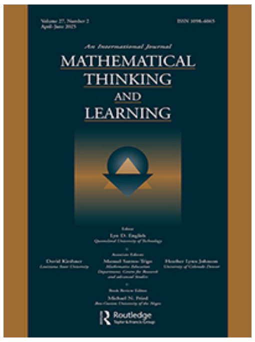

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Examining individual differences in spontaneous focusing on multiplicative relations

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ABSTRACT

Individual differences in spontaneous mathematical focusing tendencies are important predictors of mathematical development. Spontaneous mathematical focusing tendencies may support mathematical thinking in everyday situations, leading to self-initiated practice with existing mathematical skills. While much research has examined how spontaneous mathematical focusing tendencies predict later mathematical development, there is little work on these tendencies as important outcomes of mathematics instruction. Therefore, the present study examines individual differences in spontaneous focusing on multiplicative relations (SFOR) tendency in middle school students. Results reveal that formal mathematical knowledge and the ability to recognize and describe multiplicative relations when explicitly guided to do so can only predict part of individual differences in SFOR tendency, which still exist in middle school students. As well, focusing on nonspecific quantitative relations, such as “more” or “less,” may be an important step in developing SFOR tendency. These results provide a valuable understanding of potential mechanisms by which spontaneous mathematical focusing tendencies can be promoted in all students.

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Relational reasoning; spontaneous focusing on multiplicative relations; SFOR; spontaneous mathematical focusing tendencies

Mathematics curricula worldwide aim to help students use mathematics in everyday life (Finnish National Agency for Education, 2014, National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010). However, many everyday situations do not explicitly prompt students to notice and use mathematical features (Lave et al., 1984). Therefore, students need to spontaneously focus on the mathematics embedded in these situations. Researchers have been interested in how children differ in their spontaneous mathematical focusing tendencies (Hannula & Lehtinen, 2005; Verschaffel et al., 2020). For example, some children are more likely to pay attention to exact number without being told to do so, which is called spontaneous focusing on numerosity (Hannula & Lehtinen, 2005).

Recent research has examined various forms of spontaneous mathematical focusing tendencies, their relation to mathematical development, and the instructional implications of these discoveries (Braham et al., 2018; Chan & Mazzocco, 2017; Hannula & Lehtinen, 2005; McMullen et al., 2014; Perez & McCrink, 2019; Prather, 2020; Rathé et al., 2019; Wijns et al., 2020). The present study examines students’ tendency of spontaneous focusing on multiplicative relations or SFOR tendency, which is defined as the extent to which an individual spontaneously focuses on multiplicative relations, such as twice as much or half as tall, across various situations. SFOR tendency does not refer to the spontaneous development of skill or knowledge, but rather the unguided recognition and use of quantitative relations in a situation that is not explicitly mathematical. SFOR tendency plays a crucial role in relational reasoning, a key aspect of school mathematics from arithmetic to algebra (Alexander,

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2016), and is linked to mathematical skills across various ages (McMullen et al., 2016; Van Hoof et al., 2016).

Previous research has mainly examined SFOR tendency's role in supporting students' mathematical development, especially rational numbers. However, SFOR tendency should also be considered an essential outcome of mathematics instruction, one that provides a crucial link between the formal mathematics classroom and mathematical reasoning in everyday life. Since it is apparent that spontaneous mathematical focusing tendencies are influenced by social influences, such as explicit instruction (Määttä et al., 2022; McMullen, Hannula-Sormunen, et al., 2019), and formal mathematical knowledge, such as rational number and algebra knowledge (McMullen et al., 2017), it is important to investigate how SFOR itself develops. So far, little is known about potential influences on individual differences in SFOR tendency itself. Understanding these differences will better inform how promoting SFOR tendency and other mathematical focusing tendencies can be integrated into mathematics instructional practices in primary and lower secondary schools.

Quantitative relations in and out of the classroom

In the early months of life, humans demonstrate an innate ability to recognize basic quantitative relations in sets and lengths of objects (Feigenson et al., 2002; McCrink & Wynn, 2007). As children develop, their relational skills expand to include proportions, like the composition of a juice mixture, laying a foundation for formal learning, including fractions (Boyer et al., 2008; Frydman & Bryant, 1988; Sophian, 2000; Spinillo & Bryant, 1991). However, despite these early relational skills, students often struggle with fractions and rational numbers in formal education (Steffe & Olive, 2010; Van Hoof et al., 2014). Bridging this gap necessitates extensive training (Van Dooren et al., 2015).

Research suggests that connecting everyday relational experiences to classroom mathematics facilitates learning about rational numbers (Confrey et al., 2009; Määttä et al., 2022; McMullen et al., 2016), emphasizing the bidirectional relationship between intuitive, everyday understanding and formal mathematical knowledge (Hannula & Lehtinen, 2005; McMullen & Resnick, 2018). The informal meaning of quantitative relations appears helpful in supporting students' learning of the formal meaning of mathematical relations such as rational numbers (Confrey et al., 2009; McMullen et al., 2016). However, explicit connections must be made in order to capitalize on the commonalities between "formal" analytic meanings of and the "everyday" representational meanings of quantitative relations in either direction (Lobato et al., 2003; Nunes & Bryant, 2015).

Indeed, prior research on SFOR tendency shows that instruction aimed at improving students' SFOR tendency appears to also support learning about formal mathematics, such as rational numbers (Määttä et al., 2022). This intervention aimed to improve students' ability to recognize and describe multiplicative relations by making these relations more salient targets of focusing in their daily lives. However, the added benefit of later learning of formal mathematics may not be the only value in promoting SFOR tendency. Instruction that aims to support students' use of mathematical knowledge in various situations, including everyday life, is also a valuable goal of mathematics education in and of itself (Baroody, 2003). This suggests that SFOR tendency may not only be important in supporting later mathematical development but may itself be an important outcome of mathematics instruction.

Focusing on quantitative relations or not

In everyday situations in which multiplicative relations can be found and are relevant, there are often many other stimuli that can be focused on. Thus, researchers design tasks that measure SFOR tendency to have multiple features that can be focused on besides the quantitative relations of interest (e.g., multiplicative relations). For example, in previous studies, respondents could solve tasks using exact number, proto-quantitative relations (e.g., more, less), color, shape, and so on (Degrande et al., 2017; McMullen et al., 2014; Van Hoof et al., 2016). Previous research suggests that alternative features may play an important role in dictating individual differences in SFOR responses (Prather, 2020).

However, there has been little systematic examination of how school-age children respond to tasks measuring SFOR tendency, besides using exact multiplicative or additive relations (Degrande et al., 2017).

In late primary school, SFOR tendency has been measured with picture description tasks (Degrande et al., 2017; McMullen et al., 2016), akin to other studies of spontaneous mathematical focusing tendencies (Batchelor et al., 2015; Rathé et al., 2019). The most ubiquitous of these is the teleportation task (McMullen et al., 2016), in which sets of objects change in various ways (e.g., color, shape, quality), including by a constant arithmetic tranSFORmation (e.g., multiplied by 3; added 2). These tasks have primarily involved multiplicative relations (e.g., *three times as many*, *half as much*; McMullen et al., 2016), in line with the focus of these examinations on the relation between SFOR tendency and rational number knowledge. However, these tasks have also been used with additive relations (e.g., *two more*, *three fewer*), with which individual differences were also uncovered (Degrande et al., 2017).

Although individual differences in performance on such picture description tasks are related to mathematical development, no study has systematically examined patterns in the actual mathematical aspects of the tasks that students focus on. Only explicit multiplicative (or additive) relational descriptions, such as "three times as many" or "half as much," have been considered indicators of SFOR tendency, as these were the most mathematically sophisticated ways to describe the relations embedded in these tasks (McMullen et al., 2016). However, the picture description tasks used to measure SFOR tendency involve a variety of mathematical features, including exact number, additive relations, and nonspecific relations (e.g., more, less). Previously, evidence has shown that the alternative responses to tasks measuring SFOR tendency have been informative for understanding potential developmental patterns in spontaneous focusing on quantitative relations (e.g., part-whole relations) among young children (McMullen et al., 2011). The present study expands this inquiry to examine such responses in older children focusing on the main mathematical targets of focusing used in previous research with these tasks: multiplicative relations (McMullen et al., 2016), additive relations (Degrande et al., 2017), nonspecific relations (Määttä et al., 2024), and exact number (McMullen et al., 2011).

The present study

The first aim of the present study is to examine individual differences in SFOR tendency in middle school students and determine what students focus on when they do not focus on specific multiplicative relations.

Thus, I asked: *What is the nature of individual differences in performance on measures of SFOR tendency in middle school children? (Research Question 1)* To answer this question, I examined SFOR response patterns in a sample of 7th-grade students, who should have well-developed knowledge and skills with multiplicative relations. Previous evidence suggests that, in primary school children, there are substantial individual differences in SFOR tendency on both behavioral and picture description tasks (McMullen et al., 2014, 2016), even after taking into account their ability to recognize and describe the relations presented in the tasks when explicitly guided to do so. Similar patterns of individual differences should also emerge in middle school children after students had extensive instruction with different aspects of multiplicative relations, such as rational numbers and proportional reasoning.

Also, I asked: *What alternative features of SFOR tendency tasks do middle school students pay attention to in their non-SFOR responses? (Research Question 2)* To answer this question, I examined how SFOR responses are related to spontaneous descriptions of nonspecific quantitative relations (e.g., more than, less than), exact numbers (e.g., nine cans of tuna), and qualitative features of tasks (e.g., blue cans). As in previous examinations of similar task tendency (Degrande et al., 2017; McMullen et al., 2014), I expect that describing multiplicative relations and exact number will conflict with each other and, thus, be negatively related to each other. Given the relational nature of both nonspecific quantitative relations and exact multiplicative relations, it was expected that nonspecific quantitative relations and SFOR responses would not be related to each other.

The second aim of the present study is to determine how other aspects of relational reasoning may be related to individual differences in SFOR tendency, as has been found in previous studies (e.g., Van Hoof et al., 2014). Therefore, I asked: *How does SFOR relate to other concurrent aspects of relational reasoning? (Research Question 3)* In particular, I aimed to examine the interrelation of different features of relational reasoning in middle school students and examine how they explain variance in SFOR tendency. Preliminary results of a meta-analysis examining the relation between spontaneous mathematical focusing tendencies and mathematical achievement indicate a small to moderate relation (Ouyang et al., in preparation). Thus, I examined how SFOR tendency was related to rational number knowledge, the ability to recognize and describe multiplicative relations when explicitly guided to do so (referred to as *multiplicative reasoning* from hereon), and proportional and additive word problem solving. Rational number knowledge is a central aspect of mathematical development and predicts the development of SFOR tendency in late primary school (Van Hoof et al., 2016). Thus, I expected rational number knowledge to be strongly related to SFOR tendency. In younger children with behavioral imitation tasks, there have been substantial individual differences in their tendency of spontaneous focusing on quantitative relations, even among those children with similar levels of the associated formal mathematical knowledge (McMullen et al., 2014). However, it is still necessary to have sufficient multiplicative reasoning ability to provide a SFOR response. Thus, I expected that multiplicative reasoning would be closely related to SFOR tendency. Finally, both word problem solving and SFOR tendency involve mathematical modeling of an (ostensively) everyday situation (Van Dooren et al., 2010). Thus, I expected SFOR tendency and word problem-solving to also be closely related.

Methods

Participants and procedure

Participants were 167 7^{th}-grade students from one school in the southeastern United States (83 Female, 68 Male; $Mean_{Age}$ = 13 years, 9 months; SD_{Age} = 6 months). All students from two science teachers' classes were invited to take part in the study. The school had a fairly diverse population; 51% of the students were White, 28% African-American, 11% Hispanic, and 5% Asian. Additionally, 43% of students at the school were eligible for free or reduced lunch. Students were enrolled in different mathematics classes ranging from Maths 2 to Algebra, but all students had completed at least 1 year of instruction that included topics of proportional reasoning, as well, all students had completed instruction on fractions and decimals over multiple years. All participants had parental permission to participate in the study and gave their own assent before participating. The ethical board of the University of Turku approved the study. Participants completed paper-and-pencil measures of SFOR tendency, multiplicative relational knowledge, additive and proportional word problems, and rational number knowledge in a single session, led by the author, in their regular science classrooms lasting one class period of 45 min. The SFOR and multiplicative relations tasks are expected to be fairly novel tasks for participants, while the word problems and rational number knowledge measures are expected to have been encountered in their instruction prior to measurement. All data and tasks are available on the project's OSF pages at: https://osf.io/srbzj/?view_only=8a6178e12b5a48e09874be3c277883cf

Measures

SFOR tendency

SFOR tendency was measured with two picture description tasks adapted from McMullen and colleagues (2016). These paper-and-pencil tasks were presented as the first tasks in the measurement, before any mention of mathematics, and completed in a context other than a mathematics classroom (i.e., science class). Students were asked to describe differences or predict changes in situations. In these situations, mathematical relations could be seen as highly relevant, but there was no explicit guidance to pay attention to any mathematical feature of the situation. Since students were not explicitly asked to pay attention to mathematics in these tasks, any use of mathematics can be described as spontaneous. Thus, when students' responses include exact multiplicative relations, they were considered to have provided a SFOR response. All items were presented one at a time on separate pages in paper booklets, with the pictures also projected onto a screen at the front of the classroom. The *teleportation task* has been used previously in multiple studies as a measure of SFOR tendency (McMullen et al., 2016; Van Hoof et al., 2016). Students are asked to describe, in as many ways as possible, exactly how sets of items have changed (see Figure 1(a)). Students have 2 min to complete each item. A time limit was necessary to ensure all students completed the tasks within the allotted lesson but was considered sufficient for students to complete the task successfully, without extreme time pressure. After this, a new item with the same initial material but in different quantities is shown, and the students are asked to "draw what you think will arrive on Olipula, based on what happened last time." Students were told they could look back to the previous page if they needed. Colored pencils were placed on their desks before they came into class and available for use. There were a total of four items on the teleportation task, two writing, and two drawing (Figure 1).

On the *lunch task*, students were shown a picture of two meals on separate plates. The meals varied in the types and amounts of food on each plate. Students were asked to "describe in as many ways as possible how the lunches are different from each other." On the next item, the first plate had the same or a similar type of food as in the previous trial. Students were asked to draw what they expected the second plate to have, based on the previous item. There were four lunch items in total, two writing, and two drawing items.

On the written description items, students' responses were coded at the level of descriptive phrases (e.g., "three times as many," "red"). Thus, each response (and even individual phrases) could be scored with multiple codes. Categories were based on previous research (Degrande et al., 2017) and included (a) explicit multiplicative relations, which involved a specific multiplicative numerical relation between two quantities (e.g., "three times as many," "half as much"); (b) nonspecific quantitative relations, which involved a quantitative relation between two quantities without a specific numerical value ("more than"; "less"); (c) exact number, which involved an exact numerical value not used in describing a relation ("nine cans"); and (d) additive relations, which involved a specific numerical relation between two quantities that was additive in nature (e.g., "three more"; "two fewer"). For the drawn items, the number of objects or sets that were drawn in line with a correct multiplicative

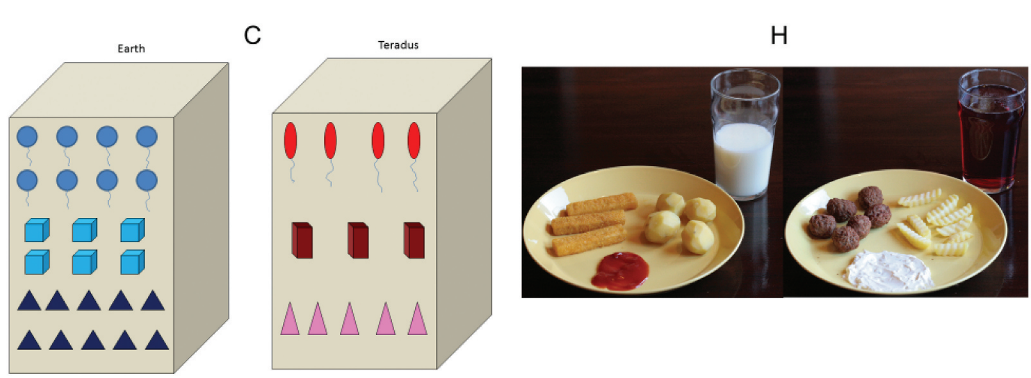

Figure 1. Example SFOR tendency task items. Letters were used instead of page numbers to avoid any numerical stimuli during testing.

relation was calculated (e.g., three times as many potatoes would be given one point). To confirm the coding scheme, the author and a trained research assistant independently scored 35 participants' responses. There was a high inter-rater reliability for these independent codings (Intraclass Correlation = .96; Koo & Li, 2016).

Multiplicative reasoning

In order to measure students' ability to recognize and describe multiplicative relations when explicitly guided to do so, students were again presented with the same items once all SFOR tasks were completed. This time, they were explicitly asked to describe or draw the multiplicative relations that were embedded in the item (e.g., "describe how the packages were divided"; "draw the number of packages that should arrive"). They had 1 min to complete each item. Since there were multiple objects or sets within each item, students were given one point for each correct multiplicative relation that was described or drawn.

Additive and proportional word problems

Students' ability to solve additive and proportional word problems was measured using an instrument derived from a previous measure by Van Dooren and colleagues (Van Dooren et al., 2010). Four word problems were presented to participants, with instructions to solve the problems. Two of them were additive (e.g., Linus and Peter are reading the same book. They read at the same speed, but Peter started before Linus. When Linus has read 4 pages, Peter has read 10 pages. When Linus has read 6 pages, how many has Peter read?), the other two were proportional (e.g., Erica and Tom are skipping rope. They started together, but Tom jumps slower. When Tom has jumped 4 times, Erica has jumped 20 times. When Tom has jumped 12 times, how many times has Erica jumped?).[1] For each item, we gave students one point for each correct response.

Rational number test

Participants' conceptual knowledge of the magnitude of rational numbers was measured with items targeting students' knowledge of fraction and decimal magnitudes. Rational number magnitude knowledge was assessed with magnitude comparison tasks previously found to be important features of rational number conceptual knowledge (Stafylidou & Vosniadou, 2004; Van Hoof, 2015). Students completed six fraction comparison items and six decimal comparison items, all of which involved comparisons that were inconsistent with whole number features (e.g., 5/9 compared with 5/7; 3.5407 compared with 3.65) and are thus have been previously argued to be valid indicators of rational number conceptual knowledge (e.g., Stafylidou & Vosniadou, 2004).

Results

Research question 1: individual differences in SFOR measure responses

Students' responses to the written and drawing items from the SFOR tasks were first examined separately. Generally, students had more drawn SFOR responses than written SFOR responses (Figure 2). However, students' SFOR drawn and written responses were reasonably consistent (Cronbach's alpha = .66; Greatest Lower Bound =.76) and thus are combined in the future analysis as a sum score that can be considered a measure of SFOR tendency.

In line with previous research (McMullen et al., 2016), SFOR tendency and Multiplicative Reasoning were correlated, r (167) = .57, $p < .001$ (Figure 3). Given the relatively high corrletions typically found between prior knowledge in mathematics ($r > .60$; Simonsmeier et al., 2021), this supports the notion that SFOR tendency and multiplicative reasoning are related, but separate

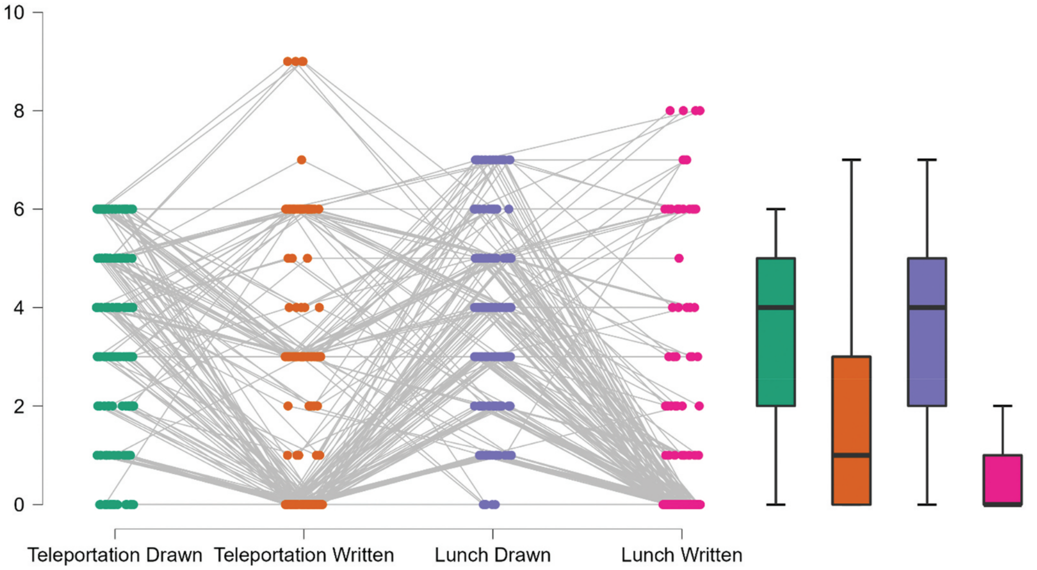

Figure 2. Raincloud and box plots of SFOR responses across item and response type.

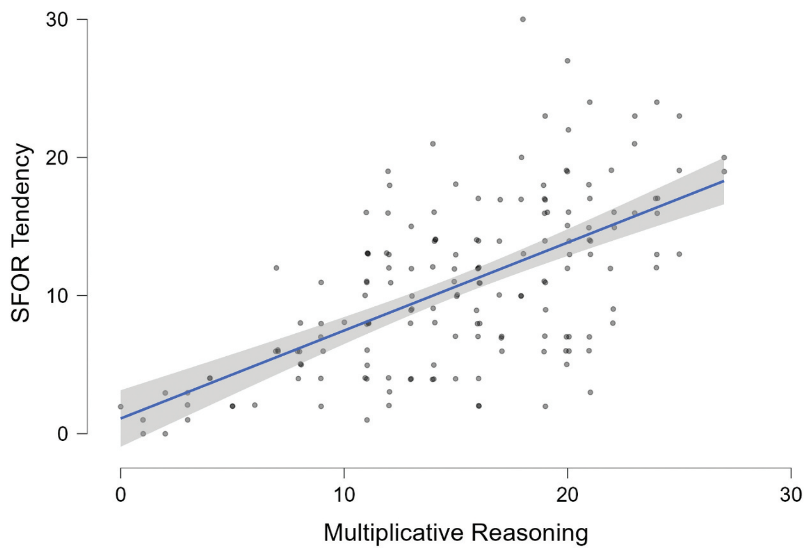

Figure 3. Scatterplot of multiplicative reasoning scores and SFOR tendency scores with regression line (and 95% confidence interval).

constructs. This determination can be seen from visual inspection of the scatter plot, as, for example, there are substantial individual differences in SFOR tendency even among those students who appear to have higher levels of Multiplicative Reasoning (e.g., <15 correct descriptions). This suggests that individual differences in SFOR tendency are not entirely explained by the ability to recognize and describe multiplicative relations when explicitly guided to do so. As well, there are few SFOR responses for those students with low levels of Multiplicative Reasoning, indicating that Multiplicative Reasoning is necessary but not sufficient for higher levels of SFOR tendency.

Figure 4 displays the raincloud plot of various responses to the written items of the SFOR tasks. As can be seen, there were substantial responses that included exact number (without a relation) and nonspecific quantitative relations (e.g. more, less). However, there were very few additive relational

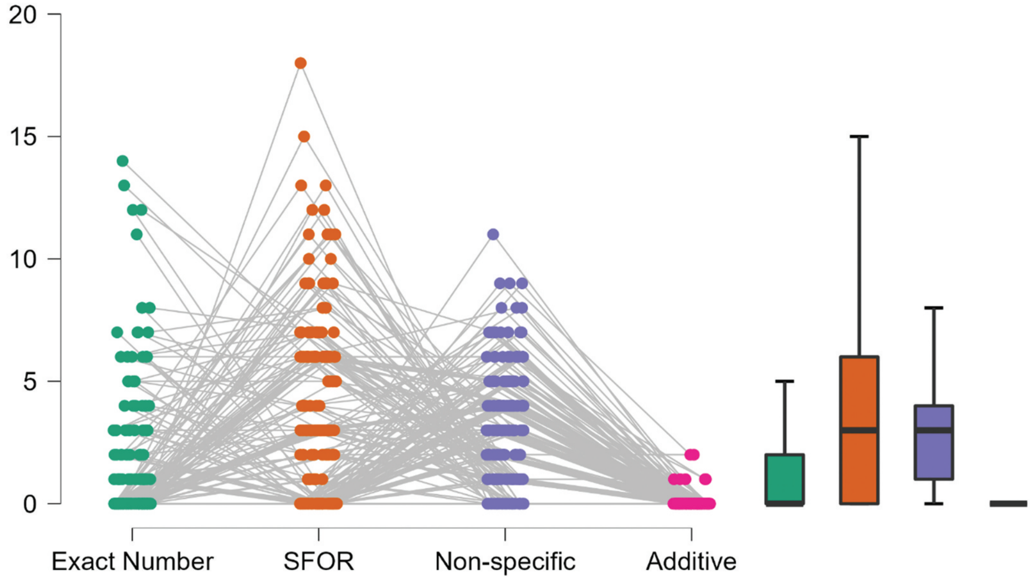

Figure 4. Raincloud and box plots of written response types on SFOR tasks (same scale used for both plots).

Table 1. Zero-order bivariate correlations between written response types on SFOR tasks.

Written Response Type	Mean	S.D.	1.	2.	3.	4.
1. SFOR	3.27	3.76	-			
2. Non-specific relation	3,09	2.34	−.37***	-		
3. Number	1.51	2.77	.02	−.12	-	
4. Additive Relation	0.05	0.08	−.01	.17*	.00	-

*$p < .05$, **$p < .01$, ***$p < .001$.

responses. Table 1 displays the correlation matrix for the response types, including qualitative responses. SFOR written responses were negatively related to nonspecific relational responses. However, there was no statistically significant relation between SFOR written responses and exact number or additive relational responses. There was a slight relation between nonspecific relational responses and additive relational responses. Overall, these results suggest that many students do spontaneously recognize the quantitative relations inherent in the SFOR tasks, especially the multiplicative relation, but not all spontaneously use exact multiplicative language to describe these relations. Given this, we consider the relation between other mathematical knowledge and both SFOR tendency and nonspecific relations in investigating research question two.

Research question 2: relation between SFOR tendency and mathematical knowledge

Basic correlational analyses indicate that all aspects of relational knowledge were related to each other (Table 2). Generally, there were positive relations between the relational reasoning variables. SFOR

Table 2. Zero-order bivariate correlations between measures.

Task	Mean	SD	1.	2.	3.	4.
1. SFOR Tendency	10.34	5.93	-			
2. Non-specific relation	3.09	2.34	−.21**	-		
3. Multiplicative Reasoning	14.68	5.80	.57***	−.23**	-	
4. Word problems	1.99	0.82	.19*	−.20*	.28***	-
5. Rational Number Magnitude	9.52	3.53	.47***	−.20**	.54***	.29***

*$p < .05$, **$p < .01$, ***$p < .001$.

Table 3. Regression analysis examining the relation of SFOR tendency to students' multiplicative reasoning, word problem solving, and rational number magnitude knowledge.

	SFOR tendency			
	β	95% CI	t	p
Multiplicative Reasoning	.42	[.27, .58]	5.41	<.001
Word problem solving	.01	[−.13, .14]	0.06	.95
Rational number magnitude knowledge	.24	[.08, .39]	3.05	.003
	F (3, 146) = 25.55, p = <.001, R^2 = .34			

tendency had a strong positive relation with rational number knowledge but only had a small positive relation with word problem-solving (e.g., Simonsmeier et al., 2021). However, there were small negative relations between nonspecific relational responses on the SFOR tasks and the other measures.

To examine how the different aspects of relational reasoning were uniquely related to SFOR tendency, a linear regression analysis was run with SFOR tendency as the dependent variable and multiplicative reasoning, word problem solving, and rational number magnitude knowledge as independent predictor variables. Overall, the model explained 34% of the variance in SFOR tendency (Table 3). Multiplicative reasoning was the strongest unique predictor of SFOR tendency, and rational number magnitude knowledge was also a significant unique predictor of SFOR tendency. However, additive and proportional word problem-solving did not uniquely predict SFOR tendency after accounting for multiplicative reasoning and rational number knowledge.

Discussion

These results extend previous research on the nature of individual differences in performance on SFOR tasks. As with younger children (McMullen et al., 2014), there were clear individual differences in middle school students' SFOR tendency. Despite most students being able to describe multiplicative relations in these tasks when explicitly guided to do so, there were substantial individual differences across the different SFOR tasks. Surprisingly, there appeared to be complex relations between SFOR tendency and the spontaneous description of other aspects of the teleportation and lunch tasks. It appears that those students who had a higher number of SFOR responses were less likely to use nonspecific quantitative relational language, such as "more" or "less," to describe the relations embedded in the written description items. However, there was no (positive or negative) relation between SFOR responses and the use of exact number (solely) or additive relations.

As in previous studies, differences in SFOR tendency appeared to be partially, but not entirely, related to individual differences in mathematical ability. Notably, many students with relatively low SFOR scores appeared to be able to describe the multiplicative relations embedded in the tasks when explicitly guided to do so. Likewise, only a minor relation between the ability to solve word problems and SFOR tendency was uncovered. In line with previous research, these results suggest that individual differences in SFOR tendency are not entirely dependent on the ability to recognize and describe the mathematics embedded in the task. Instead, the use of exact multiplicative relations on the SFOR tasks appears to capture the tendency to spontaneously recognize and describe multiplicative relations when not explicitly guided to do so.

Even though both SFOR tasks and word problems ostensibly require modeling of the world using mathematics (McMullen, Chan, et al., 2019), it appears that doing so spontaneously, without explicit guidance, is distinct from doing so in situations in which the mathematical context is made explicit even among middle school students. However, given the influence of reading competencies on word problem-solving, it may be possible that this relation is understated in the present study. Likewise, the common verbal components between the SFOR tasks and word problems may be driving the relation that is found in the present study.

The second main contribution of the present study is identifying spontaneous focusing on non-specific quantitative relations, such as *more* or *less*, as an important alternative response on the measures of SFOR tendency. These results are in contrast with previous studies of SFOR tendency, which found exact number to be the most common alternative to a SFOR response (McMullen et al., 2014). This result suggests that the importance of focusing on nonspecific quantitative relations as a possible alternative to SFOR. Previous research on SFOR or related concepts has either, (a) not differentiated from different types of quantitative relations (McMullen et al., 2014), (b) only considered multiplicative relations (Van Hoof et al., 2016), or (c) only examined exact arithmetic relations, either additive or multiplicative (Degrande et al., 2017). The present study suggests both developmental and educational implications of examining spontaneous focusing on nonspecific quantitative relations.

The level of mathematical precision may be important for providing benefits of SFOR tendency. In the present study, the relation between nonspecific relational responses on the SFOR tasks and rational number knowledge was much lower than that between the explicitly multiplicative responses (i.e., SFOR responses) and rational number knowledge. Previously, individual differences in SFOR tendency have been found to be important predictors of later mathematical development, with both rational numbers and algebra (McMullen et al., 2017; Van Hoof et al., 2016). It has been argued that rather than behaving as if mathematics is separate from their lived experiences, constrained to the mathematics classroom, students with a higher SFOR tendency appear more likely to use their mathematical knowledge in their everyday activities and reasoning (Lehtinen et al., 2017). Seeking out possibilities to apply and practice newly acquired knowledge is one characteristic of expert performers (Ericsson & Lehmann, 1996) and successful transfer of knowledge to new contexts may require the ability to recognize those situations in which existing knowledge is relevant (Lobato et al., 2003; McMullen & Resnick, 2018). The present study suggests that those students who more readily recognize that a specific multiplicative relation, such as three times as many, help describe a situation with as much precision as possible may be more likely to develop transferable and widely applicable knowledge.

The present study suggests there were important differences in the mathematical specificity of students' written relational responses on the SFOR tasks. Namely, some of the students who recognized the relational nature of the situation produced written descriptions that included exact multiplicative relations (e.g., three times as many, half), whereas other students only used nonspecific quantitative relations (e.g., more, less). Notably, most students could still describe exact multiplicative relations when explicitly guided to do so, suggesting that this was not an issue with not having the exact multiplicative language available. These results suggest that relational reasoning may be an important foundation for SFOR tendency (Alexander et al., 2020). It appears that many students already recognize the quantitative relations embedded in everyday situations, yet many fail to carry out mathematically explicit reasoning about these relations beyond nonspecific quantitative relations (such as more or less). This would be in line with previous research that suggests that, even when aware of the mathematical nature of a situation, it is not an automatic process to recognize and use the most relevant mathematical relations in everyday situations (Lave et al., 1984; Lobato et al., 2003). Harnessing the opportunities afforded by everyday situations to make relational judgments and encouraging students to be mathematically precise about the relations may be key to improving students' SFOR tendency (Määttä et al., 2021).

Limitations and future directions

Ultimately, the present study relies on students' spontaneous responses to contrived tasks that they completed in a typical, albeit non-mathematical, classroom situation. They do not represent students' own self-initiated activities in their daily lives, which limits the ability to truly capture their everyday use of mathematical relations. As well, the collection of data in science classes may have influenced students' response patterns. However, in contrast with many previous measures of spontaneous mathematical focusing tendencies (Batchelor et al., 2015; McMullen et al., 2016), the Lunch task at least involves everyday stimuli and context. Future studies could include more everyday situations as the context for picture descriptions, ideally involving more diversity in the situations, instead of only involving relational situations (i.e., comparison or changes). In the end, evidence of students' use of multiplicative relations in their everyday activities would be needed to best understand the role of SFOR tendency in mathematical development.

Additionally, the present study employed a top-down, research-lensed approach to examining the features of the SFOR tasks the participants focused on. These were based on prior studies of SFOR tendency and other spontaneous mathematical focusing tendencies (Degrande et al., 2017; McMullen et al., 2016). However, it is possible that more nuanced and broader analyses of students' responses would uncover further features of their spontaneous focusing on the tasks. Future studies may be wise to more deeply consider the specific targets of focusing, the nature of the relations described in responses (e.g., multiplicative, fractional, part-whole), and additional elements.

Conclusions

Children's own self-initiated mathematical actions, thoughts, and reasoning are central to the development and deployment of their mathematical knowledge (Ginsburg, 2006; Lehtinen et al., 2017). Spontaneous mathematical focusing tendencies and formal mathematical knowledge are in an iterative feedback loop that continuously leads each to support and develop the other (Hannula & Lehtinen, 2005; McMullen et al., 2017). The results of the present study extend these findings to show a more nuanced relation between formal mathematical knowledge and SFOR tendency. These results suggest that instead of considering spontaneous mathematical focusing tendencies as simply predictors or supporters of mathematical development, they should perhaps also be considered important in and of themselves as educational outcomes. More formal, classroom-driven knowledge, such as rational number knowledge, begets more use of this knowledge in situations outside of the classroom, an important goal of mathematics instruction. It may be a bonus that this unguided mathematical activity may also lead to more self-initiated practice with these existing mathematical skills, further supporting formal mathematical knowledge.

Note

1. The wording of the word problems appears to assume consistency in the subjects rates (e.g., number of pages read), but do not explicitly note this. However, the over 80% of participants' responses exactly match this assumption, therefore we treat this as a normative assumption in analysis.

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