DOI: 10.1007/978-3-031-83438-7_15

CITATION: Hakanen, A., Junnila, V., Laihonen, T., Miikonen, H., Yero, I. (2025). On a Tight Bound for the Maximum Number of Vertices that Belong to Every Metric Basis. In: Gaur, D., Mathew, R. (eds) Algorithms and Discrete Applied Mathematics. CALDAM 2025. Lecture Notes in Computer Science, vol 15536. Springer, Cham. https://doi.org/10.1007/978-3-031-83438-7_15

On a Tight Bound for the Maximum Number of Vertices that Belong to Every Metric Basis

Anni Hakanen[1], Ville Junnila[1], Tero Laihonen[1], Havu Miikonen[1], and Ismael G. Yero[2]

[1] Department of Mathematics and Statistics, University of Turku, Finland
[2] Department of Mathematics, Universidad de Cádiz (Algeciras Campus), Spain

Abstract. Metric bases of graphs have been widely studied since their introduction in the 1970's by Slater and, independently, by Harary and Melter. In this paper, we concentrate on the existence of vertices in a graph G that belong to all metric bases of G. We call these basis forced vertices, and denote the number of them by $\mathrm{bf}(G)$. We show that $\mathrm{bf}(G) \leq 2/3(n-k-1)$ for any connected nontrivial graph G of order n having k vertices in each metric basis. In addition, we show that this bound can be attained. Furthermore, the previous result implies the bound $\mathrm{bf}(G) \leq 2/5(n-1)$ formulated in terms of the order n of the graph for any nontrivial connected graph G. This result answers a question posed by Bagheri *et al.* in 2016. Moreover, we provide some realization results and consider some extremal cases related to basis forced vertices in a graph.

Keywords: Metric dimension · Metric basis · Basis forced vertices · Realization · Extremal graphs

AMS Subj. Class. (2020): 05C12

1 Introduction

A resolving set $R \subseteq V(G)$ in a graph $G = (V(G), E(G))$ represents a structure with the capability of uniquely recognizing all the vertices in $V(G)$ through a vector of distances to the vertices of R. The vertices of resolving sets are usually called "landmarks", and the optimization of the quantity of vertices in any resolving set has led to the notion of a metric basis, which is a resolving set of the smallest possible cardinality in G. These notions were first and separately introduced in [17] by Slater, and in [11] by Harary and Melter, but they remained almost without any attention until the work [3], which created a breaking point in the interest on them. Nowadays, the metric dimension of graphs is a very well-known topic and there is a huge amount of information about it. Some recent and interesting works on this parameter are for instance [4,5,13–16,20,21]. Moreover, for more information on this area we suggest the very interesting survey [19].

Several applications of resolving sets have been proposed in literature. Generally, the idea is to place sonar stations to some nodes of a network so that any object in a node can be located based on the distances given by the sonar stations [17]. An application of resolving sets to robot navigation already appeared in [11], and have been further developed in several other works. A recent and novel location property was presented in [18], where (certain kind of) resolving sets were used while identifying biological sequence data. In order to avoid presenting a large list of such works, any interested reader might simply check the recent survey [19], which contains a fairly complete compendium of applications, and combinatorial or computational properties of resolving sets and metric bases.

An interesting fact, regarding the metric bases of a given graph G, relates to the possible existence of vertices of G such that they are required to be landmarks in every metric basis, in order to locate or resolve the vertices of G. This fact means that a vertex satisfying this property plays a crucial role as a landmark, and thus, identifying such vertices is worthwhile. However, one cannot efficiently decide that a given vertex of a graph possesses such a property since, as proved in [9], it is a co-NP-hard problem to check whether a given vertex of a graph belongs to every metric basis of the graph. The vertices of a graph satisfying the previously mentioned property were called *basis forced vertices* in [9].

This suggests that finding tight bounds on the number of basis forced vertices in a given graph deserves attention. Hence, the aim of this work is to make significant contributions to this direction. Notice that in [1] Bagheri et al. studied graphs with unique metric bases, that is, graphs with a metric basis where all the vertices are basis forced. They provided bounds and posed a question on the maximum cardinality of a unique metric basis in any graph. In this paper, as a byproduct of Corollary 1, we are able to fully answer that open question. Some previous bounds for the number of basis forced vertices were already given in [9], and some other ones for specific families of graphs recently appeared in [10], but the former ones (and more general), were indeed not the best possible. We significantly improve them in this work.

Formally, given a connected graph $G = (V(G), E(G))$, it is said that a vertex $x \in V(G)$ *resolves* two vertices $u, v \in V(G)$ (or that u, v are *resolved* by x), if $d_G(v, x) \neq d_G(u, x)$, where $d_G(y, z)$ represents the distance between y and z, which is the number of edges in a shortest $y - z$ path in G. A set $R \subseteq V(G)$ is a *resolving set* for G if all the vertices of G are pairwise resolved by a vertex of R. A resolving set having the smallest possible cardinality in G is called a *metric basis*. The cardinality of a metric basis is the *metric dimension* of G, and denoted by $\dim(G)$.

Now, a *basis forced vertex* of a graph G is understood as a vertex $v \in V(G)$ such that it belongs to every metric basis of G. The term "basis forced vertex" was first used in [9], although it has some antecedents in the articles [1, 2], where the graphs that have a unique metric basis were studied. From now on, we represent the number of basis forced vertices of G by $\mathrm{bf}(G)$. Notice that we can have $\mathrm{bf}(G) = 0$ for many graphs. For example, a nontrivial path P has $\mathrm{bf}(P) = 0$.

Fig. 1: (a) The graph G with the resolving set $R = \{r_1, r_2\}$ and (b) the colour graph G_R.

1.1 Terminology and Notation

Throughout our exposition, the graphs G, for which $\mathrm{bf}(G)$ is studied, are nontrivial (that is, they have at least two vertices), finite, undirected and connected. The complete graph on m vertices is denoted by K_m and a path on n vertices by P_n. The complement of a graph G is denoted by $\overline{G}$.

Given a graph G and a set $R \subseteq V(G)$, in [9], the *colour graph* G_R of G (with respect to R) is defined as follows. Let $r \in R$. We denote

$$\mathcal{U}_R(r) = \{\{x,y\} \in V(G)^2 \mid d_G(r,x) \neq d_G(r,y) \text{ and } \forall\, t \in R\backslash\{r\} \colon d_G(t,x) = d_G(t,y)\}.$$

In other words, the set $\mathcal{U}_R(r)$ consists of the pairs of vertices for which r is the unique element in R that resolves the pairs.

With this in mind, the graph G_R has the vertex set $V(G_R) = V(G)$ and the edge set

$$\bigcup_{r \in R} \mathcal{U}_R(r).$$

Each $r \in R$ is assigned a colour (or a label), and we colour the edges in G_R given by $\mathcal{U}_R(r)$ with the colour associated with r. Observe that the graph G_R can be disconnected while G is connected. Moreover, note that this graph G_R might be constructed for any set of vertices $R \subseteq V(G)$, although for our purposes, we will require that such R is a resolving set (or a metric basis). If there is no edge between x and y in G_R, then there are at least two elements in a resolving set R that resolve x and y. If R is a metric basis of G, then the graph G_R has at least one edge of the colour associated with each $r \in R$. In other words, the set $\mathcal{U}_R(r)$ is nonempty for all $r \in R$.

For an example (see [9]), consider the graph G illustrated in Figure 1(a) and its resolving set $R = \{r_1, r_2\}$. For the colour graph G_R, we first form the following sets: $\mathcal{U}_R(r_1) = \{\{r_1, v_1\}, \{r_1, v_3\}, \{v_1, v_3\}\}, \{v_2, v_4\}\}$ and $\mathcal{U}_R(r_2) = \{\{r_2, v_1\}, \{r_2, v_2\}, \{v_1, v_2\}\}, \{v_3, v_4\}\}$. Then we obtain the colour graph $G_R = (V(G_R), E(G_R))$, where $V(G_R) = V(G)$ and $E(G_R) = \mathcal{U}_R(r_1) \cup \mathcal{U}_R(r_2)$. In Figure 1(b) illustrating the colour graph G_R, the edges corresponding to r_1 and r_2 are associated with black and "dashed" edges, respectively.

2 Properties of the Colour Graph and Bounds for the Metric Dimension

The colour graph G_R regarding a resolving set R in a graph G plays an important role in the proofs of this paper. Therefore, we begin by giving some properties of the graph G_R stated in the next technical lemmas. The results of the Lemma 1 are already known from [9], whereas the claims in Lemma 2 are new. In this section, we also provide (in Proposition 1) bounds on the metric dimension $\dim(G)$ when there are basis forced vertices in G. We will need these bounds for our main results in Sections 3 and 4.

Lemma 1 ([9]). *Let G be a graph and let R be a resolving set of G. Then the following properties hold for G_R.*

(i) *A colour that appears in a cycle of G_R appears at least twice in that cycle.*
(ii) *Let $x, y, z \in V(G)$. If the edges $\{x, y\}$ and $\{x, z\}$ have the same colour in G_R, then the edge $\{y, z\}$ also has the same colour in G_R.*
(iii) *If $b \in V(G)$ is a basis forced vertex of G and R is a metric basis of G, then the graph G_R has at least two edges of the colour associated with b.*
(iv) *If $b \in V(G)$ is a basis forced vertex of G and R is a metric basis of G, then the graph G_R has at least one edge $\{x, y\}$, $x, y \in V(G) \setminus R$, of the colour associated with b.*
(v) *The set of vertices R forms an independent set in G_R.*
(vi) *If there is an edge in G_R incident to $r \in R$, then the edge has the colour associated with r.*

We next show some new properties of the colour graph G_R of a graph $G = (V(G), E(G))$ with respect to some resolving set or metric basis R. If we replace a vertex u in R by a vertex $v \in V(G)$ (while keeping R otherwise intact), then we denote the new set by $R[u \leftarrow v]$. In other words,

$$R[u \leftarrow v] = (R \setminus \{u\}) \cup \{v\}.$$

Lemma 2. *Let G be a graph and let R be a resolving set of G. Then the following properties hold for G_R.*

(i) *If $b \in V(G)$ is a basis forced vertex of G and R is a metric basis of G, then the graph G_R has at least two edges $\{x, y\}$, $x, y \in V(G) \setminus R$, of the colour associated with b.*
(ii) *If R is a metric basis of G and the only edge of the colour associated with $r \in R$ in G_R is of type $\{r, x\}$, where $x \in V(G) \setminus R$, then $R[r \leftarrow x]$ is a metric basis of G.*

Proof. (i) Let us assume that $b \in V(G)$ is a basis forced vertex. By Lemma 1(iv) we know that there is at least one edge, say $\{x, y\} \subseteq V(G) \setminus R$ associated with the colour b and, by Lemma 1(iii), there is also at least one more edge, say $\{w, z\}$, of the same colour. If $\{w, z\} \subseteq V(G) \setminus R$, we are done. Clearly, by Lemma 1(v), we cannot have $\{w, z\} \subseteq R$, so it suffices to assume that $w \in R$. Moreover, $w = b$ by Lemma 1(vi). Next we consider separately two possible cases, namely, $z \in \{x, y\}$ or $z \notin \{x, y\}$.

Case 1. Assume that $z \notin \{x, y\}$. If there is a third edge of colour b, then it is enough to assume that it is $\{b, z'\}$ where $z' \in V(G) \setminus R$. However, in that case, by Lemma 1(ii) there exists an edge $\{z, z'\}$ with $z, z' \in V(G) \setminus R$ and the claim follows. Hence, we may assume that $\{x, y\}$ and $\{b, z\} = \{w, z\}$ are the only edges of colour b. The set $R[b \leftarrow x]$ cannot be a metric basis, since b is a basis forced vertex which must be in every metric basis. Therefore, $d_G(x, z) = d_G(x, b)$ as the pair z and b is the only one that cannot be resolved with respect to $R[b \leftarrow x]$ (indeed, all the other pairs of vertices in $V(G)$ apart from the pair b and z and the pair x and y were resolved by other elements of R than b). Similarly, as $R[b \leftarrow y]$ (resp. $R[b \leftarrow z]$) cannot be a metric basis, we have $d_G(y, z) = d_G(y, b)$ (resp. $d_G(z, x) = d_G(z, y)$). These three distance equations imply together that $d_G(b, y) = d_G(z, y) = d_G(z, x) = d_G(b, x)$. However, this is a contradiction, since b was the (only) vertex resolving the vertices y and x, that is $d_G(b, y) \neq d_G(b, x)$.

Case 2. Assume that $z \in \{x, y\}$. Without loss of generality, assume $z = x$. Recall that by Lemma 1(ii) there is a third edge associated with the colour b, namely, $\{b, y\}$ in G_R. If there is yet another (fourth) edge of colour b, then we can assume that it is $\{b, z'\}$ where $z' \in V(G) \setminus R$. Due to Lemma 1(ii), this implies that there is an edge $\{z', x\}$ with $x, z' \in V(G) \setminus R$ of colour b, and we are done. Hence we may assume that we have only the three edges mentioned above. Because b is a basis forced vertex, the set $R[b \leftarrow y]$ is not a metric basis and, hence, $d_G(y, b) = d_G(y, x)$. Analogously, $R[b \leftarrow x]$ cannot be a metric basis. Hence, we obtain $d_G(x, b) = d_G(x, y)$. Now we have $d_G(b, x) = d_G(x, y) = d_G(b, y)$, a contradiction, since b resolves the pair x and y.

(ii) Let R be a metric basis (when R in only a resolving set, the claim follows by an analogous argument). Assume further that $r \in R$ and $\{r, x\}$ with $x \notin R$ is the only edge of colour associated with r (clearly, r is not a basis forced vertex). If we remove r from R, then the only pair of vertices that can have the same distances to all the elements of $R \setminus \{r\}$ is r and x. But in the set $R[r \leftarrow x]$ they have different distance to x. Therefore, the set $R[r \leftarrow x]$ is a metric basis in G. □

The next result shows that if we have basis forced vertices in a graph G, then there are some limitations for the metric dimension $\dim(G)$.

Proposition 1. *If G is a graph such that* $\mathrm{bf}(G) \geq 1$*, then* $2 \leq \dim(G) \leq n - 4$.

Proof. Assume that G is a graph with at least one basis forced vertex. Recall that paths are the only graphs with $\dim(G) = 1$. Hence, as the paths do not have basis forced vertices, it immediately follows that $\dim(G) \geq 2$. Let us look at the claim $\dim(G) \leq n - 4$. Suppose to the contrary that $\dim(G) \geq n - 3$ (and $\mathrm{bf}(G) \geq 1$). Let R be a metric basis of G. By Lemma 2(i), for each basis forced vertex u, there exist at least two edges associated with the colour u in G_R between vertices of $V(G) \setminus R$. Hence, $\dim(G) \leq n - 3$ as otherwise at most one edge can occur in $V(G) \setminus R$. Thus, $\dim(G) = n - 3$ and there are exactly three vertices, say x, y and z, in $V(G) \setminus R$. In addition, we know that there exists a unique basis forced vertex in G, say b. Indeed, if $\mathrm{bf}(G) > 1$, then Lemma 2(i) would imply that $\dim(G) \leq n-4$.

By Lemma 2(i), there exist at least two edges associated with the colour b in G_R between the vertices of $V(G) \setminus R$. Without loss of generality, we may assume that $\{x, y\}$ and $\{y, z\}$ are such edges. By Lemma 1(ii), this further implies that $\{x, z\}$ is also such an edge. Let r be a vertex in $R \setminus \{b\}$. Since R is a metric basis of G, there exists at least one edge in G_R associated with the colour r. Due to the fact that the edges between x, y and z are associated with the colour b, the edges of the colour r have to be of type $\{r, w\}$, where $w \in \{x, y, z\}$ due to claims (vi) and (v) in Lemma 1. Moreover, there exists exactly one such edge since otherwise a contradiction (with the fact that the edges between x, y and z are coloured with b) follows by Lemma 1(ii). For each $w \in \{x, y, z\}$, we define

$$R_w = \{w\} \cup \{u \in R \setminus \{b\} \mid \{u, w\} \in E(G_R)\}.$$

It is immediate that the sets $\{b\}$, R_x, R_y and R_z form a partition of $V(G)$. Furthermore, we have the following observations on R_w:

(1) By Lemma 2(ii), the set $R[w \leftarrow v]$ is a metric basis for each $v \in R_w \setminus \{w\}$.
(2) For each $r \in R_w \setminus \{w\}$, we have $d_G(b, w) = d_G(b, r)$ as the vertices r and w are solely resolved by r. Thus, the basis forced vertex b has the same distance to all $u \in R_w$.

Since the vertices x, y and z are resolved from each other by the vertex b, we may without loss of generality assume that $d_G(b, x) < d_G(b, y) < d_G(b, z)$. Thus, based on the partition $\{b\}$, R_x, R_y and R_z, we have that $d_G(b, x) = 1$, $d_G(b, y) = 2$ and $d_G(b, z) = 3$. Hence, by the observation (2) above, it happens $d_G(b, r_x) = 1$, $d_G(b, r_y) = 2$ and $d_G(b, r_z) = 3$ for all $r_x \in R_x \setminus \{x\}$, $r_y \in R_y \setminus \{y\}$ and $r_z \in R_z \setminus \{z\}$.

Let $bx'y'z$ be a shortest path of length 3 from b to z. It is immediate that $x' \in R_x$ and $y' \in R_y$; note that x' and y' can be x and y, respectively. By the observation (1), $R' = R[x' \leftarrow x]$ is a resolving set of G. As above, we may deduce that the edges $\{x', y\}$, $\{x', z\}$ and $\{y, z\}$ have colour b in $G_{R'}$. Moreover, in $G_{R'}$ there exists exactly one edge of colour y', namely, $\{y, y'\}$. Therefore, by the observation (1), $R'' = R'[y' \leftarrow y]$ is a metric basis of G. Now $V(G) \setminus R'' = \{x', y', z\}$. Furthermore, $R''[b \leftarrow z]$ is a metric basis of G since $d_G(z, y') = 1$, $d_G(z, x') = 2$ and $d_G(z, b) = 3$. However, this contradicts the fact that b is a basis forced vertex. Thus, the claim follows. □

Notice that there exist graphs attaining the upper bound of the proposition. For example, the graph G of order 6 in Figure 1 satisfies $\mathrm{bf}(G) = 2 (> 1)$ and $\dim(G) = 2 = n - 4$.

3 The Main Bounds for the Maximum Number of Basis Forced Vertices

The main result of this section presents a bound for the number of basis forced vertices of a graph G in terms of the metric dimension $\dim(G)$ and the order

n of G (see Theorem 1). As a special case, another bound for such quantity is given only in terms of the order of G (see Corollary 1).

A *cactus* is a graph in which no two cycles share an edge. In what follows, we want to show that a cactus graph appears as a certain subgraph of the colour graph G_R. To make the proof of Lemma 4 simpler, we present the following lemma, the proof of which is based on an iterative idea with an algorithmic flavour. The lemma shows that given a graph with two cycles sharing an edge, we can obtain a pair of cycles such that their intersection is a path (forming a so called theta graph). Recall that we define *cycle* as a closed walk with no repeated vertices. In the next lemma, we use the term *simple cycle* to emphasise the fact that cycles have no repeated vertices. Moreover, we "abuse" the notation and use the intersection notation $C_1 \cap C_2$ of two cycles (as subgraphs) to represent the subgraph induced by the vertices of the intersection $V(C_1) \cap V(C_2)$.

Lemma 3. *Let G be a graph and let C_1 and C_2 be simple cycles in G. If $C_1 \cap C_2$ contains at least one edge, then there exist a third cycle C_3 in G such that $C_1 \cap C_3$ is a nontrivial path* (*a path of length at least* 2)*.*

Proof. The proof is given in the extended version of this article [7].

From now on, given a graph G and a resolving set R of G, we will consider a subgraph of G_R that will be "conveniently chosen" for our purposes. We consider a subgraph H_R of the colour graph G_R defined as follows. The vertex set of H_R is $V(H_R) = V(G) \setminus R$ and the edge set $E(H_R)$ is a (possibly not proper) subset of $E(G_R)$ satisfying that there are either 0 or 2 edges of each colour $r \in R$ in $E(H_R)$, i.e., there are either 0 or 2 edges of each color in H_R.

Despite the fact that several possible configurations for such subgraph H_R of G_R could exist, we next show that the connected components of each of them are bipartite cacti.

Lemma 4. *Let G be a graph and let $R \subseteq V(G)$ be a resolving set of G. Then the following claims follow for H_R.*

(i) *The graph H_R is bipartite.*
(ii) *The connected components of H_R are cacti.*

Proof. (i) Since H_R is a subgraph of G_R, if C is a cycle in H_R, then it is also a cycle in G_R. By Lemma 1(i), a colour that appears in a cycle of G_R appears at least twice in that cycle. Hence, by the choice of edges of H_R, a colour that appears in a cycle of H_R appears exactly twice in that cycle. Therefore, as all edges have a colour, all cycles in H_R have even length. It is well known that this implies that H_R is bipartite.

(ii) Suppose to the contrary that H_R is not composed of cacti, in other words, that there exist cycles in H_R that have edges in common. By Lemma 3, we may assume that C_1 and C_2 are cycles such that their intersection is a path, denoted $P = C_1 \cap C_2$. Now consider any colour r that appears on P. Clearly (as shown above), such colour r must appear exactly twice in the cycle C_1, by the choice of the edges of H_R. If r appears in $C_1 \setminus P$, then C_2 is a cycle in which r appears only once, a contradiction. Therefore, the second occurrence of r must be in P as well.

Denote the vertices of C_1 by $w_1, w_2, \ldots, w_m$ so that vertices w_i and w_{i+1} are adjacent (with the notation that $w_{m+1} = w_1$) and assume that $V(P) = \{w_1, \ldots, w_k\}$ where $k > 1$. Now consider the sequence of distances $d_G(r, w_1), d_G(r, w_2), \ldots, d_G(r, w_m), d_G(r, w_1)$ for some $r \in R$. A difference in consecutive distances $d_G(r, w_i)$ and $d_G(r, w_{i+1})$ (say, $d_G(r, w_i) - d_G(r, w_{i+1}) = j \neq 0$) corresponds to an r-coloured edge $\{w_i, w_{i+1}\}$ in $E(H_R)$. If consecutive distances are equal, the corresponding edge in H_R has a colour other than r. By the choice of the edges of H_R, there can be at most two indices i and i' where consecutive distances differ. Since the first and last elements of the sequence are equal, we must have $d_G(r, w_i) = d_G(r, w_{i+1}) + j$ and $d_G(r, w_{i'}) = d_G(r, w_{i'+1}) - j$ for some $j \neq 0$.

For colours $s \in R$ that do not appear in P, we have $d_G(s, w_1) = d_G(s, w_2) = \cdots = d_G(s, w_k)$. For colours $r \in R$ that do appear in P, we have $d_G(r, w_1) = d_G(r, w_k) + j - j = d_G(r, w_k)$, since r appears twice in P. Now the distinct vertices w_1 and w_k have $d_G(u, w_1) = d_G(u, w_k)$ for all $u \in R$, contradicting the assumption that R is a resolving set of G. Therefore, cycles in H_R do not have edges in common, in other words, H_R is composed of cacti. □

The following result regarding the largest number of edges in a graph whose components are bipartite cacti shall be needed.

Lemma 5 ([12, Lemma 2]). *If H is a bipartite graph with $|V(H)| \geq 4$ such that all of its connected components are cacti, then $\frac{3}{4}|E(H)| + 1 \leq |V(H)|$.*

With the tools above in hand, we are then able to present our main theorem. By the notation $G[V(G) \setminus S]$ we mean the graph induced by the vertices of G not including the set $S \subseteq V(G)$.

Theorem 1. *Let G be a graph of order n with $\mathrm{bf}(G) \geq 1$. Then*

$$\mathrm{bf}(G) \leq \frac{2}{3}(n - \dim(G) - 1).$$

Proof. Let $R \subseteq V(G)$ be a metric basis of G. Let H_R be a subgraph of G_R (as previously described) with vertices $V(H_R) = V(G) \setminus R$ and for the edges of H_R, we select two edges of G_R with the colour b for each basis forced vertex b. Indeed, at least two such edges exist in $G_R[V(G) \setminus R]$ by Lemma 2(i) for each basis forced vertex. By definition, $|V(H_R)| = n - \dim(G)$ and $|E(H_R)| = 2\mathrm{bf}(G)$. Since we assumed that $\mathrm{bf}(G) \geq 1$, Proposition 1 gives us that $\dim(G) \leq n - 4$, and consequently, $|V(H_R)| \geq 4$. By Lemma 4, the graph H_R is bipartite and its connected components are cacti. We can now use Lemma 5 to get

$$\frac{3}{4} \cdot 2\mathrm{bf}(G) + 1 \leq n - \dim(G) \quad \Rightarrow \quad \mathrm{bf}(G) \leq \frac{2}{3}(n - \dim(G) - 1),$$

which completes the proof. □

In [1], Bagheri *et al.* study graphs with unique metrics basis, that is, graphs G satisfying $\dim(G) = \text{bf}(G)$. Among other results, they show that for any even $k \geq 2$ there exists a graph G such that $k = \text{bf}(G) = \dim(G) = 2(n-1)/5$, where n denotes the order of G. Furthermore, they state as an open question whether the metric dimension could be larger with respect to n. In the following corollary, we answer the question by stating that this is not possible.

Corollary 1. *Let G be a graph of order n. Then*

$$\text{bf}(G) \leq \frac{2}{5}(n-1).$$

Moreover, if $\text{bf}(G) = \frac{2}{5}(n-1)$*, then G has a unique metric basis.*

Proof. The first result follows immediately by the bound of Theorem 1, due to the fact that $\text{bf}(G) \leq \dim(G)$. Moreover, if $\text{bf}(G) = \frac{2}{5}(n-1)$, then we have

$$\frac{2}{5}(n-1) = \text{bf}(G) \leq \frac{2}{3}(n - \dim(G) - 1) \quad \Longleftrightarrow \quad \dim(G) \leq \frac{2}{5}(n-1).$$

Therefore, $\dim(G) = \text{bf}(G)$ and G has a unique metric basis. □

Notice that the bound of Corollary 1 can be attained as is shown in Section 4.

4 Realization of Possible Parameters

In the previous sections, we have considered the relations between $|V(G)| = n$, $\dim(G)$ and $\text{bf}(G)$. In particular, we obtained the following lower and upper bounds for $\dim(G)$:

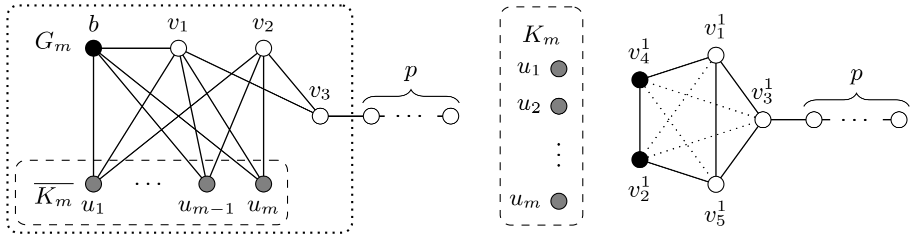
- If $\text{bf}(G) = 1$, then $2 \leq \dim(G) \leq n - 4$ by Proposition 1.
- If $\text{bf}(G) > 1$, then $\text{bf}(G) \leq \dim(G) \leq \left\lfloor n - \frac{3}{2}\text{bf}(G) - 1 \right\rfloor$ by Theorem 1.

In what follows, we show that $\dim(G)$ achieves all the values indicated by the previous inequalities when $\text{bf}(G)$ is even, or $\text{bf}(G) = 1$. For this purpose, we borrow the following result from [10].

Theorem 2 ([10, Theorem 2]). *Let G be a connected graph, and let $B \neq \emptyset$ be the set of basis forced vertices of G. Let $b \in B$, and let $v \in V(G) \setminus B$ be such that $d_G(b,v) = \max\{d_G(b,w) \mid w \in V(G) \setminus B\}$. Let P_p be the path $v_1 \dots v_p$. Let H be the graph with $V(H) = V(G) \cup V(P_p)$ and $E(H) = E(G) \cup E(P_p) \cup \{\{v, v_1\}\}$. Then, B is also the set of basis forced vertices of H and* $\dim(H) = \dim(G)$.

We use Theorem 2 to grow the order of a graph without changing its metric dimension in the proof of the following theorem. The theorem discusses the possible values of the parameters $|V(G)|$, $\dim(G)$ and $\text{bf}(G)$.

Theorem 3. *Let f, k and n be positive integers.*



(a) The graph G_m is within the dotted line. The graph $G_{m,p}$ is obtained by attaching a path to v_3.

(b) The graph $G_{q,m,p}$, with $q = 1$. The edges from and within K_m are omitted for illustrative purposes.

Fig. 2: Sketches of the constructions of Theorem 3. The black vertices are basis forced and the gray vertices are in some but not all metric bases.

– *If* $f = 1$*, then for each* $k \in \{2, 3, \ldots, n-4\}$ *there exists a graph* G *such that* $|V(G)| = n$*,* $\dim(G) = k$ *and* $\mathrm{bf}(G) = f = 1$*.*
– *If* $f > 1$ *and* f *is even, then for each* $k \in \{f, f+1, \ldots, \lfloor n - \frac{3}{2}f - 1 \rfloor\}$ *there exists a graph* G *such that* $|V(G)| = n$*,* $\dim(G) = k$ *and* $\mathrm{bf}(G) = f$*.*

Proof. We will provide a method to construct a graph G with the desired values of $|V(G)|$, $\dim(G)$ and $\mathrm{bf}(G)$.

First, we consider the case $f = 1$. By Proposition 1, we know that $2 \leq \dim(G) \leq n - 4$. Consider the graph G_m as illustrated in Figure 2(a), where $m \geq 2$. The m vertices in $\overline{K_m}$ are twins, hence a metric basis must contain at least $m-1$ of them. The set $V(\overline{K_m}) \setminus \{u_j\}$ is not a resolving set for any $j \in \{1, \ldots, m\}$ since the vertices v_1 and v_2 are not resolved. It follows that $\dim(G_m) > m - 1$. It is easy to verify that $(V(\overline{K_m}) \setminus \{u_j\}) \cup \{b\}$ is a resolving set of G_m for all $j \in \{1, \ldots, m\}$. The sets $V(\overline{K_m})$, $(V(\overline{K_m}) \setminus \{u_j\}) \cup \{v_3\}$ and $(V(\overline{K_m}) \setminus \{u_j\}) \cup \{v_i\}$ are not resolving sets for any choice of j or $i \in \{1, 2\}$ since the pairs $\{v_1, v_2\}$, $\{v_1, v_2\}$ and $\{u_j, v_3\}$ are not resolved, respectively. Therefore, $\dim(G_m) = m$, the vertex b is basis forced, and the vertex v_3 is not basis forced and has $d_G(b, v_3) = 2 = \max\{d_G(b, w) \mid w \in V(G_m) \setminus \{b\}\}$. Choosing $m = k$, and applying Theorem 2 to the vertex v_3 with the path length $p = n - k - 4$, we get the graph $G_{m,p}$ with $m + 4 + p = n$ vertices (see Figure 2(a)). By Theorem 2, the graph $G_{m,p}$ has the same metric dimension and basis forced vertices as G_m, and therefore, the graph $G_{m,p}$ has metric dimension $m = k$ and one basis forced vertex (namely, b). We assumed that $m \geq 2$ and, naturally, that $p \geq 0$. Hence, when m and p are chosen as defined, this construction realizes the values of k and n such that $k \geq 2$ and $n - k - 4 \geq 0$.

Now assume that $2 \leq f = 2q$ is even. Let $G_{q,m}$ be a graph such that $\overline{G_{q,m}} = P_5^1 \cup \cdots \cup P_5^q \cup \overline{K_m}$ (these are disjoint unions), where $V(P_5^i) = \{v_1^i, v_2^i, v_3^i, v_4^i, v_5^i\}$, $V(\overline{K_m}) = \{u_1, \ldots, u_m\}$ and $m \geq 1$. It has been shown (Lemma 22 in [9]) that all metric bases of $G_{q,m}$ are of the form $R = \bigcup_{i=1}^{q} \{v_2^i, v_4^i\} \cup (V(\overline{K_m}) \setminus \{u_j\})$ for some $j \in \{1 \ldots, m\}$. There are $2q$ basis forced vertices in $G_{q,m}$ (vertices v_2^i and v_4^i for all $i \in \{1, \ldots, q\}$), the metric dimension of $G_{q,m}$ is $2q + m - 1$, and particularly, the vertex v_3^1 is not basis forced. The vertices v_2^1 and v_3^1 satisfy the conditions for b and v in Theorem 2, respectively. Thus, we may attach a path of any length p to v_3^1 to obtain a graph $G_{q,m,p}$, illustrated in Figure 2(b). Choosing $m = k - f + 1$ and $p = n - k - \frac{3}{2}f - 1$ gives us the graph $G_{q,m,p}$ with $5q+m+p = n$ vertices, $\dim(G_{q,m,p}) = 2q+m-1 = k$, and $\mathrm{bf}(G_{q,m,p}) = 2q = f$. The construction requires that $m \geq 1$ and $p \geq 0$. The choices of m and p are possible if $k-f+1 \geq 1$ and $n-k-\frac{3}{2}f-1 \geq 0$, or equivalently, if $f \leq k \leq n-\frac{3}{2}f-1$, as promised. □

In the full version of this paper [7], we will extend the previous result to the case when $f > 1$ is odd.

Acknowledgments. Ismael G.Yero has been partially supported by the Spanish Ministry of Science and Innovation through the grant PID2023-146643NB-I00. Ville Junnila, Tero Laihonen and Havu Miikonen have been partially supported by Academy of Finland grant number 338797. Anni Hakanen was supported by Turku Collegium for Science, Medicine and Technology (TCSMT).

Disclosure of Interests. The authors have no competing interests to declare that are relevant to the content of this article.

References

1. B. Bagheri Gh., M. Jannesari, and B. Omoomi. Unique basis graphs. *Ars Comb.*, 129:249–259, 2016.
2. P. S. Buczkowski, G. Chartrand, C. Poisson, and P. Zhang. On k-dimensional graphs and their bases. *Period. Math. Hung.*, 46:9–15, 2003. https://doi.org/10.1023/A:1025745406160
3. G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann. Resolvability in graphs and the metric dimension of a graph. *Discrete Appl. Math.*, 105:99–113, 2000. https://doi.org/10.1016/S0166-218X(00)00198-0
4. M. Claverol, A. García, G. Hernández, C. Hernando, M. Maureso, M. Mora, and J. Tejel. Metric dimension of maximal outerplanar graphs. *Bull. Malays. Math. Sci. Soc.*, 44:2603–2630, 2021. https://doi.org/10.1007/s40840-020-01068-6
5. B. Foster-Greenwood, Ch. Uhl. Metric dimension of a direct product of three complete graphs. *Electron. J. Combin.*, 31:Paper No. 2.13, 2024. https://doi.org/10.37236/12399
6. A. Hakanen, V. Junnila, T. Laihonen. The solid-metric dimension. *Theor. Comput. Sci.*, 806:156–170, 2020. https://doi.org/10.1016/j.tcs.2019.02.013
7. A. Hakanen, V. Junnila, T. Laihonen, H. Miikonen, and I. G. Yero. On the Maximum Number of Vertices that Belong to Every Metric Basis. In preparation.
8. A. Hakanen, V. Junnila, T. Laihonen, and M. L. Puertas. On the metric dimensions for sets of vertices. *Discuss. Math. Graph Theory*, 43(1):245–275, 2023. https://doi.org/10.7151/dmgt.2367
9. A. Hakanen, V. Junnila, T. Laihonen, and I. G. Yero. On vertices contained in all or in no metric basis. *Discrete Appl. Math.*, 319:407–423, 2022. https://doi.org/10.1016/j.dam.2021.12.004

10. A. Hakanen, V. Junnila, T. Laihonen, and I. G. Yero. On the unicyclic graphs having vertices that belong to all their (strong) metric bases. *Discrete Appl. Math.*, 353:191–207, 2024. https://doi.org/10.1016/j.dam.2024.04.020
11. F. Harary and R. Melter. On the metric dimension of a graph. *Ars Combin.*, 2:191–195, 1976.
12. C. Hernando, M. Mora, and I. M. Pelayo. Locating domination in bipartite graphs and their complements. *Discrete Appl. Math.*, 263:195–203, 2019. https://doi.org/10.1016/j.dam.2018.09.034
13. S. Mashkaria, G. Ódor, and P. Thiran. On the robustness of the metric dimension of grid graphs to adding a single edge. *Discrete Appl. Math.*, 316:1–27, 2022 https://doi.org/10.1016/j.dam.2022.02.014
14. J. Sedlar and R. Škrekovski. Bounds on metric dimensions of graphs with edge disjoint cycles. *Appl. Math. Comput.*, 396:125908, 2021. https://doi.org/10.1016/j.amc.2020.125908.
15. J. Sedlar and R. Škrekovski. Metric dimensions vs. cyclomatic number of graphs with minimum degree at least two. *Appl. Math. Comput.*, 427:127147, 2022. https://doi.org/10.1016/j.amc.2022.127147
16. J. Sedlar and R. Škrekovski. Vertex and edge metric dimensions of unicyclic graphs. *Discrete Appl. Math.*, 314:81–92, 2022. https://doi.org/10.1016/j.dam.2022.02.022
17. P. J. Slater. Leaves of trees. *Congr. Numer.*, 14:549–559, 1975.
18. R. C. Tillquist and M. E. Lladser. Low-dimensional representation of genomic sequences. *J. Math. Biol.*, 79:1–29, 2019. https://doi.org/10.1007/s00285-019-01348-1
19. R. C. Tillquist, R. M. Frongillo, and M. E. Lladser, Getting the lay of the land in discrete space: A survey of metric dimension and its applications, *SIAM Rev.*, 65(4):919–962, 2023. https://doi.org/10.1137/21M1409512
20. J. Wang, F. Tian, Y. Liu, J. Pang and L. Miao. On graphs of order n with metric dimension $n-4$. *Graphs Combin.*, 39:Paper No. 29, 2023. https://doi.org/10.1007/s00373-023-02627-x
21. J. Wu, L. Wang, and W. Yang. Learning to compute the metric dimension of graphs. *Appl. Math. Comput.*, 432:127350, 2022. https://doi.org/10.1016/j.amc.2022.127350