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Spontaneous Focusing on Quantitative Relations and the Development of Rational Number Conceptual Knowledge

by

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To Cecilia, Juliaana, and Noah

ABSTRACT

The aim of the present set of studies was to explore primary school children's Spontaneous Focusing On quantitative Relations (SFOR) and its role in the development of rational number conceptual knowledge. The specific goals were to determine if it was possible to identify a spontaneous quantitative focusing tendency that indexes children's tendency to recognize and utilize quantitative relations in non-explicitly mathematical situations and to determine if this tendency has an impact on the development of rational number conceptual knowledge in late primary school. To this end, we report on six original empirical studies that measure SFOR in children ages five to thirteen years and the development of rational number conceptual knowledge in ten- to thirteen-year-olds. SFOR measures were developed to determine if there are substantial differences in SFOR that are not explained by the ability to use quantitative relations. A measure of children's conceptual knowledge of the magnitude representations of rational numbers and the density of rational numbers is utilized to capture the process of conceptual change with rational numbers in late primary school students. Finally, SFOR tendency was examined in relation to the development of rational number conceptual knowledge in these students.

Study I concerned the first attempts to measure individual differences in children's spontaneous recognition and use of quantitative relations in 86 Finnish children from the ages of five to seven years. Results revealed that there were substantial inter-individual differences in the spontaneous recognition and use of quantitative relations in these tasks. This was particularly true for the oldest group of participants, who were in grade one (roughly seven years old). However, the study did not control for ability to solve the tasks using quantitative relations, so it was not clear if these differences were due to ability or SFOR. Study II more deeply investigated the nature of the two tasks reported in Study I, through the use of a stimulated-recall procedure examining children's verbalizations of how they interpreted the tasks. Results reveal that participants were able to verbalize reasoning about their quantitative relational responses, but not their responses based on exact number. Furthermore, participants' non-mathematical responses revealed a variety of other aspects, beyond quantitative relations and exact number, which participants focused on in completing the tasks. These results suggest that exact number may be more easily perceived than quantitative relations. As well, these tasks were revealed to contain both mathematical and non-mathematical aspects which were interpreted by the participants as relevant.

Study III investigated individual differences in SFOR 84 children, ages five to nine, from the US and is the first to report on the connection between SFOR and other mathematical abilities. The cross-sectional data revealed that there were individual differences in SFOR.

Importantly, these differences were not entirely explained by the ability to solve the tasks using quantitative relations, suggesting that SFOR is partially independent from the ability to use quantitative relations. In other words, the lack of use of quantitative relations on the SFOR tasks was not solely due to participants being unable to solve the tasks using quantitative relations, but due to a lack of the spontaneous attention to the quantitative relations in the tasks. Furthermore, SFOR tendency was found to be related to arithmetic fluency among these participants. This is the first evidence to suggest that SFOR may be a partially distinct aspect of children's existing mathematical competences.

Study IV presented a follow-up study of the first graders who participated in Studies I and II, examining SFOR tendency as a predictor of their conceptual knowledge of fraction magnitudes in fourth grade. Results revealed that first graders' SFOR tendency was a unique predictor of fraction conceptual knowledge in fourth grade, even after controlling for general mathematical skills. These results are the first to suggest that SFOR tendency may play a role in the development of rational number conceptual knowledge.

Study V presents a longitudinal study of the development of 263 Finnish students' rational number conceptual knowledge over a one year period. During this time participants completed a measure of conceptual knowledge of the magnitude representations and the density of rational numbers at three time points. First, a Latent Profile Analysis indicated that a four-class model, differentiating between those participants with high magnitude comparison and density knowledge, was the most appropriate. A Latent Transition Analysis reveal that few students display sustained conceptual change with density concepts, though conceptual change with magnitude representations is present in this group. Overall, this study indicated that there were severe deficiencies in conceptual knowledge of rational numbers, especially concepts of density.

The longitudinal Study VI presented a synthesis of the previous studies in order to specifically detail the role of SFOR tendency in the development of rational number conceptual knowledge. Thus, the same participants from Study V completed a measure of SFOR, along with the rational number test, including a fourth time point. Results reveal that SFOR tendency was a predictor of rational number conceptual knowledge after two school years, even after taking into consideration prior rational number knowledge (through the use of residualized SFOR scores), arithmetic fluency, and non-verbal intelligence. Furthermore, those participants with higher-than-expected SFOR scores improved significantly more on magnitude representation and density concepts over the four time points. These results indicate that SFOR tendency is a strong predictor of rational number conceptual development in late primary school children.

The results of the six studies reveal that within children's existing mathematical competences there can be identified a spontaneous quantitative focusing tendency named spontaneous

focusing on quantitative relations. Furthermore, this tendency is found to play a role in the development of rational number conceptual knowledge in primary school children. Results suggest that conceptual change with the magnitude representations and density of rational numbers is rare among this group of students. However, those children who are more likely to notice and use quantitative relations in situations that are not explicitly mathematical seem to have an advantage in the development of rational number conceptual knowledge. It may be that these students gain quantitative more and qualitatively better self-initiated deliberate practice with quantitative relations in everyday situations due to an increased SFOR tendency. This suggests that it may be important to promote this type of mathematical activity in teaching rational numbers. Furthermore, these results suggest that there may be a series of spontaneous quantitative focusing tendencies that have an impact on mathematical development throughout the learning trajectory.

CONTENTS

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LIST OF EMPIRICAL STUDIES

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JM contributed to the study conception and design; and data collection, analysis, and interpretation; and was responsible for the writing of the manuscript. MMH-S contributed to the study conception and design, data collection, analysis, and interpretation, and revision of the manuscript. EL contributed to the study conception and design, data analysis and interpretation, and revision of the manuscript.

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1. INTRODUCTION

A bored child asks their parent, "*Are we there yet?*" A university student is budgeting their study time and asks, "*Will I finish in time for the movie tomorrow?*" A parent is serving dessert to a set of siblings and is greeted with the perennial response, "*That's not fair!"* None of these statements or questions is inherently mathematical, and yet in each instance there are available mathematical cues and features of the situation that could be used to more clearly answer the question or explain the situation. The bored child may be told, "*We're almost there*", "*We're one hour away*", or even "*We're halfway there.*" Differences in the use of mathematical aspects interred in these different responses may be important for understanding differences in the development of mathematical skills. A large part of practice with mathematical reasoning and skills does not happen in formal learning situations (Bransford et al., 2009). Beneficial experiences and practice with mathematical reasoning often occurs during everyday situations, such as those described above (Hannula & Lehtinen, 2005; Lobato, Rohdehamel, & Hohensee, 2012). Unlike the math classroom, everyday situations rarely involve someone providing guidance towards the relevant mathematical aspects that are present (Gunderson & Levine, 2011). Thus, it is on the individual to recognize the relevance of the mathematical aspects in these situations alone. The tendency to recognize mathematical aspects of a situation varies substantially between individuals in situations that are not explicitly mathematical (Hannula, 2005; Hannula & Lehtinen, 2001; 2005; Lehtinen & Hannula, 2006; Lobato, 2012). A higher tendency to recognize mathematical aspects as relevant may provide more opportunities for these individuals to practice mathematical reasoning, benefiting the development of formal mathematical skills.

Returning to the previous example of answers to the question, "Are we there yet?", the present dissertation is particularly concerned with the answer, "*halfway".* This answer relates the total length of the journey to the time/distance that has already passed. That this answer was provided in a situation in which the mathematical features were not inherently salient, suggests that the person who provided this answer must have recognized the relevance of the quantitative relations on her own, that is to say, spontaneously, and used these quantitative relations in formulating her response. Thus, it can be said that providing this answer required the person to first spontaneously focus on quantitative relations.

Mathematics educators around the world are increasingly under pressure to teach complex and difficult content to students, and mathematical skills are increasingly seen as an important demarcation of a successful education (Ananiadou & Claro, 2009; NCTM, 2000). Despite this, children and students continue to struggle with a number of mathematical competences across the learning continuum. Learning rational numbers is one particularly difficult topic that has received increased attention in the past few years (NMAP, 2008) and large-scale evaluation studies show that students' understanding of fractions at the end of primary school strongly predicts their later learning of more advanced mathematical content (Siegler et al, 2012). A growing body of research has aimed to investigate difficulties in the development of the number concept – as students move from reasoning about natural numbers, expanding to reasoning about integers, rational, and real numbers (Christou & Vosniadou, 2012; Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004; 2010). In particular, details of the difficulties students and adults face with learning and reasoning about rational numbers can be found throughout the literature on mathematical cognition and education (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Durkin & Rittle-Johnson, submitted; Hallet, Nunes, & Bryant, 2010; Iuculano & Butterworth, 2011; Jordan et al., 2013; Mazzocco & Devlin, 2008; Meert, Grégoire, & Noël, 2010; Merenluoto & Lehtinen, 2004; Ni & Zhou, 2005; Obersteiner, Van Hoof, Van Dooren, & Verschaffel, 2013; Siegler, Fazio, Bailey, & Zhou, 2013; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Vamvakoussi & Vosniadou, 2004; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013; Van Hoof, Verschaffel, Van Dooren, 2013). The difficulty with rational numbers arises from uncertain origins, as young children show diverse competences with pre-mathematical, quantitative relations (Boyer, Levine, & Huttenlocher, 2008; Duffy, Huttenlocher, & Levine, 2005; Frydman & Bryant, 1988; Jeong, Levine, & Huttenlocher, 2007; Mix, Levine, & Huttenlocher, 1999; Singer-Freeman & Goswami, 2001; Sophian, 2000; Sophian, Harley, & Martin, 1995; Spinillo & Bryant, 1999; Wing & Beal, 2004), which have been viewed as precursors to reasoning about rational numbers (Boyer & Levine, 2012; Confrey et al., 2009; Lesh, Post, & Behr, 1988; Sophian, 2007). One explanation for the disconnect between early competences with quantitative relations and later difficulties with rational numbers is the influence of a bias towards reasoning about natural number even when dealing with rational numbers (Ni & Zhou, 2005; Obersteiner et al., 2013; Vamvakoussi et al., 2012; Van Hoof et al., 2013). Much of the evidence surrounding students difficulties with reasoning about rational numbers suggests the over-extension of natural number knowledge to rational numbers. Evidence suggests that overcoming this bias, which leads to the successful understanding of rational numbers, requires radical conceptual change (Merenluoto & Lehtinen, 2002; 2004; Vamvakoussi & Vosniadou, 2004; 2010; Vosniadou & Verschaffel, 2004).

Despite the increase in knowledge about difficulties with rational number concepts, two limitations hamper previous studies. First, there is little evidence of the actual development of conceptual knowledge of rational numbers. In particular, while there is some evidence detailing general cognitive predictors of fraction knowledge (Jordan et al., 2013), little evidence exists that highlights actual conceptual change with rational numbers over time or the different level of difficulties students face with this development. Furthermore, some studies have investigated secondary students' development of rational and real number concepts (e.g. Merenluoto & Lehtinen, 2004). However, there is little evidence on primary school students, who are first learning about rational number concepts. The second limitation is that, until now, there has been little evidence on the developmental antecedents of rational number conceptual knowledge. As well, the existing studies on early reasoning about quantitative relations and other processes related to formal rational number concepts have solely used tasks in which the participants were aware of the mathematical nature of the tasks (e.g. Boyer at al., 2008; Jordan et al., 2013). The success of previous studies in identifying the role of individual differences in Spontaneous Focusing On Numerosity (SFON) in mathematical development (Hannula & Lehtinen, 2005) suggests that further investigations may reveal a similar tendency that could be relevant for the study of the development of rational number conceptual knowledge.

Therefore, the present dissertation aims to investigate whether individual differences in children's spontaneous focusing on quantitative relations are related to the conceptual development of rational numbers. This is done by first determining if it is possible to capture individual differences in children's spontaneous focusing on quantitative relations. As well, the present dissertation aims to identify if there is a connection between the spontaneous recognition and use of quantitative relations and the development of rational number conceptual knowledge.

The present doctoral dissertation is made up four main sections providing a theoretical and methodological framework for and a summary of the six original empirical articles. First, the following theoretical sections detail existing research on a) the development of reasoning about quantitative relations in connection with the development of rational number skills (e.g. Boyer & Levine, 2012), b) the nature of the development of rational number conceptual knowledge is explained from the perspective of a conceptual change approach (Vosniadou & Verschaffel, 2004), and c) the contribution of spontaneous quantitative focusing tendencies, in particular, Spontaneous Focusing On Numerosity (SFON) to the development of mathematical skills is described (e.g. Hannula & Lehtinen, 2005). Second, the methodology of the present works is detailed. Third, the six empirical studies are summarized with regard to the aims of the dissertation as a whole. Finally, a discussion of the main findings, theoretical and practical implications, and challenges for future research is presented.

1.1 The Development of Reasoning about Quantitative Relations

Beginning in infancy and throughout early childhood, humans express the ability to compare and relate two or more quantities to each other (Dehaene, Izard, Spelke, & Pica, 2008; Feigenson, Carey, & Spelke, 2002; Feigeson, Dehaene, & Spelke, 2004; Odic, Libertus, Feigenson, & Halberda, 2013). These early competences with reasoning about quantitative relations have been documented in a number of studies on early childhood mathematical cognition (e.g. Boyer et al., 2008). These early competences are expected to be related to later formal knowledge of rational numbers (Boyer & Levine, 2012; Confrey et al., 2009; Lesh et al., 1988; Mazzocco & Delvin, 2008; Sophian, 2000). In the present study, quantitative relations are defined as the relationship between two or more objects, sets, or symbols based on some quantifiable aspect(s). Thus, quantitative relations can refer to a number of relations found in a child's everyday environment (Frydman & Bryant, 1988; Gallistel & Gelman, 1992; Resnick, 1992; Sophian, 2000; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Wynn, 1992). These relations can include a) exact or approximate proportional relations or ratios, represented by both continuous and discrete quantities, b) additive and multiplicative relations, including formal and informal arithmetical relationships, and c) exact numerical ratios, such as fractions.

Infants have been shown to recognize the halfway point in objects, showing surprise when unsupported objects do not fall under the sway of gravity (McCrink & Wynn, 2007). The system of approximate number found in infants and even non-human animals, has features of relational reasoning, both in terms of the mental number line's logarithmic scale and the weber ratio found in magnitude comparisons (e.g. Dehaene et al., 2008). Beyond these innate capacities, children in early childhood also show a number of capabilities to solve tasks using quantitative relations (e.g. Boyer et al., 2008). A number of studies have detailed early primary school age children's ability to match proportional mixtures of juice and water, especially those represented by continuous quantities (Boyer et al., 2008; Boyer & Levine, 2012; Duffy et al., 2005; Jeong et al., 2007). Spinillo and Bryant (1999) found that six-year-olds were similarly able to match continuous proportional quantities, though discrete proportional relations were only able to be handled by seven-year-olds. However, four-year-olds have also been found to be able to match proportionally sized shapes and proportional matched sets (Sophian, 2000; Sophian et al., 1995). Likewise, Mix and colleagues (1999) determined that four- and five-year olds were able to calculate simple arithmetic problems with fractional pieces of a foam circle. Finally, Frydman and Bryant (1988) found that five-year-old children could cope with different unit sizes of pieces of candy to give fair shares.

These early competences with quantitative relations are expected to be related to later fraction knowledge. Piaget argued that it is not until relatively late that children are able to reason about proportional relations (Piaget & Inhelder, 1951/1975). However, a number of studies indicate that this is too simplistic of an understanding of children's proportional reasoning, as different aspects of proportional reasoning develop separately (Lesh et al., 1988). As well, the gradual increase in the ability to reason about the different aspects of proportional relations has been suggest to lead to formal mathematical concepts such as rational numbers (Boyer & Levine, 2012, Lesh et al., 1988 Mix et al., 2002). Confrey and colleagues (2009) propose in their recent review that partitioning and splitting objects into equal groups is a core feature of rational number reasoning and, while key to its development, is underrepresented in curricula world-wide. The early act of sharing, particularly the regrouping of objects into new units (many-as-one) is argued as the foundational to the development of the concept of fractions. Sophian (2000) as well argues that early abilities to perceive quantitative relations in or between objects may lay the foundation for the later understanding of fractions. Thus, an explicit connection between the early competences and formal fraction learning would help lessen the difficulties children experience when learning about rational numbers.

Confrey and colleagues (2009) outline the connection between different aspects of quantitative relational reasoning and the development of a wide-range of skills with relational aspects in formal mathematics., such as ratio, fractions, and decimals However, a more parsimonious model may be more beneficial for considering the connections between early reasoning about quantitative relations and the development of conceptual knowledge of rational numbers. Resnick (1992) outlines a framework for development of mathematical reasoning by focusing on the nature of the objects which can be reasoned about mathematically, from the earliest levels of intuitive reasoning about informal non-exact "protoquantities", up to the notion of operations as entities with features that can be reasoned about independent of specific numbers. In this framework, Resnick attempts to make the connection between children's existing informal competences with pre-, or proto-, mathematical aspects and the related formal mathematical ideas. This is of particular interest when considering children's own self-initiated activities with mathematical aspects of everyday situations. While Resnick focuses on reasoning about arithmetic and natural number operations, she points out that this framework is also useful for considering other mathematical topics. Thus, the framework outlined by Resnick is applied to the progression from early reasoning about quantitative relations leading up to conceptual knowledge of rational numbers (see Figure 1).

At the most basic level in Resnick's model is the *mathematics of protoquantities*, which have no explicit quantitative value, but consist of vague quantitative notions, such as "many" or "more". In fact, this level can be seen as inherently relational, as Resnick notes, protoquantitative reasoning involves the "direct perceptual comparison of objects or sets of different sizes" (1992; pg. 412). While, this level of reasoning directly relates to many aspects of mathematics, notably absent from these comparison are exact quantities, which are necessary for, among other things, the equal partitioning that is a key precursor of rational number knowledge (Confrey, 2009). While it is possible that key relational boundaries, in particular half, may be used in reasoning at this level, as has been found in infant habituation studies (e.g. McCrink & Wynn, 2007), the explicit identification of these mathematical features, as such, is not possible (cf. Spinillo & Bryant, 1999)

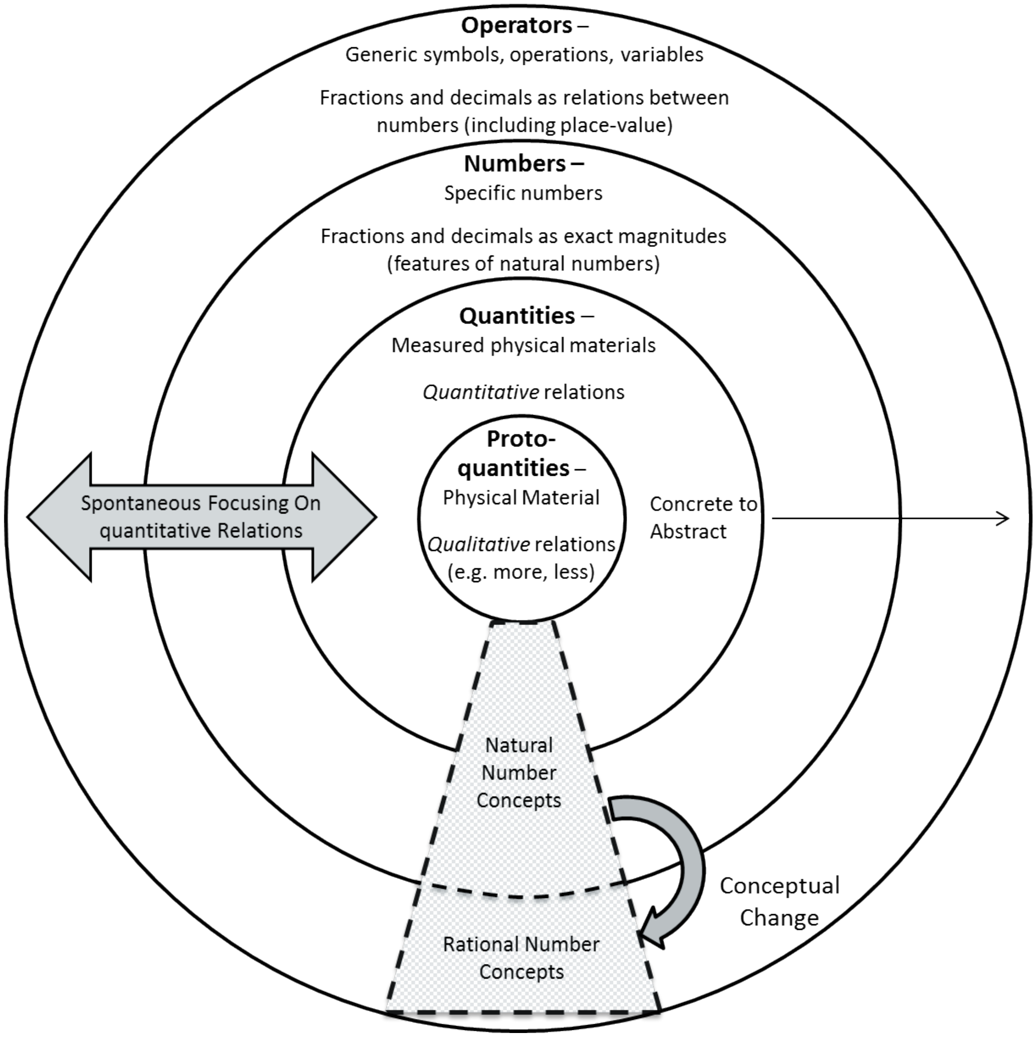

Figure 1. The development of quantitative relations and rational number knowledge. Based on model from Resnick (1992): From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge.

Resnick (1992) argues the next level of reasoning is the *mathematics of quantities*. This is the first level that involves explicit quantities, which are used in reasoning about physical material. Reasoning at this level involves the concrete application of the arithmetical operations, such as increasing, decreasing, combining, and partitioning objects using exact quantities. In this way, the quantities level contains the mathematical aspects referred to as quantitative relations in the present dissertation; the materials or objects used in the present study can be reasoned about as quantified physical material, not requiring the sole use of abstract numbers (though reasoning with formal numerical operations is possible on these tasks, it is not necessary). Importantly for the development of rational number knowledge, aspects of proportional reasoning with physical objects, which has been found in young children, also occur at this level. For instance, it is here that children are able to make the connection between the same part-whole relationship across different quantities or units (e.g. Boyer & Levine, 2012).

The next level of development defined by Resnick, the *mathematics of numbers*, is where numbers begin to act as "nouns", or "conceptual entities that can be manipulated and acted upon" (pg. 414). In terms of the development of rational numbers, this level would encompass the first skills and processes with symbolic fractions and decimals, where fractions and decimals are symbolic entities that can be acted upon and reasoned about independent of physical material. However, at this level, fractions and decimals may be seen more as special types of counting numbers that are static entities having own magnitudes, which are tied to them in a constrained manner. Thus at this level, ½ is not the relationship of 1 part to 2, but merely represents the magnitude of one-half of 1 (halfway between 0 and 1 on the number line). In this way, many features of natural numbers can be attached to fractions and decimals, often in a supportive manner (Nunes & Bryant, 2008). It may be possible to solve basic problems and possess routine skills with fractions and decimals at this level. However, it is also at this level that the natural number bias would cause problems with reasoning about fractions and decimals, through the over use of natural number features when reasoning about rational numbers (Ni & Zhou, 2005).

Thus, only in moving into the *mathematics of operations* level of reasoning do fully mathematically correct concepts of rational numbers appear. At this level, it is possible to reason about operations themselves, as operations themselves become "nouns" which "can be reasoned *about*, not just applied" independent of specific numbers (Resnick, 1992, pg. 414). Similarly, at this level, rational numbers can become entities to be reasoned about, as they are understood to represent the mathematical relations inherent in fractions (the relation between numerator and denominator) and decimals (the relation between place-value and terms). Thus, at this level, rational numbers become more than just whole numbers with non-whole values, as in the previous level, but mathematical objects that have specific features that are partially distinct from natural numbers (e.g. Vamvakoussi & Vosniadou, 2004). Thus, just as the concepts of arithmetic operations (e.g. commutativity, associativity, etc.) can be understood at this level, being that operations are now objects to be reasoned about, the new concepts of rational numbers (e.g. magnitude representations, density, etc.) can be understood at this level, being that rational numbers are now objects to be reasoned. However, reaching this level is not a simple progression as described by Resnick, but instead requires radical change in the conception of the nature of number (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004).

There are two main developmental aspects to this framework which are relevant for the present dissertation. First, children do not move uniformly from one level up to the next for all aspects of mathematics (Resnick, 1992). Thus, it is possible to be at the level of the mathematics of operations with natural numbers, but this level may actually coincide with the mathematics of numbers for rational numbers. So that, while children may be able to reason *about* the arithmetic operations with whole numbers, for example the commutativity of addition, this does not at all indicate that they are able to see fractions and decimals as the inherently relational objects they are (e.g. Siegler et al., 2013). It is precisely this feature of the development of mathematics that can cause disconnect between reasoning about natural numbers and reasoning about rational numbers that requires conceptual change (e.g. Merenluoto & Lehtinen, 2004). Also relevant for the present dissertation is the notion that the lower levels are not completely eliminated from use after moving to a higher level. As Resnick (1992, pg. 418) points out, "In passing to a higher layer of mathematical reasoning, the earlier layers are not discarded, but remain part of the individual's total knowledge system." This is particularly relevant for the discussion of spontaneous quantitative focusing tendencies. It is possible that the tendency to pay attention to quantitative relations in concrete situations (at the mathematics of quantities level) which are not explicitly mathematical may be related to the development of formal mathematical skills and knowledge.

1.2 The Natural Number Bias and Conceptual Change with Rational numbers

Preschool-aged children already can already reason about quantitative relations, included proportional relations (e.g. Boyer & Levine, 2012). Despite this, students and adults have a number of difficulties with reasoning about formal mathematical relations, especially with rational numbers. The natural number bias is presumed to be one cause of the difficulties with conceptual change with rational numbers, and is described as the inappropriate overuse of natural number features when reasoning about rational numbers (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010). The origins of the natural number bias remain a matter of controversy (Ni & Zhou, 2005). The natural number bias has been argued to arise from a number of environmental and biological sources. A number of studies have revealed the early existence of a tendency to use counting numbers inappropriately in response to proportional problems (Boyer at al., 2008; Jeong et al., 2007; Spinillo & Bryant, 1999). These results suggest that even though early competences with proportional reasoning exist, children often have trouble moving past the bias towards using counting numbers in their reasoning when it is possible to use counting numbers. It is possible that discrete representations cause more difficulties than continuous representations of quantitative relations because counting numbers are more easily recognized with discrete representations (e.g. Spinillo & Bryant, 1999).

The natural number bias may likewise be influenced by other representational issues, including innate capacities and cultural tools. For examples, despite the mental number line initially having a continuous nature (e.g. Dehaene et al., 2008), some evidence suggests that individual magnitudes on the mental number line are subsequently represented as discrete quantities (Feigenson et al., 2004; Gallistel & Gelman, 1992). However, definitive evidence of the discreteness of initial magnitude representations is lacking (Vamvakoussi et al., 2012). What is certain is that cultural tools, especially language, and everyday experience consistently nurture the discrete nature of natural numbers (Andres, Di Luca, & Presenti, 2008; Carey, 2004), while offering little support for reasoning about relational concepts (Greer, 2004). The discrete representation of numbers that remains ubiquitous throughout childhood and early school years may cause difficulties in coming to understand the dense nature of the rational number line.

In general, it appears that aspects of relational reasoning and reasoning about counting or natural numbers interact in complex ways throughout the early development of mathematical and pre-mathematical skills and processes. This interaction likewise has a complex effect on learning about rational numbers in formal mathematics, seemingly being both a blessing and a curse. So that, on the one hand students are able to apply natural number concepts correctly in reasoning about rational numbers, when they are appropriate (Nunes & Bryant, 2008), while on the other hand, children, and even adults, also tend to overuse concepts of natural number that are incompatible with rational number (Merenluoto & Lehtinen, 2004; Obersteiner et al., 2013; Vamvakoussi & Vosniadou, 2004; Vamvakoussi et al., 2012; 2013). What is clear from all of this, is that a great deal of study of the development of formal knowledge of rational numbers, beginning already in early childhood documenting relational reasoning, is needed to make sense of this complexity.

Fractions and decimal learning has been long established as a difficult topic for students. Proportional reasoning, including rational numbers, has been described by Lesh and colleagues (1988) as "the capstone of children's elementary school arithmetic … [and] the cornerstone of all that is to follow." Indeed, recent evidence suggests that rational number knowledge is an important indicator of skills with more advance mathematical topics such as algebra (Siegler et al., 2013). While students difficulties with rational numbers occurs at both the procedural and conceptual level (Hallett et al., 2010; Rittle-Johnson, Siegler, & Alibali, 2001), conceptual knowledge of rational numbers has been found to be a stronger predictor of later learning (Hallett et al., 2010) Difficulties with learning rational numbers concepts have often been viewed through the lens of learning theories of conceptual change (Merenluoto & Lehtinen, 2004; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004; Vosniadou & Verschaffel, 2004). The conceptual change view of learning rational numbers suggests that learning about concepts of fractions and decimals requires radical change in the underlying concept of number. This expansion of the concept of number from natural, counting numbers to rational numbers is not a smooth linear process, but involves significant representational and conceptual shifts in what constitutes a number and what are the fundamental features of number (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004; 2010). Thus, the development of rational number conceptual knowledge from natural number knowledge exhibits key elements of the theoretical framework of conceptual change (Vosniadou, 1994)

One possible way to operationalize the fraction and decimal concepts that must be re-conceptualized for a mature mathematical understanding of rational numbers is as, a) magnitude representations b) density, and c) operations (Stafylidou & Vosniadou, 2004; Van Hoof et al., 2013; Vamvakoussi & Vosniadou, 2004). In the present dissertation, since the focus is on the first phases of learning about rational numbers, only magnitude representations and density concepts will be addressed. The concept of *magnitude representations* of rational numbers captures the aspects of the symbolic representations of rational numbers that determine their magnitudes (Meert et al., 2010; Schneider & Siegler, 2010). Previous cross-sectional studies have revealed the importance of the density of rational numbers in developing a fully mathematical understanding of rational numbers (Merenluoto & Lehtinen, 2004; Stafylidou & Vosniadou, 2004; Vamvakoussi et al., 2011; Vamvakoussi & Vosniadou, 2004; 2010). *Density* concepts make up a key part of the understanding of the structure of the set of rational numbers.

A number of previous studies have documented students' and adults' difficulties with reasoning about the magnitude representations of rational numbers and have identified the existence of the natural number bias in the representation of magnitudes of fractions and decimals (DeWolf & Vosniadou, submitted; Durkin & Rittle-Johnson, submitted; Merenluoto & Lehtinen, 2004; Ni & Zhou, 2005; Obersteiner et al., 2013; Schneider & Siegler, 2010; Stafylidou & Vosniadou, 2004; Vamvakoussi et al., 2012; Van Hoof et al., 2012). Students face a number of difficulties with learning about fractions and decimal magnitudes; natural number magnitudes are directly perceived by symbolic representations, as well they can only be represented by one term. While with natural numbers a magnitude is represented by one and only one term (e.g. 1=1), one rational number magnitude can be represented by an infinite number of terms (e.g. 0.5=0.50=1/2=2/4=etc.). As well, unlike natural numbers, rational numbers cannot be automatically perceived.

Determining a fractions' magnitude requires understanding that the fraction represents a ratio between the two terms that make up the fraction. Less mathematically correct concepts of the magnitude representations of fractions may involve thinking about the two terms in a fraction as separate integers (Stafylidou & Vosniadou, 2004; Vamvakoussi et al, 2012; Van Hoof et al., 2012). So that, in determining the magnitude of a fraction, those with a non-mathematically correct conception of a fraction, may focus only on the magnitude of the numbers which make up the numerator and denominator. This conception of fraction magnitude lacks the understanding that fractions represent the relation between the two terms. This component based approach to the representation of fractions leads to the mistake of identifying fractions with smaller component terms as having a smaller magnitude, or vice-versa, even if this is not true. For example, one common evaluation of fraction magnitudes that reflects this component based approach is stating, for example, that 1/3 is less that 1/6, since 3 is less than 6. Surprisingly, some evidence suggests that this component based judgment of fraction magnitude may persist at an intuitive level, even among highly educated adults. For example, even mathematics experts display evidence of the natural number bias in reaction time studies of their judgments of the magnitudes of fractions (De Wolf & Vosniadou, submitted; Obersteiner et al., 2013; Vamvakoussi et al., 2012).

Magnitude representations of decimals also run counter to features of natural number (Durkin & Rittle-Johnson, submitted; Vamvakoussi et al., 2012). Decimal numbers do not follow the easily perceived feature of natural number that dictates that more digits means a larger magnitude (e.g. 65 > 7). Thus when faced with decimals that defy this feature of natural number, such as 0.65 and 0.7, students often have difficulties overcoming the natural number bias to recognize that in fact 0.7 is larger than 0.65, despite containing less terms (Vamvakoussi et al., 2012). Recent evidence suggests however that students' conceptions of the magnitude representations of decimals are even more varied, with some conceptions suggesting a misappropriation of fraction concepts to decimal concepts (Durkin & Rittle-Johnson, submitted), so that some students show the opposite bias in interpreting a decimal's magnitude, determining that those decimals with more terms are automatically smaller than those with fewer terms. This mistake may stem from the feature of fractions that a larger denominator decreases the magnitude, and therefore including a smaller place-value in a decimal indicates a smaller magnitude. Similar to judgments of fraction magnitudes, evidence suggests that the mathematically incorrect interpretation of decimal magnitudes persists on an intuitive level even into adulthood, as evidenced by reaction times (e.g. Vamvakoussi et al., 2012)

Understanding the concept of the density of rational numbers is one of the most difficult aspects of learning about fractions and decimals (e.g. Vamvakoussi & Vosniadou, 2010). Unlike natural numbers, which have a fixed order and always have a successor term, it is impossible to define the next rational number in a sequence and there are always an infinite number of rational numbers between any two rational numbers. The conceptual change needed for coming to understand these features of rational numbers is particularly difficult as the natural number sequence has been a part of most children's mathematical conception from an early age, strengthening the natural number bias towards a discrete representation of the number line (Andres et al., 2008; Ni & Zhou, 2005). So that, throughout a child's early mathematical experiences the number sequence has always been presented as a fixed, ordered, and discrete object. In contrast, there is no fixed sequence to rational numbers, with no successor being identifiable, and they are densely ordered. Even after many years of instruction and use of rational numbers, many students and adults fail to develop a mathematically mature understanding of how rational numbers are densely ordered (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004; 2010; Vamvakoussi et al., 2011; Van Hoof et al., 2013). Indeed, students display a conception of the number line as being discrete in some instances, even when recognizing its dense nature in other cases (Vamvakoussi et al., 2011). It appears that when holding these inconsistent views of the discrete or dense nature of the number line, students often have more difficulties with expressing the dense nature of fractions than decimals. These results suggest that conceptual change is needed to transverse the divide between natural numbers' discrete nature, through an incoherent conception of the nature of the number line, which includes both discrete and dense features, to a coherent understanding of the dense nature of the number line (Vamvakoussi & Vosniadou, 2010).

A number of studies have detailed the lack of understanding students and adults have of concepts of the magnitude representations and density of rational numbers (e.g. Vamvakoussi & Vosniadou, 2010). These difficulties often are a result of the inappropriate application of natural number concepts to rational number reasoning. The incongruent nature of supplementing natural number knowledge with newly learned rational number knowledge, due to fundamental differences in the nature of these types of numbers, presents a fundamental challenge in learning this material (Merenluoto & Lehtinen, 2004). While a number of studies have been successful in documenting non-mathematically correct conceptions of rational numbers, there is little evidence on the developmental trajectory of the conceptual knowledge of rational numbers (cf. Durkin & Rittle-Johnson, submitted). Since in all of these studies at least some proportion of the participants have possessed mathematically correct conceptions of rational number, it can be expected that students have different levels of success in developing mathematically correct concepts of rational numbers (Siegler, Thompson, & Schneider, 2011). Despite this, few studies have been able to capture differences in individuals' development. Most importantly, despite evidence suggesting that there is, early on, some ability to reason about quantitative relations that may be related to the later learning of formal rational number concepts (e.g. Confrey et al., 2009), there is little evidence indicating early or concurrent causes of individual differences in developmental trajectories with rational number knowledge. Filling the gaps in the research literature on the nature of the development of rational number concepts and its antecedents is one of the major aims of the present set of studies.

1.3 SFON and spontaneous quantitative focusing tendencies

It is apparent that a large part of mathematical development lies outside of situations in which mathematical aspects are made explicit (Bransford et al., 2009). In other words, a great deal of the situations in which it is possible, and even necessary, to use mathematics occur outside of formal mathematical learning situations. In these non-formal situations, it is often the case that the recognition and use of mathematical skills must be done without any outside guidance, on ones' own. Studying the necessary sub-processes triggering the use of mathematical skills is therefore crucial for how mathematical skills develop (Hannula, 2005; Hannula & Lehtinen, 2005). The tendency to recognize the mathematical aspects of these non-explicitly mathematical situations may not be equally strong in individuals. These inter-individual differences in the tendency to recognize and utilize mathematical aspects may have important consequences on the development of mathematical skills. It is therefore extremely relevant for understanding of differences in mathematical development to investigate these types of mathematical attentional tendencies, referred here to as spontaneous quantitative focusing tendencies.

The investigation of spontaneous quantitative focusing tendencies involves measuring the unguided recognition and use of mathematical aspects of a situation that is not explicitly mathematical (Hannula, 2005; Hannula-Sormunen, in press). One such spontaneous quantitative focusing tendency, introduced by Hannula and Lehtinen (2001; 2005), that has been the subject of extensive investigation in the study of young children's mathematical competences, is Spontaneous Focusing On Numerosity (SFON). Individual differences in SFON have been found in studies around the world using a number of different measures with participants from the age of three years old until adulthood (Edens & Potter 2013; Hannula et al., 2009; Hannula & Lehtinen, 2001; 2005; Hannula, Lepola, Lehtinen, 2010; Hannula, Mattinen, & Lehtinen, 2005; Hannula, Räsänen, & Lehtinen, 2007; Kucian et al., 2012; Poltz et al., 2013). SFON can be said to index inter-individual differences in the unguided recognition and use of exact number in non-explicitly mathematical situations.

The isolation of SFON as a measurable construct has been detailed in a number of studies by Hannula and colleagues (2001; 2005; 2007; 2010). Their evidence suggests that, when not guided towards these features, it cannot be assumed that all children or adults recognize with the same frequency or sensitivity the aspect of number as a salient or relevant feature of situations. Individual differences in SFON tendency have been found to be a domain specific predictor of mathematical, but not reading, development (Hannula & Lehtinen, 2005; Hannula et al., 2010). In particular, SFON tendency in four-year-olds has been found to predict mathematical skills one year later (Hannula & Lehtinen, 2005). These mathematical skills subsequently predicted SFON tendency one year later, suggesting the reciprocal development of SFON tendency and mathematical skills, each encouraging further growth in the other. Furthermore, SFON tendency in kindergarten was found to be a domain specific predictor of mathematical achievement in second grade (Hannula et al., 2010). So that, even after taking into account prior general cognitive ability measured by a non-verbal IQ test, SFON tendency predicted general mathematical skills, but not reading ability. As well, these studies suggest that SFON is not entirely explained by enumeration skills, general attentional ability, or the ability to follow directions (Hannula & Lehtinen, 2005; Hannula et al., 2007; Hannula et al., 2010). An experimental study among kindergarteners suggests that SFON tendency can be enhanced through specific interventions, which have been successful in enhancing both SFON tendency and subsequent development of enumeration skills (Hannula et al., 2005). This study also confirmed that SFON as measured in laboratory tasks was related to the spontaneous use of exact number in everyday situations, further supporting the claim that SFON tendency is a general tendency to recognize and utilize numerosities in a variety of task contexts (see also Hannula, 2005).

Spontaneous quantitative focusing tendencies, such as SFON, may tap into qualitative and quantitative differences between individuals in the spontaneous use of mathematical aspects in everyday situations. These differences in the unguided attention to mathematical (or pre-mathematical) aspects may have significant impact on the development of the related formal mathematical content. Those who have a higher tendency to spontaneously focus on certain mathematical aspects of a situation may gain more self-initiated deliberate practice with this aspect (Ericsson, 2006), contributing to more successful learning outcomes. In studies on the development of expertise in fields such as music or chess there are fundamental differences in the amount and type of deliberate practice obtained by those who become experts. In many situations, coaches or teachers may be necessary (and available) to point out the opportunity for the developing expert to practice certain skills. However, it has been found that future experts are also able to recognize, without external guidance, possibilities in everyday situations to practice their skills (Ericsson & Lehman, 1996). Those who are more successful in learning mathematical concepts and skills may be gaining qualitatively better and quantitatively more self-initiated practice with mathematical skills, through a stronger tendency to spontaneously focus on these aspects.

Deliberate practice, as defined within the study of expertise, has a number of features that distinguish it from what could be called non-deliberate practice (Ericsson, 2006; Ericsson & Lehman, 1996). One defining feature of deliberate practice is that it occurs at the edge of a budding expert's competence. Thus, at an early age, when first learning to play the piano, practicing and mastering the different musical scales may be extremely relevant for the development of expertise with piano playing. However, after a few more years of study, the relevance of practicing these basic skills will diminish, as more advance skills develop and require mastery. The same type of progression in spontaneous quantitative focusing tendencies may influence the development of mathematical competences all along the learning trajectory.

The recent finding that SFON tendency at the age of 6 years has been found to predict conceptual knowledge of rational numbers at the age of 12 years old is of particular interest to the present dissertation (McMullen, Hannula-Sormunen, & Lehtinen, in preparation).

These results suggest that spontaneous quantitative focusing tendencies may be relevant for the study of conceptual change with rational numbers. In particular, SFON tendency in early primary school may promote exploration with features of natural numbers, which leads to a conceptual understanding of the number system that helps overcome the natural number bias when learning about rational numbers. However, the same study found that concurrent SFON tendency was not significantly related to rational number conceptual knowledge at the age of 12. So while, at an early age, individual differences in SFON play a significant role in the development of mathematical skills, including enumeration skills and later rational number knowledge (Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, & Lehtinen, in preparation), individual differences in self-initiated practice with counting and enumeration alone may have a diminished effect on developmental differences, as mathematical topics increase in complexity.

When viewed from the perspective of the development of expertise, these results do not necessarily indicate that over time spontaneous quantitative focusing tendencies lose effectiveness in explaining individual differences in conceptual development of rational numbers. Instead, it is expected that a more mathematically advanced spontaneous quantitative focusing tendency increases in importance as mathematical topics taught in the classroom become more advanced. It might be that a spontaneous quantitative focusing tendency that supports self-initiated deliberate practice with quantitative relations becomes more important for the development of rational number knowledge, as enumeration processes become more habitual and well-developed. Thus, the present dissertation details a number of studies that investigate Spontaneous Focusing On quantitative Relations (SFOR), and the role of SFOR tendency in the development of rational number conceptual knowledge.

2. RESEARCH QUESTIONS/AIMS

As has been shown, more evidence is needed to better understand individual differences in the development of rational number conceptual knowledge. Not only is there a need to understand more clearly the actual development of conceptual change with rational numbers, but both theoretical and pedagogical pursuits are more well-informed with a clearer understanding of the development of differences in the learning of rational numbers (NMAP, 2008; Siegler et al., 2013). Early reasoning about quantitative relations can be expected to be related to later rational number knowledge (e.g. Boyer & Levine, 2012). However, up until now, studies of quantitative relations have focused on what children can do, when explicitly guided to do so. Previously, the investigation of not just what children *can* do, but what children *actually* do in non-explicitly mathematical situations suggests that spontaneous quantitative focusing tendencies are relevant for the development of mathematical skills (e.g. Hannula & Lehtinen, 2005). Therefore, the present study aims to:

a) delineate spontaneous focusing on quantitative relations as a distinct aspect of task performance

b) examine the development of rational number conceptual knowledge in late primary school children, and

c) explore the relationship between SFOR tendency and the development of rational number conceptual knowledge.

The present work therefore is made up of three cross-sectional and three longitudinal studies including children from the ages of five to thirteen years old. Studies I-III aim to examine the spontaneous recognition and use of quantitative relations, as a part of children's existing mathematical competences. Studies IV-VI aim to investigate the role of SFOR tendency in relation to the development of rational number conceptual knowledge.

Along with these more general goals, each study had more specific aims. Study I attempted to measure children's spontaneous recognition and use of quantitative relations in two mathematically unspecified tasks. This study focused on the individual and age related differences among young children from the ages of five to eight. Study II expanded on the previous study and examined how children's spontaneous recognition and use of quantitative relations was related to their verbalizations of reasoning in solving the same tasks, allowing for the examination of the goal-directed nature of the spontaneous recognition and use of quantitative relations in these tasks. Finally, Study III aimed to replicate the findings of Study I in a new sample of children in the USA. This study also aimed to provide evidence that individual differences in SFOR could not be entirely explained by the ability to use quantitative relations on the tasks. Finally, this study hoped to identify whether SFOR tendency was related to other more general mathematical skills.

Study IV was the first study that aimed to connect SFOR tendency with rational number conceptual knowledge. In particular, this study aimed to investigate if SFOR tendency in first grade was related to conceptual knowledge of fractions in fourth grade. This would allow for the determination of whether SFOR may play a role in the development of rational number knowledge. Study V examined in detail the conceptual change process with rational numbers in third to sixth grade students. This study aimed to identify the developmental of rational number concepts in these students, and to determine if there were individual differences in the conceptual change with these concepts. Study VI aimed to synthesize the findings of Studies I-V and present a longitudinal investigation of the contribution of SFOR tendency to the development of rational number conceptual knowledge among late primary school children. This final study provided the strongest evidence indicating how SFOR tendency might influence the development of mathematical skills.

3. METHODS

For the studies presented within this work, a mixture of longitudinal and cross-sectional methods was used. This allowed for the study of the role of SFOR tendency on the development of rational number conceptual knowledge, and as well static age-related and individual differences in SFOR and its relation to mathematical skills. The six studies that make up this doctoral dissertation are based on data from two research projects focused on children from primary school and early childhood education. The ability to look at SFOR across a wide age-range was particularly valuable in identifying how pre-existing cognitive tendencies may affect formal learning. Furthermore, the studies emanated from data collected from Finland and the USA allowing for the investigation of SFOR in multiple national contexts. All data was collected in schools or day-care centers that the participants attended, during normal times of attendance. All studies were conducted at the Centre for Learning Research (University of Turku, Finland) together with Minna M. Hannula-Sormunen and Erno Lehtinen as a part of the Academy of Finland funded DEMAS project on the mathematical development of children in early childhood and the Academy Professor project of Erno Lehtinen. In addition, the current dissertation was funded by the Multidisciplinary Doctoral Programme of Research on Learning Environments (OPMON) and the Academy of Finland.

3.1 Participants

Studies I, II, and IV

The original cross-sectional sample used in Studies I and II, included children from three day-care centers and three schools from socio-economically representative areas of a city in southwest Finland (ca. 180,000 inhabitants). Parent permission was sought for all eligible children; of the 95 written consent forms sent to parents, nine children were excluded from the study due to previously diagnosed learning impairments, a non-Finnish home language, or parent refusal. Thus, the data collection included 86 participants (43 girls) completing measures of SFOR and other mathematical skills. Children were between the ages of 4 years and 5 months and 8 years and 4 months, at the time of the first testing, and belonged to three different educational groups, kindergarten (n = 31; M_{AGE} = 5 years; 6 months), pre-school (n = 27; M_{AGE} = 6 years; 9 months), or grade one (n = 28; M_{AGE} = 7 years; 9 months).[1] Parental educational attainment for the sample was representative of adults between the ages of 25-49 in Finland.

1 In Finland, children begin formal schooling ("first grade") in the fall of the year they turn 7-years-old. Children can attend an optional pre-schooling year ("preschool") with 700 h of preschool education covering all main areas of children's academic skill development starting in the fall of the year they turn six. Before this, children can go to kindergarten ("kindergarten"), which focuses more on supporting children's overall development rather than their specific academic skills.

Study IV consisted of follow-up measures for those students in grade one at the time of the original testing. Two participants could not be located and one student was not included in the final measures because he/she had previously repeated a grade. Thus, there were 25 children included in the final sample who were from the ages of 7 years and 2 months to 8 years and 4 months (M= 7y; 8m) at the time of initial testing. As well as completing the measures of SFOR and mathematical skills in first grade as a part of Studies I and II, these participants completed a test of fraction conceptual knowledge in the fourth grade. Participants" math grades were also collected from the fall term of fourth grade.

Study III

This cross-sectional sample of children from a mid-size city (ca. 125,000) in the southeast United States included 84 kindergarten to third grade children from one economically and socially diverse school. After approval from the ethical board of the University of Turku, the School Board, and school, parental approval was received for the children. The participants (42 female) were between the ages of 5 years and 8 months and 9 years and 8 months old (M=7 years, 9 months; SD=13.7 months). Participants were grouped according to their grade-level for analysis: Kindergarten (n=23; M_{AGE}=6y; 4m), First Grade (n=20; M_{AGE} =7y; 3m), Second Grade (n=21; M_{AGE} =8y; 4m), and Third Grade (n=20; M_{AGE} =9y; 2m). Participants had no diagnosed learning, neurological, or attentional impairments. Participants all completed SFOR tasks, tasks measuring guided focusing on quantitative relations, and measures of general mathematical skills.

Studies V and VI

Participants in Studies V and VI were 263 students (141 female) from two economically and socially diverse primary schools in southwest Finland. After parents gave permission for their child's participation, all participants were informed about the nature of the study, and could refuse to participate or stop at any point during testing. At the start of the study, students were in 3rd to 5th grade, between the ages of 9 years and 2 months and 12 years and 2 months (M = 10y; 8m, SD = 10.3m) . All participants completed a SFOR task, the rational number test, and measures of mathematical and reasoning skills. For Study VI, 42 students were unable to complete all three follow-up measures, and were excluded from the final analysis. Neither excluded group differed from their peers on initial rational number knowledge.

3.2 Assessments of SFOR

The development of SFOR tasks was highly informed by the previous development of tasks that measure spontaneous focusing on numerosity (Hannula, 2005; Hannula & Lehtinen, 2005). SFOR tasks aim to measure a general tendency to focus on quantitative relations in situations that are not explicitly mathematical, including everyday situations. Key to the measurement of SFOR then is the fact that the participants should not be guided towards the mathematical aspects in the tasks, it can then be said that if they use these aspects in completing the tasks it was done spontaneously. Thus, the tasks should not have an explicit mathematical nature and should have multiple aspects, both mathematical and non-mathematical, that can be used to successfully complete the task. Before and during participation in the tasks, no mention can be made of quantitative or mathematical aspects. Either SFOR tasks should be presented before any other measures in a test battery, or there should be non-mathematical tasks prior to the presentation of SFOR tasks. During testing, there should be no discussion of any aspects of the tasks, nor should participants be given explicit feedback. Thus, the researcher should not lead participants either towards or away from the mathematical aspects of the tasks. Thus, instructions should remain open and neutral, and no explicit feedback should be given. Finally, only a small number of trials should be used when measuring a spontaneous quantitative focusing tendency, in order to preserve the spontaneous nature of participants' actions.

In order to determine if the differences in the use of quantitative relations are due to differences in spontaneous attentional processes and not due to differences in mathematical ability, the skills required to use quantitative relations successfully should be well within the competences of the participants. In other words, all participants could solve the tasks using quantitative relations when explicitly asked to do so (Hannula, 2005; Hannula & Lehtinen, 2005).

Scoring of SFOR tasks for Studies I-IV was based on video analysis of participants' utterance and actions in the task situations. Participants were scored as spontaneously responding based on quantitative relations if they gave the same total amount of bread or rice, having negotiated the different sizes of the material (e.g. a spoon three times the size of another). Thus, in order to respond with the same amount of material, the participants must have taken into account that the different unit sizes of the different sets, determined the quantitative relationship between these two units and sets, and calculated the correct response. If children made any mention of quantitative relations (e.g. "Cause one of these is equal to two of those"), they were also scored as responding based on quantitative relations, no matter the amount of bread or rice given.

In Studies I and II, the participants' responses using exact number were also coded, those in which they matched exactly the number of pieces of bread or spoonfuls of rice that the experimenter gave. A similar criterion was used for verbal responses that used exact number phrases or words. In these studies, the stimulated-recall interviews were also used to determine participants' interpretations of the tasks. After completing the SFOR tasks, the same exact tasks were repeated, but participants were asked to explain how they determined how much material to give. Participants' responses were coded with regard to their use of quantitative relational, exact number, or non-mathematical verbal responses. Furthermore, non-mathematical responses were analyzed to determine what other aspects participants found relevant in these tasks.

For Study VI, SFOR task was paper and pencil based and conducted in a whole class setting. Testing always occurred during a non-mathematical lesson, so as not to give a hint to students that the task may be mathematical in nature. This task represents the first time a spontaneous quantitative focusing tendency was measured in a group setting, presenting particular challenges for reliable assessment. One key feature of the testing situation was that participants must be strongly encouraged to think of the tasks as not having one correct answer, as the classroom setting may cause participants to think of the tasks in a more traditional manner. However, the success of these tasks suggests that SFOR measures can be presented at the group level, which presents more opportunities for both future studies of SFOR and the creation of a diagnostic tool for reliably measuring SFOR.

In the picture explanation task, participants' spontaneous relational responses were scored on a three-point scale based on an analysis of their written and drawn responses. Participants received two points if they described the exact multiplicative relation between the two groups of objects or if their drawing reflected the correct multiplicative transformation for all three sets. Participants received one point if they described a non-exact multiplicative relation (e.g. "they multiplied.") or if their drawings reflected consistent, but incorrect, multiplicative changes in the correct direction (e.g. all sets increased by a factor of two, when the correct relation was multiplied by three). For any other response participants received no points.

One of the more important considerations in developing tasks measuring spontaneous quantitative focusing tendencies is the delineation of these tendencies from the requisite skills (e.g. Hannula & Lehtinen, 2005). First, it is important to develop tasks which fall well within the cognitive, attentional, and memory capabilities of the participants. So that, if there are differences in participants responses, it is not due to these other more general capabilities, but a result of differences the spontaneous recognition and utilization of the mathematical aspects. One method for confirming that SFOR tasks are within participants' capabilities is to directly measure participants' ability to solve the tasks using the mathematical aspects when explicitly guided to do so. In Study III, the bread and rice tasks were repeated with explicit instructions to use quantitative relations for those children who did not use quantitative relations in their responses in the spontaneous conditions. In Study VI, after the spontaneous version of the task, participants were asked to describe, using multiplicative relations, how the objects changed. The use of guided versions of these tasks allow for the confirmation that differences in performance on the SFOR tasks was due to differences in SFOR, and not differences in ability to solve the tasks using quantitative relations.

3.3 Measures of conceptual knowledge of rational numbers

One goal of the present dissertation was to validate an instrument that measures students' conceptual knowledge of rational numbers. While there is a wealth of knowledge detailing students' and even adults' deficiencies with rational number knowledge (e.g. Merenluoto & Lehtinen, 2004), there are few diagnostic tools that measure this lack of knowledge (cf. Van Hoof et al., 2013). In particular, the aim was to create a measure that could capture a large range of individual differences in late primary students' conceptual knowledge, allowing for a longitudinal analysis of SFOR tendency effects on this knowledge. The first version of the test was created as a part of an earlier study of our research group (Lundman, 2009; McMullen, Hannula-Sormunen & Lehtinen, in preparation). Some of the task were similar to tasks used in studies of Martinie (2007), Merenluoto and Lehtinen (2004), Stafylidou & Vosniadou (2004), and Vamvakoussi & Vosniadou (2004). The version used in this study was developed on the basis on the detailed analysis of students' responses to the items of the original test version. The rational number test was designed especially in consideration of capturing the process of conceptual change with rational numbers undergone (or not) when first transitioning from conceptions of natural numbers to rational numbers. Thus, the Rational Number Test (RNT) used in part in Study IV, and fully in Studies V-VI, was used as a measure of rational number conceptual knowledge focusing on two aspects of rational numbers that require substantial conceptual change: magnitude representations and the density of fractions and decimals.

A number of studies (Durkin & Rittle-Johnson, submitted; Obersteiner et al., 2013; Vamvakoussi et al., 2012; Van Hoof et al., 2013) have found that when reasoning about rational numbers using natural number conceptions, certain problems cause particular difficulty. Thus, the RNT include items that aimed to capture these mathematically challenging conceptions. Fraction items involved comparing and ordering fractions in which the total magnitude and the component terms were incongruent, e.g. 3/4 compared with 5/8; a common mistake is to judge the fraction with the larger terms to have a larger magnitude. Decimal items involved comparing and ordering decimals in which the number of terms and the total magnitude were incongruent, e.g. 0.7 compared with 0.65; a common mistake is to judge those decimal numbers which have more terms as having a larger magnitude.

Likewise, difficulty with understanding the dense nature of rational numbers has been detailed in both students and adults (Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004; 2010). Density items included asking participants to state whether there were any numbers (and if so, how many) between either two fractions or two decimals; a common conceptual misunderstanding is that rational numbers, like natural numbers, are discrete in nature with no numbers between any two numbers. Responses were categorized by their level of mathematically correct conceptions. Two points were given for responses that indicated a mathematically correct conception of the density of rational numbers, one point was given for responses that recognized that there were some numbers between the two fractions or decimals, and no points were given for responses that stated that there are no numbers between the given fractions or decimals. Other density items involved asking participants to identify the largest or smallest possible fraction. While problems requiring declarative knowledge of conceptions of density may be more difficult than multiple-choice versions of those problems (Vamvakoussi & Vosniadou, 2010), the use of short answer items can capture a broader range of responses than multiple choice and can allow for a more detailed mapping of the development of conceptual change with the concept of density.

Test-retest effects are a concern when there are multiple administrations of achievement test. One way to lessen these effects is to not use exactly the same items in the different administrations. Therefore, for each instance of RNT testing the numbers were slightly altered. However, the concepts that were the target of the different questions were taken into account, so that the items were equally able to captured the participant's' concept of number across the different administrations.

3.4 Statistical Analysis

The different statistical analyses that were used in the studies presented in this dissertation can be found in Table 1. The majority of the statistical analysis was conducted in the statistical package SPSS (versions 17-19). The latent variable mixture models were conducted using Mplus 7.0 (Muthén & Muthén, 1998-2012)

Table 1. Statistical analyses used in different studies.

Statistical Analysis	Study	Purpose
Analysis of Variance		
Paired T-tests	III	Comparison of relational responses in spontaneous and guided tasks
ANOVA/ANCOVA/Repeated measures ANOVA	I-IV, VI	SFOR group comparisons
Latent variable mixture models		
Latent Profile Analysis	V	Classification by rational number knowledge
Latent Transition Analysis	V	Longitudinal modeling of rational number knowledge
Linear Regression analysis		
Calculating Residualized Scores	VI	Isolation of SFOR and rational number knowledge
Linear regression model	VI	Predictors of rational number knowledge

Analyses of variance were conducted in most studies, after testing for the normality of the data[2]. Paired sample t-tests were used in Study III to determine whether there were significant differences in the use of relational responses in the spontaneous version of the bread and rice tasks versus the guided versions of these tasks. For studying group level differences, ANOVA's were used in Studies I-III, mostly to compare age-related differences in SFOR, ANCOVA's were used in Studies III and IV to measure how SFOR tendency was related to arithmetic skills (Study III) and fraction conceptual knowledge (Study IV). Repeated measures ANOVA's were used in Study VI to track differences in rational number conceptual knowledge between two with a high SFOR tendency and a low SFOR tendency across time points. Post-hoc comparisons were used in Studies I-III to examine specific age-group and SFOR-group differences.

Latent variable mixture models, in particular Latent Profile Analysis (LPA) and Latent Transition Analysis (LTA), were used to estimate the most appropriate model for classifying students' conceptual knowledge of rational numbers in Study V. LPA models were estimated for each administration of the test independently, while the LTA model was estimated across all four time points. The use of latent variable mixture models, especially latent transition analysis, in capturing processes of conceptual change has been successful with other concepts including mental models of the earth (Straatemeier, van der Maas, & Jansen, 2008) and floating and sinking concepts (Schneider & Hardy, 2013). However, the present study is the first to report on these methods use with modeling conceptual change with rational numbers.

The use of latent variable analyses works best when it is substantiated by a theoretical framework that allows for the evaluation of models based on both statistical and theoretical grounds (Nyland, 2007; Nylund, Asparouhov, & Muthén, 2007). Both types of latent variable models were estimated with Mplus 7.0 (Muthén & Muthén, 1998-2012) using a maximum likelihood with robust standard errors estimation method (MLR). This approach to estimate better handles missing data. The LPA and LTA were carried out using mixture and longitudinal mixture models, using a 1000 random start values in the first step and 100 random start values in the second step to ensure the validity of the final solution (Geiser, 2013).

Model fit was evaluated using both information criteria (AIC, BIC, Entropy) and significance testing (BLRT), as suggested by Nylund and colleagues (2007). AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) take into account both model fit and parsimony (Geiser, Lehmann, & Eid, 2006). Both AIC and BIC can only be used to compare two or more competing models and for both, a lower value describes better model fit. AIC values typically increase with model complexity, while BIC values tend to prefer more

2 Some variables in Study VI were found to be slightly skewed and thus results of the ANOVAs were also confirmed using non-parametric tests.

simple models, suggesting that they should be used simultaneously. Entropy values signify the certainty of an estimation, and higher values approaching 1 indicate more certainty. The BLRT (Parametric Bootstrapped Likelihood Ratio Test; Nyland, 2007) estimates the models' log-likelihood difference distributions and can be used to determine if the k-class solution is a better model than the k-1-class solution, which is indicated by a significant test value ($p < .05$). Taken together with an evaluation of the appropriateness of a model, these tests and measures can inform on choosing a model over another. It is also important to consider interpretability when choosing the most appropriate classification system for LPA models (Lanza & Collins, 2008). For example, models that contain multiple classes with very low membership should be avoided if possible (Geiser, 2013). Thus, a mixture of both substantive theory and model fit criteria and tests is the best means for identifying the most appropriate LPA model for a sample.

Linear regression models were used in Study VI to a) isolate SFOR scores and rational number knowledge at Time 1, through residualized scores, and b) determine the effects of SFOR on rational number knowledge at time 4. Residualized scores can be used to separate out the variance in one variable that is explained by a second variable that is closely correlated (Cohen, Cohen, West, & Aiken, 2003; as cited in Hallett, Nunes, & Bryant, 2010). Thus, when two variables are highly related, it is possible to remove the co-variation between these two variables using residualized scores. Residualized scores are calculated by regressing the dependent variable on an independent variable, then finding the difference between the dependent variable value predicted by the regression equation and the observed dependent variable value. Thus, a positive residualized score would indicate that the observed dependent variable was higher than what would be expected based on the independent variable score. A negative residualized score would indicate that the observed dependent variable was lower than what would be expected based on the independent variable score. For example, a residualized SFOR score is the difference between their actual SFOR score and the SFOR score that is predicted by the regression equation and their rational number test score. Thus, the residualized SFOR score would indicate a participant's SFOR score above or below what would be expected based on their rational number knowledge. After confirming non-multicollinearity of the predictor variables, a linear regression model was also calculated using these residualized scores in order to look at the impact of SFOR and prior knowledge of rational numbers on later rational number knowledge.

4. OVERVIEW OF THE EMPIRICAL STUDIES

4.1 Study I

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2013). Young children's recognition of quantitative relations in mathematically unspecified settings. *Journal of Mathematical Behavior, 32*(3), 450–460. doi:10.1016/j.jmathb.2013.06.001

This article reports on the cross-sectional findings from the study of 86 Finnish-speaking children from the ages of 4.5 to 8 years old. The aim of this study was to identify if there were inter-individual and age-related differences in the spontaneous recognition and use of quantitative relations. The tasks used in this study were the first to attempt to measure the spontaneous use of quantitative relations in situations that were not explicitly mathematical.

Participants were 86 Finnish-speaking children (43 female) with no diagnosed learning impairments from two day-care centers and three schools. Participants were in either kindergarten (n=31), pre-school (n=27), or first grade (n=28). Spontaneous recognition of quantitative relations was assessed individually through two videotaped tasks, which were not explicitly mathematical. In both tasks children play fed stuffed animals either proportionally sized pieces of foam bread or with proportionally sized spoons of rice. Participants' responses could be based on quantitative relations (giving the same amount of bread or rice while taking into account the size of the pieces of bread or spoons of rice), exact number (giving the same number of pieces of bread or spoons of rice), or other non-mathematical aspects (matching neither amount nor number of pieces or spoons). No mention of the quantitative or mathematical nature of the tasks was made to children before or during their participation.

The results revealed inter-individual differences in the spontaneous recognition and use of quantitative relations on both the bread and rice tasks. Overall, 31% of participants were found to have used quantitative relations on at least one trial for the bread task and 27% did so for at least one trial on the Rice task. In particular, individual differences were found in the oldest age group made up of children in first grade; 50% of first graders used quantitative relations on at least one trial on the bread task, and 39% did so on the rice task. Furthermore, first graders responded based on quantitative relational significantly more than kindergarteners or preschoolers on both bread and rice tasks.

This study suggested that these tasks were able to capture individual differences in spontaneous recognition of quantitative relations. However, it was not clear from these results whether the inter-individual differences in relational responses on these tasks was due to differences in SFOR, or if differences in the ability to solve the tasks using quantitative relations was the cause of differences in responses. As well, more detailed analysis of participants' responses was necessary to further validate these measures as containing multiple mathematical – quantitative relations and exact number – and non-mathematical aspects. Finally, this study was unable to make a connection between spontaneous recognition of quantitative relations and formal mathematical skills. Thus, an analysis of participants' explanations of their strategies for solving the tasks was conducted using stimulated recall interviews (Study II). As well, a replication study was conducted using the same tasks, but also including measures of participants' ability to solve the tasks using quantitative relations (Study III). Finally, a follow-up study of the oldest participants in this sample was conducted, measuring their math achievement and fraction conceptual knowledge three years later (Study IV).

4.2 Study II

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2011). Young children's spontaneous focusing on quantitative aspects and their verbalizations of their quantitative reasoning. In Ubuz, B. (Ed.). *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (pp. 217-224). Ankara, Turkey: PME

In this article, we report on the same sample of 86 Finnish-speaking children as in Study I. The aim of this study was to more deeply investigate the nature of the Bread and Rice tasks as measures of the spontaneous recognition of quantitative relations. To that end, we compared children's spontaneous relational responses to their explicit verbalizations of how they interpret the bread and rice tasks.

This study utilizes the same spontaneous recognition tasks (Bread and Rice) from study I. As well, stimulated-recall interview tasks were conducted, which were a part of the second session of tasks that the children completed. In the stimulated recall tasks, children were again presented with the same material as in the original Bread and Rice tasks. However, before the child gave the bread or rice to his/her stuffed animal they were stopped and asked. "How do you know what to give…?". Children's responses were assessed as being based on a) quantitative relations, b) numerosity, or c) non-mathematical aspects, including no explanation. As well, participants' non-mathematical responses were categorized based on the content of their responses.

A 2x2x3 ANOVA [Task type x Response x Age] indicated that there were differences in the frequency of spontaneous responses based on quantitative relations and verbalizations based on quantitative relations. Participants used quantitative relations on the Bread and Rice tasks roughly as often as they reported using quantitative relations on the stimulated-recall tasks. However, there were differences in the frequency of verbalizations based on exact number and spontaneous responses based on exact number. Participants were much more likely to respond based on numerosity on the original Bread and Rice tasks than in their verbalizations on the stimulated recall versions of the tasks. Children's non-mathematical responses revealed that, besides relational and numerical aspects, participants' used non-mathematical aspects in the responses including: the manner of giving, the location of the material, the materials' (non-quantitative) size or shape, and the nature of the bread or rice.

The inclusion of verbalization measures in the study of spontaneous quantitative focusing tendencies was an attempt to investigate the validity of the Bread and Rice tasks. Results suggests that the aspects of exact number in these tasks may be more easily recognized and utilized than quantitative relations, as children were unlikely to verbalize their reasoning based on exact number. However, it is possible that those participants who more likely use quantitative relations in these unguided situations may be more able to verbalize about their mathematical reasoning in general. Finally, this study confirmed that these tasks can also be considered non-mathematical, with a number of non-mathematical aspects being seen as salient to participants.

4.3 Study III and IV

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (in press). Spontaneous focusing on quantitative relations in the development of children's fraction knowledge. *Cognition and Instruction.*

This paper reports on two studies (III and IV) with the aims of identifying SFOR as a distinct aspect in early primary school children's mathematical competences and exploring the impact of SFOR tendency on later rational number knowledge.

Study III

In this cross-sectional study, we aimed to replicate our findings from Study I regarding the spontaneous use of quantitative relations on the Bread and Rice tasks. Furthermore, we aimed to determine whether the individual differences found in the spontaneous use of quantitative relations were due to differences in the ability to use quantitative relations or if they were due to differences in SFOR. Finally, we aimed to determine whether SFOR tendency was related to existing mathematical competences, in this case symbolic arithmetic skills.

Participants of Study III were 84 English-speaking kindergarten to third grade children from a medium size city in the state of Florida. The spontaneous recognition of quantitative relations was assessed through the use of the English-language versions of the bread and rice tasks. Furthermore, those children who did not use quantitative relations in their responses on the bread or rice tasks were subsequently asked to complete these tasks again; only this time they were explicitly asked to give the same amount of bread or rice as the researcher. Thus, these children participated in versions of the bread and rice task in which their attention was explicitly guided to the quantitative relations between the different sets of bread or spoons of rice. Finally, children also completed the Woodcock-Johnson III Math Fluency test, as a measure of their symbolic arithmetic skills.

Results revealed that there were substantial individual differences in participants' SFOR, and that these differences could not be entirely explained by the ability to use quantitative relations. For example, while just over half of the kindergarteners were able to reliably use quantitative relations in the guided version of the tasks, only 13% of them did so in the spontaneous versions. As well, while almost all first graders were able to reliably use quantitative relations in the guided versions, only half did so in the spontaneous tasks. In second and third grade the differences in the spontaneous and guided use of relations was not as substantial. Finally, SFOR tendency was related to arithmetic skills ($d = 0.26$). Those children identified as having SFOR tendency (at least half of spontaneous responses being based on quantitative relations) were found to have significantly higher arithmetic skills than those who only used quantitative relations in the guided condition.

Study IV

Participants in this longitudinal follow-up study were 28 first grade students from Study I, who participated in the follow-up study three years later in fourth grade. The aims of study IV were to explore the long-term effects of SFOR tendency on later fraction knowledge. We expected that the differences in SFOR in first grade would predict later conceptual knowledge of fractions, even after controlling for general mathematical skills in first grade. This assumption arose from the idea that those children who displayed a higher SFOR tendency would gain more self-initiated practice with quantitative relational aspects of their everyday environment, leading to more well-developed conceptual knowledge of rational numbers, including fractions.

The SFOR measures used in this study were the Bread and Rice tasks reported in Study I, and were completed in the spring of the participants' first year of school (M_{age} = 7y, 8m; SD= 3.5m). Children also completed measures of non-symbolic arithmetic and number sequence elaboration in first grade. In the spring of fourth grade, students completed a test of fraction conceptual knowledge (elaborated further in Studies V and VI). Finally, participants' grades in mathematics were also collected from their teachers. An ANCOVA revealed that performance on the Rice task in first grade predicted later fraction knowledge after taking into account prior math skills. However, neither tasks predicted later math grades.

These studies provide evidence that SFOR tendency is a relevant new concept for the study of mathematical development. In other words, these studies indicate that it is possible to distinguish SFOR tendency as a distinct variant in children's mathematical competences that is not entirely explained by skill, which describes their tendency to spontaneously recognize and utilize quantitative relations in reasoning about situations that are not explicitly mathematical. SFOR tendency predicts conceptual knowledge of fractions, even while not predicting overall math achievement. That this connection was found over a three-year period suggests that SFOR tendency may play a role in the conceptual development of rational numbers.

4.4 Study V

McMullen, J., Laakkonen, E., Hannula-Sormunen, M. M., & Lehtinen, E. (2014). Modeling the developmental trajectories of rational number concept(s): A latent variable approach. *Learning and Instruction.* doi:10.1016/j.learninstruc.2013.12.004

Study V investigated the development of rational number conceptual knowledge from the perspective of conceptual change theory. In particular, we aimed to capture the developmental of conceptual change with rational numbers using latent variable mixture models, in particular, latent profile analysis (LPA) and latent transition analysis (LTA). This would give insight into the difficulties students have with learning rational number concepts.

The participants of this follow-up study were 263 third to fifth grade students from two schools in southwest Finland. Students completed a test of rational number knowledge three times over the yearlong period. The first testing was prior to their regular fraction and decimal courses in winter 2012 (Time 1), then after their fraction and decimal course in spring 2012 (Time 2), and finally prior to their fraction and decimal course the following year in winter 2013 (Time 3). The rational number test measured conceptual knowledge of magnitude representations and the density of rational numbers.

Model fit results of the Latent Profile Analysis (LPA) revealed that a 4-class model was most appropriate for classifying participants' responses at all three time points separately. This model was able to differentiate between those students who had mathematically correct conceptual knowledge of the magnitude representations of rational numbers from those who had mathematically correct knowledge of concepts of the density of rational numbers. Overall, participants had poor conceptual knowledge of rational numbers, though a limited number were successful in the magnitude representation items (roughly 10% of participants). An extremely small amount (roughly 5%) were successful on both magnitude representation and density items at any time point, suggesting that conceptual knowledge with density is very rare among this age-group. The Latent Transition Analysis results revealed that very few students displayed sustained conceptual understanding with density concepts. However, some participants who had an initial understanding of magnitude representations were found to develop a mathematically correct conception of density as well. Overall, the LTA revealed little conceptual change with rational numbers for either magnitude representations or density.

This study confirmed the difficulty students have with the conceptual understanding of rational numbers. In particular, few students either sustainably displayed or developed a fully mathematical understanding of the concepts surround the density of fractions or decimals. Results also revealed that conceptual knowledge of magnitude representations was necessary, but not sufficient, for conceptual change with density concepts. These results suggest that more studies on the development of rational number knowledge are needed to determine causes and possible boons for supporting the radical conceptual change that is needed for an understanding of rational numbers.

4.5 Study VI

Lehtinen, E., McMullen, J., & Hannula-Sormunen, M. M. (submitted). Students focusing on quantitative relations as a predictor of the long-term development of conceptual understanding of rational numbers.

The goal of this 1.5-year longitudinal study was to determine the contribution of SFOR tendency to the development of rational number knowledge in late primary school students. We expected that differences in SFOR could also be identified in older students, as it was identified in younger children. We also expected that SFOR tendency would predict the development of rational number conceptual knowledge. In particular, we expected that those students with a higher SFOR tendency would be more successful in the development of rational number conceptual knowledge than their peers with a lower SFOR tendency. Furthermore, we expected the predictive strength of SFOR tendency to not be dependent on prior knowledge of rational numbers, non-verbal intelligence, or arithmetic skills.

Participants were from the same sample of students as in Study V, totaling 263 third to fifth graders. Participants completed the rational number test (RNT) described in Study V a total of four times over the course of 1 year and 4 months. Times 1-3 were the same as in Study V, in addition to a fourth time point after students' regular fraction and decimal courses in spring 2013 (Time 4). In addition to the RNT, participants also completed a SFOR task at Time 1 The SFOR task was a picture explanation and production task, in which the students were asked to describe how sets of objects had been transformed (possibilities included shape, color, exact number, and multiplicative relations) and separately asked to draw how a second set would turn out based on the previous trial. As well, a task measuring guided focusing on quantitative relations was used to confirm that participants were able to solve the tasks using quantitative relations. Finally, participants completed measures of non-verbal intelligence (Raven's Progressive Matrices) and arithmetic skill (Woodcock-Johnson III – Math Fluency) at Time 2.

First, SFOR scores were isolated from rational number knowledge at Time 1 using residualized scores. SFOR tendency strongly predicted rational number knowledge at Time 4, even after controlling for prior knowledge of rational numbers, non-verbal intelligence, and arithmetic fluency. Furthermore, those participants with SFOR scores that were higher than would be expected based on their rational number knowledge had greater learning gains over the four time points for both magnitude representation (d=0.12) and density concepts (d=0.22). In fact, only those students with higher-than-expected SFOR scores showed any development with rational number density concepts.

These results confirm the contribution of SFOR tendency to the development of rational number conceptual knowledge. In particular, those students who more readily recognize the relevance of quantitative relations in situations that are not explicitly mathematical, and subsequently use these relations in their problem solving, have significantly greater gains in rational number conceptual knowledge during their normal lessons on rational numbers. That only those participants with a higher-than-expected SFOR tendency improved in density conceptual knowledge suggests that SFOR tendency may have an impact on successful conceptual change with rational numbers.

5. MAIN FINDINGS AND DISCUSSION

The first aim of the present dissertation was to determine if it was possible to delineate, theoretically and methodologically, children's spontaneous focusing on quantitative relations (SFOR) as an aspect of task performance that is partially distinct from the ability to reason about quantitative relations. The second major goal of the present dissertation was the investigation of the process of conceptual change with rational numbers in third to fifth grade students.. The third aim of the present dissertation was to determine the relationship between children's SFOR tendency and the development of rational number conceptual knowledge. The theoretical framework was based on the role of spontaneous quantitative focusing tendencies – such as spontaneous focusing on numerosity (SFON) – in the development of mathematical competences and the need for radical conceptual change in the development of rational number knowledge. Specifically, the aim was to link the differences in self-initiated practice with quantitative relations embodied by SFOR tendency with the development of rational number conceptual knowledge. In doing so, a number of tasks were identified that were successful in documenting individual differences in SFOR in children from the ages of 5 to 13 years old. These tasks captured differences in the unguided recognition and use of quantitative relations in situations that were not explicitly mathematical, which could not be explained by the ability to solve the tasks using these relations when explicitly guided to do so. Furthermore, latent variable mixture models, especially Latent Profile Analysis (LPA) and Latent Transition Analysis (LTA), were used to model students' conceptual knowledge of the magnitude representations and the density of rational numbers. Most importantly, the unique contribution of SFOR tendency to the development of rational number conceptual knowledge was determined.

Based on these studies, it can be determined that there indeed exists an attentional process that is referred to as Spontaneous Focusing On quantitative Relations (SFOR), which is a part of children's existing mathematical competences. SFOR is defined as the spontaneous (i.e. undirected) focusing of attention on quantitative relations and the use of these relations in situations that are not explicitly mathematical. There are substantial individual differences in SFOR among primary school children, which cannot be explained by their ability to reason about and use quantitative relations. The expression of SFOR tendency is used to describe a child's general tendency to spontaneously focus on quantitative relations in a wide variety of contexts. It is hypothesized that SFOR tendency indicates the amount of spontaneous practice with the reasoning and use of quantitative relations in everyday situations. SFOR tendency was found to be a unique contributor to the development of rational number knowledge. The impact and relevance of SFOR has been detailed in both cross-sectional and longitudinal designs in the present set of studies, indicating that SFOR tendency is an important aspect of mathematical skills and development.

One goal of the present studies was to determine the methodological feasibility of measuring SFOR. This requires tasks that capture individual differences in the spontaneous attention to and use of quantitative relations and not differences in general attention, ability to use quantitative relations, or other cognitive means. Specific findings from individual studies provide more details on the nature of SFOR tendency in both younger and older primary school students. Findings from Study I reveal differences in the spontaneous recognition and use of quantitative relations and exact number, when both were possible solution methods. This suggests that the natural number bias may be related to spontaneous quantitative focusing tendencies. The large differences in the verbalizations of quantitative relations and exact number found in Study II suggest that exact number may be more easily recognized than quantitative relations, suggesting that those who used quantitative relations on these tasks were more explicit in their actions.. Evidence from Study III indicate that the Bread and Rice tasks were able to capture SFOR, in a way that task performance was not entirely explained by the ability to use quantitative relations on the tasks. So that, even after excluding the students who were unable to use quantitative relations on the guided versions of these tasks, there remained substantial inter-individual differences in the spontaneous use of these relations. This suggests these SFOR tasks did not exceed children's procedural or mathematical skills. Thus, Study III was the first study that was able to confirm the existence of SFOR as an aspect of children's mathematical competences.

The second set of studies presented in this dissertation also provided a number of specific conclusions regarding the nature of conceptual change with rational numbers and SFOR tendency's impact on this developmental process. Study IV revealed for the first time that there was a relationship between early SFOR tendency and later fraction knowledge. These results support the conclusion that those children with a higher SFOR tendency were more likely to be successful with learning rational numbers, even though they were not more successful in mathematics class in general than their peers. Study V reported on one of the first longitudinal studies of the development of rational number conceptual knowledge. These findings capture the process of conceptual change with the number concept, though the use of Latent Transition Analysis, highlighting the particular difficulty children had with sustaining conceptual change with density concepts, along with establishing the necessity of grasping magnitude representations before successful conceptual change with density concepts can occur. The most convincing evidence of the contribution SFOR to rational number conceptual development came in Study VI, where SFOR tendency, even after controlling for prior knowledge, predicted rational number conceptual knowledge after two school years of courses on fractions and decimals. This study also suggested that the impact of SFOR tendency on the development of rational number conceptual knowledge may be particularly important during periods in which no explicit fraction or decimal instruction occurs. Put together, these studies reveal that SFOR tendency, throughout primary school, has a significant impact on success with learning rational number concepts.

5.1 Theoretical implications

The present set of studies has implications for current theories on the development of number concepts and mathematical skills, including processes of conceptual change with rational number and the role of spontaneous quantitative focusing tendencies in the development of mathematical skills. This work provides further evidence to suggest that conceptual change theories are relevant for the development of rational numbers and proposes a model of the early developmental stages of these processes. Importantly, it expands the scope of study of these development processes and the factors that influence them to include the measurement of differences in the spontaneous use of mathematical aspects in situations that are not explicitly mathematical, potentially included everyday situations. The empirical part of this work confirms that there are substantial difficulties in learning about rational number concepts. However, some 10 to 13 year olds display successful conceptual change with magnitude representations and even a few do so with density concepts. Finally, these children who are successful in developing conceptual knowledge of rational numbers seem to be the same students who are more likely to spontaneously focus on quantitative relations in non-explicitly mathematical situations. It is proposed that those with a higher SFOR tendency are more likely to recognize opportunities to acquire practice and experience with relational aspects in their everyday life. This increase in self-initiated deliberate practice with quantitative relations could account for the developmental advantage these students have in learning rational number concepts.

A number of studies have been able to detail the difficulties students and adults face when reasoning about rational number concepts that are incongruent with reasoning about natural numbers (e.g. Merenluoto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004; 2010; Obersteiner et al., 2013). The gap between concepts of natural number and rational number are so large, that it has been argued that substantial conceptual change is needed in order for this to happen. Surprisingly, until now, there has been little evidence of how the process of conceptual change with rational number actually occurs. The present studies confirm the difficulties students face when learning about the magnitude representation and density of rational numbers. As well, the clear stages of development that made up the model suggest that the framework theory of conceptual change is relevant for considering difficulties in learning about rational numbers (Vamvakoussi & Vosniadou, 2004; Vosniadou & Verschaffel, 2004). It was also discovered that there are clear differences in the development of the different concepts of rational number. In particular, understanding magnitude representations seems to be a necessary, but not sufficient, precursor to understanding density concepts. This suggests that it is not sufficient to consider rational number concepts as a monolithic set of knowledge that develops in a synchronous manner, but instead there may be a clear path through the different sub-concepts leading to a fully mathematically mature conception of rational numbers.

Hannula and Lehtinen (2005) have previously been successful in detailing the role of the spontaneous attention to and use of exact number in non-guided situations in the development of early mathematical skills (see also, Hannula, 2005; Hannula-Sormunen, in press; Hannula et al., 2010). The present study was successful in expanding this approach to the study of SFOR. Thus, it can be argued that in investigating children's mathematical reasoning with quantitative relations, it is also necessary to consider that not all children pay attention to these relations with equal frequency, when they are not explicitly guided to do so. That these differences have been found in multiple cultural contexts, across a wide age range, and using different representations of quantities (continuous and discrete) suggests that SFOR tendency may be a fairly robust. Given the variety of quantitative relational aspects of the environment, it would be important to determine which features of quantitative relations are most salient to children and adults in their everyday surroundings and how these relate to formal mathematical concepts, especially those dealing with rational numbers. The identification of these features may shed more light on how SFOR tendency impacts individual differences with rational numbers.

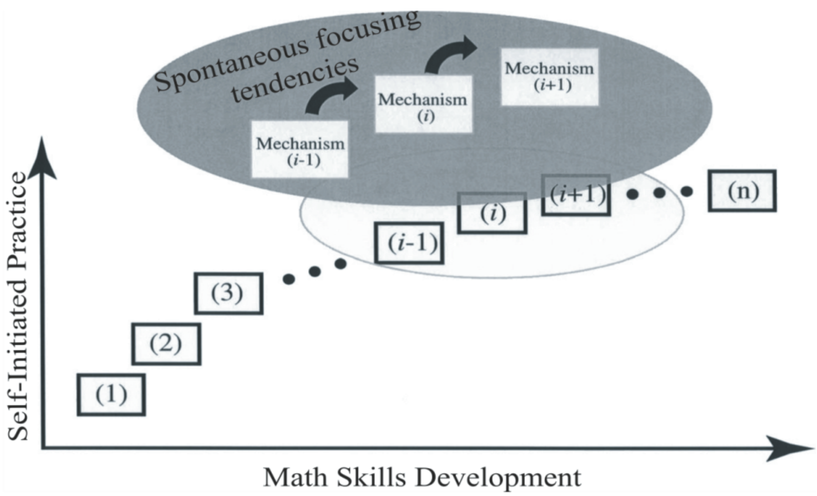

Figure 2. Role of spontaneous focusing tendencies on the development of mathematical skills. Modified from Ericsson, 2006.

Those who become experts in fields such as music or sports are more likely to notice opportunities to practice newly acquired competences than their non-future-expert counterparts (Ericsson & Lehman, 1996). Likewise, it is expected that those children who have a higher SFOR tendency may more readily recognize situations in which quantitative relational aspects may be relevant or useful for their actions. Those students with a higher sensitivity to aspects involving quantitative relations may acquire a greater amount of self-initiated deliberate practice with quantitative relations, leading to more successful development of rational number conceptual knowledge. The existence of SFOR tendency suggests that there may be a progression of spontaneous quantitative focusing tendencies, including SFON and SFOR tendencies, which increases with mathematical complexity (Figure 2). This progression of focusing tendencies may share characteristics with features of the progressive nature of deliberate practice (Ericsson, 2006). Thus, just as budding experts are always practicing skills that are at the edge of their competences, increasing in complexity and difficulty, the progression of spontaneous quantitative focusing tendencies may increase with mathematical complexity as children acquire more advanced knowledge of mathematics.

One possible explanation for the mechanism of influence that this practice offers for the development of rational number conceptual knowledge is that it helps overcome the natural number bias (Ni & Zhou, 2005). Some have argued that natural number bias may have developmental roots in the ubiquitous nature of natural number in cultural representations of number from an early age (e.g. Vamvakoussi et al., 2012). The practice acquired from a higher SFOR tendency may lead past solely paying attention to exact number when other or more advanced mathematical aspects, such as quantitative relations are present. This may lessen the impact of the ubiquity of natural number in everyday activities, leading to an easier expansion of the number concept when learning about rational numbers. Those children with a high SFOR tendency may gain quantitatively more practice with quantitative relations, leading to more variation in experience with these aspects, which support easier abstraction of these aspects into formal mathematical concepts of rational numbers (Ohlsson & Lehtinen, 1997). Not only does a higher SFOR tendency suggest quantitatively more practice with these concepts, but this practice with reasoning about everyday relations may also be qualitatively better. Practice with approximate non-exact quantitative relations, such as dealing with portions of journeys or other continuous quantities, is rarely found in classroom situations, but is common in everyday situations. SFOR tendency may be particularly beneficial in allowing the acquisition of experience in reasoning with these non-exact approximate relations outside of the classroom. Greater experience with non-exact approximate relations may provide especially strong support for learning concepts of density in recognizing that not all quantities fit neatly into common fractions or decimals.

When considering Resnick's (1992) developmental stages, SFOR tendency may act as a bridge between the upper levels of mathematics of numbers and operations and the lower level of the mathematics of quantities. As has been shown in the present dissertation, those students with a higher SFOR tendency were improved more with their conceptual knowledge of relational numbers. Thus, it may be that those who more readily recognize opportunities to reason about quantitative relations at the quantities level, as indexed by SFOR tendency, may be afforded a substantial boon when making the transition from mathematics of numbers to mathematics of operations when reasoning about rational numbers. For example, it may be that those who are more likely to recognize the proportional relations of a sharing situations (with unequal unit sizes) may gain more relevant experience with reasoning about these aspects at the mathematics of quantities level, supporting them when learning about common fractions (e.g. ½ = 2/4) at the mathematics of operations level. However, the proposed connection between different levels in Resnick's model that may be afforded by SFOR tendency requires more evidence and cannot be truly identified in the scope of the present dissertation.

Previously, little evidence has been presented that documents the development of rational number conceptual knowledge or developmental predictors of the development of this knowledge. More than anything, the present study indicates that individual differences in developmental trajectories of learning rational number concepts exist and these may not only be explained by differences in what children and students can do, but also differences in how often they focus on mathematically relevant aspects of their environment and use their mathematical skills in action. This expands the scope of explanatory factors for developmental differences in conceptual change with rational numbers to include SFOR tendency, impacting both the prediction of developmental difficulties with rational numbers, and practical, educational implications for the teaching of rational numbers.

5.2 Practical implications

The present study has practical implications for the diagnostics of early mathematical difficulties and the later teaching of rational numbers. Until now, studies of children's early quantitative relational reasoning have dealt with situations that are obviously mathematical (e.g. Boyer et al., 2008; Frydman & Bryant, 1988). However, relying on assessments of young children's reasoning about quantitative relations that merely measure what they can do ignores the fact that there may also be differences in the frequency with which they spontaneously use quantitative relations that have crucial influence on their formal mathematical development. The difficulties students face with learning rational number concepts, despite displaying the ability to reason about quantitative relations at an early age, suggests that the early warning signs of future learning difficulties with rational numbers would be beneficial for teachers and educators. Measuring children's SFOR tendency may be one further way to capture early causes of difficulties with learning formal mathematical concepts. Those children who display a low SFOR tendency may benefit from extra support, even though they are proficient with formal mathematical reasoning about quantitative relations. Thus, considering measures of SFOR to be a diagnostic tool may help capture unexplained differences in developmental differences not captured by traditional measures. A low SFOR tendency might indicate, not necessarily a lack of knowledge, but a lack of understanding of the applicability of quantitative relational concepts to everyday situations. This lack of recognition of the salience of mathematical aspects of everyday situations has been argued to be a key weak spot in mathematical curricula (Verschaffel, Greer, & De Corte, 2000), and enhancing SFOR tendency is a strong candidate for filling this gap with regards to rational number teaching.

Encouraging students to explore everyday situations in search of opportunities to use quantitative relations could be a strong boon for teachers in lessening learning difficulties with fractions and decimals. Previous research has found that interventions aimed at enhancing SFON tendency, were not only successful in increasing children's spontaneous focusing on numerosity, but subsequently lead to greater gains in enumeration skills (Hannula et al., 2005). This suggests that training spontaneous quantitative focusing tendencies may have a far-reaching influence on children's mathematical development. Providing encouragement for recognizing opportunities to apply quantitative relational reasoning in real-world situations may not only increase SFOR tendency, but lead to improvements in reasoning about rational numbers. As mathematical topics increase in abstraction, it becomes even more necessary to provide deeply meaningful real world situations in which these aspects can be found (Lehtinen & Hannula, 2006; Lobato, 2012). Developing SFOR tendency may be a key component in creating a richly connected body of mathematical knowledge that accepts the world as a fundamentally mathematical place (Hatano & Oura, 2012; Lobato, Ellis, & Munoz, 2003), which may afford the everyday opportunities needed to acquire a rich, diverse, meaningful set of experiences with quantitative relations upon which a mathematically correct conception of rational numbers can be built. The individual differences in SFOR tendency that are exposed in the present set of studies indicates that some children may already be acquiring these mathematically rich experiences, leading to more successful conceptual change with rational numbers. Increasing SFOR tendency in all students, especially those with a lower SFOR tendency, may increase the chances of them developing more mathematically correct conceptions of rational numbers.

Teaching about rational numbers should be built upon early reasoning about quantitative relations, such as equal-partitioning (Confrey et al., 2009). In particular, developing curricula that encourages reasoning about quantitative relations in a variety of non-formal situations could benefit the abstraction of rational number concepts. A curriculum which highlights opportunities to recognize and utilize non-traditional quantitative relations in everyday situations (as opposed to the more common ones such as one-half) could encourage students' spontaneous use of approximate quantitative relations. Creating such a link between these everyday situations and formal mathematical content could help emphasize the continuous nature of quantitative relations and could be particularly beneficial for aiding the conception of the density of rational numbers. Students' inability to successfully model mathematical concepts in real-world situations suggests that this sort of approach to math teaching would be necessary, even beyond rational number topics (Verschaffel et al., 2000). While it seems that the natural number bias in intuitive judgments never fully disappears (Obersteiner et al., 2013; Vamvakoussi et al., 2013), supporting children's SFOR tendency may provide more experience with overcoming this bias to reason about quantitative relations in situations when aspects of exact number are also present. Thus, SFOR enhancement as a supplement to the traditional curriculum would possibly improve the possibilities for successful learning of formal rational number concepts when they arise in the curriculum.

5.3 Limitations and challenges for future studies

The studies that make up the present dissertation represent the first attempts to conceptualize and measure SFOR and should be recognized as such. Further validation of the instruments used to measure SFOR, more replication of the studies, and continued refinement and expansion of the measures is still necessary to fully legitimatize SFOR in the mathematical development canon. The present set of studies provides a framework and justification for the further development of these research avenues, with particular holes appearing that should guide future studies. There are three main limitations to the present set of studies, which impact the conclusions that can be drawn. First, there are a number of issues with the tasks designed to measure SFOR. Second, there are issues with the use of the term of quantitative relations. Finally, there are a number of limitations in the design of the studies, which demand consideration for the future study of SFOR.

There are a number of issues to consider related to the tasks used to measure SFOR. The strength of the combination of Studies I and III is that they present a direct replication of the Bread and Rice tasks in a second country. These findings, put together, indicate that individual differences in SFOR tendency appear in multiple contexts and that the variation in SFOR is similar in these two contexts. However, only having two tasks to measure SFOR in these studies limits the conclusions that can be drawn. In fact, the present set of studies only reports on three different tasks measuring SFOR across an age range of eight years. Such a limited number of tasks risks the possibility that there may not be a general attentional tendency that can be referred to as SFOR, instead the individual differences in task performance found in these studies could be a result of idiosyncratic features of these specific tasks. However, that these tasks had similar relationships with rational number knowledge (Studies IV and VI) suggests that there may be some commonality to them. Nonetheless, future studies that both continue the replication of these tasks and involve more measures of SFOR are necessary. In particular, measures of SFOR that include different types of quantitative relations, including approximate continuous relations, are necessary for understanding what types of quantitative relations can be grouped within this term.

The Bread and Rice tasks, themselves, are also in need of serious reconsideration. While these tasks were very successful in eliciting responses based on quantitative relations and exact number, it is problematic that relatively few participants in Studies I-IV utilized non-mathematical aspects in their responses. These findings suggest that the mathematical nature of the Bread and Rice tasks may be fairly apparent, leading to a distinction between spontaneous focusing on quantitative relational aspects and solely focusing on numerosity, but not non-mathematical aspects. However, based on the interview results from Study II, there is some evidence that some participants clearly saw the tasks as non-mathematical aspects, focusing on, for instance, the qualitative shape of the pieces of bread (e.g. "*It looks like a party hat!*"). This limitation, however, did not seem to be an issue with the Teleportation task used in Study VI, which contained a number of non-mathematical aspects upon which a number participants focused. Nonetheless, in future studies measuring SFOR in younger children, more open tasks, which also contain relevant non-mathematical aspects, are necessary to more closely examine how children use quantitative relations in situations which are not explicitly mathematical.

Another issue regarding the Bread and Rice tasks that requires further attention in subsequent tasks in the nature of the quantitative relations used in the tasks. Despite that the majority of studies involving early reasoning about quantitative relations deal with proportional relations, the bread and rice tasks involve relations between two sets in which the overall amount is the same, and only the unit size differs. Thus, the Bread and Rice tasks do not involve proportional reasoning. The use of SFOR tasks which involve proportional relations with young children would give a clearer picture of how the use of the more simple form of quantitative relations in the Bread and Rice tasks influenced the results of Studies I-IV. One important consideration in the use of proportional relations is the possibility that variation in overall amount between sets may more obviously guide participants' attention to the mathematical, and possibly relational, aspects of the tasks. However, it is necessary to carefully design the tasks so that they do not make too obvious the proportional relations and mathematical aspects.

Along with these limitations based on the SFOR instruments, there are more theoretical limitations surrounding the use of the term quantitative relations. Further studies should take seriously the consideration as to whether SFOR, which makes use of the term *quantitative relations*, is a single spontaneous quantitative focusing tendency that captures the whole range of these relations. It would be important to determine exactly what types of quantitative relations make up SFOR, including proportional relations, multiplicative relations, and additive relations. It is necessary to investigate whether SFOR is truly a general quantitative relational tendency or if it is better explained as a number of partially independent focusing tendencies, which involve different aspects of quantitative relations. Furthermore, it is important to determine if there are differences in the ease of recognition of the different types of quantitative relations, which could inform pedagogical practices. As well, determining which aspects of quantitative relations are more strongly related to rational number knowledge would not only further help target pedagogical interventions, but also expose more understanding of the nature of conceptual change with rational numbers. Looking at SFOR tendency longitudinally would be necessary to determine if there are different aspects of quantitative relations that are more relevant at different points in time, and determine the stability of SFOR tendency across time (e.g., Hannula et al., 2010). Furthermore, it would be important in looking at the interaction between SFOR tendency and quantitative relational reasoning and rational number knowledge to determine if there existed a similar reciprocal developmental pattern as was found with SFON tendency and counting skills (Hannula & Lehtinen, 2005).

Finally, there are a number of limitations in the design of the present set of studies that should be taken into consideration when planning future studies of SFOR. One major concern involving the conclusions of the present set of studies is the small sample sizes in the studies in early primary school (Studies I-IV), with age groups in the different studies only having around 20 to 30 participants. While, the cross-sectional approach used in these studies allowed for a wide-angle look at SFOR tendency in a number of different ages, it lessened the possibility to capture, within age groups, more substantial individual differences. In particular, Study IV only reports on a small cohort of students; since this study provides the only evidence of early SFOR tendency being related to later fraction learning, a replication with a larger sample size is needed to fully substantiate these claims. Furthermore, this study only found a connection between the Rice task and later fraction knowledge, while there was no predictive influence of the Bread task. A larger sample, with larger statistical power, would allow for the determination of whether this effect was due to low power or an accurate representation of this relationship.

Despite our expectation that SFOR tendency would be related to SFON tendency, no age-appropriate SFON measures were included in these studies (see Hannula & Lehtinen, 2005). While SFON tendency is inherently connected with SFOR tendency on these tasks, not having separate measures of SFON and SFOR limits the understanding of how these spontaneous quantitative focusing tendencies are related. Measurement of the development of these two tendencies across time in younger children would provide more insight into not only the development of focusing tendencies, but also may shed more light on how these tendencies impact larger developmental trends related to, for example, the natural number bias.

The present set of studies provides a strong case for the importance of SFOR tendency in the development of rational number conceptual knowledge. However, no evidence can be presented on the origins of the individual differences in SFOR tendency or how SFOR tendency could be engaged in the classroom. Understanding the antecedents of SFOR tendency would allow for a better understanding of its role in the development of mathematical skills. Furthermore, determining the developmental causes of SFOR tendency will help better inform on how it can be enhanced in the classroom environment. The creation of pedagogical tools based on SFOR tendency is a crucial potential outcome of these studies and looking at predictors of SFOR tendency is necessary for the creation of these tools. Thus, the developmental factors that lead to differences in SFOR tendency should be examined. Early, seemingly innate, recognition of both approximate and exact number have been identified, and even shown to impact later mathematical skills (Mazzocco, Feigenson, & Halberda, 2011; Starr, Libertus, & Brannon, 2013). Investigating processes related to the approximate number system, which being based on a logarithmic scale, is inherently relational (Dehaene et al., 2008), could indicate whether there are individual differences at this basic representational level that impact SFOR tendency. The ubiquity of early number and quantitative relation recognition suggests that the investigation of developmental influences may be important for understanding the development of individual differences in SFOR tendency.

Not only is the present set of studies unable to identify possible developmental antecedents of SFOR tendency, they are also unable to make firm conclusions regarding SFOR tendency's influential mechanisms on the learning of rational numbers. In situ, observational evidence of SFOR in everyday situations is necessary for the determination of how SFOR tendency may affect the development of rational number knowledge. The role of SFOR tendency in mathematical development has been found to be substantial, though it is unclear exactly what this role looks like.. Only experimental evidence will allow for a more substantial indication of SFOR tendency's influence on mathematical development.

In general, the present set of studies is a useful first step in determining the nature of SFOR tendency and its role in mathematical learning, especially learning of rational number concepts. Ultimately, however, it is apparent that a number of aspects of these studies should be re-considered when approaching new investigations of SFOR. Furthermore, a number of important questions regarding the nature of SFOR tendency and its role in the development of mathematical skills cannot be addressed in this set of studies. Nonetheless, the results reported here suggest that these future studies are a fruitful pursuit.

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