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Modulus estimates of semirings with applications to boundary extension problems

Anatoly Golberg[1] · Toshiyuki Sugawa[2] · Matti Vuorinen[3]

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Abstract
In our previous paper (Golberg et al. in Comput Methods Funct Theory 20(3–4):539–558, 2020), we proved that the complementary components of a ring domain in $\mathbb{R}^n$ with large enough modulus may be separated by an annular ring domain and applied this result to boundary correspondence problems under quasiconformal mappings. In the present paper, we continue this work and investigate boundary extension problems for a larger class of mappings.

Keywords Teichmüller ring · Modulus of a ring · Boundary behavior · Directional dilatation

Mathematics Subject Classification Primary 30C65; Secondary 26B35 · 30C75 · 31B15

1 Introduction

Extremal problems of geometric function theory often lead to situations where the extremal configurations exhibit symmetry. Two classical examples of such extremal configurations are the ring domains of Grötzsch and Teichmüller which provide lower

Dedicated to the memory of Professor Lawrence Zalcman.

✉ Toshiyuki Sugawa
sugawa@math.is.tohoku.ac.jp

Anatoly Golberg
golberga@hit.ac.il

Matti Vuorinen
vuorinen@utu.fi

1 School of Mathematical Sciences, Holon Institute of Technology, 52 Golomb St., P.O.B. 305, 5810201 Holon, Israel

2 Graduate School of Information Sciences, Tohoku University, Aoba-ku, Sendai 980-8579, Japan

3 Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland

bounds for the conformal capacities of the respective two classes of ring domains and have found many important applications in the theory of quasiconformal and quasiregular mappings in $\mathbb{R}^n$, $n \geq 2$ [5, 14]. Systematic study of the capacities of these ring domains is carried out in [1]. In the planar case, the Teichmüller ring serves as an extremal case for the so-called Teichmüller theorem on the existence of an annular ring which separates the boundary components of a general ring domain. It seems, however, that the higher dimensional analogues of Teichmüller's theorem are less known.

In our previous paper [9], we have extended Teichmüller's theorem and its semiring counterpart to higher dimensions and, as examples of applications, given a conformally invariant characterization of uniformly perfect sets in $\overline{\mathbb{R}}^n$. The following theorem [9, Theorem 3.2] extends a variant of Teichmüller's theorem due to Avkhadiev and Wirths [2] to the n-dimensional case.

Theorem 1 *Let $n \geq 2$. Every ring domain $\mathcal{R}$ separating a given point x_0 in $\mathbb{R}^n$ and ∞ with $\operatorname{mod} \mathcal{R} > A_n$ contains an annular ring $\mathcal{A}$ centered at x_0 with $\operatorname{mod} \mathcal{A} \geq \operatorname{mod} \mathcal{R} - A_n$. Here A_n is the constant defined in (2.1) below and this constant A_n is sharp.*

Some other necessary results can be found in Sect. 3. For their proofs we refer to [9].

In this paper, we emphasize that our approach allows us to weaken the regularity or quasiconformality assumptions of the mappings. By definition every quasiconformal/quasiregular mapping $f : G \to \mathbb{R}^n$ of a domain $G \subset \mathbb{R}^n$ belongs to the Sobolev class $W^{1,n}_{\text{loc}}(G)$. Moreover, f is differentiable almost everywhere (a.e.) and possesses the Lusin (N)-property (preservation of zero measure sets) in G. Arbitrary homeomorphisms of Sobolev class $W^{1,p}_{\text{loc}}$ are differentiable a.e. only for $p > n - 1$, and the Lusin (N)-property holds for $p \geq n$. For some details, see Sect. 4.

Here we consider homeomorphisms of finite directional dilatations of the borderline class $W^{1,n-1}_{\text{loc}}(G)$. Our technique involves various modulus bounds for semirings which rely on Theorem 1. These results are presented in Sect. 5. In Sect. 6 we establish various sufficient conditions on boundary extensions of the mappings considered, obtaining results which guarantee Lipschitz or weak Hölder type continuity of the extended mapping.

2 Grötzsch and Teichmüller rings and related estimates

In this section we present some necessary notions connected to the conformal modulus of a curve/surface family and to the moduli of two distinguished rings named after Grötzsch and Teichmüller in $\overline{\mathbb{R}}^n$, $n \geq 2$. Here $\overline{\mathbb{R}}^n$ denotes the extended Euclidean n-space $\mathbb{R}^n \cup \{\infty\}$, which is homeomorphic to the n-sphere $\mathbb{S}^n$.

2.1 Modulus of curve/surface family

Following [18, 9.2] (cf. [4]), we recall the notion of the modulus of a k-dimensional surface family (a curve family for $k = 1$). Let ω be an open set in $\overline{\mathbb{R}}^k$, $k = 1, \ldots, n-1$. A continuous mapping $S : \omega \to \mathbb{R}^n$ is called a k-dimensional surface S in $\mathbb{R}^n$. When $k = n - 1$, it is also called a hypersurface. The number of preimages of a point y, i.e. $N(S, y) = \mathrm{card}\{x \in \omega : S(x) = y\}$ is said to be the *multiplicity function* of S at $y \in \mathbb{R}^n$.

By a k-dimensional Hausdorff area in $\mathbb{R}^n$ associated with a surface $S : \omega \to \mathbb{R}^n$, we mean

$$A_S(B) = A_S^k(B) := \int_B N(S, y)\, d\mathcal{H}^k(y)$$

for every Borel set $B \subseteq \mathbb{R}^n$. Here, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^n$ so normalized that $\mathcal{H}^k(I^k) = 1$, where $I^k = [0, 1]^k \times \{0\}^{n-k}$ is the k-dimensional unit cube embedded in $\mathbb{R}^n$. The surface S is called *rectifiable* if $A_S(\mathbb{R}^n) < \infty$.

For a Borel function $g : \mathbb{R}^n \to [0, \infty]$, its integral over S is defined by

$$\int_S g\, dA^k := \int_{\mathbb{R}^n} g(y) N(S, y)\, d\mathcal{H}^k(y).$$

Let $\Gamma = \Gamma^k$ be a family of k-dimensional surfaces S. A Borel measurable function $\varrho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ^k if

$$\int_S \varrho^k\, dA^k \geq 1$$

for every $S \in \Gamma^k$. The (conformal) modulus of Γ is defined to be

$$\mathsf{M}(\Gamma) = \inf_{\varrho} \int_{\mathbb{R}^n} \varrho(x)^n\, dm_n(x),$$

where the infimum is taken over all admissible functions ϱ on $\mathbb{R}^n$ for Γ, and m_n is the Lebesgue measure on $\mathbb{R}^n$. If a property holds for all $S \in \Gamma \setminus \Gamma_0$ for some subfamily Γ_0 of Γ with $M(\Gamma_0) = 0$ we will say that the property holds for almost every $S \in \Gamma$.

2.2 Rings

Throughout our paper, a continuum will mean a connected, compact and non-empty set. A continuum is said to be non-degenerate if it contains more than one point. A continuum $C \subsetneq \overline{\mathbb{R}}^n$ is called *filled* if $\overline{\mathbb{R}}^n \backslash C$ is connected. For a pair of disjoint filled continua C_0 and C_1 in $\overline{\mathbb{R}}^n$, the set $\mathcal{R} = \overline{\mathbb{R}}^n \backslash (C_0 \cup C_1)$ is open and connected and will be called a ring domain or, simply, a *ring* and sometimes denoted by $\mathcal{R}(C_0, C_1)$.

The ring $\mathcal{R}$ is said to have non-degenerate boundary if each component C_j is a non-degenerate continuum. We will say that $\mathcal{R}(C_0, C_1)$ separates a set E if $\mathcal{R} \cap E = \varnothing$ and if $C_j \cap E \neq \varnothing$ for $j = 0, 1$. In the sequel, when $\mathcal{R} \subset \mathbb{R}^n$, we will assume conventionally that $\infty \in C_1$ unless otherwise stated.

Let $\Gamma_{\mathcal{R}}$ be the family of all curves joining C_0 and C_1 in $\mathcal{R}$. Also, let $\Sigma_{\mathcal{R}}$ be the family of hypersurfaces S in $\mathcal{R}$ separating the boundary of $\mathcal{R}$. These are dual to each other in the sense that the relation $\mathsf{M}(\Gamma_{\mathcal{R}}) = \mathsf{M}(\Sigma_{\mathcal{R}})^{1-n}$ holds (see [6]). Then the modulus (called also the module) of $\mathcal{R}$ is defined by

$$\operatorname{mod} \mathcal{R} = \left[\frac{\omega_{n-1}}{\mathsf{M}(\Gamma_{\mathcal{R}})}\right]^{1/(n-1)} = \omega_n^{1/(n-1)} \mathsf{M}(\Sigma_{\mathcal{R}}),$$

where ω_{n-1} denotes the area of the $(n-1)$-dimensional unit sphere [25, p. IX].

For the annular (spherical) ring $\mathcal{A}(a; r_0, r_1) = \{x \in \mathbb{R}^n : r_0 < |x - a| < r_1\}$, we have $\operatorname{mod} \mathcal{A}(a; r_0, r_1) = \log(r_1/r_0)$ (see, e.g. [25, pp. 22-23]).

A ring $\mathcal{R}'$ is said to be a *subring* of a ring $\mathcal{R}$ if $\mathcal{R}' \subset \mathcal{R}$ and if $\mathcal{R}'$ separates $\overline{\mathbb{R}}^n \setminus \mathcal{R}$. By the monotonicity of the moduli of curve families, we have the inequality $\operatorname{mod} \mathcal{R}' \leq \operatorname{mod} \mathcal{R}$ in this case.

2.3 Grötzsch and Teichmüller rings

Two canonical rings are of special interest because of the extremal features of their moduli. The first one is called the Grötzsch ring $R_{G,n}(s)$, $s > 1$, and defined by

$$R_{G,n}(s) = \mathcal{R}(\overline{\mathbb{B}}^n, [se_1, \infty]).$$

Here and hereafter $\mathbb{B}^n$ denotes the unit ball centered at the origin, $\overline{\mathbb{B}}^n$ is its closure, e_1 is the unit vector $(1, 0, \ldots, 0)$ in $\mathbb{R}^n$ and $[se_1, \infty] = \{te_1 : s \leq t < \infty\} \cup \{\infty\}$. The second one is the Teichmüller ring $R_{T,n}(t)$, $t > 0$, and defined by

$$R_{T,n}(t) = \mathcal{R}([-e_1, 0], [te_1, \infty]).$$

The functions $\gamma_n(s) = \mathsf{M}(\Gamma_{R_{G,n}(s)})$ and $\tau_n(t) = \mathsf{M}(\Gamma_{R_{T,n}(t)})$ are systematically studied in [1].

Here we briefly recall the main properties of the moduli of the Grötzsch and Teichmüller rings, see, e.g. [1], [5, 5.4.1, pp. 181-182], [14, pp. 157-159].

- Both γ_n and τ_n are strictly decreasing and continuous functions.
- Let $\mathcal{R}$ be the ring $\mathcal{R}(\overline{\mathbb{B}}^n, C_1)$ for a filled continuum C_1 with $y, \infty \in C_1$ in the domain $|x| > 1$. Then $\operatorname{mod} \mathcal{R} \leq \operatorname{mod} R_{G,n}(|y|)$.
- For filled continua C_0, C_1 with $0, -e_1 \in C_0$ and $x_1, \infty \in C_1$, $\operatorname{mod} \mathcal{R}(C_0, C_1) \leq \operatorname{mod} R_{T,n}(|x_1|)$.
- The following functional identity holds for $t > 0$,

$$\operatorname{mod} R_{T,n}(t) = 2 \operatorname{mod} R_{G,n}(s), \qquad s = \sqrt{t+1}.$$

To define two important constants, we make use of two real-valued functions Φ_n and Ψ_n defined by

$$\log \Phi_n(s) = \bmod R_{G,n}(s) = \left[\frac{\omega_{n-1}}{\gamma_n(s)}\right]^{1/(n-1)},$$
$$\log \Psi_n(t) = \bmod R_{T,n}(t) = \left[\frac{\omega_{n-1}}{\tau_n(t)}\right]^{1/(n-1)}.$$

The Grötzsch (ring) constant λ_n, defined by

$$\lambda_n := \lim_{s\to\infty} \Phi_n(s)/s,$$

admits the following bounds

$$4 \le \lambda_n \le 2^{n/(n-1)} e^{n(n-2)/(n-1)}$$

and has numerous applications to various fields of Real and Complex Analysis. Note that $\lambda_2 = 4$ and the exact value of λ_n is unknown for $n \ge 3$; see [1, 5].

The quantity A_n mentioned in Theorem 1 is defined by

$$A_n = \sup_{1<t<+\infty} \left[\bmod R_{T,n}(t) - \log t\right] = \sup_{1<t<+\infty} \log \frac{\Psi_n(t)}{t}. \tag{2.1}$$

Moreover, the number A_n admits the estimate (see [9, Theorem 3.2]):

$$A_n \le 2\log \frac{(1+\sqrt{2})\lambda_n}{2} = \log \frac{(3+2\sqrt{2})\lambda_n^2}{4}.$$

When $n = 2$, it is known that $A_2 = \pi$.

3 Auxiliary results

In Introduction we have formulated the multidimensional counterpart of the Teichmüller theorem. This theorem is also crucial for the present paper and its proof can be found in our previous paper [9]. We also apply the following results of the same manuscript and provide them here for convenience of the reader.

3.1 Semirings

Following our previous paper [9], the standard model for "semiring" is the upper half of the *closed* ring

$$\mathcal{T}_R = \{x \in \mathbb{H}^n : 1 \le |x| \le R\}$$

for $1 < R < +\infty$. Here and hereafter $\mathbb{H}^n$ denotes the upper half space $\{x = (x_1, \ldots, x_n) : x_n > 0\}$. The semiring $\mathcal{T}_R$ has two distinguished boundary components

$$\partial_0 \mathcal{T}_R = \{x \in \mathbb{H}^n : |x| = 1\} \quad \text{and} \quad \partial_1 \mathcal{T}_R = \{x \in \mathbb{H}^n : |x| = R\}$$

relative to $\mathbb{H}^n$, which are homeomorphic to the $(n-1)$-dimensional open ball $\mathbb{B}^{n-1}$. Let $\Gamma(R)$ denote the family of arcs (curves) $\gamma : [0, 1] \to \mathcal{T}_R$ joining $\partial_0 \mathcal{T}_R$ and $\partial_1 \mathcal{T}_R$ in $\mathcal{T}_R$. Thanks to [25, 7.7], we obtain the formula

$$\mathsf{M}(\Gamma(R)) = \frac{\omega_{n-1}}{2} (\log R)^{1-n} . \tag{3.1}$$

Let $\Sigma(R)$ denote the family of surfaces $S : \mathbb{B}^{n-1} \to \mathcal{T}_R$ which are proper maps and the images separate $\partial_0 \mathcal{T}_R$ from $\partial_1 \mathcal{T}_R$ in $\mathcal{T}_R$. By the symmetry principle, we also have $\mathsf{M}(\Sigma(R)) = (\omega_{n-1}/2)^{1/(1-n)} \log R$.

A subset $\mathcal{S}$ of $\overline{\mathbb{R}}^n$ is called a *semiring* if it is homeomorphic to $\mathcal{T}_R$ for some $R > 1$. We denote by $\Gamma_{\mathcal{S}}$ the family of the image curves of $\Gamma(R)$ under a homeomorphism $f : \mathcal{T}_R \to \mathcal{S}$. In other words, $\Gamma_{\mathcal{S}}$ consists of curves joining the distinguished boundaries $\partial_0 \mathcal{S} = f(\partial_0 \mathcal{T}_R)$ and $\partial_1 \mathcal{S} = f(\partial_1 \mathcal{T}_R)$ in $\mathcal{S}$. Similarly, we denote by $\Sigma_{\mathcal{S}}$ the image surfaces of $\Sigma(R)$ under $f : \mathcal{T}_R \to \mathcal{S}$. Note that $\mathsf{M}(\Gamma_{\mathcal{S}})$ and $\mathsf{M}(\Sigma_{\mathcal{S}})$ do not change under conformal transformations. Moreover, as in the case of rings, by [27, Thm 3.13], we have the relation

$$\mathsf{M}(\Sigma_S) = [\mathsf{M}(\Gamma_S)]^{1/(1-n)} .$$

We now define the modulus of the semiring $\mathcal{S}$ by

$$\operatorname{mod} \mathcal{S} = \left[\frac{\omega_{n-1}}{2\mathsf{M}(\Gamma_{\mathcal{S}})} \right]^{1/(n-1)} = \left(\frac{\omega_{n-1}}{2} \right)^{1/(n-1)} \mathsf{M}(\Sigma_{\mathcal{S}}). \tag{3.2}$$

In particular, $\operatorname{mod} \mathcal{T}_R = \log R$ by virtue of (3.1).

3.2 Properly embedded semirings

Let G be a proper subdomain of $\overline{\mathbb{R}}^n$. A semiring $\mathcal{S}$ in G is said to be *properly embedded* in G if $\mathcal{S} \cap C$ is compact whenever C is a compact subset of G. That is to say, $\mathcal{S}$ is a properly embedded semiring in G if and only if some (and hence every) homeomorphism $f : \mathcal{T}_R \to \mathcal{S}$ is proper as considered to be a map $f : \mathcal{T}_R \to G$. Note that $\partial_0 \mathcal{S}$ and $\partial_1 \mathcal{S}$ are properly embedded $(n-1)$-dimensional open balls in G. (Though there is no canonical way to label $\partial_0 \mathcal{S}$ and $\partial_1 \mathcal{S}$ to the connected components of $\partial \mathcal{S}$ in G, we take the labels given by a proper embedding $f : \mathcal{T}_R \to G$ and fix them for convenience.)

From now on, we consider a semiring $\mathcal{S}$ properly embedded in $\mathbb{B}^n$ by a mapping $f : \mathcal{T}_R \to \mathcal{S} \subset \mathbb{B}^n$. Then $\mathbb{B}^n \backslash \mathcal{S}$ is an open subset of $\mathbb{B}^n$ consisting of two components

V_0 and V_1 for which $V_0 \cap \partial_1 \mathcal{S} = \varnothing$ and $V_1 \cap \partial_0 \mathcal{S} = \varnothing$. The following separation lemma [9, Lemma 4.3] will be applied in the last section of our paper.

Lemma 1 *Let $\mathcal{S}$ be a properly embedded semiring in $\mathbb{B}^n$. Then $\operatorname{mod}\mathcal{S} > 0$ if and only if the Euclidean distance $\delta = \operatorname{dist}(V_0, V_1)$ between V_0 and V_1 is positive. Moreover, in this case, the double $\hat{\mathcal{S}} := \operatorname{Int}\mathcal{S} \cup U \cup \operatorname{Int}\mathcal{S}^*$ of $\mathcal{S}$ is a ring with $\operatorname{mod}\hat{\mathcal{S}} = \operatorname{mod}\mathcal{S}$, where $\mathcal{S}^*$ is the reflection of $\mathcal{S}$ in $\partial\mathbb{B}^n$ and $U = \partial\mathbb{B}^n \backslash (\overline{V}_0 \cup \overline{V}_1)$.*

For $\xi \in \partial\mathbb{B}^n$ and $0 < r_0 < r_1 < +\infty$, we define a properly embedded semiring

$$\mathcal{T}(\xi; r_0, r_1) = \left\{ x \in \mathbb{B}^n : r_0 \leq \frac{|x-\xi|}{|x+\xi|} \leq r_1 \right\}$$

in $\mathbb{B}^n$ bounded by two Apollonian spheres. Then we have the formula $\operatorname{mod}\mathcal{T}(\xi; r_0, r_1) = \log(r_1/r_0)$ (see Lemma 4.2 in [9]). The next proposition and theorem have been obtained in [9]; see Proposition 5.3 and Theorem 5.5, respectively.

Proposition 1 *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a homeomorphism and $\xi \in \partial\mathbb{B}^n$. The mapping f extends continuously to the point ξ if*

$$\lim_{r \to 0+} \operatorname{mod} f(\mathcal{T}(\xi; r, R)) = +\infty$$

for some $R > 0$.

Theorem 2 *A homeomorphism $f : \mathbb{B}^n \to \mathbb{B}^n$ extends to a homeomorphism $f : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ if and only if for each $\xi \in \partial\mathbb{B}^n$, there is an $R = R_\xi > 0$ such that*

$$\lim_{r \to 0+} \operatorname{mod} f(\mathcal{T}(\xi; r, R)) = +\infty.$$

We also need the following separation theorem [9, Theorem 4.8].

Theorem 3 *Let $\mathcal{S}$ be a properly embedded semiannulus in $\mathbb{B}^n$. Then the connected components V_0 and V_1 of $\mathbb{B}^n \backslash \mathcal{S}$ satisfy the inequality*

$$\min\{\operatorname{diam} V_0, \operatorname{diam} V_1\} \leq Q_n \exp\left(-\frac{1}{2}\operatorname{mod}\mathcal{S}\right), \tag{3.3}$$

where $Q_n = 4\exp(A_n/2)$ and A_n is given in (2.1).

4 Quasiconformal, quasiregular mappings and their regularity properties

Due to the famous Liouville theorem, there are no conformal mappings in higher dimensions $n \geq 3$ except Möbius transformations; see, e.g. [21]. The classes of quasiconformal mappings and their non-homeomorphic counterparts - quasiregular mappings (or mappings with bounded distortion) are substantially larger than the class of conformal mappings in $\mathbb{R}^n$, $n \geq 3$, and many of the main geometric and topological properties of analytic functions in the complex plane have their counterparts in this n-dimensional function theory.

4.1 Linear mappings

Following [25, 14.1], for a real $n \times n$ invertible matrix A, we define

$$\|A\| = \sup_{h \in \mathbb{R}^n,\ h \neq 0} \frac{|Ah|}{|h|} = \max_{|h|=1} |Ah| \quad \text{and} \quad l(A) = \inf_{h \in \mathbb{R}^n,\ h \neq 0} \frac{|Ah|}{|h|} = \min_{|h|=1} |Ah|. \tag{4.1}$$

Then the quantities

$$H_I(A) = \frac{|\det A|}{l(A)^n}, \qquad H_O(A) = \frac{\|A\|^n}{|\det A|}, \qquad H(A) = \frac{\|A\|}{l(A)}$$

are called the *inner*, *outer* and *linear dilatation coefficients* of A, respectively.

By linear algebra, the following inequalities

$$H(A) \leq \min\{H_I(A), H_O(A)\} \leq H(A)^{n/2} \leq \max\{H_I(A), H_O(A)\} \leq H(A)^{n-1}, \tag{4.2}$$

hold; cf. [25, 14.3].

4.2 Quasiregular and quasiconformal mappings

Following [22, I.2], recall the definitions of quasiregular and quasiconformal mappings. A mapping $f : G \to \mathbb{R}^n$, $n \geq 2$, of a domain G in $\mathbb{R}^n$ is *quasiregular* if

(1) f is in $\mathrm{ACL}^n(G)$, and
(2) there exists a constant K, $1 \leq K < \infty$, such that

$$\|f'(x)\|^n \leq K J_f(x) \qquad \text{a.e. in } G. \tag{4.3}$$

The smallest K in (4.3) is the *outer dilatation* K_O of f in the domain G. Here and hereafter $f'(x)$ denotes the Jacobian matrix of f at x, and $J_f(x)$ denotes its determinant. Note that for continuous mappings the classes $\mathrm{ACL}^n(G)$ and $W^{1,n}(G)$ coincide; see, e.g. [15, A.5].

If f is quasiregular, then it is also true that

$$J_f(x) \leq K' l(f'(x))^n \qquad \text{a.e. in } G \tag{4.4}$$

for some $K' \geq 1$, where $l(f'(x))$ is defined in (4.1). The smallest $K' \geq 1$ in (4.4) is the *inner dilatation* $K_I(f)$ of f in G. A quasiregular homeomorphism $f : G \to f(G)$ is called *quasiconformal* [22, I.2].

In the case of a continuous, discrete and open mapping $f : G \to \mathbb{R}^n$, the linear dilatation is defined as follows. If $x \in G$, $0 < r < \operatorname{dist}(x, \partial G)$, we set

$$l(x,r) = l_f(x,r) = \inf_{|y-x|=r} |f(y) - f(x)|,$$
$$L(x,r) = L_f(x,r) = \sup_{|y-x|=r} |f(y) - f(x)|.$$

The quantity

$$H(x, f) = \limsup_{r\to 0} \frac{L(x,r)}{l(x,r)}$$

is called the *linear dilatation*.

We also say that a point x is a *regular* point of f, if f is differentiable at x and $J_f(x) \neq 0$. For a regular point $x \in G$, $H(x, f)$ equals the linear dilatation $H(f'(x))$ of the mapping $f'(x)$ (see Sect. 4.1).

Below we list some of the main properties of quasiregular/quasiconformal mappings relevant for us, see, e.g. [14, 15, 20, 22].

Let $f : G \to \mathbb{R}^n$ be a quasiregular mapping. Then

- f is differentiable a.e. in G.
- f satisfies the Lusin (N)-property, i.e. $m(E) = 0$, $E \subset G$, implies $m(f(E)) = 0$.
- f is locally Hölder continuous with exponent $K^{1/(1-n)}$ where $\max\{K_I(f), K_O(f)\} \le K$.
- If f is nonconstant, it is discrete, open and orientation-preserving.
- f belongs to $W^{1,p}_{\mathrm{loc}}(G)$ with $p = p(n, K) > n$.

In the above, the orientation-preserving property is in the topological sense. Also, it is known that $J_f > 0$ a.e. in G for a non-constant quasiregular map $f : G \to \mathbb{R}^n$. This was first shown by Martio, Rickman and Väisälä [17, Theorem 8.1] (see also [22, p. 48]).

4.3 Regularity properties of $W^{1,p}$-homeomorphisms

Following mainly [15, Chapters 2 and 4], we recall some needed regularity properties of continuous/homeomorphic mappings of the Sobolev classes $W^{1,p}$.

- Every mapping $f \in W^{1,p}$ is differentiable a.e. when $p > n$ and $n \ge 2$.
- For $p = n$ there exist mappings $f \in W^{1,p}$ which are not continuous at any point, and, therefore, differentiable nowhere.
- Every homeomorphism of $W^{1,p}$, with $p > n - 1$ for $n \ge 3$, $p \ge 1$ for $n = 2$ is differentiable a.e.
- A continuous mapping $f \in W^{1,p}$ always satisfies the Lusin (N)-property with respect to the n-dimensional Lebesgue measure when $p > n$.
- There exist continuous mappings $f \in W^{1,n}$ which fail to have the Lusin (N)-property.
- For homeomorphisms $f \in W^{1,n}$, the Lusin (N)-property holds.

- There are homeomorphisms of $W^{1,p}$, $p < n$, which do not possess the Lusin (N)-property.

For differentiability a.e., the borderline class $W^{1,n-1}$ is of special interest. We need the following statement proved in [24].

Lemma 2 *Let $G \subset \mathbb{R}^n$ be a domain for $n \geq 2$. Suppose that $f \in W^{1,n-1}_{\mathrm{loc}}(G)$ is a continuous, discrete and open mapping with (pointwise) inner dilatation $L_f(x) = H_I(f'(x))$ satisfying $L_f \in L^1_{\mathrm{loc}}(G)$. Then f is differentiable a.e. in G.*

5 Directional dilatations and modulus estimates

5.1 Directional dilatations

We recall two directional characteristics in $\mathbb{R}^n$, using the derivative of f in a direction h, $h \neq 0$, at x, given by

$$\partial_h f(x) = \lim_{t \to 0^+} \frac{f(x+th) - f(x)}{t},$$

whenever the limit exist. Note that $\partial_h f(x) = f'(x)h$ if f is differentiable at x.

Let $f : G \to \mathbb{R}^n$ be an orientation-preserving homeomorphism and $x \in G$ be a regular point of f. For a point $x_0 \in \mathbb{R}^n$, we define the *angular* and *normal dilatations* of the mapping f at $x \in G$, $x \neq x_0$ *with respect to* x_0 by

$$D_f(x, x_0) = \frac{J_f(x)}{\ell_f(x, x_0)^n}, \qquad T_f(x, x_0) = \left(\frac{\mathcal{L}_f(x, x_0)^n}{J_f(x)} \right)^{1/(n-1)},$$

respectively. Here

$$\ell_f(x, x_0) = \min_{|h|=1} \frac{|\partial_h f(x)|}{|h \cdot u|}, \qquad \mathcal{L}_f(x, x_0) = \max_{|h|=1} \big(|\partial_h f(x)| |h \cdot u| \big),$$

and $u = (x - x_0)/|x - x_0|$. The dilatations $D_f(x, x_0)$ and $T_f(x, x_0)$ are both measurable in $x \in G$.

Following the notations in [10] we denote by $L_f(x)$ and $K_f(x)$ the inner and outer dilatations of f at a regular point x of f, respectively. Namely, $L_f(x) = H_I(f'(x))$ and $K_f(x) = H_O(f'(x))$. Then the chain of inequalities

$$l(f'(x)) \leq \ell_f(x, x_0) \leq |\partial_u f(x)| \leq \mathcal{L}_f(x, x_0) \leq \|f'(x)\|$$

implies

$$\frac{1}{K_f(x)} \leq D_f(x, x_0) \leq L_f(x). \tag{5.1}$$

The normal dilatation $T_f(x, x_0)$ has tighter bounds than $D_f(x, x_0)$, since

$$\frac{1}{K_f(x)} \leq \frac{1}{L_f(x)^{1/(n-1)}} \leq T_f(x, x_0) \leq K_f(x)^{1/(n-1)} \leq L_f(x). \tag{5.2}$$

The dilatations $D_f(x, x_0)$ and $T_f(x, x_0)$ for the multidimensional case have been introduced in [10] and [6], respectively. Note that the angular and normal dilatations range both between 0 and ∞, while the classical dilatations are always greater than or equal to 1, cf. (4.2). Clearly, $L_f(x) = K_f(x) \equiv 1$ for conformal mappings f and, therefore, both directional dilatations also are equal to 1. But not vice versa. Observe also that these directional dilatations provide a reasonable kind of flexibility, although their concrete evaluations are much more complicated than those of classical ones; see, e.g. [8]. The latter fact can be illustrated by the rotation of the punctured ball $\mathbb{B}^n \setminus \{0\}$ given by

$$f(x) = (x_1 \cos\theta - x_2 \sin\theta, x_2 \cos\theta + x_1 \sin\theta, x_3, ..., x_n),$$

with $x = (x_1, ..., x_n)$ and $\theta = \log(x_1^2 + x_2^2)$. This mapping preserves the volume: $J_f(x) \equiv 1$. By a straightforward calculation, one obtains $L_f(x) = K_f(x) = (1 + \sqrt{2})^n$ for $x \neq 0$, and this mapping is quasiconformal in $\mathbb{B}^n \setminus \{0\}$. However, $D_f(x, 0) = 1$ at all points x of $\mathbb{B}^n \setminus \{0\}$; see [10].

5.2 Main Lemma

We consider the semiring $\mathcal{S} = \mathcal{S}(x_0; r, R) = \{x \in \mathbb{H}^n : r \leq |x - x_0| \leq R\}$ for $x_0 \in \partial\mathbb{H}^n$. We recall that $\operatorname{mod}\mathcal{S} = \log(R/r)$. The modulus distortion under quasiconformal and quasiregular mappings plays an essential role in geometric function theory; see, e.g. [23] and [26]. The following lemma gives upper and lower bounds for the distortions of moduli of semirings for homeomorphisms of Sobolev class $W^{1,n}$; cf. [7, Corollary 5.1].

Lemma 3 *Let f be an orientation-preserving homeomorphism of $\mathbb{H}^n$ onto a domain in $\mathbb{R}^n$. Suppose that f belongs to the Sobolev class $W^{1,n-1}_{\mathrm{loc}}(\mathbb{H}^n)$ and posesses Lusin's (N) and (N^{-1})-properties with respect to the n-dimensional Lebesgue measure $m = m_n$. Suppose further that the inner dilatation $L_f(x)$ is locally integrable in the semiring $\mathcal{S} = \mathcal{S}(x_0; r_0, r_1)$ for some $x_0 \in \partial\mathbb{H}^n$, and for almost every hypersurface $S \in \Sigma_{\mathcal{S}}$ the restriction $f|_S$ satisfies the (N^{-1})-property with respect to $(n-1)$-dimensional Hausdorff measure. Then*

$$\begin{aligned} \left(\frac{2}{\omega_{n-1}\log(r_1/r_0)} \int\limits_{\mathcal{S}} \frac{D_f(x, x_0)}{|x - x_0|^n}\, dm_n(x) \right)^{1/(1-n)} &\leq \frac{\operatorname{mod} f(\mathcal{S})}{\operatorname{mod}\mathcal{S}} \\ &\leq \frac{2}{\omega_{n-1}\log(r_1/r_0)} \int\limits_{\mathcal{S}} \frac{T_f(x, x_0)}{|x - x_0|^n}\, dm_n(x), \end{aligned} \tag{5.3}$$

and under the additional assumption $\operatorname{mod}\mathcal{S} \geq \operatorname{mod} f(\mathcal{S})$,

$$-\frac{2}{\omega_{n-1}} \int_{\mathcal{S}} \frac{T_f(x, x_0) - 1}{|x - x_0|^n}\, dm_n(x) \;\leq\; \operatorname{mod}\mathcal{S} - \operatorname{mod} f(\mathcal{S}) \\ \leq \frac{2}{\omega_{n-1}} \int_{\mathcal{S}} \frac{D_f(x, x_0) - 1}{|x - x_0|^n}\, dm_n(x). \tag{5.4}$$

Remark 1 We note that $\operatorname{mod}\mathcal{S} = \log(r_1/r_0)$ by (3.1). If we introduce the measure $d\nu_{x_0}(x) = |x - x_0|^{-n} dm_n(x)$, we have $\nu_{x_0}(\mathcal{S}) = (\omega_{n-1}/2)\log(r_1/r_0)$. Hence, the estimates (5.3) can be written in the form

$$\left(\frac{1}{\nu_{x_0}(\mathcal{S})} \int_{\mathcal{S}} D_f(x, x_0)\, d\nu_{x_0}(x)\right)^{1/(1-n)} \leq \frac{\operatorname{mod} f(\mathcal{S})}{\operatorname{mod}\mathcal{S}} \leq \frac{1}{\nu_{x_0}(\mathcal{S})} \int_{\mathcal{S}} T_f(x, x_0)\, d\nu_{x_0}(x).$$

5.3 Proof of the first inequality in (5.3)

Denote by $\mathcal{S}_0$ the set of regular points of f in $\mathcal{S}$. By virtue of Lemma 2, we find that the set $B_0 = \mathcal{S}\backslash\mathcal{S}_0$ has the n-dimensional Lebesgue measure zero, $m_n(B_0) = 0$. The Lusin (N^{-1})-property is equivalent to $J_f(x) \neq 0$ a.e.; cf. [19]. Now the (N)-property implies that also $m_n(f(B_0)) = 0$. Note also that by (5.1)–(5.2) the local integrability of $L_f(x)$ implies the same property for both directional dilatations $D_f(x, x_0)$ and $T_f(x, x_0)$.

Recall that $\Gamma_{\mathcal{S}}$ is the family of curves which join the boundaries $|x - x_0| = r_0$ and $|x - x_0| = r_1$ in $\mathcal{S}$. Let $\varrho \geq 0$ be a Borel function on $[r_0, r_1]$ such that

$$\int_{r_0}^{r_1} \varrho(t)\, dt \;=\; 1. \tag{5.5}$$

For any $y \in f(\mathcal{S}\backslash B_0) = f(\mathcal{S})\backslash f(B_0)$ we define

$$\varrho^*(y) \;=\; \varrho(|x - x_0|)\left(\frac{D_f(x, x_0)}{J_f(x)}\right)^{1/n} = \frac{\varrho(|x - x_0|)}{\ell_f(x, x_0)},$$

where $x = f^{-1}(y)$, and set $\varrho^*(y) = \infty$ for $y \in f(B_0)$, and $\varrho^*(y) = 0$ otherwise.

We now claim that $\int_{\gamma^*} \varrho^*\, dA^1 \geq 1$ for almost every curve $\gamma^* \in f(\Gamma_{\mathcal{S}}) = \Gamma_{f(\mathcal{S})}$. Then it will follow that ϱ^* is admissible for $\Gamma_{f(\mathcal{S})}$ (see e.g. [18, Theorem 9.1]). Though this claim is found in [10, Lemma 2.4], in order to see how the definition of $D_f(x, x_0)$ works, we include some details of the proof here. For almost every curve $\gamma^* = f(\gamma) \in \Gamma_{f(\mathcal{S})} = f(\Gamma_{\mathcal{S}})$, both γ and γ^* are rectifiable. We now parametrize them as $x = \gamma(s)$ and $y = \gamma^*(s^*)$ by their length parameters so that $|d\gamma(s)/ds| = 1$ and $|d\gamma^*(s^*)/ds^*| = 1$ a.e. Noting that $h_1 = d\gamma(s)/ds$ is a unit vector, we have

$$\frac{d\gamma^*(s^*)}{ds} = f'(x)\frac{d\gamma(s)}{ds} = f'(x)h_1 = \partial_{h_1} f(x)$$

as long as f is regular at $x = \gamma(s)$. Hence,

$$\frac{ds^*}{ds} = \left|\frac{d\gamma^*}{ds}\right| = |\partial_{h_1} f(x)|.$$

Since the quantity $r = |x - x_0|$ has the gradient

$$\nabla r = \frac{x - x_0}{r} =: u,$$

we have the expression

$$\frac{dr}{ds} = \nabla r \cdot \frac{d\gamma(s)}{ds} = u \cdot h_1.$$

Let a and a^* be the lengths of the curves γ and γ^*, respectively. Then

$$\begin{aligned}
\int\limits_{\gamma^*} \varrho^* \, dA^1 &= \int_0^{a^*} \varrho^*(\gamma^*(s^*))ds^* \\
&= \int_0^{a} \frac{\varrho(|x - x_0|)}{\ell_f(x, x_0)} \frac{ds^*}{ds} ds \\
&= \int_{r_0}^{r_1} \frac{\varrho(r)}{\ell_f(x, x_0)} \frac{ds^*/ds}{dr/ds} dr \\
&= \int_{r_0}^{r_1} \frac{\varrho(r)}{\ell_f(x, x_0)} \frac{|\partial_{h_1} f(x)|}{h_1 \cdot u} dr \\
&\geq \int_{r_0}^{r_1} \varrho(r) dr = 1
\end{aligned}$$

as required.

Using the chain rule and the Lusin (N)-property of f, we have

$$\mathsf{M}(\Gamma_{f(\mathcal{S})}) \leq \int\limits_{f(\mathcal{S})} \varrho^*(y)^n dm_n(y) = \int\limits_{\mathcal{S}} \varrho(|x - x_0|)^n D_f(x, x_0)\, dm_n(x).$$

Since $\varrho(t) = 1/(t \log(r_1/r_0))$ satisfies (5.5), the following estimate is obtained:

$$\mathsf{M}(\Gamma_{f(\mathcal{S})}) \leq \frac{1}{\log^n(r_1/r_0)} \int\limits_{\mathcal{S}} \frac{D_f(x, x_0)}{|x - x_0|^n}\, dm_n(x). \tag{5.6}$$

Finally, in view of the first definition in (3.2), we obtain the first inequality in (5.3).

5.4 Proof of the second inequality in (5.3)

Let ϱ be a nonnegative Borel function on the unit hemisphere $\mathbb{S}^{n-1}_+ = \{z \in \mathbb{H}^n : |z| = 1\}$ such that

$$\int\limits_{\mathbb{S}^{n-1}_+} \varrho(z)^{n-1}\, d\sigma_0(z) \;=\; 1, \tag{5.7}$$

where $d\sigma_0$ stands for the area element on the unit sphere. Similarly to the above, for any $y \in f(\mathcal{S}\backslash B_0)$, we define

$$\varrho^*(y) \;=\; \frac{1}{|x-x_0|}\,\varrho\left(\frac{x-x_0}{|x-x_0|}\right)\left(\frac{\mathcal{L}_f(x,x_0)}{J_f(x)}\right)^{1/(n-1)},$$

where $x = f^{-1}(y)$, and set $\varrho^*(y) = \infty$ for $y \in f(B_0)$ and $\varrho^*(y) = 0$ at other points. We now recall that $\Sigma_{\mathcal{S}}$ is the family of all surfaces S which separate the boundaries $|x-x_0| = r_0$ and $|x-x_0| = r_1$ in $\mathcal{S}$. Then we claim that ϱ^* is admissible for $f(\Sigma_{\mathcal{S}}) = \Sigma_{f(\mathcal{S})}$. This is due to [6, Theorem 1] but a proof is given to see how the definition of the normal dilatation $T_f(x, x_0)$ works. The proof of the claim is a little sketchy. More rigorous arguments may be found in [16]. Note also that by the assumptions of Lemma 3 and by [18, Thm 9.1] the $(n-1)$-dimensional Hausdorff area of the exceptional set B_0 for S vanishes for almost every $S^* \in \Sigma_{f(\mathcal{S})}$. To show the claim, it is enough to prove that the inequality $\int_{S^*}(\varrho^*)^{n-1} dA^{n-1} \geq 1$ for almost every $S^* \in \Sigma_{f(\mathcal{S})}$. For almost every $f(S) = S^*$ in $f(\Sigma_{\mathcal{S}}) = \Sigma_{f(\mathcal{S})}$ we may assume they are regular enough so that the following operations are allowed. Conventionally, we put $y = f(x)$ for a regular point x of f and let n and n^* be unit normal vectors of S at x and of S^* at y, respectively. If we denote by $d\sigma$ and $d\sigma^*$ the $(n-1)$-dimensional area elements of S and S^*, respectively, the rate of the volume change under f at the point x may be expressed by

$$J_f(x) = \frac{d\sigma^*(y)}{d\sigma(x)} \times |f'(x)n \cdot n^*| = \frac{d\sigma^*(y)}{d\sigma(x)} \times |\partial_n f(x) \cdot n^*|.$$

We next consider the projection $\pi : \mathbb{H}^n \to \mathbb{S}^{n-1}_+$ defined by $u = \pi(x) = (x-x_0)/|x-x_0|$. Then we have

$$\frac{d\sigma_0(u)}{d\sigma(x)} = \frac{|n \cdot u|}{|x-x_0|^{n-1}}.$$

Since $\pi(S) = \mathbb{S}^{n-1}_+$ and

$$\mathcal{L}_f(x,x_0) \geq |\partial_n f(x)||n \cdot u| \geq |\partial_n f(x) \cdot n^*||n \cdot u|,$$

we now compute

$$\begin{aligned}\int_{S^*}(\varrho^*)^{n-1}d\sigma^* &= \int_S \frac{\varrho(u)^{n-1}}{|x-x_0|^{n-1}}\frac{\mathcal{L}_f(x,x_0)}{J_f(x)}\frac{d\sigma^*(y)}{d\sigma(x)}d\sigma(x)\\ &= \int_S \varrho(u)^{n-1}\frac{\mathcal{L}_f(x,x_0)}{|\partial_n f(x)\cdot n^*|}\frac{d\sigma(x)}{|x-x_0|^{n-1}}\\ &\geq \int_S \varrho(u)^{n-1}\frac{|n\cdot u|d\sigma(x)}{|x-x_0|^{n-1}}\\ &\geq \int_{\mathbb{S}_+^{n-1}}\varrho(u)^{n-1}d\sigma_0(u)=1.\end{aligned}$$

Thus the claim has been shown.

The chain rule and the Lusin (N)-property of f now yield the inequality

$$\begin{aligned}\mathsf{M}(\Sigma_{f(\mathcal{S})}) &\leq \int\limits_{f(\mathcal{S})}\varrho^*(y)^n dm_n(y)\\ &= \int\limits_{f(\mathcal{S})}\frac{1}{|x-x_0|^n}\varrho\left(\frac{x-x_0}{|x-x_0|}\right)^n\left(\frac{\mathcal{L}_f(x,x_0)^n}{J_f(x)^n}\right)^{1/(n-1)} dm_n(y)\\ &= \int\limits_{\mathcal{S}}\varrho\left(\frac{x-x_0}{|x-x_0|}\right)^n\frac{T_f(x,x_0)}{|x-x_0|^n}dm_n(x)\,.\end{aligned}$$

Letting $\varrho(u)=(2/\omega_{n-1})^{1/(n-1)}$, which satisfies (5.7), we obtain the inequality

$$\mathsf{M}(\Sigma_{f(\mathcal{S})}) \leq \left(\frac{2}{\omega_{n-1}}\right)^{n/(n-1)}\int\limits_{\mathcal{S}}\frac{T_f(x,x_0)}{|x-x_0|^n}dm_n(x). \tag{5.8}$$

The second definition of the modulus of a semiring in (3.2) together with the inequality (5.8) yields the second inequality in (5.3).

5.5 Proof of (5.4)

To prove (5.4) we have to assume that $\operatorname{mod}\mathcal{S}\geq \operatorname{mod} f(\mathcal{S})$, which has applications in the further discussions. Hence, by (5.3),

$$\frac{\operatorname{mod}\mathcal{S}}{\operatorname{mod} f(\mathcal{S})} \leq \left(\frac{\operatorname{mod}\mathcal{S}}{\operatorname{mod} f(\mathcal{S})}\right)^{n-1} \leq \frac{2}{\omega_{n-1}\log(r_1/r_0)}\int\limits_{\mathcal{S}}\frac{D_f(x,x_0)}{|x-x_0|^n}dm_n(x).$$

This yields

$$\operatorname{mod}\mathcal{S} - \operatorname{mod} f(\mathcal{S}) \le \frac{2}{\omega_{n-1}} \frac{\operatorname{mod} f(\mathcal{S})}{\operatorname{mod}\mathcal{S}} \int\limits_{\mathcal{S}} \frac{D_f(x, x_0) - 1}{|x - x_0|^n} \, dm_n(x),$$

and the second inequality in (5.4) holds. The first one in (5.4) follows from (5.3) immediately.

Remark 2 The lower bound in (5.4) always holds no matter whether $\operatorname{mod}\mathcal{S} \ge \operatorname{mod} f(\mathcal{S})$ holds or not. Note also that in the case $\operatorname{mod}\mathcal{S} \ge \operatorname{mod} f(\mathcal{S})$, the first inequality in (5.4) is nontrivial, since $T_f(x, x_0) - 1$ can be negative.

Remark 3 Although in Lemma 3 we consider the upper half-space $\mathbb{H}^n$, obviously the modulus estimates may be easily extended to an arbitrary domain G and a properly embedded semiring $\mathcal{S} \subset G$ under a suitable regularity assumption of the boundary of G.

5.6 Lower integral bound

Here we estimate the modulus of $f(\mathcal{S})$ for a semiring $\mathcal{S}$ in terms of integrals depending on the angular dilatation $D_f(x, x_0)$.

Lemma 4 *Let $f : \mathbb{H}^n \to \mathbb{R}^n$ be an orientation-preserving homeomorphism satisfying the assumptions of Lemma 3. Then for a semiring $\mathcal{S} = \mathcal{S}(x_0; r, R)$ centered at $x_0 \in \mathbb{H}^n$*

$$\operatorname{mod} f(\mathcal{S}) \ge \int\limits_r^R \frac{dt}{t\Psi_D(t, x_0)^{1/(n-1)}}, \tag{5.9}$$

where

$$\Psi_D(t, x_0) = \frac{2}{\omega_{n-1}} \int\limits_{\mathbb{S}_+^{n-1}} D_f(x_0 + tz, x_0) \, d\sigma_0(z). \tag{5.10}$$

Proof Arguing as in the beginning of the proof of Lemma 3, we obtain

$$\mathsf{M}(f(\Gamma_{\mathcal{S}})) \le \int\limits_{\mathcal{S}} \varrho(|x - x_0|)^n D_f(x, x_0) dm_n(x). \tag{5.11}$$

Since the metric

$$\varrho(t) = \begin{cases} \left(t\Psi_D(t, x_0)^{1/(n-1)} \int\limits_r^R \frac{dt}{t\Psi_D(t, x_0)^{1/(n-1)}} \right)^{-1}, & \text{for } t \in [r, R], \\ 0, & \text{otherwise,} \end{cases}$$

satisfies (5.5), the inequality (5.11) yields

$$\mathsf{M}(f(\Gamma_S)) \leq \frac{\omega_{n-1}}{2}\left(\int_r^R \frac{dt}{t\Psi_D(t,x_0)^{1/(n-1)}}\right)^{1-n}.$$

By the first definition in (3.2), we obtain the desired estimate (5.9). □

5.7 Dominating factor

The notion of a dominating factor introduced in [12, p. 882] for the planar case, will next be extended to higher dimensions $\mathbb{R}^n$, $n \geq 2$.

A real valued function $H : [0, +\infty) \to \mathbb{R}^n$ is called a *dominating factor* if both of the following conditions hold:

(1) $H(t)$ is continuous and strictly increasing in $[t_0, +\infty)$ and $H(t) = H(t_0)$ for all $t \in [0, t_0]$ for some $t_0 \geq 0$;
(2) the function $e^{H(t)}$ is convex in $t \in [0, +\infty)$.

Note that the convexity of e^H implies that $H(t) \to +\infty$ as $t \to +\infty$. Denote by H^{-1} the inverse of H.

A dominating factor H is said to be of *divergence type* if

$$\int_1^{+\infty} \frac{H(t)\,dt}{t^{n/(n-1)}} = +\infty. \tag{5.12}$$

Otherwise, H is of *convergence type*.

An equivalent condition to (5.12) can be written as

$$\int_{\tau_1}^{+\infty} \frac{d\tau}{\left[H^{-1}(\tau)\right]^{1/(n-1)}} = +\infty, \tag{5.13}$$

for sufficiently large number τ_1.

Indeed, by the change of variables $\tau = H(t)$ and integration by parts, we have

$$\int_{\tau_1}^{\tau_2} \frac{d\tau}{\left[H^{-1}(\tau)\right]^{1/(n-1)}} = \int_{t_1}^{t_2} \frac{dH(t)}{t^{1/(n-1)}} = \frac{H(t_2)}{t_2^{1/(n-1)}} - \frac{H(t_1)}{t_1^{1/(n-1)}} + \frac{1}{n-1}\int_{t_1}^{t_2} \frac{H(t)\,dt}{t^{n/(n-1)}},$$

where $\tau_j = H(t_j)$, $j = 1, 2$. Therefore, the implication (5.12) $\Rightarrow$ (5.13) is verified immediately. In order to prove the reverse implication assume, on the contrary, that the integral in (5.13) is finite, whereas (5.12) is fulfilled. This implies that $\lim_{t\to\infty} H(t)/t^{1/(n-1)} = \infty$, i.e. $H(t) > Ct^{1/(n-1)}$ for some $C > 0$ and all sufficiently large t. Thus, $H(t)/t^{n/(n-1)} > C/t$, and we reach a contradiction, since the integral in (5.13) diverges.

The function $H(t) = \gamma t$, where γ is a positive constant, serves as an example of a dominating factor of divergence type. Indeed, $H^{-1}(\tau) = \tau/\gamma$, and the indefinite integral can be computed as

$$\int \frac{d\tau}{\left[H^{-1}(\tau)\right]^{1/(n-1)}} = \begin{cases} \gamma \log \tau, & n = 2, \\ \frac{n-1}{n-2}\gamma^{1/(n-1)}\tau^{(n-2)/(n-1)}, & n \geq 3. \end{cases}$$

Therefore,

$$\int\limits_{\tau_1}^{+\infty} \frac{d\tau}{\left[H^{-1}(\tau)\right]^{1/(n-1)}} = +\infty.$$

The following statement provides a lower bound for the modulus of $f(\mathcal{S}(x_0; r, R))$ in terms of a dominating factor. This lower bound is of independent interest, cf. [12, Lem 2.22] for the planar case.

Lemma 5 *Let $f : \mathbb{H}^n \to \mathbb{R}^n$ be an orientation-preserving homeomorphism satisfying the assumptions of Lemma 3 and $x_0 \in \partial\mathbb{H}^n$. Suppose also that a dominating factor H satisfies*

$$\int\limits_{\mathcal{S}(x_0; r_0 e^{-m}, r_0)} e^{H(D_f(x, x_0))}\, dm(x) \leq M. \tag{5.14}$$

Then

$$\operatorname{mod} f(\mathcal{S}(x_0; r_0 e^{-m}, r_0)) \geq \int\limits_{1/n}^{m} \frac{dt}{\left[H^{-1}\left(nt + \log \frac{2nM}{\omega_{n-1} r_0^n}\right)\right]^{1/(n-1)}}. \tag{5.15}$$

Proof Denoting

$$h(r) = \frac{2r^n}{\omega_{n-1}} \int\limits_{\mathbb{S}_+^{n-1}} e^{H(D_f(x_0 + rz, x_0))}\, d\sigma_0(z),$$

one can rewrite (5.14) in a form

$$\int\limits_0^m h(r_0 e^{-t})\, dt \leq \frac{2M}{\omega_{n-1}}.$$

Similarly to the proof of [12, Lem 2.22], let $T = \{t \in (0, m) : h(r_0 e^{-t}) > L\}$ for some $L > 0$. Then the length of T cannot exceed $2M/(\omega_{n-1} L)$.

Since e^H is a convex function, Jensen's inequality implies

$$e^{H(\Psi_D(r,x_0))} \le \frac{2}{\omega_{n-1}} \int\limits_{\mathbb{S}^{n-1}_+} e^{H(D_f(x_0+rz,x_0))}\, d\sigma_0(z) = \frac{h(r)}{r^n}.$$

where Ψ_D is defined in (5.10). This yields,

$$\Psi_D(r_0e^{-t}, x_0) \le H^{-1}\left(nt + \log\frac{L}{r_0^n}\right) \qquad \text{for} \quad t \in (0, m)\backslash T.$$

Now by Lemma 4 and the last upper bound,

$$\begin{aligned}
\mathrm{mod}\, f(\mathcal{S}) &\ge \int\limits_{r_0e^{-m}}^{r_0} \frac{dr}{r\Psi_D(r, x_0)^{1/(n-1)}} = \int\limits_0^m \frac{dt}{\Psi_D(r_0e^{-t}, x_0)^{1/(n-1)}} \\
&\ge \int\limits_{(0,m)\backslash T} \frac{dt}{\left[H^{-1}\left(nt + \log\frac{L}{r_0^n}\right)\right]^{1/(n-1)}} \\
&\ge \int\limits_{\frac{2M}{\omega_{n-1}L}}^{m} \frac{dt}{\left[H^{-1}\left(nt + \log\frac{L}{r_0^n}\right)\right]^{1/(n-1)}}
\end{aligned}$$

and setting finally $L = 2nM/\omega_{n-1}$, we obtain the desired bound (5.15). □

6 Boundary correspondence of mappings with finite directional dilatations

In this section we prove results about extending a mapping $f : \mathbb{H}^n \to \mathbb{H}^n$ of finite directional dilatations continuously to the boundary. Moreover, we study the modulus of continuity of the extended mapping in the cases when the extended mapping is Lipschitz or weakly Hölder continuous. Almost all our results rely on applying the Dini condition $\int_0^1 \omega_f(t)/t\, dt < \infty$. This approach has been utilized for the case when $\omega_f(t) \ge 0$ measures the difference between inner/outer dilatation and 1; e.g. [3, 11, 21]. We apply the Dini condition for essentially weaker cases when $\omega_f(t) = D_f(x, x_0) - 1$, therefore, $\omega_f(t)$ may be negative.

6.1 Homeomorphic extension to the boundary

Here we present the main result in our manuscript. Recall that $\mathcal{S}(x_0; r, R) = \{x \in \mathbb{H}^n : r \le |x - x_0| \le R\}$ for $x_0 \in \partial\mathbb{H}^n$, $0 < r < R < +\infty$.

Theorem 4 *Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be a homeomorphism satisfying the assumptions of Lemma 3 and D be a domain in the hyperplane $x_n = 0$. Suppose that the both* *integrals*

$$\int_{\mathcal{S}(t;r,R)} \frac{T_f(x,t)-1}{|x-t|^n}\, dm(x), \qquad \int_{\mathcal{S}(t;r,R)} \frac{D_f(x,t)-1}{|x-t|^n}\, dm(x) \tag{6.1}$$

are finite for each $t \in D$. *Then* f *extends to a homeomorphism of* $\mathbb{H}^n \cup D$.

Proof Assume first that $\operatorname{mod}\mathcal{S} \geq \operatorname{mod} f(\mathcal{S})$, then by (5.4), $\operatorname{mod} f(\mathcal{S}) \to +\infty$ as $r \to 0^+$, since $\operatorname{mod}\mathcal{S} \to +\infty$ as $r \to 0^+$.

In the case $\operatorname{mod}\mathcal{S} < \operatorname{mod} f(\mathcal{S})$, the same conclusion $\operatorname{mod} f(\mathcal{S}) \to +\infty$ as $r \to 0^+$ is trivial. Thus, applying a Möbius transformation from $\mathbb{H}^n$ onto $\mathbb{B}^n$ and its inverse which preserve the modulus and then the assertion of Proposition 1, one concludes that f can be extended to a homeomorphism of $\mathbb{H}^n \cup D$ into $\overline{\mathbb{H}}^n$. □

6.2 Distortion of semirings

The following result is a counterpart of Theorem 3 for the case of $\mathbb{H}^n$. In addition, the estimates of this type will be applied to studying the regularity features of mappings on the boundary. For the planar case, see [13, Theorem 2.7].

Theorem 5 *Let* $\mathcal{S}$ *be a properly embedded semiring in* $\mathbb{H}^n$ *and* V_0 *and* V_1 *be the two connected components of* $\mathbb{H}^n \setminus \mathcal{S}$ *bounded and unbounded, respectively. If, in addition,* $\operatorname{mod}\mathcal{S} > A_n$, *then for any point* $x_0 \in \partial\mathbb{H}^n \cap V_0$,

$$\sup_{y \in V_0} |y - x_0| \leq C \operatorname{dist}(x_0, V_1) e^{-\operatorname{mod}\mathcal{S}}, \tag{6.2}$$

where $C = \exp A_n$.

Proof Arguing in the same way as in Lemma 1 and applying Theorem 4 to the symmetrically extended ring $\hat{\mathcal{S}}$, there exists a annular ring $\mathcal{A} = \mathcal{A}(x_0; r, R)$, which is a subring of $\hat{\mathcal{S}}$ such that $\operatorname{mod}\mathcal{A} \geq \operatorname{mod}\mathcal{S} - A_n$. Now, since $\sup_{y \in V_0} |y - x_0| \leq r$, $\operatorname{dist}(x_0, V_1) \geq R$, and $\operatorname{mod}\mathcal{A} = R/r$, we obtain the desired estimate (6.2). □

6.3 Lipschitz continuity at the boundary

The uniform boundedness of the second integral in (6.1) provides a local Lipschitz continuity at the boundary. Recall that a mapping f is called locally Lipschitz continuous on a domain G, if for every compact subset E of G there exists a constant $C = C(E)$ such that $|f(x) - f(x_0)| \leq C|x - x_0|$ for any $x, x_0 \in E$. The next result is similar to the earlier theorem for quasiconformal automorphisms of $\mathbb{B}^n$ normalized by $f(0) = 0$ in [10].

Theorem 6 *Let* $f : \mathbb{H}^n \to \mathbb{H}^n$ *be a homeomorphism satisfying the assumptions of Theorem 4 and* D *be a domain in the hyperplane* $x_n = 0$. *Suppose that there exist*

$R > 0$ and $M > 0$ such that

$$\int\limits_{S(t;r,R)} \frac{D_f(x,t)-1}{|x-t|^n}\, dm_n(x) \ \le \ M$$

for every $t \in D$ and r, $0 < r < R$, then f extends to a homeomorphism of $\mathbb{H}^n \cup D$ into $\overline{\mathbb{H}}^n$. If $f(D) \subset \partial\mathbb{H}^n$, the boundary mapping $f : D \to \partial\mathbb{H}^n$ is locally Lipschitz continuous.

Proof The scheme of the proof follows the lines of the proof of Theorem 1.2 in [13] given for the planar case. Note that the existence of boundary extension of f to D follows from Theorem 4. Pick any point $x_0 = (x_{01}, \ldots, x_{0n})$ satisfying $x_{0n} > R$ and write $y_0 = f(x_0)$.

First assume that $\mod \mathcal{S} \ge \mod f(\mathcal{S})$. Then by (5.4)

$$\mod f(\mathcal{S}) \ \ge \ \mod \mathcal{S} - \frac{2M}{\omega_{n-1}}. \tag{6.3}$$

An appropriate choice of r_0, $\log(R/r_0) - 2\,M/\omega_{n-1} > A_n$ or, equivalently,

$$r_0 \ < \ Re^{-A_n - 2M/\omega_{n-1}},$$

allows to conclude that $\mod f(\mathcal{S}) > A_n$, and, therefore, to apply Theorem 5. Note that $f(\mathcal{S})$ separates $y_0 \in \mathbb{H}^n$ from $f(t) \in \partial\mathbb{H}^n$, thus $\operatorname{dist}(f(t), V_1) \le |f(t) - y_0|$. Here V_1 is the unbounded component of the complement $f(\mathcal{S})$ in $\mathbb{H}^n$. Choose an arbitrary point $x \in \mathbb{H}^n$ with $|x - t| < r_0$ and set $r = |x - t|$. Combining (6.2) with (6.3) we obtain

$$|f(x) - f(t)| \ \le \ C \operatorname{dist}(f(t), V_1) e^{-\mod f(\mathcal{S})} \ \le \ C_1 |f(t) - y_0||x - t|,$$

where $C_1 = e^{A_n + 2M/\omega_{n-1}}/R$.

In the case $\mod \mathcal{S} < \mod f(\mathcal{S})$, (6.2) directly yields

$$|f(x) - f(t)| \ \le \ C \operatorname{dist}(f(t), V_1) e^{-\mod f(\mathcal{S})} \ \le \ C_2 |f(t) - y_0||x - t|.$$

Here $C_2 = e^{A_n}/R$. Thus, the mapping f is locally Lipschitz continuous on D. □

6.4 Weak Hölder continuity at the boundary

The finiteness of the integral average of $(D_f(x,t)-1)/|x-t|^n$ over a half ball centered at $t \in \partial\mathbb{H}^n$ guarantees a weak Hölder continuity of a self-mapping of $\mathbb{H}^n$ with finite directional dilatations up to the boundary. By a weak Hölder continuity with exponent α of a mapping f in a domain G, we mean that there exists a constant C for every γ, $0 < \gamma < \alpha$, such that the inequality $|f(x) - f(x_0)| \le C|x - x_0|^\gamma$ holds for any $x, x_0 \in G$. For the same property of quasiconformal automorphisms of $\mathbb{B}^n$ we refer again to [10].

Theorem 7 *Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be a homeomorphism satisfying the assumptions of Theorem 4 and D be a domain in the hyperplane $x_n = 0$. Suppose that for $0 < \alpha \leq 1$ we have*

$$\limsup_{R\to 0^+} \frac{2}{\Omega_n R^n} \int\limits_{\mathcal{S}(t;0,R)} \big(D_f(x,t) - 1\big)\, dm_n(x) \ \leq \ \frac{1}{\alpha^{n-1}} - 1 \tag{6.4}$$

uniformly for every $t \in D$. Then f extends to a homeomorphism of $\mathbb{H}^n \cup D$ into $\overline{\mathbb{H}}^n$, and if $f(D) \subset \partial\mathbb{H}^n$, the boundary mapping $f : D \to \partial\mathbb{H}^n$ is locally weakly Hölder continuous on D with exponent α.

Proof Denote

$$\omega(t;R) = \frac{2}{\Omega_n R^n} \int\limits_{\mathcal{S}(t;0,R)} \big(D_f(x,t) - 1\big)\, dm_n(x) = \frac{2}{\Omega_n R^n} \int\limits_{\mathcal{S}(t;0,R)} D_f(x,t)\, dm_n(x) - 1,$$

and

$$P_f(t;r,R) = \frac{2}{\omega_{n-1}\log(R/r)} \int\limits_{\mathcal{S}(t;r,R)} \frac{D_f(x,t)}{|x-t|^n}\, dm_n(x).$$

Then arguing similarly to (2.19) in [10], we obtain

$$\big(P_f(t;r,R) - 1\big)\log(R/r) = \frac{\omega(t;R) - \omega(t;r)}{n} + \int_r^R \frac{\omega(t;s)}{s}\, ds. \tag{6.5}$$

Note now that the condition (6.4) is equivalent to

$$\lim_{R\to 0^+} \omega(t;R) \ \leq \ \frac{1}{\alpha^{n-1}} - 1$$

uniformly for $t \in D$. Thus, for a compact set $D_0 \subset D$ and arbitrary γ, $0 < \gamma < \alpha$, there exists $R > 0$ such that $\omega(t;s) \leq 1/\gamma^{n-1} - 1$ for $t \in D_0$ and $0 < s \leq R$; cf. [13, p. 960]. The function $\omega(t;s)$ is bounded from below and from above for $0 < s \leq R$, and therefore, we rewrite (6.5) as

$$\big(P_f(t;r,R) - 1\big)\log(R/r) \ \leq \ O(1) + \Big(1/\gamma^{n-1} - 1\Big)\log(R/r), \quad \text{as} \quad r \to 0,$$

which implies

$$P_f(t;r,R) \ \leq \ 1/\gamma^{n-1} + o(1), \quad \text{as} \quad r \to 0.$$

Then by the first inequality in (5.3),

$$\frac{\operatorname{mod} f(\mathcal{S})}{\log(R/r)} \geq P_f(t; r, R)^{1/(1-n)},$$

and choosing sufficiently small r_0, one can apply Theorem 5 and reach the estimate

$$|f(y) - f(t)| \leq C|y - t|^{\gamma}$$

for $y \in D$, $t \in D_0$, provided that $|y - t| \leq r_0$. □

6.5 Homeomorphic extension to $x_n = 0$

We next prove a counterpart of Theorem 2 for the case of the upper half space. The proof follows the ideas of [13, Thm 3.4].

Lemma 6 *An automorphism f of the upper half space $\mathbb{H}^n$ admits a homeomorphic extension to $\overline{\mathbb{H}}^n$ if and only if for each $t \in \partial\mathbb{H}^n$,*

$$\lim_{r \to 0^+} \operatorname{mod} f(\mathcal{S}(t; r, R)) = +\infty$$

for some $R = R(t) > 0$.

6.6 Modulus of continuity

In this subsection we establish estimates for the modulus of continuity involving a dominating factor of divergence type. First we present a sufficient condition for the continuous extension to the boundary for self-homeomorphisms of $\mathbb{H}^n$.

Theorem 8 *Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be a homeomorphism satisfying the assumptions of Lemma 3 and $x_0 \in \partial\mathbb{H}^n$. Suppose that for some positive constants γ and $M = M(x_0)$,*

$$\int_{\mathcal{S}(x_0; r_0 e^{-m}, r_0)} e^{\gamma D_f(x, x_0)}\, dm(x) \leq M.$$

Then f continuously extends to x_0.

Proof The assumption of Theorem 8 yields that f satisfies the conditions of Lemma 5 with a dominating factor of divergence type $H(t) = \gamma t$. Then, by (5.15)

$$\operatorname{mod} f(\mathcal{S}(x_0; r_0 e^{-m}, r_0)) \geq \int_{1/n}^{m} \frac{dt}{\left[H^{-1}(nt + \sigma)\right]^{1/(n-1)}},$$

where $\sigma = \log \frac{2nM}{\omega_{n-1} r_0^n}$. A straightforward calculation gives

$$\operatorname{mod} f(\mathcal{S}(x_0; r_0 e^{-m}, r_0)) \geq \gamma^{1/(n-1)} \int_{1/n}^{m} (nt+\sigma)^{1/(1-n)}\, dt$$

$$= \begin{cases} C_1 \left[(nm+\sigma)^{\mu} - (1+\sigma)^{\mu}\right], & n \geq 3, \\ C_2 \log \frac{nm+\sigma}{1+\sigma}, & n = 2, \end{cases} \tag{6.6}$$

where $C_1 = \frac{(n-1)\gamma^{1/(n-1)}}{n(n-2)}$, $C_2 = \frac{\gamma^{1/(n-1)}}{n}$ and $\mu = \frac{n-2}{n-1}$. Letting $m \to \infty$, we have

$$\lim_{m\to\infty} \operatorname{mod} f(\mathcal{S}(x_0; r_0 e^{-m}, r_0)) = +\infty$$

in both cases, therefore, the desired assertion follows from Lemma 6. □

Let $x_0 \in \partial\mathbb{H}^n$ and $x_1 \in \mathbb{H}^n$ be two arbitrary points. Denote $|x_1 - x_0| = r_0 e^{-m}$, where $m > 0$ and $r_0 > 0$ can be precisely defined later. Consider the semiring $\mathcal{S} = \mathcal{S}(x_0; r_0 e^{-m}, r_0)$ and an orientation-preserving homeomorphism $f : \mathbb{H}^n \to \mathbb{H}^n$. Then $f(\mathcal{S})$ is a properly embedded semiring in $\mathbb{H}^n$. Assume, in addition, that $\operatorname{mod} f(\mathcal{S}) > A_n$. Denoting by V_0 and V_1 the connected components of $\mathbb{H}^n \setminus f(\mathcal{S})$ ($\infty \in V_1$), Theorem 5 implies

$$\sup_{y\in V_0} |y - f(x_0)| \leq C \operatorname{dist}(f(x_0), V_1) e^{-\operatorname{mod} f(\mathcal{S})}, \tag{6.7}$$

where $C = \exp(A_n)$.

Theorem 9 *Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be a homeomorphism satisfying the assumptions of Theorem 8 and $x_0 \in \partial\mathbb{H}^n$. Then for any $x_1 \in \mathbb{H}^n$, the following modulus of continuity estimates*

$$|f(x_1) - f(x_0)| \leq \alpha \left[\log \frac{r_0}{|x_1 - x_0|}\right]^{-C_2}, \quad \text{if } n = 2, \tag{6.8}$$

$$\log |f(x_1) - f(x_0)| \leq -\beta \left[\log \frac{r_0}{|x_1 - x_0|}\right]^{\mu} + \delta, \quad \text{if } n \geq 3, \tag{6.9}$$

hold, where α and δ are constants depending on $\operatorname{dist}(f(x_0), V_1)$ and A_n, $\mu = (n-2)/(n-1)$, $\beta = C_1 n^{\mu}$, and C_1 and C_2 are defined in (6.6).

Proof Let $x_0 \in \partial\mathbb{H}^n$ and $x_1 \in \mathbb{H}^n$ be two arbitrary points. Denote $|x_1 - x_0| = r_0 e^{-m}$, where the constant $m > 0$ can be chosen in the following way. By Theorem 8,

$$\operatorname{mod} f(\mathcal{S}) \geq \gamma^{1/(n-1)} \int_{1/n}^{m} (nt+\sigma)^{1/(1-n)}\, dt$$

for the dominating factor of divergence type $H(t) = \gamma t$. Since the integral in the right-hand side tends to ∞ as $m \to \infty$, one can choose $m > 0$ such that $\operatorname{mod} f(\mathcal{S}) > A_n$. Fix such m and $r_0 = |x_1 - x_0|e^m$. The above lower bound by A_n for $\operatorname{mod} f(\mathcal{S})$ yields (6.7).

Clearly, $|f(x_1) - f(x_0)| \le \sup_{y \in V_0} |y - f(x_0)|$, therefore, using (6.6) together with (6.7) for $n = 2$ provides

$$|f(x_1) - f(x_0)| \le C \operatorname{dist}(f(x_0), V_1) e^{-C_2 \log \frac{nm+\sigma}{1+\sigma}}.$$

By a simple chain of upper bounds, we obtain

$$|f(x_1) - f(x_0)| \le \widetilde{C}(nm+\sigma)^{-C_2} \le \alpha m^{-C_2} = \alpha \left[\log \frac{r_0}{|x_1 - x_0|}\right]^{-C_2},$$

where $\alpha = \operatorname{dist}(f(x_0), V_1) \exp(A_n - C_2 \log n)$.

For higher dimensions, i.e. $n \ge 3$,

$$\begin{aligned} |f(x_1) - f(x_0)| &\le \widetilde{C} e^{-C_1[(nm+\sigma)^\mu - (1+\sigma)^\mu]} \le \widehat{C} e^{-C_1(nm+\sigma)^\mu} \\ &\le \widehat{C} e^{-C_1(nm)^\mu} = \widehat{C} e^{-\beta m^\mu}, \end{aligned}$$

where $\mu = (n-2)/(n-1)$ and $\beta = C_1 n^\mu$. Finally, by taking log,

$$\log |f(x_1) - f(x_0)| \le -\beta \left[\log \frac{r_0}{|x_1 - x_0|}\right]^\mu + \delta,$$

where $\delta = A_n + C_1(1+\sigma)^\mu + \log \operatorname{dist}(f(x_0), V_1)$. This completes the proof. □

6.7 Behavior at infinity

The following result shows that under an appropriate condition on asymptotic behavior of the integral of a Teichmüller-Wittich-Belinskiĭ type the mappings admit continuous extensions to infinity; cf. [13, Lemma 1.4].

Lemma 7 *Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be a homeomorphism satisfying the assumptions of Theorem 4 and*

$$\lim_{R \to \infty} \frac{1}{(\log R)^2} \int_{\mathcal{S}(0; r_0, R)} \frac{D_f(x, 0) - 1}{|x|^n} \, dm_n(x) = 0 \tag{6.10}$$

for some $r_0 > 0$. Then f extends continuously to infinity.

Proof In the proof of Theorem 7 we showed that

$$P_f(0; r_0, R) - 1 = \frac{2}{\omega_{n-1} \log(R/r_0)} \int_{\mathcal{S}(t; r, R)} \frac{D_f(x, 0) - 1}{|x|^n} \, dm_n(x).$$

Thus, due to (6.10), $P_f(0; r_0, R) = o(\log R)$ as $R \to +\infty$, and the desired assertion follows from (5.3) as $R \to +\infty$ and then by Lemma 6. □

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Declarations

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