of the firm's equity $f(V,T)$ can be written as:

$$f(V,T) = \max[0, V - B],$$

where V is the value of firm's assets and B is the face value of the debt. This equals the payoff structure of a call option with V being the price of the underlying security and B being the strike price. Obviously the value of the firm's debt is the asset value less the equity value. (Merton 1974, 450-454.) Because of the put-call parity this corresponds to a fair value of a risk-free zero-coupon with face value B less a put option (Bharath & Shumway 2008, 1343).

Based on the principles of the risk-neutral pricing presented in the section 2.1.3 and the Feynman-Kač formula in the equation (6) the price of the equity can be derived by discounting the expected value of the payoff at the maturity with the risk-free rate. As rhe risk-free rate is assumed to be constant in the Merton (1974) model the computation of the expectation is quite straightforward. The value of the debt can be solved by using the put-call parity as mentioned.

Therefore the value of the debt can be written as:

$$F[V,t] = V - \left[V\Phi(d_1(B,\tau)) - Be^{-r\tau}\Phi(d_2(B,\tau))\right] \tag{8}$$

$$= Be^{-r\tau}\left[y\Phi(-d_1(B,\tau)) - \Phi(d_2(B,\tau))\right],$$

where Φ is a standard normal cumulative distribution function and

$$y = \frac{V}{Be^{-r\tau}},$$

and

$$d_1(x,\tau) = \left[\ln\left(\frac{V}{x}\right) + (r - \delta + \frac{1}{2}\sigma^2)(\tau)\right] / \sigma\sqrt{\tau}, \tag{9}$$

$$d_2(x,\tau) = d_1(x,\tau) - \sigma\sqrt{\tau}$$

and

$$\tau = T - t$$

As mentioned earlier the comparison of different bonds is usually made based on yields instead of prices. Obviously in the case of prices and yields one determines the other, but as explained the yields are more useful in comparing bonds with different coupons and maturities for example. In the case of Merton (1974) model the yield of the bond can be written as follows:

$$R(\tau) - r = -\frac{1}{\tau}\ln[x\Phi(-d_1) - \Phi(d_2)] \tag{10}$$

where by definition:

$$e^{-R(\tau)\tau} = \frac{F[V,t]}{B}$$

and $R(\tau)$ is the bond's yield. Therefore $R(\tau) - r$ is the bond's risk premium or yield spread. (Merton 1974, 454.)

The above describes the Merton (1974) model in its original form. However few of the assumptions made by the original model need to relaxed before the model can be implemented for coupon-bearing bonds used in the empirical testing part of the thesis. These considerations are described in the section 4. The Merton (1974) model is also the simplest of the structural models. The subsequent models have relaxed some of the strict assumptions made by the model while keeping the fundamental idea of the contingent claim approach. These more sophisticated structural models are analysed in the next sections.

2.2.2 Default barrier models

As mentioned earlier there are some unrealistic assumptions in the original Merton model. This has been seen as the main reason for the poor performance of the model in producing bond yield spreads close to those observed in the market.

One of the unrealistic assumptions in the original Merton (1974) model is that the value of the firm can rise to an arbitrarily high level or decline into almost nothing without there being a reorganization regarding the firm's financial arrangements (Black & Cox 1976, 352). This is clearly unrealistic since the default usually occurs long before the value of the firm goes to zero and can often happen before the maturity of the debt (Longstaff & Schwartz 1995, 789). Many studies have shown that this simplification produces credit spreads much smaller than the actual ones.

Black & Cox (1976) were the first ones to consider the Merton framework with the possibility that a financial restructuring could occur before the maturity of the debt. They suggest that, if the firm's value process reaches a certain boundary, the securities of the firm will obtain a specified value. The value process could have upper boundary due to a call provision on a bond, meaning that the bond could be repurchased, if profitable, by the issuer. Specific terms of the bond can also induce a lower boundary at which the debt of the firm will be reorganized. (Black & Cox 1976, 352.) This is the more relevant one of the boundaries when considering the credit risk of the bonds.

If lower and upper boundaries are assumed, the value of the debt is constructed from four different sources. Its value at the maturity if the boundaries have not been reached, the value at the lower boundary, the value at the upper boundary and the value of the possible payouts. (Black & Cox 1976, 353.) The upper boundary is however not considered in this study since it is not relevant to credit risk and therefore the sources of the value are limited to three.

In the Black & Cox model the payoff associated with the occurrence of the default at the maturity is dependent on the value process of the firm. This is also the case in the Merton (1974) model where at the maturity the bondholders receive the minimum of the asset value and the face value of the debt. In the Black & Cox model the payoff, in the case that the default barrier is hit, is given by a contract and is dependent on time. However, Longstaff & Schwartz (1995) argue that the recovery rate, before or at the maturity, can be reliably estimated from the historical data and it can be assumed as constant. Their model therefore assumes that in the case of default the bond pays a predetermined percentage of the face value. This approach is also used in the model presented by Leland & Toft (1996).

The default barrier may be given exogenously in contracts or endogenously as a result of an optimal decision problem (Black & Cox 1976). Black & Cox (1976) assume the default barrier to be given by, for example, a safety covenant and to have a time dependence of an exponential form. However, according to Longstaff & Schwartz (1995), allowing the default barrier to depend on time and/or the risk-free interest rate or to follow a separate stochastic process makes the model more complex without adding any relevant insight. This is because it is the ratio of the asset value and the default barrier that is important, not the actual value of the default barrier. Therefore Longstaff & Schwartz (1995) model considers the barrier as a constant. Constant default barrier is also in line with the assumption of a stationary capital structure. (Longstaff & Schwartz 1995, 793.)

In the Leland & Toft model the default barrier is given endogenously as an optimal decision made by shareholders to resign the firm in the hands of the debtors. This is in contrast with Longstaff & Schwartz model where the default default barrier is defined exogenously and the default occurs when the firm is unable to pay the coupons or violates minimum net worth or working-capital requirements. Leland & Toft (1996), however, show that the default barrier as an optimal decision made by shareholders is also constant.

Although an endogenously defined default barrier might be more realistic, in this thesis the only default barrier model implemented is the Longstaff & Schwartz (1995) model. It was chosen for the implementation since, unlike Leland & Toft (1996) model, it takes into account the stochastic nature of the risk-free interest rates. In the next section the inclusion of stochastic interest rate in the structural framework is analyzed in more detail.

2.2.3 Structural models with stochastic interest rate

The original Merton (1974) model assumes the risk-free interest rate to stay constant over time. This assumption too is in odds with reality, since the interest rates evolve through time and the risk-free rates for the same maturity in different time periods can differ quite substantially. The reason why the risk-free interest rate wasn't allowed to be stochastic in the Merton (1974) model was to distinguish the effect of default risk from the term structure of interest rates (Merton 1974, 450-456). However, if one wants to make the model more realistic and generate bond prices closer to market prices, allowing the risk-free rate to vary could be useful.

A study by Shimko, Tejima & van Deventer (1993) was one of the firsts to include a stochastic interest rate into a structural credit risk model. Another study that utilizes stochastic interest rates in a structural framework is the one by Longstaff & Schwartz (1995), which has already been discussed earlier. These two differ, however, in the specification of the interest rate process.

The model developed by Longstaff & Schwartz (1995) uses the Vasicek (1977) model as the basis of the interest rate process. The Vasicek (1977) model is one of the most frequently used short rate models. The short rate models present a stochastic differential equation for the evolution of the so called short rate (i.e. risk-free rate over an infinitesimal period of time). The stochastic differential equation in the Vasicek (1977) model is of the form:

$$dr = (\xi - \beta r)dt + \eta dW, (a > 0), \tag{11}$$

where ξ can be viewed as the long term average of the short rate and β as the speed of reversion to this average level. The Vasicek (1977) model is an example of an affine term structure model.[2] (Björk 2009, 492, 507-512.)

The advantages of the Vasicek (1977) model include that it is reasonably simple and it can deliver analytical solutions for many bonds and derivatives (Veronesi 2010, 513). It is also in line with many observed properties of the interest rates (Longstaff & Schwartz 1995, 792). One of the often cited downsides of Vasicek (1977) model used to be that it allows for negative interest rates (Longstaff & Schwartz 1995, 792). However, after the fact that the long recession and years of expansionary monetary policy have infact resulted in negative interest rates[3] this might not be considered as a drawback anymore.

Another example of a short rate model is the Cox, Ingersoll, and Ross model (CIR). The CIR model is very similar to the Vasicek (1977) model but it eliminates the possibility of negative interest rates. This has been traditionally considered

[2] A term structure is said to have an affine form, if the price of a zero-coupon bond $p(t,T) = F(t, r(t); T)$, where $F(t, r; T) = e^{A(t,T)+B(t,T)r}$, and where A and B are some deterministic functions.

[3] For example the one-month Euribor has been negative from the beginning of 2015 to the beginning of 2017.

an improvement, but on the other hand the more general form makes the model slightly less tractable. With more general form the model becomes more complex and analytical formulas for some securities are not available. (Veronesi 2010, 550.) This is a common conflict in modeling in general and throughout this thesis the advantages and problems brought by the more complex approaches are considered. In most implementations of the structural models with stochastic interest rate the Vasicek (1977) model is used.

Although letting the interest rate to be stochastic makes the model in use more realistic, it is questionable whether it is necessary. Leland & Toft (1996) argue that while having a relatively small effect on credit spreads the inclusion of stochastic interest rate makes the model significantly more complex. They also argue that the stochastic interest rate produces smaller credit spreads compared to constant interest rate even though the structural models in general tend to yield credit spreads that are too small.

Leland & Toft (1996) conclude that if the correlation between the asset value changes and the changes in the risk-free short rate is strongly negative, for example -0.25, then allowing the interest rate to be stochastic narrows the credit spread by about 5–7 basis points. Since the asset returns and risk-free rates are often negatively correlated, Leland & Toft (1996) claim that allowing the interest rate to be stochastic has an adverse effect on the model's performance. They suggest that the asset payouts and the overall capital process of the firm are more important factors.

However, Acharya & Carpenter (2002) argue that this result does not generalize and that when the correlation is zero or positive the stochastic interest rate widens the spread. It therefore remains unclear whether allowing the interest rate to be stochastic improves the model or not.

The next section describes the derivation of a structural model using stochastic interest rate, namely the Longstaff & Schwartz (1995) model which is implemented in the empirical part of this thesis.

2.2.4 The Longstaff & Schwartz model

In this section the model developed by Longstaff & Schwartz (1995) is presented as an example of a structural model with a default barrier and stochastic interest rates. As mentioned the model assumes the risk-free short rate to follow the Vasicek (1977) model. The dynamics of a short rate that follows Vasicek (1977) model were presented in the equation (11).

In the case that the financial instrument is contingent on multiple assets that are driven by different Wiener processes its price process must satisfy a multidimensional form of Itô's lemma. Therefore we have that the price process $F(V, r, T)$ of an instrument that is contingent on both the stochastic risk-free interest rate and the asset value must satisfy the PDE:

$$\frac{\sigma^2}{2}V^2F_{VV} + \rho\sigma\eta VF_{Vr} + \frac{\eta^2}{2}Frr + rVF_V + (\alpha - \beta r)F_r - rF = F_T, \tag{12}$$

where α is the sum of ξ and a constant representing the market price of interest rate risk and ρ is the correlation coefficient between r and V (Longstaff & Schwartz 1995, 795). The risk-free interest rate has slightly different characteristics than any other financial instrument. Its expected profit is not in relation to any price, but is per one unit invested. Also the market price of interest rate risk is not as clear as for any other instrument and its nature is under both theoretical and empirical debate (Campbell 1986 and Ahmad & Wilmott 2007). Technically when determining the market price of interest rate risk one tries to determine the market price for market price of risk risk (Ahmad & Wilmott 2007, 69)!

In the Longstaff & Schwartz (1995) model risky bonds are assumed to have a constant recovery rate equal to $1 - w$. Therefore, the payoff function of a risky discount bond can be written as $1 - wI_{\gamma \leq T}$, where $I_{\gamma \leq T}$ is an indicator function getting a value 1, if the first passage time γ of the asset value V to the default barrier K is less than or equal to the bond's maturity T and zero otherwise. If the bond defaults before the maturity the bondholder therefore gets only a fraction of the original face value.

It follows that the price of a risky zero-coupon bond $D(X, r, T)$, where X is the ratio V/K, is given as:

$$D(X, r, T) = P(r, T) - wP(r, T)Q(X, r, T), \tag{13}$$

where $Q(X, r, T)$ is the risk-neutral probability of the first-passage time of V to K occurring before the bonds maturity. This expression for the bond price is easily comprehended. The bond price is formed as a risk-free discount bond $P(r, t)$ deducted with the discounted loss rate at default of the bond times the probability under the risk-neutral measure that the bond defaults. (Longstaff & Schwartz 1995, 796-797.)

For deriving the complete form of the bond price, however, the actual form of the probability $Q(X, r, T)$ in the pricing equation needs to be obtained. By differentiating the price function $D(X, r, T)$ in accordance with the equation (12) equation the following PDE must be satisfied by $Q(V, r, T)$:

$$\frac{\sigma^2}{2}V^2 Q_{XX} + \rho\sigma\eta X Q_{Xr} + \frac{\eta^2}{2}Qrr + (r - \rho\sigma\eta B(t,T))XQ_X$$
$$+ (\alpha - \beta r - \eta^2 B(t,T))Q_r - Q_T = 0, \tag{14}$$

subject to the condition $Q(X, r, 0) = I_{\gamma \leq T}$ i.e. the starting value of $Q(X, r, V)$ (Longstaff & Schwartz 1995, 815). The term $B(t, T)$ in the PDE is the one from the price equation for discount bond in the affine term structure models. The fact that in the affine models by definition $B_T(t, T) = 1 + \beta B(t, T) - \frac{1}{2}\psi^2$ and $A_T(t, T) = \frac{1}{2}\eta B^2(t, T) - \alpha B(t, T)$ is used in the differentiation. In the Vasicek (1977) model the parameter ψ is zero since the interest rate volatility does not depend on the current level of interest rates. (Björk 2009, 379.)

The resulting PDE resembles the PDE in the equation (12) for any contingent claim with a difference that the coefficients r and $(\alpha - \beta r)$ have been replaced by $(r - \rho\sigma\eta B(t, T))$ and $(\alpha - \beta r - \eta^2 B(t, T))$ respectively. From the Feynman-Kač formula we have that $Q(X, r, T)$ that solves the PDE is of the form $Q(X, r, T) = \mathbb{E}^Q_{T,r}[I_{\gamma \leq T}]$. Therefore, $Q(X, r, T)$ corresponds the probability that $\ln X$ goes to zero before the bonds maturity T, where probabilities are taken with respect to time-dependent processes

$$d \ln X = (r - \rho\sigma\eta B(t, T))dt + \sigma dW_t, \tag{15}$$
$$dr = (\alpha - \beta r - \eta^2 B(t, T))dt + \eta d\hat{W}_t, \tag{16}$$

where dW_t and $d\hat{W}_t$ are two correlated Wiener processes (Longstaff & Schwartz 1995, 815-816).

It is quite counter-intuitive that in the risk-neutral world the probability Q in the price process $D(X, r, T)$ is viewed so that the processes $d \ln X$ and dr follow different dynamics than they were in the equation (12). The reason for this is that Q is not itself a price process. When the risk-free interest rate is allowed to be stochastic every price process of any contingent claim is also dependent on another contingent claim i.e. the risk-free discount bond. The cross-dependency of $P(r, T)$ and $Q(X, r, T)$ affects the dynamics of Q in the pricing context, because otherwise arbitrage opportunities would exist. This the reason for measuring Q with respect to the dynamics given in the equations (15) and (16) in the pricing context.

By integrating the dynamics for r and $\ln X$ in the equations (15) and (16) and substituting for the resulting value of r in the integrated dynamics of $\ln X$ the following equation for $\ln X_T$ is obtained:

$$\ln X_T = \ln X + M(T,T) + \frac{\eta}{\beta}\int_0^T (1 - e^{-\beta(T-t)})d\hat{W} + \sigma\int_0^T dW.$$

(Longstaff & Schwartz 1995, 816.)

It is clear that $\ln X_T$ is normally distributed with the mean $\ln X + M(T,T)$ and variance $S(T)$. The joint bivariate distribution of $\ln X_T$ and $\ln X_t$ implies that $\ln X_T$, conditional on $\ln X_t = 0$ is normally distributed with mean $M(T,T) - M(t,T)$ and variance $S(T) - S(t)$. (Longstaff & Schwartz 1995, 816.) The form of the conditional variance $S(T) - S(t)$ stems from the Itô isometry.[4]

The probability $Q(X,r,T)$ in the bond pricing equation can now be constructed by integrating the first-passage density $q(0,t|\ln X,0)$ of $\ln X$ to zero at time t starting at $\ln X$ at time zero. The density $q(0,t|\ln X,0)$ is defined by the equation:

$$\Phi\left(\frac{-\ln X - M(T,T)}{S(T)}\right) = \int_0^T q(0,t|\ln X,0)\Phi\left(\frac{M(t,T) - M(T,T)}{S(T) - S(t)}\right)dt.$$

(Longstaff & Schwartz 1995, 816.)

However, there is no easy analytical solution for the expression $q(0,t|\ln X,0)$ in the above equation. For the computational purposes the above equation is, therefore, discretized to recursively obtain an approximation of the probabilities of the first-passage time for small T/n sized time intervals. The resulting approximation $Q(X,r,T,n)$ for $Q(X,r,T)$ is the sum of each of these approximated probabilities. By denoting $q_i = q(0,iT/n|\ln X,0)T/n$ it is obtained that:

$$\begin{aligned}
&Q(X,r,T,n) = \sum_{i=1}^{n} q_i,\\
&q_1 = \Phi(a_1),\\
&q_i = \Phi(a_i) - \sum_{j=1}^{i-1} q_j\Phi(b_{ij}), i = 2,3,...,n,
\end{aligned}$$

where

$$\begin{aligned}
a_i &= \frac{-\ln X - M(iT/n,T)}{\sqrt{S(iT/n)}},\\
b_{ij} &= \frac{M(jT/n,T) - M(iT/n,T)}{\sqrt{S(iT/n) - S(jT/n)}}
\end{aligned}$$

[3]The complete formula for $M(T,T)$ and $S(T)$ are presented in the appendix.

(Longstaff & Schwartz 1995, 796,816.)

This can be done since an integral is just a sum constructed with infinitesimally small subperiods. In order to achieve mathematical correctness the time intervals in the discrtetized equation would, therefore, have to be infinitesimally small. From a practical point of view, however, letting n to be sufficiently large provides a good approximation of the true value. For $n \geq 200$ these to are virtually indifferent (Longstaff & Schwartz 1995, 797).

By substituting this approximation in the equation (13) we finally get the complete formula for the price of a credit risky bond. As the default probability is viewed as a sum of default probabilities for small subperiods, valuing coupon bonds is considerably simplified, since the later coupon and principal payments do not need to be conditioned on the earlier coupon payments. Therefore, coupon bonds can be priced as simple portfolios of zero-coupons (Longstaff & Schwartz 1995, 797-798). The implementation yet requires estimating the parameters for the Vasicek (1977) model and the unobserved asset price and volatility as well as obtaining the observed parameters. The specifics of the implementation are presented in the section 4.

2.3 Parameter estimation methods for structural models

2.3.1 Proxy method

The empirical testing of the structural models is an essential part in understanding how well each model captures the reality. This however fails, if the models are not implemented properly. A common difficulty in the implementation of the structural models is that the key parameters, the value of the firm's assets and its volatility, are not directly observable and therefore have to be estimated (Li & Wong 2008, 752). There are many different methods for the estimation of these parameter and the most relevant ones of these methods are thoroughly discussed in this section.

A simple way to approach this problem is a method first used by Jones, Mason & Rosenfeld (1984), know as the proxy approach. In this method the implementation problem referred to above is solved by using the book value of total liabilities as a proxy for deriving the true unobserved market value of the firm. For example, in the study of Brockman & Turtle (2003) this method is applied by approximating the market value of the firm to be the book value of the debt plus the market value of the equity. Asset volatility is then calculated as the volatility of a time series of these proxy asset values. A method used by Jones et al. (1984) is similar but it uses the market to book value ratio of the traded debt as an estimate of market to book ratio of non-traded debt.

This method is simple and easy to use but it comes with an obvious drawback. The book values are not fair values and the book value of debt is also used in the intrinsic value of the call option in the equity pricing formula of the Merton (1974) model. The fact that the value of the option is always higher than the intrinsic value makes the proxy firm value an upwardly biased estimator as pointed out by the equation below. If the equity value is seen as a call option on the firm's assets and the firm's asset value is estimated to be equity value plus the face value of the debt the following is true:

$$C(V, B, T) = E = V_{proxy} - B < C(V_{proxy}, B, T),$$

where E is the value of equity as given by the call option formula $C(V, B, T)$ and V_{proxy} is the asset value estimated with the proxy method. (Li & Wong 2008, 753-754.)

As the call option formula is an increasing function of the asset value the inequality above implies that the proxy firm value is an upwardly biased estimator. This makes the bond price, seen as risk-free bond less put option, to be overestimated and conversely the yield spread to be underestimated. This may be the reason why structural models have been seen to underestimate bond yields according to various studies. (Li & Wong 2008, 770.)

2.3.2 Volatility restriction method

Another way to obtain the two unobservable parameters is developed by Ronn & Verma (1986). This method is often referred to as the volatility restriction (VR) approach. It has been widely used in literature and it avoids the problems with using the book values. The method is based on forming to equations from which the two unknown parameters can be solved (Ronn & Verma 1986, 878).

The first equation utilizes the fact that market value of the equity is observable and it can be set as equal to the call option formula i.e. $E = C(V, B; \sigma_V)$. The second equation is constructed by applying Itô's lemma to the pricing formula used in the first equation. The second equation is

$$\sigma_E = \sigma_V \frac{V}{E} F_V.$$

(Li & Wong 2008, 754.)

Both equations are based on the Merton's original framework and the derivation of the second equation can be made from the equation (7) presented earlier. From equation (7) one can get the percentage volatility of the contingent claim (the value of equity in this case) by dividing the term in front of the Wiener process by the value of the equity. The name of the method comes from the way the estimate of the volatility is restricted from the PDE (Li & Wong 2008, 754).

Although the volatility restriction approach has been used in various studies and it seems to produce a simple solution for obtaining the two hidden parameters, its theoretical relevance and empirical accuracy have been contested (Duan (1994); Ericsson & Reneby (2005); Li & Wong (2008)). The problem with the volatility restriction approach is that if the asset value is seen as a log-normal process and the equity is seen as a call option it follows that the equity volatility must be stochastic. However in the volatility restriction approach the equity volatility is treated as a constant in the second equation. If the equity volatility was appropriately treated as a stochastic variable, the second equation would no more form a necessary restriction for obtaining the parameters. The volatility restriction method is not statistical and cannot therefore provide the necessary statistical inferences. (Duan 1994, 156.)

The method also forces the Itô's lemma to hold at each time point. This hardly true however if the equity value abruptly changes at a certain time point. As a result the system of equations may have no solution. (Li & Wong 2008, 754.) In the empirical analysis of Li & Wong (2008) there are few of such examples.

Between the two aforementioned methods is the mixed proxy (MP) method. This method uses the second equation of the VR method, but instead computes the proxy value of the firm as in the pure proxy (PP) method presented earlier. The volatility of equity is calculated from the market values of equity after which the second equation of the VR method can be solved. (Li & Wong 2008, 754-755.)

2.3.3 Maximum likelihood estimation

As discussed in the sections 1 and 2, the maximum likelihood estimation has been suggested by many researchers to be superior to the volatility restriction approach both theoretically and empirically (Duan (1994); Ericsson & Reneby (2005); Li & Wong (2008) and Qi & Liu (2010)). The method was first presented by Duan (1994) in the context of structural models and it is based on the general statistical theory.

The maximum likelihood (ML) estimation is based on deriving a likelihood function for the occurrence of the sample given the assumption of the underlying distribution. This function is then maximized in order to determine the most probable parameter values. (Fabozzi, Focardi & Rachev 2014, 278-279.) In the case structural models the observed data are transformed values of the underlying random variables. For example in the Merton (1974) model the equity is viewed as a call option on the stochastic asset value. (Duan 1994, 156.)

The likelihood function can be thought as a density function with a difference that it is not a function of a random variable whose density is being measured, but a function of the unknown parameters. Different parameter values result in different likelihoods for the occurence of the specific values of the random variable. In the case of Merton (1974) model the values of the random variable namely the price of equity are observed on the market and it is the unknown asset value and volatility we are interested in. The term likelihood is used instead of probability, because the values of the random variable are held constant and it is a question of how likely it is that each set of parameters could produce certain observations.

Ordinarily, deriving the likelihood function for the transformed data is straightforward, but in the case of structural models there are unknown parameters in the original random variables (Duan 1994, 156). The task would be easier, if the concern was on the distributional parameters of the original random variable, for example its mean and variance. In the case of structural models, however, we know the drift and volatility of the equity price, but we are interested in the underlying parameters from which the equity prices can be transformed. According to Duan (1994) in these kinds of situations the maximum likelihood function can be difficult to form, since the inverse transformations may not have analytical solutions. In finance there are various similar problems since almost all complex financial contracts with embedded options fall into this problem category (Duan 1994, 156).

As a solution to this Duan (1994) proposes the use of standard theory of differentiable transformations to define the likelihood function of the transformed data.

For the transformed data Y with transformation $T(\cdot;\theta)$ (i.e. $Y = T(X;\theta)$), where θ is the unknown parameter, the log-likelihood function $L(Y;\theta)$ can be expressed as

$$L(Y;\theta) = L_X(T^{-1}(Y;\theta);\theta) + \ln\left|J(T^{-1}(Y;\theta);\theta)\right|, \tag{17}$$

where $L_X(\cdot;\theta)$ is the log-likelihood function of the unobserved random vector X and J stands for the Jacobian determinant of a given transformation. The Jacobian matrix is a n×n matrix containing the partial derivatives of the given function. (Duan 1994, 156-157.)

In one dimensional case the above equation would stem from the fact that by applying the chain rule of differentiation the following applies:

$$\frac{dF_Y(y)}{dy} = \frac{dF_X(T^{-1}(y))}{dx}\left|\frac{dT^{-1}(y)}{dy}\right|$$

for a strictly monotone transformation $T(.)$ with continuous and non-vanishing derivative. Here F_Y and F_X denote the cumulative distribution functions. In a multidimensional case the derivative of the transformation with respect to y is replaced by the Jacobian determinant which contains the information of all the partial derivatives. (Spanos 1999, 586-591.)

The absolute value is because otherwise negative density functions could be produced. In the case of negative values of the derivative the directions would be altered i.e. increasing the cumulation in $F_Y(y)$ would mean decreasing the cumulation in $F_X(T^{-1}(y))$. However, both $\frac{dF_Y(y)}{dy}$ and $\frac{dF_X(T^{-1}(y))}{dx}$ measure the change towards the increase of the cumulation, hence the absolute value of $\frac{dT^{-1}(y)}{dy}$ needs to be taken.

The absolute value of the Jacobian determinant measures the space change of X with respect to the space change of Y, which is why it is included in the equation. This is due to the fact that the absolute value of the determinant of a matrix equals the volume of parallelotope spanned by its rows or columns.

Since the inverse transformation for example for the call option formula would be difficult to derive explicitly, it is better to modify the equation (17) a bit. In the case of problems considered here the transformation is one-to-one differential mapping for any possible value of θ. For example a certain asset value produces only one possible value when the call option formula is applied given the parameter θ. Therefore it follows that

$$D_Y T^{-1}(Y;\theta) = \left[D_X T(X;\theta)|_{X=T^{-1}(Y;\theta)}\right]^{-1},$$

where D_X and D_Y denote the n×n first partial derivative matrix with respect to the first argument of $T(\cdot;\cdot)$. Therefore the equation (17) comes into form:

$$L(Y;\theta) = L_X(T^{-1}(Y;\theta);\theta) + \ln \left| \det\{ \left[D_X T(X;\theta)|_{X=T^{-1}(Y;\theta)} \right]^{-1} \} \right|.$$

(Duan 1994, 157.)

Since the transformation is on element-by-element basis, i.e. $y_i = T_i(x_i;\theta)$ for all i, then

$$L(Y;\theta) = L_X(T^{-1}(Y;\theta);\theta) - \sum_{i=1}^{n} \ln \left| \frac{dT_i(\hat{x}_i(\theta);\theta)}{dx_i} \right|, \tag{18}$$

where $\hat{x}_i(\theta) = T_i^{-1}(y_i;\theta)$. This is because when the transformation is on element-by-element basis, $D_X T(X;\theta)$ must be diagonal and therefore the determinant of $D_X T(X;\theta)^{-1}$ is $\prod_{i=1}^{n}(dT_i(\hat{x}_i(\theta);\theta)/dx_i)$. (Duan 1994, 158.) In the case of structural models the tansformation is on element-by-element basis since only the asset value in the first time period affects the value of the call option formula in the first time period and so on.

In the following sections the application of the theoretical concepts presented in the section 2 are examined from a more practical point of view. In the section 3 section the usability of the models for bond pricing and credit risk assessment is examined. This is analyzed based on both performance of the models in prior research and the actual application of the models by different economic agents.

3 STRUCTURAL MODELS IN PRACTICE

3.1 Debt valuation with structural models

3.1.1 Empirical performance and possible missing factors

Although structural models are appealing due to their theoretical foundation they, seem to lack the support from empirical research in bond pricing. Many of the studies considering structural models' ability to price corporate bonds have concluded that structural models produce significant pricing errors. Many reasons have been suggested for the poor performance of the structural models. Some of those reasons such as dubious parameter estimation techniques can be worked on but others are harder to overcome. The findings of the past research and its implications are thoroughly discussed in this section.

Jones et al. (1984) were one of the firsts to implement and empirically test a structural model. Model that they tested was a simple modification of the Merton (1974) model. It appeared that the model did not improve a naive model that assumed no default risk and it had no incremental explanatory power. The model appeared to significantly underestimate the credit spreads of the bonds. Jones et al. (1984) suggested that the poor performance could be due to variance estimation errors and that including stochastic interest rate and/or taxes would not improve the model significantly. (Jones et al. 1984, 624.)

Anderson & Sundaresan (2000) tested both the Merton (1974) model and an endogenous default barrier model similar to Leland & Toft (1996) model. Their result was that the endogenous default barrier model was a significant improvement compared to the Merton (1974) model.

On the other hand, a test performed by Wei & Guo (1997) showed that the Merton (1974) model can outperform the Longstaff & Schwartz (1995) model that includes the default barrier. Their explanation was that modeling the covariance between recovery rate and default probability is important and since the recovery rate is given as a exogenous constant in the Longstaff & Schwartz (1995) model, the Merton (1974) model can give more accurate results. They also show that Merton (1974) model is more general in terms of recovery rate. According to these results it seems that it is important whether the default barrier is determined to be an endogenous or exogenous variable.

Eom et al. (2004) performed a comprehensive testing of five different structural models. These included the Merton (1974) model as well as Longstaff & Schwartz

(1995) and Leland & Toft (1996) models. The result of their study did not favour structural models either. Although their result was that most of the structural models do not systematically underestimate credit spreads, they observed significant prediction errors in all the structural models they tested. The Merton (1974) model was seen to underestimate credit risk as in previous studies, but the other models generated spreads that were either unrealistically large or insignificantly small.

Previous literature had found that the structural models tend to generate too low credit spreads especially for shorter maturities. The outcome of the study by Eom et al. (2004), on the other hand, was that the maturity had no significant effect on the prediction error of structural models in general. In fact the Leland & Toft (1996) model had a tendency to produce too high spreads overall and this applied for the bonds with short maturity as well.

According to Eom et al. (2004) some extensions of the Merton (1974) model, like the addition of stochastic interest rate, caused the models to perform even worse. This is in line with opinion presented by Leland & Toft (1996) that stochastic interest rate may correct the credit risk in wrong direction. It points out that adding more complex and seemingly more realistic assumptions into the model may not necessarily improve the model.

On the other hand, Eom et al. (2004) suggest that a more accurate model than the Vasicek (1977) model could help the problem with stochastic interest rate. All in all the concern of Eom et al. (2004) was that the extensions to the Merton (1974) model managed to generate sufficiently high credit spreads but simultaneously resulted in exaggeration of credit risk resulting from asset volatility and leverage.

Chen, Fabozzi, Pan & Sverdlove (2006) compared pairs of nested models, which they argue is important in order to determine how influential certain factors are to prices. These were pairs of models, where one model is a special case of the other in terms of their assumptions. Factors that Chen et al. (2006) examined were default probability, time of default, time and amount of recovery rate, and the term structure of interest rates. This allowed them to determine what causes a model to succeed or fail. (Chen et al. 2006, 7.)

By comparing the Merton (1974) model and a model with a default barrier and random recovery rate Chen et al. (2006) came into conclusion that continuous default does not play a major role in pricing. This implies that inclusion of the default barrier may not improve the model. At the same time comparisons between other models in their study suggest that the random recovery rate and interest rate are important factors affecting prices. Therefore it possible for the Merton (1974) model to outperform the more complex Longstaff & Schwartz (1995) model because in the former random recovery rate is assumed and in the latter it is fixed. (Chen et al. 2006, 14-19.) This implies that the Merton (1974) model could explain larger part of the credit spread especially when the correlation between the asset returns and risk-free rates is negative, since in that case the stochastic interest rate in the Longstaff & Schwartz (1995) model would narrow the spread.

The study of Chen et al. (2006) differs from many of the previous studies also in that it uses credit default swap (CDS) prices rather than bond prices. The advantage is that the CDS prices reflect credit risk more purely since bond prices are affected, for example, by convexity (Chen et al. 2006, 8). The downside is that CDS data is much harder to obtain. Chen et al. (2006) also use transaction prices rather than matrix prices as was done by Eom et al. (2004). This also adds in the accuracy of their study. However, as Eom et al. (2004) they find that structural models can either underestimate or overestimate prices.

If the models could either underestimate or overestimate the credit spreads, the stochastic interest may improve the model even though it would narrow the spread. There is also no guarantee for the correlation between the asset returns and risk-free rates to be negative. The unusual interest rate environment and economic situation experienced after the financial and sovereign debt crisis could also have an effect on the relationship between the stochastic interest rate and model performance. It is therefore interesting to see how this affects the results of the empirical part of this thesis.

Ericsson & Reneby (2005) considered the choice of the parameter estimation method and its influence on the performance of a structural model. They suggested that the poor performance of the structural models, for example, in the study of Jones et al. (1984) could be a result of the chosen estimation method. They recommended the maximum likelihood method, first considered by Duan (1994) for structural models, because of its theoretical superiority. This subject was already noted by Jones et al. (1984) as they speculated about the possible errors in the estimation of unobserved parameters such as asset volatility. The result of Ericsson & Reneby (2005) supported the importance of a correct parameter estimation method since all three structural models they tested performed significantly better with maximum likelihood method compared to the simpler approaches described in the section 2.3.

The superiority of the maximum likelihood method was supported by the study of Li & Wong (2008) who tested five different structural models. The errors in prices as well as in yields in their study were surprisingly small especially when the maximum likelihood method was used. The Leland & Toft (1996) model performed best for bonds with short maturity and for medium/long maturity bonds the Longstaff & Schwartz (1995) model was the best. Therefore Li & Wong (2008) suggested a hybrid model that would use Leland & Toft (1996) model with short maturities and Longstaff & Schwartz (1995) model with medium/long maturities. Moreover they found that none of the models consistently underestimated the yields.

The results of Li & Wong (2008) are overall remarkably distinctive from the other studies in field and their results show quite strong support for the structural models in bond pricing. The study Li & Wong (2008) seems to be quite unique also in the sense that there is a lack of studies that implement structural models with maximum likelihood estimation with empirical data. For example Ericsson & Reneby (2005) only used simulated data. It could therefore be that the difficulties in parameter estimation has been causing the poor performance of the structural models and the maximum likelihood estimation might prove the models to be useful in bond pricing.

The study of Li & Wong (2008) has, however, received quite limited attention in the following literature despite the remarkable results in favor of the structural models. The question is whether the results of Li & Wong (2008) could be replicated or if they were coincidental. Unlike Eom et al. (2004) they do not present absolute percentage errors for prices or yield spreads. It could therefore be that the errors in the study of Li & Wong (2008) were actually high on average, but the positive and negative errors happened to get averaged out. The standard errors in the study of Li & Wong (2008) were somewhat larger than with Eom et al. (2004), which indicates larger dispersion in the results. One of the main focuses in this study is, therefore, to test the validity of the results of Li & Wong (2008).

Huang & Huang (2012) studied which amount of the corporate credit spreads was due to credit risk by using structural models. They used models that were previously studied in literature and in addition two extensions, one with time-varying asset risk premia and one allowing for jumps in the firm value process. The result was that the credit risk explains only a small fraction of the observed yield spreads on investment grade bonds. On speculative grade bonds the amount was larger but also did not account for the whole spread. Similar findings are also made by other researchers and the phenomenon is known as the credit spread puzzle.

Huang & Huang (2012) conclude that although features such as jumps in the value process, time-varying risk premia, endogenous default boundary or recovery risk cannot alone explain the puzzle, a combination of some of them along with stochastic volatility shows promise. For example Delianedis & Geske (2001) conducted a comparable test with structural models both with and without a jump component. They attribute the credit spreads mainly to effects of taxes, liquidity, jumps and market risk.

Influence of factors such as liquidity and taxes produce a considerable difficulties for structural models to obtain accurate bond prices and yields. Although the effect of taxes can be included in a structural framework as done in Qi & Liu (2010), the modeling becomes quite complex and differences in tax codes need to be considered. This makes the model hard to implement and unlikely to be used in practice. This is also the case with maximum likelihood estimation, where the implementation of the model becomes quite tedious.

As a result of the difficulties with structural framework described above, other approaches for credit risk modeling have been suggested. Jarrow & Protter (2004) argue that the value process of the firm and its parameters are not directly observable in the market as evidenced by the need of using estimation methods as maximum likelihood. They suggest that as the information set of an investor is more restricted than the structural framework would require a reduced-form approach is more suitable for credit risk modeling. In the reduced-form approach the default probability information is compressed in a hazard rate, which is given exogenously and not as a result of the firm value process. The other approaches than the structural approach for bond pricing and credit risk modeling are discussed in the section 3.1.2 next.

3.1.2 Alternative approaches for bond pricing

As discussed in the previous sections there are numerous risks and other factors that affect the prices of bonds, large number of which cannot directly be observed in the market. Based on the poor empirical performance of the structural models in bond pricing, at least the most basic models seem either to lack some of the important factors or to be unable to quantify some of them.

The structural approach in bond pricing has therefore not gained such popularity as the Black & Scholes (1973) model in standard option pricing although the framework is fundamentally the same. This may be because bonds are not derivatives in the purest sense compared to basic stock options for example. The usefulness of the option pricing framework applied to bonds comes out when some the risks discussed above needs to be quantified. In credit risk modeling and default probability estimation as well as in the pricing of credit derivatives the structural models have given large contributions.

Since the accurate valuation of bonds seems to require a highly sophisticated and complex structural model, the portfolio managers often use more simple approaches. The similar methods are often used as for the valuation of basic stocks. The portfolio managers often try to compare the spread of the bond with the spread of similar instruments. This technique is often referred to as relative analysis. The instruments used for the relative analysis can be the bonds of the same firm, the bonds of the similar firms or other instruments like credit default swaps (CDS)[5]. If the spread of the bond greatly differs, for example from the spread of a bond with same credit rating and maturity profile, then one of the bond's risks might be mispriced. (Fabozzi & Mann 2010, 400-401.)

In this sense the technique has the same basic idea as the structural approach as the no-arbitrage principle is the driving force in this case also. The technique in its most basic form lacks the theoretical power of the structural models and may be based more on subjective views of the portfolio manager. The bigger the institution that trades bonds the bigger the probability that at least some sort of modeling of the main risks involved is accompanied with the relative analysis. The bonds that lack appropriate benchmarks may also require more sophisticated approach. This may be for example when the bond is the only bond of the issuing firm and there yet exists no credit rating for the bond.

The problems associated with the structural models were recognized already at least in the 1990s. A new body of research governing credit risk modeling was then formed. These studies created what is now termed as the reduced-form approach to the credit risk modeling. The main difference between the structural and reduced-form approaches is the assumption concerning the observability of the default probability.

The structural approach considers the default probability to result from the firm value process and therefore to be observable from the market. The reduced-form approach, on the other hand, considers the information possessed by the investors to be limited to the knowledge about the set of state variables representing, for example, the credit rating categories. The default probability in the reduced framework changes only when the bond moves from one state to another.

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[5]The CDS with some simplifying assumptions corresponds to a bond financed with a repurchase agreement (repo) i.e. the bondholder (just like the CDS protection seller) would receive the periodic cash flows, but due to the repo-financing would have to benefit the repo-counterparty, if the bond that works as a collateral, defaults. Due to the near equivalence of these strategies, there is a linkage between bond's spread and the CDS spread. The difference between these two spreads is referred to as the CDS basis. This can be used as a tool in relative analysis. (CDS is a periodic cash flow that the protection seller receives for selling the protection.) (Fabozzi & Mann 2010, 410-413.)

The information set of the investor is therefore assumed to be more limited in the reduced-form approach. (Jarrow & Protter 2004, 8-9.)

Under reduced-form approach the price $D(t,T)$ of a default risky bond can be presented as:

$$D(t,T) = I_{\{Z>t\}}\mathbb{E}^{Q}\left[\exp\left(-\int_{t}^{T}(r(s)+\lambda(s)L(s)ds)\right)\right],$$

where λ represents a hazard rate and $L(s)$ is a loss function given as $L(s) = 1 - w(s)$, where $w(s)$ is the recovery rate. These can be modeled by arbitrary stochastic processes of their own. The multiplier I is an indicator function having value 1, if the default has not yet occurred (i.e. the random default time Z is less than t) and zero otherwise. The default in the reduced-form approach is therefore modeled as a Poisson point process. (Jarrow & Turnbull 2000, 278-281.)

The default is therefore unpredictable and occurs under the risk-neutral measure with probability of $\lambda(t)dt$ in the time interval $[t, t+dt]$. If $\lambda(t)$ was a constant the probability of default would be the same for every point in time. The default probability is, however, usually modeled as a stochastic process of its own, which can be a square root process or a jump diffusion process for example. The default probability can also depend on state variables, in which case the default process would be referred to as a Cox process. A Cox process matches roughly speaking a Poisson process conditioned on the state variables. (Jarrow & Turnbull 2000, 278-281.)

The discount factor in this setup therefore corresponds the risk-free rate adjusted with the hazard rate times the loss function. This adjustment contains the expected losses on the bond due to default, which need to be covered by the discount rate. In practice the hazard rate can also contain other than default variables like liquidity premium and tax effects. These variables are therefore included in the "default risk process". This enables positive credit spreads to be derived for also the bonds with high credit quality and short maturity, which is a characteristic often observed empirically. (Duffee 1999, 201.)

An accurate calibration of the parameters for the default probability process often requires an active market for the bond to exist. The reduced-form approach, therefore roughly speaking abandons the attempts to derive a contingent claim price for a bond based on some underlying instruments. Conversely these models attempt to specify the characteristics of the bond value and default probability processes, information which could then be used for credit derivative pricing and risk management among other purposes. The existence of sufficient information in order to derive bond prices is therefore refused and credit risk is instead modeled in more mathematically tractable way in order to fulfil other purposes than pricing of new corporate bond issues.

The derivation of the reduced-form framework therefore stems from the failure of the structural models to accurately model the bond prices and the firm value process. If a structural model that could capture all the variables affecting the bond prices and reliably infer its parameters from the stock price data could be derived, the model could be used in both bond pricing and credit risk management purposes. Even though this could be achieved, the problems with burdensome implementation procedures, could be a likely source of trouble. If we, however, cannot accurately price bonds based on other bond price data in a feasible way the reduced-form is in most applications the more preferred approach. (Jarrow & Protter 2004.)

The pricing of corporate bonds with reduced-form models could be achieved if credit ratings data for example would be used. In this case, however, the problem is that the variation in the credit quality between the bonds in the same credit rating class is quite large. In addition the credit ratings are often slow to adjust, which makes the derivation of accurate bond prices troublesome. (Kao 2000, 59-62.)

The research governing the reduced-form models' success in bond pricing is quite limited. Most of the research has been focused on developing more sophisticated models to be applied in risk management and derivatives pricing purposes. There are problems with the implementation of models on individual bond data, since the amount and quality information on the single bond level is often insufficient. Therefore an aggregate level data about default probabilities and credit migration are often used (Kao 2000, 64).

There exists some research governing the reduced-form models' ability to fit the parameters of the default probabilty and the bond price process in a way that the evolution of the bond price in time is accurately modeled. For example Duffee (1999) used Kalman filtering to infer the parameters from bond prices. The result was quite favourable for the reduced-form approach and the model was able to fit the bond prices with an average error less than 10 basis points.

The evidence was not, however, completely supportive for the model used. The autoregression test performed for the model showed some persistence in the innovations of the bond price process. This would suggest that the evolution of the price process would not be fully captured by the model, since there remains some forecastability in the innovations based on information prior to the time t. The researchers also found some instability in the parameters following the changes in the credit quality. (Duffee 1999, 218-223.) The reduced-form approach based on this is not flawless due to, for example, difficulties in inferring the model parameters based on the illiquid bond market data.

Jarrow & Protter (2004) in their study attempt to resolve the superiority of the two different approaches in a more theoretical manner. They argue that the firm value process is not observable on the market and therefore the information set of an investor assumed by the structural approach is false. They refer to studies by Duan (1994) and Ericsson & Reneby (2005) to point out that there is a consensus of the fact that the firm value process is indeed unobservable on the market. The structural framework in their opinion should be used merely for the purposes of the management of the firm itself, who might possess this information. The reduced approach they claim is the right one for the risky-debt pricing and hedging purposes (Jarrow & Protter 2004, 6-8.)

Jarrow & Protter (2004) see the maximum likelihood estimation for the structural models studied, for example, by Duan (1994) and Ericsson & Reneby (2005) as a proof of the unobservability of the firm value process. Indeed the lack of direct observability of the process is difficult to contest. However, the process might still be indirectly observable from the equity price process. If the firm value process implied by the equity price process could be satisfactorily estimated via maximum likelihood estimation, the structural models could be useful for bond pricing. This is one of the main research questions of Duan (1994), Ericsson & Reneby (2005), Li & Wong (2008) and this thesis as well. The definitiveness of the critique by Jarrow & Protter (2004) might be partly accounted for the fact that one of the researchers is the main developer of the reduced-form approach.

Another study comparing the reduced-form and structural approaches is the one by Arora, Bohn & Zhu (2005). They empirically test three different models, namely the basic Merton (1974) model, Vasicek-Kealhofer (VK) model and Hull-White (HW) model. The former two represent the structural approach and the latter applies the reduced-form approach. They test the models' performance in measuring credit risk by using CDS price data. They concluded that the more sophisticated structural model namely the VK model outperformed the HW reduced-form model especially in the default predictive power (Arora et al. 2005, 20-21). The results are not surprising regarding the fact that the study was provided by Moody's KMV Company, the developer and active user of the KMV approach, which is based on the VK model.

The empirical research governing the differences between the two approaches, especially an independent one, is quite limited. This might be partly because of the difficulties in the implementation, but also partly because of the opposition of the two approaches might not be that meaningful after all. The usefulness of either approach depends on the application and as stated earlier quite often the other one thrives when the other fails. The approaches can therefore be seen as complementing each other. Based on this it is hardly surprising that much of the recent development is also focused on hybrid models that try to combine the best of both worlds.

3.2 Credit risk assessment with structural models

3.2.1 Evidence from hedge ratios

As the comprehensive empirical studies performed by Eom et al. (2004) and Huang & Huang (2012) analyzed in the section 3.1.1 suggest, the structural models seem to be unable to accurately estimate the credit spreads of credit risky bonds. There are two plausible reasons for the models' failure. First one is that the models are unable to measure the credit exposure of the bonds. Second is that non-credit related variable not captured by the models are causing the pricing errors. (Barsotti & Viva 2015, 95.)

If the first reason is more driving, then the structural models are not really functioning in the purpose what they were designed for, i.e. in capturing the credit risk related to corporate bonds. On the other hand, if the second reason is more responsible for the models' failure, the structural models could still prove to be valuable despite the inability to produce accurate credit spreads.

The most important application of the structural models has been their usage in the credit risk management and measuring default probabilities (Bruche 2007, 2). Non-credit related variables like liquidity and taxes may affect credit spreads and prices, but they do not affect the default probabilities. Therefore, if the pricing errors are indeed due to non-credit related variables the models may still be useful in credit risk management.

To what extent the two aforementioned reasons are causing the model errors remains unclear. The models' ability to measure the credit related component can be tested by focusing on default probabilities based on the models as described in the previous section. Another way is to study the models' ability to provide accurate hedge ratios i.e. the sensitivities of debt with respect to equity as was done by Schaefer & Strebulaev (2008) and Barsotti & Viva (2015). Hedge ratios are also important since the sensitivity of debt to equity determines the composition of the replicating portfolio (Schaefer & Strebulaev 2008, 2).

The approach focusing on hedge ratios is simplified by Schaefer & Strebulaev (2008) as follows. Let the D be the actual corporate bond price observed in the market which consists of two components. The first component D_C is the credit related component. It represents the present value of the bond's cash flows taking into account the credit exposure, which are discounted with the rate consistent with the risk of the firm's equity. The second component D_{NC} contains the influence of the non-credit variables identified, for example by Huang & Huang (2012). Together these two form the actual price:

$$D = D_C + D_{NC}$$

If the second is really unrelated to credit risk, the bonds hedge ratio in relation to equity is simply the first derivative of D_C with respect to equity. Then if the structural model correctly measures the first component, it automatically produces the correct hedge ratio. (Schaefer & Strebulaev 2008, 2.)

Like the original Merton (1974) model the hedge ratio approach used by Schaefer & Strebulaev (2008) and Barsotti & Viva (2015) assumes that the firm's market value is the sum of the market values of debt and equity (i.e. $V = D + E$). The difference is that the Merton (1974) model's formula for E as a call option on the firm's assets may contain a pricing error. Since the equity and debt prices are non-linear with respect to each other, the first order derivative doesn't fully describe the change in debt value as the equity value changes. The absolute debt variation ΔD as function of the absolute equity variation ΔE can be approximated with the Taylor series. By using the second order Taylor expansion we get:

$$\Delta D \approx \frac{dD}{dE}\Delta E + \frac{1}{2}\frac{d^2 D}{dE^2}(\Delta E)^2$$

(Barsotti & Viva 2015, 96-97.)

The above equation can be expressed in terms of rates of return. By denoting the debt rate of return $(\frac{\Delta D}{D})$ by r_D the above equation comes into form:

$$r_D \approx h_E r_E + k_E r_{E_2} \tag{19}$$

where r_E is the equity rate of return, and

$$h_E = \left(\frac{1}{\delta_E} - 1\right)\left(\frac{1}{L} - 1\right)$$

$$k_E = -\frac{1}{2}\frac{\gamma_E}{\delta_E^3}\left(\frac{V}{D} - 1\right)$$

$$r_{E_2} = \frac{(\Delta E)^2}{E^2},$$

where notations $\delta_E = \frac{dE}{dV}$ and $\gamma_E = \frac{d^2E}{dV^2}$ describe the Greek Delta and Gamma of a European call option and L is the market leverage. Although it was assumed that the Equity formula as a call option on the asset value was assumed to contain a possible pricing error, the hedging coefficients h_E and k_E are assumed to be correctly measured by the model as explained above.

Therefore it is assumed that the first order derivative of equity with respect to the asset value corresponds the Greek Delta of a European call option as in the original Merton (1974) model. The coefficient h_E measures the elasticity of debt to equity value and k_E adjusts the hedging position by taking into account the convexity. (Barsotti & Viva 2015, 96-97.) The effect k_E is deemed to be much less significant than the effect of h_E, which is why Schaefer & Strebulaev (2008) in their study considered only this first order sensitivity to equity. This is also the approach taken in this thesis.

In their study Schaefer & Strebulaev (2008) implemented the original Merton (1974) model by using data from the years 1996–2003 and derived monthly cross-sectional hedge ratios based on the model. By regressing the excess bond returns to the excess equity and riskless bond returns they examined if the hedging coefficient from the model corresponds the hedging ratio derived from the market prices. Their result was that for all but AAA rating class the Merton (1974) model could not be rejected, which was a strong result in the support of the simplest structural model.

However, Schaefer & Strebulaev (2008) had the same problem that all studies on structural models have, namely the estimation of the unobserved parameters V and σ_V. If the Merton (1974) model is assumed to contain a pricing error, the maximum likelihood method cannot be straight applied since it assumes that the equity pricing formula as a call option on the firm's assets holds. Due to this

Schaefer & Strebulaev (2008) could not derive reliable hedge ratios for individual firms, but instead they used average hedge ratios for subrating classes.

Barsotti & Viva (2015) tried to overcome the problems that Schaefer & Strebulaev (2008) had and derive firm-specific hedge ratios by relaxing the assumption of normally distributed rates of return. They use two different asymmetrical namely the variance gamma and the normal inverse gaussian distributions (Barsotti & Viva 2015, 96). The use of more realistic distribution assumptions avoids the problem of multicollinearity in the original Merton (1974) model. Otherwise the original Merton (1974) model produces hedging coefficients for high grade bonds close to zero. (Barsotti & Viva 2015, 98.)

According to Schaefer & Strebulaev (2008) computing the asset volatility based on the volatility of equity and debt returns and their covariance is difficult, because the volatility of non-public debt cannot be observed. Much of the volatility of the debt prices can also be spurious due to illiquidity of corporate debt. Therefore they used average hedge ratios for subrating classes. (Schaefer & Strebulaev 2008, 9-10.) Using more realistic distribution assumptions takes better into account the probability of the extreme price fluctuations and therefore does not lead to the similar underestimation of the hedge ratios.

The results of Barsotti & Viva (2015) were not so favourable for the Merton (1974) model as the ones by Schaefer & Strebulaev (2008). Instead their result was that Merton (1974) model could produce hedge ratios for individual bonds within 10% from the observed hedge ratios in less than 6% of the cases. This is a major setback for the Merton (1974) model and it raises doubts, if the model can capture the credit risk associated with corporate bonds. On the other hand, Barsotti & Viva (2015) concluded that the model parameters like asset value and volatility had a strong impact on the size of the hedging errors. For example for the companies with smaller volatility the hedging errors were smaller on average. Therefore the problem may still lie on the difficult task of the parameter estimation. (Barsotti & Viva 2015, 95, 107.)

As mentioned earlier, when there is assumed to be a model error, in this case caused by variables such as liquidity and taxes, maximum likelihood method cannot be directly applied for the parameter estimation. This problem could be overcome by adding an error term to the equity pricing formula and using numerical techniques like importance sampling or particle filtering together with Monte Carlo simulation (Bruche (2007); Fulop & Li (2013)). This would be an extension of the likelihood-based approach.

Considering the possibility of an error in the pricing formula when estimating the parameters however requires a substantial extra effort compared to the general maximum likelihood estimation. As the latter already is in itself somewhat time-consuming, available computing power is definitely a considerable issue with the former. The methods like importance sampling or particle filtering are also beyond the scope of this study.

3.2.2 Structural models in banking regulation

Since banks and other large scale financial institutions are the primary users of sophisticated credit risk models, the concepts of this thesis are closely linked in banking and banking regulation. In this section the drivers behind the banking regulation and its recent developments are analyzed. Since the empirical part of this study utilizes European bond market data, the focus in this section is also mainly on the regulatory framework from the European perspective.

The issue of banking regulation has for a long time been of high interest among the economic agents as well as the whole public in general. The banking regulation or lack of it is often seen as major contributor to the potential new crisis. Indeed, as the subprime crisis and the collapsing CDO market in 2007–2008 drove large banks near or in default the blame was largely on the loose regulation.

The drivers behind the banking regulation are closely linked to the so-called moral hazard problem. Many large financial institutions are viewed by the regulators as too big to fail, which is why the governments and the central banks often have (explicit or implicit) guarantees on these institutions. This may well seem to be justified especially when the crisis is at hand. However, the limited downside risk may cause banks to benefit from risk taking and therefore the guarantee can actually raise the risk of a crisis. (Berger, Molyneux & Wilson 2009, 357.)

The solution to the above stated problem is often sought from regulation. A naive approach would be to impose barrier on the bank's leverage. Although easy to implement, this approach would have clear drawbacks as it does not consider the riskiness of the bank's assets. It would therefore obviously benefit the banks that invest in the higher risk assets. This simple restriction would also be easy to circumvent as the banks could take risky positions that do not appear on their balance sheets via securitizing of assets and derivatives trading. (Berger et al. 2009, 358.) Indeed, this kind of regulatory arbitrage has been one of the main reasons behind invention of such financial instruments as CDOs and CoCo bonds (Avdjiev, Kartasheva & Bogdanova 2013, 46-47 and Berger et al. 2009, 360).

Another game theoretic problem concerning the banking regulation relates to the level of authorities i.e. trade-off between national and global regulation. As seen during the last financial crisis the global financial system is strongly linked internationally, due to which reducing the systemic risk by regulating the banks benefits almost all the countries. However, there may be incentives to loosen the regulation on the national level to boost the competitive advantage of the domestic banks. (Berger et al. 2009, 358.)

This problem is one of the main reasons why national banking authorities began working on common bank capital adequacy rules, which produced the bank capital accord, now called Basel I, established by the Basel Committee on Banking Supervision. This set up the basis for common standards for calculating a measure of capital adequacy called the risk-based capital ratio.

As the risk management methods and the whole financial sector evolved the relatively simple rules of Basel I became subject of scrutiny. Due to this the Basel Committee launched its revised standards during 2004–2005 known as the Basel II. Like the Basel I, the revised framework of Basel II still requires banks to hold regulatory capital of 8% of the bank's risk-weighted assets.

One of the changes in Basel II is instead related into the calculation of those risk-weighted assets. Basel II defines the risk weighted assets as:

$$\mathrm{RWA} = \frac{1}{8\%}(s\sum_i k_i EAD_i + K_{TR} + K_{OR}),$$

where EAD is the exposure at default of instrument i (i.e. the amount under the risk of loss in the occurence of default), K_{TR} and K_{OR} are the capital charges to cover for the losses from trading and operational risks, s is a scaling factor used, for example, to provide incentives for implementation of more advanced models. The parameter k_i is the capital charge (risk weight) to cover for the i-th credit exposure. (Berger et al. 2009, 361.)

Under Basel II the risk weights are determined to be calculated so that the capital charges cover for 99.9 percentile value-at-risk (VaR) over one-year horizon (Berger et al. 2009, 363). This means that the losses from credit risk over one-year period do not exceed the capital charges with 99.9% probability. This of course requires the calculation of the default probabilities and recovery rates. The exact requirements defined by Basel II for calculating the risk weights based on these factors are discussed below.

One of the innovations in Basel II is the use of the so-called internal-ratings-based credit risk models. As the credit risk continues to account for the largest share of the required capital this can be seen as the most important innovation in Basel II. In Basel II banks are allowed to choose from two different approaches for calculating the risk-weights in the risk-weighted assets formula. The standardized approach gives predefined risk-weights for different kinds of debt types. Differing from Basel I these are allowed to depend on the credit ratings issued by external credit rating agencies. The internal-ratings-based approach (IRB), on the other hand, allows banks to use their own models to define the risk-weights. The IRB approach is yet divided to the foundation and advanced approaches. (Berger et al. 2009, 364-365.)

The difficulty faced by the regulators in allowing the banks to use their own models is that the effect of a certain asset to the risk-profile of the whole portfolio is strongly dependent on the correlations between different assets. As the modeling of these interdependencies in the portfolio is very complex, the objectivity of the banks' own model would be hard to verify. As a solution the IRB approach requires that the risk weight of a certain exposure must account only for the risk characteristics of that specific asset and not the characteristics of the portfolio. This property is called the portfolio invariance. (Berger et al. 2009, 365.)

This can be achieved, if it is assumed that there exists a single risk factor that is the only source of systematic risk in the portfolio. This allows for the usage of Merton (1974) style model accompanied with some systematic risk factor. Indeed the basic assumptions behind the Merton (1974) model are closely related to the IRB approach and the calculation of the default probabilities based on the asymptotic single risk factor (ASRF). With ASRF it is assumed that every instrument in the portfolio constitutes only a very small part of the portfolio, there exists only one source of systematic risk (i.e. risk affecting the whole market) and that the realizations of this factor are monotonically related with conditional expected losses associated with most of the risk exposures. (Berger et al. 2009, 365.)

Under these assumptions the portfolio VaR can be calculated as the sum of expected losses on the individual instruments in the portfolio conditional on the single risk factor having realization equal to the required percentile (99.9). These individual conditional expected losses can in turn be calculated as the product of the default probability of the instrument conditional on the single risk factor hitting the value at the 99.9th percentile and the loss given these same stress conditions. (Berger et al. 2009, 366.) As is evidenced by this method for capital charge calculation the estimation of individual default probabilities plays a crucial role in the functioning of the whole banking sector.

According to Bharath & Shumway (2008) the Merton (1974) model is widely used both by researchers and practitioners including numerous banks, and this is seen as a viable practice by the Basel Committee. This is why it is important to study whether the model is able to reliably estimate the probabilities of default. It is stated in the Basel II accord that the bank must satisfy its supervisors that the model used has a good predictive power (Basel Committee on Banking Supervision 2006, 93). For the risk weight issuance to be more audible the average default probabilities for the specific internally or externally defined credit category need to be used. The individual probabilities used to calculate this average can, however, be based on, for example the Merton (1974) model. (Basel Committee on Banking Supervision 2006, 102.)

The calculation of the default probabilities under the Merton (1974) model is based on so-called distance-to-default (DD) measure. The distance-to-default for one-year horizon in the Merton (1974) model is calculated simply as $d_2(D, 1)$ as presented in the equation (8), but the risk-neutral drift replaced with the drift under objective probability measure P. The actual default probability (PD) can be calculated based on the distance-to-default as $PD = \Phi(-DD)$.

The default probability is therefore simply equal to the probability of the firm value being below the book value of debt after one-year under the objective probability measure. (Bharath & Shumway 2008, 1341-1345.) When a single risk factor is assumed the default probability is affected by the correlation between the single risk factor and the individual instrument. The Merton (1974) model can also be used under Basel II framework to calculate this correlation (Conze 2015, 3).

Bharath & Shumway (2008) have made a comprehensive analysis on the Merton (1974) model's ability to accurately measure the default probability. They compared the performance of the Merton (1974) model to a naive model for which they calculated the asset value and asset volatility based on a simple rule of thumb rather than the methods commonly used by researchers for the Merton (1974) model. For the Merton (1974) model they used a method similar to the one used by Vassalou & Xing (2004), which is also used by for example KMV corporation.

The result of Bharath & Shumway (2008) was that the naive approach was performing surprisingly well against the Merton (1974) model and therefore the Merton (1974) model is not in their opinion a sufficient statistic for default probability. They, however, also implemented a reduced-form hazard rate model that used the default probability based on the Merton (1974) model as one input variable. The test showed that the default probability based on the Merton (1974) model improved the hazard rate model on a statistically significant level. Therefore the Merton (1974) model's functional form might be useful in default prediction.

The poor performance of the Merton (1974) model against naive alternative in default prediction puts the model's straightforward application in the determination of risk weights in question and therefore more research is needed. In this thesis, however, the model is not tested straight in default prediction due to the unavailability of sufficient data. The results of Bharath & Shumway (2008), however, propose that the problem with the model's poor performance might lie in the poor parameter estimation since the naively estimated parameters could produce better results. The same problem is shared in the estimation of the correlation with the single risk factor (Berger et al. 2009, 370).

Bharath & Shumway (2008) suggest that the parameter estimation used by Vassalou & Xing (2004) might be improved by applying more standard econometric estimation methods such as maximum likelihood estimation. Schaefer & Strebulaev (2008), however, argue that in the case of Merton (1974) model these two should be equal. In this thesis it is tested whether the Merton (1974) model and the more complex Longstaff & Schwartz (1995) model can be improved in bond pricing by the maximum likelihood estimation used by Li & Wong (2008) whose results showed to be promising.

Hopefully this way more evidence on the models' applicability in credit risk assessment can be gathered, since the topic seems to be relevant in the future also. The Basel regulatory framework has been for a while under development in the form of Basel III currently under implementation and the trend after financial crisis has been towards tightening of the regulation. (Basel Committee on Banking Supervision 2016, 1-15.) This is also evidential in the field of accounting standards as the new IFRS 9 standard requires the firms with material credit exposures to calculate the impairment losses by a way which requires significant modeling effort (Conze 2015, 1-5).

In the Basel III there will indeed be some more specific requirements. With respect to the credit risk and the calculation of PDs, perhaps the most important new ruling is the qualification that the rating systems should be designed so that the ratings are independent of the business cycles. Other novelties include determined minimum values for the parameters used by the model and the demand that PD should be based on the observed historical average one-year default rate, which must include data from both good and bad years.

Using the historical default rate means that the PD used in the calculation of RWA should be based on the historical default rate for that given rating category. However, the decision for which rating category the given asset is placed can be based on, for example an asset value process estimated with a Merton (1974) style framework. (Basel Committee on Banking Supervision 2016, 1-15.)

In the next section the important questions examined in the previous sections are analyzed in practice via the empirical part of the thesis. First the description of the procedures for retrieving the needed data as well as the methods for implementing the models are presented. After that the sections 5 and 6 present the actual results of the empirical part of the thesis.

4 DATA AND IMPLEMENTATION METHODS

4.1 Description of data

This section describes the methods used for the selection of the bond sample utilised in the empirical testing. The methods used for the implementation of the models are also described the section 4. There are several crucial considerations needed to be made during the implementation process and the section 4 attempts to cover the most important of these with sufficient detail. The methods used here mainly follow the ones used in the previous studies in order to achieve comparability.

In order to test whether the results of prior studies are applicable for a data from different market and period of time, prices of corporate bonds issued in Eurozone countries and denominated in euros are used here. The data was obtained mainly from Datastream for the period 2009–2014. For the part of some supplementary data the Bloomberg terminal was also used as a source of data.

The starting year was chosen due to the fact that there are no composite price data for Eurozone bonds prior 2009 on Datastream. The composite mid price is starting from July 2009 the preferred price in Datastream. According to the information from Datastream website the composite mid price better reflects the true market value of individual bonds. All of the derived data in Datastream is also based on composite mid prices starting from July 2009.

There were 1079 Eurozone issued bonds in the database, but there were only 408 euro-denominated bonds for which there was a complete composite price data available for over a one-year period. Some of the removed bonds had composite mid prices but lacked the composite ask price. Closer inspection proved that most of these bonds were clearly illiquid and therefore all of these bonds were removed from the sample.

For the implementation of the structural models, data of stock prices and some financial data are needed. Therefore the bonds examined here must be issued by companies with publicly traded stock and regular financial data. The issuing corporation was manually matched with equity issue based on the issuer name and other information as Datastream did not have any identification number for the matching purposes.

In addition the bond must have maturity of at least one year. This is because bonds with shorter maturities are highly unlikely to trade (Eom et al. 2004, 503). Following the selection criteria of Eom et al. (2004) and Li & Wong (2008) bonds issued by financial firms and public utilities were eliminated. These firms and their risk of default are strongly influenced by regulations (Eom et al. 2004, 503). Only bonds that had seniority of senior secured or senior unsecured were used in the testing. By further eliminating the bonds with floating rate coupons final sample the consisted of 97 bonds from 64 issuers. A total of 248 bond price observations could be used for the yield spread estimations. The bonds included in the final sample are presented in the appendix.

The final sample includes some bonds issued by companies under financial distress. Other researchers have considered only surviving firms, but in order to avoid the survivorship bias, the companies that went through financial distress during the sample period were also included. In the previous studies only firms that had simple capital structures were considered. Eom et al. (2004) include firms that have only one or two public bonds. Since the number of bonds in the sample would otherwise be unbearably low, this restriction is not made here. Therefore the final sample includes also issuers that have significant number of different bonds issued. There, however, appeared to be no statistically significant relation between the bond pricing results and the number of bonds issued. This is returned to in the section 5.4.

The table 1 presents the main characteristics of the final sample. The market capitalization and total liabilities figures in the table are in millions of euros while the coupon rate and the yield spread figures are in percentages. The different time period compared to the study of, Eom et al. (2004) for example, is visible in that the coupon rate in their study was much higher (7.92 %) on average. This has to do with the extremely low interest rate environment experienced during the sample used here.

The average yield spread is, however, considerably larger than in the study of Eom et al. (2004), which is a result of including few firms under financial distress as the large standard deviation points out. The overall credit quality also seems to be lower here perhaps due to more instabile credit environment. The average yield spread in the study of Eom et al. (2004) was only 93.5 basis points as here it is 364 basis points. This affects the interpretations of the results in the graphs of the section 5 where the absolute yield spread errors are plotted instead of the percentage errors. The average percentage errors and their standard deviations are, however, presented in the tables in the section 5.

The average size of the issuers is somewhat larger here and the issuers here also have higher leverage on average. As mentioned, the number of bonds issued per firm is considerably higher than in the previous studies as some of the issuers had several dozens of bonds issued.

Characteristics	**Mean**	**S.D.**
Years to maturity	5.55	3.69
Coupon rate	4.71	1.33
Market yield spread	3.64	8.03
Market capitalization	9 692	13 286
Total liabilities	12 682	15 065
Number of bonds per issuer	9.75	9.87

Table 1: Descriptive statistics of the bonds in the sample

Since the total liabilities on Datastream are reported only on a yearly-basis, the bond price data from the last day of each December are used in the implementation of the models for the bond pricing. This is also the approach chosen by for example Eom et al. (2004) and Li & Wong (2008). Schaefer & Strebulaev (2008) use monthly data, which is why in the study of the hedge ratios monthly data is used here also.

The study of Schaefer & Strebulaev (2008) heavily utilises the credit rating information on the tested bonds. For each of their regression analysis the hedging coefficients and other statistics are presented for all the major credit rating classes. Credit rating classes are also used for the calculation of asset volatility parameter in their regressions. This will be described in more detail in the section 6.

The credit rating information on Datastream or Bloomberg is, however, not very comprehensive. The ratings are supplied for only a fraction of the bonds in the sample. Since the number of bonds in this study is already significantly smaller than in the study of Schaefer & Strebulaev (2008), it is not possible to use only the bonds that have credit rating information on Datastream or Bloomberg.

This problem is avoided by using the default probabilities instead of credit ratings in the categorisation. The bonds on the sample are arranged in the different categories by using one-year default probabilities for the issuer reported by Bloomberg. These were available for practically all of the bonds in the sample. Bloomberg uses a variation of the Merton (1974) model in the calculation of the default probabilities. Therefore it would be possible to calculate comparable default probabilities based on the data used here. But since this would have little extra value, the Bloomberg reported probabilities are used to save the effort.

To make the results comparable with Schaefer & Strebulaev (2008) the default probabilities are used for the division into categories in a way that best corresponds the "real" division done by the credit rating agencies. Moody's Investors Service (2011) report, which reports the average cumulative default probabilities for different subrating categories is used here. The data of the report is from the years 1983–2010. The artificial credit rating category boundaries used here are constructed by taking the midpoint of the average default probabilities for lowest subrating category in Aaa (Aaa) and highest subrating category in Aa (Aa1) and so on.

The one-year default probability is of course an approximation of the overall credit quality of the issuer. Better approximation would use also the longer time frame default probabilities as well as other information or the actual credit ratings. However, this simplification only affects the comparability with the results of Schaefer & Strebulaev (2008) with respect to the specific rating categories. The bond pricing results and the hedge ratio results concerning broader credit quality ranges, which are the important part, should not be significantly affected. Therefore the approximation is deemed sufficient.

4.2 Model parameters

4.2.1 Recovery rate and interest rate parameters

As the original Merton model provides a solution only for the price of a zero-coupon bonds and all the bonds in the sample are coupon-bearing bonds, the model needs to be extended for the implementation purposes. More sophisticated model of Longstaff & Schwartz (1995) treats the coupons and the face value of a coupon-bearing bond as individual zero-coupon bonds. The price of the coupon-bearing bond is the price of this portfolio of zeroes. In the Longstaff & Schwartz (1995) model it is the sum of each zero-coupon bond in the portfolio. Same method is used here for the Merton model as is done in the studies by Eom et al. (2004) and Li & Wong (2008).

This approach however comes with a drawback. In the Longstaff & Schwartz (1995) model the default of the bonds is triggered as a common state variable hits a certain level, or as termed here, a default barrier. Every zero-coupon bond is priced in a way that takes into account the probabilities of default occurring at any time before the maturity. The probability of the default occurring, for example before the second coupon, is the sum of the probability of the default occurring before the first coupon and the probability of it occurring before the second coupon conditional on that no default occurred before the first coupon.

The Merton (1974) model, on the other hand, only considers default at the maturity. Summing up fair values of the payments of a coupon-bearing bond viewed as a portfolio of zeroes doesn't take into account that the default before the later payments is conditional on the default before the previous payments. A sample path of the asset value that is below the default barrier at the first coupon date and above the barrier at all the other payment dates is viewed by the model as though the bond defaulted on the first coupon, but not on the other payments. This is a violation of the model's assumption that when default happens, the bondholders immediately take over the firm. Although this proposed extension is in this aspect unrealistic, it has been used by other researchers and for the sake of comparison it is used here also.

Another adjustment made for the Merton (1974) model for implementation purposes is that at the event of default bondholders recover a fraction of the bond's face value defined by a constant recovery rate. The original model assumes that in the case of default the bondholders receive 100% of the firm value while the Longstaff & Schwartz (1995) model assumes a constant recovery rate as a fraction of the face value. In order to make the models comparable the Merton (1974) model is also implemented here by using a constant recovery rate. The assumption of a constant recovery rate was made also by Eom et al. (2004) and Li & Wong (2008).

The constant recovery rate also makes the implementation of the Merton model easier since the assumption of 100% of the firm value is based on the model's assumption of single homogeneous class of debt. This is obviously not true in our sample and in the market in general. Firms in the sample here often have multiple different debt instruments outstanding. The payment order of the debt can therefore be more complex than the original Merton model would assume. When the payments in the case of default are not distributed equally across the whole spectrum of the firm's debt the assumption of the recovery amount equal to the firm's value does not apply.

In the previous studies the recovery rate has often been assumed to be 51.31% (Eom et al. 2004, 510). This is based on the results of Keenan, Shtogrin & Sobehart (1999) who studied the historical recovery rates during the years 1977–1998 and concluded that the average recovery rate for a corporate senior unsecured debt was 51.31% (Keenan et al. 1999, 20). Another researchers like Altman & Kishore (1996) have had similar results. For comparison purposes this recovery rate is applied here for all the models.

However, more recent studies have shown that the average recovery rate has slightly dropped during recent decades. The average ultimate recovery rate during years 1987–2010 for senior unsecured debt has been 49.2%. This is the present value of recovery amount after the resolution of default. The resolution takes typically 1–2 years. If the default occurs at the maturity, the bondholders do not receive the payment immediately. The uncertainty involved in the recovery is however hard to quantify and therefore more accurate measure of recovery rate could be the one observed in the post-default trading prices. Corresponding recovery rate based on the post-default trading prices has been between 36.7%–37.4% depending on the weighting. Alternative assumptions for the recovery rate based on the above are also examined in the result section. (Moody's Investors Service 2011.)

In the original Merton (1974) model there is only one payment date, which is the maturity of the zero-coupon bond, and a risk-free interest rate for has to be determined only for this maturity. In the case of coupon-bearing bonds, on the other hand, one must determine different risk-free rates for all the payment dates since the rate varies with the time to maturity. In other words one has to estimate the yield curve i.e. the term structure of risk-free interest rates.

Here the risk-free yield curve is estimated using the Nelson & Siegel (1987) model. It is basically a regression method for constructing the curve by using rates for available maturities. It is perhaps the most applied model of this kind. (Veronesi 2010, 67-68.) A characteristic of the Nelson & Siegel (1987) model is that it smooths out the yield curve even though the original data is unsmoothed (Diebold & Rudebusch 2013, 31-35).

In the Nelson & Siegel (1987) model the following regression is estimated for available interest rates and maturities:

$$r_t(\tau) = \beta_{t0} + (\beta_{t1} + \beta_{t2})\frac{1 - e^{-\tau/\lambda_t}}{\tau/\lambda_t} - \beta_{t2}e^{-\tau/\lambda_t},$$

where $\beta_{t0}, \beta_{t1}, \beta_{t2}$ and λ_t are the parameters to be estimated from the current bond data and $r_t(\tau)$ is the risk-free interest rate at the time t for the time to maturity τ. After the model parameters have been estimated, an interest rate for an arbitrary maturity can be defined by using the above equation.

The estimation of the Nelson & Siegel (1987) model parameters was performed with the statistical program R using the package "YieldCurve" designed for this exact purpose. The method used by the package is a nonlinear least squares estimation. The risk-free interest rates and the corresponding maturities from an individual point in time are used as an input based on which the package calculates the optimal parameters. The method is similar than is described in the study of Nelson & Siegel (1987).

The benchmark yield curve is here constructed by using the German constant maturity treasury yields downloaded from the Deutsche Bundesbank (2016) website and Eurozone OIS curve downloaded from Bloomberg. The risk-free yield curves based on this data are estimated for all the required dates. In the case of maximum likelihood estimation, for example, the call option formula is fitted for 100 and 250 observation dates and for each date the risk-free yield curve is separately estimated. The short-term maturities based on the OIS curve were 1D, 2W, 3W and 1M–11M while the German constant maturity yields used were for the maturities ranging from 1 to 30 years covering each and every year.

The Longstaff & Schwartz (1995) model requires the estimation of the parameters for the Vasicek (1977) short rate model. The parameters were estimated similarly as with the Nelson & Siegel (1987) model. In the case of Vasicek (1977) model the least square functions were constructed based on the zero-coupon bond pricing formula derived from the model and these were optimized with an R package designed for nonlinear optimization.

Using multiple risk-free rates for valuing a single bond makes the concept of yield spread somewhat more complex. There is no single benchmark yield for which the bond yield could be compared. For example, if the risk-free rate corresponding the bond's final payment date is used the yield spread could well be negative, since some of the payments were valued using a smaller risk-free rate. This is of course assuming that shorter rates are smaller than longer rates as is the usual case.

For this purpose definition of the yield spread when multiple risk-free rates are used has to be made. In this study Z-spread is used as the measure of the yield spread. Z-spread could be described as the internal spread of the bond. It is constant spread that has to be added to every risk-free rate for different maturities so that the discounted payments promised by the bond match the price. Following this definition yield spread z is defined as the solution for the equation:

$$P = \sum_{i=1}^{n-1} \; e^{-(z+r_t(\tau_i))\tau_i} C_i \; + e^{-(z+r_t(\tau_n))\tau_n}(1 + C_n)$$

For a zero-coupon bond this of course corresponds the definition of the yield spread presented earlier in the section 2.1.

Of course this definition is not the only valid solution for the problem. As is visible in the equation (10), the yield spread depends on the level of the risk-free interest rate and time to maturity and does so in a nonlinear fashion. The real market yield spread could depend on those variables in even more complex way. In the above equation, on the other hand, the yield spread is a constant that is added to every risk-free rate. Theoretically every coupon could have a yield spread of its own. However, the prices for the individual coupon payments are not observable in the market and the model could not therefore be tested with the market data. This is why a simplification brought by the Z-spread is used here. The regression analysis for the hedge ratios is performed using the one-month risk-free rate as the benchmark rate as was done by Schaefer & Strebulaev (2008) and in that case the usage of Z-spread is not needed.

4.2.2 Asset payout ratio

The asset payout ratio consists of all the payment made by the firm for an investor who holds both the firm's equity and debt instruments with the ratio corresponding the firm's leverage. The equity payments consist of the dividend payments and the (net) stock repurchases made by the firm. The payments for the debtholders consist of the coupon payments and the (net) debt redemptions and paybacks. The term net means that the repurchases and redemptions should be measured as the net amount of the money transferred to bond and equity holders i.e. the issuance of new shares and the selling of the treasury shares as well as the issuance of new bonds should be subtracted.

The effect of the asset payout ratio on the implementation of the Merton (1974) and Longstaff & Schwartz (1995) models has received quite a little attention in the empirical studies referenced here. Li & Wong (2008) briefly discuss the implementation procedures concerning the asset payout ratio and try to improve the method used by, for example Eom et al. (2004), but they do not thoroughly discuss the usage of the asset payout ratio in the ML estimation. On the other hand, Schaefer & Strebulaev (2008) and Barsotti & Viva (2015) do not mention at all how the asset payouts were included in their hedge ratio tests. However, the results of this thesis show that the specification of the asset payout ratio has a significant effect in both the ML estimation and the hedge ratio tests.

Neither the original Merton (1974) model nor the Longstaff & Schwartz (1995) model include the asset payouts and assume that there is no effect from the dividends or the coupon payments on the risk-neutral asset value process. This was also an assumption made by the original Black & Scholes (1973) option pricing model. The later developments, however showed, that the inclusion of the dividends to the Black & Scholes (1973) model is quite simple.

In the Merton (1974) model for pricing risky debt the inclusion can be done in a similar manner. In this case the dividends are replaced by all the payments made for the holders of the firm's claims i.e. for the holders of portfolio equal to a fraction of V. Since both of the instruments constructing V, namely debt and equity, are modeled via the option pricing theory the inclusion of the asset payouts is in this case more complex and care should be taken about the assumptions made in the different applications of the structural models.

In the case of ML estimation, for example, the market value of equity is matched to the call option formula. If the asset payouts are included in the calculation, it must be taken into account that in this case some of the asset payouts (i.e. the dividends and repurchases benefit the equity holder) unlike in the case of the plain stock option, where the dividends go to the stockholder and not to the optionholder. The value of equity should therefore in this case be equal to the call option on the firm's assets plus the present value of the payments to the equity holders before the maturity of debt.

Based on the formulas in the appendix of the study by Li & Wong (2008) the researchers have left out the asset payout ratio from the ML estimation, but included it into the bond pricing formula. The matter is, however, not discussed in the article. This is surprising since the inclusion of the asset payouts affects the estimated parameters of the model. The inclusion of the asset payouts in the ML estimation is discussed more thoroughly in the section 4.3.

The asset payout ratio also affects the results of the hedge ratio tests. The hedge ratios implied by the Merton (1974) model are based on the delta of the call option formula i.e. the derivative of the call option formula with respect to the asset value. If the dividends are assumed to be paid continuously at a constant ratio of the asset value, as is usually done in the implementation procedures, the derivative of the call option delta is affected by the inclusion of the dividends. As will be pointed out in the section 6.2, leaving out the asset payouts completely out of the hedge ratio tests can result in hedge ratios produced by the Merton (1974) model being ridiculously low.

The effect of the asset payouts seems to be contradicting the classic Modigliani & Miller (1958) theorem about the irrelevancy of dividends to the value of equity and the value of the firm. According to the theorem the future dividends do not affect the value of equity and the capital struture should have no impact on the value of the firm. However, as the dividends are paid the value of equity as well as the value of the whole firm lowers. At the same time the dividend payment changes the capital structure of the firm and therefore the volatility of the firm should change.

If both Modigliani & Miller (1958) theory and the assumptions of the structural approach hold, the volatility of the firm should lower so that the price of the equity would be lowered exactly by the amount of the dividend and the value of the debt should stay the same. In the models used here the volatility is, however, assumed to be constant and therefore the dividends cannot be added exactly in accordance with the Modigliani & Miller (1958) theorem.

Both Merton (1974) and Longstaff & Schwartz (1995) assume that the Modigliani & Miller (1958) theorem holds and therefore the asset payouts would not need to be included. Longstaff & Schwartz (1995), for example argue that the firm has now incentive to alter its optimal capital structure and assume in their model that the coupon payments are financed with new debt.

In reality, due to the problems with asymmetric information, agency costs and the cost of liquidating the firm's assets might result in the dividends lowering the value of debt, which is a violation of the Modigliani & Miller (1958) theorem. However, in reality the firm cannot according to law pay dividends in a way that results in the default of the firm and the bondholders can also have covenants to protect their position against reckless dividend payments.

These are issues that a more realistic model would take into account, but in a large sample testing the implementation would become difficult. The assumptions about constant volatility, agency problems and dividend payments are in that way linked. All of these factors are examples of the simplifying manner by which the original Merton (1974) model treats the firm's capital process.

As many empirical studies have shown the dividend payments can indeed increase the value of equity at the cost of debtholders (Dhillon & Johnson 1994). When the dividends are included in the Merton (1974) model with constant volatility a similar transition of value happens from debtholders to equity holders. Whether the dividends can be added to the Merton (1974) model in a realistic way is, however, questionable.

Continuous dividend payments are assumed here, as has been done in the previous studies, but this does not correspond to reality as dividends are paid at discrete times, usually once per year. This could lead to inaccuracy in the model. On the other hand taking the discrete nature of dividends into account in the model requires exact information of dividend payment dates and also violates the model's assumption about continuous asset value paths.

If the asset payouts cannot be realistically added in the Merton (1974) or the Longstaff & Schwartz (1995) models, then a model that takes better into account the complex nature of the firm's capital process might be more suitable. The Leland & Toft (1996) model, for example, assumes the default barrier to be a result of an optimal decision process of the stakeholders and therefore takes into account the bankruptcy costs and the asset payouts. As the asset payouts seem to be an important factor for the model performance, their correct implementation seems crucial (Leland & Toft 1996, Anderson & Sundaresan 2000 and Acharya & Carpenter 2002). Most of the empirical studies referenced here have included the asset payout in both the Merton (1974) and the Longstaff & Schwartz (1995) model.

The effect of asset payout ratio on the model performance is studied empirically in the sections 5 and 6. The calculation of the asset payout ratio is done by using dividend yields reported in Datastream. The cash flows related to equity along with the average coupon payments for the firm's debt are retrieved from Bloomberg. The repurchases of debt are not included in the asset payout ratio since they directly affect the nominal value of the firm's debt which is used as the default boundary.

If the firm had only one instance of debt issued, the coupon payments would not need to be taken into account in the asset payout ratio. Here many of the bond issuers, however, have issued multiple bonds. The firm could also have different sorts of debts such as bank loans for which interest is also paid. The payment made for the other bonds and other debt need to be taken into account in the asset payout ratio. This is done here by using the average coupon for the firm reported in Bloomberg.

Dividend yield in Datastream is reported as the anticipated annual dividend per share divided by the current share price. At the year-end the dividend is therefore based on the dividend paid in that year and the stock price at the year's end. This is, however, an upwardly biased estimator of the dividend yield since at the year-end the dividend has already been paid and this has obviously decreased the market value of equity by the end of the year. The same goes with the stock repurchases. A simple ratio of stock repurchases to the value of equity could be done by dividing the equity-related cash flow during the year by the year-end stock price. For the same reason this would be, however, an upwardly biased estimator. (Li & Wong 2008, 762.)

Therefore a revised payment ratio for the equity payments is used. The dividend payment plus the equity related cash flows made during the year are divided by the sum of the market value of equity and the dividend and the equity-related cash flow. The asset payout ratio is calculated by weighting the equity payout ratio and the debt payout ratio according to the firm's leverage and is therefore as follows:

$$\delta = \frac{q + e_r}{1 + q + e_r} \times \frac{E}{V} + \overline{c} \times \frac{V - E}{V}, \tag{20}$$

where q is the dividend yield reported by Datastream, e_r is the equity-related cash flow during the year per the market value of equity and $\overline{c}$ is the average coupon rate.

The above formula includes the unobserved parameter V. Therefore the asset payout ratio cannot be calculated straight from the market inputs, but it is obtained as a result of parameter estimation procedures. The above formula is inserted in the parameter estimation equations and likelihood functions. The asset payout ratio can therefore differ depending on which estimation method is used.

The calculation method for the equity payout ratio is in line with the method used by Li & Wong (2008). Eom et al. (2004), however, used the reported dividend yield from Compustat as such, which according to Li & Wong (2008) makes their approach biased. Unlike Eom et al. (2004) the asset payout ratio used by Li & Wong (2008) consisted only of the payments made to the equity holders. This is clearly unrealistic since the payments made to the debtholders clearly affect the risk-neutral drift of the asset value. Therefore the asset payout ratio used here includes those payments also.

4.2.3 Model-specific considerations

As mentioned earlier, the strict assumptions of the original Merton (1974) model do not allow for the pricing of coupon-bearing bonds. Since most of the bonds traded in the market are coupon-bearing bonds the researchers implementing the model have been forced to slightly relax these assumptions. The price of a coupon-bearing bond in these extensions of the model is viewed as a collection of zero-coupon bonds.

Following the approach used by Eom et al. (2004) and Li & Wong (2008), the price of a fixed-rate bond paying semiannual coupons in the Merton (1974) model is found to be:

$$F\left[V_t, t\right] = \sum_{i=1}^{n-1} e^{-r_t(\tau_i)\tau_i} \mathbb{E}^Q \left[(c/2) I_{\{V_{T_i} \geq K\}} + \min(wc/2, V_{T_i}) I_{\{V_{T_i} < K\}}\right]$$

$$+ e^{-r_t(\tau_n)\tau_n} \mathbb{E}^Q \left[(1 + c/2) I_{\{V_T \geq K\}} + \min(w(1 + c/2), V_T) I_{\{V_T < K\}}\right],$$

where c is the coupon rate and

$$\mathbb{E}^Q\left[I_{\{V_T \geq K\}}\right] = \Phi(d_2(K,\tau)),$$

$$\min(\psi, V_{T_i})I_{\{V_{T_i}<K\}} = V_t e^{(r_t(\tau)-\delta)\tau}\Phi(-d_1(\psi,\tau) + \psi\left[\Phi(d_2(\psi,\tau) - \Phi(d_2(K,\tau)\right]$$

and $d_i(x,\tau)$, for $i = 1,2$ are given in the equations (8) and (9) in the section 2.2.1. (Li & Wong 2008, 773.)

In the above equation δ is the asset payout ratio, w is the recovery rate and K is the default barrier. The recovery amount in the equation is actually set to be the minimum of the firm value and the recovery rate times the face value. This is to ensure that the recovery amount cannot be higher than the firm value (Eom et al. 2004, 513). The book value of total liabilities are used as the default barrier.

The above extension of the original Merton (1974) model is slightly unsound since it does not take into account the fact that the default on any of the coupon payments usually results in default on the later payments. In the above simplification the asset value process could be below the default barrier at one of the coupon dates and rise back above the barrier before the maturity. The effect of this simplification, however, is not expected to have a significant effect on the performance of the model. (Eom et al. 2004 and Li & Wong 2008.)

Since the Longstaff & Schwartz (1995) model was originally developed to include coupons, a similar extension for the model is not needed. The selection of the default boundary level is, however, a consideration which needs to be made before implementing the model. According to Li & Wong (2008) there is no consensus on the correct level of the barrier. The level used by Moody's KMV is determined bond-specifically and equals the short-term debt plus half of the long-term debt, and is less than the total debt value. Empirical research have found the barrier to be less than the book value of debt and the empirical median value to be $0.738 \times B$, which was used by Li & Wong (2008). They also tried default barrier level equal to the book value of debt. The results for both of these default barrier levels are also presented here in the empirical part of the thesis in the section 5.

4.3 Likelihood functions for different models

The general theory for maximum likelihood estimation developed by Duan (1994) was presented in the section 2.3.3. The specific form of the likelihood function, however, depends on the model used. The equation (18) is in such form that likelihood functions for specific structural models can be derived from it. The log-likelihood function for the Merton (1974) model was formulated also by Duan (1994) and is presented by Li & Wong (2008) in the following form:

$$L(\mu, \sigma_V) = \sum_{i=2}^{n} \{\ln g(v_i|v_{i-1}) - \ln[V_i \times \Phi(d_1)|_{V=V_i}]\} \tag{21}$$

where μ is the asset drift, σ_V is the asset volatility, $g(.)$ is the density function of the asset value, d_1 is given in the equation (9) in the section 2.2.1 and V_i and v_i are the ith asset value and log of the asset value, respectively. (Li & Wong 2008, 772.)

The likelihood function measures, how likely it is that certain parameters for the asset value process could have produced the observed equity price path. As was shown in the section 2.3.3 the likelihood can be expressed with a likelihood of some underlying variable in case there exists one-to-one mapping between the two. The part $g(v_i|v_{i-1})$ is the likelihood of the logarithm of the asset value as a function of the asset volatility and drift conditioned on the previous logarithm of the asset value. The part $V_i \times \Phi(d_1)|_{V=V_i}$ is the derivative of the original random variable, namely the equity price with respect to the logarithm of the asset price. The fact that $\frac{\partial E_t}{\partial V_t} = \Phi(d_t)$, is used in the expression (Duan 1994, 164). The density function is given by

$$g(v_i|v_{i-1}) = \frac{1}{\sigma_V \sqrt{2\pi(t_i - t_{-1})}} \times \exp\{-\frac{[v_i - v_{i-1} - (\mu - \sigma_V^2/2)(t_i - t_{i-1})]^2}{2\sigma_V^2(t_i - t_{i-1})}\},$$

which of course is a straight application of the density function for a normally distributed variable.

The log-likelihood for the whole set of observations is then formed as a sum of the individual log-likelihoods always conditioned on the previous on the previous observation. The implementation of the model is then continued by maximizing the likelihood function in equation (21) subject to constraint $E(t_i) = C(t_i, V(t_i), \sigma_V), \forall i = 1, 2, ..., n$. This equals setting the equity price to match the call option formula at every point in time. (Li & Wong 2008, 755-772.)

The derivation of the likelihood function for the Longstaff & Schwartz (1995) model is quite more complex than for the Merton (1974) model. Since the Longstaff & Schwartz (1995) model involves a default barrier the market value of equity is viewed as a barrier option (Li & Wong 2008, 756). If the asset value drops below a certain level the bondholders take over the firm and the equity holders receive nothing. This corresponds a payoff structure of a barrier option.

The pricing of a barrier option involves "chopping off" the standard option contract function below the barrier level and therefore leaving a "chopped off" contract function, which can be priced like a contract without a barrier. For more detailed information one can refer to, for example Björk (2009). The likelihood function is derived based on the barrier option pricing formula. The derivation of the likelihood function is not presented here for compactness, but for the exact form of the likelihood function and complete derivation one can refer to Rubinstein & Reiner (1991).

Li & Wong (2008) do not specify, how they model the maturity for the firm's debt in the ML estimation. If the firm only had one homogeneous class of debt maturing at the same time, the choice of maturity would not be a problem. The problem arises when the firm has number of different bonds with different characteristics. One way would be to use bond-specific maturities. This would be in line with the pricing function for the extended Merton model, where there are several dates when the firm can default (i.e. the coupon paying dates). However, if the bond-specific maturity would be used, different bonds issued by the same firm could have different estimates for the asset value and volatility. This would be clearly unrealistic.

In their study Duan & Fulop (2009) use the same initial maturity of 10 years for all the bonds of all the firm's. This avoids using different parameters for the same firm, but is quite simplifying approach. More realistic approach would take into account the maturity profile of the firm. Therefore the average maturity for the firm's debt retrieved from Bloomberg is used here for the initial maturity. If one single point in time has to be chosen to represent the maturity profile of the firm's debt this is probably the soundest choice. This could, however, be improved even further by taking into account the distribution of the coupon payments and their effect on the duration of the firm's whole debt. The value of this improvement, however, would be insignificant and therefore the average maturity from Bloomberg is used.

Li & Wong (2008) perform their maximum likelihood estimation without the asset payout ratio. Their equity pricing formula in the MLE approach therefore corresponds the Black & Scholes (1973) formula without dividends. The absence of the asset payouts could be rationalized by considering the original theory of Merton (1974). According to theory the value of equity can be visioned as a call option on the firm's assets. In the case of a normal stock call option one would need to take into account the dividends paid for the stockholders since these lower the price of the underlying stock.

In the case of the Merton (1974) model, however, the option holder is the equity owner so the dividends are received by the option holder. Therefore the value of the equity should not be affected by the dividend policy. Otherwise the method would not be in line with the famous Modigliani & Miller (1958) theorem about the irrelevancy of the firm's dividend policy to the value of equity. In the density function the asset payout ratio would be included in the drift parameter, which is not used in the pricing formula and is therefore unimportant.

The above rationalization, however, does not work with the Merton (1974) model using a constant asset volatility. As explained in the section 4.2.2 the dividend payments in this case transfer value from the debtholders to the equity holders. This might not be unrealistic since the dividends can in reality result in this kind of value distribution (Dhillon & Johnson 1994).

The asset payout ratio affects the equity's sensitivity in relation to the value of the firm i.e. the derivative of E with respect to V. This clearly affects the estimation procedure. Therefore the approach by Li & Wong (2008) is here considered in this sense flawed and the asset payout ratio is included in the maximum likelihood estimation.

When underlying asset pays a continuous cash flow to the holder, the put-call parity used for the debt valuation comes into form:

$$e^{-rt}B - P + PV[\overline{C}] = V - (C + PV[E_r]),$$

where C and P denote the call and put option respectively, $\overline{C}$ denotes the bond's coupon payments prior maturity, E_r denotes all the equity-related payments prior the maturity of debt and PV is a present value operator.

The right hand side of the equation represents the asset value minus the value of equity and the left hand side the value of debt. The call and put prices in the equation include the asset payout ratio. This modification of the put-call-parity has not been considered by the referenced studies, where the inclusion of asset payouts has been often done without considering the effect of the present value of the payments to the debt and equity holders.

The above requires taking into account the present value of dividend and other payments made for the equity holders, since these affect the equity value. When the payments are assumed to be continuous and made at a constant rate with respect to the equity value and the risk-neutral drift of equity equals risk-free rate minus asset payouts, the present value of equity payments can be obtained as:

$$PV_t[E_r] = (1 - e^{-\delta_e(T-t)})E_t, \tag{22}$$

where δ_e is the equity payout ratio and E_t denotes the equity value at time t.

The coupon payments are not included in the total liabilities and they therefore need to be included in the estimation either via asset payout ratio or by adding them to the book value of debt. Duan & Fulop (2009) estimate the future value of total liabilities by compounding the present value of total liabilities with the risk-free interest rate. In the calculations here total liabilities are measured as their present value and the coupon payments are included in the asset payout ratio.

The number of sequential observations (n) to be used in the ML estimation needs to be chosen large enough for the result to be reliable. With small sample sizes the estimators of the parameters might not be normally distributed and therefore be bias and/or unreliable (Ericsson & Reneby 2005, 719). When a large number of bonds to be priced is in question, the computing power is a factor also needed to be taken into account. Consideration needs to be put also into the fact that observations from farther time periods may not be relevant. Using daily observations allows one to obtain large number of observations from a relatively short time period.

Li & Wong (2008) do not specify, how many sequential observations they use for their maximum likelihood estimation. Ericsson & Reneby (2005) compared the effect of different selections of n on the results of their simulation experiment. They used sample sizes of 90, 250, 500 and 750 days. When moving from 90 days sample to 250 days the efficiency measured as the standard deviation of the estimated parameters was improved by one third. Moreover the Jarque-Bera statistic, which tests the normality, decreased and therefore the rejection of the normality of the estimators became harder. Moving to even longer sample sizes improved the efficiency, but not as drastically. Overall the normality also improved as the sample size increased. (Ericsson & Reneby 2005, 724, 728–729.)

In this thesis the maximum likelihood estimation is first carried out by using 100 daily observations, which is also the number of observations used for deriving the equity price volatility. The estimation is also performed by using 250 daily observations in order to test the effect of the sample size. Therefore, the estimations consists of maximizing the log-likelihood function with 100 or 250 equality constraints.

All the constraints are of course nonlinear, which makes solving the optimization problem relatively troublesome. The statistical program R and the package

"NlcOptim" was used for solving the optimization problem. The package uses an algorithm that is based on sequential quadratic programming. In a case like this where the problem contains only equality constraints the method equals applying Newton's method to the Karush-Kuhn-Tucker conditions of the problem.

The maximum likelihood estimation was performed on the total of 174 bond observations for which there were sufficient data. The algorithm worked relatively fast since it took only about 5 minutes to estimate the parameters for all the 174 bonds using 100 observations. This equals less than two seconds per each bond observation. With a faster computer this could likely be improved significantly. Therefore, the time consumption of the estimation, in the case of Merton (1974) model, does not seem to be a problem.

In the case of Longstaff & Schwartz (1995) model, on the other hand, the estimation took several minutes per a single bond. This is partly because R rounds up extremely small (large) intermediate results to zero (infinity), which lead to usage of a special package for handling extreme numbers. This in turn, however, slows up the calculation of the likelihood function and hence resulted in the long duration of the estimation. Even though more efficient programming and computers could speed up the process, the estimation time seems to be a factor to be considered in the case of Longstaff & Schwartz (1995) model.

In the case of ML estimation for the Merton (1974) model, the optimization algorithm allowed for very wide boundaries for the parameters. At first, the asset volatility parameter obtained from the volatility restriction approach and the asset values generated based on the pure proxy approach were used as a starting point for the algorithm. Other starting points were tried also and in general the results were robust.

For some bonds, however, the algorithm couldn't converge with the original starting parameters, but could converge with slightly different starting parameters. Especially the asset volatility had a strong effect on the convergence. Therefore the algorithm was started with 6 different asset volatility parameter values ranging from $\frac{1}{3}\times$ the VR asset volatility to $2\times$ the VR volatility. The parameters that gave the largest likelihood were chosen.

This way the convergence was achieved for all the bonds without having to use different criteria for the starting parameters. This also mitigated the risk that different starting values could produce different results. The chosen parameters were also inserted back to the constraint vector to ensure that the parameters obtained by the algorithm really fulfilled the constraints.

In the case of Longstaff & Schwartz (1995) model, on the other hand, the convergence of the algorithm depended critically on the right specification of the boundaries. For most of the bonds the lower boundary for the parameters was set to $\sigma_V = 0.03$, $\mu = -1$ and $V = 0.738B$. Some bonds, however, required tighter constraints such as setting the lower boundary for the volatility as 0.05 and/or the drift as zero. Every time the boundaries were altered, the parameters from the Merton (1974) model estimation were used as a guideline to ensure that the boundaries weren't set too restrictive.

The convergence in the case of Longstaff & Schwartz (1995) model seemed to also depend more heavily on the starting values of the parameters. The upside was that the optimal parameters given by the algorithm did not seem to depend on the boundaries or the starting parameters as long as the boundaries were not too restrictive. They merely affected whether or not the convergence could be achieved. The downside was that the estimation became much more troublesome, since the convergence was harder to achieve.

In the next sections the actual results of the empirical testing of thesis are presented. The section 5 presents the result of the bond pricing tests and the section 6 the results of the hedge ratio tests. In the section 5 the extended Merton model and the Longstaff & Schwartz (1995) model are referred to as the Merton model and the LS model for brevity.

5 RESULTS OF THE BOND PRICING TESTS

5.1 Results for the Merton model estimated with PP, MP and VR methods

The table 2 presents the percentage errors in bond pricing for the extended Merton model estimated with the three simpler methods, namely the PP, MP and VR methods. The errors are presented for the bond price, yield to maturity and yield spread calculated as described in the sections 2.1.1 and 4.2.1.

The errors are always calculated as the value given by the model in question minus the market value divided by the market value. The table presents both the mean and the standard deviation for the percentage errors. The standard deviations are calculated with the usual method. A standard deviation of 10%, for example, means that on average the difference is 10 percentage points from the mean percentage error. Since the errors themselves are in percentages the standard errors are therefore also in percentages. The spread errors are bolded due to the fact that they are considered here to be the most informative measure for the estimation error as explained in the section 2.1.1.

	Pure proxy			**Mixed proxy**			**Volatility restriction**		
	Pricing error -%	Yield error -%	Spread error -%	Pricing error -%	Yield error -%	Spread error -%	Pricing error -%	Yield error -%	Spread error -%
Mean	10.48%	-21.06%	**-27.65%**	7.21%	-8.22%	**-6.08%**	11.75%	-50.14%	**-57.20%**
(S.D.)	(25.66%)	(111.75%)	**(123.80%)**	(29.23%)	(86.98%)	**(102.38%)**	(24.56%)	(56.17%)	**(64.21%)**

Table 2: Percentage errors for the Merton model estimated with PP, MP and VR methods

The results for the three simpler estimation methods are not very supportive for the Merton model in bond pricing. As in previous studies all the estimation methods seem to produce bond prices that are too high compared to the real prices observed in the market. Since an increase in yield lowers the price, overpricing means underestimation of both yield and yield spread.

Apart from the sign of the estimation error the results differ somewhat from those of the previous studies. Based on their simulation Li & Wong (2008) observed that the results from the MP approach resembled those of Eom et al. (2004) who used only one estimation method. Li & Wong (2008) concluded that the results of Eom et al. (2004) may have been mainly driven by the MP approach. The mean percentage error for spread in the study of Eom et al. (2004) was -50.42% and the standard deviation for the spread error was 71.84%. Compared to the results here these are more in line with results from VR approach.

Li & Wong (2008) did not report the percentage error for yield spread, but instead use yield differences given as the estimated yield minus the market yield. Their sample had mean yield difference of -22 (-121) basis points for VR (MP) approach and standard deviation of the yield difference of 289 (85) bp. The comparable figures here would have been -186 (-102) bp for the means and 982 (659) bp for the standard deviations. The results here for the VR approach, therefore, differ quite substantially from those of Li & Wong (2008). The mean yield spread error for the MP approach is, however, quite similar between the two.

The standard deviations imply that Li & Wong (2008) had results that were much more consistent. This may, however, be misleading since their figures are not in percentages. In our sample there are few observations were the issuer is near default and therefore the yield in these cases are extremely high. Therefore an error of 1000 bp in our sample might not be that high in relative to the market yield. For example the elimination of three abnormal issuers would drop the standard error of yield spread difference for VR approach below 300 bp.

Of the three simpler methods the MP method seems to produce prices that are closest to the market prices. The yield spread error of -8.22% and yield error of -6.08% on average are actually quite small. The mean error in yield in the study of Li & Wong (2008), for example, was -14.84% for the MP approach. The VR approach was seen to be clearly superior in terms of mean prediction errors by Li & Wong (2008). Here the VR approach produces on average much larger errors than the simplest PP approach, which is shown to be clearly upward biased in terms of prices by Li & Wong (2008).

The reason for the differing results may lie in the technique for the estimation of the total liabilities, which affects both the PP and MP approaches. Li & Wong (2008) and Eom et al. (2004) used the book value of total liabilities at the year's end as the proxy for the value of the firm's debt and calculated the historical volatility of the proxy asset value based on this (Li & Wong 2008, 754. Here the quarterly information of the firm's total liabilities is used and the book value of total liabilities is estimated to be the weighted average of the values reported in the two closest quarterly reports. Although the method in a way uses a future value of the book value of liabilities in the average this is deemed appropriate since many of the corporate actions, such as issuance of new debt, are observable on the market prior the quarterly report. The method assumes that the change in the book value of liabilities happens linearly during the rest of the quarter.

In the previous studies the volatility of the proxy asset value results purely from the movement in the market value of equity and may have made the asset volatility to be more downward biased. This in turn would result in more upward biased prices, which could explain the difference. The differences could also be related to differences in the estimation of the asset payout ratio, different interest rate environment and yield structure of the sample, different characteristics of the sample (e.g. liquidity) or to different market and period of time analyzed. Some of these factors are returned on later in the section 5.4.

Although the prices from the MP approach are not on average that far apart from the real ones compared to the previous studies, the variation of the estimation error, points out that the results for the MP approach are not very robust. The standard deviations for the percentage errors are substantially larger than those in the previous studies. Eom et al. (2004) had the standard deviation of the yield spread percentage error of 71.84% and the standard deviation of the yield error in Li & Wong (2008) was 9.84%. The standard deviation here is 102.38%, which means that the yield spreads with MP approach are often either extremely small or at least twice as high as the real yield spread. This is better illustrated in the scatter plots for the yield spreads presented in the figures 1 and 2.

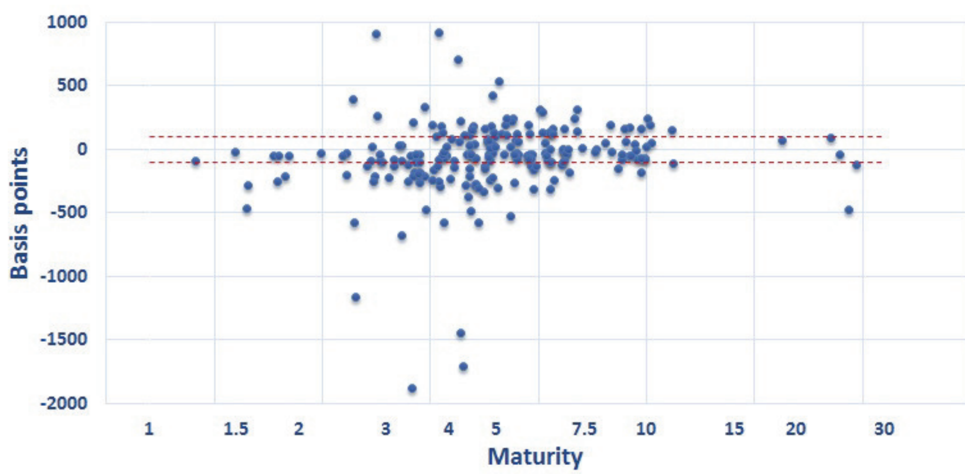

Figure 1: Yield spread errors based on the Merton model estimated with MP method

The figures above plot the basis point yield spread estimation errors against the bond's maturity. The two dotted lines depict the error of 100 basis points (i.e. 1%) and are for illustrative purposes. Few extreme observations do not fit between 1000 and -2000 basis point boundaries set for the table for illustrative purposes. These observation are caused by the issuers near default.

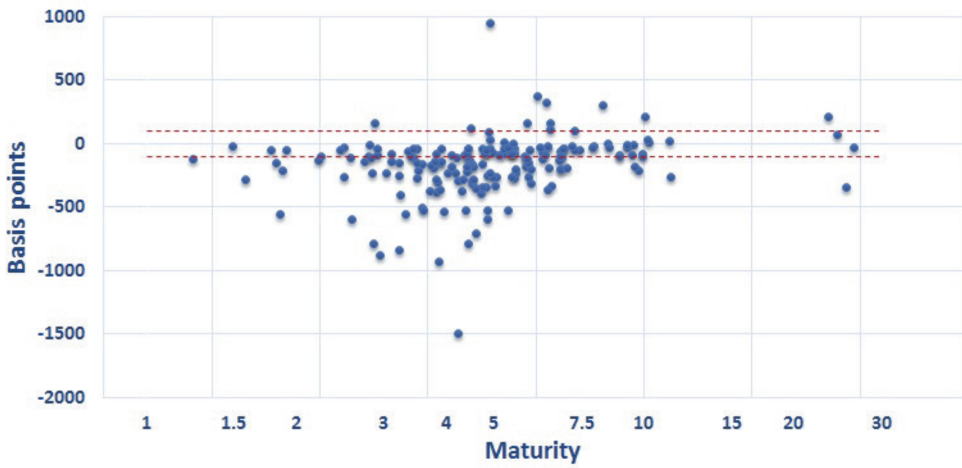

Figure 2: Yield spread errors based on the Merton model estimated with VR method

As is clear based on the figures, the accuracy of the MP approach is not as good as it would appear to be by observing the percentage errors alone. The errors from the Merton model estimated with MP approach are quite evenly distributed on both sides of zero, but the dispersion of the errors is significant. Therefore the accuracy of the approach is questionable even though the errors are relatively small on average. With VR approach, on the other hand, the underestimation of yield spreads is evident. Both approaches have errors that are quite dispersed, although with VR approach dispersion is somewhat smaller.

The same conclusion can be obtained by observing the absolute percentage errors presented in the table 3. The absolute percentage errors are calculated as the absolute value of the percentage errors, which measures the distance from the market price.

	Pure proxy			**Mixed proxy**			**Volatility restriction**		
	Pricing error-%	Yield error-%	Spread error-%	Pricing error-%	Yield error -%	Spread error-%	Pricing error-%	Yield error-%	Spread error-%
Mean	15.32%	82.97%	**99.97%**	11.99%	62.32%	**76.06%**	13.97%	62.59%	**75.79%**
(S.D.)	(23.09%)	(77.57%)	**(77.81%)**	(27.60%)	(61.10%)	**(68.61%)**	(23.36%)	(41.78%)	**(40.48%)**

Table 3: Absolute percentage errors for the Merton model estimated with PP, MP and VR methods

The mean absolute percentage error is actually worse than that of the VR approach even though the mean percentage error was substantially larger for VR approach. The absolute percentage spread errors in the study of Eom et al. (2004) were quite similar. Their absolute percentage error in spread had a mean (S.D.) of 78.02% (39.96%). These are again in line with the results from the VR approach here. Li & Wong (2008), on the other hand, did not report absolute errors.

Even though different estimation method for the total liabilities than in the previous studies was used the PP and MP approaches were not able to produce consistent estimates for the yield spreads. Perhaps due to the higher volatility resulting from the more realistic estimation of the total liabilities the Merton model was able to produce sufficiently high yield spreads unlike in many previous studies. The problem was that the spreads were either far too high or far too low. For the VR approach the underestimation of the yield spreads was quite significant.

Overall the absolute percentage spread errors were very large for all the estimation methods. On average the yield spreads differ by over three quarters of the actual yield spread for all the methods. The simplest of the structural models namely the Merton model combined with the simplest parameter estimation methods appears to be unable to accurately estimate the yield spreads of the corporate bonds.

Missing parameters like interest rate risk, liquidity or taxes cannot alone be the cause of fail of the PP and MP approaches since all these would increase the yield spread. Since both of these approaches had large number of errors that were significantly positive the inclusion of these parameters would not alone suffice for the fitting of the model. Therefore there must be either problems with either the parameter estimation or in the structural approach itself. In the case of VR approach the missing parameters might explain at least some part of the model's failure. As is implied by the rather large dispersion this may, however, be combined with a poor estimation of asset value and volatility parameters.

In the next two sections the analysis is, therefore, moved to the more sophisticated parameter estimation method, namely the maximum likelihood estimation, which is considered by Duan & Fulop (2009), Ericsson & Reneby (2005) and Li & Wong (2008) to be superior to the three approaches presented earlier. The section 5.2 presents the results for the Merton model implemented with ML estimation and the section 5.3 the results for LS model implemented with ML estimation.

5.2 Results for the Merton model with ML estimation

The percentage and absolute percentage errors from ML estimation for Merton model are presented in the tables 4 and 5. The results are given also for the LS model for comparison. These are returned to in the section 5.3. The results are given for the two different sample sizes used. The sample size in this case means the number of consecutive daily observations prior to the pricing date on which the market value of equity is matched to the equity pricing formula. The likelihoods of possible asset value paths for these time periods are then compared.

		ML estimation (100 obs.)			**ML estimation (250 obs.)**		
		Pricing error-%	Yield error-%	Spread error-%	Pricing error-%	Yield error-%	Spread error-%
Merton (extended)	Mean	16.89%	-70.29%	**-85.62%**	13.09%	-48.94%	**-62.89%**
	(S.D.)	(27.02%)	(35.14%)	**(33.85%)**	(27.44%)	(45.59%)	**(51.16%)**
Longstaff & Schwartz (K = Total liabilities)	Mean	9.39%	-32.90%	**-38.90%**	8.12%	-21.89%	**-26.33%**
	(S.D.)	(24.23%)	(53.08%)	**(64.26%)**	(25.79%)	(76.49%)	**(87.91%)**
Longstaff & Schwartz (K = 0.738 x Total liabilities)	Mean	13.07%	-39.46%	**-48.66%**			
	(S.D.)	(27.74%)	(128.85%)	**(153.41%)**			

Table 4: Percentage errors for the Merton and Longstaff & Schwatrz models with ML estimation

The results from the smaller sample size (100 obs.) are not promising regarding the Merton model's ability to produce sufficiently high credit spreads. The mean percentage spread error is actually worse than for any of the simpler approaches. A slight improvement compared to the PP, MP and VR approaches seems to be the consistency of the predictions. The standard deviations for the prediction errors are much lower than for any of the more simple approaches. This indicates that the problem may lie in the missing factors affecting the bond prices.

The mean absolute percentage spread error for the Merton model with 100 observations is of the same magnitude as the percentage spread error. The errors are therefore more of the same level across all the bonds compared to the previous approaches. If the Merton model is indeed able to capture the credit risk component of the yield spreads, it would mean that the credit risk would on average account for only 14.38% of the yield spreads. The capability of the Merton model to capture this credit risk component is studied more in the section 6.

		ML estimation (100 obs.)			ML estimation (250 obs.)		
		Pricing error-%	Yield error-%	Spread error-%	Pricing error-%	Yield error-%	Spread error-%
Merton (extended)	Mean	17.34%	72.92%	**89.17%**	14.10%	60.15%	**75.39%**
	(S.D.)	(26.74%)	(29.26%)	**(22.88%)**	(26.93%)	(29.14%)	**(29.65%)**
Longstaff & Schwartz (K = Total liabilities)	Mean	11.45%	52.22%	**64.46%**	11.27%	52.93%	**63.94%**
	(S.D.)	(23.33%)	(34.13%)	**(38.39%)**	(24.57%)	(59.30%)	**65.70%**
Longstaff & Schwartz (K = 0.738 x Total liabilities)	Mean	16.40%	80.80%	**99.42%**			
	(S.D.)	(25.91%)	(107.73%)	**(126.41%)**			

Table 5: Absolute percentage errors for the Merton and Longstaff & Schwartz models with ML estimation

The stronger similarity of the estimation errors across the bonds is further demonstrated by the figure 3

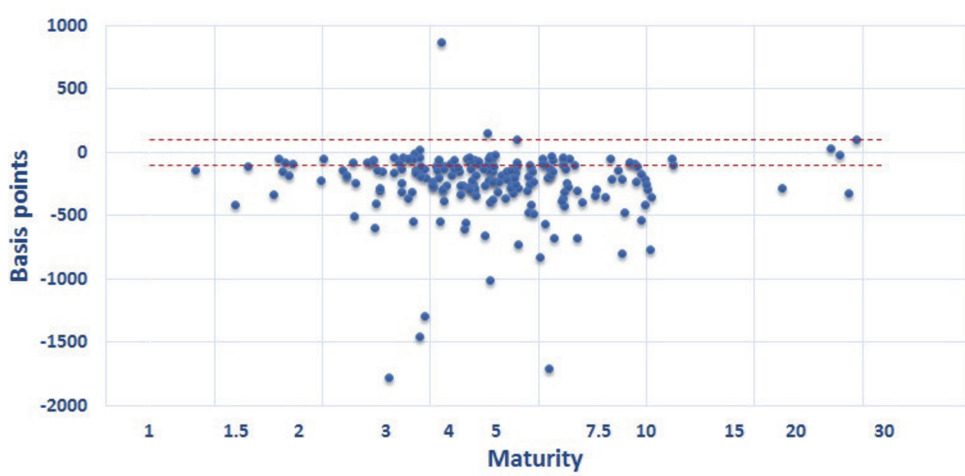

Figure 3: Yield spread errors based on the Merton model with ML estimation using 100 observations

With the VR approach there appeared to be serious underestimation of yield spreads for the bonds with maturity less than 5 years and more smaller errors or even overestimations for the bonds with maturities more than 6 years. With ML (100) estimation there are hardly any overestimation and the errors are quite packed on the same level between 0 and 500 bp apart from few outline observations. This is in line with the results of the simulation study of Ericsson & Reneby

(2005) who considered the VR approach to be biased approach for the parameter estimation.

The larger sample size (250 obs.) somewhat improves the mean percentage prediction errors. The mean errors are of the same magnitude than those of the VR approach. This might result from the fact that for the larger sample the movement in the equity value, and therefore in the asset value, caused by the volatility cannot as easily be accounted for the asset value drift. With smaller sample size the variation actually caused by the volatility might be more easily mistaken as the drift of the asset value. Indeed the higher estimated volatilities of the larger sample size lowered the bond prices and mitigated the underestimation.

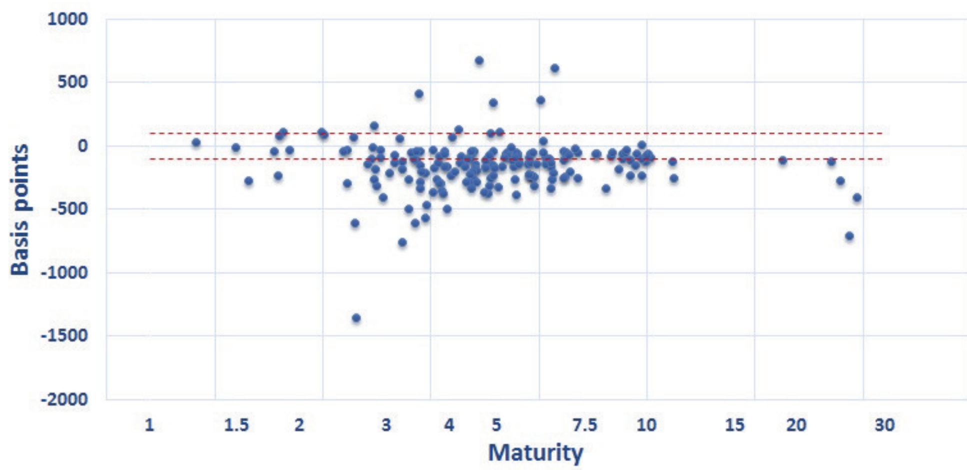

Figure 4: Yield spread errors based on the Merton model with ML estimation using 250 observations

The mean absolute percentage error for the ML (250) was smallest across all the parameter estimation methods. This is, however, only by a very small margin. The results for the Merton model in bond pricing are all in all quite poor even combined with ML estimation with larger sample size. This is in contrast with the results of Li & Wong (2008) who produced mean percentage yield errors of just -2.69% for the Merton model implemented with ML estimation.

When the recovery rate was changed to better correspond the more recent historical observations the spread errors came closer to zero on average. When the recovery rate was set as 37% of the face value MPE was -52.4%. While this is significantly smaller than with the higher recovery assumption, it does not remove the high standard deviation of the errors and therefore does not make the model much more accurate.

It was concluded in the study of Ericsson & Reneby (2005) that moving from a smaller sample size to a larger one, significantly improves the efficiency of the estimated parameters i.e. reduces the standard deviation and mean absolute percentage errors of the parameters (Ericsson & Reneby 2005, 728-729). However, the standard errors for the ML (250) errors are higher compared to the smaller sample size although they are smaller than for the any of the three simpler approaches. This is evident in the figure 4 which plots the yield spread errors for the larger sample size.

The errors are closer to zero on average, but seem to be equally or even more scattered than with the smaller sample size and there are also quite many overestimations. This seems to be contradicting the results of Ericsson & Reneby (2005) according to which the larger sample size should clearly improve the efficiency of the parameter estimators.

The study of Ericsson & Reneby (2005) was, however a simulation study, where they simulated asset value paths for 4 base scenarios where the firm's leverage and asset volatility were set either high or low. Based on the simulated asset value paths they computed the corresponding equity values in accordance with the corresponding equity pricing formula of the model in question. For these equity value paths they performed the ML estimation. This approach, therefore, relies heavily on the correctness of the given model.

The results of Ericsson & Reneby (2005) about the bias and efficiency of the estimators for the parameters would apply in reality only, if the model's assumptions were true. The assumption of a constant asset volatility, for example, might not be realistic and for the larger sample size the error related to the non-constant asset volatility would more clearly prevail. This could explain the higher standard errors for the larger sample size.

The missing parameters in the model would also make the results of Ericsson & Reneby (2005) fail in reality. In the next section the results for a more complex structural model namely the LS are presented. The LS model takes into account the possibility of default prior the maturity of the debt and the interest rate risk by allowing the risk-free rate to be stochastic. This might produce useful information for evaluating, if the problems with the Merton model are due to the unrealistic assumptions and missing variables.

5.3 Results for the Longstaff & Schwartz model

The percentage and absolute percentage errors for the LS model were presented in the tables 4 and 5. The model is implemented only by using the ML estimation. The three simpler approaches are not used for the LS-model since the ML estimation gave more consistent results in terms of standard errors for the Merton model. The mean absolute percentage error was also smallest for the ML (250) estimation.

For the three simpler approaches there were larger variation in the errors and more cases of overestimation of the yield spreads. A relevant question is whether the inclusion of default before maturity and stochastic interest rate could explain most of the yield spreads and produce accurate bond prices. Since the inclusion of default before maturity is expected to result in an increase of the yield spreads the overestimations produced by, for example, MP approach would likely be even worse if used with the LS model. The ML estimation should also be the most sound theoretically, when the assumptions of the model hold. For these reasons ML estimation is the only implementation method used for the LS model.

The LS model is implemented using two different setups for the default barrier. These are separately presented in the tables 4 and 5. The LS model implemented with the lower default barrier level somewhat improves the mean percentage errors compared to the Merton model. There are, however, few extreme overestimations of the yield spreads that bring the mean percentage error significantly closer to zero. This is visible in the large standard deviations of the percentage errors.

The large standard errors are, however, partly related to few extreme predictions for the yield spreads. The problem with ML estimation especially in the case of LS model was that some of the issuers had quite abnormal equity value paths and therefore the optimization algorithm of R did not always reach convergence. The results for these issuers could also depend heavily on the starting parameters. This was solved by starting the algorithm with various sets of starting parameters. Some of these abnormally large error's might be due to the inability of the algorithm to find the global optimum.

The estimation could be improved further by allowing more varying sets of starting parameters. This comes at the cost of estimation time. The estimation procedure for 10 different sets of starting parameters took for several hours for the whole sample. Even with faster computers and more efficient algorithm, the difficulty of the estimation procedure is likely to become a considerable factor.

Eliminating two issuers with the most extreme pricing errors would result in the standard deviation of the percentage spread error to drop in half and the mean percentage spread error to rise close to 60%. Based on this, it would appear that the LS model with lower barrier level would not significantly improve the more simple Merton model.

The abnormally large estimation errors might not be solely related to inaccuracy of the algorithm. For the barrier level of 100% there appears to be less such extreme errors. The lower barrier level may therefore be unrealistic and may have resulted in the default barrier for being unachievably low (spread underestimation) or in the estimated asset volatility that is unnaturally large (spread overestimation). Indeed the asset volatility for the lower default barrier is on average 18.4% and 17.1% for the higher default barrier.

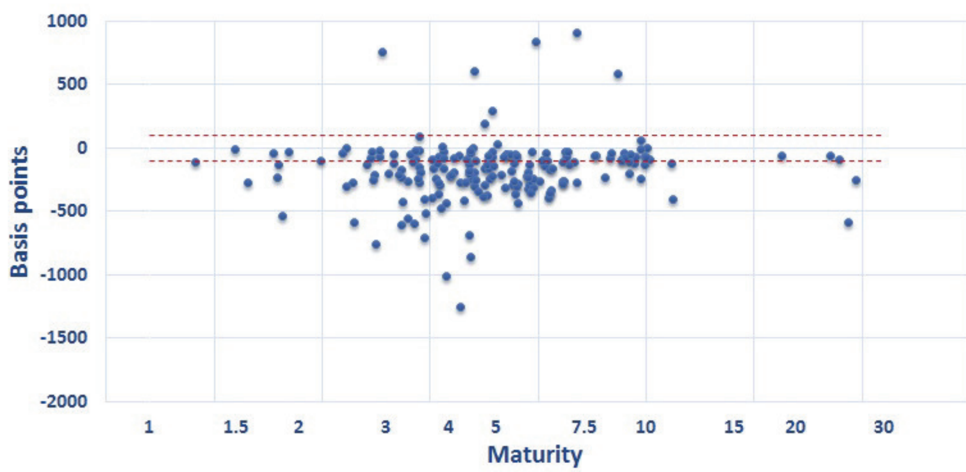

Figure 5: Yield spread errors based on the LS-model with ML estimation using 100 observations and $K = 0.738 \times B$

The figures 5, 6 and 7 plot the results for all the approaches used for the implementation of LS model. The figures point out that with the higher barrier level there are fewer extreme errors and the errors are more firmly packed as well as closer to zero. The longer sample size for the LS model with higher barrier level is not significantly improving the results. When a single extreme observation is eliminated from the larger sample, the results are almost identical for the two sample sizes with the larger sample size having slightly better results by all the measures.

The inclusion of the default before maturity and the stochastic interest rate seem to have improved the model performance and the spread errors are on average closer to zero than in the case of Merton model. Therefore the result had by Wei

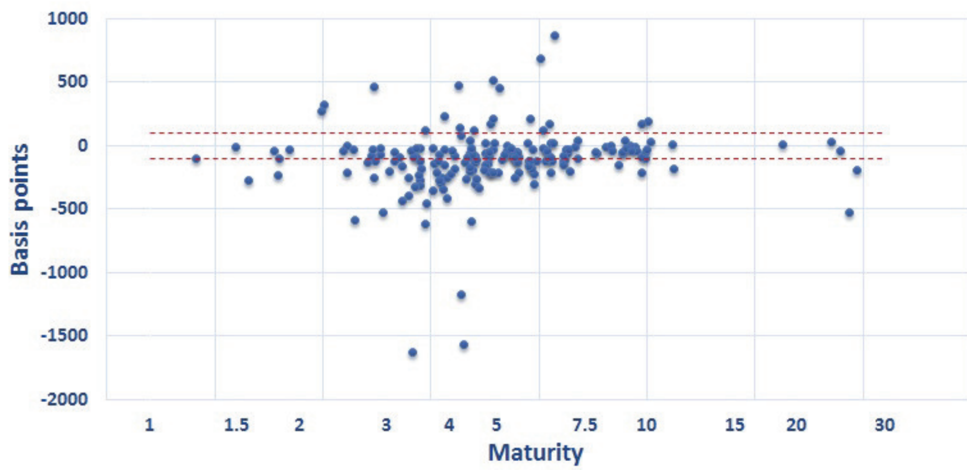

Figure 6: Yield spread errors based on the LS-model with ML estimation using 100 observations and $K = 1 \times B$

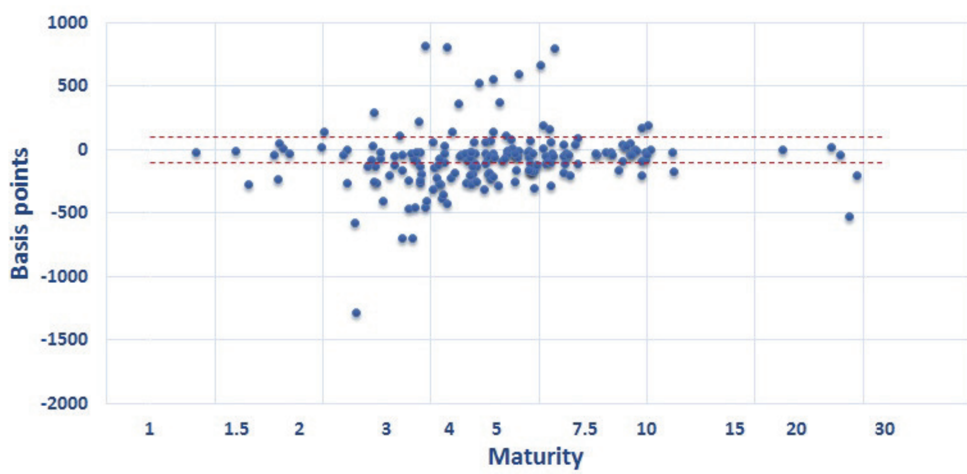

Figure 7: Yield spread errors based on the LS-model with ML estimation using 250 observations and $K = 1 \times B$

& Guo (1997) that the Merton model could explain larger part of the spreads than the LS model does not apply here. The results are more in line with the studies of Eom et al. (2004) and Li & Wong (2008) whose results indicated that the Merton model suffers from the most severe underestimation of spreads.

However, the strong results of Li & Wong (2008) do not hold in terms of the accuracy of the LS model. While the model produced spread errors closer to zero compared to the Merton model, the MPE was still -26.33% at best. When using the alternative recovery rate (37%) the MPE was actually quite close to zero (-

5.2% for the LS model with 250 observations). The high standard deviation of the errors, however, does not allow reclaiming the model as accurate. This points out that it is possible to generate high enough spreads with the LS model with reasonable assumptions, while the accuracy of the model seems to be unachievable.

When looking at the results of the LS model from different years it is revealed that in 2011 the MPE was actually positive (ca. 20–25%) when the model was implemented with the higher barrier level. This means that in 2011 there were actually quite severe overestimation of the spreads on average. The sovereign debt crisis in 2011 might have resulted in the bonds being more effected by credit risk and interest rate factors during that year.

Closer inspection revealed that the correlation between the asset returns and the risk-free rates was slightly positive that year (ca. 4%). The stochastic interest rate has therefore widened the spread, but in the year 2011 the effect appears to be too large resulting in overestimation. For the years 2012–2014 the correlation was negative, but still quite close to zero, which should still result in widening the spread but less so. During those years there were quite severe underestimation of the spreads. The LS model therefore seems to produce larger spread compared to the real ones when the correlation is larger. The accuracy of the model is, however, not that good in either of the cases.

All in all the LS model seems to bring the estimation errors closer to zero. However, the errors are still so significant that the ability of the LS model to price corporate bonds accurately is clearly questionable. One might expect the interest rate and default risk to account for the vast majority of the credit spreads. However, the results seem to indicate that these factors account for 70% at most for an average bond. The result is in contradiction with the results of Eom et al. (2004) and Li & Wong (2008) for whom the LS model had spread errors close to or clearly above zero depending on the model settings.

Other researchers, however, have found that there are variables besides those considered by the LS model that affect the credit spreads. In the next section the factors affecting the models' performance are more thoroughly analyzed. It is inspected whether these missing variables could explain the failure of both the models or whether there might still be issues with the parameter estimation methods.

5.4 Variables affecting the bond pricing results

In this section a deeper consideration is brought on the reasons why both of the models failed to produce accurate and sufficiently high credit spreads for corporate bonds. The factors that contribute to the pricing errors are analyzed via t-test similar to that performed by Eom et al. (2004). The different settings compared to the previous studies and their effect on the differences between the results are also discussed.

In the t-tests performed here it is tested whether a given variable has a statistically significant effect on the performance of the given model in bond pricing. Separate tests are done for a number of different variables possibly affecting the size of the pricing errors. In an individual test the bond sample is divided in half based on the median percentage spread error in the bond sample. For example, when considering the effect of maturity on the pricing errors, the bond sample is divided in half so that the other half contains bonds with larger spread errors and the other half contains the bonds with smaller spread errors.

In the case of maturity it is then tested whether the average maturities of those two subsamples are statistically different. The test statistic is assumed to follow a t-distribution and is calculated as the difference between the means of the two subsamples divided by the standard error of the whole sample. The null hypothesis is that the average maturity is the same for the two subsamples. The test is performed for both models implemented with ML estimation with the sample size 250.

If the large values of the variable are associated with less negative percentage prediction errors or with overpredictions, the test statistic will be large positive. If opposite is true, the statistic will be large negative value. In the case where there is no difference in a given variable between the bonds with different size of prediction errors, the test statistic will be close to zero. Critical value for the test statistic is ± 1.653 for the significance level of 5%.

The table 6 gives the t-statistics for the set of variables chosen for the testing.

All of the variables above were also consider in the t-test performed by Eom et al. (2004) except for liquidity. The default probability was not directly considered by Eom et al. (2004), but they included credit rating as a variable in their t-test, which should have a similar effect.

For the Merton model the number of years to maturity does not appear to have statistically significant effect on the pricing error. This is visible in the figures 3 and 4, where the pricing errors appear to be independent of the maturity. This

	Merton	Longstaff & Schwartz
Years to maturity	-0.20	2.69
Asset volatility	4.23	2.36
Asset payout ratio	2.85	2.62
Leverage	6.93	6.28
Default probability (1y)	4.44	1.73
Number of bonds per issuer	-1.04	1.16
Liquidity (bid-ask spread-%)	2.06	0.63
Liquidity (bid spread – ask spread)	-1.18	-0.83

Table 6: The t-statistics for the bond pricing errors

is quite surprising since in the study of Eom et al. (2004) the maturity had the strongest effect especially in the case of the Merton model with t-statistic being well over 4. Eom et al. (2004) concluded that for low leverage and low volatility bonds the underprediction with Merton model was quite severe and therefore, if the bond has a short maturity, most likely the spread generated by the model is too low. The ML estimation may have helped with this problem, since the average maturity of the firm's bonds is taken into account in the estimation of the volatility and leverage parameters.

For the LS model, on the other hand, there exists a quite clear statistical dependency between the size of the prediction error and the bond's maturity. Li & Wong (2008) concluded that the LS model works best for medium and long-term bonds. This would be in line with the results of the t-test since the test points out that larger percentage prediction errors are associated with longer maturity bonds. This is because there is underprediction with the LS model and the errors are on average negative and the larger half of the prediction errors is closer to zero. The better performance of the LS model with longer maturity bonds may be resulting from the interest rate risk component included in the model. The interest rate risk is naturally more severe for longer maturity bonds, since these have longer duration i.e. sensitivity to interest rate changes. The inclusion of the default barrier also has more time to take effect in the case of the long maturity bonds.

Since the theoretical foundation of the Merton model relies quite heavily on the assumption of simple capital structure one would expect the spread errors to be further from zero in case of firms with more complex capital structure. However, the number of bonds issued by the firm does not appear to be related to the size of the spread error. The sign of the coefficient is negative as one would expect, but it is not statistically significant.

In the case of the LS model, which is quite as simple in terms of handling the capital structure, the coefficient is opposite signed and also not statistically significant. This might be due the fact that the issuers with multiple bonds outstanding are often larger companies so the information about these companies can be of better quality. The bonds of these companies might also be more liquid. All in all it therefore seems that the number of bonds issued is not as important for the models' performance as one might expect.

The firm's leverage and asset volatility have a clearly significant effect on prediction errors for both models. This is not surprising since these are such key parameters in both models. The t-statistics for the leverage are also strikingly similar compared to the results of Eom et al. (2004) who had corresponding coefficients of 6.47 and 6.23 for the extended Merton and the LS model respectively. In both of the studies the relationship between the pricing errors and both of these parameters was clear.

If an inaccurate estimation of these parameters was the only source of pricing error, one would expect that larger pricing errors were accompanied by the larger values of these parameters. For the bonds that have low leverage or low asset volatility the pricing errors should be small because the effect of the parameters on the spread would be smaller. (Barsotti & Viva 2015, 99.) In this case the worst underestimations of spread should be related to lower leveraged and lower volatility bonds.

If there were also other factors affecting the pricing errors, it could be that for the lower credit quality bonds the credit risk accounts for larger part of the spread. These bonds usually have higher leverage and volatility and therefore the underestimation should be more serious with lower leveraged and lower volatility bonds. Many studies have indeed shown that credit risk explains only small part of the spread for the higher rated bonds (Elton, Gruber, Agrawal & Mann 2001 and Huang & Huang 2012).

The t-statistics also support this view. More serious underestimation seems to be indeed related to bonds with lower leverage and volatility. Therefore it could be that the ML estimation is able to correctly estimate these parameters, but there exists factors affecting the credit spreads that are not taken into account by the models. In line with the previous research these could be explaining larger part of the spread for the higher rated bonds. The t-statistics for the default probability are also supportive of this. The one-year default probability of the firm seems to affect the performance of both models and the underestimation seems to be more severe for the bonds with lower default probability.

However, there could still be problems with measuring the volatility and leverage correctly. For example, an erroneously high volatility could be just complementing the missing parameters. With the uncertainty of both the missing variables and the parameter estimation methods the reasons for the models' failure is difficult to explain. In order to better assess the models' ability to measure credit risk the focus should be on the modeling of the hedge ratios or default probabilities. The former of these two will be returned into in the section 6.

The liquidity of the bond is often recognized to have an effect to the bond price. Poor liquidity of an instrument often causes the market price of the instrument to deviate from the theoretical price. This is because the theoretical price is often based on the absence of arbitrage and creating a replicating portfolio. Poor liquidity makes holding the replicating portfolio impossible in practice without a significant transactions costs.

Since our models rely heavily on these assumptions the effect on liquidity on the model performance is presumably significant. The effect of liquidity has also been witnessed in many empirical studies. Bonds are also often seen as more illiquid compared to stocks. The illiquidity of European corporate bonds is also seen to be more severe than that of the US corporate bonds, which are used in many previous studies (BlackRock 2016). By glancing the bond data many of the bonds seem quite illiquid.

To measure the effect of liquidity on the spread error, the difference between the bond's bid and ask prices (i.e. bid-ask spread) is used. The calculation of the liquidity measure is done in two different ways. The first way is similar than is used by for example Huang & Huang (2015). It measures the liquidity of the bond as bid-ask spread per market price of the bond. This is the premium an investor must pay per invested Euro to buy the asset and instantly sell it again.

The t-statistic is significant for the Merton model and insignificant for the LS model, but the relation goes opposite way than one might expect. The model less underestimates the spreads for the bonds that are more illiquid. This seems contradictive, but the explanation might be that the bonds that are illiquid are often those that have also higher credit risk. As noted earlier the credit risk component often amount for a larger percentage of the spread for these bonds, which might be the reason for the sign of the relation.

The second measure of liquidity controls the size of the bond's spread. The second measure calculates the bid-ask spread as a difference of the yield spread based on the bid price and the yield spread based on the ask price divided by the average of the two. This way the bonds with different sizes of spread could be better compared in terms of liquidity. The sign of the t-statistics is this way more like one would expect, meaning that when bid and ask prices get further apart the price of the bond somewhat rises, which would cause more underprediction with illiquid bonds.

The t-statistics are, however, not significant and definitive conclusions cannot be drawn. The insignificance of the coefficients in this case may be result of bonds with higher credit risk having also weaker liquidity, which would result again in two opposite effects on the spread error. The effect of liquidity is therefore hard to measure in the case of spread errors. In the time series regressions in section 6 the effect of the changes in the bond's liquidity to the excess return of the bond was modeled, which showed that the changes in liquidity affect the excess returns on the bond.

The asset payout ratio seems to also have a significant effect on the performance of both of the models. The asset payout ratio rises the price of the put option by lowering the risk-neutral drift of the asset value and therefore lowers the bond price. On the other hand, the financially stonger companies often pay larger dividends. Therefore the relation between payout ratio and different credit classes is unclear.

When the different components of the asset payout ratio were examined separately it was revealed that the coupon payments seem to be the driving factor affecting the relationship between the asset payout ratio and the pricing errors. The dividend payments seem to have no statistically significant effect on the pricing errors while the stock repurchases have statistically significant but adverse relationship. This would be in line with the other factors affecting the pricing errors since the lower rated issuers usually pay larger coupons while financially stronger issuers make more stock repurchases and pay larger dividends.

All in all it seems plausible that the errors for both of the are caused by the variables missing from the models. It may therefore still be that the credit risk part is correctly measured by the models. However, definitive conclusions cannot be drawn according to whether the models manage to capture the credit risk component of the prices correctly and whether or not the parameters of the models are correctly estimated.

In the next section the ability of the Merton to produce correct hedge ratios and therefore to measure credit risk component of the prices is studied. Based on the results of Schaefer & Strebulaev (2008) the hedge ratio tests may produce valuable information in the model's most important application i.e. credit risk assessment.

6 RESULTS OF THE HEDGE RATIO TESTS

6.1 Regressions for market-implied hedge ratios

In this section the examination of the structural models continues with tests for the models' ability to produce correct hedge ratios. In other words the analysis is expanded to the second moment predictions. The rationale behind this approach was described in the section 3.2.1. The aim is to gain more thorough evidence of the empirical validity of the structural models and better understanding of the reasons why the models might fail in producing correct bond prices. Most of all this section attempts to measure whether the Merton (1974) model is able to capture the credit risk component of the bond prices.

The tests performed for the hedge ratios follow the study of Schaefer & Strebulaev (2008), who were the firsts to study the structural models' ability to produce correct hedge ratios. Since then the subject has also been examined by Barsotti & Viva (2015), who had fairly opposite results and left the need for more research. The focus here is more on the study of Schaefer & Strebulaev (2008), since Barsotti & Viva (2015) relaxed the assumption of normal distribution for the asset returns, which goes beyond the scope of this study. Barsotti & Viva (2015), however, report their results also by using the normal distribution and therefore the results in this section have comparability to their results also.

The first step in the hedge ratio tests performed here is made to determine linkage between the market bond returns and the underlying variables suggested by the structural models. This is done by regressing the market bond returns with the market returns on equity and risk-free interest rate. The purpose of this regression analysis is to quantify the effect of the changes in the price of equity and risk-free interest rate on the market price of bonds. This regression represents the situation in reality for which the hedge ratios suggested by the models can later be compared to.

The importance of these regressions lies in the regression coefficients of the excess equity return. If the hypothesis that the Merton model is able to produce correct hedge ratios is true, then these regression coefficients should be equal to the formula in the equation (19) presented in the section 3.2.1. The R^2 of the regressions is also of importance. The R^2 of the regressions tells what part of the excess bond returns is explained by the excess returns on equity and risk-free rate.

The changes in the risk-free interest rate are included even though the risk-free rate is considered constant by the Merton (1974) model. Some extensions of the

Merton (1974) model such as the Longstaff & Schwartz (1995) model, however, consider the risk-free rate to be stochastic, which is why it is included in the regressions by Schaefer & Strebulaev (2008).

The regression analysis is performed for the monthly excess returns over the one-month risk-free rate. In the study of Schaefer & Strebulaev (2008) one-month U.S. treasury bill is used. As this study is performed on the European corporate bonds the German government bonds are used as the basis for the risk-free return.

The rate of return is calculated for each bond j between months $t-1$ and t as:

$$r_{j,t} = \frac{P_{j,t} + AI_{j,t} + I_{j,t}C_j/N_j}{P_{j,t-1} + AI_{j,t-1}} - 1,$$

where $P_{j,t}$ is the clean price of the bond j at the end of the month t and $AI_{j,t}$ is the accrued interest rate from the last coupon payment. The indicator function ($I_{j,t}$) takes value one if the coupon falls between t and $t-1$. This is multiplied by the annual coupon (C_j) per coupon frequency (N_j). Therefore, if the coupon is paid during the month over which the return is calculated, it is taken into account in the return. The excess return is then calculated as $\overline{r}_{j,t} = r_{j,t} - rf_{1m,t}$, where $\overline{r}_{j,t}$ is the excess return and $rf_{1m,t}$ is the one-month risk-free rate (Schaefer & Strebulaev 2008, 5.)

Schaefer & Strebulaev (2008) use the excess return on equity and the 10-year treasury rate as the explanatory variables. The risk-free interest rate is therefore included in the regression in two ways. First, it is used in calculating the excess returns on equity over the one-month risk-free rate. This takes into account the changes in the overall level of the risk-free curve. Second, the excess return of the long-term rate (10-year rate) over the short-term rate (one month rate) is used as an explanatory variable. This takes into account the changes in the steepness i.e. the "twists" in the risk-free curve.

The assumption with the two risk-free interest rates made here is that at the beginning of the month the investor could invest on the one-month risk-free rate and therefore the risk-free benchmark is the yield estimated from the beginning of the period term structure. The 10-year rate, on the other hand, measures the changes in the long-term risk-free rate. Therefore the return for the 10-year bond is percentage change in the 10-year constant maturity bond price during the month.

The first regression equation is constructed based on the above as follows:

$$\overline{r}_{j,t} = \alpha_{j,0} + \alpha_{j,E}\overline{r}_{E_{j,t}} + \alpha_{j,rf}\overline{rf}_{10y,t} + \varepsilon_{j,t}.$$

In their study Schaefer & Strebulaev (2008) arrange the bonds in rating categories based on the bond's credit rating at the time of its first appearance. A different approach is, however, used here. Since the sample of bonds used here is smaller, the concern is that every rating category might not have sufficient amount of observations. Therefore the approach used here is as follows. Each continuous sequence of observations is divided in equal parts so that the length of each part is at the least the minimum number of consecutive observations.[6] Multiple criteria for the minimum number of observations is used.

Schaefer & Strebulaev (2008) use 20 consecutive observations as a minimum for the inclusion of the bond in the sample. This minimum rule is used here as one of the criteria for the division of the observations. Since many of the price series have quality price data only starting from 31.10.2013 (i.e. 15 consecutive price observations), the regressions are made also by using 15 consecutive observations as the minimum. After the division the average default probability is calculated for each part as an average of default probabilities during the sample period and the samples are rearranged into artificial credit rating categories based on that. The reason for using default probabilities instead of the exact credit ratings was explained in the section 4.1.

The approach used here makes the individual sequences, for which the regressions are made, more equal sized. The differences in the lengths of the series therefore do not affect the results. The approach used here also makes the series shorter on average. Since the credit quality of an issuer may change quite dramatically during the sample period, this approach may be, in this sense, more accurate than the one used by Schaefer & Strebulaev (2008). The small number of observations in the series might, however, affect the statistical significance of the individual regression coefficients, although the number of series to be tested is this way enlargened. The approach also allows a single firm to have multiple series and therefore individual firms could be overrepresented.

The results from this first regression are presented in the table 7. The table includes the average coefficients for the rating category and the t-statistics in parentheses as well as the coefficients of determination (R^2). The last row (N) gives the average number of consecutive observations with the number of time series observations in parentheses.

Since the number of series fulfilling the criteria is not large enough to sufficiently cover the whole range of credit rating categories the main results in the table 7

[6]Exactly equal division is not of course always possible, which is why the first divided parts of the series might have one "extra observation" so that every observation of the series is used.

are given only for the whole sample and rating categories AAA–A and BBB–C. The results are also presented by using a more loose criteria (15 consecutive price observations) and for the method used by Schaefer & Strebulaev (2008) who did not divide the time series in near equal parts as is done in this study.

	20 obs.			**15 obs.**			**20 obs. (Undivided)**
	All	**AAA-A**	**BBB-C**	**All**	**AAA-A**	**BBB-C**	**All**
α_0	0.30 (2.88)	0.15 (1.92)	0.39 (2.80)	0.35 (1.20)	0.25 (3.73)	0.42 (0.93)	0.28 (2.64)
α_{rf}	28.91 (5.40)	34.61 (7.67)	25.46 (3.69)	47.74 (2.79)	37.09 (9.19)	54.97 (2.09)	27.76 (4.84)
α_E	6.11 (5.48)	3.35 (3.02)	7.78 (5.05)	6.45 (4.83)	2.33 (2.89)	9.24 (6.02)	5.97 (6.06)
R_2	0.37	53.27	27.13	39.39	52.53	30.46	0.34
N	25.29 (45)	26.25 (17)	23.71 (28)	17.10 (89)	17.49 (36)	16.85 (53)	36.71 (31)

Table 7: The results from the regression 1 for the market-implied hedge ratios

The most important information in the table 7 lies in the regression coefficients for the excess returns on equity (α_E). The coefficient in this study for the whole sample is around 6 basis points depending on the exact testing method used. For the upper credit rating classes the coefficient is significantly lower ranging from 2.33 to 3.35 basis points depending on the estimation method. The difference compared to the lower credit classes, for which the coefficient is ranging from 7.78 to 9.24, is significant.

The difference between credit rating classes is not surprising since, as was noted in the section 5, the prices of the lower rated bonds are more strongly affected by the changes in the credit quality. This observation is also supported by many studies considering the determinants of the credit spreads. The credit spreads for the bonds in the upper credit rating classes are often seen to be consisting factors other than credit risk, for example taxes and liquidity (Elton et al. 2001 and Driessen 2005).

The average coefficient for the whole sample differs quite substantially from the one obtained by Schaefer & Strebulaev (2008). The coefficient in their study was 3.79 basis points, which is significantly lower. However, this may be explained by the overall credit quality of the samples in these two different studies. Although default probabilities are used here instead of the exact credit ratings, the credit qualities are comparable since the division here mimics the division used by credit rating agencies.

In the study of Schaefer & Strebulaev (2008) over half of the bonds belonged to the rating categories from AAA to A. Here the respective portion is just over one third. The lower overall credit quality in this study could explain why the hedging coefficient is much larger here. Indeed, the coefficients for the different credit categories are quite similar compared to the study of Schaefer & Strebulaev (2008). The difference in credit quality also corresponds the difference between the periods analyzed. After the financial crisis the uppermost credit ratings have been harder to obtain by corporate issuers.

The overall coefficient for the return on the risk-free interest rate is also closer to the coefficient for the lower rating categories in the study of Schaefer & Strebulaev (2008). In their study the overall coefficient for the returns on risk-free rate was 49.59 and was therefore much larger than here. This is also most likely due to the differences in the overall credit quality of the two samples. The unnatural interest rate environment experienced during the period analyzed in this study, might also have affected the coefficients. Due to the extremely small and even negative short-term interest rates the selection of appropriate benchmark for the short-term interest rate is not straightforward as discussed in the section 2.1.1. Constructing of the yield curve therefore involves more judgement, which might be reflected in the coefficients.

The coefficient for the risk-free interest rate is higher for the upper credit ratings in the main test which uses 20 observations as the minimum criteria. The changes in the risk-free rate therefore affect the bond price more in the case of better rated bonds, while the changes in equity price, have stronger effect on the bonds in the lower rating classes. The same result was obtained also by Schaefer & Strebulaev (2008). The interpretation of the effect is a bit more complicated than that of the effect of equity returns.

When only the changes in the risk-free rate are taken into account in the regression, the regression coefficient should be equal to the ratio of the modified durations of the bonds analyzed and the 10-year risk-free bond (Schaefer & Strebulaev 2008, 7). The difference between the rating categories might be therefore due to the different durations of the bonds in the two categories. Indeed the average duration in the AAA–A category is approximately 4.66, while in the BBB–C category the corresponding figure is 3.85. The coefficients are, however, much smaller than the ratio of the durations would imply.

The interest rates are often seen to be negatively correlated with the equity returns (Longstaff & Schwartz 1995 and Collin-Dufresne, Goldstein & Martin 2001). Due to this negative correlation the coefficient should be smaller when the equity returns are added to the regression. However, Schaefer & Strebulaev (2008) find that the coefficients for risk-free rate tend to increase, when the equity returns are included in the regression. According to Schaefer & Strebulaev (2008) this was surprising, since it contradicts the assumption of the negative correlation. In this study, however, the opposite effect is witnessed as the coefficients for risk-free rate increase when equity returns are included. The result here is therefore more supportive of negative correlation observations.

The coefficient for the lower rating category in the case of 15 observation is, however in contradiction. It seems to be higher than the ratio of the durations. The high standard error, however, makes the coefficient barely statistically significant implying that the lower number of consecutive observations might not be sufficient to obtain reliable estimates.

The negative correlation is also in line with the structural framework. In the structural models the interest rate has two opposing effects to the equity price. First, it lowers the value of the future payments as it raises the risk-neutral discount rate. Second, it raises the risk-neutral drift of the firm value and therefore raises the value of equity. The first of these effects dominates in the Merton (1974) model, while for the lower quality bonds the second effect is relatively more important compared to the higher quality bonds (Schaefer & Strebulaev 2008, 6.)

According to Hull, Predescu & White (2005) this second effect should lower the sensitivity to the overall market returns for lower quality bonds and for the higher quality bonds most of this sensitivity to market returns is explained by the first effect. For the lower category bonds this is evitable here as the coefficient for the equity return is much lower when the risk-free rate is not included in the regression. The second argument is not so clearly observed, since the coefficient for the equity return is statistically significant also for the higher quality bonds, when the risk-free rate is included. This would imply that there are other factors in the equity returns than that of resulting purely from the changes in the risk-free rate.

There are, however, none of the AAA rated bonds in the sample here, which is why this second argument of Hull et al. (2005) cannot be fully tested. The study of Schaefer & Strebulaev (2008), however, points out that the coefficient for the

AAA rated bonds is indeed statistically insignificant, when the risk-free returns are included. Since the bond returns in the uppermost rating category are not affected by equity returns it is questionable whether the Merton (1974) is suitable for analyzing these bonds.

The R^2 of the regression is somewhat lower than the one obtained by Schaefer & Strebulaev (2008). The excess returns on the risk-free asset and equity seem to account for just above one third of the variation in the excess bond returns. The respective R^2 in the study of Schaefer & Strebulaev (2008) was 0.51. This again might be accounted for the different credit qualities. The interest rate factor explains a larger portion of the excess bond returns in the higher credit categories, while the credit risk explains larger portion for the lower quality bonds. For the higher rated bonds other factors such as liquidity are more dominant, which may explain the difference between credit categories(Collin-Dufresne et al. 2001). The same relationship between R^2 and rating was also observed by Schaefer & Strebulaev (2008).

The European corporate bonds may also be more illiquid than U.S. bonds used by other researches. Indeed as the changes in the liquidity of the bond, measured by the percentage change in bid-ask spread were included in the regression as an explanatory variable the R^2 of the regression increased significantly. This implies that for the European corporate bonds the changes in liquidity are an important factor for the bond price changes. This in turn might affect the performance of structural models in generating hedge ratios.

When the risk-free returns are not included the R^2 drops significantly to near 7% level. This is not surprising since the credit risk alone is often found to explain only a small portion of the credit spreads. For the lower quality bonds the portion is larger, but still under 10%. This is in line with past research where it has been concluded that even for lower quality firms the credit risk accounts for only a small portion of the credit spread (Collin-Dufresne et al. 2001, Elton et al. (2001) and Huang & Huang (2012)).

Based on the t-statistics, all the coefficients for the excess return on equity are statistically significant at the 1% significance level. Besides from the AAA rated bonds Schaefer & Strebulaev (2008) also had statistically significant coefficients for all the rating classes. The coefficient for the AAA rated bonds in their study was not statistically different from zero. The significance of the overall coefficients in both studies demonstrates that there exists a relationship between the excess returns on the firm's bonds and its equity.

The calculation of reliable t-statistic in this case is not, however, straightforward. The coefficients reported here are averages of the individual coefficients in a particular rating class. The t-statistics cannot, however, be averaged in the same way. The individual t-statistics are not showing statistical significance on average since the number of observations in the individual series is small. After taking into account the joint probability of the coefficients of these multiple series the average coefficients for a particular rating class become significant.

Difficulty raises also from the fact that the individual series of observations are significantly overlapping. This could introduce correlation between the individual coefficients, which in turn affects the standard errors. Ignoring this overlapping nature of the series could lead to a significant underestimation of the standard errors and therefore lead to unjustified conclusions about the statistical significance. The fact that the series are only partially overlapping also needs to be taken into account. (Schaefer & Strebulaev 2008, 18.) When calculating the t-statistics the correlation between the residuals of two series is calculated only for the part that they are overlapping.

The selection of the minimum number of overlapping observations as the criteria for including the correlation in the calculation involves some judgement. Barsotti & Viva (2015) use 21 observations as the minimum in their calculation. Since the number of observations per series is smaller here, the criteria is set to 12 overlapping observations, which is corresponding a full year of observations. Choosing a smaller number as the minimum tends to raise the standard errors. All in all the t-statistics are not, however, strongly affected by the choice of the criteria and overall the coefficients remain statistically significant regardless of the selection.

The coefficient α_0 is also statistically significant for the whole sample and BBB-C category. This is quite surprising, since in the study of Schaefer & Strebulaev (2008) this constant coefficient was insignificant. The significance of the coefficient might imply the existence of some constant excess premium required from corporate bonds such as premium related to taxes or liquidity. The effect of the changes in liquidity is returned in the section 6.

In the next section the results of this first regression are used to test the validity of the Merton (1974) model. A simulation is performed on the model in order to obtain theoretical hedge ratios implied by the model. These are then compared to the results of the first regression. In the third regression the theoretical coefficients are used to test the model without using simulation and thus hopefully obtaining more thorough evidence about the validity of the model.

6.2 Simulated and formal regressions based on the Merton model

In this section the hedge ratio analysis is continued by inspecting the theoretical hedge ratios implied by the Merton (1974) model. The question is whether the Merton (1974) model is able to generate similar hedge ratios than the ones observed in reality. A simulation is first performed for the bonds used in the first regression by using bond-specific parameters. The approach again follows the study of Schaefer & Strebulaev (2008).

The simulation is performed as follows. First, the bond-specific parameters such as maturity, asset payout ratio and the book value of debt are obtained for the bonds used in the first regression. Second, the unobserved parameters in the Merton (1974) model are computed. This is done similarly as in the proxy approach. This approach is used although the maximum likelihood estimation is considered to be the most sound theoretically and it gave the most consistent results in the bond pricing tests in the section 5. The problem is that in this test the hypothesis is that the bond and equity prices might contain factors other than credit risk and therefore matching the equity price to the Merton (1974) model call option formula would be inappropriate and would potentially contaminate the results. (Schaefer & Strebulaev 2008, 9.)

The unobserved parameters are calculated by using the method used by Barsotti & Viva (2015). In order to perform the simulation the unobserved parameters asset value, mean rate of return and asset volatility are needed. Barsotti & Viva (2015) calculate the rate of return on the firm's assets as:

$$r_{V_t} = (1 - L_{t-1})r_{E_t} + L_{t-1}r_{D_t},$$

where the leverage L_{t-1} is calculated as in the pure proxy approach. The mean return and volatility are then calculated based on the return formula above. The asset value is obtained as the denominator of the L_t parameter.

The parameter r_{D_t} is the rate of return on the bond tested. The usage of the return on the bond has, however, a few drawbacks. The method can obviously be used only for bonds that are actively trading. It also assumes that the volatility of all the firm's debt is the same and since the corporate debt can be illiquid, some of the observed volatility could be spurious (Schaefer & Strebulaev 2008, 9). To test whether the choice of the parameter estimation method affects the results. The simulation is also performed by using the parameters obtained in the ML estimation and also by calculating the rate of return on asset value by using the pure proxy firm value.[7] The results of the simulation are not largely affected by the choice between these estimation methods.

The parameters are calculated based on the whole monthly time series used in the first regression. Similar division of the time series is therefore not used as in the first regression. This is because via simulation the number of observed paths becomes larger and sufficient number of observations is obtained for all the rating categories used. Therefore the division is not needed to raise the number of observations per rating class.

After obtaining the needed parameters for each bond used in the first regression and for each point in time that the necessary parameters were available a random asset value paths are generated. For each point in time an asset value path of 48 months is generated and based on this the values of the firm's equity and the firm's debt implied by the Merton (1974) model are calculated. Based on these simulated bond and equity value paths a following regression is performed:

$$r_{j,t} = \alpha_{j,0} + h_{j,E} r_{E_{j,t}} + \varepsilon_{j,t}.$$

The average regression coefficients as well as the average R^2 values are calculated for each rating category used based on this regression.

Unlike in the study of Schaefer & Strebulaev (2008) the equity volatility is calculated from daily data and then converted into a monthly volatility. Number of observations used for this was 150 days. Only the series for which the number of consecutive observations was at least 20 were used.

	All	**AAA-A**	**BBB-C**
h_E (simulation)	6.38	6.67	6.18
h_E (regression 1)	6.11	3.35	7.78
t-statistic	(0.24)	(3.00)	(-1.04)
R^2	0.80	0.82	0.78

Table 8: The hedging coefficients from the simulation

The results from the simulation are presented in the table 8. The t-statistic is against the hypothesis that the hedging coefficient from the first regression is equal to the coefficient from the simulation. Based on the results the overall coefficient is

[7]This means that the returns on the bond tested are not used, but the return is calcul ated based on changes in proxy firm value (= Book value of liabilities + Market value of equity).

strikingly similar between the real market data (regression 1) and simulated data based on the Merton (1974) model. The hedging coefficients for the two regressions are not statistically different and the overall coefficients are almost identical. The standard error for the coefficient from the first regression is 1.11 basis points, which is quite high for making robust conclusions. This is because the high standard error makes the confidence interval for the coefficient wide. The striking similarity of the coefficients is, however, very supportive for the Merton (1974) model. The same kind of similarity was obtained also by Schaefer & Strebulaev (2008).

Surprising result is the relative values for the two rating categories. The coefficient for bonds rated from AAA to A is actually higher than the coefficient for the lower quality bonds. This is exactly opposite than the result from the regression 1 and the simulation result of Schaefer & Strebulaev (2008). This would imply that the effect of the equity returns is stronger for bonds in the higher rating categories. This is opposite than the usual result that credit risk accounts for higher portion of the credit spread for the lower quality bonds. Due to the opposite relation between the two regression, the coefficient for AAA-A is statistically different for the two regressions. The coefficient for lower quality bonds is not statistically different for the two regressions, but this is mainly accounted for the high standard error (1.54) for this rating class.

The reason for this surprising and counterintuitive relationship may be found by studying the comparative statics of the hedging coefficient. The volatility of the firm's asset value lowers the call option delta (assuming that the option is in-the-money) (Schaefer & Strebulaev 2008, 7). Therefore it raises the hedging coefficient as can be seen from the equation (19) in the section 3.2.1. The asset volatility is expected to be higher for the lower quality bonds. Indeed the average asset volatility for AAA-A class is 0.12 on average and 0.14 for the BBB-C class. Based on this only, the hedging coefficient is supposed to be higher for the lower quality bonds.

The leverage, however, has two opposing effects on the hedging coefficient. First the quasi-market leverage lowers the call option delta and raises the hedging coefficient. On the other hand the leverage has an opposite effect through the multiplier $\frac{1}{L} - 1$. As leverage tends towards one that multiplier tends to zero. The option delta, however with some simplifying assumptions, cannot go below 0.5 when the option is in-the-money. Therefore the effect in the multiplier $\frac{1}{L} - 1$ dominates and the leverage lowers the hedging coefficient. The average leverage for AAA-A class is 0.46 and 0.69 for the BBB-C class. The comparative static with respect to rating class therefore depends on the relative values of the volatility and leverage parameters in the different rating classes as well as other factors.

Compared to the average parameters for different rating categories the asset volatilities and leverages calculated here differ substantially from those obtained by Schaefer & Strebulaev (2008). In their study the average asset volatility ranged from 0.21 to 0.28 depending on the rating, while the average leverage ranged from 0.1 to 0.66. Clearly compared to the study of Schaefer & Strebulaev (2008) the leverage parameter in this study has a relatively stronger effect. When asset volatility is low, its effect on the hedging coefficient is less significant. This is a potential explanation for the odd relationship between the rating and the hedging coefficient. Estimation errors in the unobserved parameters could have significantly affected the results. Due to for example stronger illiquidity of the European corporate bonds the asset volatilities could have been underestimated.

By using similar asset volatilities than used by Schaefer & Strebulaev (2008) the relationship between the rating and the hedging coefficient becomes more logical. If asset volatility of 0.20 is used for the bonds in the rating category AAA-A and 0.25 for the bonds in category BBB-C the respective average hedging coefficients are 8.64 and 12.15. Although these are much higher than in the first regression, this demonstrates that the odd relationship in the simulation might indeed be due to the weak parameter estimation method. None of the estimation methods used here, however, produced coefficients that would have matched the coefficients in the first regression.

Besides the asset volatility, there might have also been difficulties in the estimation of leverage and asset payout ratio. Indeed as discussed in the section 4.2.2 the estimation of the asset payout ratio has received surprisingly little interest in the empirical researches cited here. The calculation method of the asset payout ratio can, however, have a significant effect on the model performance especially in the low interest rate environment experienced during the sample period.

In the simulation performed here a simple method for the calculation presented in the equation (20) was used, but excluding the coupon payments. More sophisticated method would recognize the effect of the debt coupon payments, issuance of new debt and equity as well as redemptions of bonds. A straightforward inclusion of the asset payout ratio into the call option formula does not either take into account that the value of equity is raised by the present value of future dividends. The estimation of the market leverage also affects the calculation of the asset payout ratio and this was taken into account in the ML estimation, but not here. The simple method used here can therefore also responsible for the odd relationship. For example when asset payout ratio is set to zero and the volatilities from the study of Schaefer & Strebulaev (2008) are used, the resulting coefficients would appear to be quite close to the ones in the regression 1.

The assumptions concerning the leverage and the book value of debt used in the simulation are also quite simplifying. The book value of debt is assumed to be constant while the asset value changes. While in short term this might be true, in longer term the firm might aim towards a target leverage ratio. Schaefer & Strebulaev (2008) do not specify whether they use the assumption of a constant book value of debt or an assumption of a target leverage. This kind of target leverage assumption would require taking into account that the asset value would change at a rate of return specified by the bond and equity return as well as due to the additions to book value of debt.

All in all the simulation seems to work best for the bonds with an average credit rating. Indeed the coefficient for the bonds in BBB category would have been 6.00 and the respective coefficient from the simulation would have been 5.63. The leverage for these bonds is higher compared to bonds in the study of Schaefer & Strebulaev (2008) and the asset volatility is much lower. According to Barsotti & Viva (2015) the Merton (1974) model appears to work best for the big, high leveraged firms with low asset volatility, which seems to be the case here also. All the three studies also seem to point out that the model works best for the bonds in the average rating categories.

In addition to these two regressions, yet a third regression is constructed. The objective here is to test the theoretical framework of the Merton (1974) model without using simulation. While the advantage of the simulation here is that it accounts for the effects of non-stationarity, non-linearity as well as the observation frequency, the third regression is attempted to be more realistic setup to generate appropriate hedging coefficients based on the model (Schaefer & Strebulaev 2008, 8).

The third regression implemented is of the following form:

$$\overline{r}_{j,t} = \alpha_{j,0} + \beta_{j,E} h_{E_{j,t}} r_{E_{j,t}} + \alpha_{j,rf} \overline{rf}_{10y,t} + \varepsilon_{j,t},$$

where the parameter $h_{E_{j,t}}$ is the theoretical hedging coefficient implied by the model. If the theoretical assumptions of the model hold, the regression coefficient $\beta_{j,E}$ should be equal to one.

Linking the changes of the bond and equity prices via theoretical hedge ratio implied by the model is bound to provide more universal results than the simulation. However, since the value paths are now not simulated, the number of observed paths becomes much smaller. Since the real world data can also be more noisy, the correct estimation of the model parameters becomes even more vital.

By aggregating the results for rating categories the effect of abnormal observations is more limited. Therefore in order to provide even more aggregation, for this third regression the asset volatilities were estimated as the average for each rating category. The rating categories used for this aggregation were AA, A, BBB, BB and B. Since the AAA and CCC-C categories had no observations the results for these categories could not obviously be provided. The hedging coefficient was calculated based on the average volatility of the rating category in which the bond belonged to in that month. The maturity used in calculation was set to be the average for the rating category in question as was done also by Schaefer & Strebulaev (2008).

When using the assumption of normally distributed returns reliable estimates for the firm-specific hedge ratios cannot be computed due to multicollinearity problems (Barsotti & Viva 2015, 97). Therefore in the study of Schaefer & Strebulaev (2008) the estimated hedge ratios were averaged for the rating categories. The average hedge ratio for the rating category the bond belonged to at time t was then used in the regression. By using more complex distributional assumptions Barsotti & Viva (2015) were able to perform a corresponding regression by using firm-specific hedge ratios.

In the main results presented here the hedging coefficients were not averaged out, but firm-specific hedge ratios with average volatilities and maturities per rating category were used instead. The decision not to average out the hedge ratios and use the firm-specific hedge ratios instead is to illustrate the differences between individual firms in the results.

To avoid the drawbacks of using the individual bond returns in the calculation of the asset volatility, mentioned earlier, Schaefer & Strebulaev (2008) used a little different method. First by using 3 years of monthly data they calculated the equity volatility for the individual bonds. Second, they calculated the average bond returns for each rating category. Bond was included in the calculation, if it belonged to the rating category for past 15 months. They then set the volatility of the firm j's debt at month t as equal to the average volatility of that rating category in that month. Finally they calculated the correlation of the debt and equity returns as the average for each rating category. Based on this they calculated the average volatilities for each rating category for each month.

Since the length of the data and the number of bonds used here was much smaller, it was not possible to calculate the equity volatilities based on 3 years of monthly data. The volatilities were therefore calculated based on the volatility of each part of the divided sample paths. The number of observations is therefore at least 20 and approximately 25 on average.

Due to the same reason the average volatilities are calculated only for the whole sample period and not for each month. The earliest sample paths are from the year 2010 and the largest number of paths is from the years 2013–2014. The period is therefore not as wide as in the study of Schaefer & Strebulaev (2008). However, since the sovereign debt crisis in 2011 might have affected the volatilities, the usage of average volatilities for the whole sample period might have somewhat affected the results. The resulting volatilities are somewhat higher than those used in the simulation and are ranging from 0.16 to 0.31 depending on the rating category.

The results of this third regression are presented in the tables 9, 10 and 11.

	All	**AA-A**	**BBB-BB**	**AA**	**A**	**BBB**	**BB**
β_E	3.66	6.10	1.33	19.47	0.75	0.79	4.70
t-statistic	(1.20)	(1.13)	(1.24)	(1.18)	(-1.41)	(-1.11)	(2.70)
N	43	21	22	6	15	19	3

Table 9: The results from the third regression without asset payouts

	All	**AA-A**	**BBB-BB**	**AA**	**A**	**BBB**	**BB**
β_E	1.10	0.79	1.33	1.44	0.75	0.79	4.70
t-statistic	(0.36)	(-0.76)	(1.24)	(N.A.)	(-1.41)	(-1.11)	(2.70)
N	38	16	22	1	15	19	3

Table 10: The results from the third regression without asset payouts and two of the issuers removed

	All	**AA-A**	**BBB-BB**	**AA**	**A**	**BBB**	**BB**
β_E	0.26	0.18	0.34	0.19	0.18	0.23	1.04
t-statistic	(-13.1)	(-11.40)	(-8.93)	(-4.28)	(-17.53)	(-12.69)	(0.11)
N	43	21	22	6	15	19	3

Table 11: The results from the third regression with asset payouts

The regression was first performed without considering the effect of the asset payout ratio on the results. These results are presented in table 9. The t-statistics in the table are calculated against the theoretical value of one. The overall result is quite poor, but for the rating categories A and B the beta coefficients are quite close to the theoretical value one.

The poor overall performance of the Merton (1974) model is evidently caused by the beta coefficients for the upper and lower end rating categories. These are ridiculously high compared to the theoretical value. This means that the hedging coefficients produced by the model were significantly lower than the real coefficients.

The bonds in the category AA consisted mainly of four observations from bonds issued by Danone and one observation from a bond issued by JCDecaux. The relatively low leverage caused the delta for these bonds to be practically equal to one making the hedging coefficient way too small. This resulted in beta coefficients to be either ridiculously large positive or large negative.

This has also caused standard errors of the coefficient to be extremely high. This is why the coefficients are not statistically different from zero for all categories except BB despite the fact that for the categories AA-A, AA and for the overall regression the coefficients are extremely large. The coefficients are not statistically different from zero and they have therefore no statistical significance. The high standard error in this case prevents one from making robust conclusions about the overall regression result.

By eliminating these five observations from the AA category we get the results presented in the table 10. Here the overall coefficient is actually surprisingly close to the theoretical value. This may indicate that the Merton (1974) model might be able to produce accurate coefficient on average for, but only for quite restricted group of bonds.

This is supported by the fact that by including also the five extreme observations, but averaging out the hedging coefficients for each rating category the overall result is close that in table 10. When average hedging coefficients are used for each rating category the overall beta coefficient is 1.20 while the coefficients for categories AA, A, BBB and BB are 0.53, 1.02, 1.16 and 3.80 respectively. This indicates that with quite aggregated values and for quite average bonds the Merton (1974) model is able to produce realistic hedging coefficients.

The question remains that what is the reason for the failure of the model in the case of the five extreme observations. A common factor for these bonds was that the call option delta was very close to one which lead to very small hedging coefficients (i.e. large βs). Extremely low leverage estimated, for example for Danone, makes the default probability for the company to be unrealistically low. Since the companies have had quite large cash outflow in the form of dividends and stock repurchases the risk-neutral drift of the company should be lower and make the default more possible event. The inclusion of the asset payout ratio to the hedge ratio might therefore make the hedge ratios more realistic.

In table 11 the results are presented by including the asset payouts in the calculation. The asset payout ratio is included by taking into account the dividend payments, stock repurchases and the coupon payments. The present value of the expected dividend payments is also included to the call option formula by using the formula in the equation (22).

Based on the table 11, the inclusion of the asset payout ratio corrected the high beta coefficients for AA and BB categories, but overall this is a clear overcorrection. The coefficients for the categories AA, A and B have dropped close to zero and according to the t-statistics they are clearly statistically different from 1. The inclusion of the asset payouts has, however, resulted in the coefficient for BB category not being statistically different from the theoretical value. This might indicate that asset payouts are significant for producing sufficiently high hedging coefficient for lower rated bonds, but since the number of bonds in this category is low one cannot make robust conclusions.

Overall it seems, however, that the inclusion of dividends made the hedging ratios unrealistically large by lowering the risk-neutral drift too much. Based on the results it is clear that the correct determination of the asset payout ratio is crucial for the model's ability to produce realistic hedge ratios. It, however, remains unclear how this should be done in order to make the Merton (1974) model produce accurate results across the rating spectrum. The Leland & Toft (1996) model, which better takes the complexity of the firm's capital process into account, might therefore be more suitable.

All in all the tests made here showed no single method by which the Merton (1974) model could produce accurate results across all the bonds. In the case of this sample the reason for the failure can only be inspected by analyzing the characteristics of individual bonds. Based on the previous researches the reasons might be that the high rated bonds are largely unaffected by the changes in credit risk and their price movements are more determined, for example, by the changes in the risk-free term structure (Schaefer & Strebulaev 2008, 2 and Collin-Dufresne et al. 2001).

This might be the reason why the Merton (1974) model was producing either extremely large positive or extremely large negative beta coefficients and extremely small coefficients when the asset payouts were included for the high rated bonds. The fact that the changes in the equity and bond returns are possibly not correlated for these issuers makes the model unlikely to work with any implementation method.

For the bonds with lower credit quality the volatilities might have been underestimated, for example, due to the poor liquidity. The inclusion of the asset payouts seemed to correct this problem, but the results for other categories were poor and there is no theoretical reasoning for the inclusion of the asset payouts to be correct only for the lower rated bonds.

It seems that the Merton (1974) model performs best for the bonds in the average rating categories. For the far ends of the rating spectrum the model seems to fail more often. When the overall results for all the tests performed for the hedge ratios are examined, it seems that when inspecting the most average of the bonds the Merton (1974) is often able to produce quite realistic hedge ratios.

The results obtained by Schaefer & Strebulaev (2008) are somewhat different from those obtained here. Although their results also suggested that the Merton (1974) model produces more accurate hedge ratios for the firms in the average rating classes, their results were far more supportive for the model. In their study only the coefficient for AAA rated bonds was not statistically different from zero and their beta coefficients were not statistically different from one for any of the rating classes. This was a very strong result in favor of the model.

The results of Barsotti & Viva (2015) were not that encouraging for the model. Although their main results were presented using more complex distributional assumptions and firm-specific hedge ratios, they also reported the results using normal distribution and average hedge ratios. In their sample the high standard error prevented the making of robust conclusion, although the model could not be mainly rejected (Barsotti & Viva 2015, 99). They also experienced more underestimation of the hedge ratios for lower quality bonds and less so for the investment grade bonds. Their overall results suggested that the Merton (1974) model is unable to produce correct firm-specific hedge ratios. These results cannot be fully compared here due to the different assumptions about distribution of asset returns.

All in all it seems that based on results obtained here it is possible for the Merton (1974) model to produce accurate hedge ratios especially for the bonds in the average rating categories. As for the ability of the Merton (1974) model to produce correct hedge positions for the far ends of the rating spectrum and with firm-specific hedge ratios, this question needs to be left for the future research with larger sample sizes.

Even with the average bonds a significant care has to be taken regarding the selection of the model form and in the estimation of the model parameters. Since the time of the sample periods used by Schaefer & Strebulaev (2008) and Barsotti & Viva (2015) the interest rates have declined extremely low, due to which the choices regarding the estimation of risk-free drift parameter can have significant effect on the results. Care should be taken also when considering the specifics of the capital process of the firm and the asset payout ratio.

7 SUMMARY AND CONCLUSIONS

The financial world has certainly become more complex during the last decades. Part of this is due to the introduction of new complex financial products, not least of which are the credit related products such as credit default swaps (CDS) and collateralized debt obligations (CDO). While this has helped the more efficient allocation of capital and risk, there also lies significant risks in this development caused by wrong application of the financial models or by wrong kinds of incentives. This was evidenced during the financial crisis staring in 2007–2008, which has an effect still present almost a decade later.

For these reasons it is important to study and understand the theoretical concepts behind the complex financial products and risk factors as well as the applicability of those theories in practice. This is important for the people who are applying these concepts in practice, namely the bankers and investors. It is also important for the regulators who are trying to limit the risky behavior while also maintaining the efficient functioning of the economy (e.g. in the form of Basel accords) as well as for the people responsible for the supervision of these regulations, namely the chartered accountants and internal auditors.

This is why credit risk modeling was chosen as the subject of this thesis. The structural framework was chosen as the approach for credit risk modeling to be studied due to the strong theoretical background behind the approach. Indeed the foundations of the structural framework rely on the same theories as the famous Black & Scholes (1973) model for option pricing. The original Merton (1974) model views the equity as a call option on the firm's assets and based on this construction the model presents the pricing formulae for the firm's equity and its debt.

Despite the wide success of its counterpart in option pricing, the Merton (1974) model for corporate bond pricing has not gained such popularity among practitioners nor strong support from researchers. The model has not performed that well in pricing of corporate bonds in empirical testings. The usual conclusion has been that it underestimates the yield spreads and therefore overprices the bonds. This has been accounted, among other things, for the factors affecting corporate bond prices in reality, but not taken into account by the Merton (1974) model.

Due to these limitations many researchers have developed models that build on the original Merton (1974) model, but relax its strict assumptions. More recent models have relaxed, for example, the assumption that the default cannot occur prior to the maturity of the firm's debt. Other key assumption that has been modified concerns the interest rate risk. For example, in the Longstaff & Schwartz (1995) model the risk-free interest rate follows a stochastic process of its own while in the Merton (1974) model it is assumed constant.

Another notable limitation concerning the original Merton (1974) model, as well as the later improvements, is the difficulty of parameter estimation. The models are based on the firm's asset value process, but neither the value of the firm's assets nor the parameters of its stochastic process are directly observable from the market. The asset value and asset volatility parameters in the pricing formulas therefore need to be somehow estimated from other market data.

The most elementary way is to use the sum of the market value of equity and the book value of liabilities as proxy for the asset value (proxy method). A more frequently used method is to use the call option formula for the value of equity along with an application of the Itô's lemma to construct two equations from which the parameters can be solved (volatility restriction method). More recent studies have suggested more statistically coherent estimation methods. Maximum likelihood estimation for structural framework developed by Duan (1994) has been discussed by the researchers, but it lacks sufficient empirical testing.

In spite of the various attempts to improve the original Merton (1974) model none of the structural models for bond pricing with any of the parameter estimation methods have received universal acclaim from the researchers. The overall conclusion among researchers seems to be that the models cannot accurately price corporate bonds. An exception is a study by Li & Wong (2008) who implemented several structural models with several parameter estimation methods and concluded that the models could produce bond prices surprisingly close to those observed in the market when implemented with the maximum likelihood estimation. Their results, however, have not been widely noted among other researchers and therefore more evidence on the subject is needed.

Overall the researchers view that the structural models fail in corporate bond pricing due to factors not taken into account by the models. These include the illiquidity of corporate bonds, which results in that the assumptions behind arbitrage pricing do not hold since the assets cannot be continuously traded in an active market without transaction cost.

Other factors include the taxation as well as stochastic volatility and jumps in the firm value process. These both have been included in the structural models with some promising results (Qi & Liu 2010 and Bu & Liao 2014). The problem is that when the assumptions of the model are being relaxed the model becomes more and more complex. Since, for example, the Longstaff & Schwartz (1995) model already has quite tedious pricing formulae the excessive complexity of the models becomes a serious issue in the perspective of applicability. Besides according to Huang & Huang (2012), although there are promising results concerning, for example stochastic volatility, none of the factors mentioned can alone explain the poor performance of the models in bond pricing.

Despite the poor performance in empirical research, the models continue to be applied in practice, for example by many large banks (Bharath & Shumway 2008, 1341). In order to understand the reasons behind this, one must understand the different applications for structural models. Indeed while structural models might fail in corporate bond pricing, they still can turn out to be useful in credit risk assessment. The models do not have to be applied as such, but instead some of the ideas presented by the models can be used separate from the complete structural framework. In fact this seems to be the main application since the models or some of the ideas included in them are used mainly in credit risk assessment as part of wider a modeling framework.

There is also some support from the research for this kind of application. Schaefer & Strebulaev (2008) argued that structural models could well capture the credit risk component of the corporate bond prices. Their results implied that even the simplest of the structural models, namely the original Merton (1974) model, can produce accurate hedge ratios on an average level. This is quite supportive for the models' applicability in credit risk assessment since the ability to produce correct hedge ratios implies that the model is capturing the credit risk component of the bond price correctly.

There are, however, contradicting opinions as well. Barsotti & Viva (2015) performed a similar testing as Schaefer & Strebulaev (2008), but using more realistic distributional assumptions. They concluded that the firm-specific hedge ratios produced by the Merton (1974) model were close to the ones observed on the market only for very limited part of the bonds and rating categories. According to their opinion the validity of the Merton (1974) model in credit risk assessment is not certain and care should be taken before applying the model for generation of hedging positions.

Jarrow & Protter (2004) argued that it should be clear based on the research concerning structural models that the model's parameters are not observable from the market, which would in their opinion imply that a reduced-form model outperforming the structural one could be composed. The structural approach would in their opinion be suitable merely for certain types of management decisions. Bharath & Shumway (2008), on the other hand, tested the Merton (1974) model in default prediction and concluded that while the distance-to-default based on the model is not sufficient statistic for default probability the functional form of the model produces useful information for default prediction and could be used to improve the performance of a reduced-form model.

The empirical results from the bond pricing tests in this thesis were quite supportive for the prevalent view among the researchers. The Merton (1974) model implemented with various parameter estimation methods was unable to produce accurate yield spreads. The model showed similar severe underestimation of yield spreads than in the prior research.

Perhaps surprisingly the maximum likelihood estimation was the one producing the most severe underestimation of yields spreads. Instead the validity of maximum likelihood estimation lied potentially in the fact that it produced the smallest standard errors and absolute percentage errors. In the case of the simpler estimation methods the standard errors as well as the absolute percentage errors were ridiculously large indicating that while on average they produced more accurate yield spreads, the average deviation from the market yield spread was so large that the methods can hardly be considered accurate.

The consistent underestimation with maximum likelihood estimation could be more in line with the assumption that there exists factors adding up to the yield spreads not taken into account by the Merton (1974) model. Therefore the more complex Longstaff & Schwartz (1995) model was implemented using maximum likelihood estimation. While the model was not underestimating the yield spreads as severely as the Merton (1974) model, it was not able to produce accurate yield spreads. The model was implemented by trying two different default barrier assumptions.

The most accurate yield spreads on average produced the Longstaff & Schwartz (1995) model with default barrier equal to total liabilities and implemented with maximum likelihood estimation based on 250 consecutive price observations. Although the percentage error and absolute percentage error were smaller than in the case of the Merton (1974) model the standard error was larger in the case of the Longstaff & Schwartz (1995) model. The errors were also quite scattered implying that the model could not produce accurate yield spreads.

Although there were also some serious overestimations of the yield spreads the Longstaff & Schwartz (1995) model seemed to overall bring the errors closer to zero. It therefore seems that the default prior maturity and the interest rate risk are important factors for corporate bond pricing and cannot therefore be neglected. Overall there, however, seemed to be also other factors that add up to the yield spreads and are ignored by both of the models. The Longstaff & Schwartz (1995) model with aforementioned calibration settings was still underestimating the yield spreads by 26.33% on average.

The results are not supportive for the study of Li & Wong (2008) whose result was that both of the models implemented here could produce yields differing only by couple of percents from the market yields on average. This could mean that Li & Wong (2008) were able to calibrate their models in a way that produced accurate yield spreads on average, but with absolute error being high on average. The relatively large standard errors in their study could be an indication of this. All in all the results of Li & Wong (2008) could not be repeated here although different setups concerning, for example the recovery rate, were tried. The results were resembling more those obtained by Eom et al. (2004) who concluded that both of the models produces highly inaccurate yield spreads on average.

Several different factors possibly affecting the magnitude of the yield spread errors in this thesis were analyzed. Three of these factors were clearly affecting the performance of both of the models. These were leverage, asset volatility and asset payout ratio parameters for all of which the high value of the parameter was associated with less severe underestimation of the yield spread.

It is not surprising that these parameters had a clear effect on the performance of both of the models since they all are key parameters in the models and their correct implementation is far from straightforward. For the former two parameters the relationship implied that the models work better for bonds with high leverage and asset volatility. This is in line with previous research where it is often found that credit risk accounts for larger part of the yield spread for bonds with lower credit rating (Elton et al. 2001 and Huang & Huang 2012). For these bonds the mentioned parameters are obviously larger on average. For the better rated bonds the other factors than credit risk often explain the majority of the credit spread. This relationship was supported by the t-test results for default probability.

The asset payout ratio was in addition divided into different components in the t-test. The coupon payments seemed to be the dominant variable explaining the relationship with larger asset payout ratio and smaller underestimation. The amount of stock and debt repurchases was adversely related to the pricing errors. This might be resulting from that firms in a strong financial condition are doing more stock repurchases and firms in weaker financial are paying larger coupons. All of these factors therefore seem to relate to the financial position of the firm, which is in line with the previous observations of the credit spread puzzle.

The maturity of the bond seemed to be important for the performance of Longstaff & Schwartz (1995) model, but not for the Merton (1974) model. While the stochastic interest rate and default before maturity seem to be an important for the model performance it seems that this helps more the bonds with longer maturity as these factors have more time to take effect. This is in line with the results of Li & Wong (2008) who concluded that the Longstaff & Schwartz (1995) model performed better with longer maturity bonds while for shorter maturity bonds the Leland & Toft (1996) model with endogenous default barrier was better. Combining the best features of these models might be a relevant topic for the future research. This was also suggested by Li & Wong (2008).

It was shown here that it is possible for both of the models tested to produce large enough credit spreads in many occasions. Since there are many factors affecting the performance of the model it might be possible to adjust these factors and achieve accurate prices on average especially with small sample like the one in this thesis. This might have been the case in the study of Li & Wong (2008). The calibration would, however, most likely have no theoretical basis and out of sample testing would probably produce less favourable results.

All in all it seems unlikely that the structural models would be applicable in corporate bond pricing as such. Implementation of the models used here is quite tedious and there are several steps that deem a lot of consideration and which have no established procedures supported by the research. The results are also far from accurate. Adding up missing factors to the models such as stochastic volatility could improve the models, but at the same time the implementation would become even more demanding while the results are still not guaranteed to be sufficient.

The models could be useful for bond pricing if they could be utilized merely for quantifying the credit risk component of the price. This requires the models to accurately catch the credit risk component in the prices. Therefore in order to test the performance of structural models in credit risk assessment the Merton (1974) model was tested for generating hedge ratios. In this testing the model's framework could be used to model the part of the bond prices caused by the credit risk and the factors affecting the bond prices but not related to credit risk could be left outside.

First a regression analysis was composed in order to quantify the relationship between bond returns and returns on equity and risk-free rate. These were then compared to the hedge ratios produced by simulated asset value paths based on the Merton (1974) model. The hedge ratios generated based on the simulation corresponded the empirical hedge ratios surprisingly well especially for the bonds in average rating categories. For the whole sample and for the rating group BBB–C the simulated hedge ratio and the empirical hedge ratio were not statistically different.

These results are quite well in line with those of Schaefer & Strebulaev (2008) who had even stronger results in the Merton (1974) model's favor. The results obtained here were not that supportive for the model in the case of bonds in the upper rating classes. The simulated hedge ratios for the rating group AAA–A were statistically different from the empirical hedge ratios while in the study of Schaefer & Strebulaev (2008) there were better correspondence. The results imply that the functional form of the Merton (1974) model used to calculate the bond and equity returns in simulation might be useful in credit risk assessment at least for an average bond. This is line with the conclusions of Bharath & Shumway (2008) who noted that the functional form of the model can improve the performance of a reduced-form model in default prediction.

The test with hedge ratios was continued as the hedge ratios based on the models were computed without using simulation, but instead using the observed equity prices and estimated asset value and volatility parameters. The results of the test were not straightforward. When using hedge ratios based on the average maturity and asset volatility for the rating category in question together with bond-specific leverage, the beta coefficient was 3.66 compared to the theoretical coefficient one. However the difference was mainly caused by few bonds in AA rating category which had ridiculously large differences from the theoretical coefficient. By using the average hedge ratios for a rating category or by removing the outlying observations the average beta coefficient was 1.10–1.20 and was not statistically different from the theoretical value of one.

It was noted that the outlying observations had very low leverage which made the default nearly impossible according to the model and produced hedge ratios close to zero. This could be made more realistic by taking into account the asset payout ratio, which accounts for the cash outflow and decreases the risk neutral drift hence making default more likely according to the model. The inclusion of the asset payout ratio, however, produced unrealistically high hedging coefficients for all but BB category, which put in question whether the inclusion of the asset payouts was made correctly. The implementation related to asset payout ratio was not perhaps surprisingly considered by Schaefer & Strebulaev (2008) or Barsotti & Viva (2015) who performed similar testing.

It could be that the Merton (1974) model does not qualify for the firms making large dividend payments and stock repurchases. The inclusion of the asset payouts was done in a way that assumes these payments to be made continuously while in reality these cash outflows happen discretely usually couple of times per year. This makes it questionable whether the implementation could be made realistically enough for the model to work correctly and is also an example of the simplifying assumptions concerning the capital structure behind the Merton (1974) model.

Therefore a model that takes better into account the complexity of the firm's capital process, such as the Leland & Toft (1996) model, might be more suitable.

Overall it seemed that the Merton (1974) model was able to produce accurate hedge ratios, but only in a very average way. The asset payout ratio could be used to correct the hedge ratios in right direction, but in most cases this was an overcorrection. The results were quite well in line with those of Barsotti & Viva (2015) who concluded that the Merton (1974) model is only able to produce accurate hedge ratios for a very limited part of the bonds in the middle of the rating spectrum. The results here were not as favourable for the model as those of Schaefer & Strebulaev (2008) in whose test the model produced accurate hedge ratios for nearly all of the rating categories.

It seems that the usefulness of the structural models may lie only in the credit risk assessment applications and for the corporate bond pricing less complex methods are likely to be superior in terms of cost-benefits. The degree of usefulness in credit risk assessment is, however, still under question. Based on prior research the models can prove to be quite useful in this field, but the models' usage as such in either generating hedging positions or predicting default is not fully supported by the research. The results from this thesis do not distinct from this. It seems that the application of the models should not be done by fully accepting all the models' assumptions and the limitations of the whole approach should be acknowledged.

There are of course limitations in the work done in this thesis that affect the credibility of the conclusions made. First of all the data used here was quite limited compared to the reference researches. This was the case especially in the hedge ratio tests where the previous researches had used bond samples measured in thousands while in this thesis the sample was limited to few dozens of bonds. The length of the time series used were also smaller since the data for lot of the bonds included consecutive observations for a relatively small period of time. The limited availability of data with sufficient size and quality has been a common problem among researchers in this field.

Due to the restricted applicability of the models for different kinds of bonds and issuers the overall bond data was narrowed down. Since the sufficient size of data was an issue here the narrowing was not done as strictly as in previous researches. Therefore the sample was not restricted to issuers with only couple of bonds. However based on the t-tests the number of bonds per issuer did not affect the bond pricing errors on average.

The concern lies also on the quality of the bond prices itself. In some of the cases the bonds were clearly illiquid as the price stayed still for consecutive months.

Some of these observation were eliminated from the tests, but it the liquidity may be a significant issue for the remaining sample also.

The liquidity may also be even more significant issue in this thesis than in the reference studies since those studies have mainly used data from U.S. corporate bond market. One of the aims in this thesis was to test the applicability of the past research in a different market, but a limiting factor for comparison is that the European corporate bond market overall is not as liquid as the U.S. market (BlackRock 2016). The effect of liquidity was shown by taking changes in liquidity as an explanatory variable in the first regression in section 6. This indicated that at least for the European corporate bonds the changes in liquidity are an important explaining factor for price changes.

Yet another issue may be related to the fact that at the time of the largest part of the observations the interest rate environment was quite exceptional as the shortest benchmark interest rates had dropped below zero. More research would be needed on how this affects the models' performance as the negative interest rates are not supposed to occur according to the assumptions of the models.

Potential future research subjects could also lie in studying the models' applicability in default prediction and pricing of credit derivatives which seem to be the main applications of the models. These were not studied here due to the limitations in the availability of data. The estimation of the asset value and volatility parameters as well as the correct inclusion of the asset payouts in the models especially when all the assumptions of the model are not expected to hold are also matters yet to be solved.

The linkage between the structural framework and the modeling conventions acknowledged by the regulations and used by the bank also requires more research. Stronger conclusions are needed about the best practices related to the credit risk modeling approaches used by the banks as this is crucial for the stability of the global economy.

8 APPENDIX

Formulas for Longtaff-Schwartz model mean and standard deviation parameters

$$M(t,T) = \left(\frac{\alpha - \rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2} - \delta\right) t + \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3}\right) e^{(-\beta T)} \left[e^{(\beta t)} - 1\right]$$
$$+ \left(\frac{r_0}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta^2}{\beta^3}\right) \left[1 - e^{(-\beta t)}\right] - \left(\frac{\eta^2}{2\beta^3}\right) e^{(-\beta T)} \left[1 - e^{(-\beta t)}\right]$$

$$S(t) = \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2\right) t - \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{2\eta}{\beta^3}\right) \left[1 - e^{(-\beta t)}\right] + \left(\frac{\eta^2}{2\beta^3}\right) \left[1 - e^{(-2\beta t)}\right]$$

(Longstaff & Schwartz 1995, 796 and Li & Wong 2008, 774)

List of bonds used in the empirical part of the thesis

BASF 2006 4 1/2% 29/06/16
ABERTIS 2008 5.99% 14/05/38
VINCI 2012 3 3/8% 30/03/20
ABERTIS 2005 4 3/8% 30/03/20
ABERTIS 2007 5 1/8% 12/06/17
ABERTIS 2009 4 5/8% 14/10/16
ABERTIS 2012 4 3/4% 25/10/19
ABERTIS 2013 3 3/4% 20/06/23
ABERTIS 2014 2 1/2% 27/02/25
ABERTIS 2014 3 1/8% 19/03/24
DANONE SA 2013 2.6% 28/06/23
HELMA EIGENHEIMBAU 2013 5 7/8% 19/09/18
HOCHTIEF AG 2012 5 1/2% 23/03/17
NEXANS SA 2012 4 1/4% 19/03/18
AIR BERLIN PLC 2010 8 1/2% 10/11/15
AT & S AU.TCGH.& 2011 5% 18/11/16
DELHAIZE GROUP SA 2011 4 1/4% 19/10/18
ACERINOX SA 2014 5% 25/07/24
AHLSTROM 2014 4 1/8% 15/09/19
ARKEMA SA 2010 4% 25/10/17
ARKEMA SA 2012 3.85% 30/04/20
HANSEYACHTS AG 2014 8% 03/06/19
HPI AG 2011 9% 01/12/16
JCDECAUX SA 2013 2% 08/02/18
NEOPOST 2014 2 1/2% 23/06/21
ORPEA 2012 4.1% 10/01/18
ORPEA 2012 4.6% 30/05/19
ORPEA 2013 4.15% 30/11/19
DANONE SA 2013 2 1/4% 15/11/21
ERAMET SA 2013 4 1/2% 06/11/20
ERAMET SA 2014 5.29% 22/04/26
FAGRON NV 2012 4 3/4% 02/07/17
HAVAS SA 2013 3 1/8% 11/07/18
KAPSCH TRAFFICCOM 2010 4 1/4% 03/11/17
KEMIRA OYJ 2014 2 1/2% 27/05/19
KORIAN SA 2013 4 5/8% 02/08/19
LENZING AG 2010 3 7/8% 27/09/17
MELIA HOTELS INTL. 2012 7.8% 31/07/16
MOX TELECOM AG 2012 7 1/4% 02/11/17
NEOPOST 2012 3 1/2% 06/12/19
ORANGE SA 2013 3 1/8% 09/01/24
ORANGE SA 2013 3 3/4% 30/09/33
PARAGON AG 2013 7 1/4% 02/07/18
PERNOD-RICARD 2014 2 1/8% 27/09/24
PEUGEOT 2013 6 1/2% 18/01/19
PEUGEOT 2013 7 3/8% 06/03/18
PNE WIND AG 2013 8% 01/06/18
PORR AG 2013 6 1/4% 26/11/18
PUBLICIS GROUPE 2014 1 1/8% 16/12/21
PUBLICIS GROUPE 2014 1 5/8% 16/12/24
RENAULT 2013 3 5/8% 19/09/18
RENAULT 2014 3 1/8% 05/03/21
S&T AG 2013 7 1/4% 22/05/18
SAF HOLLAND SA 2012 7% 26/04/18
SANOCHEMIA PHMZTKA. 2012 7 3/4% 06/08/17
SOLAR MILLENNIUM 2010 6 1/2% 15/07/15

PERNOD-RICARD 2014 2% 22/06/20
PEUGEOT 2010 5% 28/10/16
PEUGEOT 2011 6 7/8% 30/03/16
SOLAR MILLENNIUM 2011 6% 07/03/16
STRABAG 2011 4 3/4% 25/05/18
VALEO 2011 4 7/8% 11/05/18
VOESTALPINE 2011 4 3/4% 05/02/18
VOESTALPINE 2014 2 1/4% 14/10/21
WIENERBERGER 2011 5 1/4% 04/07/18
ABERTIS 2014 3 1/8% 31/03/26
ALNO AG OLD 2013 8 1/2% 14/05/18
ALTRAN TECHNOLOGIES 2013 3 3/4% 16/07/19
ARGAN SA 2013 5 1/2% 25/11/18
ARKEMA SA 2013 3 1/8% 06/12/23
CAP GEMINI 2011 5 1/4% 29/11/16
COMPAGNIE PLASTIC O 2013 2 7/8% 29/05/20
CONSTANTIN MEDIEN A 2013 7% 23/04/18
DANONE SA 2010 3.6% 23/11/20
DANONE SA 2012 1 1/8% 27/11/17
DANONE SA 2013 1 1/4% 06/06/18
DANONE SA 2013 1 3/8% 10/06/19
SOLEN AG 2011 7 1/2% 08/04/16
SRV YHTIOT OYJ 2013 5% 18/12/18
UCB SA 2013 5 1/8% 02/10/23
VALEO 2014 3 1/4% 22/01/24
VALLOUREC SA 2014 2 1/4% 30/09/24
VILMORIN ET CIE 2014 2 3/8% 26/05/21
ATENOR GROUP SA 2012 5 3/8% 26/10/17
BASTEI LUEBBE G&CK. 2011 6 3/4% 26/10/16
BEATE UHSE AG 2014 7 3/4% 09/07/19
COMPAGNIE D ENTREP. 2012 4 3/4% 21/06/18
DELHAIZE GROUP SA 2012 3 1/8% 27/02/20
ESSILOR INTL.CMPG. 2014 1 3/4% 09/04/21
ESSILOR INTL.CMPG. 2014 2 3/8% 09/04/24
NV BEKAERT SA 2013 4 3/4% 17/10/20
PHAROL SGPS SA 2012 6 1/4% 26/07/16
ROULARTA MEDIA GP. 2012 5 1/8% 10/10/18
SAG SOLARSTROM 2010 6 1/4% 14/12/15
SAG SOLARSTROM 2011 7 1/2% 10/07/17
SINGULUS TECHS. 2012 7 3/4% 23/03/17
UCB SA 2013 4 1/8% 04/01/21

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